DEFINITE INTEGRATION

Definition :

If $\int f(x)dx = F(x)$ i.e. F(x) be an integral of f (x), then F(b) – F(a) is called the definite integral of f (x) between the limits a and b and in symbols it is written as $\int_a^b f(x)dx$ or, $[F(x)]_a^b$.

Thus if $\int_a^b f(x)dx = F(x)$ then by definition

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

a is called the lower limit or inferior limit and b is called the upper limit or superior limit.

It is clear that value of a definite integral of a function is unique and it does not depend on different forms of indefinite integral. For if

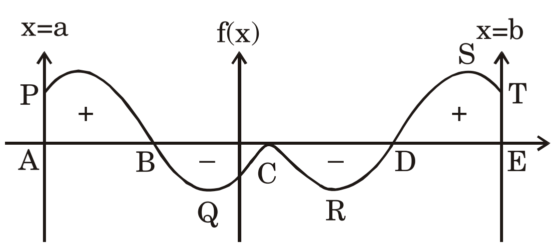
$$\int_a^b f(x)dx = [F(x)+c]_a^b = \{F(b)+c\} = F(b) - F(a)$$

Thus the value of $\int_a^b f(x)dx$ is same as when we take $\int_a^b f(x)dx = F(x)$.

Geometrical Interpretation of Definite Integral

If f(x) > 0 for all x ∈ [a, b]; then $\int_a^b f(x)dx$ is numerically equal to the area bounded by the curve y = f(x), the x-axis and the straight lines x = a and x = b

In general $\int_a^b f(x)dx$ represents algebraic sum of the areas of the figures bounded by the curve y = f(x), the x-axis and the straight lines x = a and x = b. The areas above x-axis are taken plus sign and the areas below x-axis are taken with minus sign i.e.,



i.e., $\int_a^b f(x)dx$ = area APB – area BQC – area CRD + area DSTE

Illustration 1

Find $\int_0^1 (4x^3 + 3x^2 - 2x + 1)dx$ **.**

Solution :

$$\int_0^1 (4x^3 + 3x^2 - 2x + 1)dx = 4.\frac{x^4}{3} - 2.\frac{x^2}{2} + x = x^4 + x^3 - x^2 + x$$

$$\therefore \quad \int_0^1 (4x^3 + 3x^2 - 2x + 1)dx$$

$$= (1^4 + 1^3 - 1^2 + 1) - (0 + 0 - 0 + 0) = 2 - 0 = 2$$

Illustration 2

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos 3x + 2\cos x} dx.$$

Solution :

$$\int \frac{\sin x}{\cos 3x + 3\cos x} dx = \int \frac{\sin x}{(4\cos^3 x - 3\cos x) + 3\cos x} dx$$

$$= \int \frac{\sin x}{4\cos^3 x} dx = \frac{1}{4}\int \tan x \sec^2 x dx = \frac{1}{8}\tan^2 x \text{ [Put } z = \tan x]$$

$$\therefore \quad \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos 3x + 2\cos x} dx = \frac{1}{8}[\tan^2 x]^{\frac{\pi}{4}}$$

$$= \frac{1}{8}\left(\tan^2\frac{\pi}{4} - \tan^2 0\right) = \frac{1}{8}(1 - 0) = \frac{1}{8}$$

Illustration 3

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin x}$$

Solution : $\displaystyle \int \frac{dx}{1 + \sin x} = \int \frac{dx}{1 + \cos\left(\frac{\pi}{2} - x\right)} = \int \frac{dx}{2\cos^2\left(\frac{\pi}{4} - \frac{\pi}{2}\right)}$

$$= \frac{1}{2}\int \sec^2\left(\frac{\pi}{4} - \frac{\pi}{2}\right)dx = \frac{1}{2}.\frac{\tan\left(\frac{\pi}{4} - \frac{\pi}{2}\right)}{-\frac{1}{2}} = -\tan\left(\frac{\pi}{4} - \frac{\pi}{2}\right)$$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x} = -\left[\tan\left(\frac{\pi}{4} - \frac{\pi}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \left[\tan\left(\frac{\pi}{4} - \frac{\pi}{2}\right) - \tan\frac{\pi}{4}\right] = -(\tan 0 - 1) = 1$$

2nd Method :

$$\int \frac{dx}{1+\sin x} = \int \frac{1-\sin x}{1-\sin^2 x} dx = \int \frac{1-\sin x}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx$$

$$= \int (\sec^2 x - \tan x \sec x) dx = \tan x = \sec x.$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x} = [\tan x - \sec x]_0^{\frac{\pi}{2}}$$

$$= \underset{x \to \frac{\pi}{2} - 0}{\text{Lt}} (\tan x - \sec x) - (\tan\theta - \sec 0) = \underset{x \to \frac{\pi}{2} - 0}{\text{Lt}} \frac{\sin x - 1}{\cos x} - (0 - 1)$$

$\left[\because \tan\frac{\pi}{2} \text{ and } \sec\frac{\pi}{2} \text{ are undefined}\right.$

Hence we can not take value of

$\tan x - \sec x$ at $x = \frac{\pi}{2}$

Here we take limit as $x \to \frac{\pi}{2} - 0]$

$$= \underset{h \to 0}{\text{Lt}} \frac{\sin\left(\frac{\pi}{2} + h\right) - 1}{\cos\left(\frac{\pi}{2} + h\right)} + 1 \left[\text{putting } x = \frac{\pi}{2} + h\right]$$

$$= \underset{h \to 0}{\text{Lt}} \frac{\cos h - 1}{-\sin h} + 1 = \underset{h \to 0}{\text{Lt}} \frac{1 - \cos h}{\sin h} + 1 = \underset{x \to 0}{\text{Lt}} \frac{2\sin^2 \frac{h}{2}}{\sin h} + 1$$

$$= \underset{h \to 0}{\text{Lt}} \frac{2\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2 \cdot \frac{h^2}{4}}{\frac{\sin h}{h} \cdot h} + 1 = 0 + 1 = 1$$

Note :

In the second method, numerator and denominator have been multiplied by (1 – sinx) and the value of 1 – sinx is 0 when $x = \frac{\pi}{2}$ and hence when $x = \frac{\pi}{2}$ integrand is undefined

Hence avoid multiplying numerator and denominator by an expression which becomes zero at any point of the interval [a, b] where a and b are the lower and upper limits respectively of integration.

Promblems in which integral can be found by Substitution method :

Working Rule :

When definite integral is to be found by substitution then change the lower and upper limits of integration. If substitution is $z = \phi(x)$ and lower limit of integration is a and upper limit is b then new lower and upper limits will be $\phi(a)$ and $\phi(b)$ respectively.

Illustration 4

Find the value of $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)^2}dx$**.**

Solution :

Let z = 1 + sinx, then dz = cosxdx

When x = 0, z = 1 + sin 0 = 1 + 0 = 1

and when $x = \frac{\pi}{2}, z = 1 + \sin\frac{\pi}{2} = 1 + 1 = 2$

$$\text{Now } I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)^2}dx = \int_1^2 \frac{dz}{z^2} = \int_1^2 z^{-2}dz$$

$$= \left[\frac{z^{-1}}{-1}\right]_1^2 = -\left[\frac{1}{z}\right]_1^2 = -\left[\frac{1}{2} - 1\right] = \frac{1}{2}$$

Note : Only principal value of θ is taken. For example when sinθ = 0, θ = nπ but principal value of θ is 0.

Illustration 5

Evaluate $\int_0^1 \sqrt{\frac{1-x}{1+x}}dx$**.**

Solution :

Let x = cos 2θ, then dx = –2sin2θdθ.

When x = 0, cos2θ = 0 ∴ $2\theta = \frac{\pi}{2}$ or, $\theta = \frac{\pi}{2}$

Now $I = \int_0^1 \sqrt{\frac{1-x}{1+x}}dx = \int_{\frac{\pi}{4}}^{0} \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\,(-2\sin 2\theta)d\theta$

$$= \int_{\frac{\pi}{4}}^{0} \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}(-2.2\sin\theta\cos\theta)d\theta = -\int_{\frac{\pi}{4}}^{0} 4\sin^2\theta d\theta$$

$$= -4\int_{\frac{\pi}{4}}^{0} \frac{1-\cos 2\theta}{2}d\theta = -2\int_{\frac{\pi}{4}}^{0}(1-\cos 2\theta)d\theta$$

$$= -2\left[\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{0} = -[2\theta - \sin 2\theta]_{\frac{\pi}{4}}^{0}$$

$$= -\left[(0-\sin\theta) - \left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right)\right] = -\left[(0 - \left(\frac{\pi}{2} - 1\right)\right] = \frac{\pi}{2} - 1$$

Illustration 6

Find $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$.

Solution :

Let $x = \alpha\cos^2\theta + \beta\sin^2\theta$

Then $dx = (-2\alpha\cos\theta\sin\theta + 2\beta\sin\theta\cos\theta)d\theta = 2(\beta-\alpha)\sin\theta\cos\theta d\theta$

When $x = \alpha$, $\alpha = \alpha\cos^2\theta + \beta\sin^2\theta$

or, $(\alpha-\beta)\sin^2\theta = 0$ or, $\sin^2\theta = 0$ ∴ $\theta = 0$

when $x = \beta$, $\beta = \alpha\cos^2\theta + \beta\sin^2\theta$

or, $\beta\cos^2\theta = \alpha\cos^2\theta$ or, $(\beta-\alpha)\cos^2\theta = 0$

or, $\cos^2\theta = 0$ or, $\cos\theta = 0$ or, $\theta = \frac{\pi}{2}$

Now $I = \int_a^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

$$= \int_0^{\pi/2} \frac{2(\beta-\alpha)\sin\theta\cos\theta}{\sqrt{(\alpha\cos^2\theta + \beta\sin^2\theta - \alpha)(\beta - \alpha\cos^2\theta - \beta\sin^2\theta)}}d\theta$$

$$= \int_0^{\pi/2} \frac{2(\beta-\alpha)\sin\theta\cos\theta}{\sqrt{(\beta-\alpha)\sin^2\theta(\beta-\alpha)\cos^2\theta}}d\theta = 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2} = 2\left[\frac{\pi}{2} - 0\right] = \pi$$

Illustration 7

Find $\int_1^e \frac{e^x}{x}(1 + x\log x)dx$.

Solution :

$$\int \frac{e^x}{x}(1 + x\log x)dx = \int e^x\left(\frac{1}{x} + \log x\right)dx$$

$$= \int e^x[f'(x) + f(x)]dx, \text{ where } f(x) = \log x = e^x f(x) = e^x \log x$$

$$\int_1^e \frac{e^x}{x}(1 + x\log x)dx = [e^x \log x]_1^e = e^e \log_e - e\log 1 = e^e$$

Illustration 8

Evaluate $\int_{-2}^{2}\frac{dx}{4 + x^2}$ **directly as well as by the substitution** $x = 1/t$.

Examine as to why the answer do not tally ?

