INTRODUCTION

Concept of limit : Before giving formal definition of limit we consider the following examples :

Example : Let f be a function defined by $f(x) = \dfrac{x^2 - 4}{x - 2}$

Thus f(x) is defined for all x except x = 2

At x = 2, $f(x) = \dfrac{2^2 - 4}{2 - 2} = \dfrac{0}{0}$

Thus at x = 2, f(x) is not defined because denominator can never be zero

When $x \neq 2$, $x - 2 \neq 0$ $\therefore$ $f(x) = \dfrac{x^2 - 4}{x - 2} = \dfrac{(x-2)(x+2)}{x-2} = x + 2$

Now we consider the values of f (x) when $x \neq 2$, but is very-very close to 2 and x < 2.

x	1.9	1.99	1.999	1.9999	1.99999
f(x) = x + 2	3.9	3.99	3.999	3.9999	3.99999

It is clear from the above table that as x approaches 2 i.e., as $x \to 2$ through the values less than 2, the value of f (x) approaches 4 i.e. f (x) $\to$ 4.

We will express this fact by saying that left hand limit of f (x) as $x \to 2$ exists and is equal to 4 and in symbols we shall write

$$\underset{x \to 2 - 0}{\text{Lt}} f(x) = 4 \quad \text{or} \quad \underset{x \to 2 -}{\text{Lt}} f(x) = 4$$

Again we consider the values of f (x) when $x \neq 2$, but is very-very close to 2 and x > 2.

x	2.1	2.01	2.001	2.0001	2.00001
f(x) = x + 2	4.1	4.01	4.001	4.0001	4.00001

It is clear from the table given above that as x approaches 2 i.e., as $x \to 2$ through the values greater than 2, f (x) approaches 4 i.e., f (x) $\to$ 4.

We will express this fact by saying that right hand limit of f (x) as $x \to 2$ exists and is equal to 4 and in symbols we will write.

$$\underset{x \to 2 + 0}{\text{Lt}} f(x) = 4 \quad \text{or} \quad \underset{x \to 2 +}{\text{Lt}} f(x) = 4$$

Thus we see that f (x) is not defined at x = 2 but its left hand and right hand limits as $x \to 2$ and are equal.

When Lt f (x) and Lt f (x) are equal to the same number l, we say that $\underset{x \to 2}{\text{Lt}} f(x)$ exists and is equal to l.

Here in the example considered $\underset{x \to 2-0}{\text{Lt}} f(x) = \underset{x \to 2+0}{\text{Lt}} f(x) = 4$

$\therefore$ $\underset{x \to 2}{\text{Lt}} f(x)$ exists and is equal to 4.

Right hand and Left hand limits

If x approaches a from the right, that is, from larger values of x than a, limit of f as defined before is called the right hand limit of f (x) and is written as

$$\lim_{x \to a+0} f(x) \text{ or } f(a+0)$$

The working rule for finding the right hand limit is :

"Put a + h for x in f (x) and make h approach zero".

In short, we have $f(a+0) = \lim_{h \to 0} f(a+h)$

Similarly if x approaches a from the left, that is, from smaller values of x than a, the limit of f is called the left hand limit and is written as

$$\lim_{x \to a-0} f(x) \text{ or } f(a-0)$$

In this case, we have $f(a-0) = \lim_{h \to 0} (a-h)$

If both right hand and left hand limits of f(x), as $x \to a$, exist and are equal in value, their common value, evidently, will be the limit of f(x), as $x \to a$. If, however, either or both of these limits do not exist, the limit of f(x) as $x \to a$ does not exist. Even if both these limits exist but are not equal in value then also the limit of f(x) as $x \to a$ does not exist.

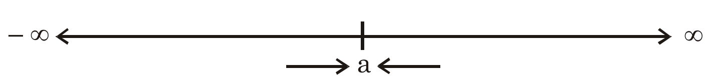
Definition of limit : We say that limit of f(x) as x tends to a exists and is equal to a real number l if as x approaches a (through the values less than or greater than a) the values of f (x) approach a definite unique real number l. In other words if for every $\in > 0$, however small, there exists $\rho > 0$, such that

$l - \in < f(x) < l + \in$ i.e. $|f(x) - l| < \in$ for all x for which

$a - \rho < x < a +$ i.e. $|x - a| < \rho$

In this case we write $\underset{x \to a}{\text{Lt}} f(x) = l$

Thus the statement $\underset{x \to a}{\text{Lt}} f(x) = l$ means that the values of f (x) will approach the number l or are equal to l as the values of x approach the number a from either direction.



- **Meaning of** $x \to \infty$: by $x \to \infty$, we mean that x is greater than any positive number however large and it is not a fixed number.

  $x \to \infty$ and and $x = \infty$ will mean the same thing.

- **Meaning of** $x \to -\infty$: by $x \to -\infty$ we mean that x is smaller than any negative number however small, $x \to -\infty$ and $x = -\infty$ will mean the same thing.

- **Meaning of** $\underset{x\to\infty}{\text{Lt}}\, f(x) = l$: We say that $\underset{x\to\infty}{\text{Lt}}\, f(x) = l$ if as x becomes larger and larger, f (x) becomes closer and closer to l.

- **Meaning of** $\underset{x\to a}{\text{Lt}}\, f(x) = \infty$: We say that $\underset{x\to a}{\text{Lt}}\, f(x) = \infty$ if as x approaches a through values less than or greater than a, f (x) becomes greater than any positive number however large.

- **Meaning of** $\underset{x\to a}{\text{Lt}}\, f(x) = -\infty$: We say that $\underset{x\to a}{\text{Lt}}\, f(x) = -\infty$ if as x approaches a through the values less than or greater than a, f (x) becomes smaller than any negative number however small.

For the existence of the limit of f (x) at x = a, it is necessary and sufficient that

(i) $f(a - 0) = f(a + 0)$ **and**

(ii) they both should be finite

Illustration 1

If $f(x) = \begin{cases} 5x - 4, & 0 < x \le 1 \\ 4x^3 - 3x, & 1 < x < 2 \end{cases}$ **show that** $\lim\limits_{x\to 1} f(x)$ **exists.**

Solution : We have,

LHL of f (x) at x = 1

$$= \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$$

$$= \lim_{h\to 0} 5(1-h) - 4 = \lim_{h\to 0} 1 - 5h = 1$$

RHL of f (x) at x = 1

$$= \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$$

$$= \lim_{h\to 0} 4(1+h)^3 - 3(1+h) = 4(1)^3 - 3(1) = 1$$

Thus RHL = LHL = 1. So $\lim\limits_{x\to 1} f(x)$ exists and is equal to 1.

Illustration 2

Evaluate the right hand limit and left hand limit of the function

$$f(x)=\begin{cases}\dfrac{|x-4|}{x-4}, & x\neq 4\\ 0, & x=4\end{cases}$$

Solution : RHL of f (x) at x = 4

$$=\lim_{x\to 4^+} f(x)=\lim_{h\to 0} f(4+h)=\lim_{h\to 0}=\frac{|4+h-4|}{4+h-4}$$

$$=\lim_{h\to 0}\frac{|h|}{h}=1$$

LHL of f (x) at x = 4

$$=\lim_{x\to 4^-} f(x)=\lim_{h\to 0} f(4-h)=\lim_{h\to 0}=\frac{|4-h-4|}{4-h-4}$$

$$=\lim_{h\to 0}\frac{|h|}{-h}=\lim_{h\to 0}\frac{h}{-h}=-1$$

This RHL ≠ LHL. So, $\lim_{x\to 4} f(x)$ does not exist.

Illustration 3

Show $\lim_{x\to 0}\dfrac{e^{1/x}-1}{e^{1/x}+1}$ does not exist.

Solution :

Let $f(x)=\dfrac{e^{1/x}-1}{e^{1/x}+1}$. Then,

LHL = $\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)$

$$=\lim_{h\to 0}\frac{e^{-1/h}-1}{e^{-1/h}+1}=\lim_{h\to 0}\frac{\left(1/e^{1/h}-1\right)}{\left(1/e^{1/h}+1\right)}=\frac{0-1}{0+1}=-1$$

$$\left[\text{as } h\to 0 \Rightarrow \frac{1}{h}\to\infty\Rightarrow e^{1/h} \Rightarrow 1/e^{1/h} \to 0\right] \qquad \text{...(i)}$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} \frac{e^{1/h}-1}{e^{1/h}+1} = \lim_{h\to 0} \frac{\left(1-1/e^{1/h}\right)}{\left(1+1/e^{1/h}\right)}$$ [Dividing numerator and denominator by $e^{1/h}$]

$$= \frac{1-0}{1+0} = 1$$ L.H.L. $\neq$ R.H.L., Hence Limit does not exit

Illustration 4

Solve $\lim\limits_{x\to 0}\left[\sin\frac{|x|}{x}\right]$, where [.] denotes the greatest integer function.

Solution : Here $\lim\limits_{x\to 0}\left[\sin\frac{|x|}{x}\right]$, since we have greatest integral function we must define function.

Now, RHL (put x = 0 + h)

$$\lim_{h\to 0}\left[\frac{\sin|0+h|}{0+h}\right],$$

we know $\frac{\sin h}{h} \to 1$ as h → 0 but less than 1 as h > sin h

$$\therefore \qquad \lim_{h\to 0} 0 = 0 \qquad \left\{\because \left[\frac{\sin h}{h}\right] = 0 \text{ as } h\to 0\right\}$$

⇒ RHL = 0

again LHL (put x = 0 – h)

$$\lim_{h\to 0}\left[\sin\frac{|0-h|}{0-h}\right],$$

we know $\frac{\sin h}{-h} \to -1$ as h → 0 but greater than – 1.

$$\therefore \qquad \lim_{h\to 0} 1 = -1 \qquad \left\{\because \left[\frac{\sin h}{h}\right] = -1 \text{ as } h\to 0\right\}$$

⇒ LHL = – 1

∴ limit does not exists as RHL = 0 and LHL = – 1.

Indeterminate Forms Or Meaningless Forms :

Following seven forms are called indeterminate forms :

1. $\frac{0}{0}$ **2.** $\frac{\infty}{\infty}$ **3.** $0.\infty$ **4.** $\infty - \infty$ **5.** 1^{∞} **6.** 0^0 **7.** ∞^0

1. **Indeterminate form $\frac{0}{0}$:** When numerator $\to 0$ and denominator $\to 0$, the form is called indeterminate form $\frac{0}{0}$. Here it should be noted that neither denominator nor numerator should be zero rather **they should tend to zero.**

   Example : $\lim\limits_{x \to 1} \frac{x^2 - 1}{x - 1}$ $\left[\frac{0}{0} \text{ form}\right]$

2. **Indeterminate form $\frac{\infty}{\infty}$:** when numerator $\to \infty$ and denominator $\to \infty$, the form is called indeterminate form $\frac{\infty}{\infty}$.

   Example : $\lim\limits_{x \to \infty} \frac{\log e^x}{x}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

3. **Indeterminate form $0.\infty$:** when one factor $\to 0$ (but not equal to zero) and other factor $\to \infty$, the form is called Indeterminate form $0.\infty$.

