MATRICES & DETERMINANTS

MATRICES

Matrix Definition :

Matrix is a rectangular array of real complex numbers in rows and columns. A matrix is denoted by the capital letters A, B, C etc. If there are m rows and n columns in the matrix, then the matrix is called a m   n matrix.

Let us consider the following system of equations

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2x - y - z &= -3 \\ 3x + 4y + 2z &= 17 \end{aligned}$$

If we arrange the coefficients of x, y and z in the order in which they occur in the given equations and enclose them in brackets, we get the following rectangular array of numbers.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 3 & 4 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 3 & 4 & 2 \end{bmatrix}$$

This type of rectangular array of numbers has been given the name matrix. The horizontal lines are called rows and the vertical lines are called columns.

Example : $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & -4 \end{pmatrix}$ or $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & -4 \end{bmatrix}$

Here A is a 2×3 matrix because A has two rows and three columns.

Element of a matrix :

The numbers occurring in the rectangular array (matrix) are called the elements of the matrix. The elements of the matrix denoted by the capital letters are usually denoted by the corresponding small letters with lower suffixes. Thus the element of the ith row and jth column of the matrix denoted by the capital letter A is usually denoted by the corresponding small letter a_{ij}. The matrix A is sometimes also written as (a_{ij}) **or** $[a_{ij}]$**.**

Definition :

A set of mn numbers (real or complex) in the form of m horizontal lines (called rows) and n vertical line (called columns), is called an m   n matrix (to be read as m and n matrix).

Type of Matrices :

(a) **Row Matrix :** If a matrix has only one row and any number of columns, is called row matrix or row vector.

For example : $A = \begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1+i & 2 & 1-i \end{bmatrix}$ are row matrices of order 1   4 and 1   3 respectively.

(b) **Column matrix or column vector :** For example, $A = \begin{bmatrix} 1+i \\ 2-i \\ 3 \\ w \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ are column matrices of orders 4   1 and 2   1 respectively.

(c) **Square matrix :** A matrix in which the number of rows is equal to the number of columns is called a Square Matrix.

Thus m n matrix A will be a square matrix if m = n, and it will be termed as a square matrix of order n or n-rowed square matrix.

Diagonal Elements : In a square matrix all those elements a_{ij} for which i = j i.e. all those elements which occur in the same row and same column namely $a_{11}.a_{22}.a_{33}$ are called the diagonal elements and the line along which they lie is called the **principal diagonal.** Also the **sum of the diagonal elements** of a square matrix A is called **trace of A.**

i.e. $a_{11} + a_{22} + a_{33} + ... = \text{Trace of A}$

In general $a_{11}, a_{22}, ... a_{nn}$ are the diagonal elements of n-rowed square matrix and $a_{11} + a_{22} + ... a = \text{Trace of A.}$

(d) **Diagonal Matrix :** A square matrix A is said to be a diagonal matrix if all its non-diagonal elements be zero.

$$\text{Thus } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix} \text{ or } \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

Above are diagonal matrices of the type 3 3. These are in short written as

Diag [1, 4, 8] or Diag [d_1, d_2, d_3]

(e) **Scalar Matrix :** A diagonal matrix [i.e. all non-diagonal elements being zero] whose all the **diagonal elements** are **equal** is called a scalar matrix.

$$\text{Thus } \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ or } \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}$$

are both scalar matrics of type 3 3.

In general for a scalar matrix,

$a_{ij} = 0 \text{ for } i \neq j$ and $a_{ij} = d \text{ for } i = j$

(f) **Unit Matrix :** A square matrix A all of whose non-diagonal elements are zero (i.e. it is a diagonal matrix) and also the **diagonal elements are unity** is called a **unit matrix** or an **identity matrix.**

$$\text{Thus } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are unit matrices of order 3 and 4 respectively. In general for a unit matrix

$a_{ij} = 0 \text{ for } i \neq j$ and $a_{ij} = 1 \text{ for } i = j$

They are generally denoted by I_3, I_4 or I_n where 3, 4, n denote the order of the square matrix. In case the order be known then we may simply denote it by I.

(g) **Zero matrix or Null Matrix :** Any m n matrix in which all the elements are zero is called a zero matrix or null matrix of the type m n and is denoted by $O_{m\ n}$.

Thus $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

All the above are zero or null matrices of the type 3   2, 3   3 and 2   4 respectively.

Trace of a Matrix : The sum of the diagonal elements of a square matrix A (say) is called the **trace** of a matrix A.

For example, If $A = \begin{bmatrix} 2 & -7 & 9 \\ 0 & 3 & 2 \\ 8 & 9 & 4 \end{bmatrix}$

Then Trace of A = 2 + 3 + 4 = 9 or (A) = 9

(h) **Horizontal Matrix :** Any matrix in which the number of columns is more than the number of rows is called a horizontal matrix.

For example, $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 8 & 9 & 7 & -2 \\ 2 & -2 & -3 & 4 \end{bmatrix}$ is a horizontal matrix.

Since here no. of columns > no. of rows.

(i) **Vertical Matrix :** Any matrix in which the number of rows is more than the number of columns is called matrix.

For example, $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \\ 8 & 9 \end{bmatrix}$ is a column matrix.

since here no. of rows > no. of columns.

(j) **Sub Matrix :** A matrix which is obtained from a given matrix by deleting any number of rows and number of columns is called a sub-matrix of the given matrix.

For example, $\begin{bmatrix} 3 & 4 \\ -2 & 3 \end{bmatrix}$ is a sub matrix of

$$\begin{bmatrix} 8 & 9 & 5 \\ 2 & 3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$$

(k) **Upper Triangular Matrix :** A square matrix in which all elements below the leading diagonal are zero, is called Upper Triangular Matrix.

For example, $\begin{bmatrix} 3 & -2 & 4 & 1 \\ 0 & 2 & -3 & 2 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ is an upper triangular matrix.

(l) **Lower Triangular Matrix :** A square matrix in which all elements above the leading diagonal are zero is called Lower Triangle Matrix.

For example, $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 8 & 6 \end{bmatrix}$ is a lower triangular matrix.

Difference between a Matrix and a Determinant

(i) A matrix can not be reduced to a number but determinant can be reduced to a number.

(ii) The number of rows may or may not be equal to the number of columns in matrices but in determinant the number of rows is equal to the number of columns.

(iii) On interchanging the rows and columns, a different matrix is formed but in determinant it does not change the value.

(iv) A square matrix A such that $|A| \neq 0$, is called a **non-singular matrix** if $|A|' = 0$, then the matrix A is called a **singular matrix.**

(v) Matrices represented by [ ], ( ), || || but determinant is represented by | |.

Illustration 1

Write down the matrix $A = [a_{ij}]_{2\times 2}$ where $a_{ij} = 2i - 3j$.

Solution : $a_{ij} = 2i - 3j$

$\therefore$ $a_{ij} = 2.1 - 3.1 = -1$, $a_{12} = 2.1 - 3.2 = 4$. $a_{13} = 2.1 - 3.3 = -7$

$a_{21} = 2.2 - 3.1 = 1$, $a_{22} = 2.2 - 3.2 = -2$, $a_{23} = 2.2 - 3.3 = -5$

$$\therefore \quad A = \begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \end{bmatrix}$$

Illustration 2

Construct a 3 3 matrix A = $[a_{ij}]$, where $a_{ij} = |2i - j|$.

Solution :

Required matrix is having 3 rows and 3 columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$a_{11} = |2 \quad 1 - 1| = 1$ $\qquad$ $a_{12} = |2 \quad 1 - 2| = 0$ $\qquad$ $a_{13} = |2 \quad 1 - 3| = |-1| = 1$

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$a_{21} = |2 \quad 1 - 1| = 3$ $\qquad a_{22} = |2 \quad 2 - 2| = 2$ $\qquad a_{23} = |2 \quad 2 - 3| = 1$

$a_{31} = |2 \quad 3 - 1| = 5$ $\qquad a_{32} = |2 \quad 3 - 2| = 4$ $\qquad a_{33} = |2 \quad 3 - 3| = 3$

Required matrix is $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \end{bmatrix}$

Illustration 3

Construct a 4 3 matrix A = [a_{ij}], where $A_{ij} = \begin{cases} i + j, & \textbf{if } i < j \\ i \times j, & \textbf{if } i = j \\ i - j, & \textbf{if } i > j \end{cases}$

Solution :

Required matrix is having 4 rows and 3 columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

$a_{11} = 1 \quad 1 = 1$ $\qquad (\therefore\ i = j)$

$a_{12} = 1 + 2 = 3$ $\qquad (\therefore\ i < j)$

$a_{13} = 1 + 3 = 4$ $\qquad (\therefore\ i < j)$

$a_{21} = 2 - 1 = 1$ $\qquad (\therefore\ i > j)$

$a_{22} = 2 \quad 2 = 4$ $\qquad (\therefore\ i = j)$

$a_{23} = 2 + 3 = 5$ $\qquad (\therefore\ i < j)$

$a_{31} = 3 - 1 = 2$ $\qquad (\therefore\ i > j)$

$a_{32} = 3 - 2 = 1$ $\qquad (\therefore\ i > j)$

$a_{33} = 3 \quad 3 = 9$ $\qquad (\therefore\ i = j)$

$a_{41} = 4 - 1 = 3$ $\qquad (\therefore\ i > j)$

$a_{42} = 4 - 2 = 2$ $\qquad (\therefore\ i > j)$

$a_{43} = 4 - 3 = 1$ $\qquad (\therefore\ i > j)$

Required matrix is $A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 4 & 5 \\ 2 & 1 & 9 \\ 3 & 2 & 1 \end{bmatrix}$

Equality of Matrices :

Comparable matrices. Two matrices are said to be comparable if their orders are same.

Equal matrices. Two matrices are said to be equal if :

(i) Their orders are same

(ii) Their corresponding elements are same

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two equal matrices, then

(i) order of A = order of B

(ii) $a_{ij} = b_{ij}$ $\forall$ i and j. i.e., $(i, j)^{th}$ element of A = $(i, j)^{th}$ element of B.

Illustration 4

$$\begin{bmatrix} x + y & 2x - y \\ y + z & 7y - z \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$$, **find** x, y **and** z.

Solution :

$x + y = 3$		and	$2x - y = 3$
Solving them we get,	$x = 2$	and	$y = 1$
$y + z = 4$		and	$7y - z = 4$
Solving them we get,	$y = 1$	and	$z = 3$

Algebra of Matrices

(i) Addition of Matrices : Two matrices A and B can be added only if A and B are of same order. Sum is obtained by adding the corresponding elements of A and B.

If $A = [a_{ij}]_{m\ \ n}$ and $B = [b_{ij}]_{m\ \ n}$,

then, $A + B = [a_{ij}]_{m\ \ n} + [b_{ij}]_{m\ \ n} = [a_{ij} + b_{ij}]_{m\ \ n}$

For example : If $A = \begin{bmatrix} 1 & 5 & 2 \\ 3 & 7 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 9 & 8 \\ 3 & 6 & 1 \end{bmatrix}$

then, since order of A and B is same, both are 2 3 matrices.

Therefore, we can add A and B.

$$A + B = \begin{bmatrix} 1 & 5 & 2 \\ 3 & 7 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 9 & 8 \\ 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1+4 & 5+7 & 2+8 \\ 3+3 & 7+6 & 6+1 \end{bmatrix} = \begin{bmatrix} 5 & 14 & 10 \\ 6 & 13 & 7 \end{bmatrix}$$

Difference of matrices A and B of same order is obtained by adding A and – B. – B is obtained by changing the sign of each element of B.

A – B can also be obtained by subtracting from the elements of A the corresponding elements of B. A and B must be of same order.

For A and B of last example,

$$A - B = \begin{bmatrix} 1 & 5 & 2 \\ 3 & 7 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 9 & 8 \\ 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1-4 & 5-9 & 2-8 \\ 3-3 & 7-6 & 6-1 \end{bmatrix} = \begin{bmatrix} -3 & -4 & -6 \\ 0 & 1 & 5 \end{bmatrix}$$

Properties of Matrix Addition :

Property I : Matrix addition is commutative i.e. if A and B be any two m n matrices, then A + B = B + A.

Property II : Matrix addition is associative i.e. if A, B and C be three m n matrices, then

$A + (B + C) = (A + B) + C$

Property III : Cancellation laws hold good for addition of matrices

i.e. if A, B, C, be any three m n matrices, then

(i) $A + B = A + C \Rightarrow B = C$ (left cancellation law)

(ii) $B + A = C + A \Rightarrow B = C$ (right cancellation law)

(i) Negative of a Matrix : If A be a given Matrix then – A is called the negative of matrix A and all its elements are the corresponding elements of A multiplied by – 1.

Thus if $A = \begin{bmatrix} 2 & 3 & -1 \\ 6 & -4 & 2 \end{bmatrix}$

then $-A = \begin{bmatrix} -2 & -3 & 1 \\ -6 & 4 & -2 \end{bmatrix}$

(ii) Scalar Multiple of a Matrix : If A be a given matrix and k is any scalar number real or complex. [We call it scalar k to distinguish it from matrix [k] which is 1 1 matrix] then by matrix kA = Ak is meant the matrix all of whose elements are k times of the corresponding elements of A.

If $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 4 \end{bmatrix}$

then $3A = \begin{bmatrix} 3.2 & 3.3 & 3.1 \\ 3.5 & 3.2 & 3.4 \end{bmatrix}$

or $3A = \begin{bmatrix} 6 & 9 & 3 \\ 15 & 6 & 12 \end{bmatrix}$

Similarly $-4A = \begin{bmatrix} -4.2 & -4.3 & -4.1 \\ -4.5 & -4.2 & -4.4 \end{bmatrix}$

$= \begin{bmatrix} -8 & -12 & -4 \\ -20 & -8 & -16 \end{bmatrix}$

II. Subtraction of Matrices :

i.e. A – B. This can be proceeded as follows A + (– B) i.e. negative of B is added to matrix B.

Illustration 5

If $X = \begin{bmatrix} 3 & 2 & 1 \\ 7 & 5 & 9 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 7 & 5 & 3 \\ 15 & 11 & 19 \end{bmatrix}$, find matrix Y.

Solution :

$$2X = 2\begin{bmatrix}3 & 2 & 1\\7 & 5 & 9\end{bmatrix} = \begin{bmatrix}6 & 4 & 2\\14 & 10 & 18\end{bmatrix}$$

Now, $$2X + Y = \begin{bmatrix}7 & 5 & 3\\15 & 11 & 19\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}6 & 4 & 2\\14 & 10 & 18\end{bmatrix} + Y = \begin{bmatrix}7 & 5 & 3\\15 & 11 & 19\end{bmatrix}$$

$$\Rightarrow \quad Y = \begin{bmatrix}7 & 5 & 3\\15 & 11 & 19\end{bmatrix} - \begin{bmatrix}6 & 4 & 2\\14 & 10 & 18\end{bmatrix}$$

$$= \begin{bmatrix}7-6 & 5-4 & 3-2\\15-14 & 11-10 & 19-18\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

Illustration 6

Find matrix X and Y such that $2X + Y = \begin{bmatrix}8 & 9 & 10\\11 & 12 & 13\end{bmatrix}$ and $X + 2Y = \begin{bmatrix}13 & 12 & 11\\10 & 9 & 8\end{bmatrix}$

Solution :

$$2X + Y = \begin{bmatrix}8 & 9 & 10\\11 & 12 & 13\end{bmatrix}$$

Multiplying both sides by 2, we get :

$$4X + 2Y = \begin{bmatrix}16 & 18 & 20\\22 & 24 & 26\end{bmatrix} \quad \text{...(i)}$$

Subtracting, $$X + 2Y = \begin{bmatrix}13 & 12 & 11\\10 & 9 & 8\end{bmatrix} \quad \text{...(ii)}$$

from equation (i) $$4X + 2Y = \begin{bmatrix}16 & 18 & 20\\22 & 24 & 26\end{bmatrix}$$

$$X + 2Y = \begin{bmatrix}13 & 12 & 11\\10 & 9 & 8\end{bmatrix}$$

$-\quad -\quad\quad -$

$$3X = \begin{bmatrix} 16 & 18 & 20 \\ 22 & 24 & 26 \end{bmatrix} - \begin{bmatrix} 13 & 12 & 11 \\ 10 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$$

$$\Rightarrow \quad X = \frac{1}{3}\begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Putting the value of X in equation (ii), we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + 2Y = \begin{bmatrix} 13 & 12 & 11 \\ 10 & 9 & 8 \end{bmatrix}$$

$$\Rightarrow \quad 2Y = \begin{bmatrix} 13 & 12 & 11 \\ 10 & 9 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \Rightarrow \quad 2Y = \begin{bmatrix} 12 & 10 & 8 \\ 6 & 4 & 2 \end{bmatrix}$$

$$\Rightarrow \quad Y = \frac{1}{2}\begin{bmatrix} 12 & 10 & 8 \\ 6 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Illustration 7

If ω is an imaginary cube root of unity, show that

$$\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix} + \begin{bmatrix} \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} + \begin{bmatrix} \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix} \text{ is a null matrix.}$$

Solution :

$$\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix} + \begin{bmatrix} \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} + \begin{bmatrix} \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ which is a null matrix.}$$

Matrix Multiplication

If $\quad A = [a_{ij}]_{m \quad p}$

$B = [b_{ij}]_{p \quad n}$ and then $C = AB = [C_{ik}]_{m \quad n}$

where $$C_{ik} = \sum_{j=1}^{p} a_{ij}\, b_{jk}$$

i.e. $$C_{ik} = a_{ij}\, b_{ik} + a_{12}\, b_{2k} + \dots\, a_{ip}\, b_{pk}$$

In other words C_{ik} = Sum of the products of nth row of A (having p elements) with kth column of B (having p elements). This is known as row by coloum multiplication of matrices.

It may be noted that in determinants we have row by row or column by column multiplication.

Illustration 8

If $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}_{2\times3}$ **and** $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}_{3\times2}$ **compute AB and show that AB ≠ BA.**

Solution :

A is 2 3 type and B is 3 2 type and hence both AB and BA are defined because the number of columns in pre-factor is equal to the number of rows in post-factor.

$$AB = \begin{bmatrix} 1.2 - 2.4 + 3.2 & 1.3 - 2.5 + 3.1 \\ -4.2 + 2.4 + 5.2 & -4.3 + 2.5 + 5.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}_{2\times2}$$

$$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}_{3\times2} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}_{2\times3}$$

$$= \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}_{3\times3}$$

Hence AB ≠ BA

If A and B be two matrices then their product is defined or in other words A is **conformable** to B for multiplication if **the number of columns of A is the same as the number of rows in B. i.e., If be** m p **and** p n**, the matrix AB will be of the type** m n**.**

Properties of matrix multiplication

(a) Multiplication of matrices is **distributive** with respect to addition of matrices.

i.e. **A (B + C) = AB + AC.**

(b) Matri multiplication is **associative** if conformability is assured.

i.e. **A (BC) = (AB) C.**

(c) The multiplication of matrices is not always commutative i.e. AB is not always equal to BA.

(d) Multiplication of a matrix A by a null matrix conformable with A for multiplication is a null matrix i.e. A0 = 0. In particular if A be a square matrix and O be square null matrix of the same order, then AO = OA = O.

(e) If AB = O then it does not necessarily mean that A = O or B = O or both are O as shown below.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

None of the matrices on the left is a null matrix whereas their product is a null matrix.

(f) **Multiplication of matrix A by a unit matrix I :** Let A be a m   n matrix and I be a square unit matrix of order n, so that A and I are conformable for multiplication, then

$$AI_n = A.$$

Similarly for IA to exist I should be square unit matrix of order m and in that case $I_m A = A$

Illustration 9

Find AB, where $\mathbf{A} = \begin{bmatrix} \mathbf{4} & \mathbf{2} \\ \mathbf{1} & \mathbf{6} \\ \mathbf{3} & \mathbf{5} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{3} & \mathbf{5} & \mathbf{1} \end{bmatrix}$

Solution :

Order of A is 3   2 and that of B is 2   3. Number of columns in A is 2 and number of rows in B is 2, therefore product AB is defined and order of AB is 3   3.

$$AB = \begin{bmatrix} 4 & 2 \\ 1 & 6 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$$

$$a_{11} = [1^{st} \text{ row of A}]\begin{bmatrix} 1^{st} \\ \text{column} \\ \text{of} \\ \text{B} \end{bmatrix} = [4 \quad 2]\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 4 \quad 1 + 2 \quad 3 = 10$$

$$a_{12} = [1^{st} \text{ row of A}]\begin{bmatrix} 2^{nd} \\ \text{column} \\ \text{of} \\ \text{B} \end{bmatrix} = [4 \quad 2]\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 4 \quad 2 + 2 \quad 5 = 18$$

$$a_{13} = [1^{st} \text{ row of A}] \begin{bmatrix} 3^{rd} \\ \text{column} \\ \text{of} \\ \text{B} \end{bmatrix} = [4 \quad 2]\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 4 \quad 4 + 2 \quad 1 = 18$$

$$a_{21} = [2^{nd} \text{ row of A}] \begin{bmatrix} 1^{st} \\ \text{column} \\ \text{of} \\ \text{B} \end{bmatrix} = [1 \quad 6]\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \quad 1 + 6 \quad 3 = 19$$

$$a_{22} = [2^{nd} \text{ row of A}] \begin{bmatrix} 2^{nd} \\ \text{column} \\ \text{of} \\ \text{B} \end{bmatrix} = [1 \quad 6]\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 1 \quad 2 + 6 \quad 5 = 32$$

$$a_{23} = [2^{nd} \text{ row of A}] \begin{bmatrix} 3^{rd} \\ \text{column} \\ \text{of} \\ \text{B} \end{bmatrix} = [1 \quad 6]\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 1 \quad 4 + 6 \quad 1 = 10$$

$$a_{31} = [3^{rd} \text{ row of A}] \begin{bmatrix} 1^{st} \\ \text{column} \\ \text{of} \\ \text{B} \end{bmatrix} = [3 \quad 5]\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3 \quad 1 + 5 \quad 3 = 18$$

$$a_{32} = [3^{rd} \text{ row of A}] \begin{bmatrix} 2^{nd} \\ \text{column} \\ \text{of} \\ \text{B} \end{bmatrix} = [3 \quad 5]\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 3 \quad 2 + 5 \quad 5 = 31$$

$$a_{33} = [3^{rd} \text{ row of A}] \begin{bmatrix} 3^{rd} \\ \text{column} \\ \text{of} \\ \text{B} \end{bmatrix} = [3 \quad 5]\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 3 \quad 4 + 5 \quad 1 = 17$$

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$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 4\times1+2\times3 & 4\times2+2\times5 & 4\times4+2\times1 \\ 1\times1+6\times3 & 1\times2+6\times5 & 1\times4+6\times1 \\ 3\times1+5\times3 & 3\times2+5\times5 & 3\times4+5\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 18 & 18 \\ 19 & 32 & 10 \\ 18 & 31 & 17 \end{bmatrix}$$

Illustration 10

If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$ find AB and BA and show that AB $\neq$ BA.

