Progression & Series

3. INTRODUCTION

A sequence is a function whose domain is the set N of natural numbers.

For example 4, 1, 3 is a sequence.

whereas **Series** is an expression achieved by adding or subtracting the terms in sequence.

i.e. if $a_1, a_2, a_3, ... a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 ... + a_n$ is a series.

A series is finite or infinite according as the number of terms in the corresponding sequence is finite or infinite.

It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some formula or its r^{th} term.

Progression are sequences whose terms follow certain patterns.

We will discuss some of the common progressions now.

3.1 Arithmetic Progression

An **arithmetic progression** is a sequence of numbers where each new term after the first is formed by adding a fixed amount called the **common difference** to the previous term in the sequence. For example the sequence 3, 5, 7, 9, 11 ... is an arithmetic progression. Note that having chosen the first term to be 3, each new term is found by adding 2 to the previous term, so the common difference is 2.

The common difference can be negative : for example the sequence 2, – 1, – 4, – 7, ... is an arithmetic progression with first term 2 and common difference – 3. In general we can write an arithmetic progression as follows :

arithmetic progression : $a, a + d, a + 2d, a + 3d, ...$

where the first term is a and the common difference is d. Some important results concerning arithmetic progressions (A.P.) now follow :

$$\boxed{\text{The } n^{th} \text{ term of an a.p. is given by : } a + (n - 1)d}$$

and in general

$$a_n = a_m + (n - m)d.$$

A finite portion of an arithmetic progression is called a **finite arithmetic progression** and sometimes just called an arithmetic progression.

$$\boxed{t_n - t_{n-1} \text{ is constant for all } n \in N. \text{ This constant } d = t_n - t_{n-1}}$$

n^{th} term from the end

If 'a' is the first term of an AP & 'd' is the common difference, then the m^{th} term from end is **$(n - m + 1)^{th}$ term** from the beginning. (where n is the no. of terms in AP).

Illustration 1

Show that the sequence defined by $a_n = 2n^2 + 1$ is not an A.P.

Solution :

We have $a_n = 2n^2 + 1$

replacing n by n + 1

$a_{n+1} = 2(n+1)^2 + 1$

According to the definition of AP the difference of two consecutive terms is always a constant.

So, if the sequence has to be in AP then the difference $a_{n+1} - a_n$ should be a constant.

Now,
$$\begin{aligned} a_{n+1} - a_n &= [2(n+1)^2 + 1] - (2n^2 + 1) \\ &= (2(n^2 + 2n + 1) + 1) - (2n^2 + 1) \\ &= 4n + 2 \end{aligned}$$

$\Rightarrow$ $(a_{n+1} - a_n)$ is not a constant, not independent of n so the given sequence is not an AP.

Illustration 2

Find the number of terms common to the two APs : 3, 7, 11, ... 407 and 2, 9, 16 ..., 709

Solution :

Let the no. of terms in two AP's be m and n.

Then nth term of I^{st} AP = 407 $\Rightarrow m = 3 + (m-1)4$

n^{th} term of 2^{nd} AP = 709 $\Rightarrow n = 2 + (n-1)7$

$\Rightarrow$ $m = 102$

& $n = 102$

So each AP consists of 102 terms.

Let p^{th} term of first AP be identical to q^{th} term of second AP.

Then
$$\begin{aligned} 3 + (p-1) &= 2 + (q-1)7 \\ \Rightarrow \quad 4p - 1 &= 7q - 5 \\ \Rightarrow \quad 4p + 4 &= 7a \\ \Rightarrow \quad 4(p+1) &= 7q \\ \Rightarrow \quad \frac{p+1}{7} &= \frac{q}{4} = K \\ \Rightarrow \quad p &= 7K - 1 \ \& \ q = 4K \end{aligned}$$

and we know that p has to be less than 102 & q also has to be less than 102

Since the total no. of terms are 102

for $p = 7K - 1 \le 102$

we get $7K \le 103$

$$K \le \frac{103}{7} \quad \Rightarrow \quad K \le 14\frac{5}{7} \qquad \text{... (i)}$$

from $q = 4K \le 102$

$\Rightarrow$ $K \le 25½$ ... (ii)

from (i) & (ii) we get

$$K \le 14 \Rightarrow K = 1, 2, 3, \ldots 14$$

3.1.3 Sum (the arithmetic series)

The sum of the numbers of a finite arithmetic progression is called an **arithmetic series.**

Express the arithmetic series in two different ways :

$$S_n = a_1 + (a_1 + d) + (a_1 + 2_d) + \ldots + (a_1 + (n-2)d) + (a_1 + (n-1)d)$$

$$S_n = (a_n - (n-1)d) + (a_n - (n-2)d) + \ldots + (a_n - 2d) + (a_n - d) + a_n.$$

Add both sides of the two equations. All terms involving d cancel, and so we're left with :

$$2S_n = n(a_1 + a_n).$$

Rearranging and remembering that $a_n = a_1 + (n-1)d$, we get

$$\boxed{S_n = \frac{n(a_1 + a_n)}{2} = \frac{n[2a_1 + (n-1)d]}{2}}$$

Guassian Trick

Let us write the sum of the natural numbers upto n in two ways as :

$$S_n = 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n$$

$$Sn = n + (n-1) + (n-2) + \ldots + 3 + 2 + 1$$

If we add these two series we get :

$$2S_n = (n+1) + (n+1) + \ldots + (n+1)$$

There are n of these (n + 1)'s, so

$$2S_n = n(n+1)$$

$$\boxed{S_n = n(n+1)/2}$$

The sum of the natural numbers from 1 to n is therefore half the product of the first term plus the last one multiplied by the number of terms.

More Properties of A.P. :

(1) If a fixed number is added (subtracted) to each term of a given AP then resulting sequence is also AP with same common difference as given AP.

$t_n = a + (n-1)d$, now let g be subtracted from each term.

$t_n = (a - g) + (n-1)d$. Therefore only the first term of AP has changed.

(2) If each term of an AP is multiplied (or divided) by fixed constant then resulting sequence is also an AP with common difference multiplied (or divided) by same constant.

(3) Sum and difference of corresponding terms of two AP's will form an AP

Let first AP be

$a_1, a_1 + d_1, a_1 + 2d_1, \ldots\ldots., a_1 + (n-1)d_1$

And second AP be

$a_2, a_2 + d_2, a_2 + 2d_2, \ldots\ldots\ldots, a_2 + (n-1)d_2$

So, $T_r = t_{1r} + t_{2r}$

$= (a_1 + (r-1)d_1 + (a_2 + (r-1)d_2)$

$= (a_1 + a_2) + (r-1)(d_1 + d_2)$

So, resulting sequence is A.P. with first term $a_1 + a_2$ and common difference $d_1 + d_2$.

(4) If we want to pick terms of an AP then convenient way of doing that is :

For three term's in AP we choose $a - d, a, a + d$.

For four terms in AP we choose $a - 3d, a - d, a + d, a + 3d$

Note : • for odd no. of terms the middle term is a & common difference is d.

• for even no. of terms the middle term is $a - d$ & $a + d$ whereas common difference is 2d.

5. In a finite AP the sum of terms equidistant from the beginning and end is always same & equal to sum of first & last term i.e.,

$a_k + a_{n-(k-1)} = a_1 + a_n$

for $k \in 1, 2, \ldots n-1$

6. $t_n = S_n - S_{n-1}$

7. If a, b, c are in AP $\Rightarrow 2b = a + c$ or $b - a = c - b$.

8. A sequence is in AP.

- iff its n^{th} term is a linear expression in n i.e. $a_n = An + B$ (where A & B are constant) & A is the common difference and
- iff sum of its n terms is of form $An^2 + Bn$, in such cases common difference is 2A.

Illustration 3

Find the number of terms in the series 20, $10\frac{1}{2}, 18\frac{2}{2}$ .... of which the sum is 300.

Solution :

Here we observe that a = 20, $d = -\frac{2}{3}$, and S_n = 300.

$$\text{So, } \frac{n}{2}\left(2 \times 20 + (n-1)\left(\frac{-2}{3}\right)\right) = 300$$

$$n^2 - 61n + 900 = 0$$

$$\Rightarrow \quad n = 25 \text{ or } 36.$$

Illustration 4

Find the sum of first 24 terms of the AP : a_1, a_2, a_3 ... if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.

