INVERSE CIRCULAR FUNCTIONS

Definition

Inverse functions relating trigonometrical ratios are called inverse trigonometric functions. The definition of different inverse trigonometric functions can be given as follows :

If $\sin\theta = x$, then $\theta = \sin^{-1}x$, provided $-1 \le x \le 1$ and $\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

If $\cos\theta = x$, then $\theta = \cos^{-1}x$, provided $-1 \le x \le 1$ and $0 \le \theta \le \pi$

If $\tan\theta = x$, then $\theta = \tan^{-1}x$, provided $-\infty < x < \infty$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

If $\cot\theta = x$, then $\theta = \cot^{-1}x$, provided $-\infty < x < \infty$ and $0 < \theta < \pi$

If $\sec\theta = x$, then $\theta = \sec^{-1}x$, provided $x \le -1$ or $x \ge 1$ and $0 \le \theta \le \pi, \theta \ne \frac{\pi}{2}$

If $\text{cosec}\,\theta = x$, then $\theta = \text{cosec}^{-1}x$, provided $x \le -1$ or $x \ge 1$ and $\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \theta \ne 0.$

Note : In the above definition restricitons on θ are due to the consideration of principal values of inverse terms. If these restrictions are removed, the terms will represent inverse trigonometric relations and not functions.

For clear understanding let us consider one example.

Let $\sin\theta = \frac{1}{2}$

$\Rightarrow \quad \sin\theta = \sin\frac{\pi}{6}$

$\Rightarrow \quad \theta = nx + (-1)^n \frac{\pi}{9}$ where $n = 0, \pm 1, \pm 2, \ldots$

Putting different values of n, we can get different values of θ.

Thus, infinite number of values of θ can be obtained. These values of θ are represented by Arc sinx..

This means Arc sinx represents the angle whose sine is equal to x.

$\because \quad -1 \le \sin\theta \le 1$ and $\sin\theta = x$ $\qquad \because -1 \le x \le 1$

Thus, Are sinx is defined only when $-1 \le x \le 1$.

Clearly for every $x \in [-1, 1]$, infinite number of values of Arc sinx will be obtained i.e., Arc sinx denotes the general value of satisfying $\sin\theta = x$.

Principal value

Numerically smallest angle is known as the principal value.

Since Inverse trigonometrical terms are in fact angles, definition of principal value of inverse trigonometrical term is the same as the definition of the principal value of angles.

Suppose we have to find the principal value of $\sin^{-1}\frac{1}{2}$

For this, let $\sin^{-1}\frac{1}{2} = \theta$ then $\sin\theta = \frac{1}{2}$

$$\Rightarrow \quad \theta = ...-\frac{11\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, ...$$

Among all these angles $\frac{\pi}{6}$ is the numerically smallest angle satisfying $\sin\theta = \frac{1}{2}$ and hence principal value of $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

Method for finding principal value

Step I. Mark the quadrant in which the angle may lie.

Step II. Mark the direction to be taken. Select anticlockwise direction for 1st and 2nd quadrants and select clockwise direction for 3rd and 4th quadrants.

Step III. Find the angles in the first circle.

Step IV. Select the numerically smallest angle. i.e., in the first rotation.

In case, two values, one with positive sign and the other with negative sign, qualify for principal value, we conventionally select the angle with positive sign as principal value.

Illustration 1

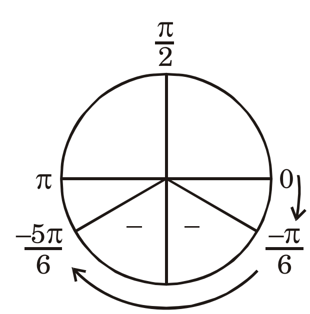
Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution :

Let $\sin^{-1}\left(-\frac{1}{2}\right) = \theta$ then $\sin\theta = -\frac{1}{2}$.

Since $\sin\theta$ is negative, θ will lie in 3rd in 4th quardrant.

Hence for principal value, we will select clockwise direction.



The angles in the first circle are $-\frac{\pi}{6}$ and $-\frac{5\pi}{6}$.

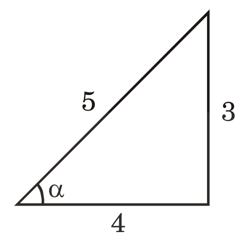
Hence principal value $\frac{-\pi}{6}$.

Illustration 2

Prove that, $\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{16}{65}$

Solution :

Let $\sin^{-1}\frac{3}{5} = \alpha$, so that $\sin\alpha = \frac{3}{5}$,

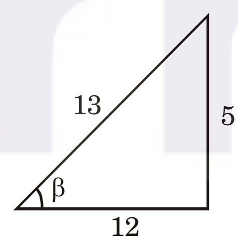


and therefore $\cos\alpha = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$

Let $\cos^{-1}\frac{12}{13} = \beta$, so that $\cos\beta = \frac{12}{13}$

and therefore, $\sin\beta = \sqrt{1-\frac{144}{169}} = \frac{5}{13}$

Let $\sin^{-1}\frac{16}{65} = \gamma$ so that $\sin\gamma = \frac{16}{65}$



We have then to prove that

$$\alpha - \beta = \gamma$$

i.e. to show that $\sin(\alpha-\beta) = \sin\gamma$

Now, $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \frac{3}{5}.\frac{12}{13} - \frac{4}{5}.\frac{5}{13} = \frac{36-20}{65} = \frac{16}{35} = \sin\gamma$$

Hence the relation is proved.

Illustration 3

Prove that $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

Solution :

Let $\tan^{-1}\frac{1}{3} = \alpha$, so that $\tan a = \frac{1}{3}$

and let $\tan^{-1}\frac{1}{7} = \beta$, so that $\tan \beta = \frac{1}{7}$

We have then to show that

$$2a + \beta = \frac{\pi}{4}$$

Now, $$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$= \frac{\frac{2}{3}}{1-\frac{1}{9}} = \frac{6}{8} = \frac{3}{4}$$

Also, $$\tan(2\alpha+\beta) = \frac{\tan 2\alpha + \tan\beta}{1-\tan 2\alpha \tan\beta}$$

$$= \frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}} = \frac{21+4}{28-3} = 1 = \tan\frac{\pi}{4}$$

$$2\alpha + \beta = \frac{\pi}{4}$$

Illustration 4

Prove that $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}$

Solution :

Let $\tan^{-1}\frac{1}{5} = \alpha$ so that $\tan\alpha = \frac{1}{5}$

Then, $$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2 a} = \frac{\frac{2}{5}}{1-\frac{1}{25}} = \frac{5}{12}$$

and $$\tan 4\alpha \frac{\frac{10}{12}}{1-\frac{25}{144}} = \frac{120}{119}$$

So that tan 4α is nearly unity, and 4α therefore, nearly $\frac{\pi}{4}$.

Let $$4\alpha = \frac{\pi}{4} + \tan^{-1} x$$

$\therefore$ $$\frac{120}{119} = \tan\left(\frac{\pi}{4} + \tan^{-1} x\right) = \frac{1+x}{1-x}$$

Hence, $$4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}$$

Illustration 5

Prove that $\mathbf{\tan^{-1}} a + \mathbf{\tan^{-1}} b = \mathbf{\tan^{-1}} \frac{a+b}{\mathbf{1}-ab}$

Solution :

Let $\tan^{-1} a = \alpha$, so that $\tan\alpha = a$,

and let $\tan^{-1} b = \beta$, so that $\tan\beta = b$,

Also, let $\tan^{-1}\left(\frac{a+b}{1-ab}\right) = \gamma$ so,that $\gamma = \frac{a+b}{1-ab}$

We have then to prove that

$$\alpha + \beta = \gamma$$

Now $$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \frac{a+b}{1-ab} = \tan\gamma$$

So, that relation is prove.

The above relation is merely the formula

$$\tan(x+y) = \frac{\tan x + \tan y}{1-\tan x \tan y}$$

expressed in inverse notation

For put $\tan x = a$, so that $x = \tan^{-1} a$,

and $\tan y = b$, so that $y = \tan^{-1} b$, Then, $$\tan(x + y) = \frac{a+b}{1-ab}$$

$$x + y = \tan^{1}\frac{a+b}{1-ab}$$

i.e. $$\tan^{-1} a + \tan^{-1} b = \tan^{-1}\frac{a+b}{1-ab}$$

In the above we have tacitly assumed that ab < 1, so that $\frac{a+b}{1-ab}$ is positive, and therefore

$\tan^{-1}\frac{a+b}{a-ab}$ lies between 0 and 90 .

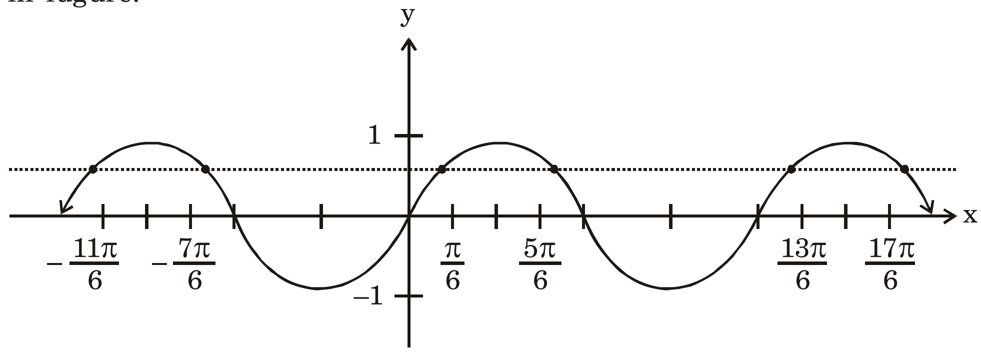
If, however, ab be > 1, then $\frac{a+b}{a-ab}$ is negative, and therefore according to our definition

$\tan^{-1}\frac{a+b}{a-ab}$ is a negative angle. Here γ is there a negative angle and, since $\tan(\pi + \gamma) = \tan\gamma$, the formula should be.

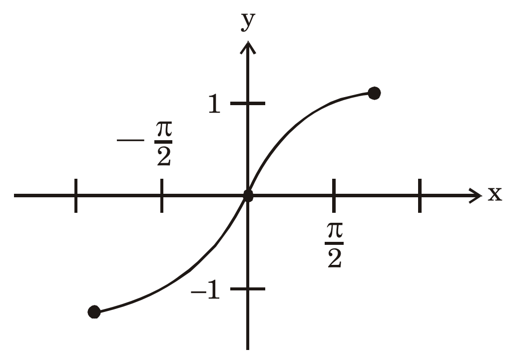
$$\tan^{-1} a + \tan^{-1} b = \pi + \tan^{-1}\frac{a+b}{1-ab}$$

Inverse Trigonometric Functions

It is evident that the sine function over the domain of all real numbers is not a one-to-one function. For example, suppose that we consider the solutions for $\sin x = \frac{1}{2}$. Certainly, $\frac{\pi}{6}$ is a solution, that there are infinitely many more solutions, such as $\frac{5\pi}{6}, \frac{13\pi}{6}\frac{17\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$, etc. as indicated in fugure.



However we can form a one-to-one function from the sine function, and not eliminate any values from its range, by restricting the domain to the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. There fore, we have a new function by the equation y = sin x with a domain of $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and a range of $-1 \le y \le 1$. (figure)



Inverse Sine Function

The inverse sine function is defined by $y = \mathbf{\sin^{-1}}\, x$ if and only if $x = \sin y$

where $\mathbf{-1 \le x \le 1}$ and $-\frac{\pi}{\mathbf{2}} \le y \le \frac{\pi}{\mathbf{2}}$

In definition, the equation $y = \sin^{-1} x$ can be read by as y is the angle whose sine is x. Therefore, $y = \sin^{-1}\frac{1}{2}$ means by y is the angle, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, inclusive, whose sin is $\frac{1}{2}$, thus $y = \frac{\pi}{6}$.

(The angle could also be expressed as 30 .)

Principal value of $\sin^{-1}x$ itself its value as inverse functions are consider as function only for a particular interval. So their is no need to define separate principal value.

Illustration 6

Evaluate $\mathbf{\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)}$

Solution :

The expression $\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$ means the cosine of the angle between $-(\pi/2)$ and $\pi/2$, inclusive, whose sine is $-1/2$. We know that the angle between $-(\pi/2)$ and $\pi/2$, inclusive, whose sine is $-1/2$ is $-(\pi/6)$, then $\cos(-\pi/6)) = \sqrt{3}/2$.

Therefore, $\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \frac{\sqrt{3}}{2}$

Inverse Consine Function :

The other trigonometric functions can also be used to introduce inverse function. In each case, a restriction needs to be place on the original domain to create a one-to-one function that contains the entire range of the original function. Then a corresponding inverse function can be defined.

