1 KINEMATICS

Kinematics is the science of describing the motion of objects using words, diagrams, numbers, graphs, and equations. The goal of any study of kinematics is to develop sophisticated mental models which serve to describe (and ultimately, explain) the motion of real-world objects.

1.1 SCALARS AND VECTORS :

The motion of objects can be described by words - words such as distance, displacement, speed, velocity, and acceleration. These mathematical quantities which are used to describe the motion of objects can be divided into two categories. The quantity is either a vector or a scalar. These two categories can be distinguished from one another by their distinct definitions:

- **Scalars** are quantities which are fully described by a magnitude alone.
- **Vectors** are quantities which are fully described by both a magnitude and a direction.

As you proceed through the lesson, give careful attention to the vector and scalar nature of each quantity.

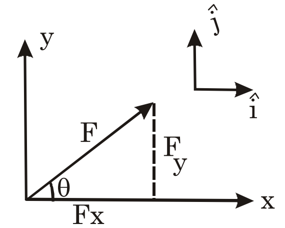
CHECK YOUR UNDERSTANDING

To test your understanding of this distinction, consider the quantities listed below. Categorize each quantity as being either a vector or a scalar.

Quantity	Category
a. 5m	Scalar
b. 30 m/sec, East	Vector. A direction is listed for it.
c. 20 degrees Celsius	scalar

Vector representation in 2-D:

The base vectors of a rectangular x-y coordinate system are given by the unit vectors $\tilde{i}$ and $\tilde{j}$ along the x and y directions, respectively.

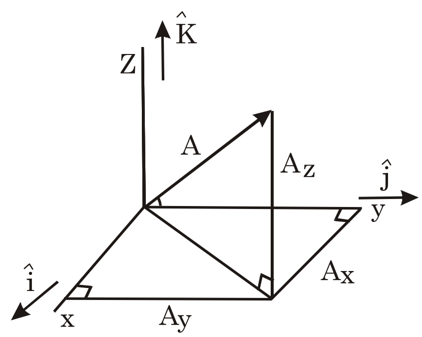


Using the base vectors, one can represent any vector **F** as

$$F = F_x\tilde{i} + F_y\tilde{j}$$

Vector representation in 3-D:

In a rectangular coordinate system the components of the vector are the projections of the vector along the x, y, and z directions. For example, in the figure the projections of vector **A** along the x, y, and zdirections are given by A_x, A_y, and A_z, respectively.



$A = A_x\tilde{i} + A_y\tilde{j} + A_z\tilde{k}$

1.2 A BODY IN MOTION :

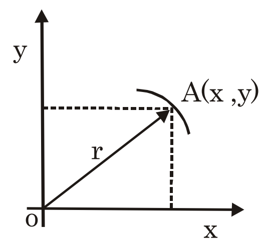
A body is said to be motion, when its changes its position with time, with respect to an observer. Similarly, if the position doesn't change with time, with respect to an observer (reference), the body is in rest.

Basic terms :

1. **Particle :** A particle is a point mass. However, in practice a body may be treated as a particle, if its size is very small compared to the distance covered by it.
2. **Position :** The position of a particle refers to its location in the space at a certain moment of time. In general the position is measured by a vector joining a fixed point (known as origin) to the moving particle. This vector is known as position vector.

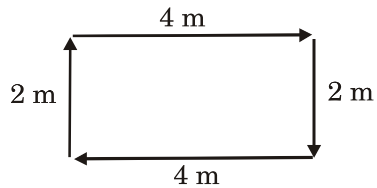
For a particle in straight line motion along X-axis, the position vector is always parallel to X-axis and hence has only X-component as non-zero. Therefore the position of a moving particle can be measured by the X-coordinate x (t) at a certain time instant t.

If a particle is moving in a curve (i.e. in a plane) the position vector can have many possible directions. The position in such a case can be measured by two numbers : X-coordinate and Y-coordinate the position vector.



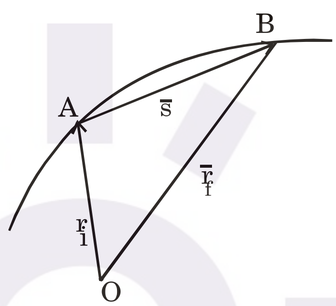
3. **Distance** is a scalar quantity which refers to "how much ground an object has covered" during its motion. It is the actual length of the path travelled by a particle.
4. **Displacement** is a vector quantity which refers to "how far out of place an object is". It is the vector joining the initial position of the particle to its final position during an interval of time. The change in the position of a moving object is known as displacement

(a) To test your understanding of this distinction, consider the m otion depicted in the diagram below. A physics teacher walks 4 meters East, 2 meters South, 4 meters West, and finally 2 meters North.



Even though the physics teacher has walked a total distance of 12 meters, her displacement is 0 meters. During the course of her motion, she has "covered 12 meters of ground" (distance = 12 m). Yet, when she is finished walking, she is not "out of place" – i.e., there is no displacement for her motion (displacement = 0 m). Displacement, being a vector quantity, must give attention to direction. The 4 meters east is canceled by the 4 meters west; and the 2 meters south is canceled by the 2 meters north.

(b) If a particle goes from A to B along a curve in some time duration and if O is the origin then



$\overline{OA}$ = initial position vector = $\bar{r}_1$

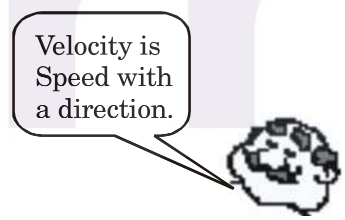
$\overline{OB}$ = final position vector = $\bar{r}_f$

AB = displacement vector = $\overline{OB} - \overline{OA}$

$\Rightarrow \quad \bar{s} = \bar{r}_f - \bar{r}_i = \Delta\bar{r}$

5. Speed: Speed is a scalar quantity which refers to "how fast an object is moving. "Rate of change of distance (x) covered by a particle, with time is called the speed of the particle. Its unit is m/s.

6. Velocity: Velocity is a vector quantity which refers to "the rate at which an object changes its position." Rate of change of displacement of the particle with time is called velocity of the particle.



7. Average speed: The average speed of a particle in a time interval is defined as the distance travelled by the particle by the time interval. If the particle travels a distance s in time t_1 to t_2 the average speed is defined as:

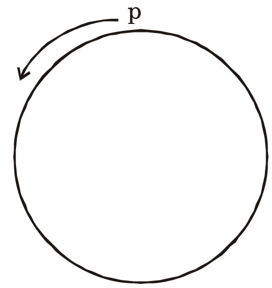
$$\text{speed}_{av} = \frac{s}{t_2 - t_1}$$

8. Average velocity: The average velocity of a particle in a time interval t_1 to t_2 is defined as its displacement divided by the time interval.

$$\vec{v}_{av} = \frac{\vec{s}}{t_2 - t_1}\left(\vec{s} = \text{displacement}\right) = \frac{\Delta x}{\Delta t}\tilde{i} + \frac{\Delta y}{\Delta t}\tilde{j}$$

Difference between average speed and average velocity :

Consider a particle moving around a circle. A particle starts at point P in the circle and covers the entire circumference of the circle and reaches back at point P (as given in the figure below):



In this case, Average speed $= \dfrac{\text{Total distance travelled}}{\text{Time taken}}$

$$= \frac{2\pi r\,(\text{circumference of the circle})}{\text{Time taken}}$$

On the contrary, magnitude of average velocity = 0, since displacement is 0.

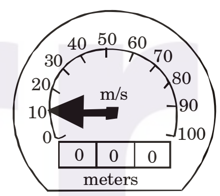
9. **Instantaneous speed**: The instantaneous speed is the speed of the particle defined for a particular instant. When we had described average speed, it meant speed of the particle defined for a time interval.

   Suppose a particle covered a distance Δs for a time interval Δt, then average speed in this case is equal to $\dfrac{\Delta s}{\Delta t}$

   Now, if the interval Δt is made extremely small, approaching to zero, the speed that is defined become for an instant. So, we can write the instantaneous speed as $\lim\limits_{\Delta t \to 0} \dfrac{\Delta s}{\Delta t} = \dfrac{ds}{dt}$.

   In general, when we talk of 'speed', it refers to the instantaneous speed.

You might think of the instantaneous speed as the speed which the speedometer reads at any given instant in time and the average speed as the average of all the speedometer readings during the course of the trip.



10. **Instantaneous velocity**: The velocity at a particular moment of time is known as instantaneous velocity. The term 'velocity' usually refers to the instantaneous velocity.
11. **Acceleration**: Rate of change of velocity with time is called acceleration. Acceleration is a vector quantity. Its unit is m/s^2.

Direction of the Acceleration Vector

Acceleration is a vector quantity so it will always have a direction associated with it. The direction of the acceleration vector depends on two factors:

- whether the object is speeding up or slowing down
- whether the object is moving in the positive (+) or negative (–) direction

The general RULE OF THUMB is :

If an object is slowing down, then its acceleration is in the opposite direction of its motion.

This RULE OF THUMB can be applied to determine whether the sign of the acceleration of an object is positive or negative, right or left, up or down, etc.

12. Average acceleration: Average acceleration is defined as the change in velocity divided by the time interval.

$$\text{Average acceleration} = \frac{\text{change in velocity}}{\text{time interval}}$$

Thus, $\vec{a}_{av} = \dfrac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$

The average acceleration depends only on the acceleration at time t_1 and t_2. The way the velocity changed in between these times does not make a difference in defining average acceleration.

13. Instantaneous acceleration: The acceleration at a given instant of time is the called the instantaneous acceleration. When we refer to acceleration, we mean 'instantaneous acceleration', i.e.,

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{\Delta \vec{v}}{dt}$$

We also know that $\vec{F} = m\vec{a}$, which means that acceleration is also decided by the net force acting on a particle. If F = 0, then a = 0 and the motion is known as uniform motion because the velocity does not change, which also means that the particle is moving with a constant velocity. While, if the force is constant, acceleration is also constant and the motion is known as **uniformly accelerated motion**, which means that acceleration acts on the particle, but it is constant. An example of such a case is the free fall motion(force and acceleration do not change).

The Big Misconception!!

We know that the acceleration of a free-falling object on Earth is 10 m/s/s. This value (known as the acceleration of gravity) is the same for all free-falling objects regardless of how long they have been falling, or whether they were initially dropped from rest or thrown up into the air. Yet the question is often asked "Doesn't a massive object accelerate at a greater rate than a less massive object?". This question is a reasonable inquiry that is probably based upon personal observations made of falling objects in the physical world. After all, nearly everyone has observed the difference in rate of fall of a single piece of paper (or similar object) and a textbook. The two objects clearly travel to the ground at different rates – with the massive book falling faster.

The answer to the question (Doesn't a massive object accelerate at a greater rate than a less massive object?) is . . . absolutely not! That is, absolutely not, if you are considering the specific type of falling motion known as free-fall. Free-fall is the motion of objects under the sole influence of gravity; free-falling objects do not encounter air resistance. Massive objects will only fall faster than less massive objects if there is an appreciable amount of air resistance present.

Example 1

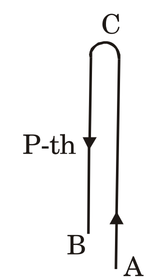
A bird flies north at 20 m/s for 15s. It rests for 5s and then flies south at 25 m/s for 10s. For the whole trip find

(a) the average speed;

(b) the average velocity;

(c) the average acceleration.

Solution.



distance traveled towards north = AC = 20 m/s   15 s = 300 m

distance traveled towards south = CB = 25 m /s   10 s = 250 m

$$\text{Average Speed} = \frac{\text{distance}}{\text{time}}$$

$$\text{Average velocity} = \text{displacement time} = \frac{300 + 250}{15 + 5 + 10}\,\text{m/s} = 18.32\ \text{m/s}$$

$$\text{Average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{300 - 250}{15 + 5 + 10} = 1.67\,\text{m/s}$$

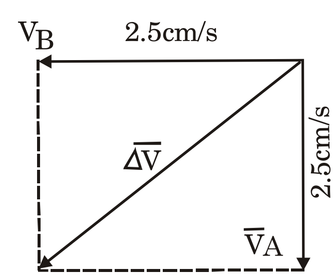
$$\text{Average Acceleration} = \vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{\Delta t} = \frac{(-25) - (+20)}{30}\,\text{m/s}^2 = 1.5\text{m/s}^2$$

Example 2

A particle goes along a quadrant AB of a circle of radius 5 cm with a constant speed 2.5 cm/s as shown. Find the average velocity and average acceleration over the interval AB.

Solution: Time taken $= \dfrac{\text{distance}}{\text{speed}} = \dfrac{3.14 \times 5}{2 \times 2.5} = 3.14\text{s}$

$$\frac{\text{displacement}}{\text{time}} = \frac{AB}{\text{time}} = \frac{\sqrt{5^2 + 5^2}}{3.14}\,\text{m/s} = 2.252\text{m/s}$$



$$\text{Average Acceleration} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_B - \vec{v}_A}{\Delta t}$$

$$= \frac{\sqrt{(2.5)^2 + (2.5)^2}}{3.14}\,\text{m/s}^2 = 1.126\,\text{m/s}^2$$

The average velocity is directed along AB and the average acceleration is directed perpendicular to AB towards O.

