Summer Intern Lectures

Time Series Analysis Team
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1 State Space Models

1.1 What is a state space model?

A state space model is a way to represent the dynamics of a set of variables. A models for observable variables $y_{i,t}$ can be thus described by a model of (possibly unobserved) $x_{i,t}$, i.e. the state variables. Any state space model can be rewritten as a system of an **observation equations**, relating the state space to the observation space, and a *transformation equation*, describing the dynamics of the state space.

In its simplest form, a state space model can be characterized by the following system:

$$y_t = G \cdot x_t \qquad \text{observation equation} \tag{1}$$

$$x_{t+1} = A \cdot x_t + C \cdot w_{t+1}, w_{t+1} \sim \mathcal{N}(0, I) \qquad \text{transition equation} \tag{2}$$

Note 1: this can be a higher-dimensional model as well, just make y_t a $k \times 1$ vector and x_t a $n \times 1$ vector.

Then what do we have:

- G is a $k \times n$ matrix known as the **output** matrix, relating state model to observable model.
- A is a $n \times n$ matrix known as the **transition** matrix, indicating the evolution of the state variable(s).
- C is a $\times m$ matrix known as the **volatility** matrix, indicating how uncertainty enters the model.
- w_{t+t1} is an $m \times 1$ vector of randomness.

  Note 2: The state and observation models can have different dimensions! This will be very useful for *Nowcasting* framework.

Example 1
The model for the variable y_t is:

$$y_{t+1} = \phi_0 + \phi_1 y_t + \phi_2 y_{t-1} \tag{3}$$

given y_0, y_{-1}. How do you convert this into a state space model?

Look at what variables enter the model. It's a constant term as well as y_y and y_{t-1}.

$$x_t := [1 \ \ y_t \ \ y_{t-1}]'.$$

Recall that the observation equation will be of the form $y_t = Gx_t$. Given our state vector x_t what is G?

$G = [0 \ \ 1 \ \ 0]$ works!

$$G \cdot x_t = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ y_t \\ y_{t-1} \end{pmatrix} = y_t.$$

What are the dynamics of the state variable? We want to be able to reproduce the initial model (3) by relating x_{t+1} and x_t as in the transition equation: $x_{t+1} = Ax_t + Cw_{t+1}$. What are A and C?

$$x_{t+1} = \begin{pmatrix} 1 \\ y_{t+1} \\ y_t \end{pmatrix} = A \begin{pmatrix} 1 \\ y_t \\ y_{t-1} \end{pmatrix}$$

Solve analytically here, or just eye-ball it! $A = \begin{pmatrix} 1 & 0 & 0 \\ \phi_1 & \phi_2 & \phi_3 \\ 0 & 1 & 0 \end{pmatrix}$ works!

Finally, what is C? There is no randomness, so it $C = \vec{0}$.

For you to do:

1) $y_{t+1} = \phi_1 y_t + \phi_2 y_{t-1} + \phi_3 y_{t-2} + \phi_4 y_{t-3} + \sigma w_{t+1}$ with $w_t \sim \mathcal{N}(0, 1)$.
2) As above but y_t is a $k \times 1$ vector (i.e. it represents k many variables at time t). ϕ_j is $k \times k$ matrix and w_t is $k \times 1$ vector. This is then a vector auto regression!