Solution :

$$I = \int_{-2}^{2}\frac{dx}{4 + x^2}$$

$$= \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2} = \frac{1}{2}[\tan^{-1}(1) - \tan^{-1}(-1)] = \frac{1}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{4} \quad \Rightarrow \quad I = \frac{\pi}{4}$$

on the other hand; if $x = 1/t$ then,

$$I = \int_{-2}^{2}\frac{dx}{4 + x^2} = -\int_{-1/2}^{1/2}\frac{dt}{t^2(4 + 1/t^2)} = -\int_{1/2}^{1/2}\frac{dt}{4t^2 + 1}$$

$$= \left[\frac{1}{2}\tan^{-1}(2t)\right]_{-1/2}^{1/2} = -\frac{1}{2}\tan^{-1}(1) - \left(-\frac{1}{2}\tan^{-1}(-1)\right) = -\frac{\pi}{8} - \frac{\pi}{8} = -\frac{\pi}{4}$$

$$\therefore \quad I = -\frac{\pi}{4} \quad \text{when } x = \frac{1}{t}$$

In above two results $I = -\pi/4$, is wrong. Since the integrand $\frac{1}{4 + x^2} > 0$ and therefore the definite integral of this function cannot be negative.

Since $x = 1/t$ is discountinuous at $t = 0$, the substitution is not valid ($I = \pi/4$).

Note : It is important the substitution must be continuous in the interval of integration.

PROPERTIES OF DEFINITE INTEGRALS

Property 1 : $\boxed{\int_a^b f(x)dx = \int_a^b f(t)dt}$

i.e. integration is independent of change of variable.

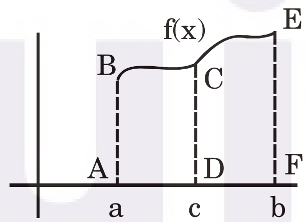
Property 2 : $\boxed{\int_a^b f(x)dx = -\int_a^b f(x)dx}$

i.e. if the limits of a definite integral are interchangable then its value becomes negative of the earlier value.

Property 3 : $\boxed{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx}$

where $a < c < b$

Logic : We know that definite integral represents the area of the function between x-axis under the given limits.



Now $\int_a^b f(x)dx =$ area of $f(x)$ i.e. area ABCEFI

$=$ area ABCDA+ area CEFDC

$$= \int_a^c f(x)dx + \int_c^b f(x)dx$$

You can prove all the above 3 properties by alzebraic method. We are leaving that part for you to do it yourself.

General form of Property-3

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + ... + \int_{c_{n-1}}^b f(x)dx$$

where $a < c_1 < c_2 < ... < c_{n-1} < b$

Working Rule

This property is used when integrand is different in different intervals. This happens in the following cases.

1. function changes or is discountinuous at some points in $[a,b]$
2. Modulus function
3. Greatest intger function & fractional part.

In each of the 3 cases we find the point where the function is different & divide the interval accordingly using property-3.

Illustration 9

(a) Evaluate $\int_{-1}^{1} f(x)dx$, **where** $f(x) = \begin{cases} \mathbf{1-2}x; & x \le \mathbf{0} \\ \mathbf{1+2}x; & x \ge \mathbf{0} \end{cases}$

(b) Evaluate $\int_{-1}^{1} f(x)dx$, **where** $f(x) = \begin{cases} \mathbf{1-2}x; & x \le \mathbf{0} \\ \mathbf{1+2}x; & x \ge \mathbf{0} \end{cases}$

Solution :

(a) The function is discontinuous at 0, at its value is changing. Hence we cannot integrate over [–1, 1]. So applying the rule.

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx$$

$$= \int_{-1}^{0} (1-2x)dx + \int_{0}^{1} (1+2x)dx$$

$$= [x - x^2]_{-1}^{0} + [x + x^2]_{0}^{1}$$

$$= [0 - (-1 - 1)] + [1 + 1 - 0] = 4$$

(b)

sign of x

–1 —— –ve —— 0 —— +ve —— –1

In case of modulus function, the value of function changes at the point where it becomes 0. Hence, breaking the interval

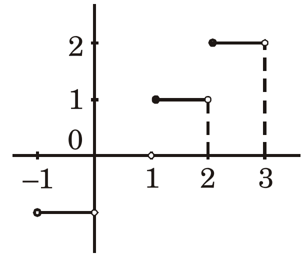
Now $\int_{-1}^{1} |x| dx = \int_{-1}^{0} |x| dx + \int_{0}^{1} |x| dx = \int_{-1}^{0} -x dx + \int_{0}^{1} x dx$

[$\because$ when $-1 < x < 0$, $|x| = -x$ and when $0 < x < 1$, $|x| = x$]

$$= -\left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} = -\left(0 - \frac{1}{2}\right) + \left(\frac{1}{2} - 0\right) = 1$$

3. $\int_{-1}^{3}[x]dx$

We know greatest integer function returns integral values only. So for every integral interval value will change.



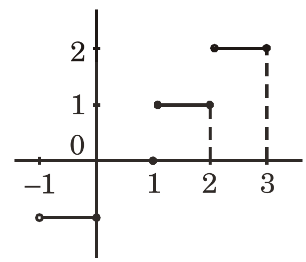
$$=\int_{-1}^{0}[x]dx+\int_{0}^{1}[x]dx+\int_{1}^{2}[x]dx+\int_{2}^{3}[x]dx$$

$$=-1+0+1+2$$

$$=2$$

4. $\int_{0}^{2}\{x\}dx$

For fractional part, let us draw the graph so for



$0<x<1 \qquad \{x\}=x$

$1<x<2 \qquad \{x\}=x-1$

(this is the reason we did such graphs in functions chapter)

$$=\int_{0}^{1}\{x\}dx+\int_{1}^{2}\{x\}dx=\int_{0}^{1}xdx+\int_{1}^{2}(x-1)dx$$

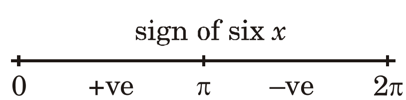
$$=\left.\frac{x^2}{2}\right|_0^1+\left.\frac{(x-1)^2}{2}\right|_1^2=\frac{1}{2}+\frac{1}{2}=1$$

Illustration 10

(a) **Find the value of** $\int_{0}^{2\pi}|\sin x|dx$ **(b)** **Evaluate** $\int_{0}^{2}|x^2+2X-3|dx$

Solution :

(a) [When $\sin x=0,\ x=n\pi$, where $n=0,\pm1,\pm2,\ldots$ out of which only p lies between lower and upper limits of integration].



Now $I=\int_{0}^{2\pi}|\sin x|dx=\int_{0}^{\pi}|\sin x|dx+\int_{\pi}^{2\pi}|\sin x|dx$

$$=\int_{0}^{\pi}\sin xdx-\int_{\pi}^{2\pi}\sin xdx=[-\cos x]_0^{\pi}-[-\cos x]_{\pi}^{2\pi}$$

$$=(\cos\pi-\cos 0)+(\cos 2\pi-\cos\pi)$$

$$=-(-1-1)+[1-(-1)]=4$$

(b) $x^2 + 2x - 3 = 0 \Rightarrow x = -3, 1$

Sign scheme for $x^2 + 2x - 3$ in [0, 2] is

$$\underset{0}{\vdash}\;\; -ve \;\; \underset{1}{+} \;\; +ve \;\; \underset{2}{\dashv}$$

Now $I = \int_0^2 |x^2+2x-3|\,dx$

$$= \int_0^1 |x^2+2x-3|\,dx + \int_1^2 |x^2+2x-3|\,dx = \int_0^1 -(x^2+2x-3)dx + \int_1^2 (x^2+2x-3)dx$$

$$= -\left[\frac{x^3}{3}+x^2-3x\right]_0^1 + \left[\frac{x^3}{3}+x^2-3x\right]_1^2$$

$$= -\left[\left(\frac{1}{3}+1-3\right)-0\right]+\left[\left(\frac{8}{3}+4-6\right)-\left(\frac{1}{3}+1-3\right)\right] = \frac{5}{3}+\frac{2}{3}+\frac{5}{3}=4$$

Illustration 11

Find the value of

(a) $\int_0^{\pi} |\cos x - \sin x|\,dx$ **(b)** $\int_0^4 \left\{\sqrt{x}\right\} dx$ **(c) Evaluate** $\int_0^{3/2} |x\cos \pi x|\,dx$

Solution :

(a) $\cos x - \sin x = 0 \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1 = \tan\frac{\pi}{4}$ $\therefore$ $x = n\pi + \frac{\pi}{4}$

where $n = 0, \pm 1, \pm 2, \ldots$ out of which only $\frac{\pi}{4}$ lies between lower and upper limits of definite integration.]

sign scheme for cos x – sin x

$$\underset{0}{\vdash}\;\; +ve \;\; \underset{\pi/4}{+} \;\; -ve \;\; \underset{\pi}{\dashv}$$

Now $I = \int_0^{\pi} |\cos x - \sin x|\,dx$

$$= \int_0^{\pi/4} |\cos x - \sin x|\,dx + \int_{\pi/4}^{\pi} |\cos x - \sin x|\,dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x)dx - \int_{\pi/4}^{\pi} (\cos x - \sin x)dx = [\sin x + \cos x]_0^{\pi/4} - [\sin x + \cos x]_{\pi/4}^{\pi}$$

$$= \left[\left(\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)-(\sin 0+\cos 0)\right]-\left[\left(\sin\pi+\cos\pi\right)-\left(\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)\right]$$

$$= \left[\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1\right)\right]-\left[(0-1)-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\right] = \sqrt{2}-1-(-1-\sqrt{2}) = \sqrt{2}-1+1+\sqrt{2} = 2\sqrt{2}$$

(b) $\int_0^4 \left\{\sqrt{x}\right\} dx$

Here also the value of fractional part will change at integral values

$$\begin{array}{ll} \sqrt{x}=1 & \text{at } x=1 \\ \sqrt{x}=2 & \text{at } x=4 \end{array}, \quad \text{which is the upper limit.}$$

so value of

$$\begin{aligned} \{x\} &= \sqrt{x}, && 0<x<1 \\ &= \sqrt{x}-1, && 1<x<4 \end{aligned}$$

$$\int_0^4 \left\{\sqrt{x}\right\} dx = \int_0^1 \sqrt{x}dx + \int_1^4 \left(\sqrt{x}-1\right) dx$$

$$= \frac{2}{3} x^{3/2}\Big|_0^1 + \frac{2}{3} x^{3/2}\Big|_1^4 - x\Big|_1^4$$

$$= \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left[(8-1)\right] - (4-1)$$

$$= \frac{7}{3}$$

(c) $x\cos \pi x = 0$ $\Rightarrow \begin{cases} x=0 \\ \cos n\pi x = 0 \text{ or } \pi x = (2n+1)\dfrac{\pi}{2}, \ n \in I \end{cases}$

$$\Rightarrow \begin{cases} x=0 \\ x=\dfrac{1}{2}, \ \text{between } 0 \text{ and } \dfrac{3}{2} \end{cases}$$

sign scheme for x cos πx is

0 +ve $\frac{1}{2}$ –ve $\frac{3}{2}$

Now $\int_0^{3/2} |x\cos \pi x| dx = \int_0^{1/2} |x\cos \pi x| dx + \int_{1/2}^{3/2} |x\cos \pi x| dx$

$$= \int_0^{1/2} x\cos \pi x dx - \int_{1/2}^{3/2} x\cos \pi x dx$$

$$= \left[\frac{x\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right]_0^{1/2} - \left[\frac{x\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right]_{1/2}^{3/2}$$

$$=\left(\frac{1}{2\pi}-\frac{1}{\pi^2}\right)-\left(\frac{3}{2\pi}-\frac{1}{2\pi}\right)$$

$$=\frac{1}{2\pi}-\frac{1}{\pi^2}+\frac{2}{\pi}=\frac{5}{2\pi}-\frac{1}{\pi^2}$$

Property 4 : $\boxed{\int_0^a f(x)dx=\int_0^a f(a-x)dx}$

Proof. Put $a-x=t$ on R.H.S.

so lower limit becomes a

and upper limit becomes 0

& $-dx=dt$

$$\therefore \int_0^a f(a-x)dx=\int_a^0 f(t)(-dt)=-\int_a^0 f(t)dt$$

$$=\int_0^a f(t)dt \qquad \{\text{using Property-2}\}$$

$$=\int_0^a f(x)dx \qquad [\text{using Property-1}]$$

Usefulness

This property is useful to convert an indefinite integral to a more easily solvable integral. This property is specially very useful in trignometric integrals. Let us see how.