By restricting the domain of the cosine function to real numbers between 0 and π, inclusive, a one-to-one function with a range between –1 and 1, inclusive, is obtained. Then the following definition creates the inverse cosine function.

The inverse cosine function or arc cosine function is defined by

$$y = \cos^{-1} x = \arccos x \text{ if and only if } x = \cos y$$

where $\mathbf{-1 \le x \le 1 \text{ and } 0 \le y \le \pi}$

Illustration 7

Solve $y = \cos^{-1}(-\sqrt{3}/2))$ for y, whre $0 \le y \le \pi$.

Solution

The expression $y = \cos^{-1}(-\sqrt{3}/2))$ can be interpreted as the angle whose consine is $-(\sqrt{3}/2)$. We know that $y = 5\pi/6$.

Illustration 8

Evaluate $\sin\left(\cos^{-1}\frac{1}{2}\right)$

Solution :

The expression $\sin\left(\cos^{-1}\frac{1}{2}\right)$ means the sine of the angle, between 0 and π, inclusive, whose cosine is 1/2. We know that $\pi/3$ is the angle whose cosine is 1/2 and we now that $\sin(\pi/3) = \sqrt{3}/2$.

Therefore, $\sin\left(\cos^{-1}\frac{1}{2}\right) = \sqrt{3}/2$

Inverse Tangent Function :

By restricting the domain of the tangent function to real numbers between $-(\pi/2)$ and $\pi/2$, $-(\pi/2)$ and $\pi/2$ are not included since the tangent is undefined at those values) a one-to-one function is obtained. Therefore, the inverse tangent function.

The inverse tangent function or arctanget function is defined by

$$y = \tan^{-1} x = \arctan x \text{ if and only if } x = \tan y$$

where $\mathbf{-\infty < x < \infty \text{ and } -(\pi/2) < y < \pi/2}$

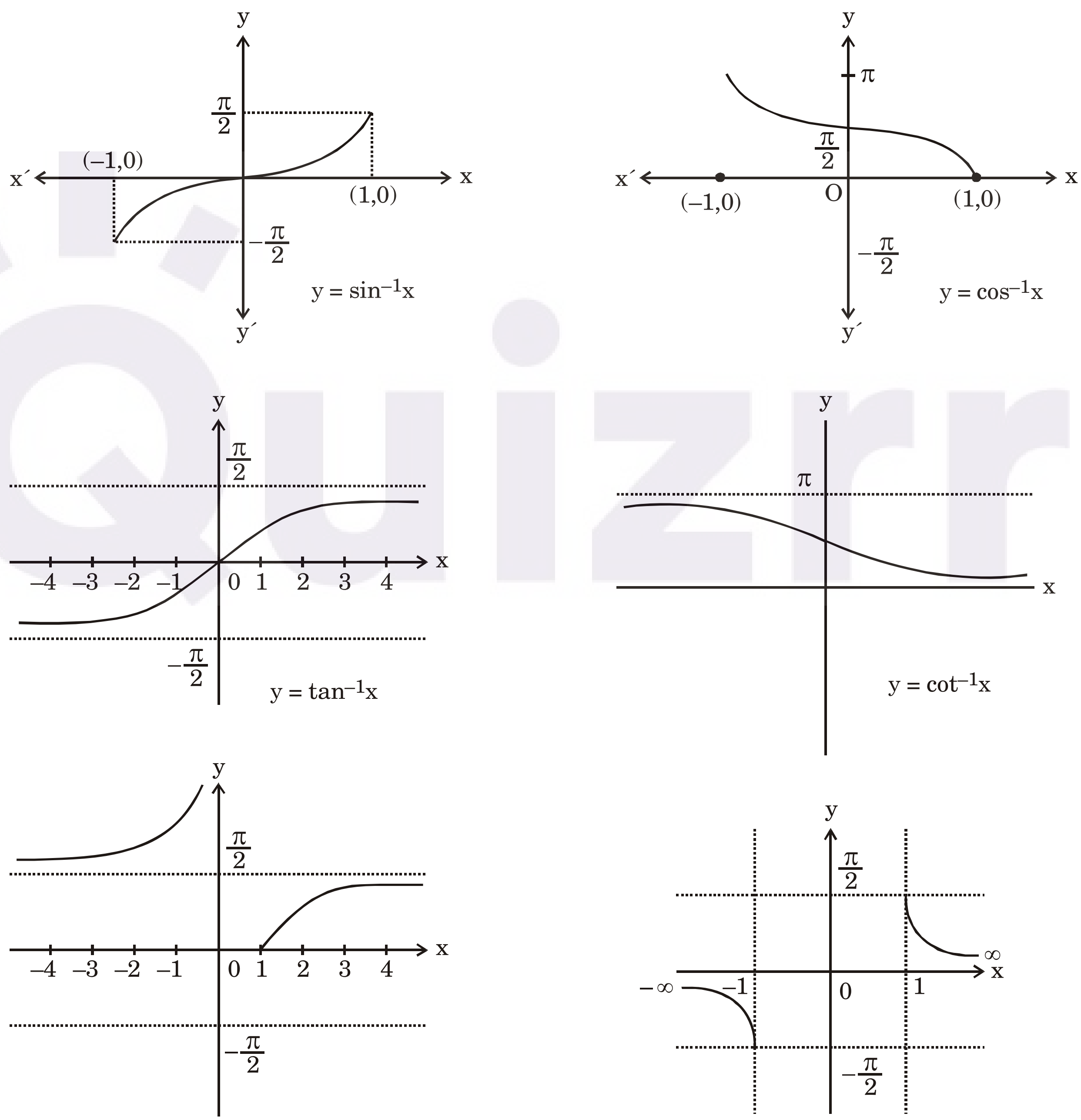
Illustration 9

Solve $y = \tan^{-1}(\sqrt{3}/3))$ **for** y**, where – 90 < y < 90 .**

Solution :

The expression $y = \tan^{-1}(\sqrt{3}/3))$ can be interpreted as the angle between –90 and 90 whose tangent is $-(\sqrt{3}/3)$. We know that y = –30 .

Graphs of Basic Inverse Trigonometric Function :



Domain and Range of Inverse Trigonometric Fuctions :

Function	Domain	Range
$y = \sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1}x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1}x$	$(-\infty, \infty)$	$(0, \pi)$
$y = \sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$y = \operatorname{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

(a) 1^{st} quadrant is common the range of all the inverse functions.

(b) 3^{rd} quadrant is not used in inverse fuctnions

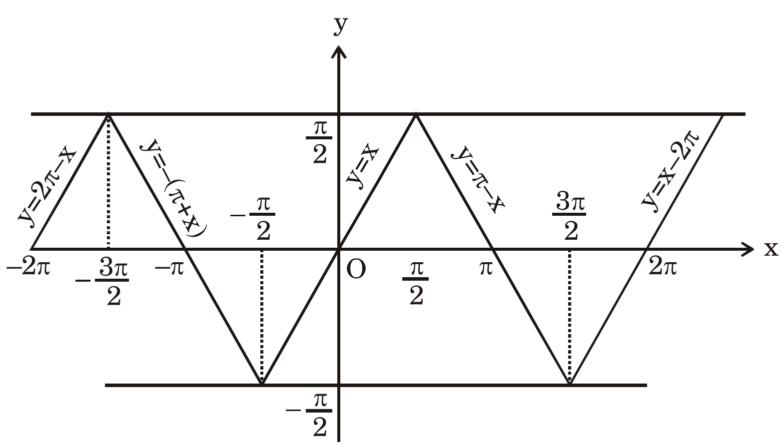
(c) 4th quadrant is used in the clockwise direction i.e. $-\frac{\pi}{2} \le y \le 0$

(d) No inverse function is periodic.

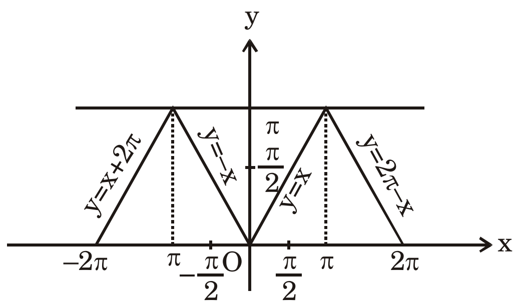
Properties or Inverse Trigonometric Functions :

PROPERTY 1 :

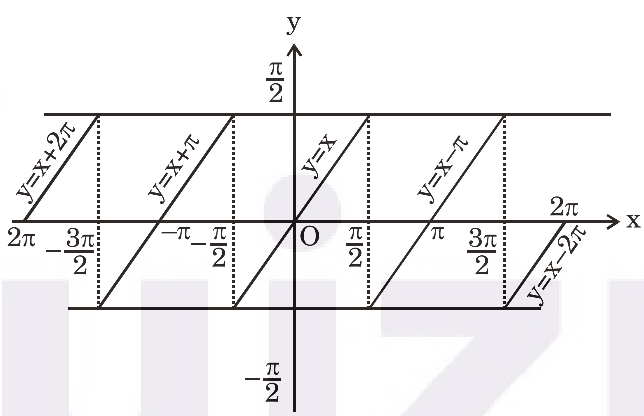
(i) $\mathbf{\sin^{-1}(\sin x) = x.}$ Provided that $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$



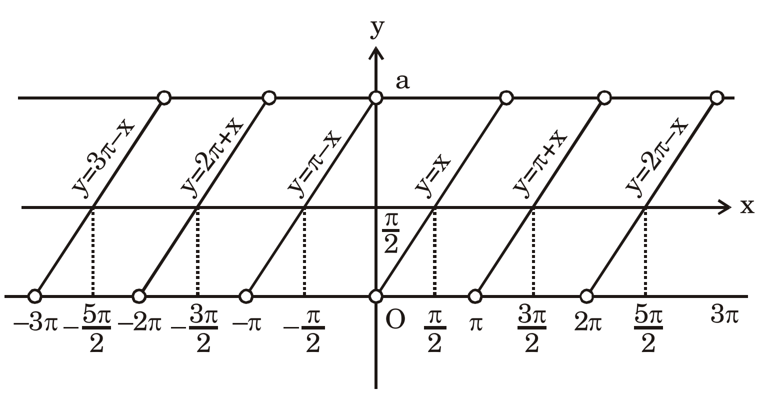
(ii) **$\cos^{-1}$ (cos x) = x,** Provided that $0 \le x \le \pi$.



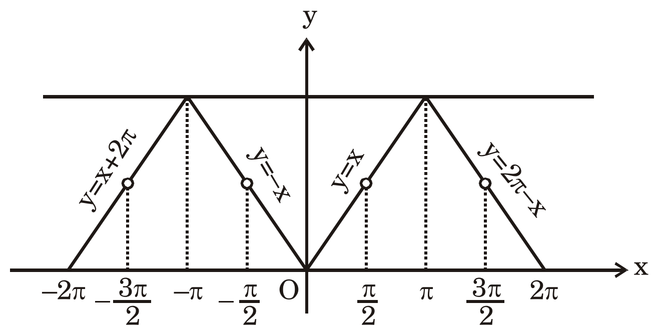
(iii) **$\tan^{-1}$ (tan x) = x,** Provided that $-\frac{\pi}{2} < x < \frac{\pi}{2}$



(iv) **$\cot^{-1}$(cot t) = x,** Provided that $0 < x < \pi$



(v) $\mathbf{sec^{-1}(sec\ x) = x,}$ Provided that $0 \le x < -\frac{\pi}{2}$ or $\frac{\pi}{2} < x \le \pi$



(vi) $\mathbf{cosec^{-1}(cosec\ x) = x}$ Provided that $-\frac{\pi}{2} \le x < 0$ or $0\ x \le \frac{\pi}{2}$

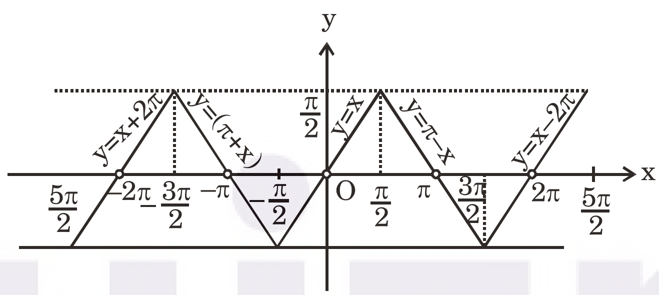


Illustration 10

Find the angle $\mathbf{\sin^{-1}\left(\sin\frac{2\pi}{3}\right)}$

Solution :

Let $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \theta$

$\Rightarrow \quad \sin\theta = \sin\frac{2\pi}{3}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$= \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$

i.e. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$

PROPERTY 2 :

(i) $\mathbf{sin\ (sin^{-1}x) = x,}$ Provided that $-1 \le x \le 1$

Let $\sin\theta = x$ .....(1)

$\theta = \sin^{-1} x$ .....(2)

Substituting the value of θ in (1) from (2), $\sin(\sin^{-1}x) = x$.