   Example : $\lim\limits_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x$ $[0.\infty \text{ form}]$

4. **Indeterminate form $\infty - \infty$** : when given expression is the difference of two functions both of whom tend to ∞, the form is called Indeterminate form $\infty - \infty$.

5. **Indeterminate form 1^{∞}** : when base $\to 1$ (but no equal to 1) and power $\to \infty$, the form is called Indeterminate form 1^{∞}.

   Example : $\mathop{\text{Lt}}\limits_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$ $[1^{\infty} \text{ form}]$

6. **Indeterminate form 0^0 :** When base $\to 0 + 0$ (but base is not equal to zero) and power $\to 0$ (but power is not equal to zero), the form is called the indeterminate form 0^0.

   Example : $\mathop{\text{Lt}}\limits_{x \to 0} (\sin x)^{\tan x}$ $[0^0 \text{ form}]$

7. **Indeterminate form ∞^0 :** When base $\to \infty$ and power $\to 0$ (but power is not equal to zero), the form is called the indeterminate form.

   Example : $\mathop{\text{Lt}}\limits_{x \to 0} (\cot x)^{\sin x}$ $[\infty^0 \text{ form}]$

Some important properties of limits

If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exists, then

1. $\lim_{x \to a} \{f(x) + g(x)\} = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} \{f(x) - g(x)\} = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} \{c.f(x)\} = c \lim_{x \to a} f(x)$, where c is a constant

4. $\lim_{x \to a} \{f(x).g(x)\} = \left\{\lim_{x \to a} f(x)\right\} . \left\{\lim_{x \to a} g(x)\right\}$

5. $\lim_{x \to a} \left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$, $g(x) \neq 0$

6. If f (x) < g(x) for all x, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

7. If f (x) ≤ g(x) for all x, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

Some Important Expansions :

1. Binomial Expansion :

(i) If n is a positive integer, then

$(1 + x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + ... + nC_nx^n$

$$= 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + ... + x^n \text{ (valid for all x)}$$

(ii) If n is a negative integer or fraction, then

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... \text{ to } \infty$$

where $-1 < x < 1$

2. $$\frac{x^n - a^n}{x - a} = \left(x^{n-1} + ax^{n-2} + a^2x^{x-3} + ... + a^{n-1}\right)$$

3. (i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ... \text{ to } \infty$, valid for all x

(ii) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - ... + \frac{x^3}{3!} + ...$ to ∞ valid for all x

4. (i) $a^x = e^x \log e^a$

(ii) $a^x = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + ...$ to ∞

5. (i) $\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$ to ∞, $-1 < x \le 1$

(ii) $\log_e (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ...$ to ∞, $-1 \le x < 1$

6. $\sin = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

7. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

8. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + ...$

Special notes on infinity

- Infinity is a very big number which is not find on number line. No variable is equal to infinity. Hence ordinary laws of algebra do not apply on ∞.

  eg. $\infty - \infty \ne 0$, $\frac{\infty}{\infty} \ne 1$

  (it is so because we do not know how big value is for ∞).

- Whenever the denominator becomes 0, the expression becomes undefined and it is not equal to ∞

  $\frac{1}{0} \ne \infty$ (undefined)

  but in case of limits

  $$\lim \left(\frac{\to \text{finite no.}}{\to \infty}\right) = 0; \lim \left(\frac{\to \infty}{\to \text{finite no.}}\right) = \infty$$

  $$\lim \left(\frac{\to 0}{\to \text{non zero}}\right) = 0 \ \& \ \lim \left(\frac{\to b}{\to a}\right) = \frac{b}{a} \quad \text{if } a \ne 0$$

  (here $\to$ means approaching)

  Do not confuse the above limits with the following limits.

Some limits which are obvious

1. $\lim\limits_{x \to 0} \left(\frac{0}{x}\right) = 0$

2. $\lim\limits_{x \to \infty} (0.x) = 0$

3. $\lim\limits_{x \to \infty} (x)^0 = 1$

4. $\lim\limits_{x \to \infty} 1^x = 1$

5. $\lim\limits_{x \to \infty} \left(x^2 - x^2\right) = 0$

6. $\lim\limits_{x \to \infty} \frac{x^2}{x^2} = 1$

(a) (i) $\frac{0}{0}$ is undefined if denominator is equal to zero

(ii) $\underset{x \to 0}{\text{Lt}} \frac{x}{0}$ does not exist if denominator is equal to zero as $\frac{x}{0}$ is undefined.

(iii) $\frac{0}{0}$ (Indeterminate) when numerator $\to 0$ and denominator $\to 0$

(b) $0.\infty = 0$

But (tends to zero) $.\infty$ is indeterminate.

(c) $1^\infty = 1$, if base is equal to 1.

But 1^∞ is indeterminate when base $\to 1$

(d) $\infty^0 = 1$, if power is equal to zero

But ∞^0 is indeterminate when power $\to 0$

(e) (i) $\underset{x \to 0}{\text{Lt}} (x)^0 = 1$, if power is equal to zero

(ii) $\underset{x \to 0+0}{\text{Lt}} (0)^x = 0$, if base is equal to zero

(iii) $\underset{x \to 0-0}{\text{Lt}} (0)^x$ does not exist if base is equal to zero as 0^x, when $x < 0$ is undefined.

(iv) 0^0 is not defined if base is equal to zero and power is equal to zero.

Properties of infinity :

(i) $\infty \pm c = \infty$,

(ii) $\infty + \infty = \infty$

(iii) $\infty.\infty = \infty$

(iv) $\infty(-\infty) = -\infty$, $(-\infty).\infty = -\infty$

(v) $\infty^{\infty} = \infty$

(vi) $c.\infty = \infty$, if $c > 0$

$= 0$, if $c = 0$

$= -\infty$ if $c < 0$

$0.\infty = 0$

But (tends to zero).∞ is indeterminate

In fact $c.\infty \to \infty$, if $c > 0$

$c.\infty = 0$, if $c = 0$

$c.\infty \to -\infty$, if $c < 0$

(vii) $c^{\infty} = \infty$, if $c > 1$

$= 0$, if $0 \le c < 1$

$= 1$, if $c = 1$

In fact $c^{\infty} \to \infty$, if $c > 1$

$c^{\infty} \to 0$, if $0 < c < 1$

$c^{\infty} = 0$, if $c = 0$

$c^{\infty} = 1$, if $c = 1$

$1^{\infty} = 1$, if base $= 1$

But (tends to 1)$^{\infty}$ is indeterminate.

Method to Evaluate the Limit of a Function :

There are a number of methods to evaluate the limit of a function but for the sake of convenience, we divide the problems in two types. If f (x) is a function of x then x is the independent variable.

Type I. We will call those problems in which the independent variable **tends to ∞ or $-\infty$ as** problems of Type I.

Type II. We will call those problems in which independent variable x **does not tend to ∞ or $-\infty$** as problems of Type II.

Examples : (i) $\underset{x \to \infty}{\text{Lt}} \left(1 + \frac{a}{x}\right)^x$ is a problem of type I, as here independent variable $x \to \infty$

(ii) $\underset{x \to 0}{\text{Lt}} \frac{a^x - bx}{x}$ is a problem of type II as here independent variable x does not tend to ∞ or $-\infty$.

Note : Here α is the independent variable and x is a constant.

Again for the sake of convenience we divide problems of each type in three categories.

Category A. Problems involving only algebraic functions.

Category B. Problems involving non-zero constant powers of sin, cos, tan, cot, sec or cosec of a variable angle.

Category C. Problems involving exponential or logarithmic functions.

Examples :

(i) $\underset{x \to a}{\text{Lt}} \dfrac{x^4 - a^4}{x - a}$. This problem is of category A of type II.

(ii) $\underset{x \to 0}{\text{Lt}} \dfrac{x + \sin x}{x}$. This problem is of category B of type II.

(iii) $\underset{x \to 0}{\text{Lt}} (\sin x)^{\tan x}$. This problem is of category C of type II.

(iv) $\underset{x \to 2}{\text{Lt}} \dfrac{\log(x-1)}{x-2}$. This problem is of category C of type II.

(v) $\underset{x \to \infty}{\text{Lt}} \dfrac{x^2 + x^2 + 1}{x^4 + 2}$. This problem is of category A of type I.

(vi) $\underset{x \to \infty}{\text{Lt}} 2n \sin \dfrac{1}{n}$. This problem is of category B of type I.

(vii) $\underset{x \to \infty}{\text{Lt}} \left(1 + \dfrac{1}{x}\right)^{2x}$. This problem is of category C of type I.

Type I. Category (A)

Problems involving algebraic expressions.

Working Rule :

1. First of all simplify the given expression in the form of $\dfrac{N}{D}$.

2. (i) Then divide each terms of the numerator and denominator by x^n where x is the independent variable and n is the highest power of x in the numerator and denominator taken together.

(ii) Then put $\dfrac{c}{x^k} = 0$, where c is a constant and $k > 0$. This is because $\dfrac{c}{x^k} \to 0$ as $x \to \infty$.

3. Alternatively, take out term containing highest power of x in the numerator and denominator as common and finally put

$x^k = 0$, if $k < 0$

$= \infty$, if $k > 0$

$= 1$, if $k = 0$

Tip : Simply check the powers in numerator & denominator.

for $\dfrac{ax^n + bx^n + \ldots}{px^d + qx^{d-1} + \ldots}$ if

(i) $n > d$; result is ∞

(ii) $n < d$; result is 0

(iii) $n = d$; limit is $\left(\dfrac{a}{p}\right)$

Illustration 5

(a) Find $\displaystyle \mathop{\mathbf{Lt}}_{x \to \infty} \frac{\mathbf{x^4 + 2x^3 + 3}}{\mathbf{2x^4 - x + 2}}$ **(b) Find** $\displaystyle \mathop{\mathbf{Lt}}_{n \to \infty} \left(\frac{\mathbf{1^2}}{\mathbf{n^3}} + \frac{\mathbf{2^2}}{\mathbf{n^3}} + \frac{\mathbf{3^2}}{\mathbf{n^3}} + \ldots + \frac{\mathbf{n^2}}{\mathbf{n^3}}\right)$

Solution :

(a) $$\mathop{\text{Lt}}_{x \to \infty} \frac{x^4 + 2x^3 + 3}{2x^4 - x + 2} = \mathop{\text{Lt}}_{x \to \infty} \frac{1 + \dfrac{2}{x} + \dfrac{3}{x^4}}{2 - \dfrac{1}{x^3} + \dfrac{2}{x^4}} = \frac{1}{2}$$

$$\left[\text{Since as } x \to \infty, \frac{2}{x} \to 0, \frac{3}{x^4} \to 0 \text{ and } \frac{2}{x^4} \to 0\right]$$

(b) $$\mathop{\text{Lt}}_{n \to \infty} \left(\frac{1^2}{n^3} + \frac{2^2}{n^3} + \frac{3^2}{n^3} + \ldots + \frac{n^2}{n^3}\right)$$

$$= \mathop{\text{Lt}}_{n \to \infty} \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3} = \mathop{\text{Lt}}_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3} = \mathop{\text{Lt}}_{n \to \infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \mathop{\text{Lt}}_{n \to \infty} \frac{n\left(2n^2 + 3n + 1\right)}{6n^2} = \mathop{\text{Lt}}_{n \to \infty} \frac{2 + \dfrac{3}{n} + \dfrac{1}{n^2}}{6} = \frac{2}{6} = \frac{1}{3}$$