Also note. ds represents the magnitude of $d\vec{r}$ but dr does not represent the same.

$$ds = |d\vec{r}|$$

$$dr \neq |d\vec{r}|$$

dr = comp. of $d\vec{r}$ along $\vec{r}$

$$= \left(d\vec{r}.\tilde{u}_r\right) = d\vec{r}.\tilde{u}_r = |d\vec{r}|\cos\alpha$$

dr = ds **[when the particle means in a straight line]**.

Example 3

The radius vector of a point depends on time t**, as**

$$\vec{r} = \vec{c}t + \frac{\vec{b}t^2}{2}$$

where $\vec{c}$ and $\vec{b}$ are constant vectors. Find the modulus of velocity and acceleration at any time t.

Solution. (i) Velocity $\vec{v} = \frac{d\vec{r}}{dt} = \vec{c} + t\vec{b}$

Modulus of velocity vector will be $|\vec{v}| = \sqrt{c^2 + b^2t^2 + 2cb\cos\theta}$, here c and b are modulus of $\vec{c}$ and $\vec{b}$ and θ is the angle between $\vec{c}$ and $\vec{b}$ which can be written as

$$c^2 = \vec{c}.\vec{c},\ b^2 = \vec{b}.\vec{b} \text{ and } cb\cos\theta = \vec{c}.\vec{b}$$

Hence, $|\vec{v}| = \sqrt{\vec{c}.\vec{c} + \vec{b}.\vec{b}\,t^2 + 2\,\vec{c}.\vec{b}\,t}$

(ii) Acceleration $\vec{a} = \frac{d\vec{v}}{dt} = \vec{b}$

Hence, $|\vec{a}| = |\vec{b}|$.

Example 4

(a) What does $\left|\frac{d\vec{v}}{dt}\right|$ and $\frac{d\left|\vec{v}\right|}{dt}$ represent ? (b) Can these be equal

(c) Can $\frac{d\left|\vec{v}\right|}{dt} = 0$ while $\left|\frac{d\vec{v}}{dt}\right| = 0$ (d) $\frac{d|\vec{v}|}{dt} \neq 0$ while $\left|\frac{d\vec{v}}{dt}\right| = 0$

Solution. (a) $\left|\frac{d\vec{v}}{dt}\right|$ is the magnitude of total acceleration. While $\frac{d\left|\vec{v}\right|}{dt}$ represents the time rate of change of speed (called the tangential acceleration, a component of total acceleration) as $|\vec{v}| = v$.

(b) These two are equal in case of one dimensional motion (without change in direction)

(c) In case of uniform circular motion speed remains constant while velocity changes.

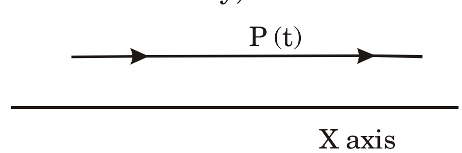
Hence, $\left|\frac{d\vec{v}}{dt}\right| = 0$ while $\frac{d|\vec{v}|}{dt} \neq 0$

(d) $\frac{d|\vec{v}|}{dt} \neq 0$ implies that speed of particle is not constant. Velocity cannot remain constant if speed is changing. Hence, $\left|\frac{d\vec{v}}{dt}\right| = 0$ cannot be zero in this case. So, it is not possible to have $\left|\frac{d\vec{v}}{dt}\right| = 0$ while $\frac{d|\vec{v}|}{dt} \neq 0$.

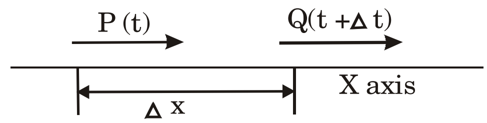
1.3 INSTANTANEOUS VELOCITY & INSTANTANEOUS ACCELERATION (USE OF DERIVATIVES)

(a) Instantaneous Velocity :

If a car is driven along a straight road for 10 km in 5 hours, the average velocity is 2 km/hr. Can we say that the car was moving with 2 km/hr at every instant in these five hours? Certainly not. As we can easily imagine, the car accelerated from rest and went through will all speeds like 0.5 km/hr, 1 km/hr etc (and it may be moving with speeds more than 2 km/hr at some instants). To deal with this, we need the concept of instantaneous velocity, which is the velocity at any instant of time.



Consider a particle moving along X-axis. Let us find the instantaneous velocity of this particle at a certain time instant t, when it passes through the point P(t) as shown. We start with the concept of average velocity. The particle passes through the point $Q(t + \Delta t)\Delta t$ seconds after it passes P(t). The ratio $\Delta X/\Delta t$ is the average velocity over the interval PQ.



$$V_{av} = \frac{\Delta x}{\Delta t} \text{ (in PQ)}$$

By taking the position $Q(t + \Delta t)$ more and more close to P(t), we can make this average velocity very closely approximate the exact instantaneous velocity at P.

Taking Q closer to P means that the interval Δt shrinks and diminishes towards zero. We usually describe this situation like this: "as Q is taken closer to P, Δt approaches zero and the average velocity in PQ approaches the instantaneous velocity at P".

As $Q \to P$, $\Delta t = 0$ and V_{av} (in PQ) $\to V_{inst}$ (at P)

$$V_{inst} = \left(\frac{\Delta x}{\Delta t} \text{ in PQ}\right)_{Q\to P} = \left(\frac{\Delta x}{\Delta t}\right)_{\Delta t\to 0}$$

In exact notation we write:

$$V_{inst} = \lim_{\Delta t\to 0}\left(\frac{\Delta x}{\Delta t}\right) = \frac{dx}{dt}$$

In words, the instantaneous velocity at a given moment (say, t) is the limiting value of the average velocity as we let Δt approach zero. The limit as $\Delta t \to 0$ is written in calculus notation as dx/dt and is called the derivative of x with respect to t.

It is important to note that we do not simply set $\Delta t = 0$, for then Δx would also be zero, and we would have an undefined number. We must consider the ratio $\Delta x/\Delta t$ as a whole; and let Δt approach zero, Δx approaches zero also; but the ratio $\Delta x/\Delta t$ approaches some definite value, which we call the instantaneous velocity.

Example 5

The position of a particle is given by the equation $x(t)=3t^3$. Find the instantaneous velocity at instants t = 2s, 4s using the definition of instantaneous velocity.

Solution. Let us find the instantaneous velocity v(t) of the particle at any time instant t. Then we can substitute t = 3s, 6s for calculating particular values. Average velocity in a time interval from t to $(t + \Delta t)$ is:

$$\frac{\Delta x}{\Delta t} = \frac{3(t+\Delta t)^3 - 3t^3}{\Delta t}$$

$$= \frac{3\Delta t\left(3t^2 + \Delta t^2 + 3t.\Delta t\right)}{\Delta t}$$

$$= 3\,(3t^2 + \Delta t^2 + 3t.\Delta t)$$

$$v(t) = \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right)$$

$$= \lim_{\Delta t \to 0} 3\left(3t^2 + \Delta t^2 + 3t.\Delta t\right)$$

$\Rightarrow$ $v(t) = 3(3t^2 + 0 + 0)$

$\Rightarrow$ $v(t) = 9t^2$

The velocity at t = 3s is v (t = 3) = 81 m/s.

The velocity at t = 6s is v (t = 6) = 324 m/s..

NOTE : The determination of instantaneous velocity by using the definition (i.e. by the limiting process as in the last example) usually involves calculation. We can find

$v = \dfrac{dx}{dt}$ by using the standard results from differential calculus.

$x = u + v + \ldots \quad \Rightarrow \quad \dfrac{dx}{dt} = \dfrac{du}{dt} + \dfrac{dv}{dt} + \ldots$

$x = \text{constant} \quad \Rightarrow \quad \dfrac{dx}{dt} = 0$ (derivative of a constant is zero)

$x = Au \quad \Rightarrow \quad \dfrac{dx}{dt} = A\dfrac{du}{dt}$ (where A is a constant)

$x = t^n \quad \Rightarrow \quad \dfrac{dx}{dt} = nt^{n-1}$

$x = \sin \omega t \quad \Rightarrow \quad \dfrac{dx}{dt} = \omega \cos \omega t$ (ω is constant)

$x = \cos \omega t \quad \Rightarrow \quad \dfrac{dx}{dt} = -\omega \sin \omega t$ (ω is constant)

$x \log t \quad \Rightarrow \quad \dfrac{dx}{dt} = \dfrac{1}{t}$

Instantaneous Acceleration:

The instantaneous acceleration of a particle is its acceleration at a particular instant of time. It is defined as the derivative (rate of change) of velocity with respect to time:

$$\bar{a} = \frac{d\bar{v}}{dt} = \lim_{\Delta t \to 0}\left(\frac{\Delta \bar{v}}{\Delta t}\right).$$

1.4 Motion in one, two and three dimensions

Motion of a block in a straight line is one dimensional (1-D) motion. Motion of a particle in a straight line can be described by only one component of its velocity or acceleration. The motion of a particle thrown vertical plane at some angle with horizontal (≠ 90) is an example of two dimensional (2-D) motion. This is called a projectile motion. Similarly a circular motion is also an example of 2-D motion. A 2-D motion takes place in a plane and its velocity (or acceleration) can be described by two components in any two mutually perpendicular direction (v_x and v_y).

Motion of a bird (or a monkey) in space is a three dimensional (3-D) motion. In a 3-D motion velocity and acceleration of a particle can be resolved in three components

(v_x, v_y, a_x, a_y and a_z). Here x, y and z are any three mutually perpendicular axes.

The position of a particle in one dimensional motion is described by one variable (say x) in a 2-D motion it involves two variables (normally x and y) and in a 3-D motion three variables are x, y and z.

Note. All we are talking above is for certain coordinate system.

1.4.1 UNIFORMLY ACCELERATED MOTION :

Equations of motion for uniformly accelerated motion $\left(\vec{a} = \text{constant}\right)$ are as under,

$$\vec{v} = \vec{u} + \vec{a}t,\ \vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2,\ \vec{v}.\vec{y} = \vec{u}.\vec{u} + 2\vec{a}.\vec{s}$$

Here $\vec{u}$ = initial velocity of particle, $\vec{v}$ = velocity of particle at time t and

$\vec{s}$ = displacement of particle in time t

Note. If initial position vector of a particle is $\vec{r}_0$, then position vector at time t can be written as

$$\vec{r} = \vec{r}_0 + \vec{s} = \vec{r}_0 + \vec{u}t + \frac{1}{2}\vec{a}t^2$$

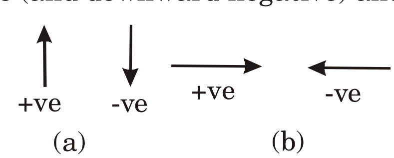
One-dimensional uniformly accelerated motion

If the motion of a particle is taking place in a straight line, there is no need of using vector addition (or subtraction) in equations of motion. We can directly use the equations.

$$v = u + at,\ s = ut + \frac{1}{2}at^2 \text{ and } v^2 = u^2 + 2as$$

Just by taking one direction as the positive (and opposite to it as negative) and then substituting u, a, etc. with sign. Normally we take vertically upward direction positive (and downward negative) and horizontally rightwards positive (or leftwards negative).

Sign convention for (a) motion in vertical direction (b) motion in horizontal direction is shown in Fig.



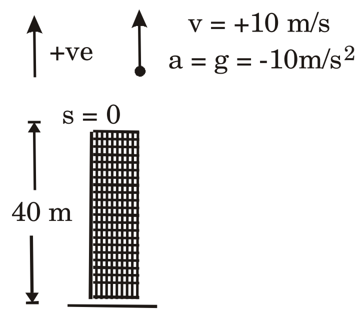
Example 6

A ball is thrown upwards from the top of a lower 40 m high with a velocity of 10 m/s. Find the time when it strikes the ground Take g = 10 m/s².

Solution. In the problem

$u = 10$ m/s, $a = -10$ m/s^2 and $s = -40$ m

(at the point where stone strikes the ground)



Substituting $s = ut + \frac{1}{2}at^2$, we have

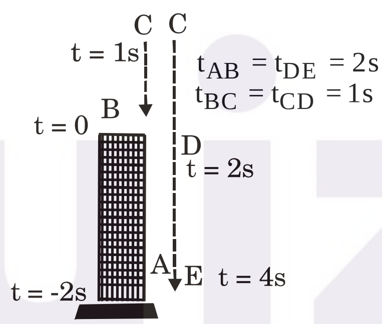
$$-40 = 10t - 5t^2$$

or $5t^2 - 10t - 40 = 0$

or $t^2 - 2t - 8 = 0$

Solving this we have t = 4s and – 2s. Taking the positive value t = 4s.

Note : The significance of t = –2s can be understood by following figure:



Example 7

A particle of mass 1 kg has a velocity of 2 m/s. A constant force of 2N acts on the particle for 1s in a direction perpendicular to its initial velocity. Find the velocity and displacement of the particle at the end of 1 second.