Here x is a number and will have no unit.

(ii) $\mathbf{cos(cos^{-1}x) = x,}$ Provided that $-1 \le x \le 1$

(iii) $\mathbf{tan(tan^{-1}x) = x,}$

Provided that $-\infty < x < \infty$

(iv) $\mathbf{cot\ (cot^{-1}x) = x,}$

Provided that $-\infty < x < \infty$

(v) $\mathbf{sec(sec^{-1})x = x,}$

Provided that $-\infty < x \le 1$ or $1 \le x < \infty$

(iv) $\mathbf{cosec\ (cosec^{-1}x) = x,}$

Provided that $-\infty < x \le -1$ or $1 \le x < \infty$

PROPERTY 3 :

(i) $\mathbf{sin^{-1}(-x) = -sin^{-1}x}$ Provided that $-1 \le x \le 1$

Let $\sin^{-1}(-x) = \theta$

or $-x = \sin\theta \Rightarrow x = -\sin\theta$ or $x = \sin(-\theta)$

or $-\theta = \sin^{-1}x$ or $\theta = -\sin^{-1}x$

$\sin^{-1}(-x) = -\sin^{-1}x$

(ii) $\mathbf{cos^{-1}(-x) = \pi - cos^{-1}x}$ Provided that $-1 \le x \le 1$

(iii) $\mathbf{tain^{-1}(-x) = -tan^{-1}x}$ Provided that $-\infty < x < \infty$

(iv) $\mathbf{cot^{-1}(-x) = \pi - cot^{-1}x}$ Provided that $-\infty < x < \infty$

(v) $\mathbf{sec^{-1}(-x) = \pi - sec^{-1}x}$ Provided that $-\infty < x \le 1$ or $1 \le x < \infty$

(vi) $\mathbf{cosec^{-1}(-x) - cosec^{-1}x}$ Provided that $-\infty < x \le -1$ or $1 \le x < \infty$

Illustration 11

Evaluate the following :

(i) $\mathbf{tan^{-1}(-1)}$ **(ii)** $\mathbf{cot^{-1}\ (-1)}$ **(iii)** $\mathbf{sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)}$

Solution :

Here we have to take principal values of the inverse functions.

(i) Let $\tan^{-1}(-1) = \theta$

Then $\tan\theta = -1$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

The only value of θ in the given interval to satisfy $\tan\theta = -1$ is $-\frac{\pi}{4}$.

Hence, $\tan^{-1}(-1) = -\frac{\pi}{4}$.

(ii) Let $\cot^{-1}(-1) = \theta$, then $\cot\theta = -1$ and $0 < \theta < \pi$.

$\Rightarrow \quad \pi = \frac{3\pi}{4}$ i.e. $\cot^{-1}(-1) = \frac{3\pi}{4}$.

(iii) Let $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \theta$

$\Rightarrow \quad \sin\theta = -\frac{\sqrt{3}}{2}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\Rightarrow \quad \theta = -\frac{\pi}{3}$ i.e. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

Illustration 12

(i) $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ **(ii)** $\sin\left[\text{arc cos}\left(-\frac{1}{2}\right)\right]$ **(iii)** $\left[\tan^{-1}\left(-\sqrt{3}\right) + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

Solution :

(i) $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$

$= \sin\left[\frac{\pi}{3} - \left\{-\sin^{-1}\left(-\frac{1}{2}\right)\right\}\right]$ $\quad [\because \sin^{-1}(-x) = -\sin^{-1}x,\ x > 0]$

$= \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right] = \sin\frac{\pi}{2} = 1.$

(ii) $\sin\left[\text{arc}\cos\left(-\frac{1}{2}\right)\right] = \sin\left[\pi - \cos^{-1}\frac{1}{2}\right]$ $\qquad \left[\because \cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\frac{1}{2}\right]$

$= \sin\left[\pi - \frac{\pi}{3}\right] = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}.$

(iii) $\sin\left[\tan^{-1}\left(-\sqrt{3}\right) + \cos^{-1}\left(-\frac{-\sqrt{3}}{2}\right)\right]$

$= \sin\left[-\frac{\pi}{3} + \pi - \frac{\pi}{6}\right] = \sin\left[\frac{\pi}{2}\right] = 1.$

Illustration 13

Evaluate $\tan\left[\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right]$

Solution :

Let $\cos^{-1}\left(\frac{\sqrt{5}}{3}\right) = 2\theta$ then $\cos 2\theta = \frac{\sqrt{5}}{3}$ and $0 \le 2\theta \le \pi$

Now, $\cos 2\pi = \frac{\sqrt{5}}{3}$

or, $\frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{\sqrt{5}}{3}$ or $\frac{1 + \tan^2\theta}{1 - \tan^2\theta} = \frac{3}{\sqrt{5}}$

or, $\frac{2\tan^2\theta}{2} = \frac{3 - \sqrt{5}}{3 + \sqrt{5}}$ [By componendo and dividendo]

or, $\tan^2\theta = \frac{3 - \sqrt{5}}{3 + \sqrt{5}} \cdot \frac{3 - \sqrt{5}}{3 - \sqrt{5}}$

or, $\tan^2\theta = \frac{\left(3 - \sqrt{5}\right)^2}{4} \Rightarrow \tan\theta = \frac{\left(3 - \sqrt{5}\right)}{2}$

But $0 < 2\theta < \pi$ $\qquad \therefore 0 \le \theta \le \frac{\pi}{2}$

$\Rightarrow$ θ lies in the first quadrant.

Hence θ is $\tan\theta = \frac{3 - \sqrt{5}}{2}$

PROPERTY 4 : Conversion Property :

(i) $\sin^{-1} = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right); -1 \le x \le 1$ **and** $\operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right), x \in \mathbf{R} - (-1,1)$

Let $\sin^{-1}x = y \quad \Rightarrow x = \sin y \Rightarrow \operatorname{cosec} y = \frac{1}{x} \quad \Rightarrow y = \operatorname{cosec}^{-1}\frac{1}{x}$

$\Rightarrow \quad \sin^{-1}x = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

(ii) $\cos^{-1}x = \sec^{-1}\left(\frac{1}{x}\right) \quad -1 \le x \le 1$ **and** $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right), x \in \mathbf{R} - (1, 1)$

(iii) $\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right) \quad x \in \mathbf{R}$ **and** $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right), x > 0 = \pi + \tan^{-1}\left(\frac{1}{x}\right) x < 0$

$$= \pi + \tan^{-1}\left(\frac{1}{x}\right) x < 0$$

PROPERTY 5 :

1. $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \quad x \in [-1,1]$

2. $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2} \quad x \in \mathbf{R}$

3. $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$

PROPERTY : 6

(1) If $x > 0$, $y > 0$ then

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \qquad \text{if } xy < 1$$

$$= \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) \qquad \text{if } xy > 1$$

(2) If $x > 0$, $y > 0$ then

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

(3) If x > 0

$$\left.\begin{array}{ll} \sin^{-1} x = \cos^{-1}\sqrt{1-x^2}, & \cos^{-1} x = \sin^{-1}\sqrt{1-x^2} \\ \sin^{-1} x = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}, & \cos^{-1} x = \tan^{-1}\dfrac{\sqrt{1-x^2}}{2} \end{array}\right\} -1 < x < 1$$

Illustration 14

Obtain the values of $\cos^{-1}\left(-\frac{3}{5}\right)+\sin^{-1}\left(-\frac{5}{13}\right)$ in terms of $\cos^{-1}$ function.

Solution :

$$\cos^{-1}\left(-\frac{3}{5}\right)+\sin^{-1}\left(-\frac{5}{13}\right)$$

$$= \pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13}\right) \qquad \left\{\text{as } \cos^{-1}\left(\frac{-3}{5}\right) = \pi - \cos^{-1}\left(\frac{3}{5}\right) \;\&\; \sin^{-1}\left(\frac{-5}{13}\right) = -\sin^{-1}\left(\frac{5}{13}\right)\right\}$$

Let $\sin^{-1}\frac{4}{5} = \alpha \Rightarrow \sin\alpha = \frac{4}{5}$

$\sin^{-1}\frac{5}{13} = \beta \Rightarrow \sin\beta = \frac{5}{13}$

consider $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{3}{5}.\frac{12}{13} - \frac{4}{5}.\frac{5}{13} = \frac{16}{65}$

$\Rightarrow \quad \alpha + \beta = \cos^{-1}\frac{16}{65}$ $(\alpha, \beta \in \text{quadrant } 1)$

$\therefore$ Given quantity $= \pi - \cos^{-1}\frac{16}{65} = \cos^{-1}\left(-\frac{16}{65}\right)$

Property-7

(i) $2\tan^{-1} x = \tan^{-1}\dfrac{2x}{1-x^2}, \; -1 < x < 1$

(ii) $2\tan x = \sin^{-1}\dfrac{2x}{1-x^2}, \; -1 < x < 1$

(iii) $2\tan^{-1} x = \cos^{-1}\dfrac{1-x^2}{1+x^2}, x > 0$

Solution :

(i) Let x = tanθ; $\frac{\pi}{4} < \theta < \frac{\pi}{4}$ (using – 1 < x < 1)

R.H.S. $= \tan^{1} \frac{2\tan\theta}{1-\tan^2\theta}$

$= \tan^{-1} \tan 2\theta$

$= 2\theta = 2\tan^{-1} x =$ L.H.S.

(ii) Let x = tanθ, $\frac{\pi}{4} < \theta < \frac{\pi}{4}$ (using – 1 < x < 1)

R.H.S. $= \sin^{-1} \frac{2\tan\theta}{1+\tan^2\theta}$

$= \sin^{-1} \sin 2\theta$

$= 2\theta \left(\because -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}\right)$

$= 2\tan^{-1}x =$ L.H.S.

(iii) Let x = tanθ, 0 < θ < π/2 (using x > 0)

R.H.S. $= \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1} \cos 2\theta$

$= 2\theta$

$= 2\tan^{-1} x$

= L.H.S.

Illustration 15

Show that $\tan^{-1} 1/3 + \tan^{-1} 1/2 = \pi/4$.

Solution :

L.H.S. $= \tan^{-1} 1/3 + \tan^{-1} 1/2$

$$= \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\frac{1}{2}}\right)\left(\because \frac{1}{3}\frac{1}{2} < 1\right)$$

$$= \tan^{-1}\left(\frac{5/6}{5/6}\right) = \tan^{-1} 1 = \frac{\pi}{4} = \text{R.H.S.}$$

Illustration 16

Prove that $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Solution :

$$2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{8}=2\tan^{-1}\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}.\frac{1}{8}}+\tan^{-1}\frac{1}{7}$$

$$=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{2.\frac{1}{3}}{1-\frac{1}{9}}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}.\frac{1}{7}}=\tan^{-1}\frac{25}{25}=\tan^{-1}1=45^\circ=\frac{\pi}{4}$$

Illustration 17

Prove that $\tan^{-1}\sqrt{\left\{\frac{a(a+b+c)}{bc}\right\}}+\tan^{-1}\sqrt{\left\{\frac{b(a+b+c)}{ac}\right\}}+\tan^{-1}\sqrt{\left\{\frac{c(a+b+c)}{ab}\right\}}=\pi$

Solution :

$$\text{L.H.S.}=\tan^{-1}\frac{\sqrt{\left\{\frac{a(a+b+c)}{bc}\right\}}+\sqrt{\left\{\frac{b(a+b+c)}{ac}\right\}}}{1-\sqrt{\left\{\frac{a(a+b+c)}{bc}\right\}}.\sqrt{\left\{\frac{b(a+b+c)}{ac}\right\}}}+\tan^{-1}\sqrt{\left\{\frac{c(a+b+c)}{ab}\right\}}$$

$$=\tan^{-1}\left(\sqrt{\frac{a+b+c}{abc}}\;\frac{(a+b)}{1-(a+b+c)/c}\right)+\tan^{-1}\sqrt{\left\{\frac{c(a+b+c)}{ab}\right\}}$$

$$=\tan^{-1}\left[\sqrt{\frac{(a+b+c)c}{ab}}\right]+\tan^{-1}\left[-\sqrt{\left[\frac{c(a+b+c}{ab}\right]}\right]$$

$$=\pi-\tan^{-1}\left[\sqrt{\frac{c(a+b+c)}{ab}}\right]+\tan^{-1}\left[-\sqrt{\left[\frac{c(a+b+c}{ab}\right]}\right]=\pi$$

PROPERTIES 8 :

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right] \quad \text{where } x \ge 0,\ y \ge 0,\ x^2 + y^2 \le 1$$

$$\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right] \quad \text{where } x \ge 0,\ y \ge 0,\ x^2 + y^2 > 1$$

$$\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right];\ 0 \le 0,\ y \le x \quad \text{where } x \ge 0,\ y \ge 0$$

$$\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left[xy - \sqrt{1-x^2}\,\sqrt{1-y^2}\right] \quad \text{where } x \ge 0,\ y \ge 0$$

$$\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left[xy + \sqrt{1-x^2}\,\sqrt{1-y^2}\right] \quad \text{where } 0 \le x \le y$$

Illustration 18

If $\sin^{-1} 6x + \sin^{-1} 6\sqrt{3}\,x = -\frac{\pi}{2}$, then find the value of x

Solution :

$$\sin^{-1} 6\sqrt{3}\,x = -\frac{\pi}{2} - \sin^{-1} 6x = -(\sin^{-1} 1 + \sin^{-1} 6x) = \sin^{-1}\left(1.\sqrt{1-(6x)^2}\right) - 6x.\sqrt{1-1})$$

$$\Rightarrow \quad -\sin^{-1}\sqrt{1-36x^2} = \sin^{-1}(-\sqrt{1-36x^2})$$

$$\Rightarrow \quad 6\sqrt{3}\,x = -\sqrt{1-36x^2}$$

$$\Rightarrow \quad 108x^2 = 1 - 36x^2 \Rightarrow 144x^2 = 1 \Rightarrow x^2 = \frac{1}{144} \Rightarrow x = \pm\frac{1}{12}$$

But only $x = -\frac{1}{12}$ satisfies the equation.

