WAVES

A wave is a disturbance which propagates energy (and momentum) from one place to the other without the transport of matter. It is well spread over a region of space without clear cut boundaries. It cannot be said to be localised here or there. It is hard to think of any mass being associated with a wave. Moreover, quantities like amplitude, wavelength, frequency and phase are used to characterise a wave which have no meaning for a particle.

The Important characteistics of a wave are :

(1) The particles of the medium traversed by a wave execute relatively small vibrations about their mean positions but the particles are not permanently displaced in the direction of propagation of the wave.

(2) Each successive particle of the medium executes the motion quite similar to its predecessors along/perpendicular to the line of travel of the wave.

(3) During wave-motion only transfer of energy takes place but not a portion of the medium.

Waves can be one, two or three dimensional according to the number of dimensions in which they propagate energy. Waves moving along strings are one-dimensional. Surfaces waves or ripples on water are two dimensional, while sound or light waves travelling radially out from a point source are 3-D.

Mechanical and Non-mechanical Waves

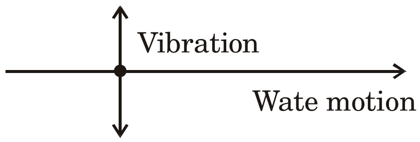
A wave may or may not require a medium for its propagation. The waves which don't require medium for their propagation are called non-mechanical, e.g., light, heat (infrared) and radio waves are non-mechanical as they can propagate through vacuum. In fact all electromagnetic waves (EMW) such as γ-rays, X-rays or microwaves are non-mechanical. On the other hand the waves which require medium for their propagation are called mechanical waves. In the propagation of mechanical waves elasticity and density of the medium play an important role. This is why mechanical waves sometimes are also referred to as elastic waves on string and springs, seismic waves or sound waves are familiar examples of mechanical waves.

Note : Apart from mechanical (elastic) and non-mechanical (electromagnetic) waves there is also another kind of waves called **'matter waves'.** These represent wavelike properties of particles and are governed by the laws of quantum physics.

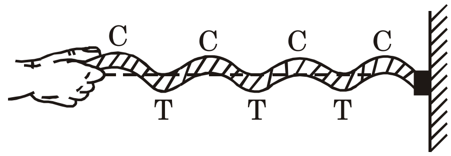
Transverse and Longitudinal Waves

Mechanical waves are further divided into two types :

(i) **Transverse waves :** If the particles of the medium vibrate at right angle to the direction of wave motion or energy propagation the wave is called transverse wave. These are propagated as crests and troughs. Waves on strings are always transverse.

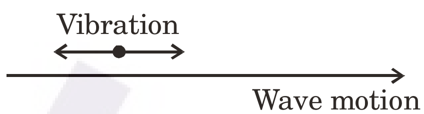


(A)

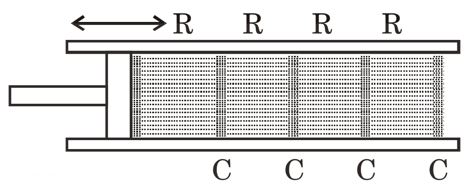


(B)

(ii) **Longitudinal waves :** If the particles of a medium vibrate in the direction of wave motion, the wave is called longitudinal. These are propagated as compressions and rarefactions and also known as pressure or compressional waves. Waves on springs or sound waves in air are examples of longitudinal waves.



(A)



(B)

Note :

(1) All non-mechanical waves are transverse

(2) In gases and liquids mechanical waves are always longitudinal e.g. sound waves in air and water.

(3) In solids, mechanical waves can be either transverse or longitudinal depending on the mode of excitation. The speeds of the two waves in the same solid are also different.

Example 1

Explain why (a) transverse mechanical waves cannot be propagated in liquids and gases while (b) waves on strings are always transverse.

Solution :

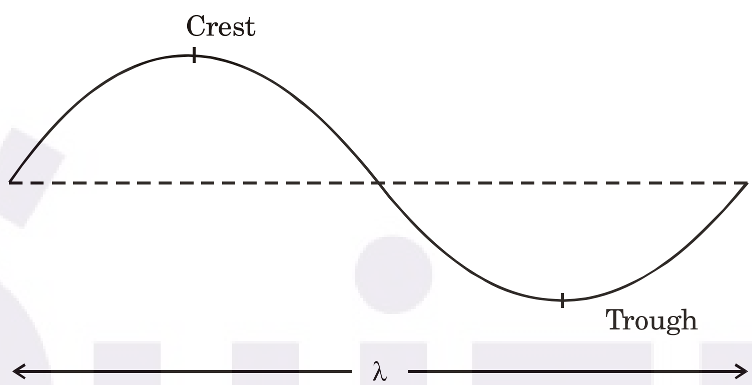
(a) To transmit a transverse mechanical wave the medium must be elastic so as to provide a restoring force when acted on by shearing stress. But liquids and gases flow when acted on by shearing stress, i.e. they cannot sustain shear stress to provide restoring force and so cannot transmit transverse mechanical waves.

(b) Longitudinal waves are pressure waves, i.e., they are transmitted as compression and rarefaction in a medium. Now as the string is non-stretchable so it can neither be compressed nor stretched, i.e., in it compression and rarefaction cannot be produced. This in turn implies that longitudinal waves cannot be propagated along a string [So the waves in a string are always transverse, that too when the string is under tension. If tension in the string is zero, transverse, mechanical waves will also not propagate as then $v = \sqrt{(T/m)} = 0$]

Some Basic Terms :

(a) **Wave frequency :** If the particles of the medium make n (also written as v) vibrations for second, n (or υ) is called the frequency of the wave. The time taken for one vibration is the wave period T and $T = \frac{1}{\upsilon}$ or $\upsilon = \frac{1}{T}$; unit – hertz (Hz).

(b) **Wavelength :** It is defined as the distance travelled by the wave in one period T unit-metre.



It can also be defined as the distance between two successive crests or between two successive troughs.

(c) **Wave velocity :** It is the distance travelled by the wave in one second symbol v or c; unit - metre/second.

If the frequency of the wave is "f" and wavelength is "λ" metres, then wave velocity v is

$$v = f\lambda \text{ m/s} \qquad ...(1)$$

wave velocity = Frequency Wavelength

Equation of Wave Motion

Some physical quantity (say y) is made to oscillate at one place and these oscillations of y propagate to other places. The y may be,

(i) displacement of particles from their mean position in case of transverse wave in a rope or longitudinal sound wave in a gas.

(ii) pressure difference (dP) or density difference ($d\rho$) in case of sound wave or

(iii) electric and magnetic fields in case of electromagnetic waves.

The oscillations of y may or may not be simple harmonic in nature. Now let us consider a one dimensional wave travelling along x-axis. In this case y is a function of position (x) and time (t). The reason is that one may be interested in knowing the value of y at a general point x at any time t. Thus, we can say that,

$$y = y\,(x,\ t)$$

But only those functions of x and t, represent a wave motion which satisfy the differential equation.

$$\frac{\partial^2 y}{\partial t^2} = k\frac{\partial^2 y}{\partial x^2}$$

Here k is a constant, which is equal to square of the wave speed, or

$$k = v^2$$

Thus, the above equation can be written as,

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2} \qquad \text{...(i)}$$

The general solution of this equation is of the form $y\,(x,\ t) = f\,(ax \pm bt)$ ...(ii)

Thus, any function of x and t which satisfies Eq. (i) or which can be written as Eq. (ii) represents a wave. The only condition is that it should be finite everywhere and at all times. Further, if these conditions are satisfied, then speed of wave (v) is given by,

$$v = \frac{\text{coefficient of } t}{\text{coefficient of } x} = \frac{b}{a}$$

The plus (+) sign between ax and bt implies that the wave is travelling along – ve x-direction and minus (–) sign shows that it is travelling along +ve x-direction.

Example 2

Which of the following functions represent a wave

(a) $(x - vt)^2$ **(b)** $\ln\,(x + vt)$ **(c)** $e^{-(x - vt)^2}$ **(d)** $\dfrac{1}{x + vt}$

Solution :

Although all the four functions are written in the form $f\,(ax \pm bt)$, only (c) among the four functions is finite everywhere at all times. Hence only (c) represents a wave.

Example 3

$\nu(x, t) = \dfrac{\mathbf{0.8}}{[(\mathbf{4}x + \mathbf{5}t)^2 + \mathbf{5}]}$ **represents a moving pulse where** x **and** y **are in metre and** t **in second. Then choose the correct alternative(s) :** **[JEE 1999]**

(a) pulse is moving in positive x**-direction**

(b) in 2 s **it will travel a distance of 2.5 m**

(c) its maximum displacement is 0.16 m

(d) it is a symmetric pulse

Solution :

(b), (c) and (d) are correct options.

The shape of pulse at x = 0 and t = 0 would be as shown in Fig.

$$y(0, 0) = \frac{0.8}{5} = 0.16 \text{ m}$$

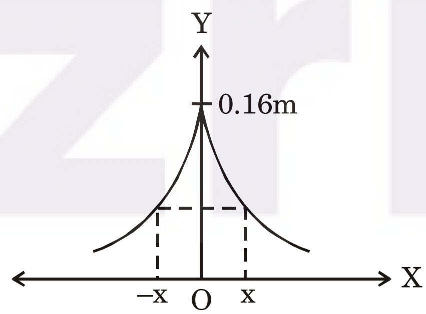
From the figure it is clear that y_{max} = 0.16 m

Pulse will be symmetric (symmetry is checked about y_{max}) if

At t = 0 $\quad y(x) = y(-x)$

From the given equation

$$\left.\begin{aligned} y(x) &= \frac{0.8}{16x^2 + 5} \\ \text{and } y(-x) &= \frac{0.8}{16x^2 + 5} \end{aligned}\right\}$$

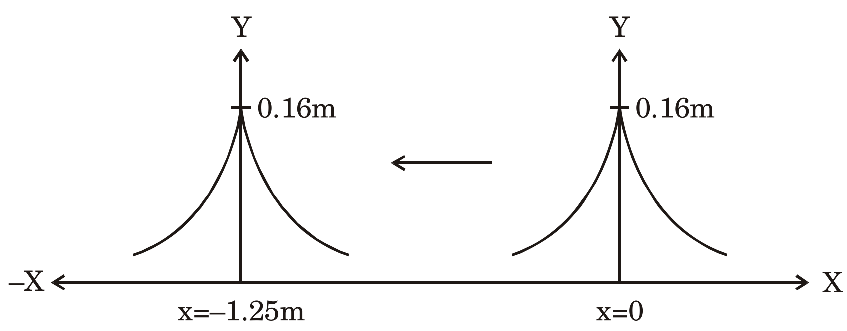


t=0

or $\quad y(x) = y(-x)$

Therefore, pulse is symmetric.

Speed of pulse : At t = 1 s and x = 1.25 m



value of y is again 0.16 m, i.e., pulse has travelled a distance of 1.25 m in 1 second in negative x-direction or we can say that the speed of pulse is 1.25 m/s and it is travelling in negative x-direction. Therefore, it will travel a distance of 2.5 m in 2 seconds. The above statement can be better understood from Fig.

Alternate method :

If equation of a wave pulse is $y = (ax \pm bt)$

the speed of wave is $\frac{b}{a}$ in negative x-direction for $y = f(ax + bt)$ and positive x-direction for $y = f(ax - bt)$. Comparing this from given equation we can find that speed of wave is $\frac{5}{4} = 1.25\,\text{m/s}$ and it is travelling in negative x-direction.

PLANE PROGRESSIVE HARMONIC WAVE

If a travelling wave is a sin or cos function of $(at - bx)$ or $(at + bx)$, the wave is said to be harmonic or plane progressive wave. Here we shall limit ourselves to 1 – D plane progressive wave which in its most general form is given by

$$y = A \sin(\omega t\ kx \mp \phi)$$

From Eqn. it is clear that a set of four parameters A, ϕ, ω and k completely describes a plane progressive wave.

(1) As the maximum value of sin or cos functions can be 1, A represents the maximum value of wave-function as is called the amplitude of the wave.

(2) The constant ϕ is called phase constant or initial phase and enables us to find the position from where time is considered. If at $t = 0$, $x = 0$, ϕ will be zero which is usually the case with a wave and implies that in wave motion time is considered when the wave was at the origin. Henceforth we shall assume $\phi = 0$ and the wave is travelling along positive x-axis unless stated otherwise.

(3) As the wave at a given position at time t' [Fig. A] will be

$$y' = A \sin(\omega t' - kx)$$

So the wave will repeat itself if $y' = y$, i.e., $t' = t + (2\pi/\omega)$ as $\sin(\theta + 2\pi) = \sin\theta$.

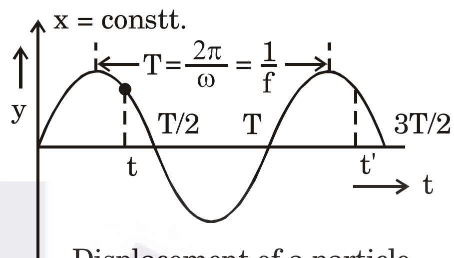
Now as the time after which a wave repeats itself is called time period, i.e.

$$T = t' - t = (2\pi/\omega)$$

Further as the rate at which the wave repeats itself is called its **frequency** f (with units Hz) so

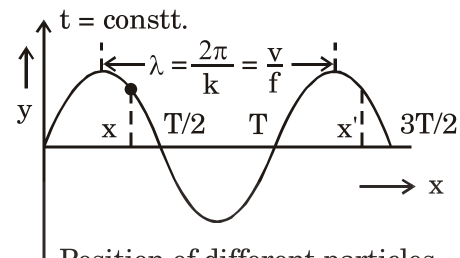
$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

or $\omega = 2\pi f$, ω is called **angular frequency** (with units rad/s). Here it is worth noting that ω, f or T are the characteristers of the source producing the wave and are independent of the nature of the medium in which the wave propagates.



Displacement of a particle at different instants

(A)



Position of different particles at same instant in a string

(B)

(4) As the wave at a given time at position x´ [Fig.(B)] will be

$$y' = A \sin(\omega t - kx')$$

So the wave will repeat itself if $y = y'$, i.e., $x' = x + (2\pi/k)$ as $\sin(\theta \pm 2\pi) = \sin\theta$.

Now as the distance after which the wave repeats itself is called **wavelength λ,**

so $$\lambda = x' - x = \frac{2\pi}{k} \quad , \quad \text{i.e.,} \quad k = \frac{2\pi}{\lambda} \qquad ...(5)$$

k is called **propagation constant or wave vector and has unit (rad/m).** The constant k or wavelength λ depends on the nature of the medium (as same source will produce waves of different wavelengths in different media) and also on the source producing the waves (as in a given medium sources of different frequencies will produce different wavelengths).

(5) If the shape of the wave does not change as the wave propagates in a medium, with increase in t, x will also increase in such a way that

$$\omega t - kx = \text{constt.} \qquad ...(6)$$

The argument of harmonic function $(\omega t - kx)$ is called **phase** of the wave and is constant if the shape of the wave remains unchanged.

Further, if we consider two points at positions x_1 and x_2 on a wave at a given instant, then

$$\phi_1 = \omega t - kx_1 \quad \text{and} \quad \phi_2 = \omega t - kx_2$$

so $$\phi_2 \sim \phi_1 = k(x_2 - x_1)$$

i.e., $$\Delta\phi = \frac{2\pi}{\lambda}\Delta x \qquad \left[\text{as } k = \frac{2\pi}{\lambda}\right] \qquad ...(7)$$

From this it is clear that if $\Delta x = \lambda$, $\Delta\phi = 2\pi$, i.e., a path difference λ corresponds to a phase change

(6) As the phase of a plane progressive wave is constt., i.e.,

$$\omega t - kx = \text{constt.}$$

the so called wave or phase velocity will be given by

$$v = \frac{dx}{dt} = \frac{\omega}{k} = \frac{2\pi f}{(2\pi/\lambda)} = f\lambda \qquad ...(8)$$

(7) As a plane progressive wave propagating along positive x-axis with t = 0 at x = 0 is given by

$$y = A \sin(\omega t - kx)$$

so the **velocity of a particle** on it will be $v_{Pa} = \frac{dy}{dt} = A\omega\cos(\omega t - kx)$ ...(10)

But as $$\cos(\omega t - kx) = \sqrt{\{1 - \sin^2(\omega t - kx)\}} = \sqrt{1 - (y/A)^2}$$

so $$v_{Pa} = \omega\sqrt{(A^2 - y^2)} \qquad ...(11)$$

It will be maximum when $y^2 = \min = 0$

i.e., $$(v_{Pa})_{max} = A\omega \qquad ...(12)$$

The acceleration of the particle is the second partial derivative of y (x, t) with respect to t,

$\therefore$ $$a_P = \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A \sin(kx - wt)$$

$$= -\omega^2 y$$

i.e. the acceleration of the particle equals $-\omega^2$ times its displacement. Thus

$$a_P = -\omega^2 \text{ (displacement)}$$

(8) A plane progressive wave (either transverse of longitudinal, mechanical, non-mechanical) in the light of above can be written in many forms such as

$$y = A \sin [\omega t - kx] \quad ...(a)$$

or $$y = A \sin 2\pi [ft - x/\lambda)] \quad [\text{as } \omega = 2\pi f \text{ and } k = (2\pi/\lambda)]$$

or $$y = A \sin 2\pi \left[\frac{t}{T} - \frac{x}{\lambda}\right] \quad [\text{as } f = 1/T]$$

or $$y = A \sin k [vt - x] \quad [\text{as in (a), } v = \omega/k]$$

or $$y = A \sin \omega [t - (x/v)] \quad [\text{as in (a), } k/\omega = 1/v]$$

Note : In solving numerical problems related to equation of plane progressive wave remember that :

(1) If the sign between t and x terms is negative the wave is propagating along positive x-axis and vice-versa.