Solution :

A is a 2   3 matrix and B is a 3   2 matrix

$\therefore$ AB is defined and it will be a 2   2 matrix.

Now $$AB = A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-8+6 & 3-10+3 \\ -8+8+10 & -12+10+5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

Since B is a 3   2 matrix and A is a 2   3 matrix

$\therefore$ BA is defined and it will be a 3   3 matrix.

Again $$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Clearly AB $\neq$ BA.

Illustration 11

If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$ and $A^2 - aA + bI = 0$, find a and b.

Solution :

$$A^2 = A.A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times 4 + 3\times 2 & 4\times 3 + 3\times 5 \\ 2\times 4 + 5\times 2 & 2\times 3 + 5\times 5 \end{bmatrix} = \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix}$$

$$-aA = -a\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} -4a & -3a \\ -2a & -5a \end{bmatrix}$$

$$bI = b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$$

Now, $A^2 - aA + bI = 0$

$$\Rightarrow \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} + \begin{bmatrix} -4a & -3a \\ -2a & -5a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 22 - 4a + b & 27 - 3a \\ 18 - 2a & 31 - 5a + b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $22 - 4a + b = 0,\ 27 - 3a = 0$

$18 - 2a = 0$ and $31 - 5a + b = 0$

Solving them, we get : $a = 9$ and $b = 14$

Illustration 12

Find the value of x if $\begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0$

Solution :

$$\begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1\times 1 + x\times 2 + 1\times 15 & 1\times 3 + x\times 5 + 1\times 3 & 1\times 2 + x\times 1 + 1\times 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{bmatrix} 2x+16 & 5x+6 & x+4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0$$

$\Rightarrow \quad [(2x + 16) \quad 1 + (5x + 6) \quad 2 + (x + 4) \quad x] = 0$

$\Rightarrow \quad [2x + 16 + 10x + 12 + x^2 + 4x] = 0$

$\Rightarrow \quad [x^2 + 16x + 28] = 0$

$\Rightarrow \quad x^2 + 16x + 28 = 0$

$\Rightarrow \quad (x + 2)(x + 14) = 0$

$\Rightarrow \quad x = -2, -14$

$\therefore \quad x = -2 \quad$ or $\quad x = -14$

Illustration 13

If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$**, show that** $A^2 = \begin{bmatrix} \cos 2\theta & \cos 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$

Solution :

$A^2 = A.A$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & \cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta - \cos\theta\sin\theta & -\sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$

Illustration 14

If $f(\theta) = \begin{bmatrix} \cos x & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$**, show that** $f(x).f(y) = f(x + y)$

Solution :

$$f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad f(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f(x+y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $$f(x).\,f(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos x\cos y - \sin x\sin y & -\cos x\sin y - \sin x\cos y & 0 \\ \sin x\cos y + \cos x\sin y & -\sin x\sin y + \cos x\cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(x+y)$$

Illustration 15

If $\mathbf{A} = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ **then show that :** $\mathbf{I_2 + A = (I_2 - A)}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Solution :

Let $\tan\frac{\alpha}{2} = t$ then

$$\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{2t}{1+t^2}$$

Now, $$\text{L.H.S.} = I_2 + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$$

$$\text{R.H.S.} = (I_2 - A)\begin{bmatrix} \frac{1-t^2}{1+t^2} & \frac{-2t}{1+t^2} \\ \frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{bmatrix}$$

$$= \left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & -t\\t & 0\end{bmatrix}\right\}\begin{bmatrix}1-t^2 & -2t\\2t & 1-t^2\end{bmatrix}\frac{1}{1+t^2} = \begin{bmatrix}1-t^2 & -2t\\2t & 1-t^2\end{bmatrix}.\frac{1}{1+t^2}$$

$$= \begin{bmatrix}1-t^2+2t^2 & -2t+t-t^3\\-t+t^3+2t & 2t^2+1-t^2\end{bmatrix}\frac{1}{1+t^2} = \begin{bmatrix}1+t^2 & -t(1+t^2)\\t(1+t^2) & 1+t^2\end{bmatrix}\frac{1}{1+t^2}$$

$$= \begin{bmatrix}1 & -t\\t & 1\end{bmatrix}\frac{1+t^2}{1+t^2} = \begin{bmatrix}1 & -t\\t & 1\end{bmatrix}$$

$$\therefore \qquad I_2 + A = (I_2 - A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\\sin\alpha & \cos\alpha\end{bmatrix}$$

Illustration 16

Three shopkeepers A, B and C go to a store to buy stationary. A purchases 12 dozen notebooks, 5 dozen pens and 6 dozen pencils, B purchases 10 dozen notebooks, 6 dozen pens and 7 dozen pencils and C purchases 11 dozen notebooks, 13 dozen pens and 8 dozen pencils. A notebook costs 40 paise, a pen costs Rs. 1.25 and a pencil costs 35 paise. Use matrix multiplication to calculate each individual's bill.

Solution :

Cost of one dozen notebooks is Rs. 12 $\times$ 0.40 = Rs. 4.80, cost of one dozen pens is Rs. 12 $\times$ 1.25 = Rs. 15 and cost of one dozen pencils is Rs. 12 $\times$ 0.35 = Rs. 4.20.

Purchases madey by A, B and C can be represented by a 3 $\times$ 3 matrix X as

	Notebook	Pen	Pencil
A	12	5	6
B	10	6	7
C	11	13	8

$$X = \begin{bmatrix}12 & 5 & 6\\10 & 6 & 7\\11 & 13 & 8\end{bmatrix}$$

Cost per dozen can be represented by a 3 $\times$ 1 matrix Y as

$$Y = \begin{bmatrix}4.80\\15\\4.20\end{bmatrix}$$

Now, product XY will give the bills of A, B and C

$$\begin{bmatrix} \text{A's bill} \\ \text{B's bill} \\ \text{C's bill} \end{bmatrix} = XY = \begin{bmatrix} 12 & 5 & 6 \\ 10 & 6 & 7 \\ 11 & 13 & 8 \end{bmatrix} \begin{bmatrix} 4.80 \\ 15 \\ 4.20 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \quad 4.80 + 5 \quad 15 + 6 \quad 4.20 \\ 10 \quad 4.80 + 6 \quad 15 + 7 \quad 4.20 \\ 11 \quad 4.80 + 13 \quad 15 + 8 \quad 4.20 \end{bmatrix} = \begin{bmatrix} 157.80 \\ 167.40 \\ 281.40 \end{bmatrix}$$

$\therefore$ A's bill is of Rs. 157.80

B's bill is of Rs. 167.40

and C's bill is of Rs. 281.40.

Various Kinds of Matrices

(i) **Idempotent Matrix :** A square matrix A is called idempotent provided it satisfies the relation $\mathbf{A^2 = A}$.

Illustration 17

Show that the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{-2} & \mathbf{-4} \\ \mathbf{-1} & \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{-2} & \mathbf{-3} \end{bmatrix}$ **is idempotent.**

Solution :

$$A^2 = A.A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2.2 + (-2).(-1) + (-4).1 & 2(-2) + (-2).3 + (-4).(-2) & 2.(-4) + (-2).4 + (-4).(-3) \\ (-1).2 + 3.(-1) + 4.1 & (-1).(-2) + 3.3 + 4.(-2) & (-1).(-4) + 3.4 + 4.(-3) \\ 1.2 + (-2).(-1) + (-3).1 & 1.(-2) + (-2).3 + (-3).(-2) & 1.(-4) + (-2).4 + (-3).(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A.$$

Hence the matrix A is idempotent.

(ii) **Periodic Matrix :** A square matrix A is called periodic, if $\mathbf{A^{k+1} = A}$, where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then k is said to be period of A. For k = 1, we get $A^2 = A$ and we called it to be **idempotent matrix.**

(iii) **Nilpotent Matrix :** A square matrix A is called Nilpotent matrix of order m provided it satisfies the relation $A^k = 0$ and A^{k-}

$^{1} \neq 0$, where k is positive integer and 0 is null matrix and k is the order of the nilpotent matrix A.

Illustration 18

Show that $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{5} & \mathbf{2} & \mathbf{6} \\ \mathbf{-2} & \mathbf{-1} & \mathbf{-3} \end{bmatrix}$ **is nilpotent matrix of order 3.**

Solution : Let

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$\therefore \quad A^3 = A^2.A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+10 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-37 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad A^3 = 0 \qquad$ i.e., $\qquad A^k = 0$

Hence A is nilpotent of order 3.

(iv) **Involutory Matrix :** A square matrix A is called involutory provided it satisfies the relation $\mathbf{A^2 = I}$**,** where I is identity matrix.

Illustration 19

Show that the matrix $\mathbf{A} = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ **is involutory.**

Solution :

$$A^2 = A.A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Hence the given matrix A is involutory.

Transpose of a matrix

Let A be any matrix then the matrix obtained by interchanging its rows and columns is called the transpose of AO and is denoted by A' or A^T. If A is a m n matrix then A' will be a n m matrix.

Example : $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \end{pmatrix}$, **then** $\mathbf{A'} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 5 \end{bmatrix}$

Note : If $A = |a_{ij}|$; $\quad i = 1, 2, ..., m$; $j = 1, 2, ..., n$

Then $\mathbf{A' = [a_{ji}]}$; $\quad j = 1, 2, ... n$; $i = 1, 2, ..., m$

Property of transpose of Matrices :

Property I. $(A + B)' = A' + B'$

Property II. If A is any matrix, then $\mathbf{(A')' = A}$

Property III. If k is any number real or complex and A be any matrix, then $(kA)' = kA'$

Property IV. If A be a m n matrix and B be a n p matrix, then $\mathbf{(AB)' = B'A'}$.

Symmetric Matrices :

A square matrix $A = [a_{ij}]$ will be called symmetric it for all values of i and j, $a_{ij} = a_{ji}$.

i.e. every i-jth element = j-ith element.

e.g. $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}_{3 \times 3}$

Property A´ = A

Skew Symmetric Matrix :

A square matrix A = $[a_{ij}]$ wil be called skew symmetric if its i-jth element is – ive of j-ith element for all values of i and j i.e. an = an for all values of i and j.

Since diagonal elements will be of the type $a_{11}.a_{22}.a_{33}$, ... a_n and by given condition $a_n = -a_n$ for all values of i

or $2a_{ii} = 0$ ∴ $a_{ii} = \mathbf{0}$

Hence the diagonal elements of a skew symmetric matrix are zero.

e.g. $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ is a skew symmetric matrix.

Property : A´ = – A.

Some results related to symmetric and skew symmetric matrices :

(i) If A is any square matrix, then A + A´ is a symmetric matrix and A – A´ is a skew symmetric matrix.

A – A´ is a skew symmetric matrix.

Proof : (A + A´) = A´ + (A´) = A´ + A + A´ [∴ (A + B)´ (A´ + B´]

Hence A + A´ is a symmetric matrix.

Again (A – A´) = A´ – (A´) = A´ – A = – (A – A´)

Hence A – A´ is a skew symmetric matrix

(ii) **Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew symmetric matrix.**

Proof : Let A be any square matrix. Then as in (i) $\frac{1}{2}$ (A + A´) will be a symmetric matrix and $\frac{1}{2}$ (A – A´) will be a skew symmetric matrix.

Let $B = \frac{1}{2}(A + A')$ and $C = \frac{1}{2}(A - A')$

Then $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = B + C$

where B is a symmetric matrix and C is a skew symmetric matrix.

To prove that the representation is unique :

If possible let A = D + E where D is a symmetric and E is a skew symmetric matrix. Then D´ = D and E´ = – E. ...(1)

Now $A = D + E \Rightarrow A' = D' + E' = D - E$ [From (1)]

$$\left.\begin{aligned} \text{Thus } A &= D + E \\ \text{and } A' &= D - E \end{aligned}\right\} \text{ and } E = \frac{1}{2}(A - A') = C.$$

Illustration 20

Express A as the sum of a symmetric and a skew symmetric matrix, where $\mathbf{A} = \begin{pmatrix} \mathbf{4} & \mathbf{2} & \mathbf{-3} \\ \mathbf{1} & \mathbf{3} & \mathbf{-6} \\ \mathbf{-5} & \mathbf{0} & \mathbf{-7} \end{pmatrix}$

Solution :

We have $$A = \begin{pmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{pmatrix}$$

$\therefore$ $$A' = \begin{pmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{pmatrix}$$

then $$A + A' = \begin{pmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{pmatrix} + \begin{pmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & 0 \end{pmatrix} \quad \text{...(1)}$$

and $$A - A' = \begin{pmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{pmatrix} - \begin{pmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 14 \end{pmatrix} \quad \text{...(2)}$$

Adding (1) and (2), we get

$$2A = \begin{pmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 14 \end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix} 4 & 3/2 & -4 \\ 3/2 & 3 & -3 \\ -4 & -3 & 0 \end{pmatrix}}_{\text{Symmetric matrix}} + \underbrace{\begin{pmatrix} 0 & 1/2 & 1 \\ 1/2 & 0 & -3 \\ -1 & 3 & 7 \end{pmatrix}}_{\text{Skew symmetric matrix}}$$

DETERMINANTS

Determinant is a number associated with every square matrix, i.e. the number of rows and columns are equal.

Determinant of a square matrix $A = [a_{ij}]$ is denoted by $|A| = |a_{ij}|$ or $\Delta = |A|$

$$\text{or} \qquad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Determinant of order 2 :

Order 2 means 2 rows & 2 columns.

$$\text{i.e.} \qquad |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

denotion a_{ij} means number associated with i^{th} row & j^{th} column.

Also notice in order 2 there are 2   2 = 4 numbers

Determinant of order 3 :

order = 3 rows   3 columns

$$\text{i.e.} \qquad |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Value of determinant of order 2 :

$$\text{for} \qquad |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= (a_{11} \quad a_{22} - a_{21} \quad a_{12})$$

i.e. cross multiply and subtract

or oder 3 :

$$\text{for} \qquad |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Here there are 2 possibilities of expanding the determinant i.e. finding its value.

One is by expanding row & the other is by expanding column.

We will solve first by expanding row.

$$|A| = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

we can achieve this by
leaving the elements of
that row & column.
Similarly for a_{12} & a_{13}.

Sign Convention :

Why there is (–) sign for a_{12} ?

How to decide :

1. multiply the term with $(-1)^{i+j}$

for example for a_{11}, it is $(-1)^{1+1} = 1$

for a_{12}, it is $(-1)^{1+2} = 1$ that is the reason for – 1 sign.

for a_{13}, it is $(-1)^{1+3} = 1$

or other way to learn sign convention is by alternate +, –

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Expanding by column

For $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

though we will generally use expension by row method.

Illustration 21

Find the value of the determinant $\Delta = \begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{3} & \mathbf{4} & \mathbf{9} \\ \mathbf{2} & \mathbf{1} & \mathbf{6} \end{vmatrix}$

Solution :

Expanding the determinant along the first row

$$\Delta = 1\begin{vmatrix}4 & 9\\1 & 6\end{vmatrix} - 2\begin{vmatrix}3 & 9\\2 & 6\end{vmatrix} + 4\begin{vmatrix}3 & 4\\2 & 1\end{vmatrix}$$

$= 1\ (24 - 9) - 2\ (18 - 8) + 4\ (3 - 8)$

$= 15 - 2 \quad 0 + 4 \quad (-5) = -5$

Illustration 22

Find the value of the determinant $\Delta = \begin{vmatrix}\mathbf{3} & \mathbf{1} & \mathbf{7}\\ \mathbf{5} & \mathbf{0} & \mathbf{2}\\ \mathbf{2} & \mathbf{5} & \mathbf{3}\end{vmatrix}$

Solution :

Expanding the determinant along the second row,

$$\Delta = -5\begin{vmatrix}1 & 7\\5 & 3\end{vmatrix} + 0\begin{vmatrix}3 & 7\\2 & 3\end{vmatrix} - 2\begin{vmatrix}3 & 1\\2 & 5\end{vmatrix}$$

$= -5\ (3 - 35) + 0\ (9 - 14) - 2\ (15 - 2)$

$= 160 - 26 = 134$

Note : Since 5 is in second row and first column so, the sign before 5 is $(-1)^{2+1} = (-1)^3 = -1$ (minus). Similarly the sign before 0 is + (plus) and that before 2 is – (minus).

Minors and cofactors :

In the determinant

$$\Delta = \begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

if we leave the row and the column passing through the element a_{ij} then the second order determinant thus obtained is called the minor of a_{ij} and it is denoted by M_{ij}. Thus we can get 9 minors corresponding to the 9 elements.

For example, in determinant (i)

The minor of the element $a_{21} = \begin{vmatrix}a_{12} & a_{13}\\ a_{32} & a_{33}\end{vmatrix} = M_{21}$

The minor of the element $a_{32} = \begin{vmatrix}a_{11} & a_{13}\\ a_{21} & a_{23}\end{vmatrix} = M_{32}$

In terms of the notation of minors if we expand the determinant along the first row, then,

$$\Delta = (-1)^{1+1} a_{11}M_{11} + (-1)^{1+2} a_{12}M_{12} + (-1)^{1+3}a_{13}M_{13}$$

$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

Similarly expanding Δ along the second column. We have,

$$\Delta = a_1M_{12} + a_{22}M_{22} - a_{32}M_{32}$$

Cofactors : The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of the element aij

If we note the cofactor of the element a_{ij}, by A_{ij}, then

Cofactor of $a_{ij} = A_{ij} = (-1)^{i+j} M_{ij}$

Cofactor of the element $a_{21} = A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

Cofactor of the element $a_{32} = A_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

So the cofactor of an element aij = $(-1)^{i+j}$ the determinant obtained by leaving the row and the column passing through that element.

In terms of the notation of the cofactors.

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$

$$\Delta = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

Also $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$

$a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$ etc.

Therefore, in a determinant the sum of the products of the elements of any row or column with the corresponding cofactors is equal value of the dterminant. Also the sum of the products of the elements of any row or column with the cofactors of the corresponding elements of any other row or column is zero.

Illustration 23

Find the determinant of minors and cofactors of the determinant $\begin{vmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{7} & \mathbf{2} & \mathbf{-5} \\ \mathbf{8} & \mathbf{-1} & \mathbf{3} \end{vmatrix}$

Solution :

Here $M_{11} = \begin{vmatrix} 2 & -5 \\ -1 & 3 \end{vmatrix}$ (Delete 1st row and first column)

$= 6 - 5$

$M_{11} = 1$

$\therefore$ $C_{11} = 1(\therefore (-1)^{1+1} = 1)$

$M_{12} = \begin{vmatrix} 7 & -5 \\ 8 & 3 \end{vmatrix}$ (Delete 1st row and 2nd column)

$= 21 - (-40)$

$M_{12} = 61$

$\therefore \quad C_{12} = 61 (\because (-1)^{1+2} = -1)$

$M_{13} = \begin{vmatrix} 7 & 2 \\ 8 & -1 \end{vmatrix}$ (Delete 1st row and 3rd column)

$= 7 - 16$

$M_{13} = -23$

$\therefore \quad C_{13} = -23 (\because (-1)^{1+3} = 1)$

$M_{21} = \begin{vmatrix} 3 & 4 \\ -1 & 3 \end{vmatrix}$ (Delete 2nd row and 1st column)

$= 9 - (-4)$

$M_{21} = 13$

$\therefore \quad C_{21} = -13 (\because (-1)^{2+1} = -1)$

$M_{22} = \begin{vmatrix} 2 & 4 \\ 8 & 3 \end{vmatrix}$ (Delete 2nd row and 2nd column)

$= 6 - 32$

$M_{22} = -26$

$\therefore \quad C_{22} = -26 (\because (-1)^{2+2} = 1)$

$M_{23} = \begin{vmatrix} 2 & 3 \\ 8 & -1 \end{vmatrix}$ (Delete 2nd row and 3rd column)

$= -2 - 24$

$M_{23} = -26$

$\therefore \quad C_{23} = 26 (\because (-1)^{2+3} = -1)$

$M_{31} = \begin{vmatrix} 3 & 4 \\ 2 & -5 \end{vmatrix}$ (Delete 3rd row and 1st column)

$= -15 - 8$

$M_{31} = -23$

$\therefore \quad C_{31} = -23 (\because (-1)^{3+1} = 1)$

$M_{32} = \begin{vmatrix} 2 & 4 \\ 7 & -5 \end{vmatrix}$ (Delete 3rd row and 2nd column)

$= -10 - 28$

$M_{32} = -38$

$\therefore \quad C_{32} = 38 (\because (-1)^{3+2} = -1)$

and $M_{33} = \begin{vmatrix} 2 & 3 \\ 7 & 2 \end{vmatrix}$ (Delete 3rd row and 3rd column)

$= 4 - 21$

$M_{33} = -17$

$\therefore \quad C_{33} = -17 (\because (-1)^{3+3} = 1)$

Hence Determinants of Minors and Cofactors are :

$$\begin{vmatrix} 1 & 61 & -23 \\ 13 & -26 & -26 \\ -23 & -38 & -17 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & -61 & -23 \\ -13 & -26 & 26 \\ -23 & 38 & -17 \end{vmatrix} \text{ are respectively.}$$

Orthogonal Matrix :

A square matrix A is called an orthogonal matrix if the product of the matrix A and as transpose A´ is an identity matrix.

i.e. $AA' = I$

Note (i) If $AA' = I$ then $A^{-1} = A'$

Note (ii) If A and B are orthogonal then AB is also orthogonal.