Solution :

We know that in AP the sum of terms equidistant from beginning & end is equal & same to the sum of first & last term. i.e. $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2}$ & so on.

So here $\quad a_1 + a_{24} = a_5 + a_{20} = a_{10} + a_{15}$ ... (i)

Now $\quad a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$

$(a_1 + a_{24}) + (a_5 + a_{20}) + (a_{10} + a_{15}) = 225$

$\Rightarrow \quad 3(a_1 + a_{24}) = 225$ {from (i)}

$\Rightarrow \quad (a_1 + a_{24}) = 75$ ... (ii)

$$\therefore \quad S_{24} = \frac{24}{2}(a_1 + a_{24})$$

from (ii)

$$S_{24} = 12\,(75) = 900$$

Illustration 5

The interior angles of a polygon are in AP. The smallest angle is 120 and the common difference is 5 . Find the number of sides of polygon.

Solution :

Let the polygon be of 'n' sides

So for a polygon of n sides, the sum of all its interior angles is given by

$S_n = (n - 2) \quad 180$ (i) (try to solve this on your own).

Hence the interior angles form an AP with a = 120 & d = 5

$\Rightarrow$ $$S_n = \frac{n}{2}\left[2 \times 120 + (n-1)5\right] \quad \text{... (ii)}$$

equating (i) & (ii)

$$\frac{n}{2}\left[2 \times 120 + (n-1)5\right] = (n-2) \times 180$$

$$n\,(5n + 240 - 5) = (n - 2) \quad 360$$

$$n\,(5n + 235) = (n - 2) \quad 360$$

$$5n^2 + 235n = 360n - 720$$

$\Rightarrow$ $$5n^2 - 125n + 720 = 0$$

$\Rightarrow$ $$n^2 - 25n + 144 = 0$$

$$(n = 16) \text{ or } n = 9$$

But the question is not over yet.

We have to check whether the values are correct or not for n = 16.

The last angle is

$a_n = a + (n - 1)d \quad \Rightarrow \quad a_{16} = 120 + (16 - 1)5 = 190$

Which is not possible (since we are looking for interior angles only)

$\therefore$ The solution is n = 9

3.1.1 Arithmetic Mean (AM)

When three quantities are in AP, then the middle one is a arithmetic mean of other two.

If a and b are two numbers and A is arithmetic mean of a and b, then a, A, b are in AP.

$\Rightarrow$ $$A - a = b - A$$

$\Rightarrow$ $$A = \frac{a+b}{2}$$

So if a, b, c are 3 numbers in AP then b is the arithmetic mean of a & c.

$\Rightarrow$ $$b - a = c - b$$

$\Rightarrow$ $$b = \frac{a+c}{2}$$

3.1.1.1 Inserting n arithmetic means between two number

$A_1, A_2, A_3, \ldots\ldots\ldots A_n$ are called n arithmetic means between two numbers a and b,

if the series a, $A_1, A_2, A_3, A_4, \ldots\ldots, A_n$, b is an AP.

For this AP, first term is a, number of terms is (n + 2), & the lastterm = b = T_{n+2}.

Let d be the common difference of this AP, then

$\Rightarrow \quad T_{n+2} = b = a + \{(n + 2) - 1\}d$

$\Rightarrow \quad b = a + (n + 1)d$

$\Rightarrow \quad d = \frac{b-a}{n+1}$

$\Rightarrow \quad A_1 = a + d = a + \frac{b-a}{n+1}$ and $A_2 = a + 2d = a + 2\frac{b-a}{n+1}$

In general k^{th} arithmetic mean is $A_k = a + kd = a + k\frac{b-a}{n+1}$

Property of AM

- Sum of all AM's inserted between two numbers a & b is $\boxed{n\left(\frac{a+b}{\mathbf{2}}\right)}$

i.e. $\boxed{\sum_{i=1}^{n} A_i = n\left(\frac{a+b}{2}\right)}$

Proof : We know $A_i = a + i\frac{(b-a)}{n+1}$

So $\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} a + \sum_{i=1}^{n} i\frac{(b-a)}{n+1}$

$$= a\sum_{i=1}^{n} 1 + \left(\frac{b-a}{n+a}\right)\sum_{i=1}^{n} i$$

$$= a(n) + \left(\frac{b-a}{n+1}\right)(1 + 2 + 3 \ldots n)$$

$$= a(n) + \left(\frac{b-a}{n+1}\right)\left(\frac{n(n+1)}{2}\right)$$

$$= an = \frac{(b-a)n}{2}$$

$$= n\left(a + \frac{b-a}{2}\right) = \frac{n(a+b)}{2}$$

Illustration 6

The sum of 2 numbers is $\frac{13}{6}$. An even number of arithmetic means are inserted between them and their sum exceeds their number by 1. Find the number of means inserted.

Solution :

Let the 2 numbers be a & b.

then, $$a + b = \frac{13}{6} \qquad \text{... (i)}$$

Suppose $A_1, A_2 \ldots A_{2N}$ be 2n arithmetic means inserted between a & b.

Then $$A_1 + A_2 + \ldots A_{2n} = 2n\left(\frac{a+b}{2}\right)$$

($\because$ We know that sum of AM's is $\frac{n}{2}(a+b)$)

putting value from (i)

$$\Rightarrow \quad A_1 + A_2 \ldots A_{2n} = n\left(\frac{13}{6}\right) \qquad \text{... (ii)}$$

Also it is given in the question that

$$A_1 + A_2 \ldots A_{2n} = 2n + 1 \qquad \text{... (iii)}$$

equating (ii) & (iii)

$$n\left(\frac{13}{6}\right) = 2n + 1$$

$$\therefore \quad 13n = 12n + 6$$

$$\Rightarrow \quad n = 6$$

$\therefore$ 12 means are inserted between a & b.

3.2 Geometric Progression

A **geometric progression** is a sequence of numbers where each term after the first is found by multiplying the previous term by a fixed number called the **common ratio.** The sequence 1, 3, 9, 27, ... is a geometric progression with first term 1 and common ratio 3. The common ratio could be a fraction and it might be negative. In general we can write a geometric progression as follows :

geometric progression : a, ar, ar^2, ar^3

where the first term is a and the common ratio is r.

Some important results concerning geometric progressions (G.P.) now follows :

The n^{th} term of a g.p. is given by ar^{n-1}.

n^{th} term from the end

The n^{th} term from the end of a finite G.P. consisting of m terms is ar^{m-n}, where a is the first term and r is the common ratio.

Note : If we are given the last term of G.P, say l, then the n^{th} term from end is given by $l\left(\frac{1}{r}\right)^{n-1}$

Illustration 7

If a, b, c **are in GP, then prove** $\frac{a^2+ab+b^2}{bc+ca+ab}=\frac{b+a}{c+b}$

Solution :

If a, b, c are in GP, then

$$b = ar$$

& $$c = ar^2$$

Now, putting these values in LHS

L.H.S

$$= \frac{a^2 + a(ar) + (ar)^2}{(ar)(ar^2) + (ar^2)(a) + a(ar)}$$

$$= \frac{a^2 + a^2r + a^2r^2}{a^2r^3 + a^2r^2 + a^2r}$$

$$= \frac{a^2(1 + r + r^2)}{a^2r(r^2 + r + 1)} = \frac{1}{r}$$

R.H.S.

$$= \frac{b+a}{c+b} = \frac{ar + a}{ar^2 + ar} = \frac{a(r+1)}{ar(r+1)}$$

$$= \frac{1}{r}$$

L.H.S. = R.H.S.

Hence proved

Illustration 8

In a finite GP the product of terms equidistant from the beginning and the end is always same and equal to the product of first and last term.

Solution :

Let $a_1, a_2 \ldots a_{n-1}, a_n$ be a finite GP with common ratio r.

$$k^{th} \text{ term from beginning } = a_1 r^{k-1}$$

$$k^{th} \text{ term from end } = a_{n-k+1}$$

$$= a_n \left(\frac{1}{r}\right)^{k-1}$$

multiplying both the terms

$$a_k \cdot a_{n-k+1} = a_1 a_n \left(r^{k+1}\right)\left(\frac{1}{r}\right)^{k-1} = a_1 a_n$$

Hence proved.