(2) The coefficient of sin or cos function, i.e., A gives the amplitude of the wave while its argument ($\omega t \mp kx$) denotes phase.

(3) The coefficient of t gives angular frequency ω [$= 2\pi f = (2\pi/T)$].

(4) The coefficient of x gives propagation constant or wave number k ($= 2\pi/\lambda$).

(5) The ratio of coefficient of t to that of x gives wave or phase velocity, i.e., $v = (\omega/k)$ and is constant for a given medium.

(6) Particle velocity will be obtained by differentiating y with respect to t, i.e., $v_{Pa} = (dy/dt) = -v$ (slope of y/x curve) and is not constt. with maximum value = $A\omega$.

(7) When a given wave passes from one medium to another its frequency does not change, so from $v = f\lambda$

$$(v_1)/(v_2) = (\lambda_1)/(\lambda_2)$$

(8) In case of vibrations of string (in its fundamental mode)

$$\lambda = 2 \text{ [length of string]}$$

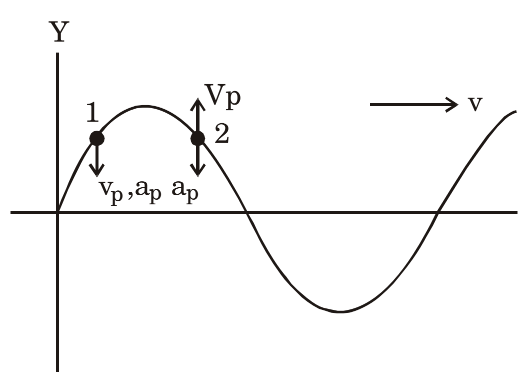
(9) While passing equation $v = f\lambda$, be careful to decide which out of v, f and λ is constant

(10) Figure shows the velocity (V_P) and acceleration (a_P) for two points 1 and 2 on a string as a sinusoidal wave is travelling in it along positive x-direction.

At 1 : Slope of the curve is positive. Hence particle velocity (V_P) is negative or downwards. Similarly displacement of the particle is positive, so acceleration will be negative or downwards.

At 2 : Slope is negative while displacement is positive. Hence V_F will be positive (upwards) and a_P is negative (downwards).

Note : The direction of V_P will change if the wave travels along negative x-direction.



Example 5

The equation of a wave is $y(x, t) = \mathbf{0.05} \sin\left[\frac{\pi}{2}(10x - 40t) - \frac{\pi}{4}\right]$m

Find (a) the wavelength, the frequency and the wave velocity

(b) the particle velocity and acceleration at $x = 0.5$ **m and** $t = 0.05$ **s.**

Solution :

(a) The equation may be written as, $y(x, t) = 0.05 \sin\left(5\pi x - 20\pi t - \frac{\pi}{4}\right)$m

Comparing this with equation of plane progressive harmonic wave,

$$y(x, t) = A \sin(kx - \omega t + \phi) \text{ we have,}$$

$$\text{wave number } k = \frac{2\pi}{\lambda} = 5\pi \text{ rad/m}$$

$\therefore$ $\lambda = 0.4$ m **Ans.**

The angular frequency is, $\omega = 2\pi f = 20\pi$ rad/s

$\therefore$ $f = 10$ Hz **Ans.**

The wave velocity is, $v = f\lambda = \frac{\omega}{k} = 4$ m/s in + x direction **Ans.**

(b) The particle velocity and acceleration are,

$$\frac{\partial y}{\partial t} = -(20\pi)(0.05)\cos\left(\frac{5\pi}{2} - \pi - \frac{\pi}{4}\right)$$

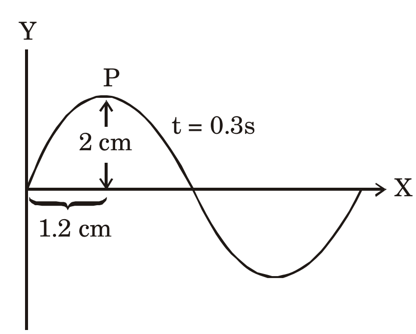
$$= 2.22 \text{ m/s}$$

$$\frac{\partial^2 y}{\partial t^2} = -(20\pi)^2(0.05)\sin\left(\frac{5\pi}{2} - \pi - \frac{\pi}{4}\right)$$

$$= 140 \text{ m/s}^2$$ **Ans.**

Example 6

Figure shows a snapshot of a sinusoidal travelling wave taken at $t = 0.3$ **s. The wavelength is 7.5 cm and the amplitude is 2 cm. If the crest P was at** $x = 0$ **at** $t = 0$**, write the equation of travelling wave.**



Solution :

Given A = 2 cm, λ = 7.5 cm

$$k = \frac{2\pi}{\lambda} = 0.84 \text{ cm}^{-1}$$

The wave has travelled a distance of 1.2 cm is 0.3 s. Hence the speed of wave

$$v = \frac{1.2}{0.3} = 4 \text{ cm/s}$$

$\therefore$ Angular frequency ω = (v) (k) = 3.36 rad/s

Since the wave is travelling along positive x-direction and crest (maximum displacement is at x = 0 at t = 0 , we can write the wave equation as,

$$y(x, t) = A \cos(kx - \omega t)$$

or $y(x, t) = A \cos(\omega t - kx)$ as $\cos(-\theta) = \cos\theta$

Therefore, the desired equation is,

$y(x, t) = (2 \text{ cm}) \cos [(0.84 \text{ cm}^{-1}) x - (3.36 \text{ rad/s})t]$ cm **Ans.**

Energy of a Plane Progressive Wave

Consider a plane wave propagating with velocity v in x-direction across an area s. An element of material medium (density = ρ kg/m^3) will have a mass ρ (sdx)

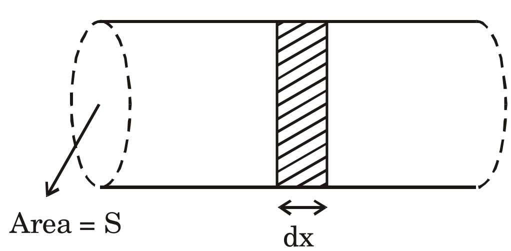
The displacement of a particle from its equilibrium position is given by the wave equation.

$$y = A \sin(\omega t - kx)$$

Total energy, $dE = \frac{1}{2} m V_{max}^2$

$$dE = \frac{1}{2}(\rho \text{ sdx})(A\omega)^2$$

$$= \rho \text{ sdx } (2\pi^2 f^2 A^2)$$



$\Rightarrow$ energy density = $\frac{dE}{sdx} = 2\pi^2 f^2 A^2 \rho \left(J/m^3\right)$

energy per unit length = $\rho s (2\pi^2 f^2 A^2)$

$\therefore$ power transmitted = $2\pi^2 f^2 A^2 \rho s v$ (J/s)

Intensity of the Wave (I)

Intensity of the wave is defined as the power crossing per unit area.

$\therefore$ $I = 2\pi^2 f^2 A^2 \rho v$ (Watt/m^2)

For wave propagation through taut string

$\rho s = \mu$, the linear density in kg/m

$\therefore$ Energy per unit length = $2\pi^2 f^2 A^2 \mu$

Note :

(1) The energy is the average value over a time period

(2) Intensity $I \propto A^2$

(μ and f are constant)

Example 7

The equation of a progressive wave is given by as $y = 0.05 \sin 2\pi \left(\frac{64t - x}{3.2}\right)$ where the amplitude and wavelength are in metres. (i) Calculate the phase velocity of the wave, (ii) also calculate x, if the phase difference between two points at a distance 0.32 m apart, along the line of propagation is π/x (iii) if the wave propagates through air (density = 1.3 kg/m^3) find the intensity of wave. (Assuming $\pi^2 = 10$)

Solution :

(i) The progressive wave is represented by $y = 0.05 \sin \frac{2\pi}{3.2}(64t - x)$

Comparing this with the standard equation $y = A \sin \frac{2\pi}{\lambda}(vt - x)$

the phase velocity (wave velocity) = 64 m/s

(ii) The phase difference of the particles separated by a distance of λ is equal to 2π.

$\therefore$ phase difference of particles separated by a distance 0.32 m = $\frac{2\pi}{3.2} \times 0.32 = \frac{\pi}{5}$ radians

$\therefore$ x = 5

(iii) The intensity of the sound wave is given by

$$I = \frac{1}{2}\rho v \omega^2 A^2 = \frac{1}{2}\left(1.3\,\frac{kg}{m^3}\right)\left(64\,\frac{m}{s}\right)\left(4\pi^2 n^2\right)(0.05m)^2$$

Here n is the frequency of the wave and is equal to $\frac{64}{3.2} = 20$ Hz

$$\therefore\ I = \frac{1}{2} \times 1.3 \times 64 \times 4\pi^2 \times 400 \times 0.0025\ W/m^2 = 1664\ W/m^2$$

VELOCITY OF WAVE PROPAGATION

The physical quantities that determine the velocity are tension in string (T), mass per unit length (μ).

Transverse Wave in a Stretched String

Consider a transverse pulse produced in a taut string of linear mass density μ. Consider a small segment of the pulse, of length Δl, forming an arc of a circle of radius R. A force equal in magnitude to the tension T pulls tangentially on this segment at each end.

Let us set an observer at the centre of the pulse, which moves along with in the pulse towards right. For the observer any small length dl of the string as shown will appear to move backward with a velocity v.

Now the small mass of the string is in a circular path of radius R moving with speed v. Therefore the required centripetal force is provided by the only force acting (neglecting gravity) is the component of tension along the radius.

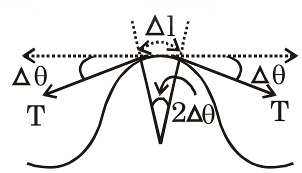
The net restoring force on the element is

$$F = 2T \sin(\Delta\theta) \approx 2T(\Delta\theta) = \frac{T\Delta l}{R}$$

The mass of the segment is $\Delta m = \mu\Delta l$

The acceleration of this element towards the centre of the circle is $a = \frac{v^2}{R}$, where v is the velocity of the pulse.

Using second law of motion $\frac{T\Delta l}{R} = (\mu\Delta l)\frac{v^2}{R}$



or $\quad v = \sqrt{\frac{T}{\mu}}$

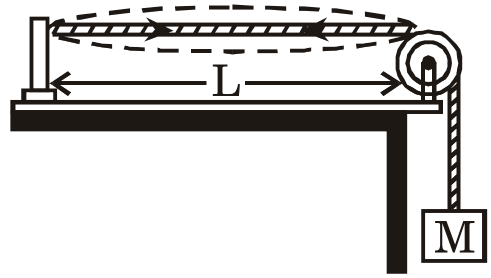
Example 8

A wire of uniform cross-section is stretched between two points 1 m apart. The wire is fixed at one end and a weight of 9 kg is hung over a pulley at the other end produces fundamental frequency of 750 Hz. (a) What is the velocity of transverse waves propagating in the wire? (b) If now the suspended weight is submerged in a liquid of density (5/9) that of the weight, what will be the velocity and frequency of the waves propagating along the wire?

Solution :

(a) In case of fundamental vibrations of string

$(\lambda/2) = L$, i.e., $\lambda = 2 \quad 1 = 2$ m



Now as $v = f\lambda$ and $f = 750$ Hz

and $v = 1500$ m/s

(b) Now as in case of a wire under tension

$$v = \sqrt{\frac{T}{m}} \text{ so } \frac{v_A}{v_B} = \sqrt{\frac{T_A}{T_B}}, \text{ i.e. } v_B = 1500 \sqrt{\frac{T_B}{T_A}}$$

$$\text{or} \quad v_B = 1500 \sqrt{\frac{Mg'}{Mg}} = 1500 \sqrt{\frac{g[1 - \sigma/\rho)]}{g}} \left[\text{as } g' = g\left(1 - \frac{\sigma}{\rho}\right)\right]$$

$$\text{or} \quad v_B = 1500 \sqrt{1 - \frac{5}{9}} = 1000 \text{ m/s}$$

Further as here λ = constt. = 2 m so from $v = f\lambda$,

$$f_B = \frac{v_B}{\lambda_B} = \frac{1000}{2} = 500 \text{ Hz}$$

i.e., in this situation,

$\lambda = 2$ m, $f = 500$ Hz and $v = 1000$ m/s

Note : Here λ = constant; so f and v will change according to the relation $v \propto f$ with $v \propto \sqrt{T}$.

Example 9

A wire of mass 9.8 10^{-3} kg per metre passes over a frictionless pulley fixed on the top of an inclined frictionless plane which makes an angle of 30 with the horizontal. Masses M_1 and M_2 are tied at the two ends of the wire. The mass M_1 rests on the plane and the mass M_2 hangs vertically downwards. The whole system is in equilibrium. Now a transverse wave propagates along the wire with a velocity of 100 m/s. Find the value of masses M_1 and M_2. ($g = 9.8$ m/s^2)

Solution :

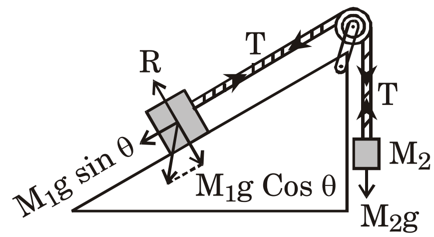
For equilibrium of M_1 along and perpendicular to the plane we have respectively :

$$M_1 g \sin\theta = T \text{ and } M_1 g \cos\theta = R \qquad ...(1)$$

And for equilibrium of M_2,

$$M_2 g = T \qquad ...(2)$$

Now as the velocity of a wave on a string is given by



$$v = \sqrt{\frac{T}{m}}$$

i.e. $$T = v^2 m$$

and as here v = 100 m/s and m = 9.8 10^{-3} kg/m

$$T = (100)^2 \quad 9.8 \quad 10^{-3} = 98\ \text{N} \qquad ...(3)$$

Substituting the value of T from Eqn. (3) in Eqns. (2) and (1),

$$M_2 = (T/g) = (98/9.8) = 10\ \text{kg}$$

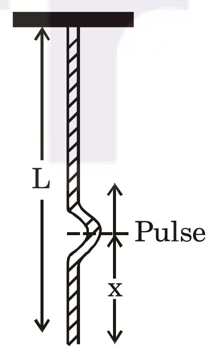
$$M_1 = (T/g\ \sin\theta) = 98/[9.8 \quad (1/2)] = 20\ \text{kg}$$

Example 10

A uniform rope of mass 0.1 kg and length 2.45 m hangs from a ceiling. (a) Find the speed of transverse wave in the rope at a point 0.5 m distant from the lower end, (b) Calculate the time taken by a transverse wave to travel the full length of the rope (g = 9.8 m/s²)

Solution :

(a) As the string has mass and it is suspended vertically, tension in it will be different at different points. For a point at a distance x from the free end, tension will be due to the weight of the string below it. So if M is the mass of string of length L, the mass of length x of the string will be (M/L)x.



$$\therefore \quad T = \left[\frac{M}{L}x\right]g$$

So $$v = \sqrt{\frac{T}{m}} = \sqrt{\frac{Mgx}{L(M/L)}} = \sqrt{gx} \qquad ...(1)$$

Here x = 0.5 m

so $$v = \sqrt{0.5 \times 9.8} = 2.21\ \text{m/s}$$

(b) From part (a) it is clear that the tension and so the velocity of the wave is different at different points. So if at point x the wave travels a distance dx in time dt,

$$v = \frac{dx}{dt} \quad \text{or} \quad \sqrt{gx} = \frac{dx}{dt} \quad \text{[From Eqn. 1]}$$

or $$\int dt = \int \frac{dx}{\sqrt{gx}}, \quad \text{i.e.,} \quad t = \frac{1}{\sqrt{g}}\int_0^L x^{-1/2}dx$$

i.e. $t = 2\sqrt{(L/g)}$ ...(2)

Hence $L = 2.45$ m so $t = 2\sqrt{(2.45/9.8)} = 1$ sec

Principle of Superposition

Two or more waves can travel simultaneously in a medium without affecting the motion of one another. Therefore, the resultant displacement of each particle of the medium at any instant is equal to the vector sum of the displacements produced by the two waves separately. This principle is called 'principle of superposition'. It holds for all types of waves, provided the waves are not of very large amplitude. We can express the superposition principle in the form

$$y(x, t) = y_1(x, t) + y_2(x, t) + \ldots + y_n(x, t)$$

or
$$y(x, t) = \sum_{J=1}^{n} y_j(x, t)$$

Here, the y_j are the individual wave functions, and their sum, the wave function $y(x, t)$ describes the resultant behaviour of the medium as a function of position and time.