Illustration 24

Verify that $\mathbf{A = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{pmatrix}}$ **is an orthogonal matrix.**

Solution :

Given $A = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{pmatrix}$

$$A' = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}$$

$$AA' = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{pmatrix} \times \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 1+4+4 & -2-2+4 & -2+4-2 \\ -2-2+4 & 4+1+4 & 4-2-2 \\ -2+4-2 & 4-2-2 & 4+4+1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1$$

Hence A is orthogonal matrix.

Adjoint of a Matrix :

If $A = [a_{ij}]$ be a n-**squared matrix** then the matrix $B = [b_{ij}]$ such that b_{ij} is the co-factor of the element a_{ji} in the determinant $|A|$ is called the adjoint of matrix A and is written as adj.A.

<u>In simple language we can say that adj.A is the transpose of the matrix formed by the co-factors of elements of $|A|$.</u>

Working rule for finding the adjoint of A.

Write down the determinant $|A|$ and the co-factors of various rows which will be columns of adj.A or replace each element in A by its co-factors and then take transpose to get adj.A.

Rule to write the cofactors of an element a_{ij}.

Cross the row and column intersecting at the element a_{ij} and the determinant which is left be denoted by D, then

$$\begin{bmatrix} \text{Cofactor of } a_{ij} = D & \text{if } i + j = \text{even} \\ \quad\quad = -D & \text{if } i + j = \text{odd} \end{bmatrix}$$

Illustration 25

$\mathbf{A} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ **find** adj.**A and show that** adj. (adj.**A**) **= A.**

Co-factor of α is δ and co-factor of β is $-\gamma$.

Co-factor of γ is $-\beta$ and co-factor of δ is α.

$\therefore$ Matrix formed by co-factors is $\begin{bmatrix} \delta & -\gamma \\ -\beta & \alpha \end{bmatrix}$ ...(1)

Adj. A = transpose of matrix (1)

$$= \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \quad ...(2)$$

You may see that δ, $-\gamma$ are the co-factors of first row of A and it forms the first column of adj.A, $-\beta$, α are the co-factors of 2nd row of A and it forms the second column of adj.A.

Rule for adjoint of 2 2 matrix : If A be 2 2 then adj.A is written by interchanging the elements of leading diagonal and changing the sign of the elemnts of other diagonal i.e. if $A = \begin{bmatrix} 3 & 4 \\ -5 & 7 \end{bmatrix}$ then adj.A = $\begin{bmatrix} 7 & -4 \\ 5 & 3 \end{bmatrix}$

...(3)

i.e. elements 3, 7 of leading diagonal have been interchanged and the sign of 4, -5 in the other diagonal have been changed.

Properties of adjoint A.

The product of a matrix and its adjoint is commutative.

(a) If A be n-rowed square matrix, then (adj.A) A = A (adj.A) = $|A|.I_n$

$$= |A|\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}|A|&0&0\\0&|A|&0\\0&0&|A|\end{bmatrix} \qquad ...(I)$$

where |A| is determinant A and I_n is the n-rowed unit matrix.

<u>Deduction</u> (a) If A is a n-squared singular matrix i.e., |A| = 0, then

A (adj.A) = (adj.A) = O (null matrix by (I)

A matrix is said to be singular if its determinant is zero i.e. |A| = 0

<u>Deduction</u> (b) **|adj.A| = |A|$^{n-1}$** if |A| is not zero.

If clearly follows from above on taking determinants of both sides in (I) (a) that

$|A|.\,|adjA| = |A|^n = |adjA|.|A|$

$|adj.A| = |A|^{n-1}$ provided |A| is not zero.

If |A| is not zero then A is said to be non-singular matrix.

Verification of the rule

(adj.**A) A = A (A**dj.**A) = |A|I$_n$**

---

if $$A = \begin{bmatrix}0&1&1\\1&2&0\\3&-1&4\end{bmatrix}$$

then $$\text{Adj } A = \begin{bmatrix}8&-5&-2\\-4&-3&1\\-7&3&-1\end{bmatrix}$$

Also |A| = – 11 by calculation

∴ A (adj.A) by actual multiplication

$$= \begin{bmatrix}-11&0&0\\0&-11&0\\0&0&-11\end{bmatrix} = \begin{bmatrix}|A|&0&0\\0&|A|&0\\0&0&|A|\end{bmatrix}$$

=diag[|A|.|A|.|A|]

Taking determinants of both sides, we get

|A adj. A| = |diag.[|A|, |A|, |A|]

or $|A||adj.A| = |A|^n$

∴ $|adj.A| = |A|^{n-1}$ provided

$|A| \neq 0$ i.e. A is non-singular.

(b) **Adj. (Adj.A) = |A|$^{n-2}$ A** if A be non-singular

Put Adj.A = B then

$B\,\text{adj}.B = |B|I_n = \text{diag.}[|B|, |B, ... |B|]$

$\therefore \quad A(B\,\text{adj}.B) = A\,\text{diag.}[|B|, |B| ... |B|]$

$= |B|A$

Now put B = adj.A in (I)

$\therefore \quad |B| = |\text{adj}A| = |A|^{n-1}$

$\therefore \quad A(\text{adj}.A)\,\text{adj}.(\text{adj}.A) = |A|^{n-1}A$

$|A|\,I_n\,\text{adj}.(\text{adj}.A) = |A|^{n-1}A$

$\therefore \quad \text{adj}.(\text{adj}.A) = |A|^{n-2}A$

Particular Case : If A be 3 3 matrix, then

$\text{adj}.(\text{adj}.A) = |A|^{n-2}A$

$= |A|^{3-2}A = |A|.A$

Minor of 2 = $\begin{vmatrix} -2 & 5 \\ 0 & 6 \end{vmatrix} = -12 - 0 = -12$

Cofactor of 2 = $(-1)^{1+2}(-12) = 12$ [$\therefore$ 2 occurs in the 1st row and 2nd column]

Inverse or reciprocal of a square matrix

Let A be a square matrix of order n. Then a matrix B (if such a matrix exists) is called the inverse of A if $AB = BA = I_n$
Inverse of the square matrix A is denoted by $\mathbf{A^{-1}}$.

Existence of the inverse :

The inverse of a square matrix A exists if and only if A is a non-singular matrix.

If part : Let A be non-singular square matrix of order n. Then $|A| \neq 0$

Let $B = \dfrac{\text{adj}.A}{|A|}$

Then $AB = \dfrac{A(\text{adj}\,A)}{|A|} = \dfrac{|A|I_n}{|A|} = I_n \left[\because A.(\text{adj}\,A) = |A|I_n\right]$ ...(1)

Hence B i.e. $\dfrac{\text{adj}\mathbf{.A}}{|\mathbf{A}|}$ **is the inverse of matrix A** (by definition of inverse)

Only if part : Let A be a square matrix of order n. Let inverse of A exist. Let B be the inverse of A.

Then by definition of inverse

$AB = I_n \Rightarrow |AB| = |In| = 1$

or $\quad |A|\,|B| = 1 [\therefore |AB| = |A|\,|B|]$

$\therefore \quad |A| \neq 0$, because product $|A|\,|B|$ is non-zero.

Hence A is non singular.

Note : (i) $A^{-1} = \dfrac{\text{adj}.A}{|A|}$ **(ii)** $AA^{-1} = I_n$ [From (1)]

Theorems :

(i) If A and B be any two non-singular matrices, then AB is also a non-singular matrix and $\mathbf{(AB)^{-1} = B^{-1} A^{-1}}$.

$\therefore$ A, B are non-singular $\therefore$ $|A| \neq 0$, $|B| \neq 0$

$\therefore$ $|AB| = |A|\ |B| \neq 0$ Hence AB is non-singular.

Now AB $(B^{-1} A^{-1})$

$= A\{B(B^{-1} A^{-1})\} = A\{BB^{-1})A^{-1}\}$ [by associative law]

$= A\{I_n A^{-1}\}$ $\{\because BB^{-1} = I_n]$

$= AA^{-1}$ $[\because I_n A^{-1} = A^{-1}]$

$= I_n$

Hence $B^{-1} A^{-1}$ is the inverse of AB

$\therefore$ $(AB)^{-1} = B^{-1} A^{-1}$

(ii) If A is a non singular matrix, then $(A^{-1})^{-1} = A$

Let A be a square matrix of order n,

Then $A^{-1}A = In$ $\therefore$ inverse of $A^{-1} = A$ $\therefore$ $(A^{-1})^{-1} = A$

(iii) $I_n^{-1} = I_n$ as $I_n^{-1} I_n = I_n$

Complex Conjugate (or Conjugaate) or a Matrix

If a matrix A is having complex numbers as its elements, the matrix obtained from A by replacing each element of A by its conjugate $(\overline{a \pm ib} = a \mp ib)$ is called the conjugate of matrix A and is denoted by A.

For example :

If $$A = \begin{bmatrix} a+ib & x-iy & p-iq \\ \lambda - i\mu & \alpha + i\beta & \gamma + i\delta \\ r - is & \mu - iv & \theta + i\phi \end{bmatrix}_{3 \times 3}$$

then $$\overline{A} = \begin{bmatrix} a-ib & x+iy & p+iq \\ \lambda + i\mu & \alpha - i\beta & \gamma - i\delta \\ r + is & \mu + iv & \theta - i\phi \end{bmatrix}_{3 \times 3}$$

Note : If all elements of A are real then $\overline{A} = A$.

Properties of Complex Conjugate of a Matrix

(i) $(\overline{\overline{A}}) = A,$ i.e., conjugate of the conjugate of a matrix is the matrix itself.

(ii) $(\overline{A + B}) = \overline{A} + \overline{B},$ i.e., the conjugate of the sum of the two matrices is the sum of their conjugates.

(iii) $(\overline{kA}) = k\overline{A},$ where k is any number.

(iv) $(\overline{AB}) = \overline{A}.\overline{B},$ where A and B being conformable to multiplication.

Conjugate Transpose of a Matrix :

The conjugate of the transpose of a matrix A is called the conjugate transpose of A and is denoted by A^θ.

Thus, A^θ = Conjugate of $A' = (\overline{A'})$

If $$A = \begin{bmatrix} 2+4i & 3 & 5-9i \\ 4 & \alpha+i\beta & 3i \\ 2 & -5 & 4-i \end{bmatrix}$$

then $$A^- = (\overline{A'}) = \begin{bmatrix} 2-4i & 4 & 2 \\ 3 & \alpha-i\beta & -5 \\ 5+9i & -3i & 4+i \end{bmatrix}$$

Properties of Transposed Conjugate Matrix

(i) For any matrix A, $(\overline{A})' = (\overline{A'})$, i.e., the transposed conjugate of a matrix is equal to the conjugate of its transpose.

(ii) For any matrix A, $(A^\theta)^\theta = A$

(iii) If A and B are two matrices conformable to addition, then $(A + B)^\theta = A^\theta + B^\theta$

(iv) For a matrix A, $(k\,A)^\theta = kA^\theta$, where k is a scalar.

(v) If A and B are two matrices conformable to the product AB, then $(AB)^\theta = B^\theta A^\theta$

Hermitian Matrix :

A square matrix A such that $\overline{A'} = A$ is called Hermitian matrix, provided aij = $a_{ij} = \overline{a}_{ji}$ for all values of i and j or A^{-1} = A.

For example :

If $$A = \begin{bmatrix} a & \lambda-i\mu & \theta+i\phi \\ \lambda-i\mu & \beta & x+iy \\ \theta-i\phi & \lambda-iy & y \end{bmatrix}$$

$\therefore$ $$A' = \begin{bmatrix} a & \lambda-i\mu & \theta-i\phi \\ \lambda+i\mu & \beta & x-iy \\ \theta+i\phi & \lambda+iy & y \end{bmatrix}$$

$\Rightarrow$ $$(\overline{A'}) = \begin{bmatrix} a & \lambda+i\mu & \theta+i\phi \\ \lambda-i\mu & \beta & x+iy \\ \theta-i\phi & \lambda-iy & y \end{bmatrix} = A$$

Hence A is Hermitian.

Skew-Hermitian Matrix :

A square matrix A such that $\overline{A'} = -A$ is called skew-hermitian matrix, provided $a_{ij} = -\overline{a}_{ji}$ for all values of i and j for $A^{\theta} = -A$.

For example :

If $$A = \begin{bmatrix} 2i & 2-3i & -2+i \\ 2-3i & -i & 3i \\ 2+i & 3i & 0 \end{bmatrix}$$

$\therefore$ $$A' = \begin{bmatrix} 2i & 2-3i & 2+i \\ -2-3i & -i & 3i \\ -2+i & 3i & 0 \end{bmatrix}$$

$\therefore$ $$(\overline{A'}) = \begin{bmatrix} -2i & 2+3i & 2-i \\ -2+3i & i & -3i \\ -2-i & -3i & 0 \end{bmatrix} = \begin{bmatrix} 2i & -2-3i & -2+i \\ 2-3i & -i & 3i \\ 2+i & 3i & 0 \end{bmatrix}$$

$\therefore$ $(\overline{A'}) = -A$

or $A^{\theta} = -A$

Hence A is Skew-Hermitian Matrix.

Properties of Hermitian and Skew-Hermitian Matrix

(i) The diagonal elements of a Hermitian matrix are necessarily real.

(ii) The diagonal elements of a Skew-Hermitian matrix are either purely imaginary or zero.

(iii) Every square matrix (with complex elements) can be uniquely expressed as the sum of a Hermitian and Skew-Hermitian matrices.

i.e., $A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$

where $\frac{1}{2}(A + A^{\theta})$ and $\frac{1}{2}(A - A^{\theta})$ are Hermitian and Skew-Hermitian parts of A.

Illustration 26

Express A as the sum of a Hermitian and a Skew-Hermitian matrix where $A = \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix}$

Solution :

We have $$A = \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix}$$

$\therefore$ $$A' = \begin{bmatrix} 2+3i & -3-i & 3-2i \\ 2 & 7 & i \\ 5 & -3-i & 2+i \end{bmatrix}$$

$$(\overline{A'}) = \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix}$$

or $$A = \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix}$$

$\therefore$ $$A + A^{\theta} = \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix} + \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1+i & 8+2i \\ -1-i & 14 & 3-2i \\ 8-2i & 3+2i & 4 \end{bmatrix} \qquad ...(1)$$

and $$A - A^{\theta} = \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix} - \begin{bmatrix} 2-3i & -3+i & 3+2i \\ -3-i & 7 & -i \\ 3-2i & 3+i & 2-i \end{bmatrix}$$

$$= \begin{bmatrix} 6i & 5-i & 2-2i \\ -5-i & 0 & 3 \\ -2-2i & -3 & 2i \end{bmatrix} \qquad ...(2)$$

adding (1) and (2), we get

$$2A = \begin{bmatrix} 4 & -1+i & 8+2i \\ -1-i & 14 & 3-2i \\ 8-2i & +2i & 4 \end{bmatrix} + \begin{bmatrix} 6i & 5-i & 2-2i \\ -5-i & 0 & 3 \\ -2-2i & -3 & 2i \end{bmatrix}$$

Hence

$$A = \begin{bmatrix} 2 & -\frac{1}{2}+\frac{i}{2} & 4+i \\ -\frac{1}{2}-\frac{i}{2} & 7 & \frac{3}{2}-1 \\ 4-i & \frac{3}{2}+i & 2 \end{bmatrix} + \begin{bmatrix} 3i & \frac{5}{2}-\frac{i}{2} & 2-2i \\ -\frac{5}{2}-\frac{i}{2} & 0 & 3 \\ -1-i & -\frac{3}{2} & i \end{bmatrix}$$

Unitary Matrix :

A square matrix A is called a unitary matrix if $AA^{\theta} = I$, where I is an identity matrix and A^{θ} is the transposed conjugate of A.

Illustration 27

Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary.

Solution :

Let $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$\therefore$ $A' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$

$\Rightarrow$ $(\overline{A'}) = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

or $A^{\theta} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$\therefore$ $AA^{\theta} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence A is unitary matrix.

Properties of Unitary Matrix

(i) If A is unitary matrix, then A´ is also unitary.

(ii) If A is unitary matrix, then A^{-1} is also unitary.

(iii) If A and B are unitary matrices then AB is also unitary.

Illustration 28

If A.B.C are three matrices such that $\mathbf{A} = \begin{bmatrix} x & y & z \end{bmatrix}, \mathbf{B} = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \mathbf{C} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ **find ABC.**

Solution :

Since associative law holds for matrix multiplication, therefore A (BC) = (AB) C which can be written as ABC.

Now $\quad BC = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax+hy+gz \\ hx+by+fz \\ gx+yf+zc \end{bmatrix}$

$\therefore \quad A(BC) = [xyz]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax+hy+gz \\ hx+by+fz \\ gx+yf+zc \end{bmatrix}$

$= [x\,(ax + hy + gz) + y\,(hx + by + fz) + z\,(gx + yf + zc)]$

$= [ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz]$

Illustration 29

Find the transpose and adjoint of the matrix A, where $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{5} & \mathbf{0} \\ \mathbf{2} & \mathbf{4} & \mathbf{3} \end{bmatrix}$

Solution :

Ist part : $A' = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 5 & 4 \\ 3 & 0 & 3 \end{bmatrix}$

2nd part :

Let B be the matrix whose elements are cofactors of the corresponding elements of the matrix A. Then B =

$\begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}$

$\therefore \quad \text{adj. } A = B' = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$

Illustration 30

Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution :

Let B be the matrix whose elements are the cofactors of the corresponding elements of A. Then

$$B = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix} \qquad \therefore \qquad \text{adj } A = B' = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = 0\,(-1) + 1.8 + 2\,(-5) = -2$$

$$\therefore \qquad A^{-1} = \frac{\text{adj}\,A}{|A|} = \frac{\text{adj}\,A}{-2} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Illustration 31

Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ **and verify that $A^{-1} A = I$**

Solution :

Let B be the matrix whose elements are co-factors of the corresponding elements of A, then

$$B = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 6 & -3 \\ -13 & 9 & -1 \end{bmatrix} \qquad \therefore \qquad \text{adj } A = B' = \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1.2 + 2(-3) + 5.5 = 21$$

$$\therefore \quad A^{-1} = \frac{\text{adj}\, A}{|A|} = \frac{\text{adj}\, A}{21} = \begin{bmatrix} 2/21 & 3/21 & -13/21 \\ -3/21 & 6/21 & 9/21 \\ 5/21 & -3/21 & -1/21 \end{bmatrix}$$

Also $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$$\therefore \quad A^{-1}A = \begin{bmatrix} \frac{2+6+13}{21} & \frac{4+9-13}{21} & \frac{10+3-13}{21} \\ \frac{-3+12-9}{21} & \frac{-6+18+9}{21} & \frac{-15+6+9}{21} \\ \frac{5-6+1}{21} & \frac{10-9-1}{21} & \frac{25-3-1}{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Illustration 32

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, prove that $A^2 - 4A - 5I = 0$, hence obtain A^{-1}.

Solution :

$$A^2 = A.A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$\text{Now } A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Thus $A^2 - 4A - 5I = O$ [Here O is the zero matrix]

$\therefore \quad A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I = A^{-1}O = O$

or $\quad (A^{-1}A)\,A - 4\,(A^{-1}\,A) - 5A^{-1} = O$; or $IA - 4I - 5A^{-1} = O$

$$\therefore \quad 5A–1 = A - 4I = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} = \begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}$$

Elemlentary operations or elementary transformations of a matrix :

Any of the following operations is called an elementary transformation.

(i) Interchanging any two rows of the given matrix. This transformation is indicated by R_{ij}, if the ith row are interchanged.

or denoted by $R_i \longrightarrow R_j$

(ii) The multiplying every element of any row of the given matrix by a non-zero number. This transformation is indicated by $R_i(k)$, if the multiplication of the ith row by a constant $k(k \neq 0)$

or denoted by $R_i \longrightarrow k.R_j$

(iii) Addition of a constant multiple of the elements of any row to the corresponding elements of any other row. This transformation is indicated by $R_{ij}(k)$, if the addition of the ith row to the elements of the jth row multiplied by constant $k(k \neq 0)$.

or denoted by $R_i \longrightarrow R_i + kR_j$

Note : Similarly we can deline the three column operations. C_{ij} $(C_i \longleftrightarrow C_j)$. $C_i(k)$ $(C_i \to kC_i)$ and $C_{ij}(k)$ $(C_i \to C_i + kC_j)$.

To compute the Inverse of a Matrix from elementary Row Transformation :

If A is reduced to I by elementary row (L.H.S.) transformation, then suppose I is reduced to P (R.H.S.) and not change A in R.H.S.

i.e., $A = I\,A$

After transformation $I = PA$

then P is the inverse of A

$\therefore \quad P = A^{-1}$

Solution of simultaneous linear equations (Matrix Method)

Let us consider the following system of n linear equations in n unknowns $x_1, x_2, ..., x_n$

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{array}\right\} \qquad \dots(1)$$

If $b_1 = b_2 = \dots = b_n = 0$, then the system of equations (1) is called a system of homogeneous linear equations and if at least one of $b_1, b_2 \dots, b_n$ is non zero, then it is called a system of non homogeneous linear equations.

Solution of a system of equations :

A set of values $\alpha_1, \alpha_2, \dots, \alpha_n$ of $x_1, x_2, \dots, x_n$ respectively which satisfy all the equations of the given system of equations is called a solution of the given system of equations.

The system of equations is said to be consistent if its solution exist otherwise it is said to be inconsistant.

Let $$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ b_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then the system of equations (1) can be written as AX = B

Here A is a n n matrix and is called the coefficient matrix. Each of X and B is a n 1 matrix.