Illustration 19

Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1} 3 = 45°$

Solution : $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}.\frac{1}{3}} = \tan^{-1} 1 = 45°$

Let $\cot^{-1} 3 = \theta$ or $\cot\theta = 3$

$$\therefore \quad \sin\theta = \frac{1}{\sqrt{10}} \qquad\qquad \therefore$$

$$\theta = \sin^{-1}\left(\frac{1}{\sqrt{10}}\right)$$

$$\therefore \quad \sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = \sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}}$$

$$= \sin^{-1}\left\{\frac{1}{\sqrt{5}}.\frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}}.\frac{2}{\sqrt{5}}\right\} = \sin^{-1}\frac{5}{\sqrt{10}} = \sin^{-1}\frac{1}{\sqrt{2}} = 45°$$

Illustration 20

$$\mathbf{\sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)}$$

Solution :

$$\text{L.H.S.} = \sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) \text{ or } \cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) \qquad \left[\because \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}\right]$$

$$\text{or} \quad \cos^{-1}\left[\frac{4}{5}\times\frac{12}{13} - \sqrt{1-\frac{16}{25}}\sqrt{1-\frac{155}{169}}\right] \text{ or } \cos^{-1}\left[\frac{4}{5}\times\frac{12}{13} - \sqrt{\frac{9}{25}}\sqrt{\frac{25}{169}}\right]$$

$$\text{or} \quad \cos^{-1}\left[\frac{48}{65} - \frac{15}{65}\right] \text{ or } \cos^{-1}\left[\frac{33}{65}\right] = \text{RHS}$$

PROPERTY-9

(i)
$$2\sin^{-1}x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2} & , \text{ if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}) & , \text{ if } \frac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}) & , \text{ if } -1 \le x \le -\frac{1}{\sqrt{2}} \end{cases}$$

Let $\sin^{-1} x = \theta$ then $\sin\theta = x$

so, $\sin 2\theta = 2\sin\theta\cos\theta$ or $\sin 2\theta = 2x\sqrt{1-x^2}$ $\left[\because \cos\theta = \sqrt{1-\sin^2\theta}\right]$

or $2\theta = \sin^{-1}(2x\sqrt{1-x^2})$ or $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

(ii) $$3\sin^{-1}x = \begin{cases} \sin^{-1}(3x-4x^3) & , \text{ if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - \sin^{-1}(3x-4x^3) & , \text{ if } \frac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x-4x^3) & , \text{ if } -1 \le x \le -\frac{1}{2} \end{cases}$$

Let $\sin^{-1}x = \theta$ then $x = \sin\theta$

Now, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ or $\sin 3\theta = 3x - 4x^3$

or $3\theta = \sin^{-1}(3x - 4x^3)$ or $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$

(iii) $$2\cos^{-1}x = \begin{cases} \cos^{-1}(2x^2-1) & , \text{ if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2-1) & , \text{ if } -1 \le x \le 0 \end{cases}$$

Let $\cos^{-1}x = \theta$ then $x = \cos\theta$

so, $\cos 2\theta = 2\cos^2\theta - 1$ or $\cos 2\theta = 2x^2 - 1$

or $2\theta = \cos^{-1}(2x^2-1)$ or $2\cos^{-1}x = \cos^{-1}(2x^2-1)$

(iv) $$3\cos^{-1}x = \begin{cases} \cos^{-1}(4x^3-3x) & , \text{ if } \frac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3-3x) & , \text{ if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 2\pi + \cos^{-1}(4x^3-3x) & , \text{ if } -1 \le x - \frac{1}{2} \end{cases}$$

Let $\cos^{-1}x = \theta$ then $x = \cos\theta$

Now, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ or $\cos 3\theta = 4x^3 - 3x$

or $3\theta = \cos^{-1}(4x^3-3x)$ or $3\cos^{-1}x = \cos^{-1}(4x^3-3x)$

Again, $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$ or $\sin 2\theta = \dfrac{2x}{1+x^2} \Rightarrow 2\theta = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$\Rightarrow$ $2\tan^{-1}x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ and $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

or $\cos 2\theta = \dfrac{1-x^2}{1+x^2}$ or $2\theta = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(v) $$3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) & , \text{ if } -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) & , \text{ if } x > \dfrac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) & , \text{ if } x < -\dfrac{1}{\sqrt{3}} \end{cases}$$

Let $\tan^{-1} x = 0$ then $x = \tan\theta$

Now, $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ or $\tan 3\theta = \dfrac{3x - x^3}{1 - 3x^2}$ or $3\theta = \tan^{-1}\left[\dfrac{3x - x^3}{1 - 3x^2}\right]$

or $3\tan^{-1} x = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$

Illustration 21

$\sin^{-1}\left(\dfrac{4x}{x^2 + 4}\right) + 2\tan^{-1}\left(-\dfrac{x}{2}\right)$ is independent at x then–

(A) $x \in [3, 4]$ **(B)** $x \in [-2, 2]$ **(C)** $x \in [-1, 1]$ **(D)** $x \in [1, \infty]$

Solution :

(B) $$\sin^{-1}\left(\frac{4x}{x^2 + 4}\right) + 2\tan^{-1}\left(-\frac{x}{2}\right) = \sin^{-1}\left(\frac{2.(x/2)}{(x/2)^2 + 1}\right) - 2\tan^{-1}\frac{x}{2} = \tan^{-1}\frac{x}{2} - 2\tan^{-1}\frac{x}{2} = 0$$

Hence, $\left|\dfrac{x}{2}\right| \le 1;\quad |x| \le 2 \Rightarrow 2 \le x \le 2$

Illustration 22

Prove that $3\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{1}{20} = \dfrac{1}{4}\pi - \tan^{-1}\dfrac{1}{1985}$

Solution :

Since $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$

$$\therefore \quad 3\tan^{-1}\frac{1}{4} = \tan^{-1}\left(\frac{3\left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)^3}{1 - 3\left(\frac{1}{4}\right)}\right) \quad \text{or} \quad 3\tan^{-1}\frac{1}{4} = \tan^{-1}\frac{47}{52}$$

$$\therefore \quad 3\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{20} = \tan^{-1}\frac{47}{52} + \tan^{-1}\frac{1}{20} = \tan^{-1}\frac{\frac{47}{52} + \frac{1}{20}}{1 - \frac{47}{52}\cdot\frac{1}{20}} = \tan^{-1}\frac{992}{993}$$

$$\text{and} \quad \frac{1}{4}\pi - \tan^{-1}\frac{1}{1985} = \tan^{-1}\frac{1 - \frac{1}{1985}}{1 + \frac{1}{1985}} = \tan^{-1}\frac{1984}{1986} = \tan^{-1}\frac{992}{993}$$

Illustration 23

To solve $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\,x)$

Solution :

$$\tan^{-1}\frac{2\cos x}{1-\cos^2 x} = \tan^{-1}(2\,\text{cosec}\,x) \quad \text{or} \quad \frac{2\cos x}{\sin x} = \frac{2}{\sin x} \quad \text{or} \quad \sin x\,(\sin x - \cos x) = 0$$

Hence, either $\sin x = 0 \quad \Rightarrow \quad x = n\pi.$

or $\quad \sin x - \cos x = 0 \quad \Rightarrow \quad x = \frac{\pi}{4} = n\pi + \frac{\pi}{4}$

Illustration 24

Prove that $\tan^{-1}\left(\frac{1}{2}\tan 2\theta\right) + \tan^{-1}(\cot\theta) + \tan^{-1}(\cot^3\theta) = 0$**, if** $\frac{\pi}{4} < \theta < \frac{\pi}{2}$

$= \pi$**, if** $0 < \theta < \frac{\pi}{4}$

Solution :

Case I : If $0 < \theta < \frac{\pi}{4}$, then $\cot\theta > 1$, $\cot^3\theta > 1$

$$\therefore \quad \tan^{-1}(\cot\theta) + \tan^{-1}(\cot^3\theta) = \pi + \tan^{-1}\left\{\frac{\cot\theta + \cot^3\theta}{1 - \cot^4\theta}\right\}$$

$$= \pi + \tan^{-1}\left\{-\frac{\cot\theta.\text{cosec}^2\theta.\sin^4\theta}{\cos^4\theta - \sin^4\theta}\right\} = \pi + \tan^{-1}\left\{\frac{-\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}\right\}$$

$$= \pi + \tan^{-1}\left\{-\frac{1}{2}\tan 2\theta\right\} = \pi - \tan^{-1}\left(\frac{1}{2}\tan 2\theta\right) \text{ since } 2\theta < \frac{\pi}{2} \text{ and } \tan 2\theta > 0$$

$$\therefore \quad \tan^{-1}\left\{\frac{1}{2}\tan 2\theta\right\} + \tan^{-1}(\cot\theta) + \tan^{-1}(\cot^3\theta) = \pi$$

Case II : If $\frac{\pi}{4} < \theta < \frac{\pi}{2}, 0 < \cot\theta < 1, 0 < \cot^3\theta < 1$

$$\therefore \quad \tan^{-1}(\cot\theta) + \tan^{-1}(\cot^3\theta) = \tan^{-1}\left\{-\frac{1}{2}\tan 2\theta\right\}$$

$$= -\tan^{-1}\left(\frac{1}{2}\tan 2\theta\right) \quad \{\text{since } 2\theta > \pi \text{ and } \tan 2\theta < 0\}$$

$$\therefore \quad \tan^{-1}\left(\frac{\tan 2\theta}{2}\right) + \tan^{-1}(\cot\theta) + \tan^{-1}(\cot^3\theta) = 0$$

Illustration 25

If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

prove that $x^2 + y^2 + z^2 + 2xyz = 1$

Solution :

Given $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z = \cos^{-1}(-z)$

$\Rightarrow \quad \cos[\cos^{-1}x + \cos^{-1}y] = \cos[\cos^{-1}(-z)]$

Let $\quad \cos^{-1}x = A$

$\quad \cos^{-1}y = B$

$\therefore \quad \cos(A + B) = \cos A \cos B - \sin A \sin B$

$\therefore \quad \cos(A + B) = xy - \sqrt{1-x^2}\sqrt{1-y^2}$

$\therefore \quad (A + B) = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$

$\Rightarrow \quad \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\} = \cos^{-1}(-z) \quad \Rightarrow \quad xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$\Rightarrow \quad (xy + z)^2 = (1 - x^2)(1 - y^2) \quad \Rightarrow \quad x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow \quad x^2 + y^2 + z^2 + 2xyz = 1$

Hence, proved.