Note : This can be taken as a standard result.

Illustration 9

If $a^x = b^y = c^z$ and x, y, z are in GP prove that $\log_b a = \log_c b$.

Solution :

Given that $a^x = b^y = c^2$

taking $\log x \log a = y \log b = 2 \log c$ ... (i)

also x, y, z are in GP

then $y = xr$ & $z = xr^2$ (where 'r' is the common ratio)

from (i)

$x \log a = y \log b$

$\Rightarrow \quad r = \dfrac{\log a}{\log b}$ ... (ii)

$y \log b = 2 \log c$

$\Rightarrow \quad xr \log b = xr^2 \log c$

$\Rightarrow \quad r = \dfrac{\log b}{\log c}$ ...(iii)

from (ii) & (iii)

$$\frac{\log a}{\log b} = \frac{\log b}{\log c}$$

$\Rightarrow \quad \log_b a = \log_c b$

$$\left\{\because \frac{\log a}{\log b} = \log_b a\right\}$$

Sum (Geometric Series)

The sum of first n terms of a GP :

$$S_n = \begin{cases} \dfrac{a\left(r^n - 1\right)}{r - 1} & \text{if } r \neq 1 \\ na & \text{if } r = 1 \end{cases}$$

Case 1 : Suppose $r = 1$

$$S_n = a + a + a + \ldots\ldots + a = na$$

Case 2 : If $r \neq 1$

$$S_n = a + ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots ar^{n-2} + ar^{n-1} + ar^n \quad \ldots \text{(i)}$$

$$rS_n = ar + ar^2 + \ldots\ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n \quad \ldots \text{(ii)}$$

So, now (1) – (2) gives

$$S_n - rS_n = ar^n$$

$$S_n(1 - r) = a(1 - r^n)$$

$$\Rightarrow \quad S_n = \frac{a\left(r^n - 1\right)}{(r - 1)}$$

Special case when $n \to \infty$ (i.e. infinite number of terms)

If $|r| < 1$ and $n \to \infty$ then $\left(\lim\limits_{n\to\infty} S_n\right) = \dfrac{a}{1 - r}$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

Now $$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a}{1 - r} - \lim_{n\to\infty} \frac{ar^n}{1 - r}$$

$$= \frac{a}{1 - r} - 0 \quad \left(\text{as } \lim_{n\to\infty} r^n = 0\right) = \frac{a}{1 - r}$$

3.2.2 Properties of G.P. :

(1) If each term of a G.P. is multiplied (or divided) by some non-zero quantity the resulting progression is G.P. with some common ratio.

Suppose the G.P. is with $t_n = ar^{n-1}$

Now this GP is multiplied by some k ($\neq 0$)

So G.P. will be

$t_n = (ak)r^{n-1}$

So, only the first term of G.P. has changed and the common ratio remains unaffected.

(2) If $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be two G.P's of common ratio r_1 and r_2 respectively, then $a_1b_1, a_2b_2, \ldots$ and $\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \ldots$ will also form G.P. common ratio will be r_1r_2 and $\frac{r_1}{r_2}$ respectively.

Let the series a_1, a_2 .... have the n^{th} term as $a_1r_1^{n-1}$ and the series b_1, b_2 .... have nth term as $b_1r_2^{n-1}$

So the series a_1b_1, a_2b_2 .... will have n^{th} term as

$(a_1r_1^{n-1})(b_1r_2^{n-1})$

$= (a_1b_1)(r_1r_2)^{n-1}$

So the common ratio now becomes r_1r_2.

(3) If we have to take three terms in G.P. we take them as $\frac{a}{r}, a, ar$ with common ratio r and four terms as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ with common ratio r^2.

(4) If a, b, c are in G.P then $b^2 = ac$

(5) If a_1, a_2, a_3 .... a_n is a G.P. then $\log a_1, \log a_2$ ...... $\log a_n$ is an AP (provided that all terms are non-zero & non-negative).

Why ?

Now a_1, a_2, a_3 ...... a_n form a G.P.

So $a_1 = a_1r^{i-1}$ (Let)

$\Rightarrow \quad \log a_i = \log a_1 + (i-1) \log r$

This is clearly term of an AP

(1) How does $\log a_1 + (i-1) \log r$ represents an AP?

Ans. Let $\log a_1 = A$

And $\log r = D$

So, $\log a_1 + (i-1)\log r = A + (i-1) D$

So, $A + (i-1) D$ is term of an AP with first term as A and common difference as D.

6. In a finite GP the product of the terms equidistant from the beginning and the end is always same and is equal to product of first & last terms.

   i.e. $a_k\, a_{n-k+1} = a_1\, a_n$ for $k = 1, 2, 3 \ldots n-1$

7. The reciprocals of the terms of a given GP, form a GP with common ratio 1/r.

   i.e. if $a_1, a_2, a_3 \ldots$ an are in GP with common ratio r then $\frac{1}{a_1}, \frac{1}{a_2}, \cdots \frac{1}{a_n}$ are in GP with common ratio $\frac{1}{r}$.

Illustration 10

Find the sum of the following series :

5 + 55 + 555 + ... n terms

Solution :

These kind of questions can be solved with a little trick after which we get the result through GP.

We have 5 + 55 + 555 + ... n terms

$$= 5\,[1 + 11 + 111 + \ldots\ldots]\text{ n terms}$$

$$= \frac{5}{9}\left[9 + 99 + 999 + \ldots\right]\text{ n terms}$$

$$= \frac{5}{9}\left[(10-1) + (100-1) + (1000-1)\ldots.\right]$$

$$= \frac{5}{9}\left[\underbrace{(10 + 100 + 1000 \ldots \text{n terms})}_{\substack{\text{This is in GP now}\\ \text{with } a = 10\ \& \\ r = 10}} - \underbrace{n\quad 1}_{\text{n times}}\right]$$

$\therefore$ sum of series

$$= \frac{5}{9}\left[\frac{10\left(10^n - 1\right)}{(10-1)} - n\right]$$

$$= \frac{5}{9}\left[\frac{10}{9}\left(10^n - 1\right) - n\right]$$

$$= \frac{5}{81}\left[10^{n+1} - 9n - 10\right]$$

Illustration 11

If S_n is the sum of first n terms of a G.P. : (a_n) and S_n is the sum of another G.P. : $(1/a_n)$, then show that : $S_n = S'_n\ a_1\ a_n$.

Solution :

$$S_n = a_1 + a_2 + a_3 + \ldots\ldots\ldots + a_n$$

$$S'_n = \frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots\ldots + \frac{1}{a_n}$$

For the first GP $a_n = a_1\, r^{n-1}$

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r}$$

where r is the common ratio.

For the second GP $\left(\frac{1}{a_n}\right)$, common ratio $= \frac{1}{r}$

$$S'_n = \frac{1}{a_1}\frac{\left(1-\frac{1}{r^n}\right)}{\left(1-\frac{1}{r}\right)} = \frac{\left(r^n-1\right)}{a_1\left(r-1\right)r^{n-1}} = \frac{r^n-1}{a_n\left(r-1\right)}$$

$$\Rightarrow \qquad S'_n = \frac{1}{a_1\, a_n}\ \frac{a_1\left(r^n-1\right)}{r-1}$$

$$\Rightarrow \qquad S'_n = \frac{1}{a_1\, a_n} S_n$$

$$\Rightarrow \qquad S'_n = \frac{1}{a_1\, a_n} S_n$$

$$\Rightarrow \qquad S_n = S'_n\ a_1 a_n$$

Illustration 12

If S_1, S_2 .... S_n are the sums of infinite geometric series whose first terms are 1, 2, 3,... n and common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots \frac{1}{n+1}$ resp., then prove $S_1 + S_2 + \ldots S_n = \frac{1}{2} n\left(n+3\right)$.

Solution :

We know the formula for an infinite series $S_\infty = \left(\frac{a}{1-r}\right)$ where a is the first term & r is common ratio.