Illustration 12

(a) $\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}$ **(b)** $\int_0^1 x(1-x)^{99}dx$

Solution :

(a) $$I=\int_0^{\pi/2}\frac{\sqrt{\sin x}dx}{\sqrt{\sin x}+\sqrt{\cos x}} \qquad ...(i)$$

see now if you solve this without any use of definite properties, as a normal indefinite question it will become a very lengthy problem.

Let us see how property comes handy in this case.

$$I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx=\int_0^{\pi/2}\frac{\sqrt{\sin\left(\pi/2-x\right)}}{\sqrt{\sin\left(\pi/2-x\right)}+\sqrt{\cos\left(\pi/2-x\right)}}$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sqrt{\cos(x)}}{\sqrt{\cos x} + \sqrt{\sin x}} \quad \text{...(ii)}$$

adding (i) and (ii)

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}dx}{\sqrt{\cos x} + \sqrt{\sin x}} + \int_0^{\pi/2} \frac{\sqrt{\cos x}dx}{\sqrt{\cos x} + \sqrt{\sin x}}$$

$$= \int_0^{\pi/2} \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right) dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_0^{\pi/2} dx$$

$$= x\Big|_0^{\pi/2} = \frac{\pi}{2}$$

$$\therefore \quad 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

So you can notice the usefulness of this property. When $\sin x$ & $\cos x$ are interchangeable by this property, use it to reduce the integral.

(b) $\int_0^1 x(1-x)^{99} dx$

Though we can solve this question by first using by parts and then applying the limits, but here we will use this property to prevent that long step.

$$I = \int_0^1 x(1-x)^{99} dx \quad \text{...(i)}$$

applying Property-4

$$I = \int_0^1 (1-x)[1-(1-x)]^{99}$$

$$= \int_0^1 (1-x)x^{99} dx$$

$$= \int_0^1 (x^{99} - x^{100}) dx$$

Now solve simply as integral of 2 functions (no need of using by parts)

$$= \frac{x^{100}}{100}\Big|_0^1 - \frac{x^{101}}{101}\Big|_0^1 = \frac{1}{100} - \frac{1}{101} = \frac{1}{10100}$$

Illustration 13

(a) $\int_0^{\pi/2} \log \tan x \, dx$ (b) $\int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cos x} dx$

(c) $\int_0^{\pi/4} \log(1+\tan x) dx$ (d) $\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$

Solution :

(a) $I = \int_0^{\pi/2} \log \tan x dx$ ...(i)

applying Property-4

$$I = \int_0^{\pi/2} \log \tan\left(\frac{\pi}{2} - 2\right) dx$$

$$= \int_0^{\pi/2} \log \cot x dx \quad \text{(ii)}$$

adding (i) and (ii)

$$2I = \int_0^{\pi/2} \log \tan x dx + \int_0^{\pi/2} \log \cot x dx$$

$$= \int_0^{\pi/2} (\log(\tan x) + \log \cot x) dx$$

Using the log property, $\log a + \log b = \log ab$

$$= \int_0^{\pi/2} \log(\tan x \times \cot x) dx$$

$$= \int_0^{\pi/2} \log 1 dx = \int_0^{\pi/2} 0.dx = 0 \qquad [\text{as } \log 1 = 0]$$

Tip. Why I thought of using this property ?

1. Using by parts , is a very long process

2. Most importantly, upper limit is $\frac{\pi}{2}$ and every trignometric, function gives it opposite pair at $\frac{\pi}{2}$ – x, hence purpose solved.

(b) $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cos x} dx$ ...(1)

Similarly in this question, a bell should ring that upper limit is $\pi/2$ and function comprises of $\sin x$ & $\cos x$ which can be interchanged.

Hence applying Property-4.

$$I = \int_0^{\pi/2} \frac{\sin(\pi/2 - x) - \cos(\pi/2 - x)}{1 + \sin(\pi/2 - x)\cos(\pi/2 - x)} dx$$

$$= \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx \qquad \text{...(ii)}$$

adding (i) and (ii)

$$2I = \int_0^{\pi/2} \left(\frac{\sin x - \cos x}{1 + \sin x \cos x} + \frac{\cos x - \sin x}{1 + \sin x \cos x} \right) dx$$

$$= \int_0^{\pi/2} 0.dx = 0$$

$\therefore \quad I = 0$

(c) $I = \int_0^{\pi/4} \log(1 + \tan x) dx$

applying Property-4

$$= \int_0^{\pi/4} \log\left(1 + \tan\left(\pi/4 - x\right)\right) dx = \int_0^{\pi/4} \log\left(1 + \frac{\tan \pi/4 - \tan x}{1 + \tan \pi/4 \tan x}\right) dx$$

$$= \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) dx \qquad \text{...(ii)}$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan x}\right) dx$$

adding (i) and (ii)

$$2I = \int_0^{\pi/4} \left(\log(1 + \tan x) + \log\left(\frac{2}{1 + \tan x}\right) \right) dx$$

$$= \int_0^{\pi/4} \log\left((1 + \tan x) \frac{2}{(1 + \tan x)} \right) dx$$

$$= \int_0^{\pi/4} (\log 2) dx$$

$$\Rightarrow \quad 2I = \log 2\int_0^{\pi/4} dx = \log 2\ x\Big|_0^{\pi/4} = \frac{\pi}{4}\log 2$$

$$\Rightarrow \quad I = \frac{\pi}{8}\log 2$$

Note. Another common mistake is the last step. Students forget that on L.H.S. it is $2I$, and they have to divide by 2 to get the answer. So keep this in mind.

(d) $\displaystyle\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$

Let $f(x) = \dfrac{x}{\sin x + \cos x}$ (i)

Then $f\left(\dfrac{\pi}{2} - x\right) = \dfrac{\dfrac{\pi}{2} - x}{\sin\left(\dfrac{\pi}{2} - x\right) + \cos\left(\dfrac{\pi}{2} - x\right)}$

or $f\left(\dfrac{\pi}{2} - x\right) = \dfrac{\dfrac{\pi}{2} - x}{\cos x + \sin x}$

(1) + (2) $\Rightarrow$ $f(x) + f\left(\dfrac{\pi}{2} - x\right) = \dfrac{\pi}{2}\dfrac{1}{\cos x + \sin x}$

$$= \frac{\pi}{2\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)}$$

$$= \frac{\pi}{2\sqrt{2}}\sec\left(x - \frac{\pi}{4}\right)$$

Now $I = \dfrac{1}{2}\displaystyle\int_0^{\pi/2}\left[f(x) + f\left(\dfrac{\pi}{2} - x\right)\right]dx$

$$= \frac{1}{2}\cdot\frac{\pi}{2\sqrt{2}}\int_0^{\pi/2}\sec\left(x - \frac{\pi}{4}\right)dx$$

$$= \frac{\pi}{2\sqrt{2}}\left[\log\left|\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right)\right|\right]_0^{\pi/2}$$

$$= \frac{\pi}{2\sqrt{2}}\left[\log\left(\operatorname{cosec}\frac{\pi}{4} + \cot\frac{\pi}{4}\right) - \log\left|\sec\frac{\pi}{4} - \tan\frac{\pi}{4}\right|\right]$$

$$= \frac{\pi}{4\sqrt{2}}\left[\log\left(\sqrt{2}+1\right) - \log\left(\sqrt{2}-1\right)\right]$$

$$= \frac{\pi}{4\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = \frac{\pi}{4\sqrt{2}}\log\left(\sqrt{2}+1\right)^2$$

$$= \frac{\pi}{2\sqrt{2}}\log\left(\sqrt{2}+1\right)$$

Property-5 $\boxed{\int_a^b f(x)dx = \int_a^b f(a+b-x)dx}$

Property-4 is a special case of Property-5 when $a = 0$ & $b = a$

Proof. Let $a + b - t = x$

$\Rightarrow$ $-dt = dx$

& at $x = a$ $t = b$

$x = b$ $t = a$

$\therefore$ $I = \int_b^a f(t)(-dt)$

by using Property-2

$I = \int_a^b f(t)dt = \int_a^b f(x)dx$

Illustration 14

(a) $\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}dx$ **(b)** $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}dx$

Solution :

(a) $I = \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}dx$

If we apply Property-5 in this integral then the denominator remains the same, which gives encouragement to use the same.

$\therefore$ applying Property-5.

$$I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{3-(3-x)} + \sqrt{3-x}}\,dx$$

$$= \int_1^2 \frac{\sqrt{3-x}\;dx}{\sqrt{x} + \sqrt{3-x}} \qquad \text{(ii)}$$

adding (i) and (ii)

$$2I = \int_1^2 \left(\frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}} + \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\right)dx$$

$$= \int_1^2 \left(\frac{\sqrt{3-x}+\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\right)dx$$

$$= \int_1^2 dx = x\Big|_1^2 = 2-1=1$$

$$\Rightarrow \quad I = \frac{1}{2}.$$

(b) $\displaystyle\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\,dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

Earlier we have seen that in case of $\left(\frac{\pi}{2} - x\right)$ transformation in trignometric integrals, the interigral was reduced to a very simple one. See, here also it is happening.

Now, $\quad \dfrac{\pi}{3} + \dfrac{\pi}{6} = \dfrac{\pi}{2}$

& property 5 replaces x by $(a+b-x)$ i.e. $\left(\frac{\pi}{2} - x\right)$.

Hence our purpose is solved.

$$\therefore \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)} + \sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \qquad \text{(ii)}$$

Now I hope you understand what we are trying to do & what we will do next. adding (i) and (ii)

$$2I = \int_{\pi/6}^{\pi/3} dx = \left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{6}$$

$$\Rightarrow \quad I = \frac{\pi}{12}$$

Property-6 $\boxed{\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}}$

Proof. From Property-3, we get

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx \qquad \text{(i)}$$

Now $\int_{-a}^{0} f(x)dx = \int_{-a}^{0} f(-t)(-dt)$ (put $x = -t$)

$$= \int_0^a f(-t)dt = \int_0^a f(-x)dx$$

$$= \begin{cases} \int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function} \\ -\int_0^a f(x)dx, & \text{if } f(x) \text{ is an odd function} \end{cases} \qquad \text{(ii)}$$

Thus, when $f(x)$ is an even function from (i) & (ii)

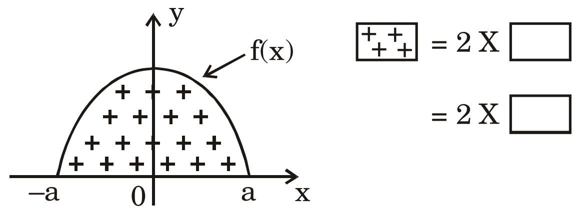
$$\Rightarrow \quad \int_{-a}^{0} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$$

and when $f(x)$ is an odd function, from (i) & (ii)

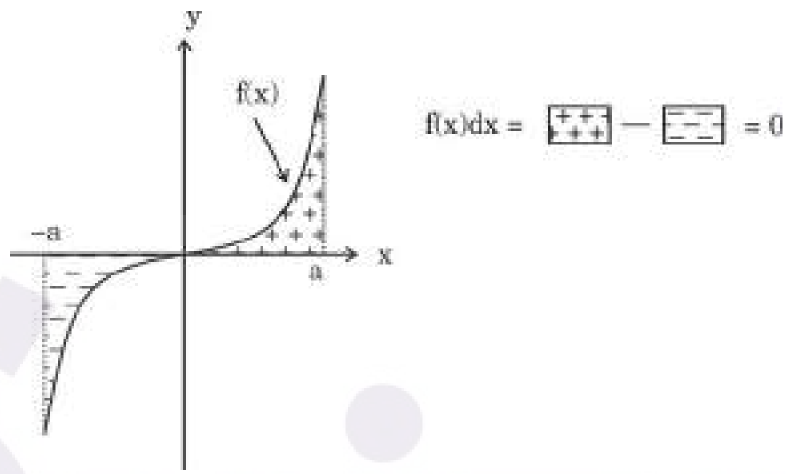
$$\Rightarrow \quad \int_{-a}^{0} f(x) = -\int_0^a f(x)dx + \int_0^a f(x)dx = 0$$

Geometrical Proof.