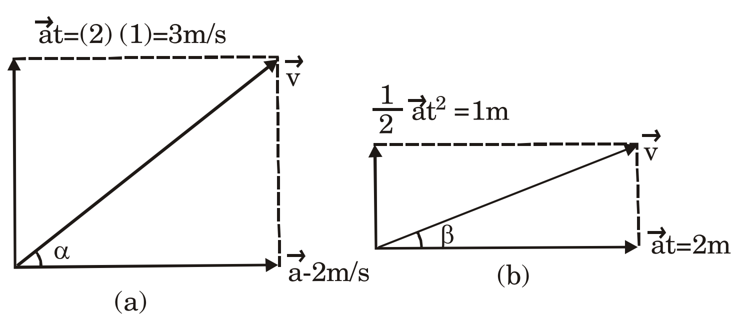
Solution. Force acting on the particle is constant. Hence, acceleration of the particle will also remain constant.

$$a = \frac{F}{m} = \frac{2}{1} = 2\text{m/s}^2$$

Since, acceleration is constant. We can apply

$$\vec{v} = \vec{u} + \vec{a}t \quad \text{and} \quad \vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$$

Refer Fig.



Here, $\vec{u}$ and $\vec{a}t$ are two mutually perpendicular vectors. So

$$|\vec{v}| = \sqrt{(|\vec{u}|)^2 + (|\vec{a}t|)^2} = \sqrt{(2)^2 + (2)^2} = 2\sqrt{2} \text{ m/s}$$

$$\alpha = \tan^{-1}\left|\frac{\vec{a}t}{\vec{u}}\right| = \tan^{-1}(1) = 45°$$

Thus, velocity of the particle at the end of 1s is $2\sqrt{2}$ at an angle of 45 with its initial velocity.

$$\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2.$$

Here $\vec{u}t$ and $\frac{1}{2}\vec{a}t^2$ are also two mutually perpendicular vectors. So,

$$\vec{s} = \sqrt{(|\vec{u}t|)^2 + \left(\left|\frac{1}{2}\vec{a}t^2\right|\right)^2} = \sqrt{(2)^2 + (1)^2} = \sqrt{5}\text{ m}$$

and $$\beta = \tan^{-1}\frac{\left|\frac{1}{2}\vec{a}t^2\right|}{|\vec{u}t|} = \tan^{-1}\left(\frac{1}{2}\right).$$

Thus, displacement of the particle at the end of 1s is $\sqrt{5}$m at an angle of $\tan^{-1}\left(\frac{1}{2}\right)$ from its initial velocity.

Example 8

Velocity and acceleration of a particle at time t = 0 are $\vec{u} = (2\tilde{i} + 3\tilde{j})$ m/s and $\vec{a} = (4\tilde{i} + 2\tilde{j})\text{m/s}^2$ respectively. Find the velocity and displacement of particle at t = 2 s.

Solution. Here, acceleration $\vec{a} = (4\tilde{i} + 2\tilde{j})\text{m/s}^2$ is constant. So, we can apply

$$\vec{v} = \vec{u} + \vec{a}t \text{ and } \vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$$

Substituting the proper values, we get

$$\vec{v} = (2\tilde{i} + 3\tilde{j}) + (2)(4\tilde{i} + 2\tilde{j}) = (10\tilde{i} + 7\tilde{j})\text{m/s}$$

and $$\vec{s} = (2)(2\tilde{i} + 3\tilde{j}) + \frac{1}{2}(2)^2\,(4\tilde{i} + 2\tilde{j}) = (12\tilde{i} + 10\tilde{j})\text{m}.$$

Therefore, velocity and displacement of particle at t = 2s are $\left(10\tilde{i}+7\tilde{j}\right)$ m/s and $\left(12\tilde{i}+10\tilde{j}\right)$ m respectively.

Following points are worthnoting in case of one dimensional motion with constant acceleration.

(i) It can be observed when either $u=0, \vec{u}\uparrow\uparrow\vec{a}$ or $\vec{u}\uparrow\downarrow\vec{a}$

(ii) In the first two cases when either $u=0$ or $\vec{u}\uparrow\uparrow\vec{a}$ motion is only accelerated.

(iii) When $\vec{u}\uparrow\downarrow\vec{a}$ a motion is first retarded (till the velocity becomes zero) and then accelerated in opposite direction.

(iv) As per our convention (vertically upward positive) acceleration due to gravity 'g' is always negative whether the particle is moving upwards or downwards. We are now left with the sign of u and s. Displacement s is measured from the point of projection.

(v) For fast calculation in objective problems, remember the following results.

(a) Maximum height attained by a particle, thrown upwards from ground

$$h=\frac{u^2}{2g}$$

(b) Velocity of particle at the time of striking the ground when released (u =0) from a height his,

$$v=\sqrt{2gh}$$

(c) In (b) time of collision with ground $t=\sqrt{\frac{2h}{g}}$

(d) Displacement of particle in nth second of its motion,

$$s_1 = u + at - \frac{1}{2}a\,.$$

(vi) **Difference between distance (d) and displacement (s)**

The 's' in equations of motion $\left(s=ut+\frac{1}{2}at^2 \text{ and } v^2=u^2+2as\right)$ is really the displacement not the distance. They have different values only when u and a are are of opposite sign or $u\uparrow\downarrow a$.

SOME COMMON MISCONCEPTIONS!!

MISCONCEPT 1 - Equations of motion are applicable every time.

CLARIFICATION - Equations of motion are applicable only when a is constant.

MISCONCEPT 2 - In eqn of motion, s gives distance

CLARIFICATION - s gives displacement, not distance. So for finding distance, we must find out instant at which velocity is reversing and then break the motion in parts.

1.4.2 MOTION WITH NON-UNIFORM ACCELERATION :

To start off with this topic it is important to have knowledge of integrals.

Short note on calculation of integrals:

We have two types of integrals : Indefinite integrals and Definite integrals. Indefinite integrals are basically anti-derivatives i.e., they are inverse of derivatives. For example, we know that derivative of $t^2 = 2t$ and this means that the indefinite integral of 2t is t^2. Similarly, the derivative of t_n is nt^{n-1} and hence the indefinite integral of nt^{n-1} is tn.

Proceeding in the similar way, we get:

$$\int t^n\,dt = \frac{t^{n+1}}{n+1}$$ Note that the derivative of $\frac{t^{n+1}}{n+1}$ is t^n.

(We can add a constant to the right hand side because the derivative of a constant is zero.)

Definite integrals are calculated over some intervals i.e., between an upper limit and a lower limit. To calculate a definite integral, first find its indefinite integral (anti-derivative) and then substitute upper and lower limits and subtract.

For example: $$\int t\,dt = \left.\frac{t^2}{2}\right] = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}$$

Some quantities defined as derivatives and integrals.

$v(t) = \frac{dx}{dt}$	v = slope of x – t graph
$(t) = \frac{dv}{dt}a$	a = slope of v – t graph
$F(t) = \frac{dp}{dt}$	F = slope of p - t graph (p = linear momentum)

$\Delta x = \int dx = \int_1^2 v(t)dt$	Δx = area under v – t graph
$\Delta v = \int dv = \int_1^2 a(t)dt$	Δv = area under a – t graph
$\Delta p = \int dp = \int_1^2 F(t)dt$	Δp = area under F - t graph
$W = \int dW = \int_1^2 F(x)dx$	W = area under F - x graph

Important result for integration

1. $\int_1^2 t^n dt = \left.\frac{t^{n+1}}{n+1}\right] = \frac{t_2^{n+2} - t_1^{n+1}}{n+1}$

2. $\int_1^2 \frac{dt}{t} = \log\frac{t_2}{t_1}$

Solving Problems Involving Non-uniform Acceleration

(a) Acceleration depends on velocity v **or time** t

By definition of acceleration, we have $a = \frac{dv}{dt}$.

After substituting the expression for acceleration in left hand separate the variables.

If a(t) is in terms of t, $\int_0^v dv = \int_0^v a(t)dt$

If a(v) is in terms of v, $\int_0^v \frac{dv}{a(v)} = \int_0^v dt$

On integrating , we get a relation between v and t.

Using $\int_0^x dx = \int_0^x v(t)dt$, x and t can also be related.

(b) Acceleration depends on velocity v or position x

$$a = \frac{dv}{dt} \quad \Rightarrow \quad a = \frac{dx}{dt}\frac{dv}{dx}$$

$$\Rightarrow \quad a = v\frac{dv}{dx}$$

This is another important expression for acceleration.

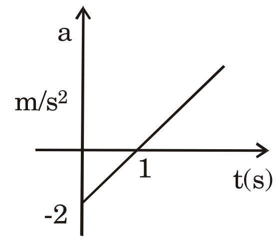
If a(x) is in terms of x, $\int_{v_0}^v vdv = \int_{x_0}^x a(x)dx$

If a(v) is in terms of v, $\int_{v_0}^v \frac{vdv}{a(v)} = \int_{x_0}^x dx$

On integrating, we get a relation between x and v.

Using $\int_{x_0}^x \frac{dx}{v(x)} = \int_0^t dt$, we can relate x and t.

Example 9



The acceleration of a particle varies with time as shown.

(a) Find an expression for velocity in terms of t **and**

(b) Calculate the displacement of the particle in the interval from t **= 2 sec. to** t **= 4 sec.**

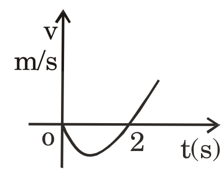
Assume that v **= 0 at** t **= 0.**

Solution.

(a) The a – t graph leads to the following expression for a(t).

$$a(t) = 2t - 2$$

$$\Rightarrow \quad \Delta v = v(t) - v(0) = \int_0^t a(t)dt;\ v(0) = 0$$



$$\Rightarrow \quad v(t) = \int_0^t (2t - 2)\,dt$$

$$= \int_0^t 2t\,dt - \int_0^t 2\,dt$$

$$= \int_0^t t\,dt - \int_0^t 2\,dt$$

$$= 2\left[\frac{t^2}{2}\right]_0^t - 2[t]_0^t$$

$$\Rightarrow \quad v(t) = t^2 - 2t$$

$\Rightarrow$ The v-t graph is a parabola.

$$\Delta x = \int_2^4 v(t)dt$$

$$= \int_2^4 (t^2 - 2t)\,dt$$

$$= \int_2^4 t^2\,dt \int_2^4 t^2\,dt$$

$$= \left.\frac{t^3}{3}\right]_2^4 - \left[\frac{t^2}{2}\right]_2^4$$

$$= \frac{4^3 - 2^3}{3} - 2\frac{(4^2 - 2^2)}{2} = \frac{20}{3}\,\text{m}.$$

Example 10

A particle of mass m is projected in a resisting medium whose resistive force is F = kv and the initial velocity is v_0.

(a) Find the expression for position and velocity in terms of time.

(b) Find the time after which the velocity becomes $v_0/2$.

Solution. Acceleration = $\frac{dv}{dt} = -\frac{kv}{m}$

$$\int_0^v \frac{dv}{v} = \int_0^v - k\frac{dt}{m}$$

$$\log\frac{v}{v_0} = \frac{-kt}{m}$$

$$\Rightarrow \quad v(t) = v_0 e^{-kt/m} \qquad \text{(i)}$$

$$\Rightarrow \quad dx = v_0 e^{-kt/m} dt$$

$$\Rightarrow \quad dx = v_0 e^{-kt/m} dt$$

$$\Rightarrow \quad \int_0^x dx = \int_0^x v_0 e^{-kt/m}\, dt$$

$$\Rightarrow \quad x = v_0\left[\frac{e^{-kt/m}}{-k/m}\right]_0^t \qquad \left[\text{using} \int e^{at}dt = \frac{e^{at}}{a}\right]$$

$$\Rightarrow \quad (t) = \frac{mv_0}{k}\left(1 - e^{-kt/m}\right) \qquad \text{(ii)}$$

(b) substituting $v = \frac{v_0}{2}$ in (i), we get

$$t = \frac{m}{k}\log 2.$$

1. We have found students often confused over the sign of 'g'. As per our sign convention (positive upwards and negative downwards) it is always negative, whether the particle is moving upwards or downwards. Now if u is upwards (i.e., $u \uparrow\downarrow g$) motion is retarded and if u is either zero or downwards $(u \uparrow\downarrow g)$ motion is accelerated.
2. Sometime the standard results are written in different manners and the students unnecessarily go on integrating or differentiating. The standard results which are usually altered are :

(i) $v = u + at$

(ii) $s = ut + \frac{1}{2}at^2$

(iii) $v^2 = u^2 + 2as$

These are the equation of motion in one dimension with constant acceleration.

(iv) $v = \omega\sqrt{A^2 - x^2}$

(v) $a = \omega^2 x$

These are the equation of simple harmonic motion.

The above point will be more clear after going through following two examples.

Example 11

Velocity of a particle moving in a straight line varies with its displacement as $v = \left(\sqrt{4 + 4s}\right)$ m/s. Displacement of particle at time $t = 0$ is $s = 0$. Find displacement of particle at time $t = 2$s.