Here let S_k = Sum of infinite series with first term

$$a = k \text{ and CR } = \frac{1}{k+1}$$

$$\therefore \qquad S_k = \frac{k}{1-\frac{1}{k+1}} = \frac{k}{\frac{(k+1)-1}{k+1}}$$

$$\Rightarrow \qquad S_k = k + 1$$

$$\therefore \qquad \text{Putting } k = 1, 2, \ldots\ldots n$$

$$\sum S = S_1 + S_2 + S_3 \ldots\ldots. S_n$$

$$= (1 + 1) + (2 + 1) + (3 + 1) \ldots (n + 1)$$

$$= (1 + 2 + 3 \ldots\ldots n) + n \quad 1$$

$$= \frac{n(n+1)}{2} + n \qquad \{\because 1 + 2 + 3 \ldots\ldots n \text{ is in AP with diff.} = 1\}$$

$$= \frac{n(n+3)}{2}$$

You can also learn this, sum of first n natural nos. = $\frac{n(n+1)}{2}$

We will cover this later though

3.2.1 Geometric Means (G.M.)

If G_1, G_2, G_3, G_4 .... G_n are GM's between a and b then a, G1, G_2, ... G_n, b are in G.P.

Now b is t_{n+2} so $b = ar^{n+1}$

Or $$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

and Thus $$G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Property of GM

(1) The product of n G.M's between a and b is equal to n^{th} power of one G.M. between a and b.

i.e. $$G_1, G_2, \ldots\ldots\ G_n = \left(\sqrt{ab}\right)^n$$

$$G_i = a\left(\frac{b}{a}\right)^{\frac{i}{n+1}}$$

$$\therefore \quad G_1, G_2, \ldots\ldots\ G_n = a^n \sum_{i=1}^{n} \left(\frac{b}{a}\right)^{\frac{i}{n+1}}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{\Sigma i}{n+1}}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{n\cancel{(n+1)}}{2\cancel{(n+1)}}}$$

$$= (ab)^{\frac{n}{2}} = \left(\sqrt{ab}\right)^n$$

(2) If a_1, a_2, ... a_n are n non-zero numbers then their G.M is given by

$$G = \left(a_1, a_2 \ldots\ldots\ a_n\right)^{\frac{1}{n}}$$

Note : You can check the analogy of formulas between Arithmetic Progression & Geometric progression.

Illustration 13

If we insert odd number (2n + 1) G.M.'s between 4 and 2916 then find the value of (n + 1)th G.M.?

Solution :

Now 4, G_1, G_2, G_3 .... G_{n+1}, ... G_{2n}, G_{2n+1}, 2916 are in G.P. So G_{n+1} will be the middle mean of (2n + 1) odd means and so it will equidistant from 1st and last term.

So, 4, G_{n+1}, 2916 are also in GP.

And thus,

$$G_{n+1} = (4.2916)^{\frac{1}{2}}$$

$$= (4 \times 9 \times 324)^{\frac{1}{2}}$$

$$= (4 \times 9 \times 4 \times 81)^{\frac{1}{2}}$$

$$= (2 \quad 3 \quad 2 \quad 9)$$

$$= 108$$

Arithmetic Means and Geometric Means (in general)

The arithmetic mean, or less precisely the average, of a list of n numbers $x_1, x_2, ..., x_n$ is the sum of the numbers divided by n :

$$\frac{x_1 + x_2 + ... + x_n}{n}$$

The geometric mean is similar, except that it is only defined for a list of nonnegative real numbers, and uses multiplication and a root in place of addition and division.

$$\sqrt[n]{x_1 . x_2 ... x_n}$$

If $x_1, x_2, ... x_n > 0$, this is equal to the exponential of the arithmetic mean of the natural logarithms of the numbers :

$$\exp\left(\frac{\ln x_1 + \ln x_2 + ... + \ln x_n}{n}\right)$$

3.3.1 The AN-GM Inequality

Restating the inequality using mathematical notation, we have that for any list of n nonnegative real numbers $x_1, x_2, ... x_n$,

$$\boxed{\frac{x_1 + x_2 + ... + x_n}{n} \geq \sqrt[n]{x_1 . x_2 ... x_n}} \text{(known as AM-GM inequality)}$$

$\Rightarrow$ AM $\geq$ GM

and that equality holds if and only if $x_1 = x_2 = ... x_n$.

3.3.2 Genralizations

There is a similar inequality for the weighted arithmetic mean and weighted geometric mean. Specially, let the nonnegative numbers x_1, x_2, ..., x_n and the nonnegative weights a_1, a_2, ..., a_n be given. Set $\alpha = \alpha_1 + \alpha_2 + ... + \alpha_n$. If $\alpha > 0$, then the inequality

$$\frac{\alpha_1 x_1 + \alpha_2 x_2 + ... + a_n x_n}{\alpha} \geq \sqrt[\alpha]{x_1^{\alpha_1}\, x_2^{\alpha_2} ...\, x_n^{a_n}}$$

holds with equality if and only if all the x_k with ak > 0 are equal. Here the convention 0 = 1 is used. If all $a_k = 1$, this reduces to the above AM-GM inequality.

Example application

Illustration 14

Consider the following function :

$$f(x, y, z) = \frac{x}{y} + \sqrt{\frac{y}{z}} + \sqrt[3]{\frac{z}{x}}$$

for x, y **and** z **all positive real numbers. Suppose we wish to find the minimum value of this function. Rewriting a bit, and applying the AM-GM inequality, we have :**

$$f(x, y, z) = 6.\frac{\frac{x}{y} + \frac{1}{2}\sqrt{\frac{y}{z}} + \frac{1}{2}\sqrt{\frac{y}{z}} + \frac{1}{3}\sqrt[3]{\frac{z}{x}} + \frac{1}{3}\sqrt[3]{\frac{z}{x}} + \frac{1}{3}\sqrt[3]{\frac{z}{x}}}{6}$$

$$\geq 6\sqrt[6]{\frac{x}{y}\cdot\frac{1}{2}\sqrt{\frac{y}{z}}\,.+\frac{1}{2}\sqrt{\frac{y}{z}}\cdot\frac{1}{3}\sqrt[3]{\frac{z}{x}}\cdot\frac{1}{3}\sqrt[3]{\frac{z}{x}}\cdot\frac{1}{3}\sqrt[3]{\frac{z}{x}}}$$

$$= 6.\sqrt[6]{\frac{1}{2.2.3.3.3}\frac{xyz}{yzx}}$$

$$= 2^{2/3}\,.\,3^{1/2}$$

Further, we know that the two sides are equal exactly when all the terms of the mean are equal :

$$f(x, y, z) = 2^{2/3}.3^{1/2} \text{ when } \frac{x}{y} = \frac{1}{2}\sqrt{\frac{y}{z}} = \frac{1}{3}\sqrt[3]{\frac{z}{x}}$$

Illustration 15

If A and G are arithmetic mean (AM) and geometric mean (GM) between two numbers a and b find the roots of the equation : $x^2 - 2Ax + G^2$

Solution :

Let α and β be the roots of the equation, then

$$\alpha + \beta = 2A \ \& \ \alpha\beta = G^2$$

Also, A is the AM between a & b and G is GM between a & b

$\Rightarrow \quad \boxed{A = \frac{a+b}{2} \ \& \ G = \sqrt{ab}}$

$\Rightarrow \quad \alpha + \beta = a + b$ and $\alpha\beta = ab$

$\Rightarrow$ The roots are a & b.

Illustration 16

If a, b, c are in A.P., x is the GM of a, b and y is GM of b, c, show that b^2 is the AM of x^2 and y^2.

Solution :

a, b, c are in AP $\Rightarrow$ $2b = a + c$ (i)

x is G.M. of a, b $\Rightarrow$ $x = \sqrt{ab}$ (ii)

y is G.M. of b, c $\Rightarrow$ $y = \sqrt{bc}$ (iii)

Squaring (ii) & (iii) and adding, we get

$\Rightarrow \quad x^2 + y^2 = ab + bc = b(a + c)$

From (i) $\quad a + c = 2b$

$\Rightarrow \quad x^2 + y^2 = 2b^2$

$\Rightarrow \quad b^2 = \frac{x^2 + y^2}{2}$

Hence b^2 is arithmetic mean (AM) of x^2 and y^2.