Illustration 26

Write in the simplest form :

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) \text{ where } -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$$

Solution :

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right)$$

$$= \tan^{-1}\left(\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right) = \frac{\pi}{4}-\frac{x}{2}$$

Differentiation and integration of inverse trigonometric functions :

To differentiate the arc tangent function, we imitate the method we used to differentiate the logarithm function. Namely, if $y = \tan^{-1}(x)$, then $\tan(y) = x$, so

$$\frac{d}{dx}\tan(y) = \frac{d}{dx}x \text{ Hence } \sec^2(y)\frac{dy}{dx} = 1$$

from which it follows that $\frac{dy}{dx} = \frac{1}{\sec^2(y)}$

Now, $\sec^2(y) = 1 + \tan^2(y) = 1 + x^2$,

so we have $\frac{dy}{dx} = \frac{1}{1+x^2}$

Hence we have demonstrated the following proposition.

Proposition : $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$

As a consequence of the proposition, we have $\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + c$

$1 + x^2$ is an irreducible polynomial. We will see more example of this type in the following examples.

Using the chain rule, we have $\frac{d}{dx}\tan^{-1}(4x^2) = \frac{8x}{1+16x^4}$

Evaluating $\int \tan^{-1}(x)\,dx$ is similar to evaluating $\int \log(x)\,dx$. That is, we will use integration by parts with

$$u = \tan^{-1}(x) \qquad v = x$$

$$du = \frac{1}{1+x^2}dx \qquad dv = dx$$

Then, $\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}dx$

Using the substitution, $u = 1 + x^2$

$du = 2x\,dx,$

we have, $\frac{1}{2}du = x\,dx,$ from which it follows that

$$\int \frac{x}{1+x^2}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\log|u| + c = \frac{1}{2}\log(1+x^2) + c$$

Thus, $\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \frac{1}{2}\log(1+x^2) + c$

Illustration 27

To evaluate $\int \frac{1}{1+4x^2}dx,$ we make the substitution

$u = 2x$

$du = 2\,dx$

Then $\frac{1}{2}du = dx,$ so $\int \frac{1}{1+4x^2}dx = \frac{1}{2}\int \frac{1}{1+u^2}du = \frac{1}{2}\tan^{-1}(u) + c = \frac{1}{2}\tan^{-1}(2x) - c$

Illustration 28

Prove that $\mathbf{4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \frac{\pi}{4}}$

Solution :

We have to prove that $4\tan^{-1}\frac{1}{5} = \frac{\pi}{4} + \tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99}$ ...(1)

Now, L.H.S. $= 4\tan^{-1}\left(\frac{1}{5}\right) = 2\tan^{-1}\left[\frac{2.\frac{1}{5}}{1-\frac{1}{25}}\right]$ $\left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$

$$= 2\tan^{-1}\frac{5}{12} = \tan^{-1}\left[\frac{2.\frac{5}{12}}{1-\frac{25}{144}}\right] = \tan^{-1}\frac{120}{119}$$

$$\tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99} = \tan^{-1}\left[\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}.\frac{1}{99}}\right]$$

$$= \tan^{-1}\left(\frac{29}{6931}\right) = \tan^{-1}\frac{1}{239}$$

$\therefore$ R.H.S. $= \frac{\pi}{4} + \tan^{-1}\frac{1}{239} = \tan^{-1}1 + \tan^{-1}\frac{1}{239}$

$$= \tan^{-1}\left[\frac{1+\frac{1}{239}}{1-\frac{1}{239}}\right] = \tan^{-1}\frac{240}{238}$$

$$= \tan^{-1}\frac{120}{119}$$

$\therefore$ L.H.S. = R.H.S. Hence the result.

Illustration 29

Prove that $\cos\left(\frac{1}{3}\sin^{-1}a\right)$ has six values and that the product of the six values is $-\frac{1}{16}(1-a^2)$.

Solution :

If $\theta = \sin^{-1}a$ and $t = \cos\frac{\theta}{3}$ then $\sin\theta = a$ and $\cos^2\theta = 1 - a^2$,

$\cos\theta = \cos 3.\frac{\theta}{3} = 4t^3 - 3t$ and t satisfies the polynomial equation of 6th degree

$16t^6 - 24t^4 + 9t^2 - 1 + a^2 = 0$

This has 6 roots and accordingly $\cos\left(\frac{1}{3}\sin^{-1}a\right)$ has six values whose product is $\frac{a^2-1}{16}$

Illustration 30

Show that $2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right] = \cos^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right]$ **for** $0 < b \le a$**, and** $x \ge 0$**.**

Solution :

$0 < b\ a,\qquad \therefore\ \sqrt{\frac{a-b}{a+b}}$ is real.

Now, L.H.S. $= 2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right]$

$$= \cos^{-1}\left[\frac{1-\frac{a-b}{a+b}\tan^2\frac{x}{2}}{1+\frac{a-b}{a+b}\tan^2\frac{x}{2}}\right] \qquad \left[\because 2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}\right]$$

$$= \cos^{-1}\left[\frac{a+b-(a-b)\tan^2\frac{x}{2}}{a+b+(a-b)\tan^2\frac{x}{2}}\right] = \cos^{-1}\left[\frac{a\left(1-\tan^2\frac{x}{2}\right)+b\left(1+\tan^2\frac{x}{2}\right)}{a\left(1+\tan^2\frac{x}{2}\right)+b\left(1-\tan^2\frac{x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\cot A\left(1+\cot^2 A\right)}{1-\cot^4 A}\right] = \tan^{-1}\left[\frac{\cot A}{1-\cot^2 A}\right]$$

$$= \tan^{-1}\left[\frac{\frac{1}{\tan A}}{1-\frac{1}{\tan^2 A}}\right] = \tan^{-1}\left(\frac{\tan A}{\tan^2 A - 1}\right)$$

$$= \tan^{-1}\left(-\frac{1}{2}.\frac{2\tan A}{1-\tan^2 A}\right) = -\tan^{-1}\left(\frac{1}{2}\tan 2A\right)$$

$\Rightarrow \quad \tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot 3A) = 0$

Case II : When $0 < A < \frac{\pi}{4}$

$\cot A > 1$ and $\cot^3 A > 1$.

$\therefore \quad \cot A.\cot^3 A > 1$

Hence, $\tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$

$$= \pi + \tan^{-1}\left[\frac{\cot A + \cot^3 A}{1 - \cot A.\cot^3 A}\right]$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right] \quad \text{if } x > 0,\ y > 0 \text{ and } xy > 1]$$

$$= \pi - \tan^{-1}\left(\frac{1}{2}\tan 2A\right) \quad \text{[as shown in case I]}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A) = \pi$$

Hence the result.

Illustration 31

If $\cos^{-1}\frac{x}{2} + \cos^{-1}\frac{y}{3} = \theta$, prove that $9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$

Solution :

Let $\cos^{-1}\frac{x}{2} = \alpha$, and $\cos^{-1}\frac{y}{3} = \beta$

$\therefore \quad \cos\alpha = \frac{x}{2}$ and $\cos\beta = \frac{y}{3}$

Given, $\alpha + \beta = \theta$ $\therefore$

$\cos(\alpha + \beta) = \cos\theta$

or, $\quad \cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos\theta$

or, $\quad \frac{x}{2}.\frac{y}{3} - \sqrt{1 - \frac{x^2}{4}}.\sqrt{1 - \frac{y^2}{9}} = \cos\theta$

or, $\quad \dfrac{xy}{6} - \dfrac{\sqrt{4-x^2}\,.\sqrt{9-y^2}}{6} = \cos\theta$

or, $\quad (xy - 6\cos\theta)^2 = (4 - x^2)\,(9 - y^2)$

or, $\quad x^2y^2 + 36\cos^2\theta - 12xy\cos\theta = 36 - 9x^2 - 4y^2 + x^2y^2$

or, $\quad 9x^2 - 12y\cos\theta + 4y^2 = 36\,(1 - \cos^2\theta)$

or, $\quad 9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta.$

Illustration 32

If $r = x + y + z$, **prove that** $\tan^{-1}\sqrt{\dfrac{xr}{yz}} + \tan^{-1}\sqrt{\dfrac{yr}{zx}} + \tan^{-1}\sqrt{\dfrac{zr}{xy}} = \pi$

Solution :

Let $\sqrt{\dfrac{xr}{yz}} = \alpha,\ \sqrt{\dfrac{yr}{zx}} = \beta$ and $\sqrt{\dfrac{zr}{xy}} = \gamma$

Then L.H.S. $= \tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma$

$$= \tan^{-1}\left[\frac{\alpha+\beta+\gamma-\alpha\beta\gamma}{1-\alpha\beta-\beta\gamma-\gamma\alpha}\right]$$

Now, $\alpha + \beta + \gamma - \alpha\beta\gamma = \sqrt{\dfrac{xr}{yz}} + \sqrt{\dfrac{yr}{zx}} + \sqrt{\dfrac{zr}{xy}} - \sqrt{\dfrac{xr}{yz}}.\sqrt{\dfrac{yr}{zx}}.\sqrt{\dfrac{zr}{xy}}$

$$= \frac{x\sqrt{r} + y\sqrt{r} + z\sqrt{r}}{\sqrt{xyz}} - \frac{r\sqrt{r}}{\sqrt{xyz}}$$

$$= \frac{\sqrt{r}\,[x+y+z]}{\sqrt{xyz}} - \frac{r\sqrt{r}}{\sqrt{xyz}}$$

$$= \frac{r\sqrt{r}}{\sqrt{xyz}} - \frac{r\sqrt{r}}{\sqrt{xyz}} = 0 \qquad [\because x + y + z = r]$$

Also, $1 - \alpha\beta - \beta\gamma - \gamma\alpha = 1 - \sqrt{\dfrac{xr}{yz}}.\sqrt{\dfrac{yr}{zx}} - \sqrt{\dfrac{yr}{zx}}.\sqrt{\dfrac{zr}{xy}} - \sqrt{\dfrac{zr}{xy}}.\sqrt{\dfrac{xr}{yz}}$

$$= 1 - \frac{r}{z} - \frac{r}{x} - \frac{r}{y}$$

$$= 1 - r\left[\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right] \neq 0 \qquad \left[\because \frac{1}{x}+\frac{1}{y}+\frac{1}{z} \neq \frac{1}{r}\right]$$

$\therefore$ L.H.S. $= \tan^{-1}0$

$= n\pi$ $\qquad [\therefore \tan^{-1}0 = \theta \Rightarrow \tan\theta = 0 \Rightarrow \theta = n\pi]$

$= -\pi, 0, \pi$ $\qquad$ [for principal values]

$= \pi$ $\qquad$ [since sum of three positive angles cannot be zero or negative]

$=$ R.H.S.

Note 1 : For principal values

$$-\frac{\pi}{2} < \tan^{-1}\alpha < \frac{\pi}{2}, -\frac{\pi}{2} < \tan^{-1}\beta < \frac{\pi}{2}, -\frac{\pi}{2} < \tan^{-1}\gamma < \frac{\pi}{2}$$

$$\therefore \quad -\frac{3\pi}{2} < \tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma < \frac{3\pi}{2}$$

Hence, we have selected $-\pi$, 0 and π only, which satisfy the above condition.

2. $\tan^{-1}\alpha = \tan^{-1}\sqrt{\frac{xr}{yz'}}$

$\therefore$ $\sqrt{\frac{xr}{yz}}$ is positive, for principal value $\tan^{-1}\sqrt{\frac{xr}{yz}}$ will represent a positive angle i.e. $\tan^{-1}\alpha$ is a positive angle. Similarly, $\tan^{-1}\beta$ and $\tan^{-1}\gamma$ are also positive angles. Sum of three positive angles is positive. Hence only π has been selected out of $-\pi$, 0 and π.

Illustration 33

If $p > q > 0$ **and** $pr < -1 < qr$**, then prove that** $\tan^{-1}\frac{p-q}{1+pq} + \tan^{-1}\frac{q-r}{1+qr} + \tan^{-1}\frac{r-p}{1+rp} = \pi$

Solution :

Since, p, q > 0, therefore for pq > 0, $\tan^{-1}\frac{p-q}{1+pq} = \tan^{-1}p - \tan^{-1}q$ ...(1)

Since $qr > -1$, $\tan^{-1}\frac{r-p}{1+qr} = \tan^{-1}q - \tan^{-1}r$ ...(2)

Since $pr < -1$ and $r < 0$ $\tan^{-1}\frac{r-p}{1+rp} = \pi + \tan^{-1}r - \tan^{-1}p$ ...(3)

On adding (1), (2) and (3) we get $\tan^{-1}\frac{p-q}{1+pq} + \tan^{-1}\frac{q-r}{1+qr} + \tan^{-1}\frac{r-p}{1+rp} = \pi$

Illustration 34

If $\tan^{-1}y = 4\tan^{-1}x$ $\left(|x| < \tan\frac{\pi}{8}\right)$, find y as an algebraic function of x and hence prove that $\tan\frac{\pi}{8}$ is a root of the equation $x^4 - 6x^2 + 1 = 0$.