Illustration II

(a) **Find** $\underset{x \to \infty}{\mathbf{Lt}} \sqrt{x}\left(\sqrt{x+c}-\sqrt{x}\right)$ **(b)** $\underset{x \to -\infty}{\mathbf{Lt}} \left(\sqrt{x^2+4x}-\sqrt{x^2-4x}\right)$

(c) **I :** $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^2+1}-\sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$

Solution :

(a) $\underset{x \to \infty}{\text{Lt}} \sqrt{x}\left(\sqrt{x+c}-\sqrt{x}\right)$

$$= \underset{x \to \infty}{\text{Lt}} \frac{\sqrt{x}\left(\sqrt{x+c}-\sqrt{x}\right)\left(\sqrt{x+c}+\sqrt{x}\right)}{\left(\sqrt{x+c}+\sqrt{x}\right)}$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{\sqrt{x}\left(x+c-x\right)}{\left(\sqrt{x+c}+\sqrt{x}\right)} = \underset{x \to \infty}{\text{Lt}} \frac{c\sqrt{x}}{\sqrt{x+c}+\sqrt{x}}$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{c}{\frac{\sqrt{x+c}}{\sqrt{x}}+\frac{\sqrt{x}}{\sqrt{x}}} = \underset{x \to \infty}{\text{Lt}} \frac{c}{\sqrt{\frac{x+c}{x}}+1}$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{c}{\sqrt{1+\frac{c}{x}}+1} = \frac{c}{1+1} = \frac{c}{2}$$

(b) $\underset{x \to -\infty}{\text{Lt}} \left(\sqrt{x^2+4x}-\sqrt{x^2-4x}\right)$

$$= \underset{x \to -\infty}{\text{Lt}} \frac{\left(\sqrt{x^2+4x}-\sqrt{x^2-4x}\right)\left(\sqrt{x^2+4x}+\sqrt{x^2-4x}\right)}{\left(\sqrt{x^2+4x}+\sqrt{x^2-4x}\right)}$$

$$= \underset{x \to -\infty}{\text{Lt}} \frac{\left(x^2+4x\right)-\left(x^2-4x\right)}{\sqrt{x^2+4x}+\sqrt{x^2-4x}} = \underset{x \to -\infty}{\text{Lt}} \frac{8x}{\sqrt{x^2+4x}+\sqrt{x^2-4x}}$$

$$= \underset{x \to -\infty}{\text{Lt}} \frac{8}{\frac{\sqrt{x^2+4x}+\sqrt{x^2-4x}}{x}}$$

$$= \underset{x \to -\infty}{\text{Lt}} \frac{8}{\sqrt{\frac{x^2+4x}{x^2}} + \sqrt{\frac{x^2-4x}{x^2}}}$$

$$\left[\text{Here } x < 0 \because x = -\sqrt{x^2} \text{ for example } -4 = -\sqrt{(-4)^2} = -\sqrt{16}\right]$$

$$\underset{x \to -\infty}{\text{Lt}} \frac{8}{\sqrt{1+\frac{4}{x}} + \sqrt{1-\frac{4}{x}}} = \frac{8}{1+1} = \frac{8}{2} = 4$$

(c) dividing the numerator & denominator by x (which is the greatest power of x possible)

$$\lim_{x \to \infty} \frac{\frac{\sqrt{x^2+1}}{x} - \frac{\sqrt[3]{x^2+1}}{x}}{\frac{\sqrt[4]{x^4+1}}{x} - \frac{\sqrt[5]{x^4+1}}{x}}$$

$$= \lim_{x \to \infty} \frac{\frac{\sqrt{x^2+1}}{x} - \sqrt[3]{\frac{x^2+1}{x}}}{\frac{\sqrt[4]{x^4+1}}{x} - \frac{\sqrt[5]{x^4+1}}{x}}$$

$$= \lim_{x \to \infty} \frac{\frac{(x^2+1)^{1/2}}{x} - \frac{(x^2+1)^{1/3}}{x}}{\frac{(x^4+1)^{1/4}}{x} - \frac{(x^4+1)^{1/5}}{x}}$$

$$= \lim_{x \to \infty} \frac{\left(\frac{x^2+1}{x^2}\right)^{1/2} - \left(\frac{x^2+1}{x^3}\right)^{1/3}}{\left(\frac{(x^4+1)}{x^4}\right)^{1/4} - \left(\frac{x^4+1}{x^5}\right)^{1/5}} = \lim_{x \to \infty} \frac{\left(1+\frac{1}{x^2}\right)^{1/2} - \left(\frac{1}{x}+\frac{1}{x^3}\right)^{1/3}}{\left(1+\frac{1}{x^4}\right)^{1/4} - \left(\frac{1}{x}+\frac{1}{x^5}\right)^{1/5}}$$

as $x \to \infty$ $\frac{1}{x^p} \to 0$ (for $p > 1$)

$$= \frac{(1+0)-0}{(1+0)-0} = 1$$

Type II. Category A.

When $x \to a$, where a is a fixed real number.

Problems in which algebraic functions occur.

Working Rule :

Limits of functions involving only algebraic functions and when independent variable does not tend to ∞ or $-\infty$ can be evaluated by using the following formula

$$\boxed{\underset{x \to a}{\textbf{Lt}} \frac{x^n - a^n}{x - a} = na^{n-1}}$$

There are other methods also to evaluate such limits.

There are 3 methods to solve these kind of questions.

(a) **Direct substitution**

for a limit, $\lim\limits_{x \to a} f(x)$, we can directly substitute $x = a$ in the limit only if the following constraints are **not** there

- $\lim\limits_{x \to a} f(x)$ is of type 1 (intermediate form)
- $\lim\limits_{x \to a} f(x)$ is undefined.

(b) **Factorisation**

Factorization method can also be used to solve these kind of questions.

for $\lim\limits_{x \to a} \frac{P(x)}{Q(x)}$, if P (a) = 0 & $\lim\limits_{x \to a} Q(a) = 0$

(note Q (a) $\neq$ 0, otherwise the function is undefined) then we can say that $(x - a)$ is a factor of both P (x) & Q (x)

$$\therefore \quad \lim_{x \to a} \frac{P(x)}{Q(x)} = \lim_{x \to a} \frac{(x-a)\,N(x)}{(x-a)\,D(x)} = \lim_{x \to a} \frac{N(x)}{D(x)}$$

repeat this procedure of cancellation until you get to a useful result.

(c) **Rationalisation**

If we get $\frac{0}{0}$ form in the problems involving roots then we must rationalise them to get the common factor, which will be cancelled out.

Illustration 6

Illustrations based on factorization

(a) $\lim_{x \to a} \left(\frac{x^3 - a^3}{x^2 - ax}\right)$ **(b)** $\lim_{x \to 3} \frac{x^3 - x^2 - 3x - 9}{x^2 - 4x + 3}$

(c) $\lim_{x \to 4} \frac{x^3 - 2x^2 - 9x + 4}{x^2 - 2x - 8}$ **(d)** $\lim_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$

Solution :

(a) given $\lim_{x \to a} \left(\frac{x^3 - a^3}{x^2 - ax}\right)$

we know $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

$$\therefore \quad \lim_{x \to a} \frac{(x-a)\left(x^2 + ax + a^2\right)}{x(x-a)}$$

$$= \lim_{x \to a} \frac{\left(x^2 + ax + a^2\right)}{x} = \frac{3a^2}{a} = 3a$$

(b) given $\lim_{x \to 3} \frac{x^3 - x^2 - 3x - 9}{x^2 - 4x + 3}$

if we put x = 3 in numerator & denominator we get 0 in both, i.e. (x – 3) is a factor of both numerator & denominator.

$\therefore$ limit becomes, $\lim_{x \to 3} \frac{(x-3)(x^2 + 2x + 3)}{(x-3)(x-1)}$

now we can put x = 3

$$= \frac{9 + 6 + 3}{2} = \frac{18}{2} = 9$$

(c) [When x = 4 numerator and denominator become zero]

$$\underset{x \to 4}{\text{Lt}} \frac{x^3 - 2x^2 - 9x + 4}{x^2 - 2x - 8}$$

$$= \underset{x \to 4}{\text{Lt}} \frac{x^3 - 4x^2 - 2x^2 - 8x - x + 4}{x^2 - 4x + 2x - 8} = \underset{x \to 4}{\text{Lt}} \frac{(x-4)\left(x^2 + 2x - 1\right)}{(x-4)(x+2)}$$

$$= \underset{x \to 4}{\text{Lt}} \frac{x^2 - 2x - 1}{x + 2} = \frac{23}{6}$$

Second Method : $\underset{x \to 4}{\text{Lt}} \dfrac{x^3 - 2x^2 - 9x + 4}{x^2 - 2x - 8}$

$$= \underset{x \to 4}{\text{Lt}} \frac{\left(\dfrac{x^3 - 4^3}{x - 4}\right)(x - 4) - 2\left(\dfrac{x^2 - 4^2}{x - 4}\right)(x - 4)\left(\dfrac{x - 4}{x - 4}\right)(x - 4)}{\left(\dfrac{x^2 - 4^2}{x - 4}\right)(x - 4) - 2\left(\dfrac{x - 4}{x - 4}\right)(x - 4)}$$

$$= \underset{x \to 4}{\text{Lt}} \frac{\dfrac{x^3 - 4^3}{x - 4} - 2\left(\dfrac{x^2 - 4^2}{x - 4}\right) - 9\left(\dfrac{x - 4}{x - 4}\right)}{\dfrac{x^2 - 4^2}{x - 4} - 2\left(\dfrac{x - 4}{x - 4}\right)}$$

$$= \frac{3.4^2 - 2.2.4^1 - 9}{2.4^1 - 2} = \frac{23}{6}$$

(d) When x = 1 numerator and denominator both become zero and hence (x – 1) is a factor of both

Now $\underset{x \to 1}{\text{Lt}} \dfrac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$

$$= \underset{x \to 1}{\text{Lt}} \frac{x^7 - x^6 + x^6 - x^5 - x^5 + x^4 - x^4 + x^3 - x^3 + x^2 - x^2 + x - x + 1}{x^3 - x^2 - 2x^2 + 2x + 2}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{x^6 (x - 1) + x^5 (x - 1) - x^4 (x - 1) - x^3 (x - 1) - x^2 (x - 1) - x (x - 1) - (x - 1)}{x^2 (x - 1) - 2x (x - 1) - 2(x - 1)}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{(x^6 + x^5 - x^4 - x^3 - x^2 - x - 1)}{x^2 - 2x - 2} = \frac{-3}{-3} = 1$$

Second Method : $\underset{x \to 1}{\text{Lt}} \dfrac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$

$$= \underset{x \to 1}{\text{Lt}} \frac{\left(\dfrac{x^7 - 1^7}{x - 1}\right)(x - 1) - 2\left(\dfrac{x^5 - 1^5}{x - 1}\right)(x - 1)}{\left(\dfrac{x^3 - 1^3}{x - 1}\right)(x - 1) - 3\left(\dfrac{x^2 - 1^2}{x - 1}\right)(x - 1)}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{7(1)^6 - 2\left(5(1)^4\right)}{3(1)^2 - 3\left(2(1)\right)} = \underset{x \to 1}{\text{Lt}} \frac{7-10}{3-6}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{-3}{-3} = 1$$