Illustration 17

If $0 < r < 1$ and m is a positive integer, show that $(2m + 1)r^{2m} (1 - r) < 1 - r^{2m + 1}$

Solution :

Using AM > GM, we have

$$\frac{1 + r + r^2 + ... + r^{2m}}{2m+1} > \left(1.r.r^2 .....r^{2m}\right)^{\frac{1}{2m-1}}$$

or $\quad \frac{1 - r2^{m+1}}{(1-r)(2m+1)} > \left(1.r.r^2\right)^{\frac{1}{2m+1}}$

or $\quad \frac{1 - r^{2m+1}}{(1-r)(2m+1)} > r^{\frac{2m(2m+1)}{2m+1}}$ or $1 - r^{2m+1} > (1-r)(2m+1)\, r^{2m}$

3.4 Harmonic Progression (H.P.) :

The sequence a_1, a_2, ... a_n is said to be a H.P. if $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \ldots\ldots \frac{1}{a_n}$ is an A.P.

The n^{th} term of a H.P. (t_n) is given by $t_n = \frac{1}{a + (n-1)d}$ where $a = \frac{1}{a_1}$ and $d = \frac{1}{a_2} - \frac{1}{a_1}$

3.4.1 Harmonic Means (H.M.) :

If H_1, H_2, H_3 .... H_n be n H.M.'s between a and b then a, H_1, H_2, H_3 .... H_n, b is a H.P.

This means $\frac{1}{a}, \frac{1}{H}, \frac{1}{H_2} \ldots\ldots \frac{1}{H_n}, \frac{1}{b}$ is a A.P.

And hence $\frac{1}{H_i} = \frac{1}{a} + \frac{i(a-b)}{(n+1)ab}$

Note : If a_1, a_2, a_3 .... an are n non-zero numbers then H.M. of these number is given by

$$H = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots\ldots + \frac{1}{a_n}}$$

a, b, c are in H.P. so, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP

And $\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow \boxed{\frac{2ac}{a+c} = b}$

Illustration 18

If the $(m + 1)^{th}$, $(n + 1)^{th}$ and $(r + 1)^{th}$ term of an A.P. are in G.P. m, n, r are in H.P., Show that ratio of the common difference to the first term in the AP is $(-2/n)$.

Solution :

Let 'a' be the first term and 'd' be common difference of the AP. Let x, y, z be the $(m + 1)^{th}$, $(n + 1)^{th}$ and $(r + 1)^{th}$ term of the A.P. then x = a + md, y = a + nd, z = a + rd. Since x, y, z are in G.P.

$\therefore \quad y^2 = xz$ i.e. $(a + nd)^2 = (a + rd)(a + md)$

$$\therefore \quad \frac{d}{a} = \frac{r + m - 2n}{n^2 - rm}$$

Now, m, n, r in H.P.

$$\Rightarrow \quad \frac{2}{n} = \frac{1}{m} + \frac{1}{r}$$

$$\Rightarrow \qquad \frac{2}{n} = \frac{m+r}{mr}$$

Hence $$\frac{d}{a} = \frac{2\left(\frac{r+m}{2} - n\right)}{n\left(n - \frac{rm}{n}\right)}$$

$$\frac{2\left(\frac{mr}{n} - n\right)}{n\left(n - \frac{rm}{n}\right)} = -\frac{2}{n}$$

Illustration 19

If a**,** b**,** c **are respectively** p^{th}**,** q^{th}**, and** r^{th} **terms of H.P., prove that**

$(q - r) + ca\ (r - p) + ab\ (p - q) = 0$

Solution :

A and D be the first term and common difference of the AP formed by the reciprocals of the given H.P.

$$\frac{1}{a} = A + (p-1)D \qquad \text{(i)}$$

$$\frac{1}{b} = A + (q-1)D \qquad \text{(ii)}$$

$$\frac{1}{c} = A + (r-1)D \qquad \text{(iii)}$$

Subtracting (iii) from (ii) we get $\frac{c-b}{bc} = (q-r)\,D$

$$\Rightarrow \quad bc\,(q-r) = -\frac{(b-c)}{D}$$

$$\text{L.H.S.} = \sum bc(q-r)$$

$$= -\sum \frac{b-c}{D} = \frac{1}{D}\sum(b-c)$$

$$= \frac{1}{D}[b - c + c - a + a - b] = 0 = \text{R.H.S.}$$

Illustration 20

If $\frac{1}{a}+\frac{1}{c}+\frac{1}{a-b}+\frac{1}{c-b}=0$**, prove that** a, b, c **are in H.P., unless** b = a + c

Solution :

$$\frac{1}{a}+\frac{1}{c}+\frac{1}{a-b}+\frac{1}{c-b}=0$$

$$\Rightarrow \quad \frac{a+c}{ac}+\frac{a+c-2b}{ac-b(a+c)+b^2}=0$$

Let $a + c = t$

$$\Rightarrow \quad \frac{t}{ac}+\frac{t-2b}{ac-bt+b^2}=0$$

$$\Rightarrow \quad act - bt^2 + b^2t + act - 2abc = 0$$

$$\Rightarrow \quad bt^2 - b^2t - 2act + 2bac = 0$$

$$\Rightarrow \quad bt\,(t-b) - 2ac\,(t-b) = 0$$

$$\Rightarrow \quad (t-b)\,(bt-2ac) = 0$$

$$\Rightarrow \quad t = b \quad \text{or} \quad bt = 2ac$$

$$\Rightarrow \quad a + c = b \quad \text{or} \quad b\,(a+c)\,2ac$$

$$\Rightarrow \quad a + c = b \quad \text{or} \quad b = \frac{2ac}{a+c}$$

$\Rightarrow$ a, b, c are in H.P. or $a + c = b$

3.4.2 Some Important Theorems :

Let A, G,H be the AM, GM, HM respectively between two positive unequal quantities.
First of all let us prove A>G

$$A = \frac{x+y}{2} \text{ and } G = \sqrt{xy}$$

So to prove $\quad \frac{x+y}{2} > \sqrt{xy}$

Now $\quad \left(\sqrt{x}-\sqrt{y}\right)^2 > 0 \;(\because\; x, y \text{ are positive})$

$$\Rightarrow \quad x + y - 2\sqrt{xy} > 0$$

$$\Rightarrow \quad \frac{x+y}{2} > \sqrt{xy}$$

Hence $\quad \boxed{\mathbf{A > G}} \quad$ .... (i)

Now let us prove G>H

Again $$H = \frac{2xy}{x+y}$$

Also $$\left(\sqrt{x} - y\right)^2 > 0$$

$\Rightarrow$ $$x + y > 2\sqrt{xy}$$

$\Rightarrow$ $$1 > \frac{2\sqrt{xy}}{x+y}$$

$\Rightarrow$ $$G > H \qquad \text{.......... (2)}$$

Combining (1) and (2) we get $\boxed{\mathbf{A > G > H}}$

Now we can prove :

$$\boxed{\mathbf{G^2 = AH}}$$

Let x, y be two numbers.

So, $$A = \frac{x+y}{2}, G = \sqrt{xy}, H = \frac{2xy}{x+y}$$

Hence $$AH = \left(\frac{x+y}{2}\right)\left(\frac{2xy}{x+y}\right)$$
$$= xy$$
$$= G^2$$

Illustration 21

If a**,** b**,** c**,** d **be four distinct positive quantities in H.P. then show that** $a + d > b + c$.

Solution :

a, b, c, d are in H.P.

Then AM > H.M.