Solution. Squaring the given equation, we get

$$v^2 = 4 + 4s$$

Now, comparing it with $v^2 = u^2 + 2as$

we get, $u = 2$m/s and $a = 2$ m/s^2

$\therefore$ Displacement at $t = 2$s is

$$s = ut + \frac{1}{2}at^2 \quad \text{or} \quad s = (2)(2) + \frac{1}{2}(2)(2)^2$$

$$s = ut + \frac{1}{2}at^2 \quad \text{or} \quad s = (2)(2) + \frac{1}{2}(2)(2)^2$$

or $s = 8$ m.

Example 12

The velocity of a particle moving in the positive direction of x-axis varies as $v = \alpha\sqrt{x}$, where a is a positive constant. Assuming that at moment $t = 0$, the particle was located at the point $x = 0$. Find

(a) the time dependence of the velocity and the acceleration of the particle

(b) the mean velocity of the particle averaged over the time that the particle takes to cover first I meters of the path.

Solution. Squaring the given equation, we have

$$v^2 = \alpha^2 x$$

Comparing this equation with $v^2 = u^2 + 2as$

we have $u = 0$ and $a = \frac{\alpha^2}{2}$

i.e., the motion is uniformly accelerated with initial velocity $u = 0$ and acceleration $a = \frac{\alpha^2}{2}$. Hence

(a) (i) $v = at$ or $v = \frac{\alpha^2 t}{2}$

(ii) $a = \frac{\alpha^2}{2} = \text{constant}$

(b) $s = \frac{1}{2}at^2 = \frac{1}{2}\frac{\alpha^2}{2}t^2$

$\therefore \quad t = \frac{2\sqrt{s}}{\alpha}$ = time taken to cover first s metres.

$\therefore \quad V_v = \frac{s}{t} = \frac{s}{\frac{2\sqrt{s}}{\alpha}}$ or $V_{av} = \frac{\alpha\sqrt{s}}{2}$.

Example 13

A particle of mass 10^{-2} kg is moving along the positive X-axis under the influence of a force $F(x) = \frac{k}{2x^2}$ where $k = 10^{-2}$ Nm^2. At time $t = 0$, it is at $x = 1.0$m and its velocity is $v = 0$.

(a) find its velocity when it reaches $x = 0.5$m

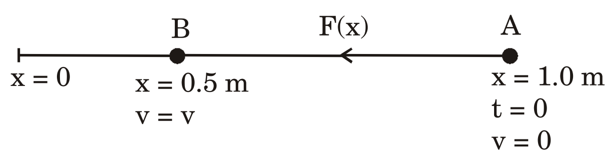
(b) find the time at which it reaches $x = 0.25$m

Solution. (a) Given $F(x) = -\frac{k}{2x^2}$

here k and x^2 are always positive. Hence F is always negative (whether x is positive or negative)

$$a(x) = \frac{F(x)}{m}$$

Substituting the values, we have



$$a = \frac{-k}{2mx^2} = -\frac{10^{-2}}{2\times 10^{-2}\times x^2} = \frac{1}{2x^2}$$

or $\quad v.\frac{dv}{dx} = -\frac{1}{2x^2}$ $\quad$ or $\quad vdv = -\frac{1}{2}\frac{dx}{x^2}$ $\quad$ or $\quad \int_0^v vdv = -\frac{1}{2}\int_{x=1.0}^{x=0.5}\frac{dx}{x^2}$ .... (i)

or $\quad \frac{v^2}{2} = \frac{1}{2}\left(\frac{1}{2}\right)_{x=1.0}^{x=0.5}$ $\quad 2 = \left[\frac{1}{0.5} - \frac{1}{1.0}\right]$ $\quad$ or $\quad v^2 = 1.0$

or $v = \pm 1.0$ m/s

so $v = 1.0$ m/s (because velocity is along negative X-direction).

(b) To find velocity of particle at x = x .

Eq. (i) can be written as

$$\frac{v^2}{2} = \frac{1}{2}\left(\frac{1}{x}\right)_{x-1.0}^{x=x} \quad \text{or} \quad v^2 = \left(\frac{1}{x} - \frac{1}{1.0}\right) = \frac{1-x}{x}$$

or $$v = \left(-\frac{dx}{dt}\right) = \sqrt{\frac{1-x}{x}} \quad \text{or} \quad \int \sqrt{\frac{x}{1-x}}dx = -\int dt$$

or $$\int_t^{0.25} \sqrt{\frac{x}{1-x}}dx = -\int_0^t dt$$

solving this, we get t = 1.48 s

Note. For integration make the substitution $x = \sin^2\theta$.

Example 14

A car starts from rest and accelerates uniformly for 10 s to a velocity of 8 m/s. It then run at a constant velocity and is brought to rest in 64 m with a constant retardation. The total distance covered by the car is 584 m. Find the value of acceleration, retardation and total time taken.

Solution: The car starts from A, accelerates from A to B, run at constant velocity from B to C and retards to rest from C to D.

From A to B

$$a = \frac{v-u}{t} = \frac{80}{10} = 0.8\,\text{m/s}^2$$

$$s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{8}(0.8)(100) = 40\text{m}$$

From B to C

s = BC = 584 – AB – CD

= 584 – 40 – 64 = 480 m

$$s = ut + \frac{1}{2}at^2$$

$\Rightarrow$ 480 = 87 + 0

$\Rightarrow$ t = 60 seconds.

From C to D

$$a = \frac{v^2 - u^2}{2s} = \frac{0^2 - 8^2}{2(64)} = 0.5\text{m/s}^2$$

$$t = \frac{v - u}{a} = \frac{0 - 8}{-0.5} = 16 \text{ seconds}$$

$\Rightarrow$ total time = t_{AB} + BC + t_{CD}

= 10 + 60 + 16 = 86 SEC

$a_{AB} = 0.8\text{m/s}^2$ and $a_{CD} = -0.5 \text{ m/s}^2$

1.4.3 GRAPHS (STRAIGHT LINE MOTION) :

With the help of graphs we visualise the variation of position (x), velocity (v), and acceleration (a) of a moving particle with time. Plotting time (t) on X-axis and x, v, a on Y-axis we get three useful graphs: (i) x–t graphs (ii) v–t graphs (iii) at graphs

Illustration I

A particle is resting on X-axis at the point (3,0). Draw its x-t graph.

The equation is: x(t) = 3.

As the particle is at rest, its X-coordinate is constant with the time and hence a horizontal line y = 3 is the x–t graph.

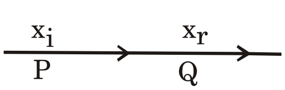
1.4.3 Graphical Interpretation of Some Quantities

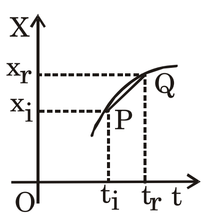
Average Velocity:

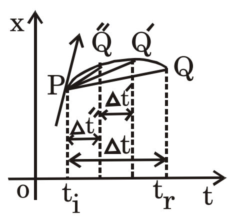
If a particle passes a point P(x_i) at time t = ti and reaches Q (x_f) at a later time instant

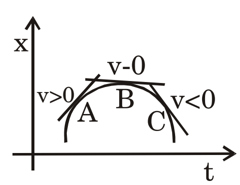
t = t_f, its average velocity in the interval PQ is $V_{av} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t}$.

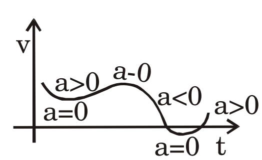
This expression suggests that the average velocity is equal to the slope of the line (chord) joining the points corresponding to P, Q on the x - t graph.











Instantaneous Velocity:

Consider the motion of the particle between the two points P and Q on the x - t graph shown. As the point Q is brought closer and closer to the point P. the time interval between PQ (At, At´, At´´,...) get progressively smaller. The average velocity for each time interval is the slope of the appropriate dotted line (PQ, PQ´, PQ´´,..). As the point Q approaches P, the time interval approaches zero, but at the same time the slope of the dotted line approaches that of the tangent to the curve at the point P. As $\Delta t \to 0$, V_{av} $(= \Delta x/\Delta t)$. $\to V_{inst}$

Geometrically, As $\Delta t \to 0$, chord $PQ \to$ tangent at P.

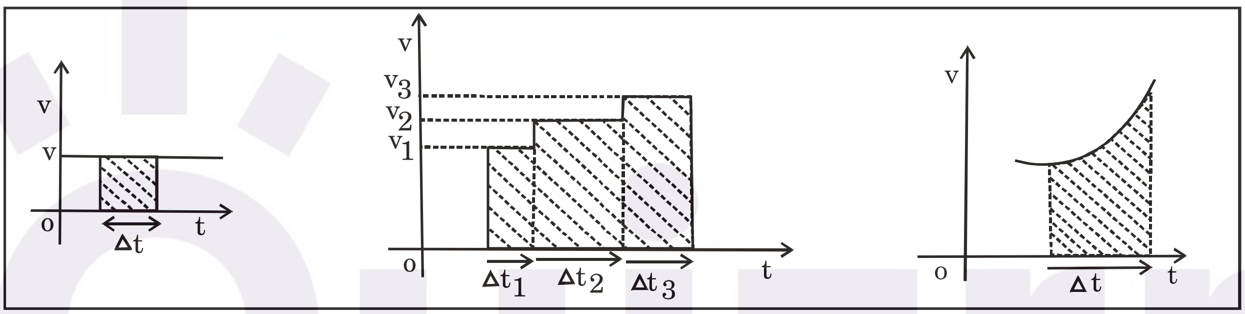
Hence the instantaneous velocity at P in the slope of the tangent at P in the x – t graph. When the slope of the x – t graph is positive, v is negative (as at the point A in figure). At C, v is negative because the tangent has negative slope. The instantaneous velocity at point B (turning point) is zero as the slope is zero.

Instantaneous Acceleration:

The derivative of velocity with respect to time is the slope of the tangent in velocity time (v - t) graph.

1.4.3.2 Displa Cement v-t Graph

Let us now discuss the problem of determining displacement from a v – t graph. For motion at constant velocity, the v – t graph is a horizontal line, as shown in the figure.



Since v $v = \dfrac{\Delta x}{\Delta t}$, the displacement Ax in a time integral Δt is given by $\Delta x = v\Delta t$.

This is just the area of the shaded rectangle of height v and width Δt.

Note that the unit of this area is (m/s) (s) = m.

$\Delta x = u\ \Delta t$ = height of rectangle x base = area of rectangle under v – t graph.

Let us now consider the case when the velocity is not constant. Let the velocity be v_1 for Δt_1 seconds, v_2 for Δt_2 seconds, v_3 for Δt_3 seconds.

Displacement

$= \Delta x = \Delta x_1 + \Delta x_2 + \Delta x_3$

$= v_1\Delta t_1 + v_2\Delta t_2 + v_3\Delta t_3$ = sum of areas of the three shaded rectangle

$\Rightarrow$ Displacement = Δx = area under v – t graph.

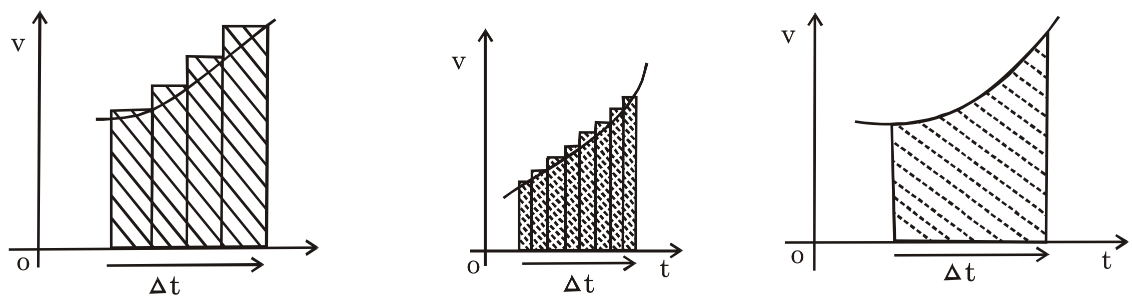
For the most general case, the v – t graph can be a curve i.e. velocity may change continuously with time as shown. To calculate displacement during an interval Δt, we divide this interval into many small intervals $(\Delta t = \Delta t_1, +\Delta t_2, +..+ \Delta t_n)$. If the number of sub-intervals is made very large (i.e. n goes on increasing), each interval becomes very small ($\Delta t_i \to 0$) and displacement during each of these sub-intervals may be taken as the area of the shaded rectangles as shown. The approximation improves as the number of rectangles (i.e. sub-intervals) is increased.

displacement = $v_1\Delta t_1 + v_2\Delta t_2 + \ldots + V_n\Delta t_n$) as $n \to \infty$, $\Delta t_i \to 0$

$\Rightarrow$ displacement = total area under the v – t curve.