If $f(x)$ is EVEN



If $f(x)$ is ODD



This property should be used only when limits are equal and opposite and the function which is to be integrated is either odd or even.

Illustration 15

(a) **Find** $\int_{-1}^{1} x^3 e^{x^4}\,dx$ **(b)** **Find** $\int_{-1}^{1} x|x|\,dx$ **(c) Evaluate** $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\,dx$

Solution :

(a) Let $f(x) = x^3 e^{x^4}$

Then $f(-x) = (-x)^3 e^{(-x)^4} = -x^3 e^{x^4} = -f(x)$

Hence $f(x)$ is an odd function.

$\therefore$ $\int_{-1}^{1} f(x)dx = 0$ or $\int_{-1}^{1} x^3 e^{x^4} dx = 0$

(b) Let $f(x) = x|x|$

Then $f(-x) = -x|-x| = -x|x| = -f(x)$ $[\because\ |x| = |-x|]$

Hence $f(x)$ is an odd function.

$$\int_{-1}^{1} f(x)dx = 0 \qquad \text{or} \qquad \int_{-1}^{1} \frac{|x|}{x} dx = 0$$

(c) Let $f(x) = x^3 \sin^4 x$

Then $f(-x) = (-x)^3 \sin^4(-x) = -x^3(-\sin x)^4$

$$= -x^3 \sin^4 x = -f(x).$$

Hence f(x) is an odd function

$$\therefore \quad \int_{-\pi/4}^{\pi/4} f(x)dx = 0$$

Illustration 16

(a) Show that $\int_{-a}^{a} f(x^2)dx = 2\int_{0}^{a} f(x^2)dx$ **(b) Evaluate** $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}dx$

Solution :

(a) $f[(-x)^2] = f(x^2)$. Hence $f(x^2)$ is an even function.

$$\therefore \quad \int_{-a}^{a} f(x^2)dx = 2\int_{0}^{a} f(x^2)dx$$

(b) $I = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}dx = \int_{-a}^{a} \frac{a-x}{\sqrt{a^2-x^2}}dx$

$$= a\int_{-a}^{a} \frac{dx}{\sqrt{a^2-x^2}} - \int_{-a}^{a} \frac{a-x}{\sqrt{a^2-x^2}}dx$$

$$= a.2\int_{0}^{a} \frac{dx}{\sqrt{a^2-x^2}} - 0 \qquad \left[\because \frac{x}{\sqrt{a^2-x^2}} \text{ is an odd function}\right]$$

$$= 2a.\left[\sin^{-1}\frac{x}{a}\right]_0^a = 2a[\sin^{-1}(1) - \sin^{-1} 0]$$

$$= 2a\left(\frac{\pi}{2} - 0\right) = \pi a$$

Property-7

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx; & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

Proof. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$

Put $x = 2a - t$ in 2nd inetgral $(dx = -dt)$ when $x = a$ then $t = a$

when $x = 2a$ then $t = 0$

$$\therefore \quad \int_a^{2a} f(x)dx = -\int_a^0 f(2a-t)dt = \int_0^a f(2a-t)dt = \int_0^a f(2a-x)dx$$

$$\therefore \quad \int_0^{2a} f(x)dx = \int_0^a f(2a-x)dx$$

If $f(2a-x) = f(x)$

then $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$

and if $f(2a-x) = -f(x)$

then $\int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx = 0$

Illustration 17

Evaluate

(a) $\int_0^{2\pi} \cos^5 x dx$ **(b)** $\int_0^{\pi} \frac{x dx}{1+\cos^2 x}$

Solution :

(a) We will first check for the property-7 conditions. For that let

$$f(x) = \cos^5 x$$

then $f(2\pi - x) = \cos^5(2\pi - x)$

$$= \cos^5 x = f(x)$$

$$\therefore \quad \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \qquad \text{if } f(2a-x) = f(x)$$

applying the rule.

$$I = 2\int_0^{\pi} \cos^5 x dx \qquad \text{...(i)}$$

Now applying property-4

$$I = 2\int_0^{\pi} \cos^5(\pi - x)dx$$

$$= 2\int_0^{\pi} -\cos^5 xdx$$

$$= -2\int_0^{\pi} \cos^5 xdx \qquad ...(ii)$$

adding (i) and (ii)

$$2I = 0$$

$$\Rightarrow \quad I = 0$$

(b) $I = \int_0^{\pi} \frac{xdx}{1+\cos^2 x}$ ...(i)

This is an interesting problem, because here. Property-7 is not visible at first. So the tip that we can derive from this question is that approach the question as given rather than going by a fixed mind.

Here we can see that property-4 is applicable so without thinking anything else I will use it first

$$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x)dx}{1+\cos^2(\pi - x)} = \int_0^{\pi} \frac{(\pi - x)dx}{1+\cos^2 x} \qquad ...(ii)$$

adding (i) and (ii)

$$2I = \int_0^{\pi} \frac{xdx}{1+\cos^2 x} + \int_0^{\pi} \frac{\pi - x}{1+\cos^2 x}dx$$

$$= \int_0^{\pi} \frac{\pi}{1+\cos^2 x}dx$$

Now if I apply property-4 back then I will have no advantage as I will get the same integral. Hence no use. But if I apply Property-7.

$$I = \frac{1}{2}\left[2\int_0^{\pi/2} \frac{\pi dx}{1+\cos^2 x}\right] \qquad \text{as } f(2a - x) = f(x)$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\pi dx}{1+\cos^2 x}$$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^{\pi} \frac{\pi dx}{1+\cos^2 x}$$

MISTAKE : Common mistake at this step, is to take it as a normal substitution integral.

i.e. $I=\frac{\pi}{2}\int_0^{\pi}\frac{\sec^2 x}{\sec^2 x+1}dx$ (dividing by $\cos^2 x$)

Now the common substiution

$x=\tan x$

$dx=\sec^2 xdx$

But wait this is a wrong step, as per the rule of substitution the function which is substituited should be continuous in the interval.

But there for $x=\tan x$, $\tan x$ is not continuous over interval $[0,\pi]$. It is discontinuous at $x=\frac{\pi}{2}$.

Therefore, it is not possible to substitute $\tan x$ in the interval $[0,\pi]$.

So next thought should be to break the interval so that we can aply the transformation.

Illustration 18

Prove $\int_0^{\pi/2}\log\sin xdx=\int_0^{\pi/2}\log\cos xdx=-\frac{\pi}{2}\log 2$

Solution : Let $I=\int_0^{\pi/2}\log\sin xdx$ ...(i)

clearly property-4 is applicable here,

$\therefore \quad I=\int_0^{\pi/2}\log\left(\frac{\pi}{2}-x\right)dx=\int_0^{\pi/2}\log\cos xdx$ ...(ii)

adding (i) and (ii)

$$2I=\int_0^{\pi/2}(\log\sin x+\log\cos x)dx$$

$$=\int_0^{\pi/2}(\log\sin x\cos x)dx \quad =\int_0^{\pi/2}\log\left(\frac{\sin 2x}{2}\right)dx$$

$$=\int_0^{\pi/2}(\log\sin 2x-\log 2)dx$$

$$=\int_0^{\pi/2}\log\sin 2xdx-\int_0^{\pi/2}\log 2dx$$

$$=\int_0^{\pi/2}\log\sin 2xdx-\log 2[x]_0^{\pi/2}$$

$$\Rightarrow \quad 2I=\int_0^{\pi/2}\log\sin 2xdx-(\log 2)\left(\frac{\pi}{2}\right) \quad ...(iii)$$

Now let us solve the integral part separately.

$$I' = \int_0^{\pi/2} (\log \sin 2x)\, dx$$

See if I apply property-4 again here I will again get I', which becomes futile as I am struck here. So what should I do ?

There are only 2 options

1. I' gives a definite value, which does not seems to be the case here.
2. express I' in terms of I to solve the question.

Now, let

$$2x = t$$

$$\Rightarrow \quad 2dx = dt$$

$$\therefore \quad I' = \int_0^{\pi} \frac{1}{2} \log(\sin t)\, dt \qquad \text{...(iv)}$$

Now (iv) is almost similar to I with the only difference being in the upper limit.

$\therefore$ applying property -

$$I' = \left(\frac{1}{2}\right) 2 \int_0^{\pi/2} \log \sin t dt \quad = \int_0^{\pi/2} \log(\sin t)\, dt = I$$

Putting this value back in (iii)

$$2I = I - \frac{\pi}{2} \log 2$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log \sin x dx = \int_0^{\pi/2} \cos x dx = -\frac{\pi}{2} \log 2$$

Now solving it as we would have done in indefinite integral case.

$$I = \pi \int_0^{\pi/2} \frac{\sec^2 x dx}{1 + \sec^2 x}$$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x dx}{2 + \tan^2 x}$$

Now obvisously we will substitute $\tan x = t$

as $x = 0 \qquad \tan x = t \to 0$

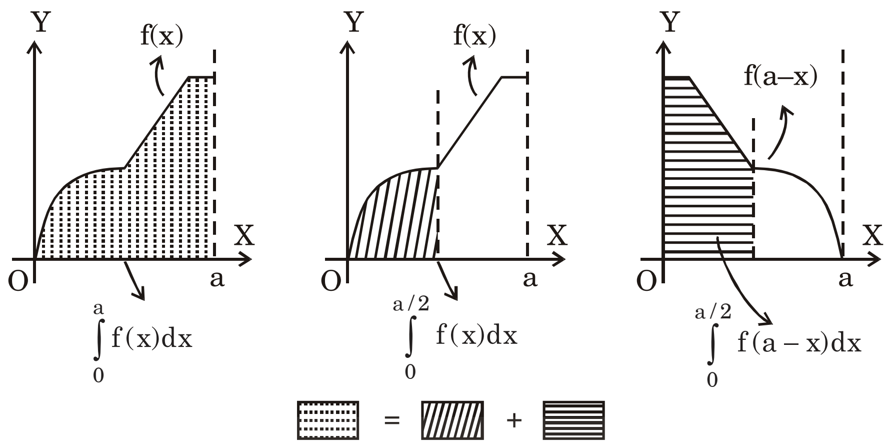
$x = \frac{\pi}{2} \qquad \tan x = t \to \infty$

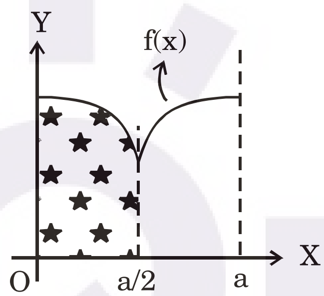
$$I = \int_0^{\infty} \frac{\pi dt}{2 + t^2} = \frac{\pi}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} \Bigg|_0^{\infty}$$

$$= \frac{\pi}{\sqrt{2}} \left(\tan^{-1} \infty - \tan^{-1} 0\right) \quad = \frac{\pi}{\sqrt{2}} \left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{2\sqrt{2}}$$

Property 8 $\boxed{\int_0^a f(x)dx = \int_0^{a/2} f(x)dx + \int_0^{a/2} f(a-x)dx}$

Geometrical Proof :





If $f(a-x) = f(x) \quad \forall x \in (0, a)$

i.e. $f(x)$ is symmetrical about $x = \dfrac{a}{2}$.

then $\int_0^a f(x)dx = 2\times \int_0^{a/2} f(x)dx$

Illustration 19

Show that $\int_0^{\pi/2} f(\sin 2x)\sin x dx = \int_0^{\pi/2} f(\sin 2x)\cos x dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x dx$

Solution :

Let $I = \int_0^{\pi/2} f(\sin 2x)\sin x dx$ ...(i)

$\Rightarrow I = \int_0^{\pi/2} f\left[\sin 2\left(\frac{\pi}{2} - x\right)\right]\sin\left(\frac{\pi}{2} - x\right)dx$ (using property-4)

$\Rightarrow I = \int_0^{\pi/2} f\left[\sin(\pi - 2x)\right]\cos x dx$

$\Rightarrow I = \int_0^{\pi/2} f(\sin 2x)\cos x dx$ ...(ii)

Hence the first part is proved.