Interference :

Consider the superposition of two sinusoidal waves of same frequency at a point. Let us assume that the two waves are travelling in the same direction with same velocity. The equation of the two waves reaching at a point can be written as,

$$y_1 = A_1 \sin(kx - \omega t)$$

and
$$y_2 = A_2 \sin(kx - \omega t + \phi)$$

The resultant displacement of the point where the waves meet is

$$\begin{aligned}
y &= y_1 + y_2 \\
&= A_1 \sin(\alpha - \omega t) + A_2 \sin(kx - \omega t + \phi) \\
&= A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t)\cos\phi + A_2 \cos(kx - \omega t)\sin\phi \\
&= (A_1 + A_2\cos\phi)\sin(kx - \omega t) + A_2 \sin\phi\cos(kx - \omega t) \\
&= A\cos\theta\sin(kx - \omega t) + A\sin\theta\cos(kx - \omega t)
\end{aligned}$$

or
$$y = A\sin(kx - \omega t + \phi)$$

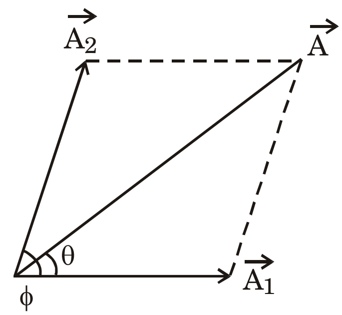
Here, $A_1 + A_2\cos\phi = A\cos\theta$

and $A_2 \sin\phi = A\sin\theta$

or
$$A^2 = (A_1 + A_2\cos\phi)^2 + (A_2\sin\phi)^2$$

or
$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi} \quad \ldots\ (1)$$

and $\tan\theta = \dfrac{A\sin\theta}{A\cos\theta} = \dfrac{A_2\sin\phi}{A_1 + A_2\cos\phi}$



The above result can be obtained by graphical method as well. Assume a vector $\vec{A}_1$ of length A_1 to represent the amplitude of first wave. Another vector $\vec{A}_2$ of length A_2, making an angle ϕ with $\vec{A}_1$ represents the amplitude of second wave. The resultant of $\vec{A}_1$ and $\vec{A}_2$ represent the amplitude of resulting function y. The angle θ represents the phase difference between the resulting function and the first wave.

Now we know that intensity of a wave is given by :

$$I = \frac{1}{2}\rho A^2\omega^2 v$$

i.e. $$I \propto A^2$$

So, if ρ, ω and v are same for both interfering waves, Eq. (i) can also be written as,

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi \qquad \text{...(ii)}$$

we see that the resulting amplitude A and intensity I depends on the phase difference ϕ between the interfering wave. Where $\cos\phi = +1$, $A = A_{max} = A_1 + A_2$

or $$I = I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$$

and the waves are said to be interfering constructively.

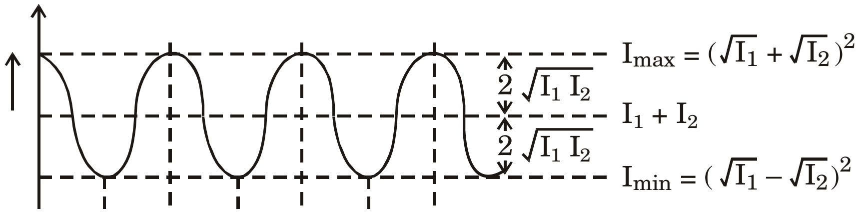
Similarly where $\cos\phi = -1$, $A = A_{min} = A_1 \sim A_2$

or $$I = I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$$

and the waves are said to be interfering destructively.

Note :

(1) All maxima are equally spaced (as path difference between two consecutive maxima is λ) and equally loud $\left[I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2\right]$. Same is also true for minima with $I_{min} = \left(\sqrt{I_1} \sim \sqrt{I_2}\right)^2$. Also interference maxima and minima are alternate as for maxima $\Delta x = 0, \lambda, 2\lambda$ etc., while for minima $\Delta x = (\lambda/2), (3\lambda/2)$, etc. This all is shown graphically.



Phase Diff.	0	π	2π	3π	4π	5π	6π	$\rightarrow$	ϕ
Phath Diff.	0	$\lambda/2$	λ	$3\lambda/2$	2λ	$5\lambda/2$	3λ	$\rightarrow$	Δx

(2) $$\frac{I_{max}}{I_{min}} = \frac{\left(\sqrt{I_1} + \sqrt{I_2}\right)^2}{\left(\sqrt{I_1} \sim \sqrt{I_2}\right)^2} \quad \frac{(A_1 + A_2)^2}{(A_1 \sim A_2)^2} \text{ with } \frac{I_1}{I_2} = \frac{A_1^2}{A_2^2}$$

So if I_1 and I_2 or A_1 and A_2 are given (I_{max}/I_{min}) can be calculated and vice-versa. From the above it is also clear than if $I_1 = I_2 = I_0$.

$$I_{max} = \left(\sqrt{I_0} + \sqrt{I_0}\right)^2 = 4I_0 \text{ and } I_{min} = \left(\sqrt{I_0} - \sqrt{I_0}\right)^2 = O$$

i.e., in maxima intensity will be 4-times that of a single wave (I_0) while intensity of minima is zero if the interfering waves are of equal intensities.

(3) In interference the intensity of maxima $\left(\sqrt{I_1} + \sqrt{I_2}\right)^2$ exceeds the sum of individual intensities $(I_1 + I_2)$ by an amount $2\sqrt{I_1 I_2}$ while of minima $\left(\sqrt{I_1} \sim \sqrt{I_2}\right)^2$ lacks $(I_1 + I_2)$ by the same amount $2\sqrt{I_1 I_2}$

Hence, we conclude that in interference energy is neither created nor destroyed but is redistributed.

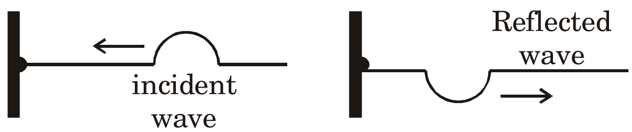
(4) Here we had assumed that the two waves from S_1 and S_2 start in the same phase. Hence, at P they have a constant phase difference $\phi = (2\pi/\lambda)\Delta x$, developed due to different paths traversed by them. Such waves are said to be '**Coherent**' and produce sustained interference effects. However, if there is an initial phase difference between the waves ϕ_0 then $\phi = \phi_0 + (2\pi/\lambda)\Delta x$ and if ϕ_0 is not constant and varies rapidly and randomly with time, at P sometimes constructive and sometimes destructive interference will take place so that $I_{av} = \frac{1}{2}(I_{max} + I_{min}) = (I_1 + I_2)$ and hence, no interference effect is observed. Such waves are called '**Incoherent**'.

So for observing interference effects waves must be coherent.

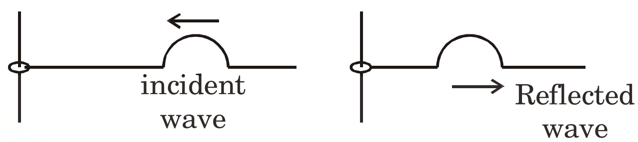
Reflection and Transmission of Waves

The nature of the reflected & transmitted wave depends on the nature of end point. There are three possibilities.

(a) **End point is fixed :** Waves on reflection from a fixed end undergoes a phase change of 180 .

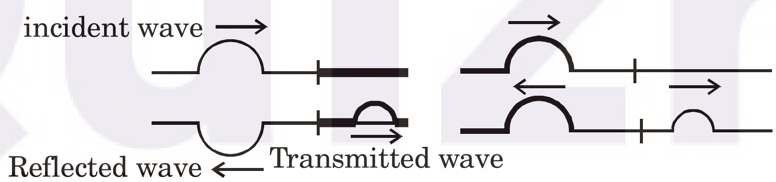


(b) **End point is free :** There is no phase change in waves on reflection.



(c) **End point is neither completely fixed nor completely free**

For example, consider a light string attached to a heavier string as shown in figure if a wave pulse is produced on the light string moving towards the junction, a part of the reflected and a part is transmitted on the heavier string. The reflected wave is inverted with to the original one (figure (a).



(a) (b)

On the other hand, if the wave is produced on the heavier string, which moves towards the junction, a part will be reflected and a transmitted, no inversion of wave shape will take place (as shown in figure (b).

So the rule is : if a wave enters a region where the wave velocity is smaller, the reflected wave is inverted. If it enters a region where the wave velocity is larger, the reflected wave is not inverted. The transmitted wave is never inverted.

Example 11

Finding the amplitude of reflected and transmitted displacement waves from a plane boundary at normal incidence, discuss the change in phase of reflected and transmitted waves if any.

Solution :

Suppose the incident wave of amplitude A_i and frequency ω is propagating along positive x-axis with velocity v_1, i.e.,

$$y_i = A_i \sin \omega [t - (x/v_1)] \quad ...(1)$$

Now as on reflection frequency does not change and for normal incidence, the reflected wave will move opposite to incident wave (i.e., along negative x-axis), but in same medium, so if A_r is its amplitude, it will be given by

$$y_r = A_r \sin \omega [t + (x/v_1)] \quad ...(2)$$

Transmitted wave will move in the direction of incident wave with same frequency in the other medium with speed v_2 and so if A_i is its amplitude it will be given by

$$y_t = A_t \sin \omega [t - (x/v_2)] \quad ...(3)$$

Now as wave is continuous so at the boundary x = 0, continuity of displacement requires

$$y_i + y_r = y_t \quad \text{for} \quad x = 0$$

Substituting Eqns. (1), (2) and (3) in the above with x = 0 and simplifying, we get

$$A_i + A_r = A_t \quad ...(4)$$

Also at boundary the slope of wave will be continuous, i.e.,

$$\frac{dy_i}{dx} + \frac{dy_r}{dx} = \frac{dy_t}{dx} \quad \text{for} \quad x = 0$$

which in the light of Eqns. (1), (2) and (3) gives

$$\frac{-A_i\omega}{v_1}\cos \omega t + \frac{A_r\omega}{v_1}\cos \omega t = \frac{-A_t\omega}{v_2}\cos \omega t$$

i.e.,
$$A_i - A_r = (v_1/v_2)A_t \quad ...(5)$$

Solving Eqns. (4) and (5) for A_r and A_t, we get

$$A_r = \frac{v_2 - v_1}{v_1 + v_2} \quad \text{and} \quad A_t = \frac{2v_2}{v_1 + v_2}A_i$$

These are the required results from these it is clear that in case of displacement waves :

(1) As A_t is always positive whatever be v_1 and v_2, the phase of transmitted wave always remains unchanged.

(2) As A_r will be positive only if $v_2 > v_1$, i.e., in case of reflection from a rare medium (or free end) there is no change in phase.

(3) As A_r will be negative if $v_2 < v_1$, i.e., in case of reflection from a denser medium (or rigid boundary or fixed end) there is a phase change of π.

Wave property	Reflection	Transmission (Refraction)
v	does not change	changes
f, T, ω	do not change	do not change
ω, k	do not change	change
A, I	change	change
ϕ	$\Delta\phi = 0$, from a rarer medium $\Delta\phi = \pi$, from a denser medium	does not change

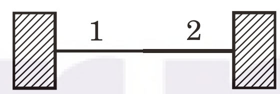
Example 12

Two strings 1 and 2 are taut between two fixed supports (as shown in figure) such that the tension in both strings is same. Mass per unit length of 2 is more than that of 1. Explain which string is denser for 1 transverse travelling wave.

Solution :

Speed of a transverse wave on a string

$$v = \sqrt{\frac{T}{\mu}} \quad \text{or} \quad v \propto \frac{1}{\sqrt{\mu}}$$



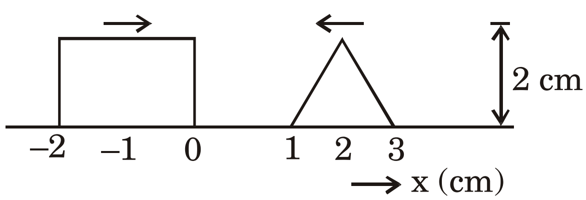
Now $\mu_2 > \mu_1$ (given)

$\therefore$ $v_2 < v_1$

i.e. medium 2 is denser and medium 1 is rarer.

Example 13

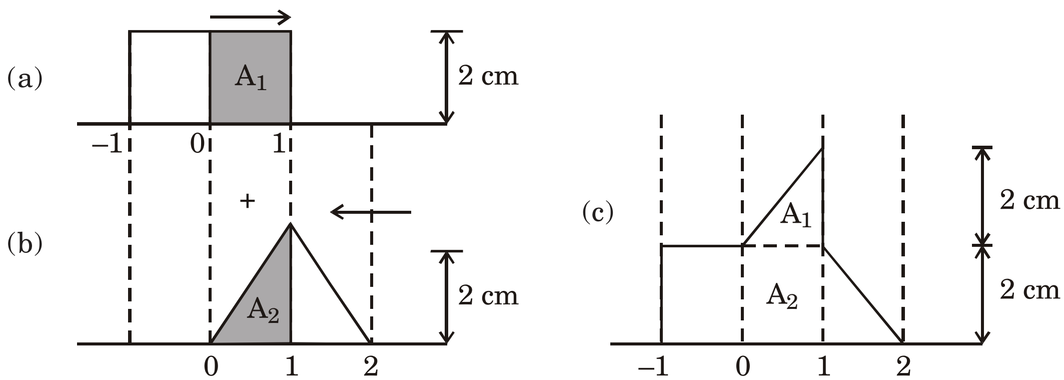
Figure shows a rectangular pulse and triangular pulse approaching each other. The pulse speed is 0.5 cm/s. Sketch the resultant pulse at t = 2 s.



Solution :

In 2 s each pulse will travel a distance of 1 cm.

The two pulses overlap between 0 and 1 cm as shown in figure. So, A_1 and A_2 can be added as shown in figure (c).



STANDING WAVES

A standing wave is formed when two identical waves travelling in the opposite directions along the same line interfere.

Consider two waves of the same frequency, speed and amplitude, which are travelling in opposite directions along a string. Two such waves may be represented by the equations.

$y_1 = A \sin(\omega t - kx)$

$y_2 = A \sin(kx + \omega t)$

Hence the resultant may be written as $y = y_1 + y_2 = A \sin(\omega t - kx) + A \sin(\omega t + kx)$

$y = 2A \sin kx \cos \omega t$

This is the equation of a standing wave.

This is the required result and from this it is clear that :

(1) As this equation satisfies the wave equation, $\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2}$ it represents a wave. However, as it is not of the form F (ax ± bt), the wave is not travelling and so is called standing or stationary wave.

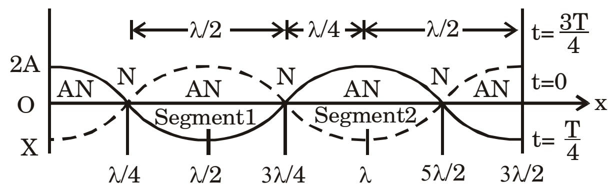
(2) The amplitude of the wave $A_s = 2A \cos kx$ is not constant but varies periodically with position (and not with time as in beats).

(3) The points for which amplitude is minimum are called nodes and for these

$\cos kx = 0,$ i.e., $kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

i.e., $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots$ $\left[\text{as } k = \frac{2\pi}{\lambda}\right]$

i.e., in a stationary wave nodes are equally spaced, and the spacing between two adjacent nodes is $(\lambda/2)$ with $A_{min} = 0$. Also for nodes, displacement $y = 0$ for all values of time (as A_s = 0), i.e., nodes are permanently at rest (through they are not physically clamped).



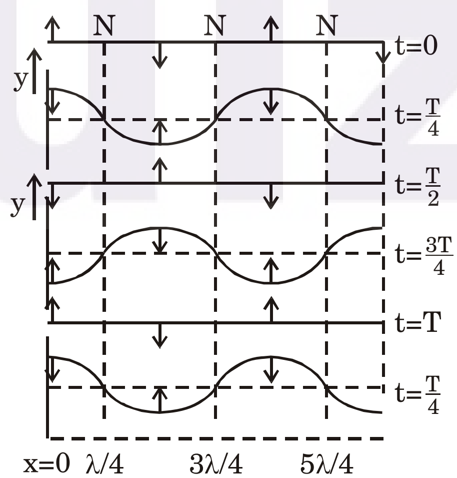
(4) The points for which amplitude is maximum are called antinodes and for these,

$\cos kx \pm 1$, i.e., $kx = 0, \pi, 2\pi, 3\pi, ...$

i.e, $$x = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, ... \qquad \left[\text{as } k = \frac{2\pi}{\lambda}\right]$$

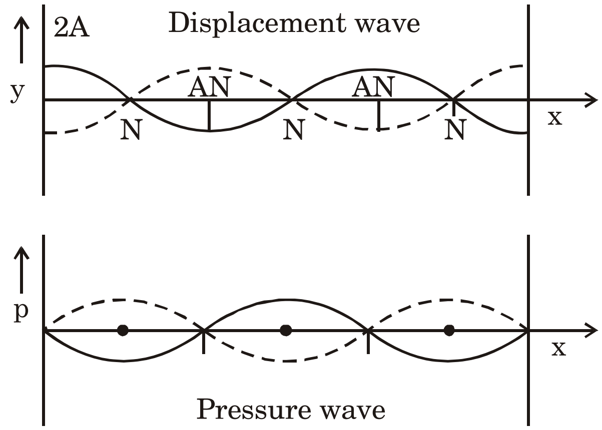
i.e., like nodes, antinodes are also equally spaced with spacing ($\lambda/2$) and $A_{max} = \pm 2A$. Furthermore, nodes and antinodes are alternate with spacing ($\lambda/4$).

(5) The nodes divide in the medium into segments (or loops). All the particles in a segment vibrate in same phase, but in opposite phase with the particles in the adjacent segment. Twice in one period all the particles pass through their mean position simultaneously with maximum velocity ($A_s\omega$), the direction of motion being reversed after each cycle.



(6) Since antinodes have always maximum displacement, their velocity is also maximum compared to other points and velocity at nodes is zero.

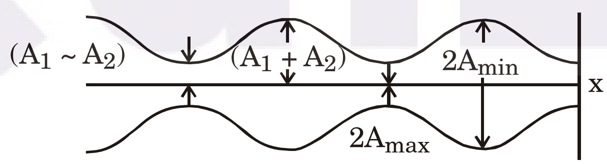
(7) Standing waves can be transverse or longitudinal, e.g., in strings (under tension) if reflected wave exists, the waves are transverse-stationary, while in organ pipes waves are longitudinal-stationary. In case of of longitudinal waves are pressure and displacement waves have a phase difference of ($\pi/2$) at nodes where displacement is min pressure will be max while at



antinodes where displacement is max pressure will be min, i.e. in case of longitudinal-stationary waves, nodes are points of max pressure (min displacement) while antinodes of minimum pressure (max displacement).