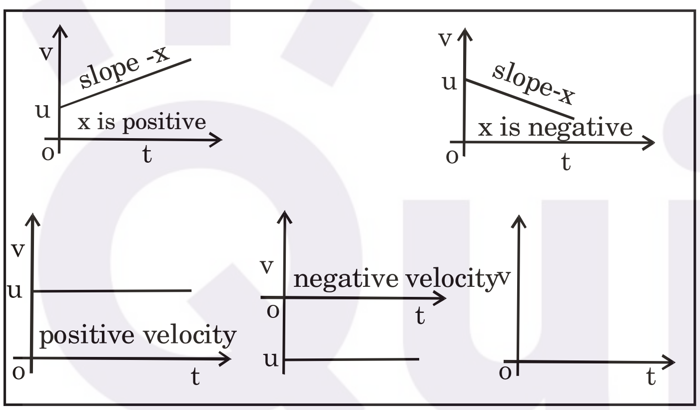
Since a negative velocity causes a negative displacement, areas below the time axis are taken negative.

In a similar way, we can see that $\Delta v = a\Delta t$ leads to the conclusion that area under a – t graph gives the change in velocity Av during that interval.



As the number of sub-intervals is increased ($n \to \infty$), the sum of the areas of the rectangles approaches the area under the curve.

1.4.3.3 Motion with Uniform Velocity u :



Consider a particle moving along X-axis with uniform velocity u starting from the point $x = x_i$ at t =0. Equations of x , v, a are:

$x(t) = x_i + ut$; $v(t) = u$; $a(t) = 0$

x-t graph is a straight line of slope u through x_i

- as velocity is constant, v–t graph is a horizontal line.
- a–t graph coincides with time axis because a = 0 at all time instants.

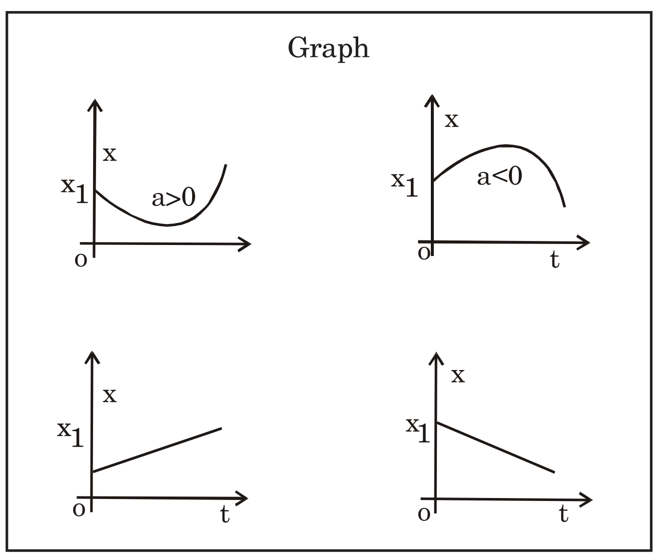
1.4.3.4. Uniformly Accelerated Motion (a≠0)

$$x(t) = x_i + ut + \frac{1}{2}at^2$$

$v(t) = u + at$

$a(t) = a$

- x(t) is a quadratic polynomial in terms of t. Hence x-t graph is a parabola.
- v(t) is a linear polynomial in terms of t. Hence v-t graph is a straight line of slope a.
- a-t graph is a horizontal line because a is constant.



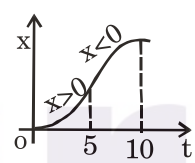
1.4.3.5 Important Points to Remember

- For uniformly accelerated motion ($a \neq 0$), x–t graph is a parabola (opening upwards if $a > 0$ and opening downwards if $a < 0$). The slope of tangent at any point of the parabola gives the velocity at that instant.
- For uniformly accelerated motion ($a \neq 0$), v–t graph is straight line whose slope gives the acceleration of the particle.
- In general, the slope of tangent in x–t graph is velocity and the slope of tangent in v–t graph is the acceleration.
- The area under a–t graph gives the change in velocity.
- The area between the v–t graph and the time-axis gives the distance traveled by the particle, if we take all areas as positive; shaded area = distance covered in the time interval $t = t_1$ to $t = t_2$. (see example 9)
- Area under v–t graph gives displacement if areas below the t-axis are taken negative. (see example 9)

Example 15

A car accelerates from rest at the rate of 1 m/s^2 for 5 seconds and then retards at the same rate till it comes to rest. Draw the x–t, v–t and a–t graphs.

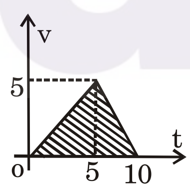
Solution. Velocity acquired after 5 sec. $u + at = 0 + 1(5) = 5$ m/s.

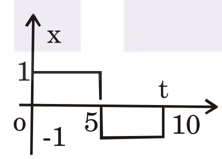


time taken to come to rest $= \dfrac{0-5}{-1} = 5$ sec.

$\Rightarrow$ car starts at $t = 0$ and accelerates till $t = 5$s.

The car starts slowing down at $t = 5$s and comes to rest at $t = 10$s.





In the v-t graph the area of shaded triangle = distance covered

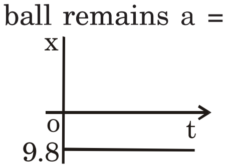
$= \dfrac{1}{2}(5)(10) = 25\,\text{m}$

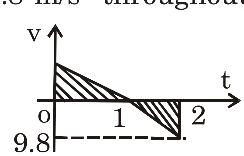
Example 16

A ball is thrown vertically upwards with a speed of 9.8 m/s from the ground. Draw the x–t, v–t and a–t graph for its motion.

Solution. As the acceleration of the ball remains $a = -9.8$ m/s^2 throughout the motion,

a–t graph is a horizontal line.

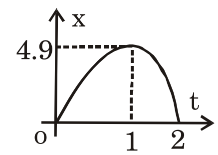




time taken by the ball to reach top

$$= \frac{0 - 9.8}{-9.8} = 1 \text{ sec \& the height attained}$$

$$= \frac{0^2 - 9.8^2}{2(-9.8)} = 4.9\,\text{m}$$



As a < 0, x-t graph is a parabola opening downwards. The x-coordinate of the ball is zero initially and finally when it reaches back after 2 seconds.

NOTE:

(i) velocity at the top = 0

slope of the tangent to the x-t graph at t =1 is zero.

(ii) distance covered = shaded area (taking area above or below t-axis as positive)

$$= 2\left(\frac{1}{2} \times 1 \times 9.8\right) = 9.8\,\text{m}$$

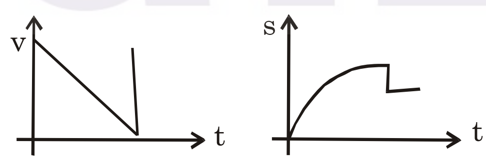
displacement = area (taking area below t-axis as negative) = 4.9 – (4.9) = 0 m

(iii) slope of line in v–t graph = acceleration = – 9.8 m/s^2.

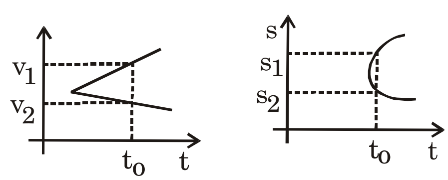
(iv) between t = 0 & t = 1, velocity is + ve i.e., the ball is going up between t = 1 & t = 2, velocity is – ve i.e., the ball is coming down.

Some More Points

1. Slopes of v–t or s–t graphs can never be infinite at any point, because infinite slope of v—t graph means infinite acceleration. Similarly, infinite slope of s–f graph means infinite velocity. Hence, the following graphs are not possible:



2. At one time, two values of velocity or displacement are not possible. Hence, the following graphs are not acceptable:



3. Different values of displacements in s–t graph corresponding to given v–t graph can be calculated just by calculating areas under v–f graph. There is no need of using equations like v = u + at, etc.

Example 17

A car accelerates from rest at a constant rate at a constant rate α for some time, after which it decelerates at a constant rate β, to come to rest. If the total time elapsed is 1 sec evaluate (a) the maximum velocity reached and (b) the total distance traveled.

Solution. (a) Let the car accelerates for time t_1 and decelerates for time t_2. Then

$$t = t_1 + t_2 \qquad \text{(i)}$$

and corresponding velocity-time graph will be as shown in Fig.

From the graph

$$\alpha = \text{slope of line OA} = \frac{V_{max}}{t_1}$$

or $$t_1 = \frac{V_{max}}{\alpha} \qquad \text{(ii)}$$

and $$\beta = -\text{ slope of line AB} = \frac{V_{max}}{t_2}$$

or $$t_2 = \frac{V_{max}}{\beta} \qquad \text{(iii)}$$

From equation (i), (ii) and (iii), we get

$$\frac{V_{max}}{\alpha} + \frac{V_{max}}{\beta} = t \text{ or } V_{max}\left(\frac{\alpha+\beta}{\alpha\beta}\right) = t$$

or $$V_{max} = \frac{\alpha\beta t}{\alpha+\beta}$$

(b) Total distance = displacement

= area under v – t graph

$$= \frac{1}{2} \times t \times \frac{\alpha\beta t}{\alpha+\beta}$$

or Distance $$= \frac{1}{2}\left(\frac{\alpha\beta t^2}{\alpha+\beta}\right).$$

Note. This problem can also be solved by using equations of motion (v = u + at, etc.). Try it yourself.

1.5 CIRCULAR MOTION

Uniform Circular Motion

A particle moving along a circular path with a constant speed is said to be an uniform circular motion.

Angular velocity (ω)

When a particle moves along a circle, it covers somearc length (= s) along the circumference in time t sec. The angle subtended by this arc at the centre or the angle rotated by the radius vector is known as angular displacement in time t sec.

If the particle goes from A to B

angular displacement = $\angle AOB = \theta$

distance covered = arc AB = s

By definition of θ in radian,

$$\theta = \frac{s}{r}$$

The angular displacement per unit time is known as angular velocity.

$$\omega = \frac{\text{angular displacement}}{\text{time}}$$

$$\omega = \frac{\theta}{t} = \frac{s/r}{t} = \frac{1}{r}\left(\frac{s}{r}\right)$$

now s/t = v = speed of the particle.

$$\Rightarrow \quad \omega = \frac{v}{r} \qquad \Rightarrow \quad v = r\omega$$

Time period of revolution (T)

If the particle completes one revolution, $\theta = 2\pi$

Let time for one revolution = T,

$$\Rightarrow \quad \omega = \frac{2\pi}{T}$$

Let n = frequency of revolution

= number of revolution completed in one second

$$\Rightarrow \quad n = \frac{1}{T} = \frac{\omega}{2\pi}$$

$$\Rightarrow \quad \omega = \frac{2\pi}{T} = 2\pi n \text{ and}$$

$$v = r\omega = \frac{2\pi r}{T} = 2\pi rn$$

Velocity and acceleration in circular motion :

The magnitude of velocity does not change with time but the direction of the velocity (along tangent) keeps on changing from moment to moment. As the velocity vector changes direction with time, the acceleration is non-zero in uniform circular motion.

Let v = speed of moving particle (same at all points) and r = radius of the circle.

Let the particle start from A at t = 0. After time t, it reaches point P where the position vector is :

$$\bar{r}(t) = \overline{OP} = r\cos\omega t\,\tilde{i} + r\sin\omega t$$

$$\Rightarrow \qquad \bar{v}(t) = \frac{d\bar{r}}{dt} = -r\omega\sin\omega t\,\tilde{i} + r\omega\cos\omega t\,\tilde{j}$$

$$\text{acceleration} = \bar{a}(t) = \frac{d\bar{v}}{dt} = r\omega^2\cos\omega t\,\tilde{i} - r\omega^2\sin\omega t^2\,\tilde{j}$$

$$= -\omega^2\left(r\cos\omega t\,\tilde{i} - r\omega^2\sin\omega t\,\tilde{j}\right)$$

$$\Rightarrow \qquad \bar{a} = -\omega^2\bar{r}$$

The magnitude of the accelration is ω^2 r and it is directed towards the centre O.

The figure shows the direction of velocity and acceleration for different positions of moving particle on the circle. As the acceleration is directed toward the centre, it is known as **centripetal acceleration** or **radial acceleration** (along the radius).

$\Rightarrow$ centripetal acceleration = $\omega^2 r = v^2/r$

NOTE : So far we have observed that in uniform circular motion, the magnitude of velocity (v) and magnitude of acceleration (v^2/r) are constant, while the direction of the velocity (along the tangent) and the direction of acceleration (along the radius) keep on changing with time.

Non-uniform Circular Motion

If the speed of the particle rotating in the circle changes with time, it is said to be in **non-uniform circular motion.** The acceleration of the particle in that case has two components :

(1) A radical (or centripetal) component which causes the changes in the direction of the velocity. It is directed towards the centre and has a magnitude a_r given as : ar = v^2/r

Note that this component is also present in uniform circular motion.

(2) A tangential component which causes the change in magnitude of velocity. It is directed along the tangent and its magnitude is decided by the net tangential force acting on the particle. Its magnitude is given by a_t as :

$$a_t = \frac{dv}{dt}$$ where v is the speed of the particle.

The tangential acceleration is in the direction of motion if the particle speeds up and opposite to the direction of motion if the particle slows down.

NOTE : In uniform circular motion, tangential component $a_t = 0\ m/s^2$ because speed does not change.