$I = \int_0^{\pi/2} f(\sin 2x)\sin x dx$

$$= \int_0^{\pi/4} f(\sin 2x)\sin x dx + \int_0^{\pi/4} f\left[\sin 2\left(\frac{\pi}{2} - x\right)\right]\sin\left(\frac{\pi}{2} - x\right)dx \quad \text{(using Property-5)}$$

$$= \int_0^{\pi/4} f(\sin 2x)\sin x dx + \int_0^{\pi/4} f(\sin 2x)\cos x dx$$

$$= \int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)dx$$

$$= \int_0^{\pi/4} f\left[\sin 2\left(\frac{\pi}{4} - x\right)\right]\left[\sin\left(\frac{\pi}{4} - x\right) + \cos x\left(\frac{\pi}{4} - x\right)x\right]dx \quad \text{(using property-4)}$$

$$= \int_0^{\pi/4} f(\cos 2x)\left[\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right]dx$$

$$= \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x dx$$

Hence the second part is also proved.

Some more algebraic properties :

Property-9 $\boxed{\int_a^b f(x)dx = (b-a)\int_0^1 f((b-a)x + a)dx}$

for the proof of this property we will approach from R.H.S.

Put $z = (b-a)x + a$ in R.H.S.

$\Rightarrow \quad dz = (b-a)dx$

& when $x = 0$, $\quad z = a$

$x = 1$, $\quad z = b$

$\therefore$ new integral becomes

$$\int_a^b (b-a)f(z)\frac{dz}{(b-a)} = \int_a^b f(x)dx = \text{L.H.S.}$$

some other properties.

1. $\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x)dx$

2. $\int_a^b f(x)dx = \int_{a-c}^{b-c} f(x+c)dx$

   or $\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x-c)dx$

3. $\int_a^b f(x)dx = \frac{1}{c}\int_a^b f\left(\frac{x}{c}\right)dx$

Illustration 20

Evaluate $\int_{-4}^{-5} e^{(x+5)^2} dx + 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2} dx$

Solution :

Note : Here we know $\int e^{x^2} dx$ cannot be evaluated by idefinite integral

Thus, $I_1 = \int_{-4}^{-5} e^{(x+5)^2} dx$

$$= (-5+4)\int_0^1 e^{((-5+4)x-4+5)^2} dx$$

$\therefore$ $I_1 = -\int_0^1 e^{(x-1)^2} dx$ ...(i)

again, let

$$I_2 = \int_{1/3}^{2/3} e^{9(x-2/3)^2} .dx$$

$$= \left(\frac{2}{3}-\frac{1}{3}\right)\int_0^1 e^{9\left[\left(\frac{2}{3}-\frac{1}{2}\right)x+\frac{1}{3}-\frac{2}{3}\right]^2} .dx$$

$$= \frac{1}{3}\int_0^1 e^{(x-1)^2} .dx$$

$$= \frac{1}{3}(-I_1) \quad \text{...(ii)}$$

where, $I = I_1 + 3I_2$

$$= I_1 + 3\left(-\frac{I_1}{3}\right)$$

$$= I_1 - I_1$$

$$I = 0$$

$\therefore$ $\int_{-4}^{-5} e^{(x+5)^2} dx + 3\int_{1/3}^{2/3} e^{9(x-2/3)^2} dx = 0$

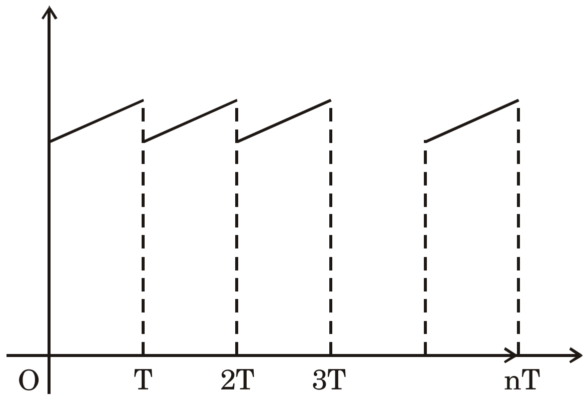
PROPERTIES RELATED TO PERIODICITY

Property-10 : I $f(x)$ is a periodic function with period T then $\int_0^{nT} f(x)dx = n\int_0^{T} f(x)dx$

The proof of this property is really easy one.

Geometrical Proof.

If $f(x)$ is periodic then it will repeat (the curve also) after an interval of T.



Since the area under the curve for an interval of T is same everytime.

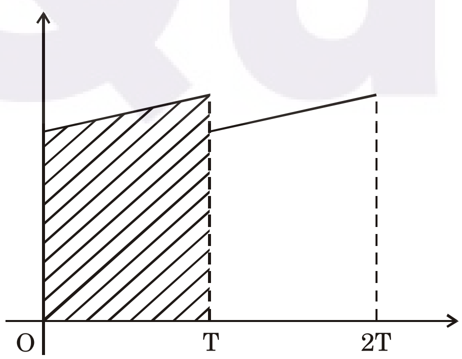
Total area = $n \times (\text{curve under one interval})$

$$\therefore \quad \int_0^{nT} f(x)dx \backslash = n\int_0^T f(x)dx$$

Property-11 $\boxed{\int_a^{a+T} f(x)dx = \int_0^T f(x)dx}$

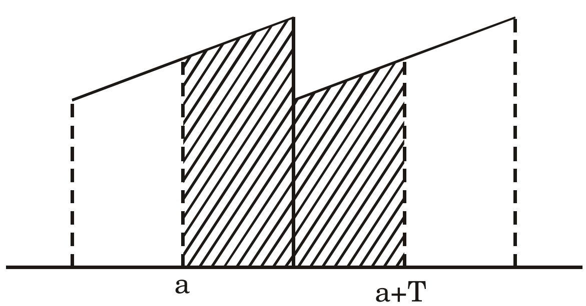
Let us see the proof using geometry

if



$\int_0^T f(x)dx$

now



so observation only we can say

$$\int_0^T f(x)dx = \int_a^{a+T} f(x)dx$$

Property-12 : Generalization of the above property is

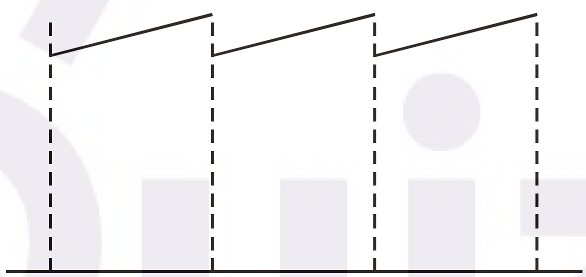
$$\boxed{\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx}$$

$$\int_a^{a+nT} f(x)dx = \int_0^{nT} f(x)dx = n\int_0^T f(x)dx$$

We can also use gemoetrical proof here.

Property-13 : $\boxed{\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx,}$ $n, m \in I$

Geometrical Proof.



We can write,

$$\int_{mT}^{nT} f(x)dx = \int_0^{nT} f(x)dx - \int_0^{mT} f(x)dx$$

i.e. area of curve between = area of curve uptil nT - area of curve uptil mT

$$\therefore \qquad \int_{mT}^{nT} f(x)dx = n\int_0^T f(x)dx - m\int_0^T f(x)dx$$

$$= (n-m)\int_0^T f(x)dx$$

Some other properties deduced from earlier properties

1. $\int_{nT}^{a+nT} f(x)dx = \int_0^a f(x)dx$

2. $\int_{a+mT}^{a+nT} f(x)dx = \int_{mT}^{nT} f(x)dx$

3. $\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx$

Illustration 21

(a) Prove that $\int_0^{10}(x-[x])dx = 5$. **(b)** $\int_0^{100} e^{x-[x]}dx = 100(e-1)$ **(c)** $\int_0^{400\pi}\sqrt{1-\cos 2x}dx = 800\sqrt{2}$

Solution :

(a) Since $x-[x]$ is a periodic function with period one unit. Therefore

$$\int_0^{10}(x-[x])dx = 10\int_0^1(x-[x])dx = 10\left[\int_0^1 x dx - \int_0^1 [x]dx\right]$$

$$= 10\left[\left[\frac{x^2}{2}\right]_0^1 - 0\right] = \frac{10}{2} = 5$$

(b) Since $x-[x]$ is a periodic function with period one unit, therefore so is $e^{x-[x]}$, and hence

$$\int_0^{100} e^x - [x]dx = 100\int_0^1 e^{x-[x]}dx = 100\int_0^1 e^{x-0}dx$$

$$= 100\int_0^1 e^x.dx = 100(e-1)$$

(c) $\int_0^{400\pi}\sqrt{1-\cos 2x} = \int_0^{400\pi}\sqrt{2}|\sin x|\,dx$

$$= \sqrt{2}\times 400\int_0^{\pi}|\sin x|dx \qquad [\because\ |\sin x| \text{ is periodic with period } \pi\ ]$$

$$= 400\sqrt{2}\int_0^{\pi}\sin x dx = 400\sqrt{2}\left[-\cos x\right]_0^{\pi} = 800\sqrt{2}$$

Illustration 22

(a) Evaluate $\int_0^{4\pi}|\cos x|\,dx$ **(b) Evaluate** $\int_0^{32\pi/3}\sqrt{1+\cos 2x}dx$

Solution :

(a) Note that $|\cos x|$ is a periodic with period π.