(8) As in stationary waves nodes are permanently at rest, so no energy can be transmitted across them. However, this energy oscillates between elastic potential energy and kinetic energy of the particles of the medium. When all the particles are at their extreme position KE is minimum while elastic PE is max and when all the particles (simultaneously) pass through their mean position KE will be maximum while elastic PE minimum. The total energy confined in a segment (elastic PE + KE), always remains the same.

(9) In standing wave if the amplitudes of component waves are not equal, then as $A_{min} \neq 0$ i.e., node will not be permanently at rest and so some energy will pass across the node and the wave will be partially standing.



In such situations we estimate the extent to which the resultant wave is standing by the term **standing wave ratio** defined as

$$SWR = \frac{A_{max}}{A_{min}} = \frac{(A_1 + A_2)}{(A_1 \sim A_2)}$$

$$[\text{as } A_{max} = A_1 + A_2 \text{ and } A_{min} = A_1 \sim A_2]$$

So that for a progressive wave SWR = (min) = 1 (as $A_2 = 0$) while for perfectly standing wave SWR = (max) = ∞ (as $A_1 = A_2$). The value of SWR for all other waves will lie between these limits (i.e., 1 and ∞).

Differences between a Travelling Wave and a Standing Wave

(1) In a travelling wave, the disturbance produced in a region propagates with a definite velocity but in a standing wave, it is confined to the region where it is produced.

(2) In a travelling wave, the motion of all the particles is similar in nature. In a standing wave, different particles move with different amplitudes.

(3) In a standing wave, the particles at nodes always remain at rest. In travelling waves, there is no particle, which always remains in rest.

(4) In a standing wave, all the particles cross their mean position together. In a travelling wave, there is no instant when all the particles are at the mean position together.

(5) In a standing wave, all the particles between two successive nodes reach their extreme positions together, thus moving in phase. In a travelling wave, the phases of nearby particles are always different.

(6) In a travelling wave, energy is transmitted from one region of space to other but in a standing wave, the energy of one region is always confined in that region.

Example 14

The vibrations of a string of length 60 cm fixed at both ends are represented by the equation $y = 4\sin\left[\frac{\pi x}{15}\right]\cos(96\pi t)$ where x and y are in cm and t in sec. (a) What is the maximum displacement at x = 5 cm ? (b) Where are the nodes located along the string ? (c) What is the velocity of the particle at x = 7.5 cm and t = 0.25 s ? (d) Write down the equations of component waves whose superposition gives the above wave.

Solution :

(a) For x = 5, $y = 4\sin(5\pi/15)\cos(96\pi t)$

or $$y = 2\sqrt{3}\cos(96\pi t)$$

So y will be max when

$\cos(96\pi t) = \max = 1$, i.e. $(y_{max})_{x=5} = 2\sqrt{3}$ cm

(b) At nodes amplitude of wave is zero,

i.e. $$4\sin\left[\frac{\pi x}{15}\right] = 0 \text{ or } \frac{\pi x}{15} = 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots$$

So $x = 0, 15, 30, 45, 60$ cm [as length of string = 60 cm]

(c) As $$y = 4\sin(\pi x/15)\cos(96\pi t)$$

$$\frac{dy}{dt} = -4\sin\left[\frac{\pi x}{15}\right]\sin(96\pi t)\times(96\pi)$$

So the velocity of the particle at x = 7.5 cm and t = 0.25 s,

$$v_{pa} = -\,384\pi \sin(7.5\pi/15)\sin(96\,\pi \quad 0.25)$$

i.e., $$v_{pa} = -\,384\pi \quad 1 \quad 0 = 0$$

(d) As 2 sin A cos B = sin (A + B) + sin (A – B)

So $$y = 4\sin\left[\frac{\pi x}{15}\right]\cos(96\pi t)$$

$$= 2\left[\sin\left(\frac{\pi x}{15} + 96\pi t\right) + \sin\left(\frac{\pi x}{15} - 96\pi t\right)\right]$$

or $$y = 2\sin\left[96\pi t + \frac{\pi x}{15}\right] - 2\left[96\pi t - \frac{\pi x}{15}\right]$$ [as sin (– θ) = – sin θ]

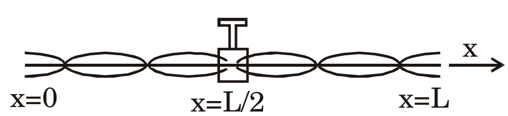
i.e., $$y = y_1 + y_2 \text{ with } y_1 = 2\sin\left[96\pi t - \frac{\pi x}{15}\right]$$

Example 15

A metallic rod of length 1 m is rigidly clamped at its mid point. Longitudinal stationary waves are set up in the rod in such a way that there are two nodes on either side of the mid-point. The amplitude of an antinode is 2 $\quad 10^{-6}$ m. Write the equation of motion at a point 2 cm from the mid-point and those of constituent waves in the rod. (Y = 2 $\quad 10^{11}$ N/m^2 and ρ = 8 $\quad 10^3$ kg/m^3)

Solution :

In rods, like strings, clamped point is a node while the free antinode; so the situation in accordance with given condition is as shown in Fig.



Now as distance between two consecutive nodes is λ/2 while between a node and an antinode is λ/4

$$4\times\left[\frac{\lambda}{2}\right] + 2\left[\frac{\lambda}{4}\right] = L \text{ i.e., } \lambda = \frac{2\times 1}{5} = 0.4 \text{ m}\ldots \qquad \ldots(1)$$

Also as $$Y = 2 \quad 10^{11}\ \text{N/m}^2 \text{ and } \rho = 8 \quad 10^3\ \text{kg/m}^3$$

$$v = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{2\times 10^{11}}{8\times 10^3}} = 5000 \text{ m/s}$$

So from $\qquad v = f\lambda,\ f = (v/\lambda) = (5000/0.4) = 12500\ \text{Hz}$ ...(2)

Now if incident and reflected waves along the rod are $y_1 = A \sin(\omega t - kx)$ and $y_2 = A \sin(\omega t + kx + \phi)$, resultant wave will be :

$$y_1 = y_1 + y_2 = A\,[\sin(\omega t - kx) + \sin(\omega t + kx + \phi)]$$

But as $\qquad \sin C + \sin D = 2 \sin \dfrac{(C+D)}{2} \cos \dfrac{(C \sim D)}{2}$

$$y = 2A \cos\left[kx + \frac{\phi}{2}\right] \sin\left[\omega t + \frac{\phi}{2}\right]$$

Now as free end of the rod is an antinode, i.e., amplitude is max at $x = 0$, so that

$$\cos\left[k \times 0 + \frac{\phi}{2}\right] = \max = 1, \quad \text{i.e.} \quad \phi = 0$$

and $\qquad A_{max} = 2A = 2 \quad 10^{-6}$ m (given)

So $\qquad y = 2 \quad 10^{-6} \cos kx \sin \omega t$

or $\qquad y = 2 \quad 10^{-6} \cos\left[\dfrac{2\pi}{\lambda}\right] \sin 2\pi ft \qquad \left[\text{as } k = \dfrac{2\pi}{\lambda} \text{ and } \omega = 2\pi f\right]$

Above equation in the light of Eqns. (1) and (2) reduces to

$$y = 2 \quad 10^{-6} \cos 5\pi x \sin 25000\pi t \qquad ...(3)$$

Now as for a point 2 cm from the mid-point $x = (0.50 \pm 0.02)$,

$$y = 2 \quad 10^{-6} \cos 5\pi\ (0.50 \pm 0.02) \sin 25000\pi t$$

This is the required result.

Now as $\qquad 2 \cos A \sin B = \sin(A + B) - \sin(A - B)$

the resultant wave $y = 2 \quad 10^{-6} \cos(5\pi x) \sin(25000\pi t)$ can be written as

$$y = 10^{-6}\,[\sin(5\pi x + 25000\pi t) - \sin(5\pi x - 25000\pi t)]$$

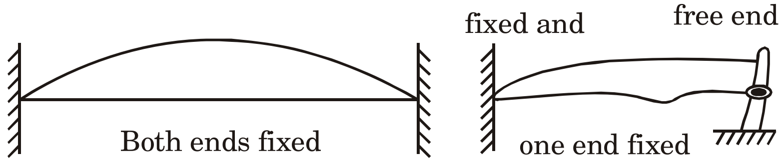
i.e. $\quad y = 10^{-6} \sin[25000\pi t + 5\pi x] + 10^{-6} \sin[25000\pi t - 5\pi x] \qquad [\text{as } \sin(-\theta) = -\sin\theta]$

or $\quad y = y_1 + y_2$ with $y_1 = 10^{-6} \sin[25000\pi t + 5\pi x]$

and $\quad y_2 = 10^{-6} \sin[25000\pi t - 5\pi x]$

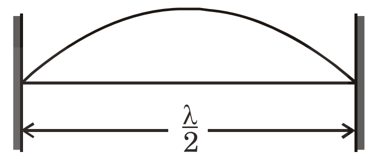
Stationary waves in Strings

When a string capable of vibrating (under tension) is set into vibration, transverse harmonic waves will propagate along it. It gets reflected at the other fixed end. The incident and the reflected waves interfere to produce a stationary transverse wave in which the ends are always nodes, if both ends are fixed, whereas if there is a free end and one fixed end, then free end will be an antinode as here displacement will be maximum.



(a) **Fundamental Mode :** In the simplest form, the string vibrates in one loop in which the ends are the nodes and the centre is the antinode. This mode of vibration is known as the fundamental mode and frequency of vibration is known as the fundamental frequency or first harmonic.

Since the distance between consecutive nodes is



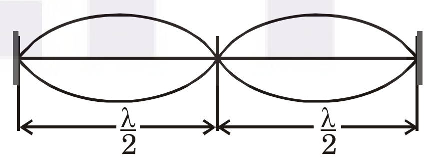
$$L = \frac{\lambda_1}{2} \quad \therefore \lambda_1 = 2L$$

If f_1 is the fundamental frequency of vibration, then the velocity of transverse waves is given as, $v = \lambda_1 f_1$

or $$f_1 = \frac{v}{2L} \quad \text{...(i)}$$

(b) The same string under the same conditions may also vibrate in two loops, such that the centre is also the node

$\therefore$ $$L = 2\lambda_2/2 \quad \therefore \lambda_2 = L$$



If f_2 is frequency of vibrations

$\therefore$ $$f_2 = \frac{v}{\lambda_2} = \frac{v}{L}$$

$\therefore$ $$f_2 = \frac{v}{L} \quad \text{...(ii)}$$

The frequency f_2 is known as **second harmonic or first overcome**.

(c) The same string under the same conditions may also vibrate in three segments.

$\therefore$ $$L = \frac{3\lambda_3}{2}$$

$\therefore$ $$\lambda_3 = \frac{2}{3}L$$

If f_3 is the frequency in this mode of vibration, then,

$$f_3 = \frac{3v}{2L} \qquad \text{...(iii)}$$

The frequency f_3 is known as **third harmonic** or **second overtone**.

Thus a stretched string vibrates with frequencies, which are integral multiples of the fundamental frequencies. These frequencies are known as harmonics.

The velocity of transverse wave in stretched string is given as $v = \sqrt{\frac{T}{\mu}}$. Where T = tension in the string.

μ = linear density or mass per unit length of string. If the string fixed at two ends, vibrates in its fundamental mode, then v = 2Lf

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

Now as in case of waves $v = f\lambda$, i.e., $f = v/\lambda$ and for waves along a string $v = \sqrt{(T/m)}$, the possible frequencies of vibration of the string in the light of Eqn. (1) will be :

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{m}} \text{ with } n = 1, 2, 3, ...$$

i.e., in case of vibrations of strings, number of natural frequencies are possible and if we take

$$f = \frac{1}{2L}\sqrt{\frac{T}{m}},\ f_n = nf$$

i.e., $\qquad f_1 = f, \quad f_2 = 2f, \quad f_3 = 3f, ...$

So in case of vibrations of strings higher frequencies are integral multiples of f, i.e., forms a harmonic series.

The frequency will be minimum when n = min = 1, i.e.

$$f_{min} = f_1 = f = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

Harmonic	Mode	Antinodes or loops	Nodes	Frequency	Wavelength
First	Fundamental	1	2	f	(2L/1)
Second	I overtone	2	3	2f	(2L/2)
Third	II overtone	3	4	3f	(2L/3)
...	...	...	...	...	...
nth	(n – 1)th overtone	n	(n + 1)	nf	(2L/n)

Regarding frequency of a vibrating string it is worth noting that :

(1) As a string has many natural frequencies (all integral multiples of fundamental frequency), so when it is excited with a tuning fork (or a vibrating body), the string will be in resonance with the given body if any one of its natural frequencies coincides with that of the body.

(2) As for a sting $f = (1/2L)\left(\sqrt{T/m}\right)$

(a) $f \propto (1/L)$ if T and m are constant

(b) $f \propto \sqrt{T}$ if L and m are constant

(c) $f \propto \left(1/\sqrt{m}\right)$ if T and L are constant

These laws of vibration of string are known as Mersenne's laws of vibration of string and according to these the frequency of a string can be changed by changing its length, tension or mass per unit length.

(3) If M is the mass of a string of length L, m = (M/L)

So $$f = \frac{1}{2L}\sqrt{\frac{T}{m}} = \frac{1}{2L}\sqrt{\frac{T}{(M/L)}} = \frac{1}{2}\sqrt{\frac{T}{ML}} \quad ...(4)$$

Also if the radius of string is r and its density ρ, $m = \pi r^2 \rho$.

So $$f = \frac{1}{2L}\sqrt{\frac{T}{\pi r^2 \rho}} \quad \text{i.e.,} \quad f = \frac{1}{2Lr}\sqrt{\frac{T}{\pi\rho}} \quad ...(5)$$

(4) If the string is vibrating in nth harmonic, its frequency will be nf, the number of loops in the string or antinodes will be n, while total number of nodes (including two at the ends) will be (n + 1), e.g., in case of 3rd harmonic of a vibrating string, frequency of vibration will be 3f, antinodes will be 3 while total nodes = 3 + 1 = 4 and λ = (2L/3).

(5) In case of vibrations of composite string (i.e., string made up by joining two strings of different lengths, cross-sections and densities) having same tension through out, the joint is a node while lowest common fundamental frequency of the string will be

$$f_C = n_1 f_1 = n_2 f_2$$

Here higher harmonics will be integral multiples of common frequency f_C.

Example 16

The fundamental frequency of a sonometer wire increases by 6 Hz if its tension is increased by 44% keeping the length constant. Find the change in the fundamental frequency of the sonometer wire, when the length of the wire is increased by 20% keeping the original tension in the wire.

Solution :

If case of vibration of a string, fundamental frequency is given by

$$f = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

So if length of given wire is kept constant,

$$(f'/f) = (T'/T)^{1/2}$$

and as here $f' = f + 6$ and $T' = T + 0.44\ T = 1.44\ T$,

$$\frac{(f+6)}{f} = \sqrt{\frac{1.44T}{T}} \quad \text{i.e.} \quad f = 30 \text{ Hz}$$

Now it keeping the original tension (T), the length of given wire is changed.

$$\frac{f''}{f} = \frac{l}{l''} = \frac{1}{1.20} \qquad [\text{as } l'' = l + 0.20\,l = 1.20\,l]$$

so $$f'' = \frac{30}{1.2} = 25 \text{ Hz}$$

and hence $$\Delta f = f'' - f = 25 - 30 = -5 \text{ Hz}$$

i.e, fundamental frequency will decrease by 5 Hz. **Ans.**

Example 17

Two strings A (length L_1) and B (length L_2) are made of steel and are kept under the same tension. If A has a radius twice that of B, what should be value of L_2/L_1 for them to have the same fundamental frequencies? What should be the value of L_2/L_1 if the first overtone of the former should equal the third harmonic of the latter?

Solution :

	String A	**String B**	
Tension	T	T	(same tension)
Linear mass density	$\pi r^2 \rho$	$\pi\left(\frac{r}{2}\right)^2 \rho$	ρ = density of steel
Fundamental frequency	f	f	(same)
Lengths	L_1	L_2	

For the string A, $f = \dfrac{1}{2L_1}\sqrt{\dfrac{T}{\pi r^2 \rho}}$

For the string B, $f = \dfrac{1}{2L_2}\sqrt{\dfrac{T}{\pi\left(\frac{r}{2}\right)^2 \rho}} = \dfrac{1}{2L_2}\sqrt{\dfrac{4T}{\pi r^2 \rho}}$

Since both have the same fundamental frequency.

$$\frac{1}{2L_1}\sqrt{\frac{T}{\pi r^2 \rho}} = \frac{1}{2L_2}\sqrt{\frac{4T}{\pi r^2 \rho}}$$

$$\frac{L_2}{L_1} = 2$$

If f be the fundamental frequency of string A with length L_1

$$f = \frac{1}{2L_1}\sqrt{\frac{T}{\pi r^2 \rho}}$$

The first overtone of A = second harmonic = 2n

$$\therefore \quad f' = \frac{1}{2L_2}\sqrt{\frac{4T}{\pi r^2 \rho}}$$

The third harmonic of this is 3n´

$$3f' = \frac{3}{2L_2}\sqrt{\frac{4T}{\pi r^2 \rho}}$$

Since 2f = 3f

$$\frac{1}{L_1}\sqrt{\frac{T}{\pi r^2 \rho}} = \frac{3}{2L_2}\sqrt{\frac{4T}{\pi r^2 \rho}} = \frac{3}{L_2}\sqrt{\frac{T}{\pi r 2\rho}}$$

$$\frac{L_2}{L_1} = 3$$

Example 18

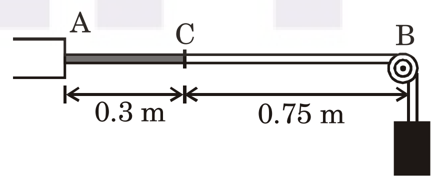
Two metallic strings A and B of different materials are connected in series forming a joint. The strings have similar cross-sectional area. The length of A is l_A = 0.3 m and that of B is l_B = 0.75 m. One end of the combined string is tied with a support rigidly and the other end is loaded with a block of mass m passing over a frictionless pulley. Transverse waves are set up in the combined string using an external source of variable frequency. Calculate (i) the lowest frequency for which standing waves are observed such that the joint is a node and (ii) the total number of antinodes at this frequency. The densities of A and B are 6.3 10^{-3} kg m^{-3} and 2.8 10^{-3} kg m^{-3} respectively.