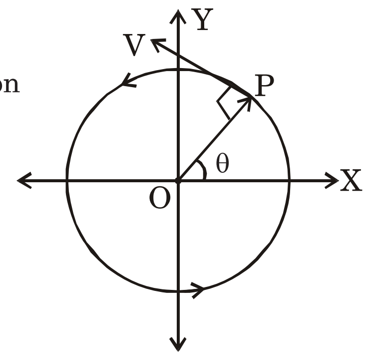
Circular Motion :

1. Circular motion is a special case of curvillinear motion where radius of curvature is same at all points of trajectory.

$$a_t = \left(\frac{dv}{dt}\right),\ a_r = \left(\frac{V^2}{r}\right)$$

a_t = tangential accderation

a_r = radial acceleration



$$\vec{r} = (R\cos\theta)\tilde{i} + (R\sin\theta)\tilde{j}$$

$$\vec{V} = \left(\frac{d\vec{r}}{dt}\right)$$

Also, $\left(\vec{V}.\vec{R} = 0\right) \Rightarrow \left(\vec{V} \perp \vec{R}\right)$

$$\vec{r} = (R\cos\theta)i + (R\sin\theta)j$$

$$\vec{V} = \frac{d\vec{r}}{dt} = R(-\sin\theta)\left(\frac{d\theta}{dt}\right)\tilde{i} + R\cos\theta\left(\frac{d\theta}{dt}\right)\tilde{j}, \qquad \left\{\because \frac{d\theta}{dt} = w\right\}$$

$$= \overline{V}\ Rw(-\sin\theta)\tilde{i} + Rw\,(\cos\theta)\tilde{j}$$

$$= \bar{a}\left(\frac{d\tilde{v}}{dt}\right) = \frac{d}{dt}\left(\omega R\,[-\sin\theta\tilde{i} + \cos\theta\tilde{j}]\right)$$

$$= R\left(-\sin\theta i + \cos\theta j\right)\frac{d\omega}{dt} + (R\omega)\left(-\cos\theta i - \sin\theta j\right)\frac{d\omega}{dt}$$

$$= \alpha R\left(-\sin\theta i + \cos\theta j\right) + \omega^2 R\left(-\cos\theta i - \sin\theta j\right) = \alpha R\,(\tilde{u}_\theta) + w^2 R\,(-\tilde{u}_r) \qquad \left\{\because\ \alpha = \frac{d\omega}{dt}\right\}$$

Let the unit vector along tangent and radius be

$$\tilde{u}_\theta = \vec{V} = (-\sin\theta)i + (\cos\theta)\tilde{j}$$

$$\tilde{u}_r = \frac{\vec{r}}{|\vec{r}|} = (\cos\theta)\,i + (\sin\theta)\,j$$

$$\vec{v} = \omega R\,(\tilde{u}_\theta)$$

$$\vec{a} = \alpha R\left(\tilde{u}_\theta\right) + \omega^2 R\left(-\tilde{u}_r\right)$$

$$\boxed{\vec{a} = \vec{\alpha} \times \vec{R} + \vec{\omega} \times \vec{v}}$$

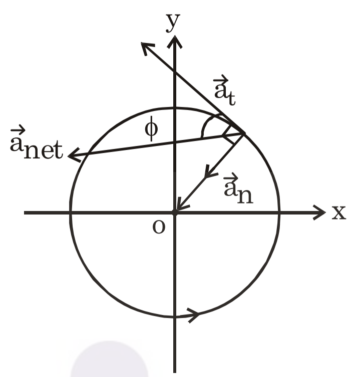
$$\vec{a} = \left(\vec{a} + \vec{a}_r\right)$$

$$a_t = (\alpha R) = \frac{d}{dt}(\omega R) = \left(\frac{dv}{dt}\right)$$

$$a_t = \left(\frac{dv}{dt}\right) = \left(\frac{vdv}{ds}\right)$$

$$a_t = \alpha R = \frac{R\,d\omega}{dt} = R\left(\frac{\omega d\omega}{d\theta}\right)$$

$$a_r = \omega r = \frac{v^2}{r}$$



$\boxed{\tan\phi = \left(\frac{a_n}{a_t}\right)}$ generally angle b/w net acceleration is asked.

Rotational eqns. can be derived as

$$\omega = \omega_0 + \alpha t \qquad \omega = \left(\frac{dv}{dt}\right) \text{ .... (1)}$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \qquad \alpha = \left(\frac{d\omega}{dt}\right) \text{ .... (2)}$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta \qquad \alpha = \left(\omega\frac{d\omega}{d\theta}\right) \text{ .... (3)}$$

In vector form, we have following relations :

(1) $\vec{V} = \vec{\omega} \times \vec{r}$

(2) $\vec{a}_r = -(\omega)^2\vec{r} = \left(\vec{\omega} \times \vec{v}\right)$

(3) $\vec{a}_t = \left(\vec{\alpha} \times \vec{r}\right)$

Example 18

If a Particle is moving along a circular path of radius R with velocity V = $a_o t$. Then find out acceleration of the particle when it has covered nth fraction of the circle.

Solution $a_t = \left(\frac{dv}{dt}\right) = a_0$ ..... (1)

$v^2 = 0 + 2\,a_0\,(s)$ [$S = (2\pi r)n$

$v^2 = 2a_o\,(2\pi rn)$

$v^2 = (4\pi rna_0)$

$\therefore \quad a_r = \frac{v^2}{r} = 4\pi na_0$ ... (2)

Resultant acceleration

$$= \sqrt{a_t^2 + a_r^2} = \sqrt{a_0^2 + 16a^2n^2a_0^2} = \left|\vec{a}\right| = a_0\sqrt{\left(1 + 16a^2n^2\right)}$$

$\tan\phi = \left(\frac{a_r}{a_t}\right) = 4\pi n$ with tangential acceleration.

Example 19

A particle moves along a circular path of radius R, with retardation, such that at any moment the tangitial and normal acceleration are equal in magnitude.

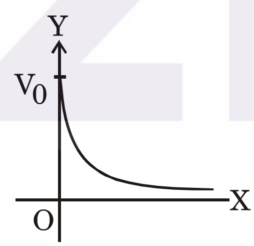
(1) If initial speed of the particle is V_0, then find the speed of the particle as a function of time t and distance s. Also plot it.

(2) Find the total accleration in terms of S. Plot it.

Solution

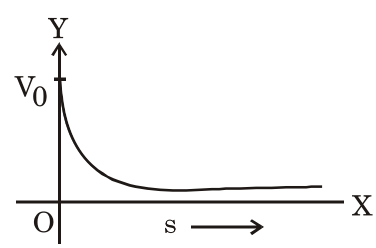
(1) $|\vec{a}_t| = |\vec{a}_r|$

$\Rightarrow \quad -\left(\frac{dv}{dt}\right) = \left(\frac{v^2}{r}\right)$ $\quad \int_{v_o}^{v(t)} -\frac{dv}{V^2} = \frac{1}{R}\int_o^t dt$



$\Rightarrow \quad \frac{1}{V(t)} - \frac{1}{V_0} = \frac{t}{R}$ $\quad V(t) = \left(\frac{V_0 R}{V_0 t + R}\right)$

Also, $\overline{V} - \frac{dv}{ds} = \left(\frac{v^2}{r}\right)$



$-\frac{dv}{v} = \left(\frac{ds}{R}\right)$

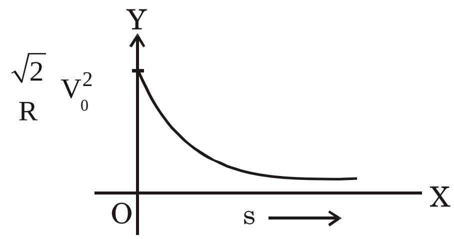
$-\int_{v_0}^{v_s} \frac{dv}{v} = \frac{1}{R}\int_0^s ds$

$\log\left(\frac{V_s}{V_o}\right) = -\frac{S}{R} \quad \Rightarrow \quad V_s = V_o e^{-S/R}$

(2) $a_{net} = \sqrt{a_r^2 + a_t^2}$

$= \sqrt{a_r^2 + a_r^2} = \left(\sqrt{2}\right) a_r$

$\therefore \quad a_{net} = \sqrt{2}\,\frac{v^2}{R} = \frac{\sqrt{2}}{R} V_0^2\, e^{-2S/R}$



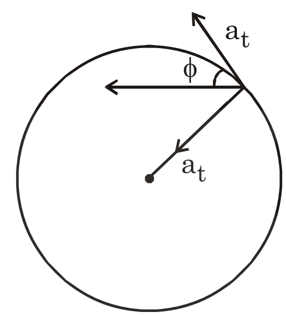
Example 20

A particle moves along a circle of radius R with speed = $\alpha\sqrt{s}$ where s is the distance covered.

(a) Find the angle between velocity and acceleration. as a function of distance covered.

(b) Find the distance covered when the angle is $\pi/_4$.

Solution



(a) $\tan\phi = \left(\frac{a_r}{a_t}\right)$ $\qquad \left(v = \alpha\sqrt{s}\right)$

$= \frac{\alpha^2 s}{R}\left(\frac{2}{\alpha^2}\right)$ $\qquad a_r = \frac{v^2}{R} = \left(\frac{\alpha^2 s}{R}\right)$

$\tan\phi = \frac{2s}{R}$ $\qquad a_t = v\frac{dv}{ds} = \frac{\alpha^2}{2}$

(b) When $\phi = \pi/_4$, $s = (R/_2)$

Example 21

A particle moves in X-Y plane with const-tangential acceleration = a_0 and a_r = bt^4, where b is const. Find total acceleration as a function of distance covered s. (Initial speed = 0)

Solution.

$a_r = \left(\frac{v^2}{r}\right) \Rightarrow \boxed{r = \left(\frac{v^2}{a_r}\right)}$

$v^2 = (2a_0 s)$

Now, $a_r = bt^4$

$s = ut + \frac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{2s}{a_0}} \quad \therefore \quad a_r = b\left(\sqrt{\frac{2s}{a_0}}\right)^4$

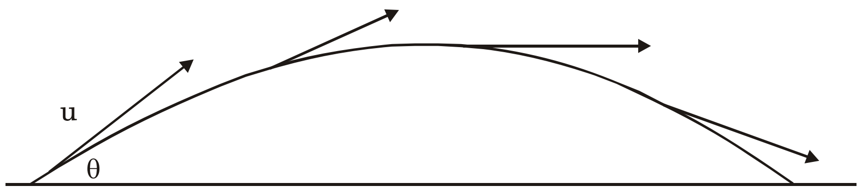
radius of curvature, $r = \frac{2a_0 s\, a_0^2}{b\, 4s^2}$

$\boxed{r = \left(\frac{a_0^3}{2bs}\right)}$

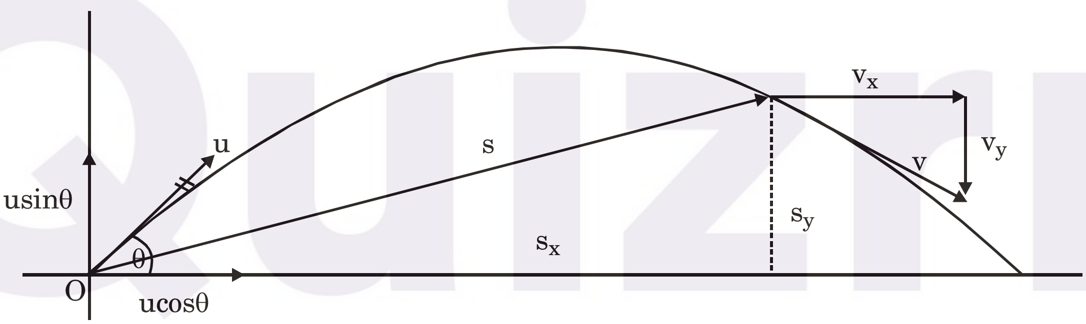
$a_t = \sqrt{a_0^2 + a_r^2}$ $\qquad \boxed{a_t = \sqrt{a_0^2 + b^2\left(\frac{16s^4}{a_0^4}\right)}}$

1.6 PROJECTICE MOTION

We have already seen that when a particle is given a vertical velocity in earth's gravitational field, it moves along a vertical line. Imagine the motion of a particle when it is given an initial velocity u directed at an angle θ of with the horizontal and $\theta \neq 90$.



- Such a particle will move horizontally and as well as vertically i.e. along a curve.
- For convenience, we will take origin at the point from where the particle is thrown and X-axis, Y-axis as horizontal and vertical respectively.
- The velocity of particle at any instant is directed along the tangent to the path and can have horizontal and vertical components.
- The only force acting on the particle is its weight (mg) directed downwards. Hence acceleration is g directed vertically downwards.
- As acceleration does not change with time, the projectile motion is a uniformly accelerated motion. At all time instants, $a_x = 0$ and $a_y = -g$.



Horizontal motion of projectile :

$a_x = 0$

$v_x = u_x + 0(t)$

$v_x = u \cos \theta$ at all time instants, i.e. the horizontal velocity is constant.

s_x = horizontal component of displacement in a time interval of t sec.