Question based on formula

Illustration 7

(a) $\lim\limits_{x \to 5} \dfrac{x^4 - 625}{x^3 - 125}$

(b) $\lim\limits_{x \to 64} \dfrac{x^{1/6} - 2}{x^{1/3} - 4}$

(c) $\lim\limits_{x \to 0} \dfrac{(x+h)^{1/n} - x^{1/n}}{h}$

(d) $\lim\limits_{x \to 0} \dfrac{(1+x)^5 - 1}{3x + 5x^2}$

Solution :

(a) given $\lim\limits_{x \to 5} \dfrac{x^4 - 625}{x^3 - 125}$

if we write it like $\lim\limits_{x \to 5} \dfrac{x^4 - 5^4}{x^3 - 5^3}$

$$= \lim_{x \to 5} \frac{\dfrac{x^4 - 5^4}{x - 5}}{\dfrac{x^3 - 5^3}{x - 5}}$$

& now we can use the formula $= \dfrac{4.5^3}{3.5^2} = \dfrac{20}{3}$

(b) $\lim\limits_{x \to 64} \dfrac{x^{1/6} - 2}{x^{1/3} - 4}$

$$\underset{x \to 64}{\text{Lt}} \frac{x^{\frac{1}{6}} - 2}{x^{\frac{1}{3}} - 4} = \underset{x \to 64}{\text{Lt}} \frac{x^{\frac{1}{6}} - (64)^{\frac{1}{6}}}{x^{\frac{1}{3}} - (64)^{\frac{1}{3}}} \qquad \left[\because (64)^{\frac{1}{6}} = 2, (64)^{\frac{1}{3}} = 4\right]$$

$$= \underset{x \to 64}{\text{Lt}} \frac{\dfrac{x^{\frac{1}{6}} - (64)^{\frac{1}{6}}}{x - 64}}{\dfrac{x^{\frac{1}{3}} - (64)^{\frac{1}{3}}}{x - 64}} = \frac{\frac{1}{6}(64)^{\frac{1}{6}-1}}{\frac{1}{3}(64)^{\frac{1}{3}-1}} = \frac{1}{2}(64)^{-\frac{5}{6}+\frac{2}{3}}$$

$$= \frac{1}{2}(64)^{-\frac{1}{6}} = \frac{1}{2}\left(2^6\right)^{-\frac{1}{6}} = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$$

(c) $\underset{h \to 0}{\text{Lt}} \dfrac{(x+h)^{\frac{1}{n}} - x^{\frac{1}{n}}}{h}$

$$= \underset{(x+h) \to x}{\text{Lt}} \frac{(x+h)^{\frac{1}{n}} - x^{\frac{1}{n}}}{(x+h) - x} \qquad [\because \text{ when } h \to 0,\ x + h \to x]$$

$$= = \frac{1}{n} x^{\frac{1}{n} - 1} = \frac{1}{n} x^{\frac{1-n}{n}}$$

Note : Here h is variable and x is a constant.

(d) $\underset{x \to 0}{\text{Lt}} \dfrac{(1+x)^5 - 1}{3x + 5x^2}$

$$= \underset{x \to 0}{\text{Lt}} \frac{\dfrac{(1+x)^5 - 1^5}{(1+x)-1}}{x(3+5x)}.x = \underset{x \to 0}{\text{Lt}} \frac{\dfrac{(1+x)^5 - 1^5}{(1+x)-1}}{3+5x} = \frac{5.1^4}{3} = \frac{5}{3}$$

Second Method : $\underset{x \to 0}{\text{Lt}} \dfrac{(1+x)^5 - 1}{3x + 5x^2}$

$$= \underset{x \to 0}{\text{Lt}} \frac{\left(1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5\right) - 1}{x(3+5x)}$$

$$= = \underset{x \to 0}{\text{Lt}} \frac{\left(5x + 10x^2 + 10x^3 + 5x^4 + x^5\right)}{x(3+5x)}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\left(5x + 10x + 10x^2 + 5x^3 + x^4\right).x}{x(3+5x)} = \frac{5}{3}$$

Illustration 8

Illustrations based on rationalization

(a) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x^2}-\sqrt{1+x}}{x}$ **(b)** $\lim\limits_{x \to 2} \dfrac{x-2}{\sqrt{x^2-4}+\sqrt{x-2}}$ **(c)** $\mathop{\mathrm{Lt}}\limits_{x \to a} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$, $a \neq 0$

Solution :

(a) [Given function $\sqrt{1+x^2}-\sqrt{1+x}$ is of the form $\sqrt{a}-\sqrt{b}$]

$$\mathop{\mathrm{Lt}}_{x \to 0} \frac{\sqrt{1+x^2}-\sqrt{1+x}}{x}$$

$$= \mathop{\mathrm{Lt}}_{x \to 0} \left(\frac{\sqrt{1+x^2}-\sqrt{1+x}}{x}\right)\left(\frac{\sqrt{1+x^2}+\sqrt{1+x}}{\sqrt{1+x^2}+\sqrt{1+x}}\right)$$

$$= \mathop{\mathrm{Lt}}_{x \to 0} \frac{1+x^2-(1+x)}{x\left(\sqrt{1+x^2}+\sqrt{1+x}\right)} = \mathop{\mathrm{Lt}}_{x \to 0} \frac{x^2-x}{\left(\sqrt{1+x^2}+\sqrt{1+x}\right)x}$$

$$= \mathop{\mathrm{Lt}}_{x \to 0} \frac{x(x-1)}{x\left(\sqrt{1+x^2}+\sqrt{1+x}\right)} = \mathop{\mathrm{Lt}}_{x \to 0} \frac{x-1}{\left(\sqrt{1+x^2}+\sqrt{1+x}\right)}$$

$$= \frac{-1}{\sqrt{1}+\sqrt{1}} = -\frac{1}{2}$$

(b) $\mathop{\mathrm{Lt}}\limits_{x \to 2} \dfrac{x-2}{\sqrt{x^2-4}+\sqrt{x-2}} \left[\dfrac{0}{0} \text{ form}\right]$

$$= \mathop{\mathrm{Lt}}_{x \to 2} \frac{\left(\dfrac{x^1-2^1}{x-2}\right)(x-2)}{\sqrt{\left(\dfrac{x^2-2^2}{x-2}\right)(x-2)}+\sqrt{\dfrac{x^1-2^1}{x-2}(x-2)}} = \mathop{\mathrm{Lt}}_{x \to 2} \frac{\left(\dfrac{x^1-2^1}{x-2}\right).(x-2)}{\left(\sqrt{\dfrac{x^2-2^2}{x-2}}+\sqrt{\dfrac{x^1-2^1}{x-2}}\right).\sqrt{x-2}}$$

$$= \mathop{\mathrm{Lt}}_{x \to 2} \frac{\left(\dfrac{x^1-2^1}{x-2}\right).\sqrt{x-2}}{\left(\sqrt{\dfrac{x^2-2^2}{x-2}}+\sqrt{\dfrac{x^1-2^1}{x-2}}\right)} = \frac{1.0}{\sqrt{2.2^1}+1} = \frac{0}{3} = 0$$

Second Method

$$= \underset{x \to 2}{\text{Lt}} \frac{x-2}{\sqrt{x^2-4}+\sqrt{x-2}} = \underset{x \to 2}{\text{Lt}} \frac{x-2}{\sqrt{(x-2)(x+2)}+\sqrt{x}-2}$$

$$= \underset{x \to 2}{\text{Lt}} \frac{x-2}{\sqrt{(x-2)}+\sqrt{x+2+1}} \underset{x \to 2}{\text{Lt}} \frac{\sqrt{x-2}}{\sqrt{x+2}+1} = \frac{0}{2+1} = 0 \quad \textbf{[I.I.T. 78]}$$

(c) Required limit

$$= \underset{x \to a}{\text{Lt}} \frac{\frac{\sqrt{a+2x}-\sqrt{3a}}{(a+2x)-3a}.\,2(x-a)-\frac{\sqrt{3x}-\sqrt{3a}}{3x-3a}.3(x-a)}{\frac{\sqrt{3a+x}-\sqrt{4a}}{(3a+x)-4a}.\,(x-a)-2\left(\frac{\sqrt{x}-\sqrt{a}}{x-a}\right)(x-a}$$

$$= \underset{x \to a}{\text{Lt}} \frac{\frac{(a+2x)^{\frac{1}{2}}-(3a)^{\frac{1}{2}}}{a+2x-3a}.\,2-\frac{(3x)^{\frac{1}{2}}-(3a)^{\frac{1}{2}}}{3x-3a}.3}{\frac{(3a+x)^{\frac{1}{2}}-(4a)^{\frac{1}{2}}}{(3a+x)-4a}.-2\left(\frac{x^{\frac{1}{2}}-a^{\frac{1}{2}}}{x-a}\right)}$$

Type II. Category B

Problems in which non-zero constant powers of sin, cos, tan, cot, sec, cosec of variable angle occur (problems involving trigonometrical expressions).

Working Rule :

1. First of all see whether independent variable tend to zero or not. If the independent variable $x \to a$, where $a \neq 0$, then put $x = a + h$. Then go on simplifying only those factors of the numerator and denominator which tend to zero till $\sin\theta$ or $\tan\theta$ occurs as a factor where $\theta \to 0$.