Solution :

The frequency of transverse waves in a stretched string is given by $n = \frac{p}{2l}\sqrt{\frac{T}{m}}$

where the string is vibrating with p loops, T = tension in the string and m = mass per unit length of the string.

As the frequency of the wave in both string (A vibrating with p loops B vibrating with q loops) must be the same, so



$$\frac{p}{2l_A}\sqrt{\frac{T}{m_A}} = \frac{q}{2l_B}\sqrt{\frac{T}{m_B}}$$

$$\therefore \quad \frac{p}{q} = \frac{l_A}{l_B}\sqrt{\frac{m_A}{m_B}} = \frac{l_A}{l_B}\sqrt{\frac{\rho A}{\rho B}}$$

$$= \frac{0.3}{0.75}\sqrt{\frac{6.3}{2.8}} = \frac{3}{5}$$

So, p = 3, q = 5

No. of antinode = p + q = 3 + 5 = 8 **Ans.**

Note : (i) The frequency cannot be calculated as tension is not provided.

(ii) Total no. of nodes including the two at the ends will be = 9

Example 19

An aluminium wire of cross-sectional area 1 10^{-6} m^2 is joined to a copper wire of the same cross-section. This compound wire is stretched on a sonometer, pulled by a weight of 10 kg. The total length of the compound wire between the two bridges is 1.5 m of which the aluminium wire is 0.6 m and the rest is the copper wire. Transverse vibrations are set up in the wire by using an external force of variable frequency. Find the lowest frequency of excitation for which standing waves are formed, such that the joint in the wire is a node. What is the total number of nodes observed at this frequency excluding the two at the ends of the wire ? The density of aluminium is 2.6 10^3 kg/m^3 and that of copper 1.0401 10^4 kg/m^3.

Solution :

As the total length of the wire is 1.5 m and out of which L_A = 0.6 m, so the length of the copper wire L_C = 1.5 – 0.6 = 0.9 m. The tension in the whole wire is same (= Mg = 10 g N) and as fundamental frequency of vibration of string is given by

$$f = \frac{1}{2L}\sqrt{\frac{T}{m}} = \frac{1}{2L}\sqrt{\frac{T}{\rho A}} \qquad [\text{as } m = \rho A]$$

$$\text{So} \quad f_A = \frac{1}{2L_A}\sqrt{\frac{T}{\rho_A A}} \quad \text{and} \quad f_C = \frac{1}{2L_C}\sqrt{\frac{T}{\rho_C A}} \qquad ...(1)$$

Now as in case of composite wire, the whole wire will vibrate with fundamental frequency

$$f = n_A f_A = n_C f_C \qquad ...(2)$$

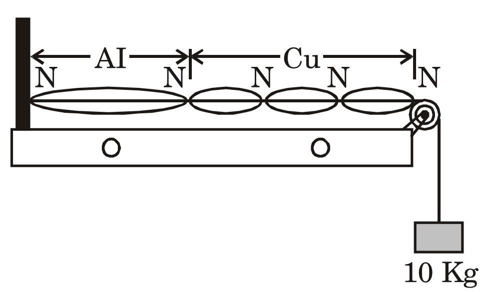
Substituting the values of f_A and f_C from Eqn. (1) in (2),

$$\frac{n_A}{2 \times 0.6}\sqrt{\frac{T}{A \times 2.6 \times 10^3}} = \frac{n_C}{2 \times 0.9}\sqrt{\frac{T}{A \times 1.0401 \times 10^4}}$$

$$\text{i.e.,} \quad \frac{n_A}{n_C} = \frac{2}{3}\sqrt{\frac{2.6}{10.4}} = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

So that for fundamental frequency of composite string, n_A = 1 and n_C = 3, i.e., aluminium string will vibrate in first harmonic (one loop), i.e., fundamental mode while copper string in third harmonic (3 loops), i.e., II overtone as shown

$\therefore$ $f = f_A = 3f_C$



This is turn implies that total number of nodes in the string will be 5 and so number of nodes excluding the nodes at the ends = 5 – 2 = 3,

$$\text{and} \quad f = f_A = \frac{1}{2 \times 0.6}\sqrt{\frac{10 \times 9.8}{10^{-6} \times 2.6 \times 10^3}} \simeq 161.8 \text{ Hz} \left(= 3f_C\right)$$

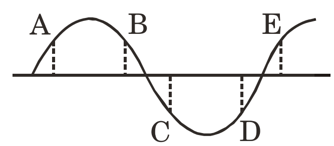
Example 20

A string 120 cm in length sustains a standing wave, with the points of string at which the displacement amplitude is equal to $\sqrt{2}$ mm being separated by 15.0 cm. Find the maximum displacement amplitude. Also find the harmonic corresponding to this wave.

Solution :

From Fig. points A, B, C, D and E are having equal displacement amplitude.

Further, $x_E - x_A = \lambda = 4 \quad 15 = 60$ cm



As $\lambda = \dfrac{2l}{n} = \dfrac{2 \times 120}{n} = 60$

$\therefore \quad n = \dfrac{2 \times 120}{60} = 4$

So, it corresponds to 4th harmonic.

Also, distance of node from A = 7.5 cm as distance between B and C = 15 cm and node is between them. Taking node at origin, the amplitude of stationary wave can be written as,

$$a = A \sin kx$$

Here $a = \sqrt{2}$ mm; $k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{60}$ and $x = 7.5$ cm

$$\therefore \quad \sqrt{2} = A \sin\left(\frac{2\pi}{60} \times 7.5\right) = A \sin\frac{\pi}{4}$$

Hence, $A = 2$mm

SOUND

Sound waves are mechanical waves. They require a medium for their propagation i.e. they cannot propagate in vaccum.

Sound is produced in a material by a vibrating source. Sound waves constitute alternate compression and rarefaction pulses travelling in the medium. The compression travels in the medium at a speed, which depends on the elastic and inertial properties of the medium.

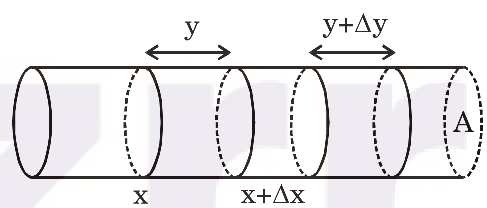
The description in terms of pressure wave is more appropriate than the description in terms of the displacement wave as far as sound properties are concerned.

Sound as Pressure Wave

A longitudinal wave in a fluid is described either in terms of the longitudinal displacements suffered by the particles of the medium or in terms of the excess pressure generated due to the compression or rarefaction.

Consider a wave going in the x-direction in a fluid. Suppose that at a time t, the particle at the undisturbed position x suffers a displacement y in the x-direction.

$$y = y_0 \sin\omega\left(t - \frac{x}{v}\right) \quad \text{...(i)}$$



A is cross-sectional area.

Increase in volume of this element at time t is

$$\Delta V = A\ dy$$

$$= Ay_0\left(\frac{-\omega}{v}\right)\cos\omega\left(t - \frac{x}{v}\right)\Delta x$$

where Δy has been obtained by differentiating equation (i) with respect to t.

$$\Rightarrow \quad \text{volume strain is } \frac{\Delta V}{V} = -\frac{Ay_0\omega\cos\omega\left(t - \frac{x}{v}\right)\Delta x}{vA\Delta x} = \frac{y_0\omega}{v}\cos\omega\left(t - \frac{x}{v}\right) = \frac{\delta y}{\delta x}$$

$$\therefore \quad \text{volume strain} = \frac{\delta y}{\delta x}$$

The corresponding stress i.e., the excess pressure developed in the element at x, at time t is

$p = B\left(\frac{-\Delta V}{V}\right)$ where B is the bulk modulus of the material.

$$\Rightarrow \quad p = \frac{By_0\omega}{v}\cos\omega\left(t - \frac{x}{v}\right) \qquad \text{...(ii)}$$

$$\therefore \quad P = -\frac{B\delta y}{\delta x}$$

Comparing equations (i) and (ii), the relation between the pressure amplitude P_0 and the displacement amplitude s_0 is

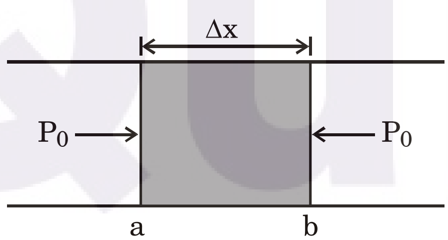
$$p_0 = \frac{B\omega}{v} y_0 = Bky_0 \Rightarrow y_0 = \frac{p_0\lambda}{2\pi B} \text{ where k is a wave number.}$$

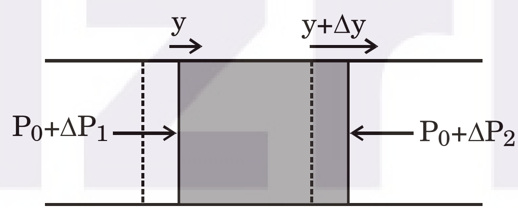
As observed from equations (i) and (ii), pressure wave is 'cos θ' type, if displacement is described as 'sin θ' type.

Thus, the pressure-maxima occur where where the displacement is zero and displacement maxima occur where the pressure is at its normal level.

Speed of a Longitudinal Wave

First we calculate the speed at which a longitudinal pulse propagates through a fluid. We will apply Newton's second law to the motion of an element of the fluid and from this we derive the wave equation.





Consider a fluid element 'ab' confined to a tube of cross sectional area S as shown in figure. The element has a thickness Δx. We assume that the equilibrium pressure of the fluid is P_0. Because of the disturbance, the section 'a' of the element moves a distance y from its mean position and section 'b' moves a distance y + Δy to a new position b´. The pressure on the left side of the element becomes $P_0 + \Delta P_1$ and on the right side it becomes $P_0 + \Delta P_2$. If ρ is the equilibrium density, the mass of the element is $\rho S \Delta x$. (When the element moves its mass does not change, even though its volume and density do change).

The net force acting on the element is,

$$F = (\Delta P_1 - \Delta P_2)\, S$$

and its acceleration is
$$a = \frac{\partial^2 y}{\partial t^2}$$

Thus, Newton's second law applied to the motion of the element is

$$(\Delta P_1 - \Delta P_2)\, S = \rho S \Delta x \frac{\partial^2 y}{\partial t^2}$$

Next we divide both sides by Δx and note that in the limit as $\Delta x \to 0$ we have $(\Delta P_1 - \Delta P_2) / \Delta x$

$\to \partial P/\partial x$, Eq. (i) then takes the form $-\dfrac{\partial P}{\partial x} = \rho \dfrac{\partial^2 y}{\partial t^2}$ ...(ii)

The excess pressure ΔP may be written as $\Delta P = -B\dfrac{\partial y}{\partial x}$

When this is used in Eq. (ii), we obtain the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{B}.\frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial t^2} = \frac{B}{\rho}\frac{\partial^2 y}{\partial x^2}$$

Comparing this equation with the wave equation

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

We have $v = \sqrt{\dfrac{B}{\rho}}$ (speed of longitudinal wave in a fluid)

This is the speed of longitudinal waves within a gas or a liquid.

When a longitudinal wave propagates in a solid rod or bar, the rod expands sideways slightly when it is compressed longitudinally and the speed of a longitudinal wave in a rod is given by

$$v = \sqrt{\frac{Y}{\rho}} \quad \text{(speed of a longitudinal wave in a solid rod)}$$

Velocity of Sound In An Ideal Gas

The motion of sound wave in air is adiabatic. In the case of an ideal gas, the relation between pressure P and volume V during an adiabatic process is given by

$$PV^{\gamma} = \text{constant}$$

Where γ is the ratio of the heat capacity at constant pressure to that at constant volume.

After differentiating, we get

$$V^{\gamma}\frac{dP}{dV} + \gamma P V^{\gamma-1} = 0$$

Since $B = -\frac{vdP}{dV} = \gamma P$

using the gas equation $\frac{P}{\rho} = \frac{RT}{M}$ where M is the molar mass.

Thus $v = \sqrt{\frac{\gamma RT}{M}}$ (T = temperature is Kelvin).

Note :

(i) **Effect of temperature :** If the specific volume of gas is v. The velocity of sound = $\sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$

If c_1 and c_2 be the velocities of sound in a gas at temperatures $t_1^\circ C$ and $t_2^\circ C$ and P_1 and P_2 the respective pressures and V_1 and V_2 the specific volumes at these temperatures, ratio of the two velocities of sound is

$\frac{v_1}{v_2} = \sqrt{\frac{P_1 V_1}{P_2 V_2}} = \sqrt{\frac{RT_1}{RT_2}}$ where T and T_2 are the absolute temperatures.

Hence, $v \propto \sqrt{T}$

$\therefore \quad \frac{v_1}{v_2} = \sqrt{\frac{273 + t_1}{273 + t_2}}$

If v_t and v_0 are the velocities at t C and 0 C, then $\frac{v_t}{v_o} = \sqrt{\frac{273 + t}{273}}$

$$\Rightarrow \quad v_t = v_0 \left(1 + \frac{t}{273}\right)^{1/2}$$

when t is small

$$\frac{v_t}{v_0} \approx \left(1 + \frac{t}{546}\right)$$

$$\Rightarrow \quad v_t = v_0 \left(1 + \frac{t}{546}\right)$$

Put v_0 = 332 m/s at 0 C

v_t = (332 + 0.61 t) m/s

This implies that for small temperature variations at 0 C, velocity of sound changes by 0.61 m/s when temperature changes by 1 C.

(ii) **Effect of pressure :** In a gas; $v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$ change in pressure has no effect on velocity of sound in a gas, so long as temperature is constant because; $\frac{P}{\rho}$ = constant; as long as temperature is constant.

(iii) **Effect of relative humidity :** When humidity increases, there is an increase in the relative number of water molecules and hence a decrease in molar mass, and the speed of sound increases.

(iv) The speed of sound in air is not affected by amplitudes, frequency, phase, boundness, pitch of quality.

Energy, Power and Intensity of Sound

If a sound wave given by y = A sin (ωt − kx) is propagating through a medium, the particle velocity will be $v_{Pa} = \frac{dy}{dt} = A\omega\cos(\omega t - kx)$

So if ρ is the density of the medium, kinetic energy of the wave per unit volume will be

$$= \frac{1}{2}\rho\left[\frac{dy}{dt}\right]^2 = \frac{1}{2}\rho A^2\omega^2\cos^2(\omega t - kx)$$

and its maximum value will be equal to energy per unit volume [as $(KE)_{max} = (PE)_{max} = E$], i.e., energy density U.So.

$$U = \frac{1}{2}\rho A^2\omega^2 S\Delta x \qquad ...(1)$$

So the energy associated with a volume S Δx will be $\Delta E = U\,\Delta V = \frac{1}{2}\rho A^2\omega^2 S\,\Delta x$

So, **power** (rate of transmission of energy) will be

$$P = \frac{\Delta E}{\Delta t} = \frac{1}{2}\rho v\omega^2 A^2 S \qquad \left[\text{as } \frac{\Delta x}{\Delta t} = V\right] \qquad ...(3)$$

Now as **Intensity** is defined as average energy transmitted per unit normal area per sec., i.e., power per unit area, so

$$I = \frac{\Delta E}{S\,\Delta t} = \frac{P}{S} = \frac{1}{2}\rho v\omega^2 A^2 \qquad ...(4)$$

Further as in case of sound wave displacement amplitude is related to pressure amplitude through the relation $p_0 = \rho v A\omega$, so

$$I = \frac{1}{2}\rho v\omega^2\left[\frac{p_0}{\rho v\omega}\right]^2 = \frac{1}{2}\frac{p_0^2}{\rho v} \qquad ...(5)$$

Eqns. (4) and (5) give intensity of sound in terms of displacement and pressure amplitude respectively and according to these for a given source and medium

$$I \propto A^2 \text{ (or } p_0^2\text{)} \qquad ...(6)$$

Note : In case of vibrating string, as ρS will represent mass per unit length m, so from Eqn. (3) the average rate of transport of energy, i.e., power transmitted by a vibrating string will be

$$P = \frac{1}{2}mv\omega^2A^2 \quad \text{with} \quad m = \frac{\text{mass}}{\text{length}} \qquad ...(7)$$

The SI unit of intensity is W/m^2. However; as human ear responds to sound intensities over a wide range, i.e., from 10^{-12} W/m^2 to 1 W/m^2, so instead of specifying intensity of sound in W/m^2, we use a logarithmic scale of intensity called the sound level defined as

$$SL = 10\log\left[\frac{I}{I_0}\right] \qquad ...(8)$$

where I_0 is the threshold of human ear, i.e., 10^{-12} W/m^2. The sound level defined in this way is expressed in decibel (dB). A sound of intensity I_0 has an SL = 10 log (I_0/I_0) = 0dB while sound at the upper range of human hearing called threshold of pain has a intensity of 1 W/m^2 or a SL = 10 log $(1/10^{-12})$ = 120 dB.