$s_x = u_x t + \frac{1}{2} a_x t^2$

$= u \cos \theta\ (t)$ (as a_x is equal to zero)

horizontal displacement = (horizontal velocity) (time)

Vertical motion of projectile :

$a_y = -g$

$v_y = u_y - gt$

$s_y = v_y - \frac{1}{2} gt^2$

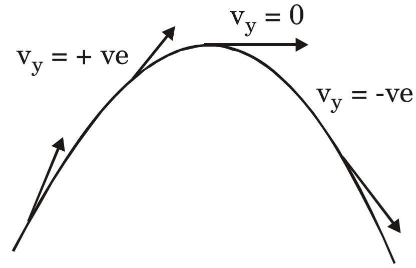
$v_y^2 = u_y^2 - 2gs_y$

u_y = the vertical component of initial velocity

s_y = vertical component of displacement during t sec.

v_y = vertical component of final velocity

Vertical component of velocity at any time



(i) is zero, if the particle moving horizontally (at the highest point)

(ii) is + ve if it is going up.

(iii) is _ ve if it is coming down.

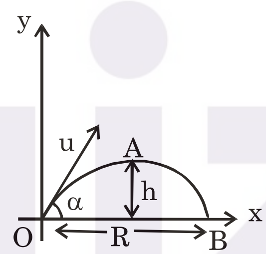
Time of Flight (T)

Refer Fig. Here, x and y-axes are in the directions shown in figure. Axis x is along horizontal direction and axis y is vertically upwards. Thus,

$u_x = u\cos\alpha$, $u_y = u\sin\alpha$, $a_x = 0$ and $a_y = -g$

At point B, $s_y = 0$. So, applying

$s_y = u_y t + \frac{1}{2}a_y t^2$, we have



$$0 = (u\sin\alpha)\ t - \frac{1}{2}gt^2$$

$$t = 0, \frac{2u\sin\alpha}{g}$$

Both $t = 0$ and $t = \frac{2u\sin\alpha}{g}$ correspond to the situation where $s_y = 0$. The time $t = 0$ corresponds to point O and time $t = \frac{2u\sin\alpha}{g}$ corresponds to point B. Thus, time of flight of the projectile is :

$$T = t_{OAB} \quad \text{or} \quad \boxed{T = \frac{2u\sin\alpha}{g}}$$

Horizontal Range (R)

Distance OB is the range R. This is also equal to the displacement of particle along x-axis in time $t = T$. Thus, applying $s_x = u_x t + \frac{1}{2}a_x t^2$, we get

$$R = (u\cos\alpha)\left(\frac{2u\sin\alpha}{g}\right) + 0$$

as $a_x = 0$ and $t = T = \frac{2u\sin\alpha}{g}$

$\therefore \quad R = \dfrac{2u^2 \sin\alpha \cos\alpha}{g} = \dfrac{u^2 \sin 2\alpha}{g}$

or $\quad \boxed{R = \dfrac{u^2 \sin 2\alpha}{g}}$

Here, two points are important regardingthe range of a projectile.

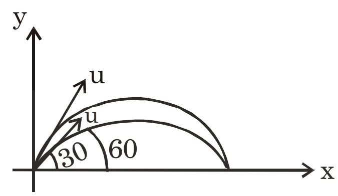
(i) Range is maximum where sin 2α = 1 or α = 45 and this maximum range is :

$$R_{max} = \frac{u^2}{g} \qquad \text{(at } \alpha = 45\text{)}$$

(ii) For given value of u range at α and range at 90 – α are equal although times of flight and maximum heights may be different. Because

$$R_{90^\circ - \alpha} = \frac{u^2 \sin 2(90^\circ - \alpha)}{g} = \frac{u^2 \sin(180^\circ - 2\alpha)}{g}$$

$$= \frac{u^2 \sin 2\alpha}{g} = R_\alpha$$



So, $\quad R_{30} = R_{60}$ or $R_{20} = R_{70}$

Maximum Height (H)

At point A vertical component of velocity becomes zero, i.e. $v_y = 0$. Substituting the proper values in

$$v_y^2 = u_y^2 + 2a_y s_y$$

we have $\quad 0 = (u \sin \alpha)^2 + 2(-g)(H)$

$\therefore \quad \boxed{H = \dfrac{u^2 \sin^2 \alpha}{2g}}$

Equation of trajectory :

It is the equation of the curve along which the particle moves.

Let the particle move from O to an arbitary point P on the curve in time t.

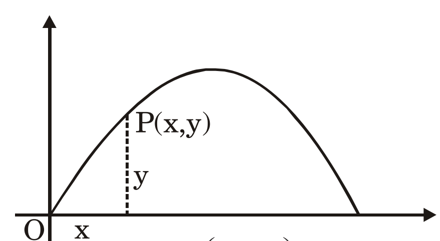
If the coordinates of P are (x, y)

$s_x = x$

$s_y = y$

$\Rightarrow \quad x = (u \cos\theta)\, t \quad \& \quad y = (u \sin\theta)\, t - 1/2\, g\, t^2$

eliminating t from two equations;



$$y = u\sin\theta\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2 \quad \Rightarrow \quad y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} \quad \text{or} \quad y = x\left(1 - \frac{x}{R}\right)\tan\theta$$

This is the equation of the curve along which the particle moves. This is called as the equation of the trajectory of the projectile. As y is quadratic polynomial in terms of x i.e.of the form $y = ax - bx^2$, the curve followed by the projectile is a parabola.

Example 22

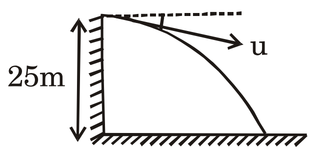
A projectile is given an initial velocity of 5 m/s at an angle 30 below horizontal from the top of a building 25 m high. Find :

(a) the time after which it hits the ground

(b) the distance from the building where it strikes the ground. (Take g = 10 m/s^2)

Solution :

The projectile is thrown from O and lands at A on the ground.



From O to A :

$s_y = -25$ m; $u_y = -5 \sin 30 = -2.5$ m/s.
(– ve becuse vertical component is downwards)

(i) $a_y = -g = -10$ m/s$_2$

$s_y = u_y t + 1/2\, a_y t^2$

$-25 = -2.5\, t - 1/2\, (10)t^2$

$10\, t^2 + 5t - 50 = 0$

on solivng, we get : t = 2 s, – 2.5 s

$\Rightarrow$ the relevant time = 2s

(ii) the distance of A from the building = s_x

$s_x = u_x t = (5 \cos 30\ )\ 2 = 5\sqrt{3}$ m

Example 23

Find the angle of projection of a projectile for which the horizontal range and maximum height are equal.

Solution. Given, R = H

$\therefore \quad \dfrac{u^2 \sin 2\alpha}{g} = \dfrac{u^2 \sin^2 \alpha}{2g}$

or $\quad 2\sin\alpha \cos\alpha = \dfrac{\sin^2\alpha}{2}$

or $\quad \dfrac{\sin\alpha}{\cos\alpha} = 4$ or $\tan\alpha = 4$

$\therefore \quad \alpha = \tan^{-1}(4)$ **Ans.**

Example 24

Prove that the maximum horizontal range is four times the maximum height attained by the projectile; when fired at an inclination so as to have maximum horizontal range.

Solution. For θ = 45 , the horizontal range is maximum and is given by

$$R_{max} = \frac{u^2}{g}$$

Maximum height attained

$$H_{max} = \frac{u^2 \sin^2 45°}{2g} = \frac{u^2}{4g} = \frac{R_{max}}{4}$$

or $\quad R_{max} = 4\, H_{max}$ **Proved.**

Example 25

There are two angles of projection for which the horizontal range is the same. Show that the sum of the maximum heights for these two angles is independent of the angle of projection.

Solution. There are two angles of projection α and $90 - \alpha$ for which the horizontal range R is same.

Now, $$H_1 = \frac{u^2 \sin^2 \alpha}{2g}$$

and $$H_2 = \frac{u^2 \sin^2 (90° - \alpha)}{2g} = \frac{u^2 \cos^2 \alpha}{2g}$$

Therefore, $$H_1 + H_2 = \left(\sin^2 \alpha + \cos^2 \alpha\right) = \frac{u^2}{2g}$$

Clearly the sum of the heights for the two angles of projection is independent of the angles of projection.

Example 26

Show that there are two values of time for which a projectile is at the same height. Also show mathematically that the sum of these two times is equal to the time of flight.

Solution. For vertically upward motion of a projectile

$$y = (u \sin\alpha)\, t - \frac{1}{2} gt^2$$

or $$\frac{1}{2} gt^2 - (u \sin\alpha)\, t + y = 0$$

This is a quadratic equation in t. Its roots are

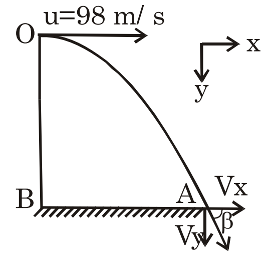
$$t_1 = \frac{u \sin\alpha - \sqrt{u^2 \sin^2 \alpha - 2gy}}{g}$$

and $$t_2 = \frac{u \sin\alpha + \sqrt{u^2 \sin^2 \alpha - 2gy}}{g}$$

$\therefore$ $$t_1 + t_2 = \frac{2u\sin\alpha}{g} = T \quad \text{(time of flightof the projectile)}$$

Example 27

A projectile is fired horizontally with a velocity of 98 m/s from the top of a hill 490 m high. Find :



(a) the time taken by the projectile to reach the ground

(b) the distance of the point where the particle hits the ground from foot of the hill and

(c) the velocity with which the projectile hits the ground ($g = 9.8\ m/s^2$)

Solution. In this problem we cannot apply the formulae of R, H and T directly. We will have to follow the three steps discussed in the theory. Here, it will be more convenient to choose x and y directions as shown in figure.

Here, $u_x = 98$ m/s. $a_x = 0$, $u_y = 0$ and $a_y = g$

(a) $s_y = 490$m. So, applying

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

$\therefore$ $$490 = 0 + \frac{1}{2}(9.8)t^2$$

$\therefore$ $$t = 10 \text{ s}$$

(b) $$BA = s_x = u_x t + \frac{1}{2} a_x t^2$$

or $$BA = (98)(10) + 0$$

or $$BA = 980 \text{ m}$$ **Ans.**

(c) $$v_x = u_x = 98 \text{ m/s}$$

$$v_y = u_y + a_y t = 0 + (9.8)(10) = 98 \text{ m/s}$$

$\therefore$ $$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(98)^2 + (98)^2} = 98\sqrt{2} \text{ m/s}$$

and $$\tan\beta = \frac{v_y}{v_x} = \frac{98}{98} = 1$$

$\therefore$ $$\beta = 45$$

Thus, the projectile hits the ground with a velocity $98\sqrt{2}$ m/s at an angle of $\beta = 45$ with horizontal as shown in Fig **Ans.**

Example 28

A stone is thrown with a velocity of 19.6 m/s at an angle of 30 above horizontal from the top of a building 14.7 m high. Find :

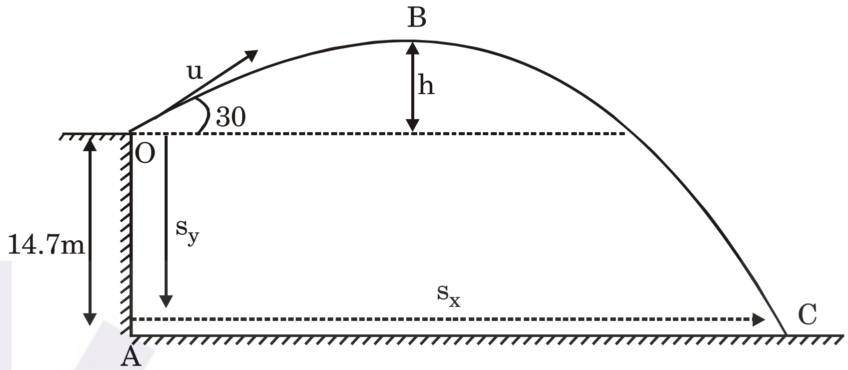
(i) the time after which the stone strikes the ground

(ii) the distance of the landing point of the stone from the building.

(iii) the velocity with which the stone hits the ground.

(iv) the maximum height attained by the stone above the ground.

Solution.



Consider the interval from O to C

u_x = 19.6 cos 30 = 9.83 m/s a_x = 0 m/s^2 u_y = 19.6 sin 30 = 9.8 m/s; a_y = 9.8 m/s^2

(i) $s_y = u_y\, t + 1/2\, a_y\, t^2$

$-14.7 = 9.8\, t + 1/2\,(-9.8)\, t^2$

$\Rightarrow$ $4.9\, t^2 - 9.8\, t - 14.7 = 0$

$\Rightarrow$ $t = -1,\ 3$ s

stone lands at C after 3 seconds.