Hence the given integral,

$$I = 4\int_0^{\pi}|\cos x|\,dx$$

$$= 4\left\{\int_0^{\pi/2}\cos x dx - \int_{\pi/2}^{\pi}\cos x dx\right\}$$

$$= 4\left\{(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\right\} = 4(1+1) = 8.$$

(b) $\int_0^{32\pi/3}\sqrt{1+\cos 2x}dx=\sqrt{2}\int_0^{10\pi}|\cos x|dx+\sqrt{2}\int_{10\pi}^{32\pi/3}|\cos x|dx$

$$=10\sqrt{2}\int_0^{\pi}|\cos x|dx+\sqrt{2}\int_0^{2\pi/3}|\cos x|dx$$

$$=10\sqrt{2}\left[\int_0^{\pi/2}\cos xdx-\int_{\pi/2}^{\pi}\cos xdx\right]+\sqrt{2}\left[\int_0^{\pi/2}\cos xdx+\int_{\pi/2}^{2\pi/3}-\cos xdx\right]$$

$$=10\sqrt{2}[1+1]+\sqrt{2}\left[1-\frac{\sqrt{3}}{2}+1\right]$$

$$=20\sqrt{2}+\sqrt{2}\left(2-\frac{\sqrt{3}}{2}\right)=22\sqrt{2}-\sqrt{\frac{3}{2}}$$

Illustration 23

Show $\int_0^{n\pi+V}|\sin x|dx=(2n+1)-\cos V$**, where** n **is positive integer. and** $0\le V<\pi$**.** [IIT-1994]

Solution :

$$\int_0^{n\pi+V}|\sin x|dx=\int_0^{V}|\sin x|dx+\int_V^{n\pi+V}|\sin x|dx$$

$$=\int_0^{V}\sin xdx+n\int_0^{\pi}|\sin x|dx \quad \text{(Using Property-IX)}$$

$$=(-\cos x)_0^V+n\int_0^{\pi}\sin xdx$$

$$=(-\cos V+1)+n(-\cos x)_0^{\pi}$$

$$=-(\cos V)+1+n(1+1)$$

$$=(2n+1)-\cos V$$

$$\Rightarrow \quad \int_0^{n\pi+V}|\sin x|dx=(2n+1)-\cos V$$

where n is positive integer and $0\le V<\pi$.

PROPERTIES INCLUDING INEQUALITIES :

1. If $f(x)\le 0$ on an interval $[a,b]$, then

$$\int_a^b f(x)dx\le 0$$

or if $f(x)\ge 0$, then $\int_a^b f(x)dx\ge 0$

The reason is very obvious because if $f(x) \geq 0$ the area will be above x-axis i.e. positive & for $f(x) < 0$ it will be negative.

2. Property-14 :

$$\boxed{\text{If } f(x) \leq g(x) \text{ on } [a,b], \text{ then } \int_a^b f(x)dx \leq \int_a^b g(x)dx}$$

The proof is similar to the reasoning given in the above case.

3. Property-15 : If m and M are the smallest & largest values of function $f(x)$ defined on an interval $[a,b]$ then

$$\boxed{m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)}$$

Proof. It is given that

$$m \leq f(x) \leq M$$

integrating both sides

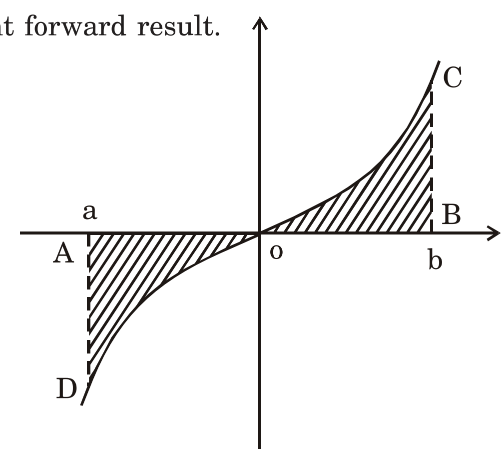
$$\int_a^b m dx \leq \int_a^b f(x)dx \leq \int_a^b M dx$$

$$\Rightarrow \quad m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

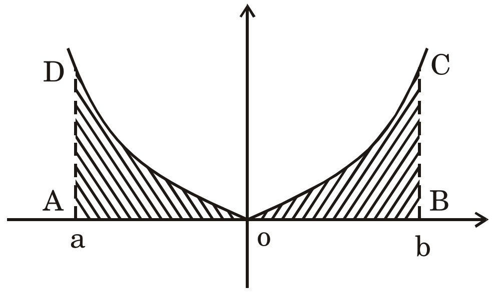
Property-16 : If $f(x)$ is defined over $[a,b]$ then

$$\boxed{\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx}$$

This is very straight forward result.



So $\left|\int_a^b f(x)dx\right| = |\text{area OBC - area OAD}|$ ...(i) whereas



$\therefore \quad \int_a^b |f(x)|\,dx = \text{area OBC} + \text{area OAD}$ ...(ii)

compare (i) and (ii) to get to the result.

Property-17 : If $f^2(x)$ & $g^2(x)$ are integrable over $[a,b]$ then

$$\boxed{\left|\int_a^b f(x)g(x)dx\right| \le \left(\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)\right)^{1/2}}$$

Illustration 24

(a) Show that $\left|\int_{10}^{19} \frac{\sin x}{1+x^8}dx\right| < \frac{1}{10^7}$ **(b)** Prove that $\frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x}} \le \frac{\pi}{4\sqrt{2}}$

Solution :

(a) $\left|\int_{10}^{19} \frac{\sin x}{1+x^8}dx\right| \le \int_{10}^{19}\left|\frac{\sin x}{1+x^8}\right|dx$ $\left[\because \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx\right]$

$$= \int_{10}^{19}\frac{|\sin x|}{1+x^8}dx$$

$$\le \int_{10}^{19}\frac{1}{1+x^8}dx \le \int_{10}^{19}\frac{1}{x^8}dx \qquad [\because |\sin x| \le 1]$$

$$= \left[-\frac{1}{7x^7}\right]_{10}^{19} = \frac{1}{7.(10)^7} - \frac{1}{7.(19)^7} \qquad < \frac{1}{7.(10)^7} < \frac{1}{10^7}$$

(b) $0 \le x \le 1$

$\therefore \quad 4-x^2 \ge 4-x^2-x^3 \ge 4-x^2-x^2$ $[\because x^2 > x^3]$

$\Rightarrow \quad 4-x^2 \ge 4-x^2-x^3 \ge 4-2x^2 > 1$

$\Rightarrow \quad 4-x^2 \ge 4-x^2-x^3 \ge \sqrt{4-2x^2}$

$\Rightarrow \quad \sqrt{4-x^2} \ge \sqrt{4-x^2-x^3} > \sqrt{4-2x^2}$

$$\Rightarrow \quad \frac{1}{\sqrt{4-x^2}} \le \frac{1}{\sqrt{4-x^2-x^3}} \le \frac{1}{\sqrt{4-2x^2}}$$

$$\Rightarrow \quad \int_0^1 \frac{dx}{\sqrt{4-x^2}} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{1}{\sqrt{2}}\int_0^1 \frac{dx}{\sqrt{2-x^2}}$$

$$\Rightarrow \quad \left[\sin^{-1}\frac{x}{2}\right]_0^1 \le I \le \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1$$

$$\Rightarrow \quad \frac{\pi}{6} \le I \le \frac{\pi}{4\sqrt{2}}$$

Illustration 25

(a) **Prove that** $4 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30}$ **(b)** **Prove that** $\int_0^1 \sqrt{(1+x)(1+\pi^3)}\,dx \le \sqrt{\frac{15}{8}}$.

Solution :

(a) Let $y = \sqrt{3+x^3}$, then $\frac{dy}{dx} = \frac{-3x^2}{2\sqrt{3+x^3}} > 0$

$\therefore$ y is an increasing function

$\therefore$ $1 \le x \le 3$

$$\Rightarrow \quad \sqrt{3+1^3} \le \sqrt{3+x^3} \le \sqrt{3+3^3}$$

$$\Rightarrow \quad 2 \le \sqrt{3+x^3} \le \sqrt{30}$$

$$\Rightarrow \quad \int_1^3 2dx \le \int_1^3 \sqrt{3+x^3}\,dx \le \sqrt{30}\int_1^3 dx$$

$$\Rightarrow \quad 4 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30}$$

(b) Let $f(x) = \sqrt{1+x}$ and $g(x) = \sqrt{1+x^3}$

If $f^2(x)$ and $g^2(x)$ and $f(x)g(x)$ are integrable functions on [a,b], then

$$\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)}$$

$$\therefore \quad \int_0^1 \sqrt{(1+x)(1+x^2)}\,dx \le \sqrt{\left(\int_0^1 (1+x)dx \int_0^1 (1+x^3)dx\right)}$$

$$= \sqrt{\left[x+\frac{x^2}{2}\right]_0^1 \left[x+\frac{x^4}{4}\right]_0^1}$$

$$= \sqrt{\frac{3}{2}.\frac{5}{4}} = \sqrt{\frac{15}{8}}$$

Thus, $\int_0^1 \sqrt{(1+x)(1+x^3)}dx \le \sqrt{\frac{15}{8}}$.

Leibnitz's Rule for differentiation

Case-I. If the limits are function of the variable whose derivative is taken.

i.e. $$\frac{d}{dx}\underbrace{\left[\int_{g(x)}^{h(x)} f(t)dt\right]}_{\text{independent of x}} = f\left[h(x)\right]\times h'(x) - f\left[g(x)\right]\times g'(x)$$

a very common case is

$$\frac{d}{dx}\int_0^x f(t)dt = f(x)$$

Case-2. If the integrand is the function of variable whose derivative is taken, i.e.

$$\frac{d}{dx}\left[\int_a^b f(x,t)dt\right] = \int_a^b \frac{d}{dx} f(x,t)dt$$

taking t as a constant while differentiating.

Case-3. General Case :

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t,x)dt = \int_{g(x)}^{h(x)} \frac{d}{dx} f(t,x)dt + f\left[h(x)\right]h'(x) - f\left[g(x)\right]\times g'(x)$$

You can see that its a combination of both the results.

TIP :

Learn the formulas by variable of differentiation, integrand & limit of the integral not by cases.

Property-18. Let a function $f(x,a)$ be continuous for $a \le x \le b$ and $c \le \alpha \le d$. Then for any $\alpha \in [c,d]$,

if $I(\alpha) = \int_a^b f(x,\alpha)dx$, then

$$\boxed{\frac{dI(\alpha)}{d\alpha} = \int_a^b \frac{\partial(f(x,\alpha))}{\partial\alpha}dx}$$

Illustration 26

Differentiate the following w.r.t. x

(a) $\int_0^{x^2}(\cos t^2)dt$ (b) $\int_{1/x}^{\sqrt{x}}\sin t^2 dt$

Solution :

(a) We have to find $I = \frac{d}{dx}\left(\int_0^{x^2}\cos t^2 dt\right)$

This is an example of case-1 where only limits are a function of x.

$$\therefore \quad I = \left(\cos(x^2)^2\right)\times\frac{d}{dx}(x^2) - \cos(0).0$$

$$\Rightarrow \quad I = 2x\cos x^4$$

(b) $I = \frac{d}{dx}\int_{1/x}^{\sqrt{x}}\sin t^2 dt$

This is again an example of case-1

$$I = \left(\sin\left(\sqrt{x}\right)^2\right)\times\frac{d}{dx}\left(\sqrt{x}\right) - \left(\sin\left(\frac{1}{x}\right)^2\right)\times\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \sin x \times \frac{1}{2\sqrt{x}} - \sin\left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{2\sqrt{x}}\sin x + \frac{1}{x^2}\sin\left(\frac{1}{x^2}\right)$$

Illustration 27

Find the points of maxima / minima of $\int_0^{x^2}\frac{t^2-5t+4}{2+e^t}dt$ **.**

Solution. We will apply the normal rules of maxima/minima & for maxima/minima we differentiate

$$\therefore \text{ if} \quad f(x) = \int_0^{x^2}\frac{t^2-5t+4}{2+e^t}dt$$

$$\text{then} \quad f'(x) = \frac{d}{dx}\int_0^{x^2}\left(\frac{t^2-5t+4}{2+e^t}\right)dt$$

again case-1 example, only limit is a function of x.