For first three terms

$\therefore$ $$\frac{a+c}{2} > b$$

$\Rightarrow$ $$a + c > 2b \qquad \text{...... (1)}$$

And for last three terms

$$\frac{b+d}{2} > c$$

$\Rightarrow$ $$b + d > 2c \qquad \text{........ (2)}$$

From (1) and (2)

$$a + c + b + d > 2b + 2c$$

$\Rightarrow$ $$a + d > b + c$$

Illustration 22

Prove that $\left(\dfrac{a+b}{\mathbf{2}}\right)^{a+b} > a^b b^a$**,** $ab \in \mathbf{N}$ **&** $a \neq b$**.**

Solution

By just seeing the question we get the feel that it is kind of AM > GM

So, we get the idea from R.H.S.

a repeated b times

& b repeated a times

Now applying AM > GM

$$\dfrac{(a + a + ... \text{ b times}) + (b + b + ... \text{a times})}{\underset{\left(\substack{\text{since no. of terms} \\ \text{are } a+b}\right)}{a+b}} > \left[(a.a.a...)\,(b..b..b.)\right]^{\frac{1}{a+b}}$$

$$\Rightarrow \qquad \dfrac{ab+ab}{a+b} > \left(a^b\, b^a\right)^{\frac{1}{a+b}}$$

$$\Rightarrow \qquad \dfrac{2ab}{a+b} > \left(a^b\, b^a\right)^{\frac{1}{a+b}} \qquad \text{... (i)}$$

Now, we know that AM > HM

i.e. $$\dfrac{a+b}{2} > \dfrac{2ab}{a+b} \qquad \text{... (ii)}$$

So from (i) & (ii)

$$\dfrac{a+b}{2} > \left(a^b\, b^a\right)^{\frac{1}{a+b}}$$

$$\Rightarrow \qquad \left(\dfrac{a+b}{2}\right)^{a+b} > a^b\, b^a$$

Arithmetic mean of m^th power

Let a_1, a_2 ...... a_n be n positive real numbers and let m be a real number, then

$$\dfrac{a_1^m + a_2^m + ... a_n^m}{n} \geq \left(\dfrac{a_1 + a_2 + ... a_n}{n}\right)^m \quad \text{if } m \in R - [0,\ 1]$$

for $m \in (0,\ 1)$, then $$\dfrac{a_1^m + a_2^m + ... a_n^m}{n} \leq \left(\dfrac{a_1 + a_2 + ... a_n}{n}\right)^m$$

and for m = 0 & 1 $$\dfrac{a_1^m + a_2^m + ... a_n^m}{n} = \left(\dfrac{a_1 + a_2 + ... a_n}{n}\right)^m$$

Illustration 23

Prove that $a^4 + b^4 + c^4 \geq abc\ (a + b + c)\ [a_1 b_1\ c > 0]$

Solution :

Using m^{th} power inequality

$$\frac{a^4 + b^4 + c^4}{3} \geq \left(\frac{a+b+c}{3}\right)^4 \qquad \text{... (i)}$$

Now we have $\left(\frac{a+b+c}{3}\right)^4$ in RHS

using AM > GM

$$\Rightarrow \quad \frac{a+b+c}{3} \geq (abc)^{1/3}$$

$$\Rightarrow \quad \left(\frac{a+b+c}{3}\right)^3 \geq abc$$

but we require $\left(\frac{a+b+c}{3}\right)^4$, so multiplying both sides by $\frac{(a+b+c)}{3}$

$$\left(\frac{a+b+c}{3}\right)^4 \geq \frac{abc(a+b+c)}{3} \qquad \text{... (ii)}$$

From (i) & (ii) we get

$$\frac{a^4 + b^4 + c^4}{3} \geq \frac{abc(a+b+c)}{3}$$

$$\Rightarrow \quad a^4 + b^4 + c^4 \geq abc\ (a + b + c)$$

Hence proved.

Summation of Series

To find the summation of a series we follow 2 basic steps.

Step 1 : Find the n^{th} term (T_n)

Step 2 : Find the sum using formula (for summation, using T_N)

To find the n^{th} term of a series we try to find out the differences between successive terms. If the difference is constant then they are in A.P and sum can be calculated by using a formula.

But many times you will find different differences between successive terms.

For ex. $\begin{array}{ccccccc} 1 & & 3 & & 6 & & 10 \\ & 2 & & 3 & & 4 & \end{array}$ (difference is increasing)

In such cases,

we find second order difference (i.e. we again find difference of differences obtain)

$$\begin{array}{ccccccc} 1 & & 3 & & 6 & & 10 \\ & 2 & & 3 & & 4 & \\ & & 1 & & 1 & & \end{array}$$

This process of finding the n^{th} order difference continues unless we get a constant difference.

$$\begin{array}{ccccccc} 1 & & 3 & & 6 & & 10 \\ & 2 & & 3 & & 4 & \end{array} \longrightarrow \text{We Call this second order difference}$$

Also, we can divide them into levels.

1. S = 1, 3, 5, 7 ... Level 0 (typical AP with constant difference)

2. S′ = $\begin{array}{ccccccc} 1, & & 3, & & 6, & & 10 \\ & 2 & & 3 & & 4 & \end{array}$ ⟵ Level 1 (difference are in AP)

3. S = $\begin{array}{ccccccccc} 1 & & 2 & & 5 & & 11 & & 21 \\ & 1 & & 3 & & 6 & & 10 & \\ & & 2 & & 3 & & 4 & & \end{array}$ ⟵ Level 2 (difference of differences are in AP)

So now we can generalize and find the results.

Type of series	General term
Level 0	$a + br$
Level 1	$ar^2 + br + c$
Level 2	$ar^3 + br^2 + cr + d$
& this can go on.	

for ex. S = 1, 3, 6, 10

We know this is level 1.

& for level 1, $T_r = ar^2 + br + c$

∴ $T_1 = a + b + c = 1$ .... (a)

$T_2 = 4a + 2b + c = 3$ .... (b)

$T_3 = 9a + 3b + c = 6$ .... (c)

Using the following [(b) – (a) & (c) – (b), we get $3a + b = 2$ & $5a + b = 3$

Subtracting we get $a = ½$

Similarly we can find b & c now which comes out to $b = ½$, $c = 0$

$$\therefore \quad T_r = \frac{r^2}{2} + \frac{r}{2} = \frac{r}{2}(r+1)$$

We can easily find the sum of this series now using formula for $\sum n^2$ & $\sum n$.

Lets take another example, this time of level 2.

We have S = 1, 2, 5, 11, 21

We know it is level 2, so

$$T_r = ar^3 + br^2 + cr + d$$

$$\therefore \quad T_1 = 1 = a + b + c + d$$

$$T_2 = 2 = 8a + 4b + 2c + d$$

$$T_3 = 5 = 27a + 9b + 3c + d$$

$$T_4 = 11 = 64a + 16b + 4c + d$$

solving eqn. we get $a = {}^1/_6$, $b = 0$, $c = -{}^1/_6$, $d = 1$

$$\therefore \quad T_r = \frac{r^3}{6} - \frac{r}{6} + 1$$

This is how we solve these kind of questions, now we will move to questions where differences are in G.P.

3.5.2 Difference in G.P

These are kind of questions in which differences successive terms is in GP instead of AP.

There are 2 methods to solve these kind of problems.

Method 1 : It is the method of finding T_r term with the help of levels (formulas)

Level 0

(Simple GP)

2 4 8 16

2 2 2

(multiplied by 2)

Here the series is clearly in GP,

Level 1

1 3 7 15 31

difference 2 4 8 16

again in simple GP
with factor 2.

Level 2

1 2 5 12

1 3 7

2 4

(now in simple GP
with factor 2)

So,

Type of series	General Term
Simple GP	ab^r
(Level 1) difference in GP	$a + bc^r$
(Level 2) difference of difference in GP	$a + br + cd^r$

Method 2 :

This method involves the rearrangement of the given series in such a way that we get a simple GP series in between the given series.

Let us take an example to clear the concept.

We have, $S_n = 1 + 3 + 7 + 15 + ... + T_n$

now shift the values of S_n & subtract it from S_N only.

$$
\begin{aligned}
S_n &= 1 + 3 + 7 + 15 + ... T_n \\
S_n &= \quad 1 + 3 + 7 + ... T_{n-1} + T_n \\
\hline
0 &= \underbrace{1 + 2 + 4 + 8 \quad ..... (n-1) \text{ terms}}_{\substack{\text{now this is a general} \\ \text{GP with } r = 2}} - T_n
\end{aligned}
$$

$$\Rightarrow \qquad T_n = \underbrace{1 + [2 + 4 + 8 + ... (n-1) \text{ terms}]}_{\text{this becomes a simple GP for n terms.}}$$

$$\therefore \qquad T_n = \frac{1(2^n - 1)}{2 - 1} = 2^n - 1$$

$$\text{or} \qquad T_r = 2^r - 1$$

$$\therefore \qquad \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (2^r - 1) = \sum_{r=1}^{n} 2^r - \sum_{r=1}^{n} 1$$

$$\Rightarrow \qquad S_n = \frac{2(1-2)^n}{1-2} n = 2^{n+1} - 2 - n$$

3.6 Arithmetic Geometric Series

Consider the series :

$$S_n = 1 + 2r + 3r^2 + 4r^3 + ... + nr^{(n-1)}$$

This series is neither arithmetic (the differences between the terms isn't constant) nor geometric (the ratio of successive terms isn't constant), yet it seems to be something of both.