2. Then write $\sin\theta = \frac{\sin\theta}{\theta}.\theta$ and $\tan\theta = \frac{\sin\theta}{\theta}.\theta$ and use the formula

$$\underset{\theta \to 0}{\textbf{Lt}} \frac{\sin\theta}{\theta} = \mathbf{1},\ \underset{\theta \to 0}{\textbf{Lt}} \frac{\tan\theta}{\theta} = \mathbf{1} \quad \text{whichever is required.}$$

Illustration 9

(a) **Find** $\underset{\theta \to 0}{\textbf{Lt}} \dfrac{\sin a\theta}{\tan b\theta}$ **(b)** $\underset{x \to 0}{\textbf{Lt}} \dfrac{\tan x^\circ}{\tan x}$

Solution :

(a) $$\underset{\theta \to 0}{\text{Lt}} \frac{\sin a\theta}{\tan b\theta} = \underset{\theta \to 0}{\text{Lt}} \frac{\left(\dfrac{\sin a\theta}{a\theta}\right)a\theta}{\left(\dfrac{\tan b\theta}{b\theta}\right)b\theta}$$

$$= \underset{\theta \to 0}{\text{Lt}} \frac{\left(\dfrac{\sin a\theta}{a\theta}\right)a}{\left(\dfrac{\tan b\theta}{b\theta}\right)b} = \frac{1.a}{1.b} = \frac{a}{b}$$

(b) $x^\circ = \dfrac{\pi}{180}x$ radian

$$\text{Now } \underset{x \to 0}{\text{Lt}} \frac{\tan x^\circ}{x} = \underset{x \to 0}{\text{Lt}} \frac{\tan \dfrac{\pi x}{180}}{x} = \underset{x \to 0}{\text{Lt}} \frac{\dfrac{\tan \dfrac{\pi x}{180}}{\dfrac{\pi x}{180}} \cdot \dfrac{\pi x}{180}}{x} = \frac{\pi}{180}$$

Illustration 10

(a) **Find** $\underset{x \to 0}{\textbf{Lt}} \dfrac{x(\cos x + \cos 2x)}{\sin x}$ **(b)** $\underset{x \to 0}{\textbf{Lt}} \dfrac{\cos x - \cos 3x}{x(\sin 3x - \sin x)}$

(c) $\underset{x \to 0}{\textbf{Lt}} \dfrac{\tan x - \sin x}{1 - \cos x}$ **(d)** $\underset{x \to \frac{\pi}{2}}{\textbf{lim}} \dfrac{\sin(\cos x)\cos x}{\sin x - \operatorname{cosec} x}$

Solution :

(a) $$= \underset{x \to 0}{\text{Lt}} \frac{x(\cos x + \cos 2x)}{\sin x} = \underset{x \to 0}{\text{Lt}} \frac{x(\cos x + \cos 2x)}{\dfrac{\sin x}{x}.x}$$

[Here factor (cos x + cos 2x), does not tend to zero, hence it is not necessary to simplify it]

$$= \underset{x \to 0}{\text{Lt}} \frac{\cos x + \cos 2x}{\dfrac{\sin x}{x}} = \frac{1+1}{1} = 2$$

(b) $$\lim_{x \to 0} \frac{\cos x - \cos 3x}{x(\sin 3x - \sin x)} = \lim_{x \to 0} \frac{2\sin\frac{x+3x}{2}\sin\frac{3x-x}{2}}{x.2\cos\frac{x+3x}{2}\sin\frac{3x-x}{2}}$$

$$= \lim_{x \to 0} \frac{2\sin 2x \sin x}{2x\cos 2x \sin x} = \lim_{x \to 0} \frac{\sin 2x}{x\cos 2x}$$

$$= \lim_{x \to 0} \frac{\frac{\sin 2x}{2x}.2x}{x\cos 2x} = \lim_{x \to 0} \frac{\frac{\sin 2x}{2x}.2}{\cos 2x} = \frac{2}{1} = 2$$

(c) $$\lim_{x \to 0} \frac{\tan x - \sin x}{1 - \cos x} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{1 - \cos x}$$

$$= \lim_{x \to 0} \frac{\sin x - \sin x \cos x}{\cos x\,(1 - \cos x)} = \lim_{x \to 0} \frac{\sin x(1 - \cos x)}{\cos x(1 - \cos x)}$$

$$= \lim_{x \to 0} \tan x = 0$$ [By the definition of limit because the form is not indeterminate]

(d) $$\lim_{x \to \frac{\pi}{2}} \frac{\sin(\cos x)\cos x}{\sin x - \operatorname{cosec} x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sin(\cos x)\cos x \sin x}{\sin^2 x - 1}$$

$$= \lim_{x \to \frac{\pi}{2}} -\frac{[\sin(\cos x)\sin x]}{\cos x} = -\lim_{x \to \frac{\pi}{2}} \frac{\sin(\cos x)\sin x}{\cos x}$$

as $x \to \frac{\pi}{2}$ $\cos x \to 0$,

$\therefore$ we can use $\frac{\sin x}{x}$ rule

$$= -\lim_{x \to \frac{\pi}{2}} \sin x = -1$$

Illustration 11

$\left(\text{Using } \lim\limits_{x \to 0} \frac{\sin x}{x} = 1\right)$ **Evaluate the following limits :**

(a) $\lim\limits_{x \to \frac{\pi}{3}} \frac{\tan x - \sqrt{3}}{9x^2 - \pi^2}$ **(b)** $\mathop{Lt}\limits_{x \to 0} \frac{\tan x - \sin x}{x^3}$

(c) $\mathop{Lt}\limits_{x \to y} \frac{\tan x - \tan y}{x - y}$ **(d)** $\lim\limits_{x \to a} \frac{a \sin x - x \sin a}{ax^2 - a^2 x}$

Solution :

(a) $\lim\limits_{x \to \frac{\pi}{3}} \frac{\tan x - \sqrt{3}}{9x^2 - \pi^2} = \lim\limits_{x \to \frac{\pi}{3}} \frac{\tan x - \tan\frac{\pi}{3}}{9x^2 - \pi^2}$

Using $\tan A - \tan B = \frac{\sin(A - B)}{\cos A \cos B}$ we get,

$$\lim_{x \to \frac{\pi}{3}} \frac{\sin\left(x - \frac{\pi}{3}\right)}{\cos x \cos\frac{\pi}{3}(3x - \pi)(3x + \pi)}$$

$$= \frac{1}{3} \frac{1}{\cos\frac{\pi}{3}\cos\frac{\pi}{3}(\pi + \pi)} \left(\text{using } \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right) = \frac{2}{3\pi}$$

(b) $\mathop{Lt}\limits_{x \to 0} \frac{\tan x - \sin x}{x^3} = \mathop{Lt}\limits_{x \to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3}$

$$= \mathop{Lt}_{x \to 0} \frac{\sin x(1 - \cos x)}{x^3 \cos x} = \mathop{Lt}_{x \to 0} \frac{\sin x . 2\sin^2\frac{x}{2}}{x^3 \cos x}$$

$$= \mathop{Lt}_{x \to 0} \frac{2\left(\frac{\sin x}{x}\right)x.\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\left(\frac{x}{2}\right)^2}{x^3 \cos x} = \mathop{Lt}_{x \to 0} \frac{2\left(\frac{\sin x}{x}\right)x.\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2.\frac{1}{4}}{\cos x} = \frac{2.1.1^2\frac{1}{4}}{1} = \frac{1}{2}$$

(c) [Here independent variable x is not tending to zero rather x is tending to y, hence put x = y + h]

Let x = y + h, then as $x \to y$, $h \to 0$

$$\text{Now} \quad \underset{x \to y}{\text{Lt}} \frac{\tan x - \tan y}{x - y} = \underset{h \to 0}{\text{Lt}} \frac{\tan(y+h) - \tan y}{y + h - y}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{1}{h}\left[\frac{\sin(y+h)}{\cos(y+h)} - \frac{\sin y}{\cos y}\right]$$

$$= \underset{h \to 0}{\text{Lt}} \frac{\sin(y+h)\cos y - \cos(y+h)\sin y}{h\cos(y+h)\cos y}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{\sin(y+h-y)}{h\cos(y+h)\cos y}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{\sin h}{h} \cdot \frac{1}{\cos(y+h)\cos y} = 1\,\frac{1}{\cos^2 y} = \sec^2 y$$

(d) $\displaystyle \lim_{x \to a} \frac{a\sin x - x\sin a}{ax^2 - a^2 x}$

$$= \lim_{x \to a} \frac{a\sin x - x\sin x + x\sin x - x\sin a}{ax(x-a)}$$

$$= \lim_{x \to a} \frac{(a-x)\sin x + x(\sin x - \sin a)}{ax(x-a)}$$

$$= \lim_{x \to a} \frac{(a-x)\sin x}{ax(x-a)} + \lim_{x \to a} \frac{\sin x - \sin a}{a(x-a)}$$

$$= -\frac{\sin a}{a^2} + \lim_{x \to a} \frac{2\cos\left(\frac{x+a}{2}\right)}{2a}\left[\frac{\sin\frac{(x-a)}{2}}{\frac{(x-a)}{2}}\right] = -\frac{\sin a}{a^2} + \frac{\cos a}{a}$$

Illustration 12

(a) **If** $f(x) = \dfrac{\tan 2x - x}{3x - \sin x}$, **find** $\underset{x \to 0}{\text{Lt}}\, f(x)$

(b) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha x}{\cos 2\beta x - \cos 2\alpha x} \cdot x$

(c) $\underset{h \to 0}{\text{Lt}} \dfrac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

(d) $\underset{x \to 0}{\text{Lt}} \dfrac{1 - \cos x \cos 2x \cos 3x}{\sin^2 x}$

Solution :

(a) $$\underset{x\to 0}{\text{Lt}}\, f(x) = \underset{x\to 0}{\text{Lt}}\, \frac{\tan 2x - x}{3x - \sin x}$$ [IIT-71]

$$= \underset{x\to 0}{\text{Lt}}\, \frac{\left(\frac{\tan 2x}{2x}\right)2x - x}{3x - \left(\frac{\sin x}{x}\right)x} = \underset{x\to 0}{\text{Lt}}\, \frac{\left(\frac{\tan 2x}{2x}\right)2 - 1}{3 - \frac{\sin x}{x}} = \frac{2-1}{3-1} = \frac{1}{2}$$

(b) $$\underset{x\to 0}{\text{Lt}}\, \frac{\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha x}{\cos 2\beta x - \cos 2\alpha x}.x$$

$$= \underset{x\to 0}{\text{Lt}}\, \frac{\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha x}{2\sin(\alpha+\beta)x.\sin(\alpha-\beta)x}.x$$

$$= \underset{x\to 0}{\text{Lt}}\left[\frac{\frac{\sin(\alpha+\beta)x}{(\alpha+\beta)x}.(\alpha+\beta)x + \frac{\sin(\alpha+\beta)x}{(\alpha-\beta)x}.(\alpha-\beta)x + \frac{\sin 2\alpha x}{2\alpha x}.2\alpha x}{2\frac{\sin(\alpha+\beta)x}{(\alpha+\beta)x}.(\alpha+\beta)x.\frac{\sin(\alpha-\beta)x}{(\alpha-\beta)x}.(\alpha-\beta)x}\right]x$$

$$= \underset{x\to 0}{\text{Lt}}\, \frac{\frac{\sin(\alpha+\beta)x}{(\alpha+\beta)x}.(\alpha+\beta) + \frac{\sin(\alpha-\beta)x}{(\alpha-\beta)x}.(\alpha-\beta) + \frac{\sin 2\alpha x}{2\alpha x}.2\alpha}{2\frac{\sin(\alpha+\beta)x}{(\alpha+\beta)x}.(\alpha+\beta).\frac{\sin(\alpha-\beta)x}{(\alpha-\beta)x}.(\alpha-\beta)}$$

$$= \frac{1.(\alpha+\beta) + 1.(\alpha-\beta) + 1.2\alpha}{2.1(\alpha+\beta)\,1.(\alpha-\beta)} = \frac{4\alpha}{2(\alpha^2-\beta^2)} = \frac{2\alpha}{\alpha^2-\beta^2}$$

$$= \alpha \cos \alpha + \sin \alpha = \sin \alpha - \alpha \cos \alpha$$

(c) $$\underset{h\to 0}{\text{Lt}}\, \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$$ [IIT-79]

$$= \underset{h\to 0}{\text{Lt}}\, \frac{(a^2 + 2ah + h^2)\sin(a+h) - a^2 \sin a}{h}$$

$$= \underset{h\to 0}{\text{Lt}}\left[\frac{a^2(\sin(a+h) - \sin a}{h}\right] + \frac{(2ah + h^2)\sin(a+h)}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \left[\frac{a^2.2\cos\frac{2a+h}{2}\sin\frac{h}{2}}{h} + (2a+h)\sin(a+h) \right]$$

$$= \underset{h \to 0}{\text{Lt}} \frac{2a^2 \cos\frac{2a+h}{2} \frac{\sin\frac{h}{2}}{\frac{h}{2}} . \frac{h}{2}}{h} + \underset{h \to 0}{\text{Lt}} (2a+h)\sin(a+h) \quad = a^2\cos a + 2a \sin a$$