Solution :

We have $\tan^{-1}y = 4\tan^{-1}x = 2\tan^{-1}\dfrac{2x}{1-x^2}$ (as $|x| < 1$)

$$= \tan^{-1}\frac{\frac{4x}{(1-x^2)}}{1 - \frac{4x^2}{(1-x^2)^2}} = \tan^{-1}\frac{4x(1-x^2)}{x^4 - 6x^2 + 1} \qquad \left(\text{as } \left|\frac{2x}{1-x^2}\right| < 1\right)$$

$$\Rightarrow \quad y = \frac{4x(1-x^2)}{x^4 - 6x^2 + 1}$$

If $x = \tan\frac{\pi}{8} \Rightarrow \tan^{-1}y = 4\tan^{-1}x = \frac{\pi}{2} \Rightarrow y = \infty \Rightarrow x^4 - 6x^2 + 1 = 0$

Illustration 35

Show that $(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = \alpha\pi^3$ has no real solutions for $\alpha < 1/32$.

Solution :

LHS $= (\sin^{-1} x + \cos^{-1} x)\,((\sin^{-1} x)^2 - (\cos^{-1} x)(\sin^{-1} x) + (\cos^{-1} x)^2) = \alpha\pi^3$

$$\Rightarrow \quad \frac{\pi}{2}((\sin^{-1} x)^2 - (\cos^{-1} x)(\sin^{-1} x) + (\cos^{-1} x)^2) = \alpha\pi^3$$

$$\Rightarrow \quad \frac{\pi}{2}\left(\frac{\pi^2}{4} - 3\cos^{-1} x \sin^{-1} x\right) = \alpha\pi^3 \Rightarrow \frac{\pi^2}{8} - \frac{3}{2}.\frac{\pi}{2}\sin^{-1} x + \frac{3}{2}(\sin^{-1} x)^2 = \alpha\pi^2$$

$\Rightarrow \quad 12y^2 - 6\pi y + \pi^2(1 - 8\alpha) = 0$, where $y = \sin^{-1} x$ ...(1)

In order to eq. (1) to have real roots we must have

$$D \geq 0 \Rightarrow 36\pi^2 - 4.12\pi^2(1 - 8\alpha) \geq 0 \Rightarrow 3 - 4 + 32\alpha \geq 0\ \alpha \geq \Rightarrow \frac{1}{32}$$

Illustration 36

Find the positive integral solutions x and y of the equation.

$$\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)+\cos^{-1}\frac{y}{\sqrt{1+y^2}}=\tan^{-1}(3)$$

Solution :

We have, $\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)+\cos^{-1}\frac{y}{\sqrt{1+y^2}}=\tan^{-1}(3)$

$$\tan^{-1}x+\tan^{-1}\frac{1}{y}=\tan^{-1}3$$

$$\tan^{-1}x-\tan^{-1}3=\tan^{-1}\frac{1}{y}$$

$$\tan^{-1}3-\tan^{-1}x=\tan^{-1}\frac{1}{y}\Rightarrow\tan^{-1}\frac{3-x}{1+3x}=\tan^{-1}\frac{1}{y}$$

since x and y are positive integers $\Rightarrow x = 1, 2 \Rightarrow x = 1, y = 1$

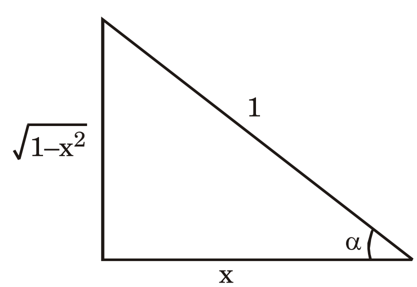
$x = 2, y = 7$

Illustration 37

Solve $\cos^{-1}x\sqrt{3}+\cos^{-1}x=\frac{\pi}{2}$

Solution :

Given, $\cos^{-1}x\sqrt{3}+\cos^{-1}x=\frac{\pi}{2}$ ...(i)



or, $\cos^{-1}x\sqrt{3}=\frac{\pi}{2}-\cos^{-1}x$

or, $\cos\cos^{-1}x\sqrt{3}=\cos\left(\frac{\pi}{2}-\cos^{-1}x\right)$

or, $x\sqrt{3}=\sin\cos^{-1}$

or, $x\sqrt{3}=\sin\sin^{-1}\sqrt{1-x^2}$

or, $x\sqrt{3} = \sqrt{1 - x^2}$

Squaring we get $3x^2 = 1 - x^2$

or, $4x^2 = 1 \Rightarrow x = \pm\frac{1}{2}$

Check : When $x = \frac{1}{2}$,

L.H.S. of eqn. (i) $= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{6} + \frac{\pi}{3} + \frac{\pi}{2}$ = R.H.S. of equation (i)

When $x = -\frac{1}{2}$,

L.H.S. of equation (i) $= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$

$= \pi - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \pi - \cos^{-1}\left(\frac{1}{2}\right)$

$= \pi - \frac{\pi}{6} + \pi - \frac{\pi}{3} = \frac{3\pi}{2} \neq$ R.H.S. of equation (i)

$\therefore$ $x = \frac{1}{2}$ is the only solution.

Note :

1. As squaring has been done during the solution process it is necessary to check roots.

2. While solving inverse trigonometric equations roots must be checked.

3. Here $x = -\frac{1}{2}$ is an extraneous root.

Illustration 38

The greater of the angles **[IIT – 89]**

$\mathbf{A = 2\tan^{-1}\left(2\sqrt{2} - 1\right)}$ **and**

$\mathbf{B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\frac{3}{5}}$ **is...**

Solution :

$$A = 2\tan^{-1}\left(2\sqrt{2}-1\right)$$

$= 2\tan^{-1}(2 \quad 1.414 - 1)$

$= 2\tan^{-1}(1.828)$

$= 2 \quad (> 60\ )$ $\quad \left[\because \tan 60^\circ = \sqrt{3} = 1.732 \Rightarrow \tan^{-1}(1.732) = 60^\circ\right]$

> 120

$$B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$$= \sin^{-1}\left[3\times\frac{1}{3} - 4\left(\frac{1}{3}\right)^3\right] + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{23}{27}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$= \sin^{-1}(0.852) + \sin^{-1}(0.60)$

$= (< 60\ ) + (< 45\ )$

< 105

$\Rightarrow$ A > B. Hence greater angle is A.

[Note : $\sin^{-1}\dfrac{\sqrt{3}}{2} = \sin^{-1}(0.86) = 60^\circ$

$\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \sin^{-1}(.71) = 45^\circ$

$\Rightarrow \quad \sin^{-1}(0.852) < 60$ and $\sin^{-1}(0.60) < 45$]

Illustration 39

Prove that $\mathbf{\tan^{-1}\left(\dfrac{a_1x - y}{x + a_1y}\right) + \tan^{-1}\left(\dfrac{a_2 - a_1}{1 + a_2a_1}\right) + \tan^{-1}\left(\dfrac{a_3 - a_2}{1 + a_3a_2}\right) + \ldots}$

$$\mathbf{\tan^{-1}\left(\frac{a_n - a_{n-1}}{1 + a_na_{n-1}}\right) + \tan^{-1}\left(\frac{1}{a_n}\right) = \tan^{-1}\frac{x}{y}}$$

Solution :

When ever we have to sum trigonometric inverse terms we try to express each term as difference of two inverse terms and then add.

Here, $$\tan^{-1}\left(\frac{a_1x - y}{x + a_1y}\right) = \tan^{-1}\left(\frac{a_1 - \frac{y}{x}}{1 + a_1\frac{y}{x}}\right) = \tan^{-1}a_1 - \tan^{-1}\frac{y}{x}$$

$$\tan^{-1}\left(\frac{a_2 - a_1}{1 + a_2 a_1}\right) = \tan^{-1} a_2 - \tan^{-1} a_1$$

$$\tan^{-1}\left(\frac{a_3 - a_2}{1 + a_3 a_2}\right) = \tan^{-1} a_3 - \tan^{-1} a_2$$

$$\tan^{-1}\left(\frac{a_n - a_{n-1}}{1 + a_n a_{n-1}}\right) = \tan^{-1} a_n - \tan^{-1} a_{n-1}$$

$$\tan^{-1}\left(\frac{1}{a_n}\right) = \cot^{-1} a_n$$

Adding we get L.H.S. $= \tan^{-1} a_n + \cot^{-1} a_n - \tan^{-1} \frac{y}{x}$

$$= \frac{\pi}{2} - \tan^{-1}\frac{y}{x} \qquad \left[\because \tan^{-1} a_n + \cot^{-1} a_n = \frac{\pi}{2}\right]$$

$$= \cot^{-1}\frac{y}{x} = \tan^{-1}\frac{x}{y} = \text{R.H.S.}$$

Illustration 40

Find the sum : $\cot^{-1} 2 + \cot^{-1} 8 + \cot^{-1} 18 + \ldots$ to infinity.

Solution :

Let t_n denote the nth terms of the series.

then, $t_n = \cot^{-1} 2n^2$

or, $t_n = \cot^{-1}(2n - 1) - \cot^{-1}(2n + 1)$ ...(1)

[$\therefore$ $\cot^{-1}(2n - 1) - \cot^{-1}(2n + 1)$

$$= \cot^{-1}\left[\frac{(2n-1)(2n+1)+1}{(2n+1)-(2n-1)}\right]$$

$$= \cot^{-1}\left(\frac{(4n^2 - 1 + 1)}{2}\right) = \cot^{-1} 2n^2\,]$$

Putting n = 1, 2, 3, ... etc. in (1), we get

$t_1 = \cot^{-1} 1 - \cot^{-1} 3$

$t_2 = \cot^{-1} 3 - \cot^{-1} 5$

$t_3 = \cot^{-1} 5 - \cot^{-1} 7$

$t_n = \cot^{-1} (2n - 1) - \cot^{-1} (2n + 1)$

adding, $S_n = \cot^{-1} 1 - \cot^{-1} (2n + 1)$

as $n \to \infty$, $\cot^{-1} (2n + 1) \to 0$

Hence the required sum = $\cot^{-1} 1 = \dfrac{\pi}{4}$

Illustration 41

Show that the function $y = 2\tan^{-1} x + \sin^{-1} \dfrac{2x}{1 + x^2}$ is a constant for $x \geq 1$. Find the value of this constant.

Solution :

Note that since $x > 1$, we can not write

$$\sin^{-1}\left(\frac{2x}{1 + x^2}\right) = 2\tan^{-1} x, \text{ (for principal values)}$$

Working Rule : To prove that the given function is a constant, any one of the following methods can be used.

1. Express both terms in the same inverse function and simplify. If y is independent of x, then it will be a constant.

II. Since $y = f(x)$

$\therefore$ if $\dfrac{dy}{dx} = 0$, y will be a constant.

Here we will use method I.

Case I : For $x = 1$, the given function is

$$y = 2\tan^{-1} 1 + \sin^{-1} \frac{2.1}{1+1} = 2.\frac{\pi}{4} + \frac{\pi}{2} = \pi$$

Case II : for $x > 1$, $2\tan^{-1} x = \pi - \sin^{-1} \dfrac{2x}{1 + x^2}$

$$\Rightarrow \quad 2\tan^{-1}x + \sin^{-1} \frac{2x}{1 + x^2} = \pi$$

Thus, for $x \geq 1$, $2\tan^{-1}x + \sin^{-1} \dfrac{2x}{1 + x^2} = \pi$

Illustration 42

Using Mathematical Induction, prove that **[IIT–91]**

$$\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\ldots+\tan^{-1}\left(\frac{1}{n^2+n+1}\right)=\tan^{-1}\left(\frac{n}{n+2}\right)$$

Solution :

We have to prove that $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\ldots+\tan^{-1}\left(\frac{1}{n^2+n+1}\right)=\tan^{-1}\left(\frac{n}{n+2}\right)$ ...(1)

when n = 1

L.H.S. of (1) = $\tan^{-1}\left(\frac{1}{3}\right)$

R.H.S. of (1) = $\tan^{-1}\left(\frac{1}{1^2+1+1}\right)=\tan^{-1}\left(\frac{1}{3}\right)$

Hence result (1) is true for n = 1 ...(A)

Suppose that the result (1) is true for n = m

i.e. $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\ldots+\tan^{-1}\left(\frac{1}{m^2+m+1}\right)=\tan^{-1}\left(\frac{m}{m+2}\right)$

Adding $\tan^{-1}\frac{1}{(m+1)^2+(m+1)+1}$ to both sides, we get

Now, $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\ldots+\tan^{-1}\left(\frac{1}{m^2+m+1}\right)+\tan^{-1}\left[\frac{1}{(m+1)^2+(m+1)+1}\right]$

$$=\tan^{-1}\left(\frac{m}{m+2}\right)+\tan^{-1}\left[\frac{1}{(m+1)^2+(m+1)+1}\right]$$

$$=\tan^{-1}\left(\frac{m}{m+2}\right)+\tan^{-1}\left[\frac{1}{m^2+3m+3}\right]$$

$$=\tan^{-1}\left(\frac{m}{m+2}\right)+\tan^{-1}\left[\frac{2}{2m^2+6m+6}\right]$$

$$= \tan^{-1}\left(\frac{m}{m+2}\right) + \left[\tan^{-1}\left(\frac{m+1}{m+3}\right) - \tan^{-1}\left(\frac{m}{m+2}\right)\right]$$

$$\tan^{-1}\left(\frac{m+1}{m+3}\right) = \tan^{-1}\left(\frac{m+1}{m+1+2}\right)$$

$\Rightarrow$ the result (1) is true for n = m + 1 also it is true for n = m ... (B)

From (A) & (B), we can say that result (1) is true for any natural number n.