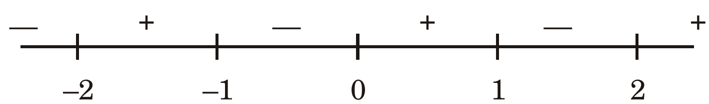
$$\therefore \quad f'(x) = \frac{(x^2)^2-5(x^2)+4}{2+e^{x^2}}.\frac{d}{dx}(x^2) - 0$$

$$= 2x\left[\frac{(x^2)^2 - 5(x^2) + 4}{2 + e^{x^2}}\right]$$

$$= \frac{2x(x^2-4)(x^2-1)}{2+e^{x^2}}$$

$$= \frac{2x.(x-2)(x+2)(x-1)(x+1)}{2+e^{x^2}}$$

equality it to zero.



Hence the points of maxima (i.e. where sign = -1, 1 changes from +ve to -ve) & points of minma (where sign changes from -ve to +ve) = -2,0,2

Illustration 28

(a) If $y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x.\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta$, find $\frac{dy}{dx}$ at $x = \pi$.

(b) Evaluate $\int_0^{\pi/2} \log(1+\sin\alpha\sin^2 x)\mathrm{cosec}^2 x dx$.

Solution :

(a) $y = \int_{\pi^2/16}^{x^2} (\cos x).\frac{\cos\sqrt{\theta}}{1+\sin^2\theta}d\theta$

here $\cos x$ is a constant in integration, so it can be moved out of integral & this is the trick here.

$$y = \cos x.\int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta$$

now differentiate using product rule.

$$\frac{dy}{dx} = \frac{d}{dx}(\cos x)\times\int_{\pi^2/16}^{x^2}\frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} + \cos x.\frac{d}{dx}\int_{\pi^2/16}^{x^2}\frac{\cos\sqrt{\theta}d\theta}{1+\sin^2\sqrt{\theta}}$$

the derivative of integral is an example of our case-1 i.e. integrand is not the function of x, only limits are

$$\frac{dy}{dx} = -\sin x.\int_{\pi^2/16}^{x^2}\frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta + (2x)\times(\cos x)\frac{\cos\sqrt{x^2}}{1+\sin^2\sqrt{x^2}} + 0$$

$$= \frac{2x\cos^2 x}{1+\sin^2 x} - \sin x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta \qquad \text{(i)}$$

now solving the integral by substituting

$$\theta = t^2$$

$\Rightarrow$ $\qquad d\theta = 2tdt$

& limits at $\quad \theta = x^2 \qquad t = x$

$$\theta = \frac{\pi^2}{16} \qquad t = \frac{\pi}{4}$$

$\therefore$ Integral $\quad I = \int_{\pi/4}^{x} \frac{\cos t}{1+\sin t} 2tdt$

But stop, we are solving in futile. We need to find the value of $\frac{dy}{dx}$ at $x = \pi$ & ahead of this integral is $\sin x$ which is 0 at $x = \pi$, so no need of solving the integral

$$\therefore \quad \left.\frac{dy}{dx}\right]_{x=\pi} = \frac{2\pi\cos^2(\pi)}{1+\sin^2(\pi)} = 2\pi\cos^2(\pi) \quad = 2\pi$$

(b) $\quad I = \int_0^{\pi/2} \log(1+\sin\alpha\sin^2 x)\operatorname{cosec}^2 x dx$

now this is question based on property-18.
Here I is a function of α, so according to property

$$\frac{dI}{d\alpha} = \int_0^{\pi/2} \frac{\partial}{\partial\alpha}\left(\log(1+\sin\alpha\sin^2 x)\right)\operatorname{cosec}^2 x dx$$

∂ means differentiating the function containing α only & taking all other variables as constant while differentiating.

$$\therefore \quad \frac{\partial I}{\partial\alpha} = \int_0^{\pi/2} \frac{1}{(1+\sin\alpha\sin^2 x)} \times \sin^2 x\cos\alpha.\operatorname{cosec}^2 x dx$$

$$= \int_0^{\pi/2} \frac{\cos\alpha dx}{(1+\sin\alpha\sin^2 x)} = \int_0^{\pi/2} \frac{\cos\alpha\sec^2 x dx}{\operatorname{cosec}^2 x + \sin\alpha\tan^2 x}$$

$$= \int_0^{\pi/2} \frac{\cos\alpha\sec^2 x dx}{1+(1+\sin\alpha)\tan^2 x}$$

Put $\tan x = t$

$$\sec^2 x dx = dt$$

& limits at $x = 0$ $t = 0$

$x = \frac{\pi}{2}$ $t = \infty$

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{\cos\alpha\, dt}{1+(+\sin\alpha)t^2}$$

$$= \frac{\cos\alpha}{(1+\sin\alpha)}\int_0^\infty \frac{dt}{t^2+\frac{1}{\sin\alpha}}$$

$$= \frac{\cos\alpha}{1+\sin\alpha}\tan^{-1} t\sqrt{1+\sin\alpha}\Big|_0^\infty \left\{\sqrt{1+\sin\alpha}\right\}$$

$$= \frac{\cos\alpha}{1+\sin\alpha}\left(\sqrt{1+\sin\alpha}\right)\left(\frac{\pi}{2}-0\right)$$

$$= \frac{\pi.\cos\alpha}{2(1+\sin\alpha)}\sqrt{1+\sin\alpha}$$

$$= \frac{\pi}{2}\frac{\cos\alpha}{\sqrt{1+\sin\alpha}}$$

$$= \frac{\pi}{2}\frac{(\cos^2\alpha/2-\sin^2\alpha/2)}{\sin\alpha/2+\cos\alpha/2}$$

as $1+\sin\alpha = \left(\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}+2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\right)$

$$= \frac{\pi}{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)$$

$\therefore$ $\frac{dI(\alpha)}{d\alpha} = \frac{\pi}{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)$

now integrating to get the value of I

$$I = \frac{\pi}{2}\int\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)dx$$

$$I = \frac{\pi}{2}\left(2\sin\frac{\alpha}{2}+2\cos\frac{\alpha}{2}\right)+C$$

$$I = \pi\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)+C \quad \text{...(i)}$$

for value of C put $\alpha = 0$ in the equation

$$I(0) = \pi(\sin 0+\cos 0)+C$$
$$= \pi + C \quad \text{... (ii)}$$

& $I(0)$ can be found out using the original function.

& $I(a) = \int_0^{\pi/2} \log(1+\sin\alpha\sin^2 x)\operatorname{cosec}^2 x dx$

$$I(0) = \int_0^{\pi/2} (\log(1)).\operatorname{cosec}^2 x dx = 0$$

Putting this in (ii)

$$I(0) = \pi + C$$

$\Rightarrow$ $0 = \pi + C$

$\Rightarrow$ $C = -\pi$

putting this value in (i)

$$I(\alpha) = \pi\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)-\pi$$

Property-19 : If $f(t)$ is an odd function, then $g(x) = \int_a^x f(t)dt$ is an even function.

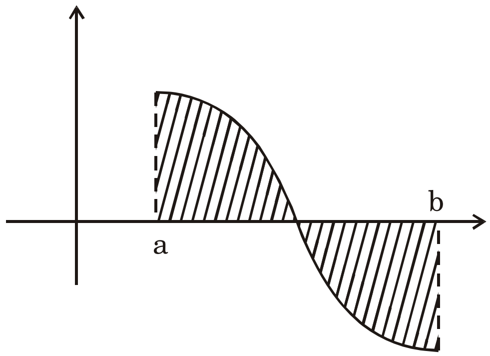
Property-20 : If $f(t)$ is an even function, then $g(x) = \int_0^x f(t)dt$ is an odd function.

NOTE : For $\int_a^x f(t)dt$ might not be an odd function. So apply the above property taking special care of limits.

Property-21 : If $f(x)$ is a continuous function on $[a,b]$ then there exists a point $c \in (a,b)$ such that $\int_a^b f(x)dx = f(c)(b-a)$. This is known as Mean Value Theorem of Integration.

Property-22 : If $f(x)$ is continuous in $[a,b]$ & $\int_a^b f(x) = 0$ then the equation $f(x) = 0$ has atleast one root in (a,b).

Proof of this property is very simple.



The area can be zero only iff there is some part of $f(x)$ below the x-axis (i.e. negative area). And for that to happen for a continuous function $f(x)$, $f(x)$ must cross the $y=0$ line at atleast one point.

IMPROPER INTEGRAL

If $f(x)$ is continuous on $[a,\infty]$, then $\int_a^{\infty} f(x)dx$ is called as improper integral and

$$\boxed{\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^{b} f(x)dx}$$

If the Right Hand Limit of integral exists then the improper integral is convergent otherwise it is divergent.

GAMMA FUNCTION

It is defined by the improper integral, by $\int_0^{\infty} e^{-x}x^{n-1}dx$ and is denoted by Γn

$\therefore \quad \Gamma n = \int_0^{\infty} e^{-x}x^{n-1}dx$ where x is a positive rational number.

Properties of Gamma function

1. $\Gamma 1 = 1,$ $\Gamma 0 = \infty$ and $\Gamma(n+1) = n\Gamma n$
2. if $n \in N$, $\Gamma(n+1) = n!$
3. $\Gamma(1/2) = \sqrt{\pi}$

Useful extensions of gamma function :

1. $\int_0^{\pi/2} \sin^2 x dx = \int_0^{\pi/2} \cos^n x dx$

$$= \begin{cases} \dfrac{n-1}{n}.\dfrac{n-3}{n-2}.\dfrac{n-5}{n-4}\ldots\dfrac{3}{4}.\dfrac{1}{2}.\left(\dfrac{\pi}{2}\right), & \text{if n is even} \\ \dfrac{n-1}{n}.\dfrac{n-3}{n-2}.\dfrac{n-5}{n-4}\ldots\dfrac{4}{5}.\dfrac{2}{3}.1, & \text{if n is odd} \end{cases}$$

2. $\int_0^{\pi/2} \sin^m x \cos^n x dx = \int_0^{\pi/2} \sin^n x \cos^m x dx$

$$= \frac{(m-1)(m-3)\ldots(1)(n-1)(n-3)\ldots(1)}{(m+n)(m+n-2)\ldots 2}.\frac{\pi}{2}$$ when both m & n belong to even integer

$$= \frac{(m-1)(m-3)\ldots(1 \text{ or } 2)(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots 1 \text{ or } 2}$$ when either of m or n belong to odd integer

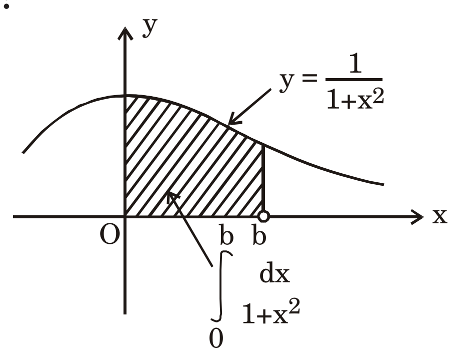
3. $\int_0^{\pi} \sin^m x \cos^n x dx = 0,$ if n is odd

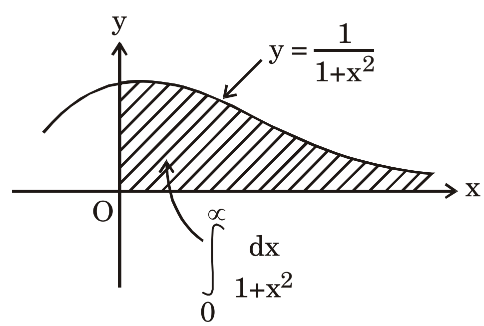
$$= 2\int_0^{\pi/2} \sin^m x \cos^n x dx,$$ if n is even

Illustration 29

Evaluate the integral $\int_0^{+\infty} \dfrac{dx}{1+x^2}$

Solution :





By the defintion of an improper integral we find

$$\int_0^{+\infty} \frac{dx}{1+x^2} + \lim_{b\to+\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b\to+\infty} \tan^{-1} x\Big|_0^b = \lim_{b\to+\infty} \tan^{-1} b = \frac{\pi}{2}.$$

Illustration 30

Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$

Solution :

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2}$$

The second integral is equal to $\frac{\pi}{2}$. Compute the first integral :

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{\alpha \to +\infty} \int_{\alpha}^{0} \frac{dx}{1+x^2} = \lim_{\alpha \to -\infty} \tan^{-1} x \Big|_{\alpha}^{0} = \lim_{\alpha \to -\infty} (\tan^{-1} 0 - \tan^{-1} \alpha) = \frac{\pi}{2}$$

Therefore

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

In many cases, it is sufficient to determine, whether the given intergal converges or diverges, and to estimate its value.