(d) $\cos x \cos 2x \cos 3x$

$= \frac{1}{2}$ $(\cos 2x \cos 3x \cos 2x)$

$= \frac{1}{2}$ $[(\cos 2x + \cos 4x) \cos 2x]$

$= \frac{1}{4}$ $[2 \cos^2 2x + 2 \cos 4x \cos 2x]$

$= \frac{1}{4}$ $[1 + \cos 4x + \cos 2x + \cos 6x]$

Now $\underset{x \to 0}{\text{Lt}} \frac{1 - \cos x \cos 2x \cos 3x}{\sin^2 2x}$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - \frac{1}{4}(1 + \cos 2x + \cos 4x + \cos 6x)}{\sin^2 2x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - \cos 2x + 1 - \cos 4x + 1 - \cos 6x}{4\sin^2 2x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2\sin^2 x + 2\sin^2 + 2\sin^2 3x}{4\sin^2 2x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2\left(\frac{\sin x}{x}\right)^2 .x^2 + 2\left(\frac{\sin 2x}{2x}\right)^2 .4x^2 + 2\left(\frac{\sin 3x}{3x}\right)^2 .9x^2}{4\left(\frac{\sin 2x}{2x}\right)^2 .4x^2} = \frac{28}{16} = \frac{7}{4}$$

Type I. Category B.

Trigonometric Problems in which variable tends to infinity.

There are no formulas as such for this type.

Illustration 13

(a) $\underset{x\to\infty}{\text{Lt}}\left(\frac{\sin x}{x}\right)$ (b) $\underset{x\to\infty}{\text{Lt}}\ 2x\tan\left(\frac{1}{x}\right)$

Solution :

(a) We have $\underset{x\to\infty}{\text{Lt}}\frac{\sin x}{x}$

we know $-1 \le \sin x \le 1$, for all values of $x \in R$ & and as $x\to\infty, \frac{1}{x}\to 0$

$\therefore$ Limit be comes

$\underset{x\to\infty}{\text{Lt}}\ (\to 0)$ (a number between (–1, 1)

$= 0$

Note : **You can though remember this limit.**

(b) $\underset{x\to\infty}{\text{Lt}}\ 2x\tan\left(\frac{1}{x}\right)$

put $x=\frac{1}{h}$, so as $x\to\infty$ $h\to 0$

$\underset{h\to 0}{\text{Lt}}\ \frac{2}{h}\tan(h)$

$= 2$ $\left\{\text{as } \underset{h\to 0}{\text{Lt}}\frac{\tan h}{h}=1\right\}$

Illustration 14

Find the Limit, $\underset{x\to\infty}{\text{Lt}}\left(\frac{x+\cos x}{x+\sin x}\right)$

Solution :

Given Limit is $\underset{x\to\infty}{\text{Lt}}\frac{x+\cos x}{x+\sin x}$

$$= \underset{x\to\infty}{\text{Lt}} \frac{1+\frac{\cos x}{x}}{1+\frac{\sin x}{x}}$$

$$= \underset{x\to\infty}{\text{Lt}} \frac{1+0}{1+0} \qquad \left\{ \underset{x\to\infty}{\textbf{Lt}} \frac{\textbf{cos}\, x}{x} = \mathbf{0} \quad \text{using the same concept as in } \frac{\sin x}{x} \right\}$$

$= 1.$

Type II. Category C

Problems containing exponential and logarithmic functions :

Working Rule :

Using the following formulae whichever is required

(i) $$\underset{f(x)\to 0}{\textbf{Lt}} \frac{a^{f(x)} - \mathbf{1}}{f(x)} = \mathbf{log}_e\, a$$

This formula should be used only when base is a constant and power is a variable.

Special Case :

$$\underset{f(x)\to 0}{\text{Lt}} \frac{e^{f(x)} - 1}{f(x)} = 1$$

(ii) $$\underset{f(x)\to 0}{\textbf{Lt}} [\mathbf{1} + k f(x)]^{\frac{\mathbf{1}}{f(x)}} = e^k$$

This formula should be used when both base and powers are variables.

(iii)

This formula should be used in case of logarithmic function.

(iv) $$\underset{f(x)\to 0}{\textbf{Lt}} \frac{\mathbf{log}\{\mathbf{1} + f(x)\}}{f(x)} = \mathbf{1}$$

$$= \underset{x\to a}{\text{Lt}}\, e^{g(x)[f(x)-1]}$$

Here $x \to a,\ f(x) \to 1$ and $g(x) \to \infty$

This formula should be used only when indeterminate form is 1^{∞}.

Illustration 15

(a) $\underset{x\to 1}{\text{Lt}}\left(\frac{\log x}{x-1}\right)$ (b) $\underset{x\to 0}{\text{Lt}}\frac{e^x-1-x}{x^2}$

(c) $\underset{h\to 0}{\text{Lt}}\left(\frac{\log(x+h)-\log x}{h}\right)$

Solution :

(a) given limit is $\underset{x\to 1}{\text{Lt}}\left(\frac{\log x}{x-1}\right)$

replacing x by h + 1, {limit also changes}

$$= \underset{h\to 0}{\text{Lt}}\frac{\log(1+h)}{1+h-1}$$

$$= \underset{h\to 0}{\text{Lt}}\frac{\log(1+h)}{h} = 1$$

(b) $\underset{x\to 0}{\text{Lt}}\frac{e^x-1-x}{x^2}$

in this question if we try to use the formula $\left(\frac{e^x-1}{x}\right)$, then it will not be solved, why ?

Because we will get zero in numerator & denominator, which becomes unsolvable

$$\underset{x\to 0}{\text{Lt}}\frac{\frac{e^x-1}{x}-\frac{1}{x}}{x} = \underset{x\to 0}{\text{Lt}}\frac{(\to 1)-1}{x} = \frac{0}{0} \text{ form}$$

so either we use L' hospitals or we go for expansion series.
Here we will go for expansion series

$$= \underset{x\to 0}{\text{Lt}}\frac{\left(1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}\ldots\right)-1-x}{x^2}$$

$$= \underset{x\to 0}{\text{Lt}}\left(\frac{x^2}{2!}+\frac{x^2}{3!}+\ldots\right) = \underset{x\to 0}{\text{Lt}}\left(\frac{1}{2}+\frac{x}{3!}+\frac{x^2}{4!}\ldots\right)$$

putting x = 0 in the rest = $\frac{1}{2}$

(c) Given limit is,

This is a simple limit, just use log properties i.e. $\log a - \log b = \log \frac{a}{b}$

$$= \underset{h \to 0}{\mathrm{Lt}} \frac{\log\left(\frac{x+h}{x}\right)}{h}$$

$$= \underset{h \to 0}{\mathrm{Lt}} \frac{\log\left(\frac{1+h}{x}\right)}{h} = \underset{h \to 0}{\mathrm{Lt}} \frac{\log\left(\frac{1+h}{x}\right)}{\frac{h}{x} \times x}$$

$$= \left(\frac{1}{x}\right) \qquad \left\{\because \underset{h\to 0}{\mathrm{Lt}} \log \frac{(1+h)}{h} = 1\right\}$$

Illustration 16

(a) Evaluate $\lim\limits_{x \to 0} \frac{(ab)^x - a^x - b^x + 1}{x^2}$ **(b) Evaluate** $\lim\limits_{x \to 0} \frac{e^{\tan x} - e^x}{\tan x - x}$

(c) $\lim\limits_{x \to 0} \frac{3^x - 5^x}{x}$ **(d)** $\lim\limits_{x \to 0} \frac{6^x - 2^x - 3^x + 1}{\sin^2 x}$

Solution :

(a) $$\lim_{x \to 0} \frac{(ab)^x - a^x - b^x + 1}{x^2}$$

$$= \lim_{x \to 0} \frac{a^x b^x - a^x - b^x + 1}{x^2}$$

$$= \lim_{x \to 0} = \frac{a^x (b^x - 1) - (b^x - 1)}{x^2}$$

(c) $$\lim_{x \to 0} \frac{3^x - 5^x}{x} = \lim_{x \to 0} \left[\frac{3^x - 1}{x} - \frac{5^x - 1}{x}\right]$$

$$= \log 3 - \log 5 = \log \frac{3}{5}$$

$$= \lim_{x \to 0} = \frac{(a^x - 1) \times (b^x - 1)}{x} = \lim_{x \to 0} \frac{a^x - 1}{x} \times \lim_{x \to 0} \frac{(b^x - 1)}{x} = \mathbf{log}\ a \quad \mathbf{log}\ b$$

(b) $$\lim_{x \to 0} \frac{e^{\tan x} - e^x}{\tan x - x} = \lim_{x \to 0} \frac{e^x \times \left[e^{(\tan x - x)} - 1\right]}{(\tan x - x)}$$

$$= \lim_{x \to 0} \frac{e^x \left\{e^{\tan x - x} - 1\right\}}{(\tan x - x)} = e^0 \quad 1 \qquad [\text{as } x \to 0, \tan x - x \to 0]$$

$= 1 \quad 1 = 1$

(d) $\lim\limits_{x \to 0} \dfrac{6^x - 2^x - 3^x + 1}{\sin^2 x}$

$$= \lim_{x \to 0} \frac{(2^x - 1)(3^x - 1)}{x^2} \frac{x^2}{\sin^2 x} = \lim_{x \to 0} \frac{2^x - 1}{x} \lim_{x \to 0} \frac{3^x - 1}{x} \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^2$$

$$= \log_e 3 \log_e 2$$

Type 1. Category C

Problems involving exponential & logarithmic functions.

1. If power is variable, express the given expression as power of e. Use the formula, $a^x = e^{x \log a}$

2. Use expansion series where required

3. Use the formula $\mathop{\mathrm{Lt}}\limits_{f(x) \to 0} \{1 + k f(x)\}^{\frac{1}{f(x)}} = e^k$ where k is constant

Illustration 17

Find the limit

(a) $\mathop{\mathrm{Lt}}\limits_{x \to \infty} \left(1 + \dfrac{a}{x}\right)^x$ **(b)** $\mathop{\mathrm{Lt}}\limits_{x \to \infty} x\left(e^{\frac{1}{x}} - e^{\frac{-1}{x}}\right)$

Solution :

We try to convert these questions to type 2 only so that we can use formulas.