I. **<u>When system of equations is non-homogeneous</u> :**

(i) If $|A| \neq 0$, the system of equations is consistent and has a unique solution given by $X = A^{-1} B$

Proof : Since $|A| \neq 0$, therefore A^{-1} exist.

Now $AX = B$ \ $A^{-1}(AX) = A^{-1} B$

or, $(A^{-1} A) X = A^{-1}B$ [by associative law for matrix multiplication]

or $I_n X = A^{-1}B$

or $X = A^{-1}B$ [$\because I_n X = X$]

(ii) If $|A| = 0$, the system of equations has no solution or an infinite number of solutions according as (adj A).B is non-zero or zero.

II. **<u>When system of equations is homogeneous</u> :**

(i) If $|A| \neq 0$, the system of equations has only trivial solution and number of solutions is one.

(ii) If $|A| = 0$, the system of equations has non-trivial solution and the number of solutions is infinite.

If the system of homogeneous linear equations has number of equations less than the number of unknowns, then it has non-trivial solution.

Illustration 34

Solve the following equations by matrix method :

$5x + 3y + z = 16$

$2x + y + 3z = 19$

$x + 2y + 4z = 25$

Solution :

Let $A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 16 \\ 19 \\ 15 \end{bmatrix}$

Then the matrix equation of the given system of equations become AX = B

Now $|A| = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} = 5(4-6) - 3(8-3) + 1(4-1) = -22 \neq 0$

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1}B$

Let C be the matrix whose elements are the cofactors of the corresponding elements of A, then

$$C = \begin{bmatrix} -2 & -5 & 3 \\ -10 & 19 & -7 \\ 8 & -13 & -1 \end{bmatrix} \quad \therefore \text{adj } A = C' = \begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-22}\begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2/22 & 10/22 & -8/22 \\ 5/22 & -19/22 & 13/22 \\ -3/22 & 7/22 & 1/22 \end{bmatrix}\begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \quad \therefore x = 1, y = 2, z = 5$$

Illustration 35

Find the product of two matrices A and B where $\mathbf{A} = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ **and use it for solving the equations**

$x + y + 2z = 1$
$3x + 2y + z = 7$
$2x + y + 3z = 2$

Remark. For reducing a 3   3 matrix, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ to identify matrix by using elementary row transformation, it is advisable to follow the order shown below :

$$\begin{bmatrix} a & b & c \\ \text{i} & \text{ix} & \text{viii} \\ d & e & f \\ \text{ii} & \text{iii} & \text{vii} \\ g & h & i \\ \text{iv} & \text{v} & \text{vi} \end{bmatrix}$$

(i) first reduce a to 1,

(ii) then reduce d to 0,

(iii) then reduce e to 1 (by dividing),

iv) then reduce g to 0 (by using R_1),

(v) then reduce h to 0 (by using R_2),

(vi) then reduce i to 1 (by dividing),

(vii) then reduce f to 0 (by using R_3),

(viii) then reduce c to 0 (by using R_2).

Illustration 33

Find the inverse of the matrix, $A \begin{bmatrix} 0 & 0 & -1 \\ 3 & 4 & 5 \\ -2 & -4 & -7 \end{bmatrix}$ by using elementary row transformations.

Solution : We have

$$A = I_3A$$

$$\Rightarrow \qquad \begin{bmatrix} 0 & 0 & -1 \\ 3 & 4 & 5 \\ -2 & -4 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \leftrightarrow R_3$, we get

$$\begin{bmatrix} -2 & -4 & -7 \\ 3 & 4 & 5 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_2$, we get

$$\begin{bmatrix} 1 & 0 & -2 \\ 3 & 4 & 5 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Quizrr

Applying $R_2 \to R_2 - 3R_1$, we get

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 11 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -2 & -3 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_2 \to \frac{1}{4}R_2$, we get

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{11}{4} \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -\frac{2}{3} & -\frac{3}{4} \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_3 \to -R_3$, we get

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{11}{4} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -\frac{2}{4} & -\frac{3}{4} \\ -1 & 0 & 0 \end{bmatrix} A$$

Applying $R_2 \to R_2 - \frac{11}{4}R_3$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ \frac{11}{4} & -\frac{2}{4} & -\frac{3}{4} \\ -1 & 0 & 0 \end{bmatrix} A$$

Applying $R_1 \to R_1 + 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ \frac{11}{4} & -\frac{2}{4} & -\frac{3}{4} \\ -1 & 0 & 0 \end{bmatrix} A$$

$$\Rightarrow \qquad A^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ \frac{11}{4} & -\frac{2}{4} & -\frac{3}{4} \\ -1 & 0 & 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$$

Solution : $AB = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -5+3+6 & -5+2+3 & -10+1+9 \\ 7+3-10 & 7+2-5 & 14+1-15 \\ 1-3+2 & 1-2+1 & 2-1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 4I_3 \qquad \text{....(1)}$$

Also given system of equations in matrix from is BX = C ....(2)

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$

From (2), $X = B^{-1}C$ [multiplying both sides of (2) by B^{-1} as $B^{-1}B = I$]

From (1), $AB = 41_3 \quad \therefore \frac{A}{4} . B = I_3$

$$\therefore \quad B^{-1} = \frac{A}{4} = \begin{bmatrix} -5/4 & 1/4 & 3/4 \\ 7/4 & 1/4 & -5/4 \\ 1/4 & -1/4 & 1/4 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{5}{4}+\frac{7}{4}+\frac{6}{4} \\ \frac{7}{4}+\frac{7}{4}-\frac{10}{4} \\ \frac{1}{4}-\frac{7}{4}+\frac{2}{4} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$\therefore \quad x = 2,\ y = 1,\ z = -1$

Rank of a matrix

Consider the matrix $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}_{2 \times 3}$

Above is 2 3 matrix.

Sub-matrix of order *r* :

If we retain any r rows and equal number of r columns we will have a square submatrix of order r whose determinant is called minor of order r.

In the above mat~~rix we cannot have a square sub matrix of order 3 because there are no 3 rows though 3 columns are there. We~~ can at the most have minors of order 2 say $\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix}, \begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix}$

or we can have minors of order 1 which means each element of the matrix is a minor of order 1. Similarly if a matrix A is 3 4 then we can have minors of order 3, 2 and 1 only.

The rank of a given matrix A is said to be *r* if

(a) Every minor of A of order r + 1 is zero.

(b) There is at least one minor of A of order r which does not vanish.

The rank r of matrix A is written as $\rho(A) = r$

Note : From above it clearly follows that the rank of a null matrix i.e., zero matrix all of whose elements are zeroes is zero.

Also the rank of a **singular square matrix** of order n cannot be n because minor of highest order i.e., $|A| = 0$ and there is only one minor of highest order.

Working rule :

Calculate the minors of highest possible order of a given matrix A. If it is not zero then the order of the minor is the rank. If it is zero and all other minors of the same order be also zero then calculate minors of next lower order and if at least one of them is not zero then this next lower order will be the rank. If, however, all the minors of next lower order are zero then calculate minors of still next lower order and so on.

Illustration 36

Find the rank of the matrix $\mathbf{A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}}$

Solution :

First method :

$$|A| = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix} = 1\,(40 - 42) - 2\,(20 - 21) + 3\,(12 - 12) = 0$$

There will be square submatrices of A of order 2. Now we consider the determinants of these submatrices.

(i) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 4 - 4 = 0$ (ii) $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = 7 - 6 = 1 \neq 0$

Thus rank of A = 2

[Because there is at least one square submatrix of order 2 whose determinant is non-zero and determinants of all square submatrices of A of order greater than 2 are zero. Only one such matrix exist and that is the matrix A itself.

Second method : We put A in echelon form

Now $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{cases} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{cases}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} [R_3 \to R_3 - R_2]$$

This is the echelon form of matrix A.

$\therefore$ Rank A = number of non-zero rows in echelon form = 2

Solution of Equations

We have already introduced the following terms in relation to solution of equations.

i.e. Consistent (having solution). Inconsistent (having no solution), unique (only one solution) infinite (many solutions). Trivial (all variables zero i.e. unique.

Representation of equations in matrix form

I. Intersecting lines. Unique solution. Consistent

$x + 2y = 4$

$3x + y = 2$

or $$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \text{ or } AX = B$$

A is called **coefficient matrix** and $C = [A, B] = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix}$ is 2 3 matrix. C is called **augmented matrix.**

Nature of solution

(a) Rank A = Rank C = n = 2 the number of unknown. Solution is unique and is obtained as under.

Here $|A| = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = 1 - 6 = -5 \neq 0$

$\therefore$ A is non-singular and A^{-1} exists and

$$A^{-1} = \frac{1}{|A|}(\text{adj.}A) = \frac{1}{-5}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

Multiplying both sides by A^{-1} we have

$A^{-1}(AX) = A^{-1}B$ or $X = A^{-1}B$

or $\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$$= -\frac{1}{5}\begin{bmatrix} 0 \\ -10 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$\therefore$ $x = 0$

$y = 2$

2nd Method by Cramer's rule

$$\frac{x}{D_1} = \frac{y}{D_2} = \frac{1}{D}$$

or $\dfrac{x}{\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix}} = \dfrac{y}{\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}} = \dfrac{1}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}}$ or $\dfrac{x}{0} = \dfrac{y}{-10} = \dfrac{1}{-5}$

(b) Homogeneous equations. Unique solution. Trivial.

$x + 2y = 0$

$3x + 2y = 0$ or $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$AX = B$

$$C = [A, B] = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

Here as above Rank A = Rank C = n = 2 the number of unknown variables and hence the solution will be unique and is given by $X = A^{-1} B$

or $\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ $x = 0, y = 0$

II. Coincident lines. Infinite solutions. Consistent

(c) $x + 2y = 4$

$3x + 6y = 12$

or $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$

or AX = B $C = [A, B] = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 6 & 12 \end{bmatrix}$

Here $|A| = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = 0$

so that matrix A is singular and A^{-1} will not exist

Apply $R_2 - 3R_1$

$$C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Clearly Rank C = Rank A = 1 which is less than n = 2 the number of variables and hence the system of equations are consistent and will have infinite solutions. As a matter of fact we have only one equation on two variables x + 2y = 4 (The second equation on dividing by 3 becomes the same as x + 2y = 4)

$\therefore$ We can have infinite number of points on a line Choosing x = c we have $y = \dfrac{4 - c}{2}$

(d) Homogeneous equations

x + 2y = 0

3x + 6y = 0 or $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or Ax = B

Arguing as in (c) Rank C = Rank A = 1, $|A| = 0$. As a matter of fact we have only one equation x + 2y = 0. Rank C = Rank A = 1, < n = 2 the number of variables.

Choosing y = c, x = – 2c $\therefore$ (c, – 2c) constitute infinite solutions.

III. Parallel lines. Inconsistent. No solution.

(e) x + 2y = 4

3x + 6y = 7 or $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

or AX = B $C = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 6 & 7 \end{bmatrix}$

$|A| = 0$ i.e., A is singular.

Clearly Rank A = 1 but Rank C = 2

Rank A < Rank C $\therefore$ Inconsistent

As a matter of fact the two equations are $x + 2y = 4$ and $x + 2y = \frac{7}{3}$ which are clearly inconsistent and have no solution.

Gist

I. Rank A = Rank C = n the number of variables then equations are consistent and have unique solution. $A \neq 0$ sec (a), (b).

II. Rank A = Rank C = r < n the number of variables the equations are consistent and have infinite solutions, $|A| = 0$ sec (c), (d).

III. Rank A < Rank C

$\therefore$ Inconsistent, $|A| = 0$ sec (e).

Non-homogeneous equation : We will illustrate the method by two examples. We have discussed this topic in determinants. Here we illustrate the method by use of matrices.

Illustration 37

Solve the following equations

$5x - 6y + 4z = 15$

$7x + 4y - 3z = 19$

$2x + y + 6z = 46$

Solution :

The above equations can be written in matrix form as $\begin{bmatrix} 5 & -6 & 4 \\ 7 & 6 & -3 \\ 2 & 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$

or $AX = B$

$$|A| = D = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}$$

Apply $C_1 - 2C_2$, $C_3 - 6C_2$ to make two zeros in R_3

$$D = \begin{vmatrix} 17 & -6 & 40 \\ -1 & 4 & -27 \\ 0 & 1 & 0 \end{vmatrix}$$

$= -[-459 + 40] = 419 \neq 0.$

$\therefore$ The matrix A is non-singular or rank of matrix A is 3. We will have a unique solution and the equations are consistent.

By Cramer's rule.

$$\frac{x}{D_1} = \frac{y}{D_2} = \frac{z}{D_3} = \frac{1}{D}$$

where D_1 is obtained from D by replacing the first column of D by b's i.e., 15, 19, 46.

$$D_1 = \begin{vmatrix} 15 & -6 & 4 \\ 19 & 4 & -3 \\ 46 & 1 & 6 \end{vmatrix}$$

$$= 15\ (27) - 19\ (-40) + 46\ (2) = 1257$$

$$D_2 = \begin{vmatrix} 5 & 15 & 4 \\ 7 & 19 & -3 \\ 2 & 46 & 6 \end{vmatrix} = -\begin{vmatrix} 15 & 5 & 4 \\ 19 & 7 & -3 \\ 46 & 2 & 6 \end{vmatrix}$$

$$= -\{15\ (48) - 19\ (22) + 46\ (-43) = 1676$$

$$D_3 = \begin{bmatrix} 5 & -6 & 15 \\ 7 & 4 & 19 \\ 2 & 1 & 46 \end{bmatrix}$$

$$= 15\ (-1) - 19\ (17) + 46\ (62) = 2514$$

$\therefore \quad \dfrac{x}{1257} = \dfrac{y}{1676} = \dfrac{z}{2514} = \dfrac{1}{419}$

$\therefore \quad x = 3,\ y = 4,\ z = 6.$

2nd Method :

AX = B.

Since A is non-singular and hence its inverse exists. Multiplying both sides by A^{-1} we get

$A^{-1}\ AX = A^{-1}B \quad$ or $\quad IX = A^{-1}\ B$

or $\quad X = A^{-1}\ B$

Now $\quad A^{-1} = \dfrac{adj.A}{|A|}$ where

$$|A| = \begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix} = 419$$

Cofactors of elements of first row of |A| are 27, – 48, – 1

Cofactors of elements of 2nd row of |A| are 40, – 22, – 17

Cofactors of elements or 3rd row of |A| are 2, 43, 62

Above will be respective columns of adj.A

$\therefore$ $$\text{adj.A}=\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

$\therefore$ $$A^{-1}=\frac{\text{Adj.A}}{|A|}=\frac{1}{419}\begin{vmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{vmatrix}$$

$\therefore$ $X = A^{-1} B$

or $$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}\begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

or $$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{419}\begin{bmatrix} 27.15+40.19+2.46 \\ -48.15+22.19+43.46 \\ -1.15-17.19+62.46 \end{bmatrix}$$

$$=\frac{1}{419}\begin{bmatrix} 1257 \\ 1676 \\ 2514 \end{bmatrix}=\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$\therefore$ $x = 3,\ y = 4,\ z = 6$

Illustration 38

Solve the equations

$x + y + z = 6$

$x - y + z = 2$

$2x + y - z = 1$

Solution :

In matrix form the given equations are $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$ or $AX = B$

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

$\therefore \quad |A| = D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$

Apply $R_2 - R_1$, $R_3 - 2R_1$,

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & 3 \end{vmatrix} = 6 \neq 0$$

i.e. matrix A is non-singular.

We will have unique solution and the equations are consistent.

$\therefore$ By Cramer's rule,

$$\frac{x}{D_1} = \frac{y}{D_2} = \frac{z}{D_3} = \frac{1}{D}$$

where D_1 is obtained from D by replacing the first column of D by b's i.e., 6, 2, 1

$$\therefore \quad D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 6 & 1 & 1 \\ -4 & -2 & 0 \\ 7 & 2 & 0 \end{vmatrix}$$

By $R_2 - R_1$ and $R_3 + R_1$

$$\therefore \quad D_1 = \begin{vmatrix} -4 & -2 \\ 7 & 2 \end{vmatrix} = -8 + 14 = 6$$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 1 \\ 0 & -4 & 0 \\ 0 & -11 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} -4 & 0 \\ -11 & -3 \end{vmatrix} = 12$$

$$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 6 \\ 0 & -2 & -4 \\ 0 & -1 & -11 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & -4 \\ -1 & -11 \end{vmatrix} = 22 - 4 = 18$$

$$\therefore \quad \frac{x}{6} = \frac{y}{12} = \frac{z}{18} = \frac{1}{6}$$

$\therefore \quad x = 1, y = 2, z = 3.$

Above gives the unique solution.

Illustration 39

Investigate for what values of λ, μ the simultaneous equations

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$

or $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

have, (a) no solution, (b) a unique solution, (c) an infinite number of solutions.

Solution :

(a) For no solution

Rank A $\neq$ Rank C

(b) For unique solution i.e., coefficient matrix is non-singular.

Rank A = Rank C = n

(c) For infinite number of solutions,

Rank A = Rank C = r where r < n

Augmented Matrix

$$C = [A, B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix} \qquad \text{Apply } R_3 - R_2$$

or $$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix} \qquad \text{Apply } R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix}$$

Also $$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda - 3 \end{bmatrix}$$

(a) $\lambda = 3, \mu \neq 10.$

In this case Rank A = 2 whereas Rank C = 3.

$\therefore$ Rank A $\neq$ Rank C and hence no solution.

(b) In case $\lambda \neq 3$ then coefficient matrix A is non-singular Rank A = Rank C = 3, the number of variables. Hence we have a unique solution which can be found by Cramer's rule or by the help of inverse or by equivalent system of equations.

(c) In case $\lambda = 3$ and $\mu = 10$ then the ranks of both A and C will be $2 < n = 3$ and the equations will be consistent. We shall assign arbitrary value to $3 - 2 = 1$ variable and remaining $r = 2$ variables shall be found in terms of these.

The equivalent system of equations when $\lambda = 3$ and $\mu = 10$ are

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

or $x + y + z = 6,\ y + 2z = 4,\ 0 = 0$

Choose $z = k \quad \therefore \quad y = 4 - 2k$

$\therefore \quad x = 6 - y - z = 6 - (4 - 2k) - k = 2 + k.$

Homogeneous Equations :

Refer chapter of determinants. If $D = |A| \neq 0$ then the system of homogeneous equations have trivial solution i.e., $x = 0$, $y = 0$ and $z = 0$. If however $|A| = 0$ then the system of equations will have non-trivial solution. We will illustrate the same by following examples.

Illustration 40

Solve completely the equations

$x + 3y - 2z = 0$

$2x - y + 4z = 0$

$x - 11y + 14z = 0$

Solution :

$AX = O$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

Reduce the matrix to Echelon form by applying elementary row, column operation

$$|A| = \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix} \text{ by } R_2 - 2R_1,\ R_3 - R_1$$

$$= -112 + 112 = 0$$

Since $|A| = 0$ the system of equations with have non-trivial solution

Above is Echelon form of A and its rank is 2 the number of non-zero rows. Rank $A = r < n$ where $n = 3$. Hence the system has non-trivial solution. We shall assign arbitrary values to $n - r = 3 - 2 = 1$ variable and remaining $r = 2$ variables shall be found in terms of these.

The equivalent system of equations is $AX = O$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

$x + 3y - 2z = 0,\ -7y + 8z = 0$ and $-14y + 16z = 0$

The last two are identical.

Thus we have only two equations in three variables i.e., $x + 3y - 2z = 0$ and $y = 8z/7$.

We choose $z = k$ $\therefore$ $y = 8k/7$

and hence $x = 2z - 3y = 2k - \dfrac{24}{7}k = \dfrac{10}{7}k$

Illustration 41

Discuss for all values of k the system of equations

$x + (k + 4)y + (4k + 2)z = 0$ **or** $AX = O$

$2x + 3ky + (3k + 4)z = 0$

$x + 2(k + 1)y + (3k + 4)z = 0$

Solution :

The given equations can be written as $\begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$

$$|A| = \begin{vmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{vmatrix}$$

Apply $R_2 - 2R_1$, $R_3 - R_1$

$$\therefore \quad |A| = \begin{vmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{vmatrix}$$

$= (k - 8)(-k + 2) + 5k(k - 2)$

or $\quad -k^2 + 2k + 8k - 16 + 5k^2 - 10k = 4k^2 - 16$

Now $|A| = 0$ when $k^2 = 4$ $\therefore$ $k = \pm 2$

Hence when $k = \pm 2$ then $|A| = 0$ so that the matrix A is singular and hence the system will have non-trivial solution. But when $k \neq \pm 2$ then $|A| \neq 0$ that the system will have only trivial solution i.e. $x = 0, y = 0, z = 0$.

For $k = 2$ the equivalent system of equations will be $\begin{vmatrix} 1 & 6 & 10 \\ 0 & -6 & -10 \\ 0 & 0 & 0 \end{vmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$

Clearly rank of A is 2 and we will assign arbitrary values to $n - r$ i.e. $3 - 2 = 1$ variable and remaining $r = 2$ variables shall be found in terms of these

$x + 6y + 10z = 0,\ -6y - 10z = 0,\ 0 = 0$

Only two equations in three variables.

Choose $z = c$ $\therefore$ $y = -5c/3$ and $x = 0$

For $k = -2$ the equivalent system of equations will be

$$\begin{bmatrix} 1 & 2 & -6 \\ 0 & -10 & 10 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

or $$\begin{bmatrix} 1 & 2 & -6 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$x + 2y - 6z = 0$, $y - z = 0$ and $y - z = 0$

Choose $z = k$ $\quad\therefore\quad$ $y = k$ and $x = 4k$

PROPERTIES OF DETERMINANTS

Property I. The value of a determinant is not changed when rows are changed into corresponding columns. Naturally when rows are changed into corresponding columns, then columns will be changed into corresponding rows.