The Integral of a Discontinuous function :

A function $f(x)$ is defined and continuous when $a \le x < c$, and either not defined or discontinuous when $x = c$. In this case, one cannot speak of the integral $\int_a^c f(x)dx$ as the limit of integral sums, because $f(x)$ is not continuous on the interval $[a, c]$, and for this reason the limit may not exist.

The integral $\int_a^c f(x)dx$ of the function $f(x)$ discontinuous at the point c is defined as follows :

$$\int_a^c f(x)dx = \lim_{b \to c-0} \int_a^b f(x)dx$$

If the limit on the right exist, the integral is called an important convergent integral, otherwise it is divergent.

If the function $f(x)$ is dicsontinuous at the left extremity of the interval $[a, c]$ (that is, for $x = a$), then by defintion $\int_a^c f(x)dx = \lim_{b \to a+0} \int_b^c f(x)dx$

if the function $f(x)$ is discontinuous at some point $x = x_0$ inside the interval $[a, c]$, we put

$$\int_a^c f(x)dx = \int_a^{x_0} f(x)dx + \int_{x_0}^c f(x)dx$$

if both imporper integral on the right side of the equation exist.

Illustration 31

(a) Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x}}$. **(b) Evaluate the integral** $\int_{-1}^1 \frac{dx}{x^2}$

Solution :

(a) $\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{b\to 1-0} \int_0^b \frac{dx}{\sqrt{1-x}} = -\lim_{b\to 1-0} 2\sqrt{1-x}\,\Big|_0^b$

$$= -\lim_{b\to 1-0} 2\left(\sqrt{1-b}-1\right) = 2$$

(b) Since inside the interval of integration there exists a point $x=0$ where the integrad is dicsontinuous, the interal must be represented sum of two terms :

$$\int_{-1}^1 \frac{dx}{x^2} = \lim_{\varepsilon_1\to -0} \int_{-1}^{\varepsilon_1} \frac{dx}{x^2} + \lim_{\varepsilon_2\to +0} \int_{\varepsilon_2}^1 \frac{dx}{x^2}$$

Calculate each limit separately :

$$\lim_{a\to -0} \int_{-1}^{\varepsilon_1} \frac{dx}{x^2} = -\lim_{\varepsilon_1\to -0} \frac{1}{x}\Bigg|_{-1}^{\varepsilon_1} = -\lim_{\varepsilon_1\to -0}\left(\frac{1}{\varepsilon_1} - \frac{1}{-1}\right) = \infty$$

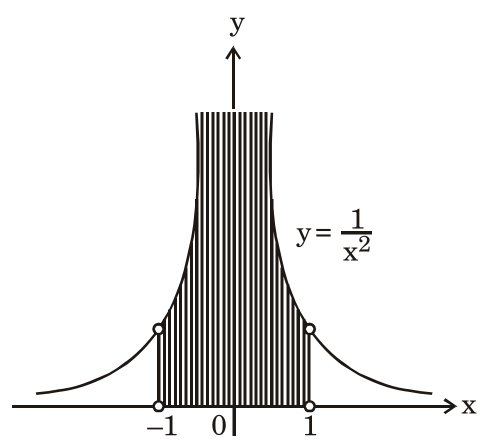
Thus, the integral diverges on the interval $[-1, 0]$

$$\lim_{\varepsilon_1\to +0} \int_{\varepsilon_1}^1 \frac{dx}{x^2} = -\lim_{\varepsilon_1\to +0}\left(1 - \frac{1}{\varepsilon_2}\right) = \infty$$

And this means that the integral also diverges on the interval [0,1]. Hence, the given integral diverges on the entire interval $[-1,1]$. It should be noted that if we had begun to evaluate the given integral without paying attention to the discontinuity of the intgerand at the point $x=0$, the result would have been wrong.

Indeed $\int_{-1}^1 \frac{dx}{x^2} = -\frac{1}{x}\Bigg|_{-1}^1 = -\left(\frac{1}{1} - \frac{1}{-1}\right) = -2$

which is impossible (fig.)



SUMMATION OF SERIES USING DEFINITE INTEGRAL AS A LIMIT OF SUM

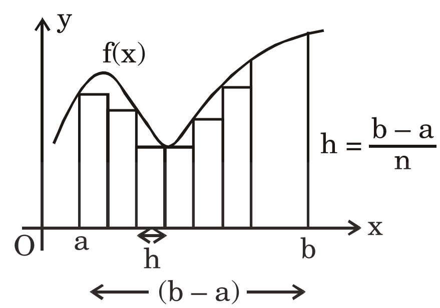
If $f(x)$ is an integrable function defined on $[a,b]$ then

$$\lim_{h\to 0}\left[h\left\{f(a)+f(a+h)+f(a+2h)+\ldots+f(a+(n-1)h\right\}\right]$$

is called the definite integral of $f(x)$ between limits a and b.

$$\therefore \quad \int_a^b f(x)dx = \lim_{h\to 0} h\left[\left\{f(a)+f(a+h)+f(a+2h)+\ldots+f(a+(n-1)h\right\}\right]$$

$$= \lim_{h\to 0}\left|\sum_{r=0}^{n} f(a+rh)\right|$$



It should be noted that as $b \to 0,\ a \to \infty$

$$nh = b - a$$

Putting $a = 0,\ b = 1$, so that $h = \frac{1}{n}$

We get $\int_0^t f(x)dx = \lim_{n\to\infty} \frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$

WORKING RULE

Step-1. Express the series in the form, $\lim_{n\to\infty}\left[\frac{1}{n}\sum f\left(\frac{r}{n}\right)\right]$

Step-2. Replace Σ by $\int$,

$\frac{r}{n}$ by x and

$\left(\frac{1}{n}\right)$ by dx

Step-3. Obtain the lower & upper limits of the integral by computing $\lim_{n\to\infty}\left(\frac{r}{n}\right)$ for the least & greatest value of r respectively i.e. put the starting & ending values of r to get the limits.

Illustration 32

Show that $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+...+\frac{1}{6n}\right)=\log 6$ [IIT – 81]

Solution :

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+...+\frac{1}{6n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+...+\frac{1}{n+5n}\right)$$

[writing last term in the same form as the 1st, 2nd, 3rd, ... terms are]

$$=\lim_{n\to\infty}\sum_{r=1}^{5n}\frac{1}{n+r}=\lim_{n\to\infty}\sum_{r=1}^{5n}\frac{1}{n}\left(\frac{n}{n+r}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{5n}\frac{1}{1+\frac{r}{n}}$$

$\because$ lower limit of $r=1$

$\therefore$ lower limit of integration $=\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$

$\because$ upper limit of $r=5n$

$\therefore$ upper limit of integration $=\lim_{n\to\infty}\frac{5n}{n}=5$

Hence from (i) required limit

$$=\int_0^5\frac{dx}{1+x}=\left[\log(1+x)\right]_0^5$$

$$=\log 6-\log 1=\log 6 \qquad [\because \log 1=1]$$

Illustration 33

Evaluate $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+...+\frac{1}{2n}\right)$

Solution : $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+...+\frac{n}{n^2+n^2}\right)$

[Writing last term in the same form as first, 2nd, 3rd,.... term are]

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{n}{n^2 + r^2} \right) = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{n^2}{n^2 + r^2}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1 + \left(\frac{r}{n}\right)^2} \quad \text{...(i)}$$

$\because$ lower limit of $r = 1$

$\therefore$ lower limit of integration = $\lim_{n\to\infty} \frac{1}{n} = 0$

$\because$ upper limit of $r = n$

$\therefore$ upper limit of integration = $\lim_{n\to\infty} \frac{n}{n} = 1$

Hence from (i), required limit

$$= \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Illustration 34

Evaluate $\lim_{n\to\infty} \left[\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2 - 1^2}} + \frac{1}{\sqrt{n^2 - 2^2}} + \ldots + \frac{1}{\sqrt{n^2 - (n-1)^2}} \right]$

Solution : $\lim_{n\to\infty} \left[\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2 - 1^2}} + \frac{1}{\sqrt{n^2 - 2^2}} + \ldots + \frac{1}{\sqrt{n^2 - (n-1)^2}} \right]$

$$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{\sqrt{n^2 - r^2}} = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} \frac{n}{\sqrt{n^2 - r^2}}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} \frac{1}{\sqrt{1 - \left(\frac{r}{n}\right)^2}}$$

$\because$ lower limit of $r=0$ $\quad\therefore$ lower limit of integration $= \lim_{n\to\infty}\frac{y}{n}=0$

$\because$ upper limit of $r=n-1$

$\therefore$ upper limit of integration $= \lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1$

$\therefore$ from (i), required limit

$$=\int_0^1\frac{dx}{\sqrt{1-x^2}}=\left[\sin^{-1}x\right]_0^1=\sin^{-1}(1)-\sin^{-1}(0)=\frac{\pi}{2}-0=\frac{\pi}{2}$$

Illustration 35

Evaluate $\lim_{n\to\infty}\sum_{r=1}^{r=n}\frac{r^2}{r^3+n^3}$.

Solution :

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{r^3+n^3}=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r^2n}{r^3+n^3}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{\left(\frac{r}{n}\right)^2}{\left(\frac{r}{n}\right)^3+1} \quad ...(i)$$

lower limit of $r=1$ $\quad\therefore$ lower limit of integration $=\lim_{n\to\infty}\frac{1}{n}=0$

upper limit of $r=n$ $\quad\therefore$ upper limit of integration $=\lim_{n\to\infty}\frac{n}{n}=1$

$\therefore$ from (i), required limit $=\int_0^1\frac{x^2}{x^3+1}dx$ ...(ii)

Let $z=x^3+1$, then $dz=3x^2dx$

when $x=0,\ z=1$ and when $x=1, z=2$

$\therefore$ from (ii), required limit $=\frac{1}{3}\int_1^2\frac{dz}{z}=\frac{1}{3}[\log z]_1^2 = \frac{1}{3}(\log 2-\log 1)=\frac{1}{3}\log 2$.