(a) Given, $\mathop{\mathrm{Lt}}\limits_{x \to \infty} \left(1 + \dfrac{a}{x}\right)^x$

Now here it is a type II question only. Compare it with formula $\mathop{\mathrm{Lt}}\limits_{x \to 0} (1 + ax)^{\frac{1}{x}}$.

if you put $x = \dfrac{1}{h}$, limit changes to

$$= \mathop{\mathrm{Lt}}\limits_{h \to 0} (1 + ah)^{\frac{1}{h}}$$

$$= e^a \quad \{\text{using } \mathop{\mathrm{Lt}}\limits_{x \to 0} (1 + x)^{\frac{1}{x}} = e\}$$

(b) $\underset{x \to 0}{\text{Lt}}\ x\left(x^{\frac{1}{x}} - e^{-\frac{1}{x}}\right)$

again putting x = $x = \frac{1}{h}$

$$\underset{h \to 0}{\text{Lt}}\ \frac{1}{h}\left(e^{h} - e^{-h}\right)$$

$$= \underset{h \to 0}{\text{Lt}}\left[\frac{e^{h} - 1}{h} - \frac{(e^{-h} - 1)}{h}\right] = \underset{h \to 0}{\text{Lt}}\left[\frac{e^{h} - 1}{h}\right] - \underset{h \to 0}{\text{Lt}}\left[\frac{e^{-h} - 1}{h}\right]$$

in second limit put h = – h

$$= 1 - \underset{h \to 0}{\text{Lt}}\left[\frac{e^{h} - 1}{-h}\right] = 1 + 1 \left\{\because \underset{h \to 0}{\text{Lt}}\ \frac{e^{h} - 1}{h} = 1\right\}$$

$$= 2$$

Illustration 18

(a) $\lim\limits_{x \to 0}(1 - 2x)^{\frac{1}{x}}$

(b) $\lim\limits_{x \to \infty}\left(1 - \frac{1}{x^{2}}\right)^{x}$

(c) $\lim\limits_{x \to \infty}\left(\frac{x + 1}{x - 2}\right)^{2x - 1}$

(d) $\lim\limits_{x \to 1} x^{\cot 2x}$

Solution :

(a) $\lim\limits_{x \to 0}(1 - 2x)^{\frac{1}{x}}$

using formula

$$= \lim_{x \to 0}(1 - 2x)^{\frac{-1}{2x}}\left[\frac{1}{x} \times -2x\right]$$

$$\Rightarrow \quad e^{-\lim\limits_{x \to 0}\frac{2x}{x}} = e^{-2}$$

(b) $\lim\limits_{x \to \infty}\left(1+\frac{1}{x^2}\right)^x$

our formula is $\lim\limits_{x \to 0}(1+f(x))^{\frac{1}{f(x)}} = e$ where f (x) $\to$ 0

here $f(x) = \frac{1}{x^2}$ which as $x \to \infty$, $f(x) \to 0$, hence we can apply the same formula.

$$\Rightarrow \quad \lim_{x \to \infty}\left(1+\frac{1}{x^2}\right)^x = \lim_{x \to \infty}\left(1+\frac{1}{x^2}\right)^{x^2}\left[x \times \frac{1}{x^2}\right]$$

$$= e^{\lim\limits_{x\to\infty}\frac{x}{x^2}} = e^{\lim\limits_{x\to\infty}\frac{1}{x}}$$

$$= e^0 = 1$$

(c) $\lim\limits_{x \to \infty}\left(\frac{x+1}{x-2}\right)^{2x-1}$

$$= \lim_{x \to \infty}\left(1+\frac{x+1}{x-2}-1\right)^{2x-1} = \lim_{x \to \infty}\left(1+\frac{3}{x-2}\right)^{2x-1}$$

again we can apply the concept as used in previous question

$$= e^{\lim\limits_{x\to\infty}\frac{3}{x-2}\times 2x-1} = e^6$$

(d) $\lim\limits_{x \to 1} x^{\cot \pi x}$

$$= \lim_{x \to 1}(1+(x-1))^{\cot zx}$$

$$= e^{\lim\limits_{x \to 1}(x-1)\cot zx} = e^{\lim\limits_{x \to 1}\frac{x-1}{\tan \pi x}} \quad \text{or} \quad = e^{\lim\limits_{x \to 1}\frac{x-1}{\tan(\pi - zx)}}$$

$$= e^{\lim\limits_{x \to 1}\frac{x-1}{\tan \pi(1-x)}}$$

$$= e^{\lim\limits_{x \to 1}\frac{-1}{\pi}\frac{\pi(1-x)}{\tan z(1-x)}} \qquad \left\{\text{as } \operatorname*{tim}_{\theta \to 0}\frac{\tan\theta}{\theta} = 1\right]$$

$$= e^{\frac{-1}{\pi}}$$

Some confusing limits

Illustration 19

(a) $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ **(b)** $\lim\limits_{x \to 0} [x - 3]$

(c) $\lim\limits_{x \to 0} \sin\dfrac{1}{x}$ **(d)** $\lim\limits_{x \to 0} x \sin\left({}^{1}/_{x}\right)$

Solution :

(a) for the limit to exist LHL & RHL should be equal lets take LHL first

$$\lim_{x \to 0} \frac{|x|}{x} \qquad \left\{\begin{matrix}|x| \text{ gives } (-x) \text{ for negative nos. \& since x is} \\ \text{approaching from negative side, } |x| \text{ gives } (-x)\end{matrix}\right\}$$

$$= \lim_{x \to 0} \frac{-x}{x} = -1 \qquad \text{...(i)}$$

Now, RHL

$$\lim_{x \to 0^+} \frac{|x|}{x} \qquad \left\{\text{as } x \to 0^+ \text{ i.e. from positive side } |x| \text{ returns } + x\right\}$$

$$= \lim_{x \to 0} \frac{x}{x} = 1 \qquad \text{...(ii)}$$

from (i) & (ii) LHL ≠ RHL, hence limit does not exist.

(b) $\lim\limits_{x \to 0} [x - 3]$

again we will find LHL & RHL for this question

LHL $\lim\limits_{x \to 0^-} [x - 3]$

$$= \lim_{h \to 0} [0 - h - 3]$$

$$= \lim_{h \to 0} \underbrace{[-h - 3]}$$

This will be a number between (– 4, – 3) and we know that for this the value of greatest integer function is $^-$ 4.

$\therefore$ LHL = ${}^{-}4$

RHL $\lim\limits_{x \to 0^+} [x - 3]$

$= \lim\limits_{h \to 0} [0 + h - 3]$

$= \lim\limits_{h \to 0} \underbrace{[h - 3]}$

this will be a number between (– 3, – 2) and hence greatest integer function returns – 3

$\therefore$ RHL = – 3

Now, LHL $\neq$ RHL

Hence limit does not exist.

(c) $\lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$

now as $x \to 0$ $\left(\frac{1}{x}\right) \to \infty$

but for $\sin\left(\frac{1}{x}\right)$ or sin (∞) is not a finite value. In fact it is a oscillatory value between [– 1, + 1] because we dont know the value of ∞.

Note : Some of the students get confused in this, in fact some think that sin (∞) $\to \infty$ which is absolutely wrong as sin x can never return a value other than [– 1, 1].

Since the limit is not finite, limit does not exists.

(d) $\lim\limits_{x \to 0} x\sin\left(\frac{1}{x}\right)$

This is a very important limit. Let us solve it.

We already solved the part $\lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$ in the above question. It is a value between [– 1, 1] but $x \to 0$, x approaches 0.

Hence limit becomes

$= \to 0$ (a number between [– 1, 1]

= 0

Hence limit exists and is equal to 0.

(You can check by equating LHL & RHL)

Illustration 20

Evaluate the following limits :

(a) $\underset{x \to 1}{\textbf{Lt}} (1 - x) \tan \frac{\pi x}{2}$ (b) $\underset{x \to \pi/2}{\textbf{Lt}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right) \cot x}$

Solution :

(a) These type of limits are solved by substituting the limit.

See now in the limit,

$\underset{x \to 1}{\text{Lt}} (1 - x) \tan \frac{\pi x}{2}$, $\tan \frac{\pi x}{2}$ approaches ∞ so somehow we need to remove this. If it can be converted to $\cot \frac{\pi x}{2}$, then the limit will be solved as $\cot \frac{\pi x}{2}$ will approach 0.

And we know $\cot \frac{\pi x}{2} = \tan \left(\frac{\pi}{2} - \frac{\pi}{2} x\right)$ or $\tan \frac{\pi}{2} (1 - x)$

now, do you see something

putting x as (1 – x) solves the question

(1 – x) becomes x & lim changes to

$$\lim_{x \to 0} x \cot \frac{\pi x}{2} = \lim_{x \to 0} \frac{x}{\sin \frac{\pi x}{2}} = \frac{2}{\pi} \qquad [\text{as } \cos \theta = 1]$$

(b) $\underset{x \to \pi/2}{\text{Lt}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right) \cot x}$

Note : In these type of questions you will get the clue of what to substitute from the question itself.

Like in this question we will substitute $\frac{\pi}{2} - x$ for x, hence limit becomes

$$\therefore \quad \lim_{x \to 0} \frac{1 - \sin\left(\frac{\pi}{2} - x\right)}{x \cot\left(\frac{\pi}{2} - x\right)} = \lim_{x \to 0} \frac{1 - \cos x}{x \tan x}$$

$$= \lim_{x \to 0} \left(\frac{1 - \cos x}{x^2}\right)\left(\frac{x}{\tan x}\right) = \frac{1}{2} \times 1 \qquad \left\{\textbf{as } \lim_{x \to 0} \frac{\mathbf{1 - \cos x}}{x^2} = \frac{\mathbf{1}}{\mathbf{2}}\right\}$$

you can learn this

Some questions on expansion series

Generally expanding makes the question a bit easy. Lt us see how.

Illustration 21

Evaluate $\underset{x \to \infty}{\text{Lt}} \dfrac{x^5}{e^x}$

Solution : $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

putting this back in the limit

$$\underset{x \to \infty}{\text{Lt}} \frac{x^5}{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots}$$

dividing by x^5

$$\underset{x \to \infty}{\text{Lt}} \frac{1}{\frac{1}{x^5} + \frac{1}{x^4} + \frac{1}{2!x^3} + \frac{1}{3!x^2} + \frac{1}{4!x} + 1 + \frac{x}{6!} + \ldots}$$

we can see that denominator is approaching ∞

Hence the limit becomes $\underset{x \to \infty}{\text{Lt}} \dfrac{1}{\to\infty}$ which is 0.

Illustration 22

Find $\underset{x \to 0}{\text{Lt}} \dfrac{\log(1 + x)}{3^x - 1}$

Solution :

Here we will apply expansion series of both log (1 + x) & a^x which is

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$$\& \ a^x = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + \ldots$$

using these

$$\underset{x \to 0}{\text{Lt}} \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}}{1 + x\log 3 + \frac{x^2}{2!}(\log 3)^2 \ldots - 1}$$

$$= \lim_{x \to 0} \frac{x\left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} \ldots\right)}{x \log 3\left(1 + \frac{x}{2}\log 3 + \ldots\right)}$$

Now we can put x = 0 in the limit also

Hence limit is $\frac{1}{\log 3}$

LIMIT BY L' 'HOSPITAL'S RULE

L'Hospital's rule is applicable only when the form is $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

In case of other indeterminate forms, first of all they should be changed to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then L Hospital's rule should be applied.

'Hospital's rule : Let f (x) and g (x) be differentiable functions at x = a.

Let $f'(x), f''(x), f'''(x), \ldots, f^n(x)$ denote the first, second, third, ... nth derivatives respectively of f (x) and $g'(x), g''(x), g'''(x), \ldots, g^n(x)$ denote the first, second, third, ..., nth derivatives respectively of g(x).