Proof : Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding the determinant along first row,

$$a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2) \qquad \text{...(i)}$$

Example : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix} = 1 (15 - 4) - 2 (10 - 0) + 3 (2 - 0) = -3$

and $\Delta' = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & 4 & 5 \end{vmatrix} = 1 (15 - 4) - 2 (10 - 3) + 0 (8 - 9) = -3$

Clearly $\quad \Delta' = \Delta$

Property II. If any two rows or columns of a determinant are interchanged, the sign of the determinant is changed, but its value remains the same.

Example : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix} = 1 (15 - 4) - 2 (10 - 0) + 3 (2 - 0) = -3$

and $\Delta' = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 0 & 1 & 5 \end{vmatrix} [R_1 > R_2]$

$= 2 (10 - 3) - 3 (5 - 0) + 4 (1 - 0) = 3$

Property III.

The value of a determinant is zero if any two rows or columns are identical

Example : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} = 1\,(9-8) - 2\,(6-4) + 3\,(4-3) = 0$

Property IV.

A common factor of all elements of any rwo (or of any column) may be taken outside the sign of the determinant. In other words if all the elements of the same row (or the same column) are multiplied by a certain number, then the determinant becomes multiplied by that number.

Example : $\begin{vmatrix} 32 & 24 & 16 \\ 8 & 3 & 5 \\ 4 & 5 & 3 \end{vmatrix} = 8 \begin{vmatrix} 4 & 3 & 2 \\ 8 & 3 & 5 \\ 4 & 5 & 3 \end{vmatrix}$ [taking 8 common from Ist row]

$= 8 \times 4 \begin{vmatrix} 1 & 3 & 2 \\ 2 & 3 & 5 \\ 1 & 5 & 3 \end{vmatrix}$ [taking 4 common from the Ist column]

Property V.

If every element of some column or (row) is the sum of two terms, then the determinant is equal to the sum of two determinants; one containing only the first term in place of each sum, the other only the second term. The remaining element of both determinants are the same as in the given determinant.

Proof : We have to prove that $\begin{vmatrix} a_1 + \alpha_1 & b_1 & c_1 \\ a_2 + \alpha_2 & b_2 & c_2 \\ a_3 + \alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$

Note : $\begin{vmatrix} a_1 + b_1 & c_1 + d_1 & e_1 \\ a_2 + b_2 & c_2 + d_2 & e_2 \\ a_3 + b_3 & c_3 + d_3 & e_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & e_1 \\ a_2 & c_2 & e_2 \\ a_3 & c_3 & e_3 \end{vmatrix} + \begin{vmatrix} a_1 & d_1 & e_1 \\ a_2 & d_2 & e_2 \\ a_3 & d_3 & e_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & e_1 \\ b_2 & c_2 & e_2 \\ b_3 & c_3 & e_3 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 & e_1 \\ b_2 & d_2 & e_2 \\ b_3 & d_3 & e_3 \end{vmatrix}$

Property VI.

The value of a determinant does not change when any row or column is multiplied by a number or an expression and is then added to or subtracted from any other row or column.

Here it should be noted that if the row or column which is changed is multiplied by a number, then the determinant will have to be divided by that number.

Example : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{vmatrix} = -7,\ \Delta' = \begin{vmatrix} 5 & 2 & 13 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{vmatrix} [R_1 \to R_1 + 2R_3]$

$= 5\ (15 - 0) - 2\ (10 - 8) + 13\ (0 - 6)$

$= 75 - 4 - 78 = -7$

$$\underset{R_1 \to 3R_1 + 2R_3}{\Delta' = \frac{1}{3}} \begin{vmatrix} 7 & 6 & 19 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{vmatrix}$$

[Here Δ'' has also been obtained from Δ applying $R_1 \to 3R_1 + 2R_3$]

$$= \frac{1}{3}[7\ (15 - 0) - 6\ (10 - 8) + 19\ (0 - 6)] = \frac{1}{3}(-21) = -7$$

In obtaining Δ'' from Δ, R_1 has been changed and R_1 has been multiplied by 3, therefore, the determinant has been divided by 3

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + mb_1 & b_1 & c_1 \\ a_2 + mb_2 & b_2 & c_2 \\ a_3 + mb_3 & b_3 & c_3 \end{vmatrix}$$

Evaluation of determinant :

While evaluating the detrminants we use some following notations :

For example :

(1) R_i denotes the i^{th} row & C_i denotes the i^{th} column

(2) $R_i \leftrightarrow R_j$ denotes the exchange of i^{th} and j^{th} row and $C_i \leftrightarrow C_j$ denotes the exchange of i^{th} & j^{th} column.

(3) $R_i \to R_i + \lambda R_j$ denotes the addition of λ times the element of j^{th} row.

Similarly $C_i \to C_i + \lambda C_j$

(4) $R_i \to \lambda R_i$ denotes the multiplication of all elements of i^{th} row by λ.

Trick : In evaluating determinants we always try to bring as many zeros as possible by using the properties we did earlier.

Product of two determinants

Let $\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

then product of Δ_1 and Δ_2 is defined as

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1x_1 + a_2x_2 + a_3x_3 & a_1y_1 + a_2y_2 + a_3y_3 & a_1z_1 + a_2z_2 + a_3z_3 \\ b_1x_1 + b_2x_2 + b_3x_3 & b_1y_1 + b_2y_2 + b_3y_3 & b_1z_1 + b_2z_2 + b_3z_3 \\ c_1x_1 + c_2x_2 + c_3x_3 & c_1y_1 + c_2y_2 + c_3y_3 & c_1z_1 + c_2z_2 + c_3z_3 \end{vmatrix}$$

Example : Let $\Delta_1 = \begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 5 \\ 0 & 2 & 1 \end{vmatrix} = \Delta_2 = \begin{vmatrix} 3 & 4 & -1 \\ 0 & -1 & 2 \\ -3 & 2 & 6 \end{vmatrix}$

Then $$\Delta_1\Delta_2 = \begin{vmatrix} 1.3 + 2.4 + 3(-1) & 1.0 + 2(-1) + 3.2 & 1(-3) + 2.2 + 3.6 \\ (-2).3 + 3.4 + 5(-1) & (-2).0 + 3(-1) + 5.2 & (-2)(-3) + 3.2 + 5.6 \\ 0.3 + 2.4 + 1(-1) & 0.0 + 2(-1) + 1.2 & 0(-3) + 2.2 + 1.6 \end{vmatrix}$$

$$= \begin{vmatrix} 8 & 4 & 19 \\ 1 & 7 & 42 \\ 7 & 0 & 10 \end{vmatrix}$$

Note : (i) Here we have multiplied rows by rows. Since value of a determinant does not change when rows and columns are interchanged, therefore, while finding the product of two determinants, we can also multiply rows by columns or columns by columns.

Note : (ii) $\Delta_1\Delta_2 = \Delta_2\Delta_1$ (since in a determinant rows and columns can be interchanged)

An important result

If $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ then $\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \Delta^2$

Where capital letters denote the cofactors of corresponding small letters in Δ i.e. A_i = cofactor of a_i, B_i = cofactor of b_i and c_i = cofactor of c_i in the determinant Δ. We have seen that

$a_1A_1 + a_2A_2 + a_3A_3 = \Delta$

$b_1B_1 + b_2B_2 + b_3B_3 = \Delta$

$c_1C_1 + c_2C_2 + c_3C_3 = \Delta$

and $a_1B_1 + a_2B_2 + a_3B_3 = 0$

$b_1A_1 + b_2A_2 + b_3A_3 = 0$

$a_1C_1 + a_2C_2 + a_3C_3 = 0$

$c_1A_1 + c_2A_2 + c_3A_3 = 0$

$b_1C_1 + b_2C_2 + b_3C_3 = 0$

$c_1B_1 + c_2B_2 + c_3B_3 = 0$

Case I. When $\Delta \neq 0$

Let $\Delta_1 = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

Now $\Delta\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

$$\begin{vmatrix} a_1A_1 + a_2A_2 + a_3A_3 & a_1B_1 + a_2B_2 + a_3B_3 & a_1C_1 + a_2C_2 + a_3C_3 \\ b_1A_1 + b_2A_2 + b_3A_3 & b_1B_1 + b_2B_2 + b_3B_3 & b_1C_1 + b_2B_2 + b_3C_3 \\ c_1A_1 + c_2A_2 + c_1A_3 & c_1B_1 + c_2B_2 + a_3B_3 & c_1C_1 + c_2C_2 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

Thus $\Delta\Delta_1 = \Delta^3 \quad \Rightarrow \quad \Delta_1 = \Delta^2 \quad [\because \Delta \neq 0]$

Case II. When $\Delta = 0$

Sub. Case (i) When all elements of the determinant Δ are zero. In this case cofactors of each element of the detreminant Δ will be zero and hence $\Delta_1 = 0$

$\therefore \quad \Delta_1 = \Delta^2 \qquad$ [Here $\Delta_1 = \Delta^2 = 0$]

Sub. Case (ii) When at least one element of the determinant Δ is non-zero. There is no loss of generality in assuming that $a_1 \neq 0$.

Now $$\left.\begin{aligned} a_1A_1 + a_2A_2 + a_3A_3 &= \Delta = 0 \\ a_1B_1 + a_2B_2 + a_3B_3 &= 0 \\ a_1C_1 + a_2C_2 + a_3C_3 &= 0 \end{aligned}\right\} \qquad ...(i)$$

(i) is system of homogeneous linear equations in a_1, a_2, a_3 where $a_1 \neq 0$, therefore it has nontrivial solution.

$$\therefore \quad \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = 0 \Rightarrow \Delta_1 = 0$$

Thus $\Delta_1 = \Delta^2 \quad [\therefore \Delta = 0 \text{ and } \Delta_1 = 0]$

Illustration 42

Show that $\begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{vmatrix} = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2$

Solution :

Let $D = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}$

Cofactors of Ist Row of D are

$x^2 + a^2$, $ab + cx$, $ac - bx$

Cofactors of 2nd Row of D are

$ab - cx$, $x^2 + b^2$, $ax + bc$

and cofactors of 3rd row of D are :

$ac + bx$, $bc - ax$, $x^2 + c^2$

$\therefore$ Determinant of cofactors of D is

$$D^c = \begin{vmatrix} x^2+a^2 & ab+cx & ac-bx \\ ab-cx & x^2+b^2 & ax+bc \\ ac+bx & bc-ax & x^2+c^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & ax+bc & c^2+x^2 \end{vmatrix} \quad \text{(Rows interchanging into columns)}$$

$$= D^2$$

$$= \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2 \quad (\because D^c = D^2, \text{ D is third order determinant})$$

Hence, $\begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & ax+bc & c^2+x^2 \end{vmatrix} = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2$

Express a Determinant into Product of two Determinants :

This method will be clear from few examples.

Illustration 43

Prove that $$\begin{vmatrix} \mathbf{2} & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & \mathbf{2}(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & \mathbf{2}\alpha\beta\gamma\delta \end{vmatrix} = \mathbf{0}$$

Solution :

$$\text{L.H.S.} = \begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ \alpha+\beta & \gamma+\delta & 0 \\ \alpha\beta+\gamma\delta & \gamma\delta & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ \gamma+\delta & \alpha+\beta & 0 \\ \gamma\delta & \alpha\beta & 0 \end{vmatrix} \qquad \text{(Row by Row)}$$

$= 0 \quad 0$

$= 0$

$=$ R.H.S.

Illustration 44

Prove that $$\begin{vmatrix} \cos(\mathbf{A}-\mathbf{P}) & \cos(\mathbf{A}-\mathbf{Q}) & \cos(\mathbf{A}-\mathbf{R}) \\ \cos(\mathbf{B}-\mathbf{P}) & \cos(\mathbf{B}-\mathbf{Q}) & \cos(\mathbf{B}-\mathbf{R}) \\ \cos(\mathbf{C}-\mathbf{P}) & \cos(\mathbf{C}-\mathbf{Q}) & \cos(\mathbf{C}-\mathbf{R}) \end{vmatrix} = \mathbf{0}$$

Solution :

$$\text{L.H.S.} = \begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix} \qquad \text{(Row by Row)}$$

$= 0 \quad 0$

$= 0$

$=$ R.H.S.

Illustration 45

Prove that

$$\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 \end{vmatrix} = 2\,(a_1-a_2)(a_2-a_3)(a_3-a_1)(b_1-b_2)(b_2-b_3)(b_3-b_1)$$

Solution :

$$\text{L.H.S.} = \begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1^2-2a_1b_1+b_1^2 & a_1^2-2a_1b_2+b_2^2 & a_1^2-2a_1b_3+b_3^2 \\ a_2^2-2a_2b_1+b_1^2 & a_2^2-2a_2b_2+b_2^2 & a_2^2-2a_2b_2+b_3^2 \\ a_3^2-2a_3b_1+b_1^2 & a_3^2-2a_3b_2+b_2^2 & a_3^2-2a_3b_3+b_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & -2b_1 & b_1^2 \\ 1 & -2b_2 & b_2^2 \\ 1 & -2b_3 & b_3^2 \end{vmatrix} \text{ (Row by Row)}$$

$$= (-1)\begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} \times (-2)\begin{vmatrix} 1 & b_1 & b_1^2 \\ 1 & b_2 & b_2^2 \\ 1 & b_3 & b_3^2 \end{vmatrix}$$

$$= (2)\begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} \times \begin{vmatrix} 1 & b_1 & b_1^2 \\ 1 & b_2 & b_2^2 \\ 1 & b_3 & b_3^2 \end{vmatrix}$$

$$= 2\,(a_1-a_2)(a_2-a_3)(a_3-a_1)(b_1-b_2)(b_2-b_3)(b_3-b_1) = \text{R.H.S.}$$

Illustration 46

Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution :

Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Now $\Delta = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$ $[R_1 \to R_1 - R_2, R_2 \to R_2 - R_3]$

$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$

$= (a-b)(b-c)(b+c-a-b)$

$= (a-b)(b-c)(c-a)$

Note : Determinant $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ is called a circular determinant and its value can be directly used in solving problems.

Illustration 47

Let a, b, c **be positive and not all equal. Show that the value of the determinant** $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ **is negative.** **[IIT–81]**

Solution :

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} \qquad [C_1 \to C_1 + C_2 + C_3]$$

$$= (a + b + c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

[Taking (a + b + c) common from first column]

$$= (a + b + c) \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \; [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$= (a + b + c) \{\{c - b\} (b - c) - (a - b) (a - c)\}$$

$$= (a + b + c) \{bc + ca + ab - a^2 - b^2 - c^2\}$$

$$= -(a + b + c) (a^2 + b^2 + c^2 - bc - ca - ab)$$

$$= \frac{1}{2} (a + b + c) [(a^2 + b^2 - 2ab) + (b^2 + c^2 - 2bc) + (c^2 + a^2 - 2ac)]$$

$$= \frac{1}{2}(a + b + c) [(a - b)^2 + (b - c)^2 + (c - a)^2] \; ...(i)$$

$\therefore$ a, b, c are positive $\qquad \therefore$ $a + b + c > 0$

Again since a, b, c are unequal $\qquad \therefore$ $(a - b)^2 + (b - c)^2 + (c - a)^2 > 0$

$\therefore$ from (i), $\Delta < 0$

Illustration 48

Prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$

Solution :

$$\text{L.H.S.} = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$$

Multiplying C_1, C_2, C_3 by a, b, c respectively

$$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}$$

Now taking common a, b, c from R_1, R_2, R_3 respectively

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix} \qquad [C_1 \to C_1 + C_2 + C_3]$$

$$= (1 + a^2 + b^2 + c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1 + a^2 + b^2 + c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$= (1 + a^2 + b^2 + c^2)\ (1.1.1)$$

$$= 1 + a^2 + b^2 + c^2 = \text{R.H.S.}$$

Illustration 49

Solve the equation $\begin{vmatrix} \mathbf{15} - x & \mathbf{1} & \mathbf{10} \\ \mathbf{11} - \mathbf{3}x & \mathbf{1} & \mathbf{16} \\ \mathbf{7} - x & \mathbf{1} & \mathbf{13} \end{vmatrix} = \mathbf{0}$

Solution :

Given $\begin{vmatrix} 15 - x & 1 & 10 \\ 11 - 3x & 1 & 16 \\ 7 - x & 1 & 13 \end{vmatrix} = 0$

or $\begin{vmatrix} 15-x & 1 & 10 \\ -4-2x & 0 & 6 \\ -8 & 0 & 3 \end{vmatrix} = 0$ $[R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$

or $(-1)(-12 - 6x + 48) = 0$

or $-36 + 6x = 0 \therefore \quad x = 6$

Illustration 50

If $a + b + c = 0$, solve the equation $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = \mathbf{0}$

Solution :

From given equation, $\begin{vmatrix} a+b+c-x & c & b \\ a+b+c-x & b-x & a \\ a+b+c-x & a & c-x \end{vmatrix} = 0$ $[C_1 \to C_1 + C_2 + C_3]$

or $(a + b + c - x)\begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix} = 0$

or $(-x)\begin{vmatrix} 1 & c & b \\ 0 & b-c-x & a-b \\ 0 & a-c & c-b-x \end{vmatrix} = 0$ $[R_2 \to R_2 - R_1; R_3 \to - R_1]$

$[\therefore a + b + c = 0]$

or $x[(b - c - x)(c - b - x) - (a - c)(a - b)] = 0$

or $x(x^2 - b^2 - c^2 + 2bc - a^2 + ab + ca - bc) = 0$

or $x(x^2 - a^2 - b^2 - c^2 + ab + bc + ca) = 0$

$\therefore \quad x = 0$

or $x^2 = a^2 + b^2 + c^2 - (ab + bc + ca)$

$= (a^2 + b^2 + c^2) - \frac{1}{2}\{(a + b + c)^2 - (a^2 + b^2 + c^2)\}$

$= \frac{3}{2}(a^2 + b^2 + c^2)$ $[\therefore a + b + c = 0]$

$\therefore \quad x = 0$, or $x = \pm\sqrt{\left\{\frac{3}{2}(a^2 + b^2 + c^2)\right\}}$

Illustration 51

Show without expanding that $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

Solution :

$$\text{L.H.S.} = \frac{1}{a^2b^2c^2}\begin{vmatrix} a^3 & a^2bc & a^3bc \\ b^3 & ab^2c & ab^3c \\ c^3 & abc^2 & abc^3 \end{vmatrix} \quad \begin{bmatrix} R_1 \to a^2R_1 \\ R_2 \to b^2R_2 \\ R_3 \to c^2R_3 \end{bmatrix}$$

$$= \frac{(abc)\,(abc)}{a^2b^2c^2}\begin{vmatrix} a^3 & a & a^2 \\ b^3 & b & b^2 \\ c^3 & c & c^2 \end{vmatrix} = -\begin{vmatrix} a & a^3 & a^2 \\ b & b^3 & b^2 \\ c & c^3 & c^2 \end{vmatrix} \qquad [C_2 \leftrightarrow C_2]$$

$$= \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} \quad [C_2 \leftrightarrow C_3]$$

Illustration 52

Evaluate : $\Delta = \begin{vmatrix} 1 + a_1 & a_2 & a_3 \\ a_1 & 1 + a_2 & a_3 \\ a_1 & a_2 & 1 + a_3 \end{vmatrix}$

Solution :

$$\Delta = \begin{vmatrix} 1 + a_1 + a_2 + a_3 & a_2 & a_3 \\ 1 + a_1 + a_2 + a_3 & 1 + a_2 & a_3 \\ 1 + a_1 + a_2 + a_3 & a_2 & 1 + a_3 \end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$$

$$= (1 + a_1 + a_2 + a_3)\begin{vmatrix} 1 & a_2 & a_3 \\ 1 & 1 + a_2 & a_3 \\ 1 & a_2 & 1 + a_3 \end{vmatrix}$$

$$= (1 + a_1 + a_2 + a_3)\begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & a_2 & 1 + a_3 \end{vmatrix}\begin{bmatrix} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{bmatrix}$$

$$= (1 + a_1 + a_2 + a_3)\begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix}$$

$$= (1 + a_1 + a_2 + a_3)\,(1 - 0) = 1 + a_1 + a_2 + a_3$$

Illustration 53

Show that $\begin{vmatrix} a + b + \mathbf{2}c & a & b \\ c & b + c + \mathbf{2}a & b \\ c & a & c + a + \mathbf{2}b \end{vmatrix} = \mathbf{2}\,(a + b + c)^{\mathbf{3}}$

Solution :

$$\Delta = \begin{vmatrix} 2\,(a + b + c) & a & b \\ 2\,(a + b + c) & b + c + 2a & b \\ 2\,(a + b + c) & a & c + a + 2b \end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$$

$$= 2\,(a + b + c)\begin{vmatrix} 1 & a & b \\ 1 & b + c + 2a & b \\ 1 & a & c + a + 2b \end{vmatrix}$$

[taking 2 (a + b + c) common from first column]

$$= 2\,(a + b + c)\begin{vmatrix} 0 & -(b + c + a) & 0 \\ 0 & (b + c + a) & -(a + b + c) \\ 1 & a & c + a + 2b \end{vmatrix} \quad [R_1 \to R_1 - R_2 \text{ and } R_2 \to R_2 - R_3]$$

$$= 2\,(a + b + c)\,(a + b + c)^2\begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & a & c + a + 2b \end{vmatrix}$$

[taking (a + b + c) common from first and second rows]

$$= 2\,(a + b + c)^3\; 1\,\{(-1)\,(-1) - 1.0\} = 2\,(a + b + c)^3.$$

Illustration 54

Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Solution :

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} [R_1 \to R_1 + R_2 + R_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & -b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad \text{[taking (a + b + c) common from first row]}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix} \quad [C_2 \to C_2 - C_1;\ C_3 \to -C_1]$$

$$= (a+b+c)\begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix} \quad \text{[Expanding along } R_1]$$

$$= (a+b+c)\,[-(b+c+a) \quad -(c+a+b)]$$

$$= (a+b+c)(a+b+c)^2 = (a+b+c)^3$$

Illustration 55

Prove that $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = (y-z)(z-x)(x-y)(yz+zx+xy)$

Solution :

$$\Delta = \frac{1}{x.y.z}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix} \quad [C_1 \to xC_1,\ C_2 \to yC_2,\ C_3 \to zC_3]$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} x^2 & y^2 & z^2 \\ 1 & 1 & 1 \\ x^3 & y^3 & z^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x^2 & y^2 - x^2 & z^2 - x^2 \\ x^3 & y^3 - x^3 & z^3 - x^3 \end{vmatrix} \quad [C_2 \to C_2 - C_1; \to C_3 - C_1]$$

$$= 1 \begin{vmatrix} (y-x)(y+x) & (z-x)(z+x) \\ (y-x)(y^2+xy+x^2) & (z-x)(z^2+zx+x^2) \end{vmatrix}$$

$$= (y-x)(z-x) \begin{vmatrix} y+x & z+x \\ y^2+xy+x^2 & z^2+zx+x^2 \end{vmatrix}$$

[taking $(y - x)$ and $(z - x)$ common from first and second columns respectively

$$= (y-x)(z-x) \begin{vmatrix} y+x & z-y \\ y^2+xy+x^2 & (z^2-y^2)+zx-xy \end{vmatrix} \quad [C_2 \to C_2 - C_1]$$

$$= (y-x)(z-x) \begin{vmatrix} y+x & z-y \\ y^2+xy+x^2 & (z-y)(x+y+z) \end{vmatrix}$$

$$= (y-x)(z-x)(z-y) \begin{vmatrix} y+x & 1 \\ y^2+xy+x^2 & (x+y+z) \end{vmatrix}$$

$$= (y-x)(z-x)(z-y)[(y+x)(x+y+z) - (y^2+xy+x^2)]$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx)$$

Illustration 56

If x, y, z are all different and if $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, **prove that $xyz = -1$.**

Solution :

$$\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= -\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \qquad [C_2 \leftrightarrow C_3 \text{ in first det.}]$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad [1 + xyz]$$

$[C_1 \leftrightarrow C_2]$

$= (x - y)(y - z)(z - x)(1 + xyz)$ [From value of a circular determinant]

$\therefore \quad \Delta = 0 \Rightarrow (x - y)(y - z)(z - x)(1 + xyz) = 0$

$\Rightarrow \quad 1 + xyz = 0$

$[\because x \neq y, y \neq z, z \neq x]$

$\Rightarrow \quad xyz = -1$

Illustration 57

Evaluate $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Solution :

$$\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \qquad [R_1 \to R_1 - R_2 - R_3]$$

$$= \frac{1}{c}\begin{vmatrix} 0 & -2c & -2b \\ 0 & c(c+a-b) & b(c-a+b) \\ c & c & a+b \end{vmatrix} \qquad [R_2 \to cR_2 - bR_3]$$

$$= \frac{1}{c}\,[c(-2bc)[c - a - b - (c + a - b)]]$$

$$= (-2bc)(-2a) = 4abc$$

Illustration 58

If a**,** b**,** c **are all positive and** p^{th}**,** q^{th} **and** r^{th} **terms of a G.P., then prove that**

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$$

Solution :

Let 'A' be the first term and 'R' common ratio of G.P.