We also use dB as a relative measure to compare different sounds with one another, rather than with reference intensity; as for two intensities I_1 and I_2.

$$SL_1 - SL_2 = 10\log\frac{I_1}{I_0} - 10\log\frac{I_2}{I_0}$$

or

$$SL_1 - SL_2 = 10\log\frac{I_1}{I_2} \qquad ...(9)$$

e.g., two sounds whose intensity ratio is 2 differ in SL by 10 log 2 = 3 dB. Here it must be kept in mind that ratio of two intensities corresponds to difference in their sound level (and not ratio).

Note : While solving problems related to intensity of sound along with the above, also remember that :

(i) As intensity, $$I = \frac{\Delta E}{S\,\Delta t} \text{ while } U = \frac{\Delta E}{\Delta V}$$

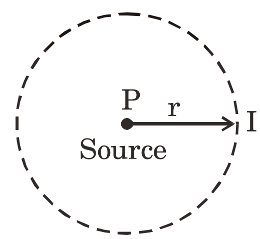
So $$\frac{I}{U} = \frac{\Delta E}{S\,\Delta t} \times \frac{S\,\Delta L}{\Delta E} = v \left[\text{as } \Delta V = S\,\Delta L \text{ and } \frac{\Delta L}{\Delta t} = v\right]$$

or intensity I (energy flux) = U v

$$= \text{Energy density} \quad \text{velocity} \qquad \text{...(10)}$$

(ii) With increase in distance from the source the total energy or power transmitted remains the same but intensity decreases. For an isotropic point source of power P, intensity I at a distance r from it will be

$$I = \frac{P}{S} = \frac{P}{4\pi r^2} \left[\text{as } S = 4\pi r^2\right] \qquad \text{...(11)}$$



Now as for a given medium and source,

$$I \propto A^2 \qquad \text{...(12)}$$

So from Eqns. (11) and (12)

$A^2 \propto (1/r^2)$, i.e., for spherical waves

$I \propto (1/r^2)$ and $A \propto (1/r)$

(iii) In case of electromagnetic waves (e.g. light or radio waves),

$$I = \frac{1}{\mu_0}(EB) \text{ with } \frac{E}{B} = c \text{ and } c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Example 21

(a) The power of sound from the speaker of a radio is 20 mW. By turning the knob of volume control the power of sound is increased to 400 mW. What is the power increase in dB as compared to original power ?

(b) How much more intense is an 80 dB sound than a 20 dB whisper ?

Solution :

(a) As intensity is power per unit area, for a given source $P \propto I$, so

$SL_2 - SL_1 = 10 \log (I_2/I_1)$

i.e. $$\Delta SL = 10 \log \frac{P_2}{P_1} = 10 \log \frac{400}{20}$$

i.e. $$\Delta SL = 10[\log 20) \simeq 13 \text{ dB}$$

(b) By definiton of sound level,

$$SL_2 - SL_1 = 10 \log (I_2/I_1)$$

So $$80 - 20 = 10 \log (I_2/I_1)$$

or $$6 = \log (I_2/I_1), \quad \text{i.e.,} \quad (I_2/I_1) = 10^6$$

Example 22

An observer is at a distance of one metre from a point of light source whose power output is 1 kW. Calculate the magnitude of electric and magnetic fields assuming that the source is monochromatic, it radiates uniformly in all directions and that at the point of observation it behaves like a travelling plane wave. Given that $(\mu_0/4\pi) = 10^{-7}$ H/m and c = 3 $\quad 10^8$ m/s.

Solution :

By definition of intensity,

$$I = \frac{P}{S} = \frac{P}{4\pi r^2} = \frac{10^3}{4\pi \times 1^2} = \frac{10^3}{4\pi} \text{ W/m}^2$$

Now in case of electromagnetic waves, as

$$I = \frac{1}{\mu_0} EB \qquad \text{and} \qquad \frac{E}{B} = c$$

so $$I = \frac{1}{\mu_0} E \times \frac{E}{c}, \qquad \text{i.e.,} \qquad E = \sqrt{I\mu_0 c}$$

$$\therefore \quad E = \sqrt{[10^3/4\pi] \times \left(4\pi \times 10^{-7}\right) \times 3 \times 10^8} = 100\sqrt{3} = 173 \text{ V/m}$$

and $$B = \frac{E}{c} = \frac{100\sqrt{3}}{3 \times 10^8} = \frac{1}{\sqrt{3}} \times 10^{-6} = 5.77 \times 10^{-7} \text{ Web/m}^2$$

Note :

(i) As $B = \mu H$, $H = \dfrac{5.77 \times 10^{-7}}{4\pi \times 10^{-7}} = 0.46 \text{ A/m}$

(ii) The peak values of fields will be $E_0 = (\sqrt{2})E$ and $B_0 = (\sqrt{2})B$ $\quad \left[\text{as } E = E_0/\sqrt{2}\right]$

Characteristics of Sound

Sound is characterised by the following three parameters :

(a) **Loudness :** It is the sensation received by the ear due to intensity of sound.

$$L \propto \log I$$

i.e. greater the amplitude of vibration, greater will be the intensity I ($\propto A^2$) and so louder will be the sound as in a shout and lesser the intensity.

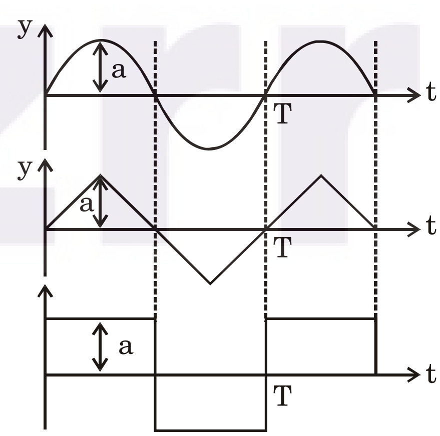
The unit of loudness is decibels (dB) and $L = 10 \log_{10} \dfrac{I}{I_0}$ (in dB) Here, I_0 is constant i.e., minimum intensity (= 10^{-12} W/m^2) just double at intermediate frequencies.

(b) **Pitch :** It is the sensation received by the ear due to frequency and is the characteristic which distinguishes a shrill (or sharp) sound from a grave (or flat) sound. As pitch depends on frequency, higher the frequency higher will be the pitch and shriller will be the sound. Regarding pitch it is worth noting that :

(1) The buzzing of a bee or humming of a mosquito has high pitch but low loudness while the roar of a lion has large loudness but low pitch.

(2) Due to more harmonic usually the pitch of female voice is higher than male.

(c) **Quality (or Time) :** It is the sensation received by the ear due to waveform. Two sounds of same intensity and frequency as shown in fig. will produce different sensation on the ear if their waveforms are different. Now as waveform depends upon overtones present, quality of sound depends on number of overtones, i.e., harmonics present and their relative intensities. The dependence of quality on phase is controversial. Regarding quality it is worth noting that :



(1) We can recognise a person (without seeing) by listening to his sound as it has a definite quality.

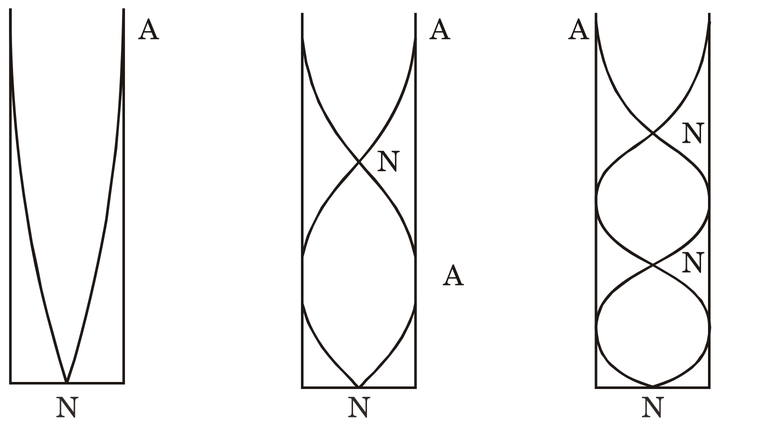
(2) If same note is played on different instruments say sitar and veena at same loudness it produces different sensation on the ear due to their quality.

(3) Sometimes it becomes difficult to recognise a person by listening to his sound on telephone or tape due to poor quality of sound.

Stationary Waves in Air Column

Closed Pipe

A stationary wave pattern can be maintained in a closed tube containing a gas only for a frequency, which has one of the values making the length of the column a whole number of quarter wavelengths. It should be noted that the open end is always an antinode and the closed end a node. According to this condition there arises a number of standing waves as shown in figure. The wave pattern, which has the lowest frequency, is called fundamental and the others are called overtones.



Fundamental First overtone Second overtone

(a) First Mode of vibration (Fundamental mode)

The length of air column L is equal to $\frac{\lambda}{4}$

$\therefore \quad \lambda = 4L$

$v = f\lambda$

$f = \frac{v}{\lambda} = \frac{v}{4L}$

where f is the frequency of fundamental mode.

(b) Second Mode of Vibration (First overtone)

$L = \frac{3\lambda_1}{4}$

$\lambda_1 = \frac{4L}{3}$

$\therefore$ frequency $f_1 = \frac{v}{\lambda_1} = \frac{3v}{4L} = 3f$

The frequency of first overtone is 3 times the value of fundamental.

(c) Third Mode of Vibration (Second overtone)

Here $L = \frac{5\lambda_2}{4}$

$$\lambda = \frac{4L}{5}$$

$\therefore$ frequency $f_2 = \frac{v}{\lambda_2} = \frac{5v}{4L} = 5f$

When an air column is excited the fundamental and a number of possible overtones are present in the vibration. Of these the loudest is the fundamental and overtones progressively becomes weaker in intensity. The overtones whose frequencies are integral multiples of fundamental are called harmonics. The fundamental with frequency f itself is taken as first harmonic. The overtone with frequency 2f is called second harmonic and the overtone with frequency 3f is called third harmonic and so on.

In the case of closed type indicated above all odd harmonics are present and even harmonics are absent.

End correction : In the above discussion it is assumed that the position of antinode coincides with the opened of pipe exactly. This is not however true and it is found that antinode is a little bit displaced above the open end. If e is the end correction, then for fundamental mode.

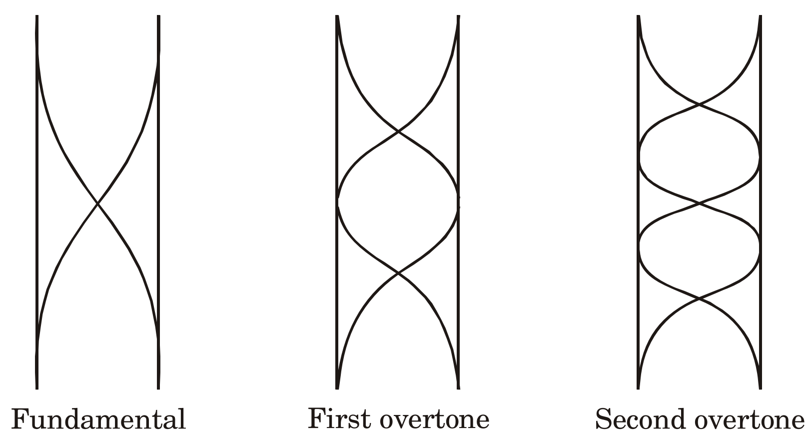
$$\frac{\lambda_1}{4} = (L + e)$$

For the first overtone $\frac{3\lambda_2}{4} = (L + e)$ and so on.

The end correction depends upon the diameter of the pipe. If d is the diameter, the end correction e = 0.3 d.

Open Pipes

A pipe with both ends open is called open pipe. The first three modes of vibrations, starting from fundamental in open pipes are shown in figures.



(a) **First mode of vibration** (Fundamental mode) : In the fundamental mode there is a node between antinodes at each end.

$$\therefore \quad L = \frac{\lambda}{2} \text{ or } \lambda = 2L$$

$$f = \frac{v}{\lambda} = \frac{v}{2L}$$

(b) **Second mode of vibration** (First overtone) : If λ_1 and f_1 are the wavelength and frequency of the first overtone in open pipe.

$$\lambda_1 = L$$

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{L} = \frac{2v}{2L} = 2f$$

The frequency of first overtone is twice that of fundamental. It corresponds to second harmonic.

(c) **Second overtone :** If λ_2 and f_2 be the wavelength and frequency of second overtone in the open pipe.

$$L = \frac{3\lambda_2}{2}$$

$$\lambda_2 = \frac{2L}{3}$$

$$f_2 = \frac{v}{\lambda_2} = \frac{3v}{2L} = 3f$$

This corresponds to third harmonic of the vibrating system.

In an open pipe all the harmonics, both odd and even are present.

Free, Damped and Forced Vibrations

A body capable of vibration, if excited, and set free, vibrates freely in its own natural way. The frequency of such free vibration depends on the mass, elastic property and dimensions of the body. The frequency is called free frequency or natural frequency of the body.

Damped Vibrations

The amplitude of free vibrations of a body gradually diminishes and finally the vibrations die away after sometime. This is due to the vibratory motion being damped by forces internal and external to the body.

Forced Vibrations

If an external periodic force is applied to a body which is capable of vibration and if the frequency of the applied periodic force is not the same as the frequency of the body, the body begins to vibrate initially with its own natural frequency but these vibrations die down quickly and the body ultimately vibrates with the frequency of the external periodic force. Such vibrations are called forced vibrations.

Example

Where will a person hear maximum sound at (displacement) node or antinode ?

Solution :

Perception of sound is due to pressure variations and as at node displacement is minimum, pressure will be maximum while at antinode as displacement is maximum, pressure will be minimum. So sound will be maximum at displacement nodes (which is actually pressure-antinode).

Example 23

A tuning fork having frequency of 340 Hz is vibrated just above a cylindrical tube. The height of the tube is 120 cm. Water is slowly poured in. What is the minimum height of water required for resonance ? (v = 340 m/s)

Solution :

As the tuning fork is in increase with air column in the pipe closed at one end.

$$f = n\frac{v}{4L} \quad \text{with } n = 1, 3, 5, \ldots$$

So length of air column in the pipe $L = \frac{nv}{4f} = n\frac{340 \times 100}{4 \times 340} = 25n \text{ cm}$ with $n = 1, 3, 5, \ldots$

i.e., $L = 25 \text{ cm}, 75 \text{ cm}, 125 \text{ cm}, \ldots$

Now as the tube is 120 cm, so length of air column must be lesser than 120 cm, i.e, it can be only 25 cm or 75 cm. Further if h is the height of water filled in the tube,

$$L + h = 120 \text{ cm} \quad \text{or} \quad h = 120 - L$$

So h will be minimum when L = max = 75

$\therefore$ $h_{min} = 120 - 75 = 45$ cm **Ans.**

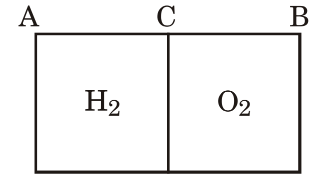
Example 24

AB is a cylinder of length 1 m fitted with a thin flexible diaphragm C at middle and two other thin flexible diaphragms A and B at the ends. The positions AC and BC contain hydrogen and oxygen gases respectively. The diaphragms A and B are set into vibrations

of the same frequency. What is the minimum frequency of these vibrations for which diaphragm C is a node ? Under the conditions of the experiment the velocity of sound in hydrogen is 1100 m/s and oxygen 300 m/s.

Solution :

As diaphragm C is a node, A and B will be antinodes (as in an organ pipe either both ends are antinode or one end node and the other antinode), i.e., each part will behave as a closed end organ pipe so that



$$f_H = \frac{v_H}{4L_H} = \frac{1100}{4 \times 0.5} = 550 \text{ Hz}$$

And $$f_O = \frac{v_0}{4L_O} = \frac{330}{4 \times 0.5} = 150 \text{ Hz}$$

As the two fundamental frequencies are different, the system will vibrate with a common frequency f_C such that

$$f_C = n_H f_H = n_O f_O$$

i.e., $$\frac{n_H}{n_O} = \frac{f_O}{f_H} = \frac{150}{550} = \frac{3}{11}$$

i.e., the third harmonic of hydrogen and 11th harmonic of oxygen or 9th harmonic of hydrogen and 33rd harmonic of oxygen will have same frequency. So the minimum common frequency

$$f = 3 \quad 550 \quad \text{or} \quad 11 \quad 150 = 1650 \text{ Hz}$$

(as 6th harmonic of H and 22nd of O will not exist.)

Example 25

A 'pop' gun consists of a tube 25 cm long closed at one end by a cork and at the other end by a tightly fitted piston. The piston is pushed slowly in. When the pressure rises to one and half times the atmospheric pressure, the cork is violently blown out. Calculate the frequency of the 'pop' caused by its ejection. (v **= 340 m/s)**

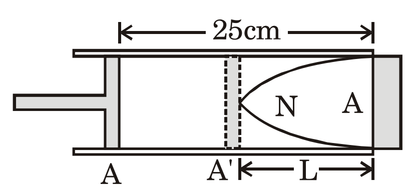
Solution :

Assuming the cross-section to be A and compression to be isothermal (as the process is slow), from

$$P_1V_1 = P_2V_2$$

$$P \quad 25 \quad A = (3/2) \quad P \quad L \quad A, \text{ i.e., } L = (50/3) \text{ cm}$$

Now after the ejection of cork, for oscillating air node will be at piston (rigid boundary) while antinode will be at the open end and as minimum distance between node and antinode is $(\lambda/4)$.



so $$\frac{\lambda}{4} = L = \frac{50}{3}\text{cm}, \quad \text{i.e.,} \quad \lambda = \frac{2}{3}\text{m}$$

and hence $$f = \frac{v}{\lambda} = \frac{340 \times 3}{2} = 510 \text{ Hz}$$

Example 26

The water level in a vertical glass tube 1.0 m long can be adjusted to any position in the tube. A tuning fork vibrating at 660 Hz is held just over the open top end of the tube. At what position of the water level will there be resonance. Speed of sound is 330 m/s.