Hence the result.

Note : $\tan^{-1}\left(\frac{m+1}{m+3}\right) - \tan^{-1}\left(\frac{m}{m+2}\right)$

$$= \tan^{-1}\left[\frac{\frac{m+1}{m+3} - \frac{m}{m+2}}{1 + \left(\frac{m+1}{m+3}\right)\cdot\frac{m}{m+2}}\right] = \tan^{-1}\left[\frac{(m+1)(m+2) - m(m+3)}{(m+2)(m+3) + m(m+1)}\right]$$

$$= \tan^{-1}\left[\frac{m^2 + 3m + 2 - (m^2 + 3m)}{(m^2 + 5m + 6) + m^2 + m}\right] = \tan^{-1}\left(\frac{2}{2m^2 + 6m + 6}\right)$$

$$= \tan^{-1}\left(\frac{1}{m^2 + 3m + 3}\right)$$

Illustration 43

If x_1, x_2, x_3, x_4 are the roots of the equation $x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos\beta - \sin\beta = 0$, prove that $\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4 = n\pi + \frac{\pi}{2} - \beta$. Where n is an integer.

Solution :

Since x_1, x_2, x_3, x_4 are the roots of the equation $x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos\beta - \sin\beta = 0$

$$\therefore \quad \sum x_1 + x_2 + x_3 + x_4 = -\frac{(-\sin 2\beta)}{1} = \sin 2\beta$$

$\Sigma x_1 x_2 = \cos 2\beta$

$\Sigma x_1 x_2 x_3 = \cos\beta$ and $x_1 x_2 x_3 x_4 = -\sin\beta$

Now, $\tan\left[\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4\right]$

$$= \frac{\sum x_1 - \sum x_1 x_2 x_3}{1 - \sum x_1 x_2 + x_1 x_2 x_3 x_4} = \frac{\sin 2\beta - \cos\beta}{1 - \cos 2\beta - \sin\beta}$$

$$= \frac{2\sin\beta\cos\beta - \cos\beta}{2\sin^2\beta - \sin\beta} = \frac{\cos\beta\,(2\sin\beta - 1)}{\sin\beta\,(2\sin\beta - 1)}$$

$$= \cot\beta$$

or, $\tan(\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4) = \tan\left(\frac{\pi}{2} - \beta\right)$

$\Rightarrow \quad \tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4 = n\pi + \frac{\pi}{2} - \beta.$

Where x = 0, ± 1, ± 2, ... i.e. x is an integer.

[$\therefore \quad \tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$]

Note

1. If x_1, x_2, x_3, x_4 are the roots of equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ then,

sum of roots is $x_1 + x_2 + x_3 + x_4 = -\frac{b}{a}$.

i.e. $\sum x_1 = \frac{-b}{a}$

sum of the product of roots taken two at a time i.e. $\sum x_1 x_2 = \frac{c}{a}$

sum of the product of roots taken three at a time i.e. $\sum x_1 x_2 x_3 = \frac{-b}{a}$

Product of roots i.e. $x_1 x_2 x_3 x_4 = \frac{e}{a}$

2. We can mark the similarity of results from solution of quadratic equation.

If $ax^2 + bx + c = 0$ and x_1 and x_2 be the roots of this equation then sum of roots $= x_1 + x_2 = \frac{-b}{a}$.

Product of roots $= x_1 x_2 = \frac{c}{a}$

3. $\tan(a_1 + a_2 + a_3 + a_4) = \frac{s_1 - s_3}{1 - s_2 + s_4}$

Where $s_1 = \Sigma\tan\theta_1 = \tan\theta_1 + \tan\theta_2 + \tan\theta_3 + \tan\theta_4$.

$s_2 = \Sigma\tan\theta_1 \tan\theta_2$

$s_3 = \Sigma\tan\theta_1 \tan\theta_2 \tan\theta_3$

$s_4 = \Sigma\tan\theta_1 . \tan\theta_2 . \tan\theta_3 . \tan\theta_4$

Illustration 44

If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$ and $x + y + z = \frac{3}{2}$, then prove that $x = y = z$.

Solution :

Let $\cos^{-1} x = \alpha$, $\cos^{-1} y = \beta$, $\cos^{-1} z = \gamma$

$\Rightarrow$ $\cos\alpha = x$, $\cos\beta = y$, $\cos\gamma = z$

Also, $\alpha + \beta + \gamma = \pi$

From equation, $x + y + z = \frac{3}{2}$

or, $\cos\alpha + \cos\beta + \cos\gamma = \frac{3}{2}$ ...(1)

Let $z = \cos\alpha + \cos\beta + \cos\gamma$ and angle γ be fixed

then $z = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + \cos\gamma$

$= 2\sin\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + \cos\gamma$ $\left[\because \frac{\alpha+\beta}{2} = \frac{\pi}{2} - \frac{\gamma}{2}\right]$

Since γ is fixed $\cos\gamma$ and $\sin\frac{\gamma}{2}$ are fixed. Only changing term is $\cos\frac{\alpha-\beta}{2}$

Clearly, z will be maximum if $\frac{\cos\alpha-\beta}{2} = 1$ i.e., $\alpha = \beta$

Thus, when angle γ is fixed, z will be maximum if $\alpha = \beta$

Similarly when angle β is fixed, z will be maximum if $\gamma = \alpha$

and when angle α is fixed, z will be maximum if $\beta = \gamma$

$\Rightarrow$ z will be maximum if $\alpha = \beta = \gamma = 60$ $[\therefore \alpha + \beta + \gamma = \pi]$

$\Rightarrow$ $z_{max} = \cos 60 + \cos 60 + \cos 60 = \frac{3}{2}$

Thus, the maximum value of $\cos\alpha + \cos\beta + \cos\gamma = \frac{3}{2}$ and is possible only when $\alpha = \beta = \gamma$

from (1), $\cos\alpha + \cos\beta + \cos\gamma = \frac{3}{2}$, which is the maximum value

$\therefore$ $\alpha = \beta = \gamma$

$x = y = z$

Illustration 45

Convert the trigonometric function sin [2cos^{-1} {cot (2tan^{-1} x)}] into an algebraic function f (x). Then from the algebraic function find all the values of x for which f (x) is zero. Express the values of x in the form $a \pm \sqrt{b}$ where a and b are rational numbers.

Solution : Given expression

$$= \sin [2 \cos^{-1} \{\cot (2 \tan^{-1} x)\}]$$

$$= \sin\left[2\cos^{-1}\left\{\cot\tan^{-1}\frac{2x}{1-x^2}\right\}\right]$$

$$= \sin\left[2\cos^{-1}\left(\cot\cot^{-1}\frac{1-x^2}{2x}\right)\right]$$

$$= \sin\left[2\cos^{-1}\frac{1-x^2}{2x}\right]$$

$$= \sin.\sin^{-1}\left[2.\frac{1-x^2}{2x}.\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}\right]$$

[Let $\cos^{-1}\frac{1-x^2}{2x} = \theta \Rightarrow \cos\theta = \frac{1-x^2}{2x}$

and $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}$]

Now, $\sin 2\theta = 2 \sin\theta \cos\theta \quad \Rightarrow \quad 2\theta = \sin^{-1}(2\sin\theta \cos\theta)$

$$= \sin^{-1}\left[2.\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}.\frac{1-x^2}{2x}\right]$$

$$= 2\frac{1-x^2}{2x}.\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}$$

From question, $f(x) = 2\frac{1-x^2}{2x}\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}$

when f(x) = 0, we have

$$2\frac{1-x^2}{2x}.\sqrt{1-\left(\frac{1-x^2}{2x}\right)} = 0$$

or, $(1-x^2).\sqrt{4x^2-1-x^4+2x^2} = 0$

or, $(1-x^2).\sqrt{6x^2-1-x^4} = 0$

$\Rightarrow$ either $1 - x^2 = 0$ or, $\sqrt{6x^2-1-x^4} = 0$

$\Rightarrow$ $x = \pm 1$ or, $x^4 - 6x^2 + 1 = 0$

$\Rightarrow$ $x^2 = \frac{6 \pm \sqrt{36-4.1.1}}{2}$

$= 3 \pm 2\sqrt{2}$

$= (1 \pm \sqrt{2})^2$

$\Rightarrow$ $x = (1 \pm \sqrt{2})$

$\therefore$ $x = \pm 1, (1 \pm \sqrt{2})$

Illustration 46

Solve the equation sin (2 cos^{-1} (cot (2 tan^{-1} x))) = 0

Solution :

$\sin(2\cos^{-1}(\cot(2\tan^{-1}x))) = 0$

$\Rightarrow$ $2\cos^{-1}(\cot(2\tan^{-1}x)) = n\pi,\ n \in I$

$\Rightarrow$ $\cos^{-1}(\cot(2\tan^{-1}x)) = \frac{n\pi}{2}, n \in I$

Since we are interested only in principal values $\Rightarrow$ n = 0, 1, 2

$\therefore$ $\cos^{-1}(\cot(2\tan^{-1}x)) = 0, \pi/2, \pi$

$\Rightarrow$ $\cot(2\tan^{-1}x) = 1, 0, -1 \Rightarrow 2\tan^{-1}x = m\pi + \frac{\pi}{4}, m\pi + \frac{\pi}{2}, m\pi - \frac{\pi}{4}$

Hence again we are interested in principal values.

i.e. $-\pi < m\pi + \frac{\pi}{4} < \pi \Rightarrow m = 0, -1 \Rightarrow 2\tan^{-1} x = \frac{\pi}{4}$ and $-\frac{3\pi}{4}$ and correspondingly

$$x = \tan\left(\frac{\pi}{8}\right), \tan\left(\frac{-3\pi}{8}\right)$$

Similarly, $-\pi < m\pi - \frac{\pi}{4} < \pi \Rightarrow m = 0, -1 \Rightarrow$ $x = \tan\left(\frac{-\pi}{8}\right), \tan\left(\frac{3\pi}{8}\right)$

$$-\pi < m\pi + \frac{\pi}{2} < \pi \Rightarrow m = 0, -1 \Rightarrow x = \tan\left(\pm\frac{\pi}{4}\right)$$

Hence the results are $x = \pm\tan\left(\frac{\pi}{4}\right), \pm\tan\left(\frac{\pi}{8}\right), \pm\tan\left(\frac{3\pi}{8}\right)$

Illustration 47

Given $0 \le x \le \frac{1}{2}$ then the value of $\tan\left[\sin^{-1}\left\{\frac{x}{\sqrt{2}} + \frac{\sqrt{1-x^2}}{\sqrt{2}}\right\} - \sin^{-1} x\right]$ is

(a) -1 **(b)** 1

(c) $\frac{1}{\sqrt{3}}$ **(d)** $\sqrt{3}$

Solution :

Ans. (b). Put $x = \sin\theta$

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right) = \sin^{-1}\sin\left(\theta + \frac{\pi}{4}\right) = \theta + \frac{\pi}{4}$$

$$\therefore \quad E = \tan\left[\theta + \frac{\pi}{4} - \theta\right] = \tan\frac{\pi}{4} = 1$$

Illustration 48

If $\cos^{-1} p + \cos^{-1} q + \cos^{-1} r = \pi$, then prove that $p^2 + q^2 + r^2 + 2pqr = 1$

Solution : $\cos^{-1} p + \cos^{-1} q + \cos^{-1} r = \pi$

$$\Rightarrow \quad \cos^{-1}\left[pq - \sqrt{1-p^2}.\sqrt{1-q^2}\right] = \pi - \cos^{-1} r = \cos^{-1}(-r)$$