Then, $t_p = AR^{p-1} = a$

$t_q = AR^{q-1} = b$

$t_r = AR^{r-1} = c$

Now, $\log a = \log AR^{p-1} = \log A + \log R^{p-1} = \log A \ \ (p-1)\log R$

Similarly, $\log b = \log A + (q-1)\log R$

$\log c = \log A + (r-1)\log R$

$$\therefore \quad \Delta = \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} + \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix}$$

$$= \log A \begin{vmatrix} 1 & p & 1 \\ 1 & q & 1 \\ 1 & r & 1 \end{vmatrix} + \log R \begin{vmatrix} p-1 & p & 1 \\ q-1 & q & 1 \\ r-1 & r & 1 \end{vmatrix}$$

$$= \log A \quad (0) + \log R \left[\begin{vmatrix} p & p & 1 \\ q & q & 1 \\ r & r & 1 \end{vmatrix} - \begin{vmatrix} 1 & p & 1 \\ 1 & q & 1 \\ 1 & r & 1 \end{vmatrix} \right]$$

$$= \log R\, [0 - 0] = 0$$

Illustration 59

If a, b, c **are in A.P., show that** $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$

Solution :

If a, b, c are in A.P., then $2b = a + c$

$$\text{Let} \qquad \Delta = \begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

Applying $R_1 \to R_1 + R_3$

$$\Delta = \begin{vmatrix} 2x+4 & 2x+6 & 2x+a+c \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

Applying $R_1 \to R_1 - 2R_2$

$$\Delta = \begin{vmatrix} 0 & 0 & a+c-2b \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} \quad [\because 2b = a + c]$$

$$= 0$$

Illustration 60

Solve $\begin{vmatrix} \mathbf{3}x-\mathbf{8} & \mathbf{3} & \mathbf{3} \\ \mathbf{3} & \mathbf{3}x-\mathbf{8} & \mathbf{3} \\ \mathbf{3} & \mathbf{3} & \mathbf{3}x-\mathbf{8} \end{vmatrix} \mathbf{= 0}$

Solution :

$$\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} 3x-2 & 3 & 3 \\ 3x-2 & 3x-8 & 3 \\ 3x-2 & 3 & 3x-8 \end{vmatrix} = 0$$

Taking $3x - 2$ common from C_1

$$(3x - 2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3x-8 & 3 \\ 1 & 3 & 3x-8 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$(3x-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3x-11 & 0 \\ 0 & 0 & 3x-11 \end{vmatrix} = 0$$

Expanding by C_1

$\Rightarrow$ $(3x-2)(3x-11)^2 = 0$

$\Rightarrow$ $3x - 2 = 0$ or $3x - 11 = 0$

$\Rightarrow$ $x = 2/3$ or $x = 11/3$

Differential coefficient of a determinant

let $y = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$

where $f_i(x)$, $g_i(x)$, $h_i(x)$, $i = 1, 2, 3$ are differentiable function of x.

$$\frac{dy}{dx} = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

Illustration 61

If f, g and h are differentiable functions of x and

$$\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2 f)'' & (x^2 g)'' & (x^2 h)'' \end{vmatrix} \quad \textbf{prove that} \quad \Delta' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

Solution :

$$\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2 f)'' & (x^2 g)'' & (x^2 h)'' \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ x^2 f'' + 4xf' + 2f & x^2 g'' + 4xg' + 2g & x^2 h'' + 4xh' + 2h \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2f'' + 4xf' & x^2g'' + 4xg' & x^2h'' + 4xh' \end{vmatrix}$$

by $R_3 - 2R_2$ and $R_2 - R_1$

Now apply $R_3 - 4R_2$

$$\Delta = x.x^2 \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix}$$

Now differentiate as explained in (b)

$$\therefore \quad \Delta' = \begin{vmatrix} f' & g' & h' \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

In the second det. $R_2.R_3$ become identical after taking x^3 common from R_3.

Illustration 62

Let α be a repeated root of quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$, $C(x)$ be polynomial of degree 3, 4 and 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$, where dash denotes the derivative.

Solution :

$$\text{Set} \quad P(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} \qquad \text{...(1)}$$

$$\text{Then} \quad P'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} + 0 + 0 \qquad \text{...(2)}$$

Since f(x) = 0 is a quadratic having a repeated root α, we can write $f(x) = a(x-\alpha)^2$ where a is a constant.

Now P(x) will be divisible by f(x) if P(x) and P´(x) are divisible by $(x-\alpha)$ i.e. $P(\alpha) = 0$ and $P'(\alpha) = 0$ which is obvious by (1) and (2) because of two identical rows.

Illustration 63

If $\Delta = \begin{vmatrix} \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \sin(\theta+\alpha) & \sin(\theta+\beta) & \sin(\theta+\gamma) \\ p+\theta\sin\alpha & p+\theta\sin\beta & r+\theta\sin\gamma \end{vmatrix}$ **then prove that Δ is independent of θ.**

Solution :

$\Delta = f(\theta)$. If will be independent of θ if $\Delta'(\theta) = 0$

$$\Delta' = 0 + 0 + \begin{vmatrix} \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \sin(\theta+\alpha) & \sin(\theta+\beta) & \sin(\theta+\gamma) \\ \sin\alpha & \sin\beta & \sin\gamma \end{vmatrix}$$

$$\Delta' = \Sigma \sin\alpha \sin(\theta+\gamma-\theta-\beta)$$

$$= \Sigma \sin\alpha \sin(\beta-\gamma) = 0$$

$\therefore\quad \Delta$ = constant i.e. independent of θ.

Illustration 64

If $\alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 = \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 = 0$

$\gamma_1^2+\gamma_2^2+\gamma_3^2 = 1$ **and** $\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = \dfrac{\sqrt{3}}{2}\sqrt{(\alpha_1^2+\alpha_2^2+\alpha_3^2)(\beta_1^2+\beta_2^2+\beta_3^2)}$ **, then show that**

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}^2 = \frac{1}{4}(\alpha_1^2+\alpha_2^2+\alpha_3^2).(\beta_1^2+\beta_2^2+\beta_3^2)$$

Solution :

We have $\quad \Delta^2 = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$

$$= \begin{vmatrix} \sum\alpha_1^2 & \sum\alpha_1\beta_1 & \sum\alpha_1\gamma_1 \\ \sum\beta_1\alpha_1 & \sum\beta_1^2 & \sum\beta_1\gamma_1 \\ \sum\gamma_1\alpha_1 & \sum\gamma_1\beta_1 & \sum\gamma_1^2 \end{vmatrix}$$

$$= \begin{vmatrix} \sum \alpha_1^2 & \frac{\sqrt{3}}{2}\sqrt{()\,()} & 0 \\ \frac{\sqrt{3}}{2}\sqrt{()\,()} & \sum \beta_1^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{(Using the given relation)}$$

$$= (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)(\beta_1^2 + \beta_2^2 + \beta_3^2) - \frac{3}{4}(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)(\beta_1^2 + \beta_2^2 + \beta_3^2)$$

$$= \frac{1}{4}(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)(\beta_1^2 + \beta_2^2 + \beta_3^2)$$

Illustration 65

Prove that the area of the triangle enclosed by the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ is $\dfrac{1}{2}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 \div (a_1b_2 - a_2b_1)(a_2b_3 - a_3b_2)(a_3b_1 - a_1b_3)$

Solution :

Consider the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

If Δ' is the det. formed by the cofactors of Δ, then we know that $\Delta' = \Delta^2$ i.e.,

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 \qquad \text{...(1)}$$

Now solving $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$, we get

$$\frac{x_1}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{y_1}{\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}$$

i.e., $\dfrac{x_1}{A_1} = \dfrac{y_1}{B_1} = \dfrac{1}{C_1}$

Hence point of intersection of the above two lines is $\left[\frac{A_1}{C_1}.\frac{B_1}{C_1}\right]$

This gives once of the vertices of the triangle

Similarly other two vertices are $\left[\frac{A_2}{C_2},\frac{B_2}{C_2}\right]$ and $\left[\frac{A_3}{C_3},\frac{B_3}{C_3}\right]$. Hence area of triangle

$$= \frac{1}{2}\begin{vmatrix} A_1/C_1 & B_1/C_1 & 1 \\ A_2/C_2 & B_2/C_2 & 1 \\ A_3/C_3 & B_3/C_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} + (C_1 C_2 C_3)$$

$$= \frac{1}{2}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + (a_2b_2 - a_3b_2)(a_3b_1 - a_1b_3)(a_1b_2 - a_2b_1)$$

[Using (1) and substituting the values of C_1, C_2, C_3]

Integration of a Determinant

If $$f(x) = \begin{vmatrix} f & g & h \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

when f, g, h are the function of x and a, b, c, a_1, b_1, c_1 are constants, then

$$\int_{\lambda}^{\mu} F(x)\,dx = \begin{vmatrix} \int_{\lambda}^{\mu} f\,dx & \int_{\lambda}^{\mu} g\,dx & \int_{\lambda}^{\mu} h\,dx \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

Also if $$F(x) = \begin{vmatrix} f & a & a_1 \\ g & b & b_1 \\ h & c & c_1 \end{vmatrix}$$

then $$\int_{\lambda}^{\mu} F(x)\,dx = \begin{vmatrix} \int_{\lambda}^{\mu} f\,dx & a & a_1 \\ \int_{\lambda}^{\mu} g\,dx & b & b_1 \\ \int_{\lambda}^{\mu} h\,dx & c & c_1 \end{vmatrix}$$

Illustration 66

If $f(x) = \begin{vmatrix} \sin^5 x & \text{In} \sin x & \dfrac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \\ n & \sum_{k=1}^{n} k & \prod_{k=1}^{n} k \\ \dfrac{8}{15} & \dfrac{\pi}{2}\ln\left(\dfrac{1}{2}\right) & \dfrac{\pi}{4} \end{vmatrix}$ **find the value of** $\int_0^{\pi/2} f(x)\,dx$

Solution :

$$\int_0^{\pi/2} f(x)\,dx = \begin{vmatrix} \int_0^{\pi/2} \sin^5 x\,dx & \int_0^{\pi/2} \text{In} \sin x\,dx & \int_0^{\pi/2} \dfrac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \\ n & \sum_{k=1}^{n} k & \prod_{k=1}^{n} k \\ \dfrac{8}{15} & \dfrac{\pi}{2}\text{In}\left(\dfrac{1}{2}\right) & \dfrac{\pi}{4} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{4}{5}\cdot\dfrac{2}{3} \text{ (By Walli's formula)} & -\dfrac{\pi}{2}\text{In}\,2 & \dfrac{\pi}{4} \\ n & \sum_{k=1}^{n} k & \prod_{k=1}^{n} k \\ \dfrac{8}{15} & \dfrac{\pi}{2}\text{In}\left(\dfrac{1}{2}\right) & \dfrac{\pi}{4} \end{vmatrix} = \begin{vmatrix} \dfrac{8}{15} & \dfrac{\pi}{2}\text{In}\left(\dfrac{1}{2}\right) & \dfrac{\pi}{4} \\ n & \sum_{k=1}^{n} k & \prod_{k=1}^{n} k \\ \dfrac{8}{15} & \dfrac{\pi}{2}\text{In}\left(\dfrac{1}{2}\right) & \dfrac{\pi}{4} \end{vmatrix}$$

$= 0$ (since R_1 and R_2 are identical)

Illustration 67

Let $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$ **then find the value of** $\int_0^{\pi/2} f(x)\,dx$.

Solution :

Applying $C_2 \to C_2 - \cos^2 x\, C_1$

then $f(x) = \begin{vmatrix} \sec x & 0 & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x - \cos^4 x & \operatorname{cosec}^2 x \\ 1 & 0 & \cos^2 x \end{vmatrix}$ (Expan. along C_2)

$$= (\cos^2 x - \cos^4 x) \begin{vmatrix} \sec x & \sec^2 x + \cot x \operatorname{cosec} x \\ 1 & \cos^2 x \end{vmatrix}$$

$$= (\cos^2 x - \cos^4 x)\ (\cos x - \sec^2 x - \cot x \operatorname{cosec} x)$$

$$= \cos^2 x\ (1 - \cos^2 x) \left(\cos x - \frac{1}{\cos^2 x} - \frac{\cos x}{\sin^2 x}\right)$$

$$= \cos^2 x \sin^2 x \left(\cos x - \frac{1}{\cos^2 x} - \frac{\cos x}{\sin^2 x}\right)$$

$$= \cos^3 x \sin^2 x - \sin^2 x - \cos^3 x$$

$$= -\cos^3 x\ (-\sin^2 x + 1) - \sin^2 x$$

$$f(x) = -\cos^5 x - \sin^2 x$$

$\therefore$ $$\int_0^{\pi/2} f(x)\,dx = -\int_0^{\pi/2} \cos^5 x\,dx - \int_0^{\pi/2} \sin^2 x\,dx$$

$$= -\left(\frac{4}{5}.\frac{2}{3}.1\right) - \left(\frac{1}{2}.\frac{\pi}{2}\right)$$

$$= -\left(\frac{8}{15} + \frac{\pi}{4}\right)$$ [By Walli's formule]

System of liner Equations :

Definition 1 :

A system of linear equations in n unknowns $x_1, x_2, x_3, ... x_n$ is of the form :

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{array}\right\} \quad ...(A)$$

If $b_1, b_2, ... b_n$ are all zero, the system is called **homoeneous** and non-homogeneous if at least one b_i is non-zero.

Definition 2.

The **solution set** of the system (A) is an n tuple (α_1, α_2, ..., α_n) of real numbers (or complex numbers if the coefficients are complex) which satisfy each of the equations of the sysem.

Definition 3.

A system of equations is called **consistent** if it has at least one solution; **inconsistent** if it does not have any solution; determinate if it has a unique solution; **indeterminate** if it has more than one solution.

Gist of discussion in simple language :

1. **Consistent :** Solution exists whether unique or infinite number of solutions.
2. **Inconsistent :** Solution does not exist.
3. **Homogeneous Equations :** Constant terms zero.
4. **Trivial solution :** All variables zero i.e., $x = 0$, $y = 0$, $z = 0$
5. **Non-trivial Solution :** Infinite number of solutions.

As a matter of fact on division by 2 the second equation reduces to first. Thus we have got only one line $2x + 3y = 10$ on which lie infinite number of points. Thus there are infinite number of solutions and the system is consistent.

Solution of System of Linear Equations by Determinants (Cramer's Rule)

System of linear equations in two variables :

Consider the system $a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

Solving these equations, we get

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

These can be exppressed in terms of determinants as :

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$\Rightarrow$ $x = \dfrac{D_x}{D}$, $y = \dfrac{D_y}{D}$ where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is the determinant of coefficient of x and y,

$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ is obtained by replacing coefficient of x by constants c_1, c_2 in D and

$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ is obtained by replacing coefficient of y by constants c_1, c_2 in D.

System of linear equations in three variables :

Consider the system
$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then, $xD = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$

Applying $C_1 \rightarrow C_1 + yC_2 + zC_3$

$$xD = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = D_x$$

$\Rightarrow$ $x = \dfrac{Dx}{D}$

Similarly, $y = \dfrac{D_y}{D}$ and $z = \dfrac{D_z}{D}$

where, $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, is the determinant of the coefficients of x, y and z.

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

is obtained by replacing coefficients of x by constants d_1, d_2, d_3 in D.

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

is obtained by replacing coefficients of y by constants d_1, d_2, d_3 in D.

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

is obtained by replacing coefficients of z by constants d_1, d_2, d_3 in D.

Criterion for consistency

For system of lienar equations in two variables :

(i) If $D \neq 0$, then the system is consistent and has a unique solution.

(ii) If $D = 0$, $D_x = 0$ and $D_y = 0$ then the system is consistent and has infinite number of solutions.

(iii) If $D = 0$ and any one of D_x or D_y is non-zero, then the system is inconsistent (no solutions).

For system of lienar equations in three variables :

(i) If $D \neq 0$, then the system is consistent and has a unique solution.

(ii) If $D = 0$, $D_x = 0$ and $D_y = 0$ and $D_z = 0$, then the system is consistent and has infinite number of solutions.

(iii) If $D = 0$ and any one of D_x, D_y or D_z is non-zero, then the system is inconsistent.

Illustration 68

Solve the following system of equations by Cramer's rule : $x + y = 5$, $y + z = 3$, $x + z = 4$

Solution :

Given system of equations is $x + y = + 0z = 5$, $0x + y + z = 3$, $x + 0y + z = 4$

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1\,(1 - 0) - 1\,(0 - 1) = 1 + 1 = 2$$

$$D_x = \begin{vmatrix} 5 & 1 & 0 \\ 3 & 1 & 1 \\ 4 & 0 & 1 \end{vmatrix} - 5\,(1 - 0) - 1\,(3 - 4) = 5 + 1 = 6$$

$$D_y = \begin{vmatrix} 1 & 5 & 0 \\ 0 & 3 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 1\,(3 - 4) - 5\,(0 - 1) = -1 + 5 = 4$$

$$D_z = \begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 0 & 4 \end{vmatrix} = 1\,(4 - 0) - 1\,(0 - 3) + 5\,(0 - 1) = 4 + 3 - 5 = 2$$

Now, $x = \dfrac{D_x}{D} = \dfrac{6}{2} = 3$; $y = \dfrac{D_y}{D} = \dfrac{4}{2} = 2$ and $z = \dfrac{D_z}{D} = \dfrac{2}{2} = 1$

$\therefore$ $x = 3$, $y = 2$ and $z = 1$

How to solve when $D_1 = D_2 = D_3 = 0$

1. Put any variable let us say $z = k$
2. Consider any 2 equations and solve them for x & y using Cramer's rule.

Illustration 69

Solve the following system of equations by Cramer's rule :

$x - y + 3z = 6,$
$x + 3y - 3z = -4,$
$5x + 3y + 3z = 10$

Solution :

$$D = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 3 & -3 \\ 5 & 3 & 3 \end{vmatrix}$$

$$= 1\,(9 + 9) + 1\,(3 + 15) + 3\,(3 - 15) = 18 + 18 - 36 = 0$$

$$D_x = \begin{vmatrix} 6 & -1 & 3 \\ -4 & 3 & -3 \\ 10 & 3 & 3 \end{vmatrix}$$

$$= 6\,(9 + 9) + 1\,(-12 + 30) + 3\,(-12 - 30) = 108 + 18 - 126 = 0$$

$$D_y = \begin{vmatrix} 1 & 6 & 3 \\ 1 & -4 & -3 \\ 5 & 10 & 3 \end{vmatrix}$$

$$= 1\,(-12 + 30) - 6\,(3 + 15) + 3\,(10 + 20) = 18 - 108 + 90 = 0$$

$$Dz = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 3 & -4 \\ 5 & 3 & 10 \end{vmatrix}$$

$$= 1\,(30 + 12) + 1\,(10 + 20) + 6\,(3 - 15) = 42 + 30 - 72 = 0$$

$\therefore$ $D = D_x = D_y = D_z = 0$

Hence, system of equations is consistent and has infinite solutions.