Solution :

Resonance corresponds to a pressure antinode at closed end and pressure node at open end. Further, the distance between a pressure node and a pressure antinode is $\frac{\lambda}{4}$, the condition of resonance would be,

$$\text{length of air column } l = n\frac{\lambda}{4} = n\left(\frac{v}{4f}\right)$$

Here, n = 1, 3, 5, ...

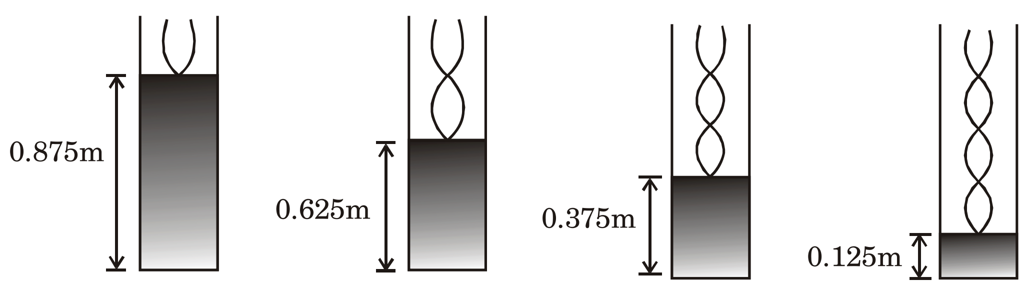
$$l_1 = (1)\left(\frac{330}{4 \times 660}\right) = 0.125 \text{ m}$$

$$l_2 = 3l_1 = 0.375 \text{ m}$$
$$l_3 = 5l_1 = 0.625 \text{ m}$$
$$l_4 = 7l_1 = 0.875 \text{ m}$$
$$l_5 = 9l_1 = 1.125 \text{ m}$$

Since $l_5 > 1$ m (the length of tube), the length of air columns can have the values from l_1 to l_4 only. Therefore, level of water at resonance will be

$$(1.0 - 0.125) \text{ m} = 0.875 \text{ m}$$
$$(1.0 - 0.375) \text{ m} = 0.625 \text{ m}$$
$$(1.0 - 0.625) \text{ m} = 0.375 \text{ m}$$

and $$(1.0 - 0.875) \text{ m} = 0.125 \text{ m}$$ **Ans.**



In all the four cases shown in figure, the resonance frequency is 660 Hz but first one is the fundamental tone or first harmonic. Second is first overtone or third harmonic and so on.

Resonance :

Resonancee is a special case of forced vibration. If the frequency of the external periodic force is the same as the natural frequency of the body, the body responds to the forced vibrations more willingly and there is a gain in the amplitude of its vibrations. This is called resonance.

Resonance has vast application in acoustics, electrical circuits and electronics.

Resonance in Air Columns-Resonance Tubes :

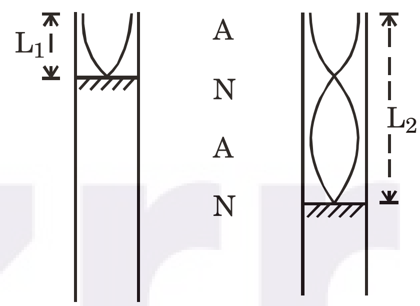
Suppose the length of air column in a long tube can be adjusted either by dipping the tube in a reservoir of water or by allowing the water level to occupy a desired position in the tube by pressure flow; the column can be made to vibrate in resonance with an excited tuning fork kept over the mouth of the tube.

For two lengths of air column L_1 and $L_2 \simeq 3L_1$, the resonance would occur and the positions corresponds to the fundamental mode and the first overtone respectively.

If λ be the wavelength of sound in air and v the velocity of sound in air, then

$$L_1 + e = \frac{\lambda}{4}$$

$$L_2 + e = \frac{3\lambda}{4}$$



where e is the end correction

From the above equations we get

$$\frac{\lambda}{2} = L_2 - L_1 \quad \text{or} \quad \lambda = 2\,(L_2 - L_1)$$

$$v = f\lambda = 2f\,(L_2 - L_1)$$

where f is the frequency of vibration of the air column which is in resonance with the tuning fork of same frequency.

Example 27

A tube of a certain diameter and length 48 cm is open at both ends. Its fundamental frequency is found to be 320 Hz. The velocity of sound in air is 320 m/s. Estimate the value of end correction in cm.

Solution :

Let the length of the open tube be L. The end correction on both sides is e. The tube vibrates in its fundamental. Then

$$\frac{\lambda}{2} = L + 2e \text{ or } \lambda = 2\,(L + 2e)$$

If v be the velocity of sound in air the fundamental frequency is given by

$$f = \frac{v}{\lambda} = \frac{v}{2(L+2e)}$$

f = 320 Hz; v = 320 m/s or; $320 = \frac{320}{2(L+2e)}$

or $L + 2e = 0.5$ m

$2e = 0.5$ m $- 0.48$ m $= 0.02$ m

$e = 0.01$ m $= 1$ cm.

BEATS

When two sound waves of nearly equal (but not exactly equal) frequencies travel in same direction, at a given point due to their superposition, intensity alternately increases and decreases periodically. This periodic waxing and waning of sound at a given position is called beats.

Calculation of beat frequency

Suppose two waves of frequencies f_1 and f_2 ($< f_1$) are meeting at some point in space. Let the oscillations at some point in space (say $x = 0$) due to two waves be $y_1 = A_1 \sin 2\pi f_1 t$,

$$y_2 = A_2 \sin 2\pi f_2 t$$

If they are in phase at some time t_s then

$2\pi f_1 t = 2\pi f_2 t$ or $f_1 t = f_2 t$

They will be again in phase at time $(t + T)$,

$2\pi f_1 (t + T) = 2\pi f_2 (t + T) + 2\pi$

$\Rightarrow$ $f_1(t + T) = f_2 (t + T) + 1$

$\Rightarrow$ $\boxed{T = \frac{1}{f_1 - f_2}}$

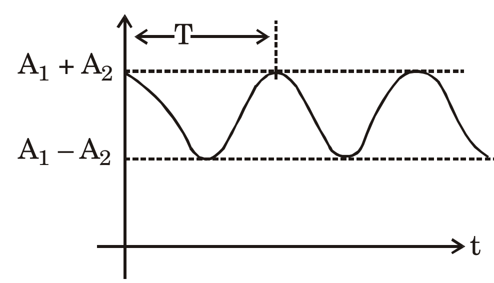
where T is the time period of the beat or $\boxed{\text{Beat frequency } f = f_1 - f_2}$

Note : If the waves are in phase at some time ($t = 0$) will be constructive and the resultant amplitude will be $A_1 + A_2$, where A_1 and A_2 are the amplitudes of indivisual sound waves.

But at some time ($t = t_0$) because the frequencies are different, the waves will be out of phase or the interference will be destructive and resultant amplitude will be $A_1 - A_2$.

and $T = 2t_0 = \frac{1}{f_1 - f_2}$

Where t_0 is the time between constructive and destructive interference.



Alternative Method :

$y_1 = A \sin (2\pi f_1 t), \quad y_2 = A \sin 2\pi f_2 t$

Now as $\sin C + \sin D = 2 \sin \left(\frac{C + D}{2}\right) \cos \left(\frac{C - D}{2}\right)$

$$y = 2A \cos 2\pi \frac{(f_1 - f_2)t}{2} \sin 2\pi \frac{(f_1 + f_2)t}{2}$$

or $\quad y = 2A \cos 2\pi f_A t \sin 2\pi f_{av} t \quad$ with $\quad f_A = \frac{f_1 \sim f_2}{2}$ and $f_{av} = \frac{f_1 + f_2}{2}$

or $\quad y = A_b \sin 2\pi f_{av} t \quad$ with $\quad A_b = 2A \cos (2\pi f_A t)$

Thus, the resultant wave is a harmonic progressive wave of frequency f_{av} i.e., $\frac{f_1 + f_2}{2}$ and amplitude A_b which is periodic in time.

Also, it can be seen that a beat, that is maximum and minimum intensity, will occur when

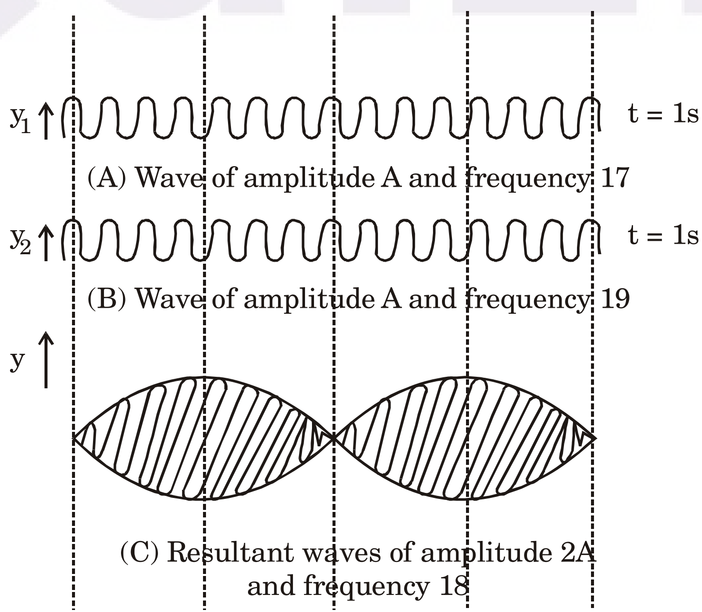
$I \propto A_b^2 = \max$ (or min)

or $\quad \cos 2\pi f_A t = \pm 1$ or zero

or $\quad 2\pi f_A t = 0, \pi, 2\pi \ldots$ or $\pi/2, 3\pi/2, 5\pi/2$

i.e. $\quad t = 0, \frac{1}{2f_A}, \frac{2}{2f_A}, \frac{3}{2f_A}$ or $\frac{1}{4f_A}, \frac{3}{4f_A}, \frac{5}{4f_A}$

$\Delta t = \frac{1}{2f_A} \quad$ beat frequency $= \; f_b = \frac{1}{\Delta t} = f_1 \sim f_2$



(A) Wave of amplitude A and frequency 17

(B) Wave of amplitude A and frequency 19

(C) Resultant waves of amplitude 2A and frequency 18

Example 28

A column of air and a tuning fork produce 4 beats per second when sounded together. The tuning fork gives the lower note. The temperature of air is 15 C. When the temperature falls to 10 C the two produces 3 beats per second. The frequency of fork is f, then find the value of 5f.

Solution :

Let λ be the wavelength and n be the frequency of fork

At 15 C, $\frac{v_{15}}{\lambda} - f = 4$ or $\frac{v_{15}}{\lambda} = f + 4$

At 10 C, $\frac{v_{10}}{\lambda} - f = 3$ or $\frac{v_{10}}{\lambda} = f + 3$

$$\therefore \quad \frac{v_{15}}{v_{10}} = \frac{f+4}{f+3}$$

But $\frac{v_{15}}{v_{10}} = \sqrt{\frac{273+15}{273+10}} = \sqrt{\frac{288}{283}}$ $\quad \therefore \quad \sqrt{\frac{288}{283}} = \frac{f+4}{f+3}$

$$\left(1 + \frac{5}{283}\right)^{1/2} = \frac{f+4}{f+3} \qquad 1 + \frac{5}{566} = \frac{f+4}{f+3}$$

$$\frac{5}{566} = \frac{f+4-f-3}{f+3} = \frac{1}{f+3}$$

$5f + 15 = 566$

$5f = 551$

Example 29

Two radio stations broadcast their programmes at the same amplitude A, and at slightly different frequencies ω_1 and ω_2 respectively, where $\omega_2 - \omega_1 = 10^3$ Hz. A detector receives the signals from the two stations simultaneously. It can only detect signals of intensity > $2A^2$. (a) Find the time-interval between successive maxima of the intensity of the signal received by the detector. (b) Find the time for which the detector remains idle in each cycle of the intensity of the signal.

Solution :

If the detector is at x = 0, the two radio-waves at the site of detector in accordance with given conditions (i.e., $A_1 = A_2 = A$ and $f_1 = \omega_1$ and $f_2 = \omega_2$) will be

$y_1 = A \sin 2\pi\omega_1 t$ and $y_2 = A \sin 2\pi\omega_2 t$

So by principle of superposition,

$= y_1 + y_2 = A \sin 2\pi\omega_1 t + A \sin 2\pi\omega_2 t$

But as $\sin C + \sin D = 2 \sin \frac{(C+D)}{2} \cos \frac{(C-D)}{2}$

$$y = 2A \cos 2\pi\frac{(\omega_2 - \omega_1)}{2}t \sin 2\pi \frac{(\omega_1 + \omega_2)}{2}t$$

or $\quad y = A' \sin 2\pi\frac{(\omega_1 + \omega_2)}{2}t$ with $A' = 2A \cos 2\pi \frac{(\omega_2 - \omega_1)}{2}t$

So that $1 \propto (A')^2 \propto 4A^2 \cos^2\pi(\omega_2 - \omega_1)t$ ...(1)

(a) So I will be maximum when

$\cos^2\pi(\omega_2 - \omega_1)t = \max = 1$, i.e., $\cos \pi(\omega_2 - \omega_1)t = \pm 1$

or $\quad \pi(\omega_2 - \omega_1)t = 0, \pi, 2\pi, ...,$

i.e., $t = 0; [1/(\omega_2 - \omega_1)], [2/(\omega_2 - \omega_1)], ...$

So time interval between two consecutive maxima

$$T = t_2 - t_1 = \frac{1}{\omega_2 - \omega_1} = \frac{1}{10^3} = 10^{-3}\text{ s}$$

(b) As $1 \propto 4A^2 \cos^2\pi(\omega_2 - \omega_1)t$ it will be $2A^2$ when

$2A^2 = 4A^2 \cos^2\pi(\omega_2 - \omega_1)t$

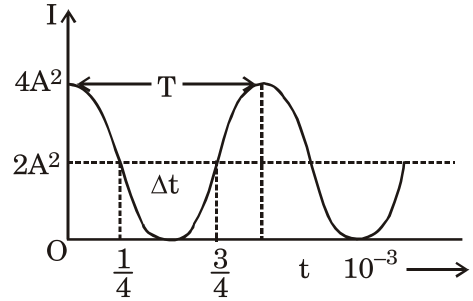
i.e., $\cos \pi(\omega_2 - \omega_1)t = \pm \left(1/\sqrt{2}\right)$

or $\quad \pi(\omega_2 - \omega_1)t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, ...$

i.e., $\quad t = \frac{1}{4(\omega_2 - \omega_1)}, \frac{3}{4(\omega_2 - \omega_1)}, \frac{5}{4(\omega_2 - \omega_1)}, ...$

So time interval between two successive positions for which intensity remains $\leq 2A^2$ as shown in Fig.

$$\Delta t = t_n - t_{n-1} = \frac{1}{2(\omega_2 - \omega_1)} = \frac{1}{2 \times 10^3} = 5 \times 10^{-4} \text{ s}$$



Example 30

There are three sources of sound of equal intensities with frequency 400, 401 and 402 Hz. What is the beat frequency heard if all are sounded simultaneously ?

Solution :

As intensities are equal, amplitudes of waves will be equal and it for simplicity we consider the waves at x = 0 with 401 = f, $y_1 = A \sin 2\pi(f - 1)t$; $y_2 = A \sin 2\pi ft$ and $y_3 = A \sin 2\pi(f + 1)t$

So by principle of superposition,

$y = A \sin 2\pi(f - 1)t + A \sin 2\pi ft + A \sin 2\pi (f + 1)t$

Taking first and last terms together,

$y = 2A \cos 2\pi t \sin 2\pi ft + A \sin 2\pi ft$

or $y = A[2 \cos 2\pi t + 1] \sin 2\pi ft$

or $y = A' \sin 2\pi ft$ with $A' = A[1 + 2 \cos 2\pi t]$

So $I \propto (A')^2 \propto A^2 (1 + 2 \cos 2\pi t)^2$ ...(1)

For I to be max or min,

$$\frac{dI}{dt} = 0, \quad \text{i.e.} \quad \frac{d}{dt}(1 + 2\cos 2\pi t)^2 = 0$$

i.e. $2(1 + 2 \cos 2\pi t)(2 \sin 2\pi t) \quad 2\pi = 0$

i.e., either $\sin 2\pi t = 0$ or $1 + 2 \cos 2\pi t = 0$

So if $1 + 2 \cos 2\pi t = 0$, i.e., $\cos 2\pi t = (-\frac{1}{2})$

or $2\pi = 2\pi n \pm \left(\frac{2\pi}{3}\right)$,..., with n = 0, 1, 2 ....

i.e., $t = \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \ldots$ ...(2)

and for these values of t [i.e., $\cos 2\pi t = (-\frac{1}{2})$], I = 0, i.e., intensity is minimum.

