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1 Endogenous Grid Method

These notes are an adaptation of Josep Pijoan-Mas' notes with a few steps explained more in detail as well as the addition of some Julia code.

1.1 Simple consumption/savings model

Consider the dynamic programming problem

$$\begin{aligned} v(a,y) = \max_{a',n} \ & \{u(c,n) + \beta E_y \left[v(a',y')\right]\} \\ \text{s.t. } & a' + c \leq aR + nyw, \\ & a' \geq -b, \, c \geq 0. \end{aligned}$$

where the income process is a Markov chain, i.e. y' only depends on y and some stochastic factor. Substituting the (binding) budget constraint, gives us the FOCs are:

$a')$

$$-u_1(c,n) + \beta E_y \left[v_1(a',n')\right] = 0$$

$n)$

$$u_1(c,n)yw + u_2(c,n) = 0$$

and the Envelope condition is:

$a)$

$$v_1(a,y) = u_1(c,n)R.$$

Including the Envelope condition in the first FOC and assuming we can isolate n from the second FOC gives us:

$$u_1(c,n) = \beta R E_y \left[u_1(c',n')\right] \tag{1}$$

$$n = n(c,y) \tag{2}$$

1.2 Euler Equation Iteration

Here follow the steps for the Euler Equation iteration.

1. Guess a policy function for consumption $\boldsymbol{c = g_0^c(a, y)}$. This will be updated and will determine our convergence.

2. Substitute this rule on the right-hand side of (1) and get:

$$u_1(c) = \beta R E_y \left[u_1(g_0^c(a', y'), n') \right],$$

Assuming $u_1(\cdot)$ is invertible, invert the equation just derived in order to get consumption:

$$c = u_1^{-1}\left(\beta R E_y \left[u_1(g_0^c(a', y'), n') \right] \right) =: \tilde{g}_0^c(a', y)$$

Notice that $\tilde{g}_0^c(\cdot)$ depends on a' because, in the above expression those are known at at 0 and it depends on y because the expectation is taken over income and next period income only depends on current income.
This formulation tells us that consumption today is a function of assets tomorrow and income today. So, knowing y and a', we can pin down c.

3. If the asset grid is $A = \{a_1, \ldots a_m\}$ and the income process is discrete, i.e. $Y = \{y_1, \ldots, y_n\}$ and Markovian (as we are assuming) with transition matrix Γ, we can write

$$c = \tilde{g}_0^c(a_i, y_j) = u_1^{-1}\left(\beta R \sum_l \Gamma_{j,l} u_1(g_0^c(a_i, y_l), n') \right)$$

Notice further that A is the asset grid for tomorrow's assets.

Next, using the (binding) budget constraint we can deduce current assets given tomorrow's assets and current income, namely

$$a_i + c = a_{i,j} R + n y_j w$$
$$\Rightarrow a_{i,j}^* = \frac{a_i + c - n y_j w}{R}$$

If hours worked were exogenous we would be done, however, we can, making some separability assumptions, say as before that $n = n(c, y_j) = n(\tilde{g}_c^0(a_i, y_j), y_j) =: \tilde{g}_0^n(a_i, y_j)$ and so

$$a_{i,j}^* = \frac{a_i + \tilde{g}_c^0(a_i, y_j) - \tilde{g}_0^n(a_i, y_j) y_j w}{R} \tag{3}$$

Eq. (3) gives us a set of new asset grids $A_j^* = \{a_{1,j}^*, \ldots a_{n,j}^*\}$, one for each possible income state. Furthermore, we have consumption at each one of these asset positions, or in other words, the new policy function for the new asset grid is $g_c^1(a_{i,j}^*, y_j) = \tilde{g}_c^0(a_i, y_j)$.

4. Next we need to map the new policy function, defined on the grid A_j^*, onto the old asset grid A, i.e. we need to find $g_c^1(a_i, y_j)$. We do so as follows:

   (a) If $a_i \leq a_{1,j}^*$ then assets next period are very low which means that this period the agent will be at the borrowing constraint, i.e. $a_i = a_1$ and so, from the budget constraint we get:

   $$g_1^c(a_i, y_j) = a_i R + n y_j w - a_1 = a_i R + g_1^n(a_i, y_j) y_j w - a_1.$$

   (b) If $a_i > a_{1,j}^*$ on the other hand a_i must lie within some $\left[a_{k,j}^*, a_{k+1,j}^* \right]$ interval. Simply interpolate $g_1^c(a_{i,j}, y_j)$ from $g_1^c(a_{k,j}, y_j)$ and $g_1^c(a_{k+1,j}, y_j)$.

5. Finally, the updated policy function is:

$$g_c^1(a_i, y_j) = \begin{cases} a_i R + n y_j w - a_1 = a_i R + g_1^n(a_i, y_j) y_j w - a_1 & \text{if } a_i \leq a_{i,j}^* \\ interp\left(g_1^c(a_{k,j}, y_j), g_1^c(a_{k+1,j}, y_j) \right) & \text{if } a_i > a_{i,j}^* \end{cases} \tag{4}$$

2 Julia example

Here is the example...