Putting $z = k$ and considering first two equations, we get

$x - y = 6 - 3k,\ x + 3y = -4 + 3k$

Again, $D = \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} = 3 + 1 = 4$

$$D_x = \begin{vmatrix} 6-3k & -1 \\ -4+3k & 3 \end{vmatrix} = 18 - 9k - 4 + 3k = 14 - 6k$$

$$D_y = \begin{vmatrix} 1 & 6-3k \\ 1 & -4+3k \end{vmatrix} = -4 + 3k - 6 + 3k = -10 + 6k$$

Now, $x = \dfrac{Dx}{D} = \dfrac{14-6k}{4} = \dfrac{7-3k}{2}$

and $y = \dfrac{D_y}{D} = \dfrac{-10+6k}{4} = \dfrac{-5+3k}{2}$

$\therefore\quad x = \dfrac{7-3k}{2},\ y = \dfrac{5+3k}{2}$ and $z = k$

By giving arbitrary values to k, we find that the given system has infinite number of solutions.

Illustration 70

Prove that the system of equation

$3x - y + 4z = 3$

$x + 2y - 3z = -2$

$6x + 5y + \lambda z = -3$

has at least one solution for any real λ. Find the set of solutions when $\lambda = -5$.

[IIT – 84]

Solution :

$$\Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} = 3\,(2\lambda + 15) + 1\,(\lambda + 18) + 4\,(5 - 12) = 7\,(\lambda + 5)$$

Case I :

When $\lambda \neq 5$. $\lambda + 5 \neq 0$. In this case $\Delta \neq 0$, therefore given system of equations has unique solution.

Case II :

When $\lambda = -5$. $\lambda + 5 = 0$. In this case $\Delta = 0$

Also $\Delta_1 = \begin{vmatrix} 3 & -1 & 4 \\ -2 & 2 & -3 \\ -3 & 5 & -5 \end{vmatrix} = 0,\ \Delta_2 = \begin{vmatrix} 3 & 3 & 4 \\ 1 & -2 & -3 \\ 6 & -3 & -5 \end{vmatrix} = 0$

and $\Delta_1 = \begin{vmatrix} 3 & -1 & 3 \\ 1 & 2 & -2 \\ 6 & 5 & -3 \end{vmatrix} = 0$

Hence in this case given system of equations has infinitely many solutions. Thus the given system of equations has at least one solution.

Set of solutions : Putting the value of λ, given equations become

$3x - y + 4z = 3$ ...(i)

$x + 2y - 3z = -2$ ...(ii)

and $6x + 5y - 5z = -3$ ...(iii)

multiplying equation (i) by 2 and adding it to (ii), we get

$$7x + 5z = 4 \quad \text{or} \quad z = \frac{4 - 7x}{5}$$

From (i), $y = 3x + 4z - 3$

$$= 3x + \frac{16 - 28x}{5} - 3 = \frac{1 - 13x}{5}$$

Thus solution is given by

$$\left.\begin{aligned} x &= t \\ y &= \frac{1 - 13t}{5} \\ z &= \frac{4 - 7t}{5} \end{aligned}\right\} \text{where } t \text{ is an arbitrary number.}$$

Illustration 71

For what values of p and q, the system of equations

$2x + py + 6z = 8$

$x + 2y + qz = 5$

$x + y + 3z = 4$

has (i) no solution (ii) a unique solution (ii) infinitely many solutions.

Solution :

$$\Delta = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} \begin{aligned} &= 2(6 - q) - p(3 - q) + 6(1 - 2) \\ &= 12 - 2q - 3p + pq - 6 \end{aligned}$$

$$= pq - 2q - 3p + 6 = (p - 2)(q - 3)$$

$$\Delta_1 = \begin{vmatrix} 8 & p & 6 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix}$$

$= 2\,(6 - q) - p\,(15 - 4q) + 6\,(5 - 8)$

$= 48 - 8q - 15p + 4pq - 18 = 4pq - 8q - 15p + 30$

$= 4q(p - 2) - 15(p - 2) = (4q - 15)\,(p - 2)$

$$\Delta_2 = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & q \\ 1 & 4 & 3 \end{vmatrix} = 2(15 - 4q) - 8(3 - q) + 6(4 - 5) = 0$$

$$\Delta_3 = \begin{vmatrix} 2 & p & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix} = 2(8 - 5) - p(4 - 5) + 8(1 - 2)$$
$$= 6 + p - 8 = p - 2$$

Case I : When $\Delta \neq 0$ i.e. $p \neq 2$, $q \neq 3$, given system of equations has unique solution.

Case II : $p = 2$, $\Delta = 0$, $\Delta_1 = 0$, $\Delta_2 = 0$, $\Delta_3 = 0$

$\therefore$ given system of equations has infinitely many solutions.

when $q = 3$, $p \neq 2$, $\Delta = 0$, $\Delta_1 \neq 0$

therefore, given system of equations has no solution.

Thus the given system of equation has

(i) unique solution when $p \neq 2$, $q \neq 3$

(ii) infinite number of solutions when $p = 2$

(iii) no solution when $p \neq 2$, $q = 3$.

Illustration 72

Solve using Cramer's rule :

$$\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = 4, \quad \frac{2}{x} + \frac{1}{y} - \frac{3}{z} = 0, \quad \frac{3}{x} + \frac{1}{y} + \frac{1}{z} = 6$$

Solution :

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 3 & 1 & 1 \end{vmatrix} = 1\,(1 + 3) + 1\,(2 + 9) + 1\,(2 - 3)$$

$= 4 + 11 - 1 = 14$

$$D_{1/x} = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 1 & -3 \\ 6 & 1 & 1 \end{vmatrix}$$

$= 4\ (1 + 3) + 1\ (0 + 18) + 1\ (0 - 6) = 16 + 18 - 6 = 28$

$$D_{1/y} = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 0 & -3 \\ 3 & 6 & 1 \end{vmatrix}$$

$= 1\ (0 + 18) - 4\ (2 + 9) + 1\ (12 - 0) = 18 - 44 + 12 = -14$

$$D_{1/z} = \begin{vmatrix} 1 & -1 & 4 \\ 2 & 1 & 0 \\ 3 & 1 & 6 \end{vmatrix}$$

$= 1\ (6 - 0) + 1\ (12 - 0) + 4\ (2 - 3) = 6 + 12 - 4 = 14$

Now, $\frac{1}{x} = \frac{D_{1/x}}{D} = \frac{28}{14} = 2 \quad \Rightarrow \quad x = 1/2;$

$\frac{1}{y} = \frac{D_{1/y}}{D} = \frac{-14}{14} = -1 \quad \Rightarrow \quad y = -1;$

$\frac{1}{z} = \frac{D_{1/z}}{D} = \frac{14}{14} = 1 \quad \Rightarrow \quad z = 1$

$\therefore \quad x = 1/2,\ y = -1$ and $z = 1$

Illustration 73

If $f(x) = ax^2 + bx + c$ and $f(0) = 6$, $f(2) = 11$, $f(-3) = 6$, find a, b, c and determine the quadratic function $f(x)$ using determinants.

Solution :

$f(0) = 6 \quad \Rightarrow \quad c = 6$

$f(2) = 11 \quad \Rightarrow \quad 4a + 2b + c = 11$

$f(-3) = 6 \quad \Rightarrow \quad 9a - 3b + c = 6$

$\therefore$ We have

$0a + 0b + c = 6$

$4a + 2b + c = 11$

$9a - 3b + c = 6$

$$D = \begin{vmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \\ 9 & -3 & 1 \end{vmatrix} = 0 + 0 + 1\,(-12 - 18) = -30$$

$$D_a = \begin{vmatrix} 6 & 0 & 1 \\ 11 & 2 & 1 \\ 6 & -3 & 1 \end{vmatrix} = 6(2 + 3) + 1\,(-33 - 12) = 30 - 45 = -15$$

$$D_b = \begin{vmatrix} 0 & 6 & 1 \\ 4 & 11 & 1 \\ 9 & 6 & 1 \end{vmatrix} = -6(4-9) + 1(24-99) = 30 - 75 = -45$$

$$D_c = \begin{vmatrix} 0 & 0 & 6 \\ 4 & 2 & 11 \\ 9 & -3 & 6 \end{vmatrix} = 6(-12-18) = -180$$

Now, $a = \dfrac{D_a}{D} = \dfrac{-15}{-30} = \dfrac{1}{2}$; $\quad b = \dfrac{D_b}{D} = \dfrac{-45}{-30} = \dfrac{3}{2}$

and $c = \dfrac{D_c}{D} = \dfrac{-180}{-30} = 6$

$\therefore$ $f(x) = ax^2 + bx + c = \dfrac{1}{2}x^2 + \dfrac{3}{2}x + 6$

Illustration 74

Let λ and α be real. Find the set of all values of λ for which the system :

$x\lambda + (\sin\alpha)y + (\cos\alpha)z = 0$

$x + (\cos\alpha)y + (\sin\alpha)z = 0$

$-x + (\sin\alpha)y - (\cos\alpha)z = 0$

has a non-trivial solution. For $\lambda = 1$, find all values of α. **[IIT – 93]**

Solution :

For nontrivial solution,

$$\Delta = 0 \text{ or } \begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

or $\lambda(-\cos^2\alpha - \sin^2\alpha) - (-\sin\alpha\cos\alpha - \sin\alpha\cos\alpha) - (\sin^2\alpha - \cos^2\alpha) = 0$ [Expanding along C_1]

or $-\lambda + 2\sin\alpha\cos\alpha + (\cos^2\alpha - \sin^2\alpha) = 0$

or $\lambda = \sin 2\alpha + \cos 2\alpha$

When $\lambda = 1$, $\sin 2\alpha + \cos 2\alpha = 1$

or $\sin 2\alpha = 1 - \cos 2\alpha$

or $2\sin\alpha\cos\alpha = 2\sin^2\alpha$

or $2\sin\alpha(\cos\alpha - \sin\alpha) = 0$

$\therefore$ $\sin\alpha = 0$ or $\cos\alpha - \sin\alpha = 0$

$\therefore$ $\sin\alpha = 0$ or $\tan\alpha = 1$.

Hence $\alpha = n\pi$ or $\alpha = n\pi + \dfrac{\pi}{4}$; $n \in I$

Illustration 75

For a fixed positive integer n, if $D=\begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$ **then show that** $\frac{D}{(n!)^3}-4$ **is divisible by n.** **[IIT – 92]**

Solution :

$$D = n!\,(n+1)!\,(n+2)!\begin{vmatrix} 1 & n+1 & (n+1)(n+2) \\ 1 & n+2 & (n+2)(n+3) \\ 1 & n+3 & (n+3)(n+4) \end{vmatrix}$$

$$= n!\,(n+1)!\,(n+2)!\begin{vmatrix} 1 & n+1 & (n+1)^2 \\ 1 & n+2 & (n+2)^2 \\ 1 & n+3 & (n+3)^2 \end{vmatrix} \quad [C_3 \to C_3 - C_2]$$

$$= n!\,(n+1)!\,(n+2)!\,(-1)(-1)\,2 \left[\because \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)\right]$$

$$\therefore \quad \frac{D}{(n!)^3} = \frac{2.n!\,(n+1)!\,(n+2)!}{n!\,n!\,n!}$$

$$= 2(n+1)(n+1)(n+2) = 2(n^2+2n+1)(n+2)$$

$$= 2(n^3+4n^2+5n+2) = 2n^3+8n^2+10n+4$$

or $\quad \frac{D}{(n!)^3} - 4 = n(2n^2+8n+10),$ which is divisible by n.

Illustration 76

Evaluate $\begin{vmatrix} {}^x\mathbf{C_1} & {}^x\mathbf{C_2} & {}^x\mathbf{C_3} \\ {}^y\mathbf{C_1} & {}^y\mathbf{C_2} & {}^y\mathbf{C_3} \\ {}^z\mathbf{C_1} & {}^z\mathbf{C_2} & {}^z\mathbf{C_3} \end{vmatrix}$

Solution :

We know that ${}^nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$, where $\underline{|n} = 1.2.3\ldots n$ and $\underline{|0} = 1$

$$\therefore \quad {}^xC_1 = \frac{\lfloor x}{\lfloor 1 \lfloor x-1} = \frac{x}{\lfloor 1},\ {}^xC_2 = \frac{x(x-1)}{\lfloor 2},\ {}^xC_3 = \frac{x(x-1)(x-2)}{\lfloor 3}$$

Now $$\Delta = \begin{vmatrix} \frac{x}{\lfloor 1} & \frac{x(x-1)}{\lfloor 2} & \frac{x(x-1)(x-2)}{\lfloor 3} \\ \frac{y}{\lfloor 1} & \frac{y(y-1)}{\lfloor 2} & \frac{y(y-1)(y-2)}{\lfloor 3} \\ \frac{z}{\lfloor 1} & \frac{z(z-1)}{\lfloor 2} & \frac{z(z-1)(z-2)}{\lfloor 3} \end{vmatrix}$$

$$= \frac{xyz}{\lfloor 2 \lfloor 3} \begin{vmatrix} 1 & x-1 & (x-1)(x-2) \\ 1 & y-1 & (y-1)(y-2) \\ 1 & z-1 & (z-1)(z-2) \end{vmatrix}$$

$$= \frac{xyz}{12} \begin{vmatrix} 1 & x-1 & (x-1)^2 \\ 1 & y-1 & (y-1)^2 \\ 1 & z-1 & (z-1)^2 \end{vmatrix} \qquad [C_3 \to C_3 + C_2]$$

$$= \frac{xyz}{12} (x-y)(y-z)(z-x) \quad \left| \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \right.$$

$$= (a-b)(b-c)(c-a)]$$

Illustration 77

If $a \neq p$**,** $b \neq q$**,** $c \neq r$ **and** $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = \mathbf{0}$**, then find the value of** $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$ **[IIT 91]**

Solution :

Given $$\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

or $$\begin{vmatrix} p-a & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0 \qquad \begin{bmatrix} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{bmatrix}$$

or $(p-a)[(q-b)r-b(c-r)]-(b-q)\ [0-a(c-r)]=0$

or $(p-a)(q-b)r-b(p-a)(c-r)+a(b-q)(c-r)=0$

or $(p-a)(q-b)r+b(p-a)(r-c)+a(q-b)(r-c)=0$

or $$\frac{r}{r-c}+\frac{b}{q-b}+\frac{a}{p-a}=0$$

or $$\frac{r}{r-c}+\left(\frac{b}{q-b}+1\right)+\left(\frac{q}{p-a}+a\right)=0+1+1$$

or $$\frac{r}{r-c}+\frac{q}{q-b}+\frac{p}{p-a}=2 \quad \text{or} \quad \frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2$$

Illustration 78

Show that

$$\begin{vmatrix} (x-a)^2 & b^2 & c^2 \\ a^2 & (x-b)^2 & c^2 \\ a^2 & b^2 & (x-c)^2 \end{vmatrix} = x^2\,(x-2a)(x-2b)(x-2c)\left(x+\frac{a^2}{x-2a}+\frac{b^2}{x-2b}+\frac{c^2}{x-c^2}\right)$$

Solution :

$$\Delta=\begin{vmatrix} x(x-2a) & x(2b-x) & 0 \\ 0 & x(x-2b) & x(2c-x) \\ a^2 & b^2 & (x-c)^2 \end{vmatrix} \begin{bmatrix} R_1 \to R_1 - R_2, \\ R_2 \to R_2 - R_3 \end{bmatrix}$$

$$= x^2\begin{vmatrix} x-2a & -(x-2b) & 0 \\ 0 & x-2b & -(x-2c) \\ a^2 & b^2 & (x-c)^2 \end{vmatrix} \begin{bmatrix} \text{taking } x \text{ common} \\ \text{from } R_1 \text{ and } R_2 \end{bmatrix}$$

$$= x^2\,(x-2a)(x-2b)(x-2c)\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \dfrac{a^2}{x-2a} & \dfrac{b^2}{x-2b} & x+\dfrac{c^2}{x-2c} \end{vmatrix}$$

$$= x^2\,(x-2a)(x-2b)(x-2c)\left(x+\frac{a^2}{x-2a}+\frac{b^2}{x-2b}+\frac{c^2}{x-2c}\right)$$

$$= \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ \dfrac{a^2}{x-2b} & \dfrac{b^2}{x-2b} & 1 \end{vmatrix} \quad [C_3 \to C_3 + C_2 + C_1]$$

$$= x^2 (x-2a)(x-2b)(x-2c)\left(x + \frac{a^2}{x-2a} + \frac{b^2}{x-2b} + \frac{c^2}{x-2c}\right)$$

Illustration 79

If a > 0, d > 0, find the value of the determinant

$$\begin{vmatrix} \dfrac{1}{a} & \dfrac{1}{a(a+d)} & \dfrac{1}{(a+d)(a+2d)} \\ \dfrac{1}{a+d} & \dfrac{1}{(a+d)(a+2d)} & \dfrac{1}{(a+2d)(a+3d)} \\ \dfrac{1}{a+2d} & \dfrac{b}{(a+2d)(a+3d)} & \dfrac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

[IIT–96]

Solution :

$$\Delta = \frac{1}{(a+d)^2 (a+2d)^3 (a+3d)^2 (a+4d)}$$

$$\begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & a+3d & a+d \\ (a+3d)(a+4d) & a+4d & a+2d \end{vmatrix}$$

$$= \frac{1}{(a+d)^2 (a+2d)^3 (a+3d)^2 (a+4d)}$$

$$\begin{vmatrix} (a+d)(a+2d) & 2d & a \\ (a+2d)(a+3d) & 3d & a+d \\ (a+3d)(a+4d) & 4d & a+2d \end{vmatrix} \quad [C_2 \to C_2 - C_3]$$

$$= \frac{1}{a(a+d)^2 (a+2d)^3 (a+3d)^2 (a+4d)}$$

$$\begin{vmatrix} (a+d)(a+2d) & 2d & a \\ (a+2d)2d & 0 & d \\ (a+3d)2d & 0 & d \end{vmatrix} \quad [R_3 \to R_3 - R_2,\ R_2 \to R_2 - R_2]$$

$$= \frac{-2d}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}[2d^2(a+2d-a-3d)]$$

$$= \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

Illustration 80

If $2s = a + b + c$, show that $\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = 2s^3(s-a)(s-b)(s-c)$

Solution :

Let $\alpha = s - a$, $\beta = s - b$, $\gamma = s - c$, then

$\beta + \gamma = 2s - (b + c) = a$

$\gamma + \alpha = b$ and $\alpha + \beta = c$, $\alpha + \beta + \gamma = 3s - (a + b + c) = 3s - 2s = s$

Now L.H.S. $= \begin{vmatrix} (\beta+\gamma)^2 & \alpha^2 & \alpha^2 \\ \beta^2 & (\gamma+\alpha)^2 & \beta^2 \\ \gamma^2 & \gamma^2 & (\alpha+\beta)^2 \end{vmatrix}$

$= 2\alpha\beta\gamma(\alpha + \beta + \gamma)^3$

$= 2(s-a)(s-b)(s-c)s^3 = 2s^3(s-a)(s-b)(s-c)$

Illustration 81

Find the value of θ lying between 0 and $\pi/2$ and satisfying the equation :

$$\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\sin 4\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\sin 4\theta \\ \cos^2\theta & \sin^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

[IIT – 88]

Solution :

Given equation becomes : $\begin{vmatrix} 2+4\sin 4\theta & \sin^2\theta & 4\sin 4\theta \\ 2+4\sin 4\theta & 1+\sin^2\theta & 4\sin 4\theta \\ 2+4\sin 4\theta & \sin^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$

or $(2+4\sin 4\theta)\begin{vmatrix} 1 & \sin^2\theta & 4\sin 4\theta \\ 1 & 1+\sin^2\theta & 4\sin 4\theta \\ 1 & \sin^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$

or $2(1+2\sin 4\theta)\begin{vmatrix} 1 & \sin^2\theta & 4\sin 4\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$ [Applying $R_2 \to R_2 - R_1$ & $R_3 \to R_1 - R_1$]

or $2(1 + 2\sin 4\theta) = 0$ [Expanding along R_1]

or $\sin 4\theta = \frac{1}{2}$ $\therefore$ $4\theta = \left(\pi + \frac{\pi}{6}\right) = \frac{7\pi}{6}$ $\therefore$ $\theta = \frac{7\pi}{24}$

Illustration 82

Let m **and** p **be two positive integers such that** $m \geq p$ **+ 2.**

If $\Delta(m, p) = \begin{vmatrix} {}^{m}C_{p} & {}^{m}C_{p+1} & {}^{m}C_{p+2} \\ {}^{m+1}C_{p} & {}^{m+1}C_{p+1} & {}^{m+1}C_{p+2} \\ {}^{m+2}C_{p} & {}^{m+2}C_{p+2} & {}^{m+2}C_{p+2} \end{vmatrix}$

Show that $\Delta(m, p) = \frac{{}^{m+2}C_3}{{}^{p+2}C_3}\Delta(m-1, p-1)$. **Hence or otherwise, prove that**

$$\Delta(m, p) = \frac{({}^{m+2}C_3)({}^{m+1}C_3)\ldots({}^{m-p+3}C_3)}{({}^{p+2}C_3)({}^{p+1}C_3)\ldots({}^{3}C_3)}$$

Solution :

$r.{}^{n}C_{r} = n^{n-1}C_{r-1}$

$\therefore$ ${}^{n}C_{r} = \frac{n}{r}.{}^{n-1}C_{r-1}$

Now $\Delta(m,\ p) = \begin{vmatrix} \dfrac{m}{p}.\,^{m-1}C_{p-1} & \dfrac{m}{p+1}.\,^{m-1}C_{p} & \dfrac{m}{p+2}.\,^{m-1}C_{p+1} \\ \dfrac{m+1}{p}.\,^{m}C_{p-1} & \dfrac{m+1}{p+1}.\,^{m}C_{p} & \dfrac{m+1}{p+2}.\,^{m}C_{p+1} \\ \dfrac{m+2}{p}.\,^{m+1}C_{p-1} & \dfrac{m+2}{p+1}.\,^{m+1}C_{p} & \dfrac{m+2}{p+2}.\,^{m+1}C_{p+1} \end{vmatrix}$