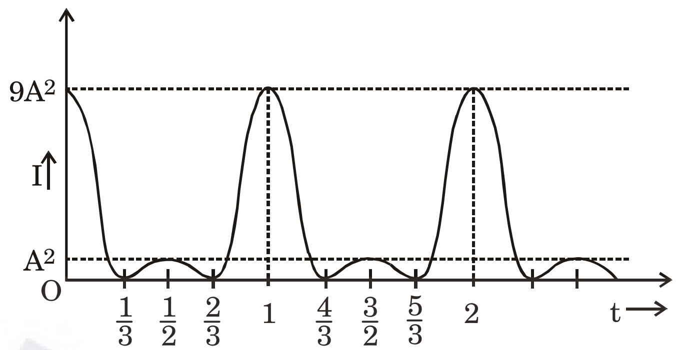
and if $\sin 2\pi t = 0$, i.e., $2\pi t = n\pi$ with n = 0, 1, 2 ...

i.e., $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ ...(3)

I [from Eqn. (1)] will be $9A^2, A^2, 9A^2, A^2, \ldots$

i.e., intensity is maximum (with two different values).

So from Eqns. (2) and (3) it is clear that in one second we get two minima (and two maxima of different intensities) and hence beat frequency (i.e., number of beats per sec) is two.



DOPPLER EFFECT

When a sound source and an observer are in relative motion with respect to the medium in which the waves propagate, the frequency of waves observed is different from the frequency of sound emitted by the source. This phenomenon is called Doppler effect. This is due to the wave-nature of sound propagation and is therefore applicable to light waves also. The apparent change of colour of a star can be explained by this principle.

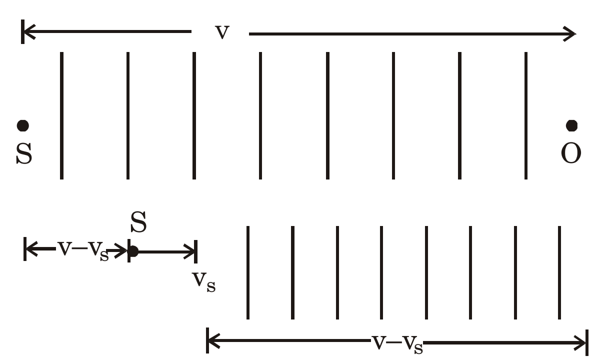
Calculation of Apparent Frequency

Suppose v is the velocity of sound in air, v_0 is the velocity of the observer (O) and f is the frequency of the source.

(i) **Source moves towards stationary observer :** If the source S were stationary the f waves sent out in one second towards the observer O would occupy a distance v, and the wavelength would be v/f.

If S moves with a velocity v_s towards O, the f waves sent out occupy a distance $(v - v_s)$ because S has moved a distance v_s towards O in 1 s. So the apparent wavelength would be

$$\lambda' = \left(\frac{v - v_s}{f}\right)$$

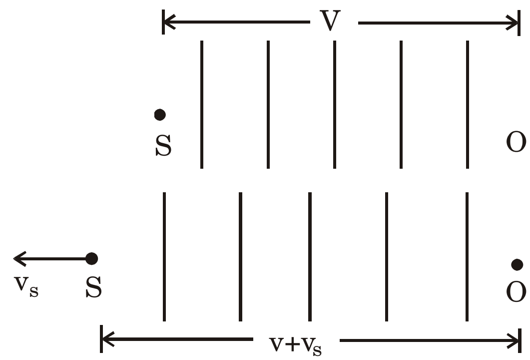


Thus, apparent frequency

$$f' = \frac{\text{velocity of sound relative to O}}{\text{wavelength of wave reaching O}}$$

$$f' = \frac{v}{\lambda'} = f\left(\frac{v}{v - v_s}\right)$$

(ii) **Source moves away from stationary observer :** Now, apparent wavelength



$$\lambda' = \frac{v + v_s}{f}$$

$\therefore$ Apparent frequency

$f' = v/\lambda'$

or $f' = f\left(\frac{v}{v + v_s}\right)$

(iii) **Observer, moves towards stationary source**

$$f' = \frac{\text{velocity of sound relative to O}}{\text{wavelength of wave reaching O}}$$

Here, velocity of sound relative to O = $v + v_0$

and wavelength of waves reaching O = v/f

$$\therefore \quad f' = \frac{v + v_0}{v/f} = f\left(\frac{v + v_0}{v}\right)$$

(iv) **Observer moves away from the stationary source**

$$f' = \frac{v - v_0}{v/f} = f\left(\frac{v - v_0}{v}\right)$$

(v) **Source and observer both moves towards each other**

$$f' = \frac{(v + v_0)}{\left(\frac{v - v_s}{f}\right)} = f\left(\frac{v + v_0}{v - v_s}\right)$$

(vi) **Both moves away from each other**

$$f' = f\left[\frac{v - v_0}{v + v_s}\right]$$

(vii) **Source moves towards observer but observer moves away from source**

$$f' = f\left(\frac{v - v_0}{v - v_s}\right)$$

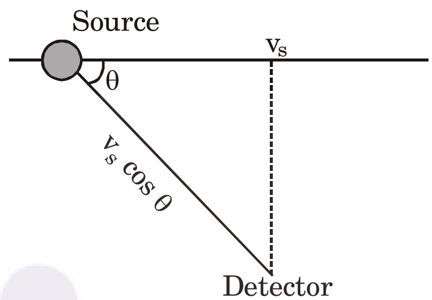
(viii) **Source moves away from observer but observer moves towards source**

$$f' = f\left[\frac{v + v_0}{v + v_s}\right]$$

Discussion

(1) There is always an increase in frequency or pitch if source moves towards detector or detector moves towards source or both move towards each other while a decrease in frequency if either or both move away. The change in frequency or pitch depends on speed of source and detector and not on distance between them, e.g., if an engine is approaching a stationary listener at constant velocity, increase in pitch, by Eqn. (2) will be same when the engine is either at a distance of 1 km or 10 m from the listener. However, intensity will be different in the two cases as $I \propto (1/r^2)$.

(2) If the motion is along some other direction, the components of velocities along the line joining source and detector are considered for v_S and v_D, e.g., if at any instant the line joining the moving source and stationary detector makes an angle θ with the direction of motion of source, $v_S \rightarrow v_S \cos\theta$



and so $$f_{A\rho} = f\left[\frac{v}{v - v_S \cos\theta}\right] \quad ...(3)$$

In such situations $f_{A\rho}$ is not constant and depends on θ and may be greater, equal to or less than f as θ < = or > 90 .

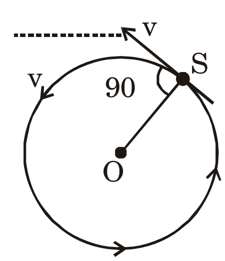
(3) If the medium is at rest relative to earth, v_S, v_D and v will refer to the speeds of source, detector and sound relative to earth which is usually the case. However, if the medium (air) itself starts moving with respect to given frame of reference (say earth), appropriate changes must be made in Eqn. (2), i.e., if wind blows at a speed w from the source to the detector $v \rightarrow v + w$ and if in opposite direction (i.e., from detector to source) $v \rightarrow v - w$.

(4) There will be no Doppler effect, i.e., no change in frequency.

(a) If source and detector both move in same direction with same speed, i.e., if $v_S = v_P = u$

$$f' = f\left[\frac{v-u}{v-u}\right] = f$$

(b) If one is at the centre of a circle while the other is moving on it with uniform speed. In this situation component of u along the ling of sight, i.e., radius, will be u cos 90 = 0; so

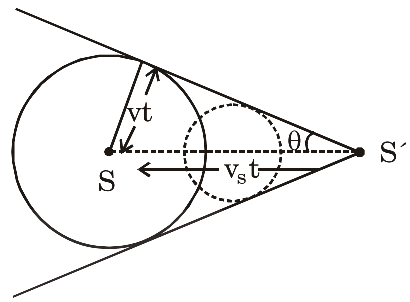


$$f' = f\left[\frac{v \pm 0}{v \mp 0}\right] = f$$

(c) If both are at rests and wind blows at speed w. In this situation

$$f' = f\left[\frac{(v \pm w) \pm 0}{(v \pm w) \mp 0}\right] = f$$

(5) Speed of detector or source becomes equal to or greater than that of sound, Doppler formula (2) does not apply (as it was derived by assuming v_D and $v_S < v$). For example if $v_D > v$ and the detector is moving away from the source, the sound will never reach it and if $v_S > v$ the source gets ahead of the wave in its direction of motion.



When the speed of source (v_S) is greater than the speed of sound (v) is called supersonic speed and the ratio (v_S/v) **Mach number.** In this situation a conical wavefront of high-energy pressure waves [with source at its apex and semicone angle $\theta = \sin^{-1}(v/v_S) = \sin^{-1}$ (1/Mach No.)] called **'shock-waves'** is continuously produced and when we intercept it, a loud bang of sound called **sonic boom** is heard which can break windows and even cause damage to buildings. Here it is worthy to note that shock-waves are produced not only when source crosses the sound barrier (a misconception) but are generated continuously as long as $v_S > v$.

Example 31

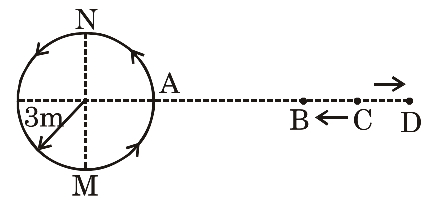
A source of sound is moving along a circular orbit of radius 3 m with an angular velocity of 10 rad/s. A sound detector located far away from the source is executing linear simple harmonic motion along the line BD with amplitude BC = CD = 6 m. The frequency of oscillation of the detector is **(5/π) per sec. The source is at the point A when the detector is at the point B. If the source emits a continuous sound wave of frequency 340 Hz, find the maximum and the minimum frequencies recorded by the detector [velocity of sound = 330 m/s].**

Solution :

Time period of circular motion $T = (2\pi/\omega) = (2\pi/10)$ is same as that of SHM, i.e., $T = (1/f) = (\pi/5)$, so both will complete one periodic motion in same time. Further more as source is moving on a circle, its speed

$$v_S = r\omega = 3 \quad 10 = 30 \text{ m/s}$$

and as detector is executing SHM



$$v_D = \omega\sqrt{A^2 - y^2} = 10\sqrt{6^2 - y^2}$$

i.e., $(v_D)_{max} = 60$ m/s when $y = 0$

i.e., detector is at C. Now in the case of Doppler effect,

$$f_{A\rho} = f\left[\frac{v \pm v_D}{v \mp v_S}\right]$$

So $f_{A\rho}$ will be maximum when both move towards each other.

$$f_{max} = f\left[\frac{v + v_D}{v - v_S}\right] \text{ with } v_D = \text{max}$$

i.e., the source is at M and detector at C moving towards B, so

$$f_{max} = 340\left[\frac{330 + 60}{330 - 30}\right] = 442 \text{ Hz}$$

Similarly $f_{A\rho}$ will be minimum when both are moving away from each other, i.e.,

$$f_{min} = f\left[\frac{v - v_D}{v + v_S}\right] \quad \text{with } v_D = \text{max}$$

i.e., the source is at N and detector at C but moving towards D, so

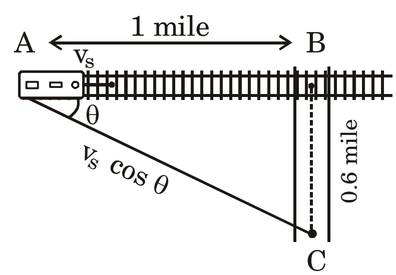
$$f_{min} = 340\left[\frac{330 - 60}{330 + 30}\right] = 225 \text{ Hz}$$

Example 32

A locomotive approaching a crossing at a speed of 80 miles/hr., sounds a whistle of frequency 400 Hz when 1 mile from the crossing. There is no wind, and the speed of sound in air is 0.200 mile/s. What frequency is heard by an observer 0.60 miles from the crossing on the straight road which crosses the railroad at right angles ?

Solution :

The situation is shown in Fig.



Here as $AC = \sqrt{1^2 + 0.6^2} = 1.166$

$$\cos\theta = \frac{AB}{AC} = \frac{1}{1.166} = 0.857$$

So speed of source along the line of sight

$$v_S \to v_S \cos\theta = \frac{80}{60 \times 60} \times 0.857 = 0.019 \text{ mile/s}$$

So $$f_{A\rho} = f\left[\frac{v}{v - v_S}\right] = 400\left[\frac{0.2}{0.2 - 0.019}\right] = 442 \text{ Hz}$$

Example 33

A sonometer wire under tension of 64 N vibrating in its fundamental mode is in resonance with a vibrating tuning fork. The vibrating portion of the sonometer wire has a length of 10 cm and mass 1 g. The vibrating tuning fork is now moved away from the vibrating wire at a constant speed and an observer standing near the sonometer hears one beat per sec. Calculate the speed with which the tuning fork is moved, if the speed of sound in air is 300 m/s.

Solution :

As the frequency of a vibrating string

$$f_S = \frac{1}{2L}\sqrt{\frac{T}{m}} = \frac{1}{2}\sqrt{\frac{T}{ML}} \qquad \left[\text{as } m = \frac{M}{L}\right]$$

so $$f_S = \frac{1}{2}\sqrt{\left[\frac{64}{10^{-3} \times 10^{-1}}\right]} = 400 \text{ Hz}$$

Now, as initially sonometer wire is in resonance with tuning fork, the frequency of tuning fork.

$$f = f_S = 400 \text{ Hz}$$

When the tuning fork is moved away from the observer standing near the sonometer at a constant speed u the apparent frequency of tuning fork will be

$$f_R = f\left[\frac{v}{v + u}\right]$$

As f_R is producing beats with f, f_R is nearly equal to f, i.e., u << v so that

$$f_R = f\left[1 + \frac{u}{v}\right]^{-1} = f\left[1 - \frac{u}{v}\right]$$

So beat frequency $\Delta f = f - f_R = f\left[\frac{u}{v}\right]$

and substituting given data, $u = v\left[\frac{\Delta f}{f}\right] = 300\left[\frac{1}{400}\right] = 0.75 \text{ m/s}$ **Ans.**

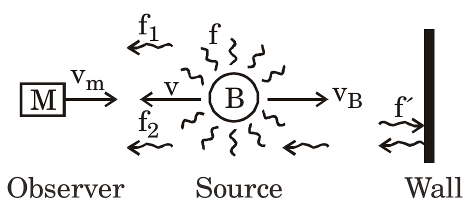
Example 34

A band playing music at a frequency f is moving towards a wall at a speed v_B. A motorist is following the band with a speed v_m. If v is the speed of sound, obtain an expression for the beat frequency heard by the motorist.

Solution :

The situation is shown in Fig. As the motorist (observer) is following the band (source), he will hear two frequencies one directly from the band while the other reflected from the wall.

Taking the direction of sound from source to observer to be positive, the frequency of



$$f_1 = f\left[\frac{v-(-v_m)}{v-(-v_B)}\right] = f\left[\frac{v+v_m}{v+v_B}\right] \quad ...(1)$$

Now as the frequency of sound reaching the wall towards which the band (source) is moving,

$$f' = f\left[\frac{v}{v-v_B}\right] \quad ...(2)$$

The frequency of reflected sound from the wall, heard by motorist (observer) who is moving towards the wall (stationary source) will be

$$f_2 = f'\left[\frac{v+v_m}{v}\right] = f\left[\frac{v+v_m}{v-v_B}\right] \quad \text{[from Eqn. (2)]}$$

So the heat frequency heard by the motorist

$$\Delta f = f_2 - f_1 = f\left[\frac{v+v_m}{v-v_B}\right] - f\left[\frac{v+v_m}{v+v_B}\right] = f\,\frac{(v+v_m)}{(v^2-v_B^2)} \times 2v_B \quad \textbf{Ans.}$$

Example 35

A train approaching a hill at a speed of 40 km/hr sounds a whistle of frequency 580 Hz when it is at a distance of 1 km from the hill. A wind with speed 40 km/hr is blowing in the direction of motion of the train. Find (a) the frequency of the whistle as heard by an

observer on the hill (b) the distance from the hill at which the echo from the hill is heard by the driver and its frequency. [Velocity of sound in air = 1200 km/hr]

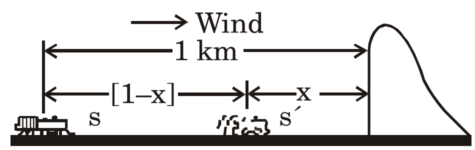
Solution :

(a) For observer at rest (on hill) and source [engine] moving towards the observer,

$$f_A = f\left[\frac{v}{v - v_S}\right]$$

Now as wind is blowing from source to observer $v \to v + w$

$$\therefore \qquad f_A = \frac{(v+w)}{(v+w) - v_S}$$



Substituting the given data,

$$f_A = 580\left[\frac{1200 + 40}{(1200 + 40) - 40}\right] = 599.3 \text{ Hz}$$

(b) If x is the required distance from the hill, the distance moved by the train will be (1 – x) and hence the time taken by the train to travel this distance is (1 – x)/40.

In this time sound travels a distance 1 km at speed (1200 + 40) and comes back a distance x at speed (1200 – 40); so

$$\frac{1 - x}{40} = \frac{1}{1240} + \frac{x}{1160}, \text{ i.e., } x = \frac{29}{31} \text{ km} = 933.3 \text{ m}$$

Now the engine will act as observer and hill as source; so the frequency heard by the moving observer towards the stationary source will be

$$f_2 = f_A\left[\frac{v + v_D}{v}\right]$$

But in this situation as wind is blowing opposite to the direction of motion of sound;

$$v \to v - w$$

so $$f_2 = f_A\left[\frac{(v - w) + v_D}{(v - w)}\right] = 599.3 \times \frac{1200}{1160} = 620 \text{ Hz}$$