$\therefore \quad pq - \sqrt{(1-p^2)(1-q^2)} = -r$

or $\quad (pq + r)^2 = (1 - p^2)(1 - q^2)$

or $\quad p^2q^2 + r^2 + 2pqr = 1 - p^2 - q^2 + p^2q^2$

or $\quad p^2 + q^2 + r^2 + 2pqr = 1$

Illustration 49

If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = 3\pi/2$, then the value of $x^{100} + y^{100} + z^{100} - \dfrac{9}{x^{101} + y^{101} + z^{101}}$ is

(a) 0 **(b)** 1

(c) 2 **(d)** 3

Solution :

Ans. (a). We know that $|\sin^{-1} x| \le \pi/2$

Hence from the given relation we observe that each of $\sin^{-1} x$, $\sin^{-1} y$ and $\sin^{-1} z$ will be $\pi/2$ so that $x = y = z = \sin(\pi/2) = 1$ $\therefore$

$3 - \frac{9}{3} = 0$

Illustration 50

$\tan^{-1} \dfrac{2MN}{M^2 - N^2} + \tan^{-1} \dfrac{2pq}{p^2 - q^2} = \tan^{-1} \dfrac{2MN}{M^2 - N^2}$ **where** $M = mp - nq$, $N = np + mq$

Solution :

Dividing $\dfrac{2MN}{M^2 - M^2}$, $\dfrac{2pq}{p^2 - q^2}$ and $\dfrac{2MN}{M^2 - N^2}$ by m^2, p^2 and M^2 respectively, $2\tan^{-1} x = \tan^{-1} \dfrac{2x}{1 - x^2}$

$$\text{L.H.S.} = 2\tan^{-1} \frac{n}{m} + 2\tan^{-1} \frac{q}{p}$$

$$= 2\tan^{-1} \frac{pn + mq}{pm - nq} = 2\tan^{-1} \frac{N}{M} = \text{R.H.S.}$$

Illustration 51

If $\cos^{-1}$ (p/a) + $\cos^{-1}$ (q/b) = α, then $\frac{p^2}{a^2} - \frac{2pq}{ab}\cos\alpha + \frac{q^2}{b^2} = \sin^2\alpha$

Solution :

We have $\cos^{-1}\left[\frac{p}{a}.\frac{q}{b} - \sqrt{\left(1-\frac{p^2}{a^2}\right)}\sqrt{\left(1-\frac{q^2}{b^2}\right)}\right] = \alpha$

or $\frac{pq}{ab} - \sqrt{\left(1-\frac{p^2}{a^2}\right)}\sqrt{\left(1-\frac{q^2}{b^2}\right)} = \cos\alpha$

$\therefore$ $\left(\frac{pq}{ab} - \cos\alpha\right)^2 = 1 - \frac{p^2}{a^2} - \frac{q^2}{b^2} + \frac{p^2q^2}{a^2b^2}$

or $\frac{p^2q^2}{a^2b^2} + \cos^2\alpha - \frac{2pq}{ab}\cos\alpha$

$= 1 - \frac{p^2}{a^2} - \frac{q^2}{b^2} + \frac{p^2q^2}{a^2b^2}$

or $\frac{p^2}{a^2} - \frac{2pq}{ab}\cos\alpha + \frac{q^2}{b^2} = 1 - \cos^2\alpha = \sin^2\alpha$

Illustration 52

Solve the equation : $\tan^{-1} 2x + \tan^{-1} 3x = n\pi + (3\pi/4)$.

Solution :

L.H.S. = $\tan^{-1}\frac{2x+3x}{1-2x.3x} = n\pi + \frac{3\pi}{4}$

or $\frac{5x}{1-6x^2} = \tan\frac{3\pi}{4} = -1$

or $6x^2 - 5x - 1 = 0$ $\Rightarrow$ $(x - 1)(6x + 1) = 0$

$\Rightarrow$ $x = 1, -1/6$

Illustration 53

Find whether x = 2 satisfies the equation $\tan^{-1}\frac{x+1}{x-1}+\tan^{-1}\frac{x-1}{x}=\tan^{-1}(-7)$.

If not, then how should the equation be re-written ?

Solution :

$$\tan^{-1}\frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{x+1}{x-1}\cdot\frac{x-1}{x}}=\tan^{-1}(-7)$$

$\therefore \quad \frac{2x^2-x+1}{1-x}=7$ or $2x^2-x+1=-7+7x$

or $2x^2-8x+8=0$ or $x^2-4x+4=0$

or $(x-2)^2=0$ $\therefore$ $x=2$.

But if we put x = 2 in the given equation the L.H.S. is + ive and R.H.S. is – ive. Hence x = 2 does not satisfy. We will have to write the equation as $\tan^{-1}\frac{x+1}{x-1}+\tan^{-1}\frac{x-1}{x}=\pi+\tan^{-1}(-7)$.

Now x = 2 will make both sides + ive.

Note : Here $xy=\frac{x+1}{x-1}\frac{x-1}{x}=\frac{x+1}{x}>1$

$\therefore$ R.H.S. $=\pi+\tan^{-1}(-7)$

Illustration 54

Solve for x, sin [2 cos^{-1} **cot (2 tan**$^{-1}$ **x)] = 0.**

Solution :

$$2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}=\cot^{-1}\frac{1-x^2}{2x}$$

$$\therefore \quad \cot(2\tan^{-1}x)=\frac{1-x^2}{2x}$$

$$\therefore \quad \text{L.H.S.}=\sin\left[2\cos^{-1}\frac{1-x^2}{2x}\right]=0$$

Now $2\cos^{-1}z=\cos^{-1}(2z^2-1)$

$$\therefore \quad 2\cos^{-1}\frac{1-x^2}{2x} = \cos^{-1}\left[2.\frac{(1-x^2)^2}{4x^2} - 1\right]$$

$$= \cos^{-1}\left[\frac{x^4 - 4x^2 + 1}{2x^2}\right]$$

$$\therefore \quad \text{L.H.S.} = \sin\left[\cos^{-1}\frac{x^4 - 4x^2 + 1}{2x^2}\right] = 0 \qquad ...(1)$$

Again $\sin\cos^{-1} t = \sin\sin^{-1}\sqrt{1-t^2} = 0$,

$\therefore \quad 1 - t^2 = 0$

Hence from (1), we have

$(x^4 - 4x^2 + 1)^2 - (2x^2)^2 = 0$

or $\quad (x^4 - 4x^2 + 1 - 2x^2)(x^4 - 4x^2 + 1 + 2x^2) = 0$

or $\quad (x^4 - 2x^2 + 1)(x^4 - 6x^2 + 1) = 0$

From Ist factor $(x^2 - 1)^2 = 0$, $x = \pm 1$

From 2nd factor $x^4 - 6x^2 + 9 = -1 + 9$

or $\quad (x^2 - 3)^2 = 8$

$$x^2 = 3 \pm 2\sqrt{2} = (1 \pm \sqrt{2})^2$$

$$\therefore \quad x = \pm(1 \pm \sqrt{2})$$

Illustration 55

Prove that $\tan^{-1}\dfrac{c_1x - y}{c_1y + x} + \tan^{-1}\dfrac{c_2 - c_1}{1 + c_2c_1} + \tan^{-1}\dfrac{c_3 - c_2}{1 + c_3c_2} + ... + \tan^{-1}\dfrac{1}{c_n} = \tan^{-1}\dfrac{x}{y}$

Solution :

$$T_1 = \tan^{-1}\frac{x/y - 1/c_1}{1 + (x/y).(1/c_1)}$$

$$= \tan^{-1}\frac{x}{y} - \tan^{-1}\frac{1}{c_1} \text{ etc.}$$

$\therefore \quad$ L.H.S.

$$= \left(\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{1}{c_1}\right) + \left(\tan^{-1}\frac{1}{c_1} - \tan^{-1}\frac{1}{c_2}\right) + ... - \tan^{-1}\frac{1}{c_n} = \tan^{-1}\frac{x}{y}$$

Illustration 56

$$\tan^{-1}\sqrt{\left\{\frac{x\,(x+y+z)}{yz}\right\}}+\tan^{-1}\sqrt{\left\{\frac{y\,(x+y+z)}{zx}\right\}}+\tan^{-1}\sqrt{\left\{\frac{z\,(x+y+z)}{xy}\right\}}=\pi$$

Solution :

Put $x + y + z = r$

$$\therefore \quad \tan^{-1}\sqrt{\left(\frac{rx}{yz}\right)}+\tan^{-1}\sqrt{\left(\frac{ry}{zx}\right)} = \tan^{-1}-\frac{\sqrt{r/xyx)}\,(x+y)}{(1-r/z)}$$

$$= \tan^{-1}-\frac{\sqrt{(rz)/(xy)}\,(x+y)}{-(x+y)} = \pi+\tan^{-1}\left\{-\sqrt{\left(\frac{rz}{xy}\right)}\right\} = \pi-\tan^{-1}\sqrt{\left(\frac{rz}{xy}\right)}$$

$$\therefore \quad \tan^{-1}\left(\frac{rx}{yz}\right)+\tan^{-1}\sqrt{\left(\frac{ry}{zx}\right)}+\tan^{-1}\sqrt{\left(\frac{rz}{xy}\right)}=\pi$$

Illustration 57

Solve for x **the following equations**

$$\sec^{-1}\frac{x}{a}-\sec^{-1}\frac{x}{b}=\sec^{-1}b-\sec^{-1}a$$

Solution :

From the given equation we have $\cos^{-1}\frac{a}{x}+\cos^{-1}\frac{1}{a}=\cos^{-1}\frac{1}{b}+\cos^{-1}\frac{b}{x}$

or $$\cos^{-1}\left[\frac{a}{x}.\frac{1}{a}-\left\{\sqrt{\left(1-\frac{a^2}{x^2}\right)\left(1-\frac{1}{a^2}\right)}\right\}\right]$$

$$=\cos^{-1}\left[\frac{1}{b}.\frac{b}{x}-\sqrt{\left(1-\frac{1}{b^2}\right)\left(1-\frac{b^2}{x^2}\right)}\right]$$

or $$\frac{1}{x}-\frac{\sqrt{x^2-a^2}\,\sqrt{a^2-1}}{ax}=\frac{1}{x}-\frac{\sqrt{b^2-1}\,\sqrt{x^2-b^2}}{bx}$$

or $b^2\,(a^2-1)\,(x^2-a^2)=a^2\,(b^2-1)\,(x^2-b^2)$

or $x^2\,(a^2b^2-b^2-a^2b^2+a^2)=a^2b^2\,(a^2-1-b^2+1)$

or $x^2\,(a^2-b^2)=a^2b^2\,(a^2-b^2)$ $\quad \therefore \; x = ab.$

Illustration 58

Sum the following series :

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}+\ldots n \text{ and } \infty \text{ or}$$

$$\tan^{-1}\frac{1}{1+1+1^2}+\tan^{-1}\frac{1}{1+2+2^2}+\tan^{-1}\frac{1}{1+3+3^2}+\ldots$$

Solution :

$$T_n = \tan^{-1}\frac{1}{1+n+n^2}=\tan^{-1}\frac{(n+1)-n}{1+(n+1)n}$$

$$= \tan^{-1}(n+1) - \tan^{-1} n$$

Putting n = 1, 2, 3, ..., n and adding, we get

$$S_n = \tan^{-1}(n+1) - \tan^{-1} 1$$

$$\therefore \quad S_\infty = \tan^{-1}\infty - \frac{\pi}{4}=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

Illustration 59

$$\cot^{-1}\left(1+\frac{3}{4}\right)+\cot^{-1}\left(2^2+\frac{3}{4}\right)+\cot^{-1}\left(3^2+\frac{3}{4}\right)+\ldots\infty$$

Solution :

$$T_n = \cot^{-1}\left(n^2+\frac{3}{4}\right)=\cot^{-1}\frac{4n^2+3}{4}$$

$$= \tan^{-1}\frac{4}{4\left(n^2+\frac{3}{4}\right)}=\tan^{-1}\frac{1}{1+\left(n^2-\frac{1}{4}\right)} = \tan^{-1}\frac{\left(n+\frac{1}{2}\right)-\left(n-\frac{1}{2}\right)}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}$$

$$= \tan^{-1}\left(n+\frac{1}{2}\right)-\tan^{-1}\left(n-\frac{1}{2}\right)$$

Putting n = 1, 2, 3, ..., n and adding

$$S_n = \tan^{-1}\left(n+\frac{1}{2}\right)-\tan^{-1}\frac{1}{2}$$

$$\therefore \quad S_\infty = \frac{\pi}{2}-\tan^{-1}\frac{1}{2}=\cot^{-1}\frac{1}{2}=\tan^{-1}2$$