It looks like something that is familiar yet alien.

If we know :

$$\frac{1}{(1-r)^2} = (1-r)^{(-2)} = 1 + 2r + 3r^2 = 4r^3 + ... + nr^{(n-1)} + ...$$

then we know the sum to infinity of the series is $(1 - r)^{-2}$, if $|r| < 1$ so the series converges. However, this doesn't tell us the sum to n terms.

Consider

$$S_n = 1 + 2r + 3r^2 + 4r^3 + ... + nr^{(n-1)} \qquad \text{(i)}$$

And using our trick from the geometric series, multiply this by r :

$$r\,S_n = r + 2\,r^2 + 3\,r^3 + 4\,r^4 + ... + n\,r^n \qquad \text{(ii)}$$

Subtract Equations (i) & (ii) :

$$S_n\,(1-r) = 1 + r + r^2 + ... + r^{n-1} - nr^n$$

Nothing we know the formula for the geometric series, and using it :

$$S_n\,(1-r) = \frac{(1-r^n)}{(1-r)} - n\,r^n$$

Bringing it all together under one denominator :

$$S_n\,(1-r) = \frac{\left(1 - r^n - n\,r^n + n\,r^{(n+1)}\right)}{(1-r)}$$

Rounding up like terms, gives us the formula :

$$S_n = \frac{\left(1 + n\,r^{(n+1)} - (n+1)\,r^n\right)}{\left(1-r^2\right)}$$

Therefore :

$$1 + 2r + 3r^2 ...\, n\,r^{(n-1)} = \frac{\left(1 - (n+1)\,r^n + n\,r^{(n+1)}\right)}{(1-r)^2}$$

If we multiply throughout by a constant, a we get :

$$a + 2\,a\,r + 3\,a\,r^2 \ldots.\ n\,a\,r^{(n-1)} = \frac{\left(1-(n+1)r^n + nr^{(n+1)}\right)}{(1-r)^2}$$

The sum to infinity of this series, when n tends to infinity (and $|r| < 1$, is :

$$\frac{1}{(1-r)^2} = 1 + 2r + 3r^2 + 4r^3 + \ldots. + nr^{(n-1)} + \ldots$$

3.6.1 General Arithmetic Geometric Series

In this series, which is neither geometric nor arithmetic, has the form :

$$a + (a + d)\,r + (a + 2d)r^2 + (a + 3d)\,r^3 + \ldots.. + (a + (n-1)d)\ ^{r(n-1)}$$

The simple arithmetic-geometric series is a special case of this, where a = 1

If we expand this series, we get :

$$(a + ar + \ldots\ldots + ar^{(n-1)} + (d\ r\ (1 + 2\ r + 3r^2 \ldots\ldots + (n-1)\ r^{(n-2)})$$

Naturally, we note the first bit is a normal geometric series, and the second bit is our simple arithmetic geometric series.

That is :

$$a + (a + d)\,r + (a + 2\,d)\,r^2 + (a + 3\,d)\,r^3 + \ldots. + (a + (n-1)\,d\ r(n-1)\,d\ r^{(n-1)}$$

$$= \quad a\frac{\left(1-r^n\right)}{(1-r)} + rd\,\frac{(1-nr)^{(n-1)} + (n-1)r^n)}{(1-r)^2}$$

Sum of infinite arithmetic – Geometric sequence

for $|r|<|$ & $n \to \infty$, r^n approaches O

putting these values in general term obtained above.

$$\boxed{S\,\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}}$$

3.8 Some Special Series

(1) $\sum n = 1 + 2 + 3 + \ldots.. + n = \dfrac{n(n+1)}{2}$ (i.e. sum of first n natural numbers)

This is an AP with a = 1 and d = 1

So, $\sum n = \dfrac{n}{2}\left(2.1 + (n-1)1\right)$

$$\boxed{\sum n = \frac{n(n+1)}{2}}$$

(2) $\sum n^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

Proof :

$\therefore \quad (x + 1)^3 - x^3 = 3x^2 + 3x + 1$

Putting x = 1, 2, 3, 4, ... n then

$2^3 - 1^3 = 3.1^2 + 3.1 + 1$

$3^3 - 2^3 = 3.2^2 + 3.2 + 1$

$4^3 - 3^3 = 3.3^2 + 3.3 + 1$

$5^3 - 4^3 = 3.4^2 + 3.4 + 1$

$(n + 1)^3 - n^3 = 3.n^2 + 3n + 1$

Adding all we get,

$(n + 1)^3 - 1^3 = 3(1^2 + 2^2 + ... + n^2) + 3(1 + 2 + ... + n) + (1 + 1 .... n \text{ times})$

$\Rightarrow \quad n^3 + 3n^2 = 3n = 3\ \Sigma n^2 = 3\ \Sigma n + n$

$\therefore \quad 3\sum n^2 = n^3 + 3n^2 + 3n - 3\frac{n(n+1)}{2} - n$

$\Rightarrow \quad \boxed{\sum n^2 = \frac{n(n+1)(2n+1)}{6}}$

(3) $\sum n^3 = 1^3 + 2^3 + 3^3 .... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Proof :

$\therefore \quad (x + 1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$

Putting x = 1, 2, 3, 4 .... n then,

$2^4 - 1^4 = 4.1^3 + 6.1^2 + 4.1 + 1$

$3^4 - 2^4 = 4.2^3 + 6.2^2 + 4.2 + 1$

$4^4 - 3^4 = 4.3^3 + 6.3^2 + 4.3 + 1$

$(n + 1)^4 - n^4 = 4.n^3 + 6.n^2 + 4.n + 1$

Adding all we get,

$(n + 1)^4 - 1^4 = 4(1^3 + 2^3 + ... + n^3) + 6\ (1^2 + 2^2 + ... + n^2) + 4\ (1 + 2 + ... + n) +$
$(1 + 1 + ... + n \text{ times})$

$\Rightarrow \quad n^4 + 4n^3 + 6n^2 + 4n = 4\sum n^3 + 6\sum n^2 + 4\sum n + n$

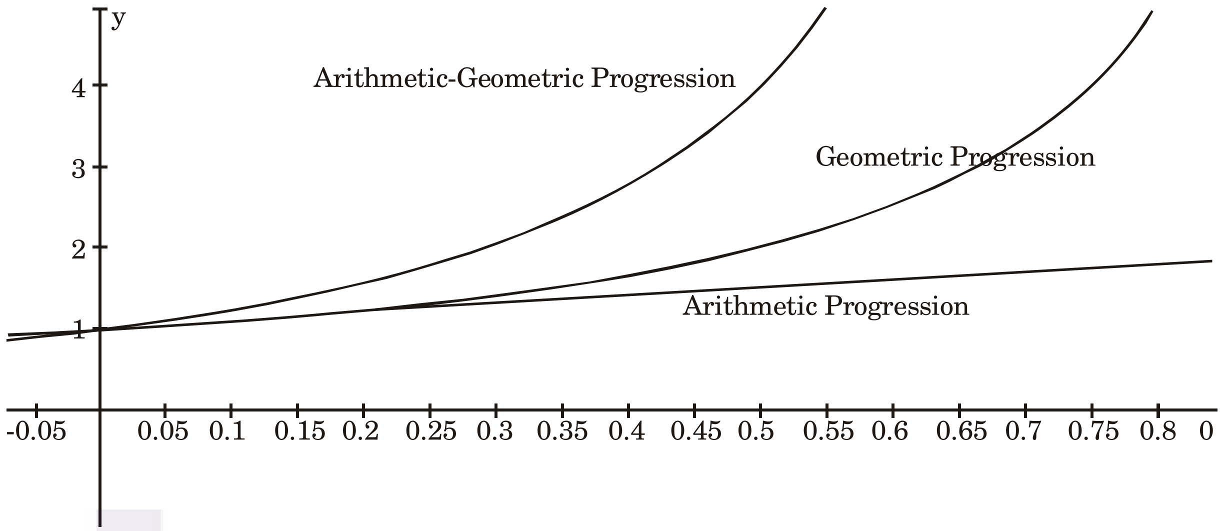
$\Rightarrow \quad 4\sum n^3 + \frac{6.n(n+1)(2n+1)}{6} + \frac{4.n(n+1)}{2} + n$