According to L'Hospital's rule

1. $$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)}\left[\frac{0}{0} \text{ form}\right] = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

In general if $f'(x), f''(x), \ldots, f^{n-1}(x) \to 0$ and $g'(x), g''(x), \ldots g^{n-1}(x) \to 0$ as $x \to a$ and $\lim_{x \to a} f^n(x)$ and $\lim_{x \to a} g^n(x)$ are simultaneously zero, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}\left[\frac{0}{0} \text{ form}\right] = \lim_{x \to a} \frac{f^n(x)}{g^n(x)}$$

2. $$\lim_{x \to a} \frac{f(x)}{g(x)}\left[\frac{\infty}{\infty} \text{ form}\right] = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

In general if $f'(x), f''(x), \ldots, f^{n-1}(x) \to \infty$ and $g'(x), g''(x), \ldots g^{n-1}(x) \to \infty$ as $x \to a$ and $\lim_{x \to a} f^n(x)$ and $\lim_{x \to a} g^n(x)$ are simultaneously ∞, then

$$\boxed{\lim_{x \to \infty} \frac{f(x)}{g(x)}\left[\frac{\infty}{\infty} \text{ form}\right] = \lim_{x \to \infty} \frac{f^n(x)}{g^n(x)}}$$

How to change the indeterminate forms to form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

1. **When the form is $0.\infty$,** bring the suitable factor in the denominator.

   The form will be now $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

   Example : $\underset{x \to 1}{\text{Lt}} (1 - x) \tan \frac{\pi x}{2}$ **[$0.\infty$ form]**

   $$= \underset{x \to 1}{\text{Lt}} \frac{1 - x}{\cot \frac{\pi x}{2}} \quad \left[\frac{0}{0} \text{ form}\right]$$

2. **When the form is $\infty - \infty$:** Go on simplifying until it reduces to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

3. **When the form is $1^\infty, \infty^0, 0^0$:** Let the required limit be P, then take logarithm and proceed.

Illustration 23

Find $\underset{x \to 1}{\text{Lt}} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$

Solution :

$$\underset{x \to 1}{\text{Lt}^-} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} \qquad \left[\frac{0}{0} \text{form}\right]$$

$$= \underset{x \to 1}{\text{Lt}} \frac{7x^6 - 10x^4 + 0}{3x^2 - 6x + 0} \quad \text{[by L 'Hospital's rule]}$$

$$= \frac{7 - 10}{3 - 6} = \frac{(-3)}{(-3)} = 1$$

Illustration 24

Find $\underset{x \to \alpha}{\text{Lt}} \frac{x \sin \alpha - \alpha \sin x}{x - \alpha}$

[Here x is the variable and a is a constant, therefore we will have to differentiate w.r.t. to x.]

Solution : $\underset{x \to \alpha}{\text{Lt}} \frac{x \sin \alpha - \alpha \sin x}{x - \alpha} \qquad \left[\frac{0}{0} \text{ form}\right]$

$$= \underset{x \to \alpha}{\text{Lt}} \frac{1.\sin \alpha - \alpha \cos x}{1 - 0} = \sin \alpha - \alpha \cos \alpha$$

Illustration 25

Find $\underset{x \to 0}{\textbf{Lt}} \dfrac{\tan x - \sin x}{x^3}$

Solution : $\underset{x \to 0}{\text{Lt}} \dfrac{\tan x - \sin x}{x^3} \qquad \left[\dfrac{0}{0} \text{ form}\right]$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sec^2 x - \cos x}{3x^2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2\sec x \sec x \tan x + \sin x}{6x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2\sec^2 x.\sec^2 x + 2\tan x \,.\, 2\sec x \sec x \tan x + \cos x}{6}$$

$$= \frac{2 + 0 + 1}{6} = \frac{1}{2}$$

Illustration 26

Find $\underset{h \to 0}{\textbf{Lt}} \dfrac{(a + h)^2 \sin(a + h) - a^2 \sin a}{h}$

[Here h is the variable]

Solution :

$$\underset{h \to 0}{\text{Lt}} \frac{(a + h)^2 \sin(a + h) - a^2 \sin a}{h} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \underset{h \to 0}{\text{Lt}} \frac{(a + h)^2 \cos(a + h)(0 + 1) + \sin(a + h)2(a + h)(0 + 1) - 0}{1}$$

$= a^2\cos a + 2a\sin a$ [by L'Hospital's rule]

Miscellaneous Forms :

(I) **0^0 form :** When $\lim_{x \to a} f(x) \neq 1$ but f(x) is positive in the neighbourhood of x = a.

In this case we write, $\{(f(x)\}^{g(x)} = e^{\log_e \{f(x)\}^{g(x)}}$

$$\Rightarrow \qquad \lim_{x \to a} [f(x)]^{g(x)} = e^{\lim_{x \to a} g(x) \log_e f(x)}$$

Illustration 27

Evaluate $\lim_{x \to 0^+} (\sin x)^x$

Solution :

Let $A = \lim_{x \to 0^+} (\sin x)^x$

$$\Rightarrow \quad \log A = \lim_{x \to 0^+} x \log(\sin x)$$

$$\log A = \lim_{x \to 0^+} \frac{\log(\sin x)}{1/x} \qquad \text{[By L' Hospital's rule]}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{\sin x} . \cos x}{-\frac{1}{x^2}} = -\lim_{x \to 0^+} x^2 \cot x$$

$$= -\lim_{x \to 0^+} \frac{x^2}{\tan x} = 0$$

$$\Rightarrow \quad A = 1 \quad \text{or} \quad \lim_{x \to 0^+} (\sin x)^x = 1$$

Illustration 28

Evaluate $\lim_{x \to 0} (\operatorname{cosec} x)^x$

Solution :

Let $A = \lim_{x \to 0} (\operatorname{cosec} x)^x \qquad (\infty^0 \text{ form})$

$$\log A = \lim_{x \to 0} x \log(\operatorname{cosec} x)$$

$$= \lim_{x \to 0} \frac{\log(\operatorname{cosec} x)}{\frac{1}{x}} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{1}{\operatorname{cosec} x} \cdot \frac{(-\operatorname{cosec} x \cot x)}{-\frac{1}{x^2}} \qquad \text{[By L'Hospital's rule)}$$

$$= \lim_{x \to 0} \frac{x^2}{\tan x} = 0$$

$\therefore \quad \log A = 0 \quad \text{or} \quad A = 1$

$\Rightarrow \quad \lim_{x \to 0} (\operatorname{cosec} x)^x = 1$

Illustration 29

Evaluate $\lim_{x \to 0} e^{\frac{1}{x \log x}}$

Solution :

Let $A = \lim_{x \to 0} e^{\frac{1}{x \log x}}$

$$\log A = \lim_{x \to 0} \frac{1}{x \log x} . \log e \quad = \lim_{x \to 0} \frac{1/x}{\log x} . \log e \qquad \left(\frac{\infty}{\infty} \text{ form}\right) \qquad \text{[By L' Hospital's rule]}$$

$$= \lim_{x \to 0} \frac{-1/x^2}{1/x} = -\infty$$

$\log_e A = -\infty$

$\Rightarrow \quad A = e^{-\infty} \text{ or } \lim_{x \to 0} \frac{1}{e^{x \log x}} = 0$

Illustration 30

Evaluate $\lim_{x \to 0} |x|^{\sin x}$

Solution :

$$\lim_{x \to 0} |x|^{\sin x} = \lim_{x \to 0} e^{\sin x \log_e |x|} = e^{\lim_{x \to 0} \frac{\log_e |x|}{\operatorname{cosec} x}}$$

$$= e^{\lim_{x\to 0} \frac{1/x}{-\operatorname{cosec} x \cot x}}$$

$$= e^{\lim_{x\to 0} -\frac{\sin^2 x}{x\cos x}} = e^{\lim_{x\to 0} -\left(\frac{\sin x}{x}\right)^2 \cdot \left(\frac{x}{\cos x}\right)}$$

$$= e^{-(1).(0)} = e^0 = 1$$

Illustration 31

Solve $\lim_{x\to 0^+} \log_{\sin x} \sin 2x$

Solution :

Here, $\lim_{x\to 0^+} \log_{\sin x} \sin 2x$

$$= \lim_{x\to 0^+} \frac{\log \sin 2x}{\log \sin x} \qquad \left(\frac{-\infty}{-\infty}\text{ form}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\sin 2x}.\,2\cos 2x}{\frac{1}{\sin x}.\cos x} = \lim_{x\to 0^+} \frac{\left(\frac{2x}{\sin(2x)}\right)\cos 2x}{\left(\frac{x}{\sin x}\right)\cos x} \qquad \text{[By L' Hospital's]}$$

$$= \lim_{x\to 0^+} \frac{\cos 2x}{\cos x} = 1$$

Illustration 32

Solve $\lim_{x\to 0^+} (\sin x)^{\tan x}$

Solution :

Here $\qquad \lim_{x\to 0^+} (\sin x)^{\tan x} \qquad (0^0 \text{ form})$

let $\qquad A = \lim_{x\to 0^+} (\sin x)^{\tan x}$

Taking log on both sides, we get

$$\log_e A = \lim_{x\to 0^+} \tan x \log(\sin x)$$

$$= \lim_{x\to 0^+} \frac{\log(\sin x)}{\cot x} \qquad \left(\frac{-\infty}{-\infty}\text{ form}\right) \qquad \text{[By L' Hospital's]}$$

Applying L-Hospital's rule $= \lim_{x \to 0^+} \frac{\frac{1}{\sin x}}{-\operatorname{cosec}^2 x} . \cos x = \lim_{x \to 0^+} -\sin x . \cos x = 0$

$\therefore \quad \log_e A = 0$

$\Rightarrow \quad A = e^0 = 1 \qquad \Rightarrow \qquad A = 1$

Illustration 33

Evaluate $\lim_{n \to \infty} (\pi n)^{2/n}$

Solution :

Here; $\quad A = \lim_{n \to \infty} (\pi n)^{2/n} \quad (\infty^0 \text{ form})$

$$\log A = \lim_{n \to \infty} \frac{2 \log (\pi n)}{n} \qquad \left(\frac{\infty}{\infty} \text{form}\right)$$

$$= \lim_{n \to \infty} \frac{2 . \frac{1}{\pi n} . \pi}{1} = \lim_{n \to \infty} \frac{2}{n} = 0$$

$\log_e A = 0 \quad \Rightarrow \quad A = 1$

Illustration 34

Evaluate $\lim_{n \to \infty} \left(\frac{e^n}{\pi}\right)^{1/n}$

Solution :

Here, $\quad A = \lim_{n \to \infty} \left(\frac{e^n}{\pi}\right)^{1/n} \quad (\infty^0 \text{ form})$

$$\therefore \quad \log A = \lim_{n \to \infty} \frac{1}{n} \log \left(\frac{e^n}{\pi}\right)$$

$$= \lim_{n \to \infty} \frac{n \log e - \log \pi}{n} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{n \to \infty} \frac{\log e - 0}{1}$$

$$= \log e$$

$\Rightarrow \quad A = e$