Taking m common from R_1, (m + 1) from R_2, (m + 2) from R_3, 1/p from C_1, 1/(p + 1) from C_2 and 1/(p + 2) from C_3, we get

$$\Delta(m,\ p) = \frac{m(m+1)(m+2)}{p(p+1)(p+2)}\begin{vmatrix} ^{m-1}C_{p-1} & ^{m-1}C_{p} & ^{m-1}C_{p+1} \\ ^{m}C_{p-1} & ^{m}C_{p} & ^{m}C_{p+1} \\ ^{m+1}C_{p-1} & ^{m+1}C_{p} & ^{m+1}C_{p+1} \end{vmatrix}$$

$$= \frac{^{m+2}C_3}{^{p+2}C_3}\,\Delta\,(m-1,\ p-1)$$

$$= \frac{^{m+2}C_1}{^{p+2}C_3}.\frac{^{m+1}C_3}{^{p+1}C_3}\,\Delta\,(m-2,\ p-2)$$

$$= \frac{^{m+2}C_3}{^{p+2}C_3}.\frac{^{m+1}C_3}{^{p+1}C_3}.\frac{^{m}C_3}{^{p}C_3}\,\Delta\,(m-3,\ p-3)$$

$$= \frac{(^{m+2}C_3)(^{m+1}C_3)\ldots(^{m-p+3}C_3)}{(^{p+2}C_3)(^{p+1}C_3)\ldots(^{3}C_3)} \qquad \ldots(i)$$

Now $\quad \Delta\,(m-p,\,0) = \begin{vmatrix} ^{m-p}C_0 & ^{m-p}C_1 & ^{m-p}C_2 \\ ^{m-p+1}C_0 & ^{m-p+1}C_1 & ^{m-p+1}C_2 \\ ^{m-p+2}C_0 & ^{m-p+2}C_1 & ^{m-p+2}C_2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & m-p & \dfrac{(m-p)(m-p-1)}{2} \\ 1 & m-p+1 & \dfrac{(m-p+1)(m-p)}{2} \\ 1 & m-p+2 & \dfrac{(m-p+2)(m-p+1)}{2} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & m-p & \frac{(m-p)(m-p-1)}{2} \\ 0 & 1 & m-p \\ 0 & 1 & m-p+1 \end{vmatrix} = 1 \text{ [Applying } R_3 \to R_3 - R_2 \text{ and } R_2 \to R_2 - R_1]$$

Hence from (i), $\Delta(m, p) = \dfrac{({}^{m+2}C_3)({}^{m+1}C_3)\ldots({}^{m-p+3}C_3)}{({}^{p+2}C_3)({}^{p+1}C_3)\ldots({}^{3}C_3)}$

Illustration 83

If α, β and γ are such that $\alpha + \beta + \gamma = 0$, then prove that $\begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix} = 0$

Solution :

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ \cos\gamma & \sin^2\gamma & \cos\alpha - \cos\beta\cos\gamma \\ \cos\beta & \cos\alpha - \cos\beta\cos\gamma & \sin^2\beta \end{vmatrix} \begin{bmatrix} C_2 \to C_2 - \cos\gamma C_1 \\ C_3 \to C_3 - \cos\beta C_1 \end{bmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \cos\gamma & \sin^2\gamma & -\sin\gamma\sin\beta \\ \cos\beta & -\sin\gamma\sin\beta & \sin^2\beta \end{vmatrix} \begin{bmatrix} \because \alpha = -(\beta+\gamma) \therefore \cos\alpha = \cos(\beta+\gamma) \\ \therefore \cos\alpha = \cos\beta\cos\gamma - \sin\beta\sin\gamma \end{bmatrix}$$

$= 1\ (\sin^2\gamma\ \sin^2\beta - \sin^2\beta\ \sin^2\gamma) = 0$

Illustration 84

If A, B, C are the angles of a triangle, show that $\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} = 0$

Solution :

$$\Delta = \begin{vmatrix} 2\sin A\cos A & \sin C & \sin B \\ \sin C & 2\sin B\cos B & \sin A \\ \sin B & \sin A & \sin C\cos C \end{vmatrix}$$

$$= -\cos\alpha\cos\beta\cos\gamma = \begin{vmatrix} \tan\alpha - \tan\gamma & \tan\beta - \tan\gamma & \tan\gamma \\ 0 & 0 & 1 \\ \sin^2\alpha - \sin^2\gamma & \sin^2\beta - \sin^2\gamma & \sin^2\gamma \end{vmatrix}$$

[Applying $C_1 \to C_1 - C_3$ & $C_2 \to C_2 - C_3$]

$$= -\cos\alpha\ \cos\beta\ \cos\gamma \begin{vmatrix} \dfrac{\sin(\alpha-\gamma)}{\cos\alpha\ \cos\gamma} & \dfrac{\sin(\beta-\gamma)}{\cos\beta\ \cos\gamma} & \tan\gamma \\ 0 & 0 & 1 \\ \sin(\alpha+\gamma)\sin(\alpha-\gamma) & \sin(\beta+\gamma)\sin(\beta-\gamma) & \sin^2\gamma \end{vmatrix}$$

$$\left[\because \tan\alpha - \tan\gamma = \frac{\sin\alpha}{\cos\alpha} - \frac{\sin\gamma}{\cos\gamma} = \frac{\sin(\alpha-\gamma)}{\cos\alpha\ \cos\gamma} \text{ etc.}\right]$$

$$= -\cos\alpha\ \cos\beta\ \cos\gamma.\ \sin(\alpha-\gamma)\ \sin(\beta-\gamma) \begin{vmatrix} \dfrac{1}{\cos\alpha\cos\gamma} & \dfrac{1}{\cos\beta\cos\gamma} & \tan\gamma \\ 0 & 0 & 1 \\ \sin(\alpha+\gamma) & \sin(\beta+\gamma) & \sin^2\gamma \end{vmatrix}$$

$$= -\cos\alpha\ \cos\beta\ \cos\gamma\sin(\alpha-\gamma)\ \sin(\beta-\gamma).\ (-1)\left\{\frac{\sin(\beta+\gamma)}{\cos\alpha\ \cos\gamma} - \frac{\sin(\alpha+\gamma)}{\cos\beta\ \cos\gamma}\right\}$$

$$= \sin(\alpha-\gamma)\ \sin(\beta-\gamma)\ [\cos\beta\ \sin(\beta+\gamma) - \cos\alpha\ \sin(\alpha+\gamma)]$$

$$= \sin(\alpha-\gamma)\ \sin(\beta-\gamma)\ \left[\frac{1}{2}\{\sin(2\beta+\gamma)+\sin\gamma\} - \frac{1}{2}\{\sin(2\alpha+\gamma)+\sin\gamma\}\right]$$

$$= \frac{1}{2}\ \sin(\alpha-\gamma)\ \sin(\beta-\gamma)\ [\sin(2\beta+\gamma) - \sin(2\alpha+\gamma)]$$

$$= \frac{1}{2}\sin(\alpha-\gamma)\ \sin(\beta-\gamma).\ 2\cos(\alpha+\beta+\gamma)\ \sin(\beta-\alpha)$$

$$= \sin(\alpha-\beta)\ \sin(\beta-\gamma)\ \sin(\gamma-\alpha)\ \cos(\alpha+\beta+\gamma)$$

Illustration 85

If $\mathbf{D_r} = \begin{vmatrix} r & x & \dfrac{n(n+\mathbf{1})}{\mathbf{2}} \\ \mathbf{2}r-\mathbf{1} & y & n^{\mathbf{2}} \\ \mathbf{3}r-\mathbf{2} & z & \dfrac{n(\mathbf{3}n-\mathbf{1})}{\mathbf{2}} \end{vmatrix}$ **show that** $\sum_{r=1}^{n} \mathbf{D_r} = \mathbf{0}$

Solution :

$$\sum_{r=1}^{n} D_r = D_1 + D_2 + \ldots + D_n$$

$$= \begin{vmatrix} \sum_{r=1}^{n} r & x & \frac{n(n+1)}{2} \\ \sum_{r=1}^{n} (3r-1) & y & n^2 \\ \sum_{r=1}^{n} (3r-2) & z & \frac{n(3n+1)}{2} \end{vmatrix} = \begin{vmatrix} 1+2+\ldots+n & x & \frac{n(n+1)}{2} \\ 1+3+5+\ldots.+2n-1 & y & n^2 \\ 1+4+7+\ldots.+3n-2 & z & \frac{n(3n-1)}{2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n(n+1)}{2} & x & \frac{n(n+1)}{2} \\ n^2 & y & n^2 \\ \frac{n(3n-1)}{2} & z & \frac{n(3n-1)}{2} \end{vmatrix}$$

$= 0$ [Since C_1 and C_3 are identical.]

Illustration 86

Without expanding the determinant show that the value of $\begin{vmatrix} -5 & 3+5i & \frac{3}{2}-4i \\ 3-5i & 8 & 4+5i \\ \frac{3}{2}+4i & 4-5i & 9 \end{vmatrix}$ **is real.**

Solution :

Let $z = \begin{vmatrix} -5 & 3+5i & \frac{3}{2}-4i \\ 3-5i & 8 & 4+5i \\ \frac{3}{2}+4i & 4-5i & 9 \end{vmatrix}$

Then $\bar{z} = \begin{vmatrix} -5 & 3-5i & \frac{3}{2}+4i \\ 3+5i & 8 & 4-5i \\ \frac{3}{2}-4i & 4+5i & 9 \end{vmatrix}$

$$= \begin{vmatrix} -5 & 3+5i & \frac{3}{2}-4i \\ 3-5i & 8 & 4+5i \\ \frac{3}{2}+4i & 4-5i & 9 \end{vmatrix}$$ [Changing rows into corr esponding columns]

$= z$

$\therefore \quad \bar{z} = z$ and hence z is real.

Illustration 87

If $p(x)$, $q(x)$ and $r(x)$ are three polynomials of degree 2, then prove that

$$\begin{vmatrix} p(x) & q(x) & r(x) \\ p'(x) & q'(x) & r'(x) \\ p''(x) & q''(x) & r''(x) \end{vmatrix}$$ **is a independent of x.**

Solution :

Let $p(x) = a_1x^2 + b_1x + c_1$, $q(x) = a_2x^2 + b_2x + c_2$

and $r(x) = a_3x^2 + b_3x + c_3$

then $p'(x) = 2a_1x + b_1$, $q'(x) = 2a_2x + b_2$, $r'(x) = 2a_3x + b_3$

and $p''(x) = 2a_1$, $q''(x) = 2a_2$, $r''(x) = 2a_3$

Now $\Delta = \begin{vmatrix} p(x) & q(x) & r(x) \\ p'(x) & q'(x) & r'(x) \\ p''(x) & q''(x) & r''(x) \end{vmatrix}$

$$= \begin{vmatrix} a_1x^2+b_1x+c_1 & a_2x^2+b_2x+c_2 & a_3x^2+b_3x+c_3 \\ 2a_1x+b_1 & 2a_2x+b_2 & 2a_3x+b_3 \\ 2a_1 & 2a_2 & 2a_3 \end{vmatrix}$$

$$= 2\begin{vmatrix} a_1x^2+b_1x+c_1 & a_2x^2+b_2x+c_2 & a_3x^2+b_3x+c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$ [$R_2 \to R_2 - xR_3$ and [taking 2 common from R_3]

$$= 2\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$ [$R_1 \to R_1 - x^2 R_3 - xR_2$]

Clearly Δ is independent of x.

Illustration 88

If $f_r(x)$, $g_r(x)$, $h_r(x)$, where $r = 1, 2, 3$ are polynomial in x such that $f_r(a) = g_r(a) = h_r(a)$, $r = 1, 2, 3$ and $\mathbf{F}(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$ then find $F'(a)$. **[IIT – 85]**

Solution :

$$F(x) = \begin{vmatrix} f'_1(x) & f'_2(x) & f'_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g'_1(x) & g'_2(x) & g'_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f'_2(x) & f'_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h'_1(x) & h'_2(x) & h'_3(x) \end{vmatrix}$$

$$\therefore \quad F'(a) = \begin{vmatrix} f'_1(a) & f'_2(a) & f'_3(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g'_1(a) & g'_2(a) & g'_3(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix}$$

$$+ \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h'_1(a) & h'_2(a) & h'_3(a) \end{vmatrix} = 0 + 0 + 0 = 0.$$

Since $g_r(a) = h_r(a)$, hence in first determinant R_2 & R_3 are identical;
Since $f_r(a) = h_r(a)$, hence in second determinant R_1 & R_3 are identical;
Since $f_r(a) = g_r(a)$, hence in third determinant R_1 & R_2 are identical.

Illustration 89

Let α be a repeated root of the quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$, $C(x)$ be polynomials of degree 3, 4 and 5 respectively. Show that $\Delta(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$, where dash denotes the derivative.

Solution :

Since $f(x) = 0$ is a quadratic equation with repeated root α, therefore $f(x) = a_r(x - \alpha)^2$ where a is a constant.

Clearly, $\Delta(x)$ is a polynomial of degree at most 5.

Eqn. $\Delta(x) = 0$ will behave two roots equal to α if $\Delta(\alpha) = 0$ & $\Delta'(\alpha) = 0$

$$\text{Now, } \Delta'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} \quad [\because A(\alpha), A'(\alpha) \text{ etc. are constants}]$$

$$\therefore \quad \Delta(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0 \quad [\because R_1 \,\&\, R_2 \text{ are identical}]$$

$$\text{And} \quad \Delta'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0 \quad [\because R_1 \,\&\, R_3 \text{ are identical}]$$

Thus, $\Delta(\alpha) = 0$ and $\Delta'(\alpha) = 0$.

Hence equation $\Delta(x) = 0$ has two roots equal to α.

Therefore, $\Delta(x) = (x - \alpha)^2 \phi(x)$ where $\phi(x)$ is a polynomial of degree at most 3.

Now $\Delta(x) = a(x-\alpha)^2 . \dfrac{\phi(x)}{a} = a(x-\alpha)^2 . g(x)$, where $g(x) = \dfrac{\phi(x)}{a}$

Thus $\Delta(x) = f(x).g(x)$, where $g(x)$ is a polynomial in x.

Hence $\Delta(x)$ is divisible by $f(x)$.

Illustration 90

If $u_n = \displaystyle\int_0^{\pi/2} \frac{1-\cos 2nx}{1-\cos 2x} dx$, then find the value of the determinant $\Delta = \begin{vmatrix} \frac{\pi}{2} & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{vmatrix}$

Solution :

We have, $u_1 = \displaystyle\int_0^{\pi/2} \frac{1-\cos 2x}{1-\cos 2x} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$

Now $2u_{n+1} - (u_n + u_{n+2}) = \displaystyle\int_0^{\pi/2} \frac{2\cos(2n+2)x - \{\cos(2n+4)x + \cos(2nx)\}}{1-\cos 2x} dx$

$$= \int_0^{\pi/2} \frac{2\cos(2n+2)x - 2\cos(2n+2)x \cos 2x}{1-\cos 2x} dx$$

$$= \int_0^{\pi/2} 2\cos(2n+2)x\,dx = \left[\frac{\sin(2n+2)x}{n+1}\right]_0^{\pi/2} = 0$$

Hence $u_n + u_{n+2} - 2u_{n+1} = 0$

$$\text{Now } \Delta = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{vmatrix} = \begin{vmatrix} u_1 - 2u_2 + u_3 & u_2 & u_3 \\ u_4 - 2u_5 + u_6 & u_5 & u_6 \\ u_7 - 2u_8 + u_9 & u_8 & u_9 \end{vmatrix}$$

Applying $C_1 \to C_1 - 2C_2 + C_3$, we obtain

$$= \begin{vmatrix} 0 & u_2 & u_3 \\ 0 & u_5 & u_6 \\ 0 & u_8 & u_9 \end{vmatrix} = 0 \qquad [C_1 \to C_1 - 2C_2 + C_3]$$

Illustration 91

Consider the system of linear equations in x, y, z

$(\sin 3\theta)\ x - y + z = 0$

$(\cos 2\theta)\ x + 4y + 3z = 0$

$2x + 7y + 7z = 0$

Find the value of θ for which this system has non-trivial solutions.

Solution :

The system will have a non-trivial solution if $\begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$

or $(28 - 21) \sin 3\theta - (-7 - 7) \cos 2\theta + 2(-3 - 4) = 0$

or $7 \sin 3\theta + 14 \cos 2\theta - 14 = 0$

or $3 \sin\theta - 4 \sin^3\theta + 2(1 - 2\sin^2\theta) - 2 = 0$

or $4\sin^3\theta + 4\sin^2\theta - 3\sin\theta = 0$

or $\sin\theta\,(2\sin\theta - 1)(2\sin\theta + 3) = 0$

$\therefore$ $\sin\theta = 0$ which gives $\theta = n\pi$,

or $\sin\theta = \dfrac{1}{2}$ which gives $\theta = n\pi + (-1)^n.\pi/6$,

where n is an integer

[Note that $\sin\theta \neq -3/2$]

Illustration 92

Three vectors a**,** b**,** c **are given by**

(a) $a = i - \mathbf{3}j + \mathbf{2}k$**,** $b = \mathbf{2}i - \mathbf{4}j - \mathbf{4}k$**,** $c = \mathbf{3}i + \mathbf{2}j - k$

(b) $a = i + j + k$**,** $b = \mathbf{2}i + \mathbf{3}j - k$**,** $c = -i - \mathbf{2}j + \mathbf{2}k$

Determine whether the systems of vectors in (i) and (ii) are linearly independent or dependent.

Solution :

Consider the linear relation

$$xa + yb + zc = 0 \qquad \text{...(i)}$$

If, x, y, z are all zero, then it is L.I. and if x, y, z are not all zero, then it is L.D.

On putting the values of a, b, c as given and combining the terms of i, j, k, we have

(a) $(x + 2y + 3z)\, i + (-3x - 4y + 2z)\, j + (2x - 4y - z)\, k = 0$

We know that i, j, k being non-coplanar represent at a L.I. system. Hence all the scalars in the above must be zero.

$$\therefore \quad x + 2y + 3z = 0$$
$$-3x - 4y + 2z = 0$$
$$2x - 4y - z = 0$$

Above represents a homogeneous system of equations

$$\text{Its} \quad \Delta = \begin{vmatrix} 1 & 2 & 3 \\ -3 & -4 & 2 \\ 2 & -4 & -1 \end{vmatrix} \begin{array}{l} \text{Apply } C_2 - 2C_1 \\ \text{and } C_3 - 3C_1 \end{array}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ -3 & 2 & 11 \\ 2 & -8 & -7 \end{vmatrix} \begin{array}{l} = -14 + 88 \\ = 74 \neq 0 \end{array}$$

Henec the above system has only trivial solution i.e. all x, y, z will be zero. Hence the vectors a, b, c are L.I.

(b) In this case proceeding as above

$$\Delta = \begin{vmatrix} -1 & 2 & 1 \\ -2 & 3 & 1 \\ 2 & -1 & 1 \end{vmatrix} \text{Apply } C_3 - C_2$$

The columns C_1 and C_3 become identical so that $\Delta = 0$. Hence the system has non-trivial solution i.e. all x, y, z are not zero. Hence the vectors a, b, c are L.D.

Illustration 93

$\mathbf{A} = \begin{bmatrix} a & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & c & b \\ \mathbf{1} & d & b \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} a & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & d & c \\ f & g & h \end{bmatrix}$, $\mathbf{U} = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$, $\mathbf{V} = \begin{bmatrix} a^2 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$ **. If there is vector matrix X, such that AX = U has infinitely many solutions, then prove that BX = V cannot have a unique solution. If** $afd \neq 0$ **then prove that BX = V has no solution.** **[IIT – 2004]**

Solution :

$\Delta \neq 0$, Unique solution (Intersecting lines)

$\Delta = 0$, $\Delta_1 = 0$, $\Delta_2 = 0$, Infinite solutions (Identical lines)

$\Delta = 0$, $\Delta_1 \neq 0$, $\Delta_2 \neq 0$, No solution (Parallel lines)

AX = U has infinite many solutions.

$\therefore \quad \Delta = 0, \Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 0$

$|A| = 0, |A_1| = 0, |A_2| = 0, |A_3| = 0.$

$$|A| = 0 \quad \Rightarrow \quad \begin{vmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{vmatrix} = 0$$

or $\quad ab(c - d) + 1(d - c) = 0$

or $\quad (c - d)(ab - 1) = 0$

$\therefore \quad ab = 1$ or $c = d$ ...(1)

$$|A_1| = 0 \quad \Rightarrow \quad \begin{vmatrix} f & 0 & 1 \\ g & c & b \\ h & d & b \end{vmatrix} = 0$$

Above is possible if g = h and c = d ...(2)

as in that case R_2 and R_3 become identical.

$$|A_2| = 0 \quad \Rightarrow \quad \begin{vmatrix} a & f & 1 \\ 1 & g & b \\ 1 & h & b \end{vmatrix} = 0$$

Above is possible if g = h ...(3)

as in that case R_2 and R_3 become identical

$$|A_3| = 0 \quad \Rightarrow \quad \begin{vmatrix} a & 0 & f \\ 1 & c & g \\ 1 & d & h \end{vmatrix} = 0$$

by virtue of relations in (2)

$$BX = V$$

$|B| \neq 0$ for unique solution

and $|B| = 0$ for no unique solution.

$|B| = \begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix} = 0$ as C_2 and C_3 are identical by virtue of relations in (2).

BX = V has no solution, then $|B| = 0$, $|B_1| = 0$ but $|B_2| = a^2 cf = a^2 df = a\,(adf) \neq 0$ as $adf \neq 0$.

Similarly $|B_3| = a^2 df \neq 0$

Hence there is no solution.

Quizrr