$\Rightarrow \quad 4\sum n^3 = n^4 + 4n^3 + 6n^2 + 4n - n(n+1)(2n+1) - 2n(n+1) - n$

$= n^2\ (n + 1)^2$

$\therefore \quad \boxed{\sum n^3 = \left\{\frac{n(n+1)}{2}\right\}^2}$

3.9 Graph of Arithmetic, Geometric and Arithmetic-Geometric Progression



Summation of Series (Method-2)

The following method works in the case when the n^{th} term of a series can be expressed as the difference of two quantities.

i.e $\quad T_r = f(r) - f(r-1)$

or $\quad T_r = f(r) - f(r+1)$

Steps to be followed in such a case

- find the general term (T_r)
- Express the general term as the difference of two consecutive terms
- In summation part (from 0/1 to n) (n) goes to the greater term & (1) goes to termwise smaller term

Let us solve some questions to understand the rule.

Illustration 26

Find S_n, where $S_n = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4}$...(n terms)

Solution :

Step 1 : find general term

$$T_r = \frac{1}{r(r+1)}$$

Step 2 : expressing T_r as difference of 2 terms here methods of partial fraction can be used

$$T_r = \frac{1}{r} - \frac{1}{r+1}$$

Step 3 : $S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right)$

now the rule says that place n in greater term (termwise not by value) which is r + 1 here.

$$\Rightarrow \quad \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} \frac{1}{r} - \frac{1}{r+1} = \left(1 - \frac{1}{n+1}\right)$$

$$= \frac{n}{n+1}$$

$$\therefore \quad S_n = \frac{n}{n+1}$$

Illustration 27

Find S_n where $S_n = \frac{1}{1.2.3} + \frac{1}{2.3.4} ...(n\ \text{terms})$

Solution : finding general term first

$$T_r = \frac{1}{r(r+1)(r+2)}$$

this can be rewritten as

$$T_r = \frac{1}{2}\left(\frac{(r+2) - r}{r(r+1)(r+2)}\right)$$

$$= \frac{1}{2}\left(\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right)$$

Note here that though in numerator we could have many options like [(r + 1) – r] etc. but we use the above one (i.e. ((r + 2) – r)) because after that we get terms which contains consecutive termwise terms (like r (r + 1) & (r + 1) (r + 2). The rule is valid for consecutive terms only.

here $\frac{1}{(r+1)(r+2)}$ is obtained by replacing r by r + 1 in $\frac{1}{r(r+1)}$, which is what we require for rule to be valid.

Now, $S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} \frac{1}{2}\left(\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right)$

Now applying step 3

$$S_n = \frac{1}{2}\sum_{r=1}^{n} \frac{1}{r\,(r+1)} - \frac{1}{(r+1)\,(r+2)}$$

$$\Rightarrow \quad S_n = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right)$$

Illustration 19

Find S_n, where $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \ldots (n$ terms)

Solution : $T_r = \dfrac{1}{(2r+1)\,(2r+1)(2r+3)}$

again we will try to divide the term in 2 parts

$$T_r = \frac{1}{4}\left[\frac{(2r+3)-(2r-1)}{(2r-1)\,(2r+1)\,(2r+3)}\right]$$

$$T_r = \frac{1}{4}\left\{\frac{1}{(2r-1)\,(2r+1)} - \frac{1}{(2r+1)\,(2r+3)}\right]$$

Now, we know that the second term can be achieved by replacing r by r + 1 in first term.

$$\therefore \quad S_n = \sum_{r=1}^{n} Tr = \frac{1}{4}\sum_{r=1}^{n}\left(\frac{1}{(2r-1)(2r+1)} - \frac{1}{(2r+1)\,(2r+3)}\right)$$

according to step 3

$$S_n = \frac{1}{4}\left(\frac{1}{3} - \frac{1}{(2n+3)\,(2n+3)}\right)$$

Illustration 29

Find S_n, where $\frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} \ldots (n$ terms)

Solution : $T_r = \dfrac{1}{r(r+2)}$

$$= \frac{1}{2}\frac{(r+2)-r}{r(r+2)}$$

$$= \frac{1}{2}\left[\frac{1}{r}-\frac{1}{r+2}\right] \qquad \text{.... (i)}$$

now still we don't have termwise consecutive terms, so we will try to achieve that adding & subtracting $\frac{1}{r+1}$ in (i)

$$T_r = \frac{1}{2}\left[\left(\frac{1}{r}-\frac{1}{r+1}\right)+\left(\frac{1}{r+1}-\frac{1}{r+2}\right)\right]$$

now the 2 block above are termwise consecutive respectively.

$$S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\frac{1}{2}\left[\left(\frac{1}{r}-\frac{1}{r+1}\right)+\left(\frac{1}{r+1}-\frac{1}{r+2}\right)\right]$$

$$= \frac{1}{2}\left[\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)+\sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)\right]$$

$$= \frac{1}{2}\left[\left(1-\frac{1}{n+1}\right)+\left(\frac{1}{2}-\frac{1}{n+2}\right)\right]$$

(applying step 3)

Illustration 30

Find S_n, where S_n = 1.3.5 + 3.5.7 + 5.7.9 + ... (n terms)

Solution : Here, $T_r = (2r-1)(2r+1)(2r+3)$

Now here too we have to divide T_r into 2 terms.

Trick : Let a be the term before $(2r-1)$ i.e. $(2r-3)$ & b be the term after $(2r+3)$ i.e. $(2r+5)$ multiply and divide by $(b-a)$

$$\therefore \quad T_r = \frac{1}{8}\left[(2r+5)-(2r-3)\right](2r-1)(2r+1)(2r+3)$$

$$= \frac{1}{8}\left[\underbrace{(2r+5)(2r-1)(2r+1)(2r+3)}_{\text{1st block}} - \underbrace{(2r-3)(2r-1)(2r+2)(2r+3)}_{\text{2nd block}}\right]$$

Here the 2nd block can be achieved by replacing r by $(r-1)$ in the first block

$\therefore$ Ist block is termwise greater than 2nd block

$\Rightarrow$ We can now apply step 3

$$S_n = \sum_{r=1}^{n} T_r$$

$$= \sum_{r=1}^{n} \frac{1}{8}\left[(2r+5)(2r+3)(2r+1)(2r-1)-(2r+3)(2r+1)(2r-1)(2r-3)\right]$$

$$= \frac{1}{8}\left[(2n+5)(2n+3)(2n+1)(2n-1)-(5)(3)(1)(-1)\right]$$

$$= \frac{1}{8}\left[(2n+5)(2n+3)(2n+1)(2n-1)+15\right]$$

Illustration 31

Find S_n, where $S_n = \frac{1}{1.3} + \frac{1}{1.3.5} + \frac{1}{1.3.5.7}$ ...(n terms)

Solution : firstly, $T_r = \frac{r}{1.3.5...(2r+1)}$

now we will try to break T_r into 2 termwise consecutive terms

$$T_r = \frac{1}{2}\left[\frac{2r}{1.3.5...(2r+1)}\right]$$

$$= \frac{1}{2}\left[\frac{(2r+1)-1}{1.3.5...(2r+1)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1.3.5...(2r-1)} - \frac{1}{1.3.5...(2r+1)}\right]$$

($\because$ Previous term of (2r + 1) is (2r – 1)

Now we have termwise consecutive terms

Applying step 3

$$Sn = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} \frac{1}{2}\left[\frac{1}{1.3.5...(2r-1)} - \frac{1}{1.3.5...(2r+1)}\right]$$

($\therefore$ The second term is termwise greater)

$$\therefore \quad S_n = \frac{1}{2}\left[1 - \frac{1}{1.3.5...(2n+1)}\right]$$