(ii) From O to C, the horizontal displlacement

= Sx

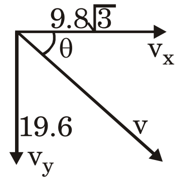
$S_x = u_x t = (19.6 \cos 30\ )\quad 3 = 50.92$ m

distance of C from the building = AC

= 50.92 m

(iii) the horizontal velocity remains constant

Hence at C,



$v_x = u_x = 9.8\sqrt{3}$ m/s

$v_y = u_y = + a_y\, t$

$v_y = 19.6\quad \sin 30\ - 9.8\quad 3$

$v_y = -19.6$ m/s

v_y is – ve because the stone is moving down when it hits the ground.

resultant velocity = $\sqrt{v_x^2 + v_y^2}$

$$= \sqrt{\left(98\sqrt{3}\right)^2 + (19.6)^2} = 9.8\sqrt{7}\ \text{m/s}$$

velocity is directed at an angle θ given by :

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{19.6}{9.8\sqrt{3}}\right)$$

$$= \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)\ \text{below horizontal}$$

(iv) maximum height attained above ground

= height of B above point of projection + height of building

= h + 14.7

= $(u^2/2g)\sin^2 30\ + 14.7$

= 19.6 m

Example 29

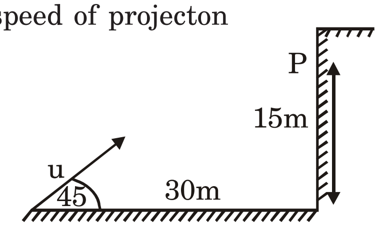
A projectile shot at an angle of 45 above the horizontal strikes a building 30 m away at a point 15 m above the point of projection. Find :

(a) the speed of projection

(b) the magnitude and direction of velocity of projectile when it strikes the building.

Solution :

Let u = speed of projecton



(a) Let P be the point on the building where projectile hits it.

Taking point of projection as origin, coordinates of P are (30, 15). Using the equation of trajectory.

$$y = x \tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

$$\Rightarrow \quad 15 = 30 \tan 45 - \frac{g(30)^2}{2u^2\cos^2 45^\circ}$$

$\Rightarrow$ u = 24.2 m/s

(b) at P, $v_x = u_x = 24.2 \cos 45 = 17.11$ m/s.

$$v_y^2 = u_y^2 + 2a_y s_y$$

$$v_y^2 = u^2 \sin^2 45^\circ - 2g(15)$$

$$v_y^2 = (60g)(0.5) - 30g = (\ )$$

$\Rightarrow$ $v_y = 0$; At P, projectile is at its heighest point and hence moving horizontally.

Example 30

A rifle with a muzzle velocity of 100 m/s shoots a bullet at a small target 30 m away in the same horizontal line. how high above the target must the gun be aimed so that the bullet will hit the target ?

Solution :

Let θ = angle of projection of bullet

Let the rifle be at O and target at T

As OT is horizontal range = 30 m

$\Rightarrow$ $(u^2/g)\sin 2\theta = 30$

$\Rightarrow$ $\sin 2\theta = 0.03$

As $\sin 2\theta$ is small, so take $\sin 2\theta \approx 2\theta$

$2\theta = 0.03 \Rightarrow \quad \theta = 0.015$

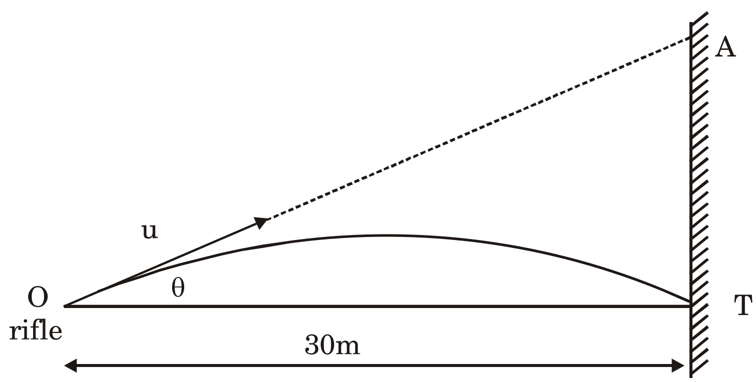
The rifle is aimed at A.

The height of A above the target = AT

$\Rightarrow$ AT = 30 tan θ (from triangle OAT)

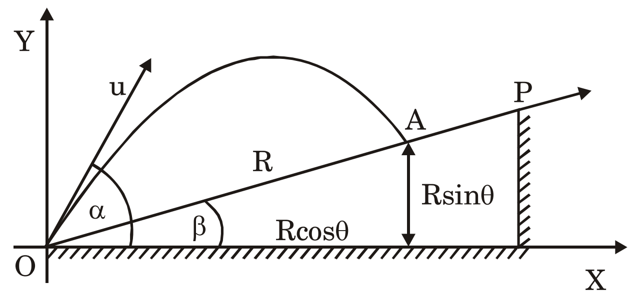
AT $\approx 30(\theta) = 30 \quad (0.015)$

$\Rightarrow$ AT = 45 cm.



Range of a Projectile on Inclined Plane

Let OP be an inclined plane making an angle β with the horizontal line OX (X-axis). Let a projectile be projected from O with a speed u at an angle α with the horizontal. It meets the inclined plane at the point A. The distance OA is the range on the inclined plane.



Let R = OA

⇒ A has the coordinates : (R cos β, R sin β)

The equation of trajectory is :

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

$$= \frac{x}{u\cos\alpha} - \frac{x}{u\cos\beta}$$

$$= \frac{x(\cos\beta - \cos\alpha)}{u\cos\alpha\,\cos\beta}$$

$$= \frac{2u^2\,(\cos\beta - \cos\alpha)}{t(\tan\alpha + \tan\beta)\,u\cos\beta\,\cos\alpha}$$

$$= \frac{2u(\cos\beta - \cos\alpha)}{g\,\sin(\alpha+\beta)}$$

$$= \frac{2u}{g}\,2\sin\frac{\alpha+\beta}{2}\,\sin\frac{\alpha-\beta}{2}$$

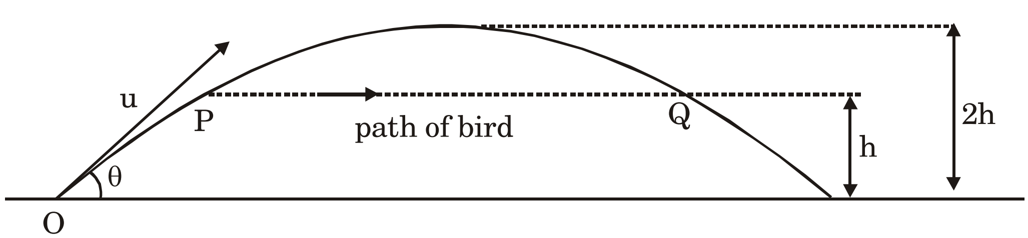
$$= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$= \frac{2u^{\sin\left(\frac{\alpha-\beta}{2}\right)}}{g\,\cos\left(\frac{\alpha+\beta}{2}\right)}$$

Example 31

A stone is projected from the ground in such a direction so as to hit a bird on the top of a telegraph post of height h and attains the maximum height of 2 h above the ground. If at the instant of projection, the bird were to fly away horizontally with a uniform speed, find the ratio between the horizontal velocities of bird and the stone, if the stone hits the bird while descending.

Solution. Let the u and θ be the velocity and the angle of projection of the stone.



As 2 h is the maximum height attained,

$$2h = \frac{u^2 \sin^2\theta}{2g}$$

$\Rightarrow$ $u^2 \sin^2\theta = 4\,gh$

The bird starts from point P at the instant when the stone is thrown and goes to Q where it gets hit by the stone.

Hence the time taken by brd to go from P to Q = time taken by the stone from O to Q = t_{OQ}

Let v = horizontal velocity of the bird

$$\Rightarrow \quad v = \frac{PQ}{t_{OQ}}$$

time taken by the stone to go from P to Q = time from O to Q – time from O to P = $t_{OQ} - t_{OP}$

horizontal velocity of stone = $u\cos\theta = \dfrac{PQ}{t_{OQ} - t_{OP}}$

As the stone is at a height h at t_{OP} and t_{OQ}, these time instants are the roots of the equation :

$$h = (u \sin\theta)\, t - \frac{1}{2}gt^2$$

solving and using $u \sin\theta = \sqrt{4gh}$ we get,

$$t_{OQ}, t_{OP} = \frac{2u\sin\theta \pm \sqrt{4u^2\sin^2\theta - 8gh}}{2g}$$

$$= \frac{4\sqrt{gh} \pm 2\sqrt{2}\sqrt{gh}}{2g}$$

$$\Rightarrow \quad t_{OQ} = \frac{\left(2+\sqrt{2}\right)}{g}\sqrt{gh};\ t_{OP} = \frac{\left(2-\sqrt{2}\right)}{g}\sqrt{gh}$$

Hence the ratio of horizontal velocities = $\dfrac{v}{u\cos\theta}$

$$= \frac{t_{OQ} - t_{OP}}{t_{OQ}} = \frac{\dfrac{2\sqrt{2}}{g}\sqrt{gh}}{\dfrac{2+\sqrt{2}}{g}\sqrt{gh}}$$

$$= \frac{2\sqrt{2}}{2+\sqrt{2}} = \frac{2}{\sqrt{2}+1}$$

Example 32

With what minimum speed must a particle be projected from origin so that it is able to pass through a given point P (a, b) ?

Solution :

Let u and α be the velocity and angle of projection respectively.

For the projectile to pass through P (a, b)

$$b = a\tan\alpha - \frac{ga^2}{2u^2\cos^2\alpha}$$

$$b = a\tan\alpha - \frac{ga^2}{2u^2}\left(1+\tan^2\alpha\right)$$

$\Rightarrow$ $ga^2\tan^2\alpha - 2a\,u^2\tan\alpha + (ga^2 + 2bu^2) = 0$

The projectile will pass through P (a, b) if this equation (quadratic in $\tan\alpha$) gives some real value of α.

i.e., its discriminant ≥ 0

$4a^2u^4 - 4ga^2\,(ga^2 + 2bu^2) \geq 0$

$u^4 - 2gbu^2 - g^2a^2 \geq 0$

$u^4 - 2gbu^2 + b^2g_2\; b^2g^2 + a^2g^2$

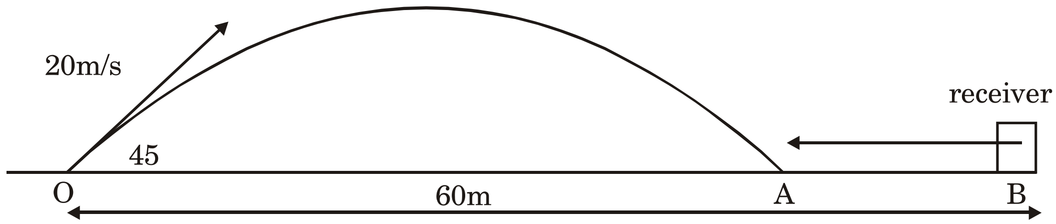
$(u^2 - bg)^2\; (b^2 + a^2)\; g^2$

$$\Rightarrow \quad u \geq \sqrt{bg + g\sqrt{a^2+b^2}}$$

Example 33

A football is kicked off with a initial speed of 20 m/s at a projection angle of 45 . A receiver on the goal-line at a distance of 60 m away in the direction of the kick, starts running to meet the ball at that instant. What must be his speed if he is to catch the ball before it hits the ground ? (Take g = 10 m/s^2)

Solution.



Let u = 20 m/s; θ = 45

v = speed of the receiver

The ball is projected from O and the receiver runs from B to catch the ball at A. In any problem involving motion of two bodies, usual steps to be followed are :

(i) let t be the time after they meet,

(ii) express the relation between the magnitude of displacement in terms of t.

Let t = time after which the receiver meets the ball hence t = time taken by the ball to go from O to A and t = time taken by the receiver to go from B to A.

$$\Rightarrow \quad OA = \text{range} = \frac{u^2}{g}\sin 2\theta$$

and AB = distance covered by receiver = vt

$$OB = 60 \Rightarrow \quad \frac{u^2}{g}\sin 2\theta + vt = 60 \quad \text{.... (1)}$$

Also t = time of flight $\Rightarrow \quad t = \dfrac{2u\sin\theta}{g}$

Solving (1) & (2), we get v and t. Eliminating t :

$$\frac{u^2}{g}\sin 2\theta + v\left(\frac{2u\sin\theta}{g}\right) = 60$$

$$v = \frac{60g - u^2\sin 2\theta}{2u\sin\theta}$$

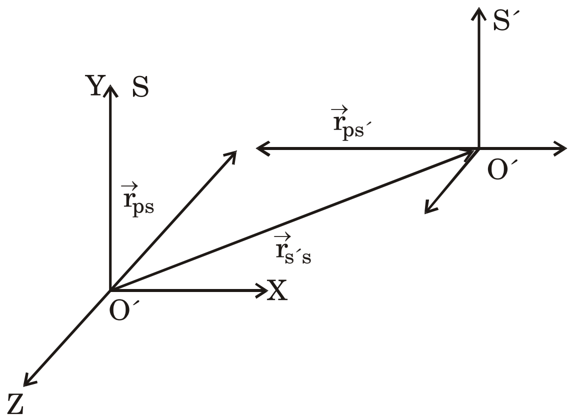
$$v = \frac{600 - 400}{2(20)}\sqrt{2}$$

$$\Rightarrow \quad v = 5\sqrt{2} \text{ m/s}$$

1.7 RELATIVE VELOCITY

Consider two frames of reference S and S´ and suppose the particle P is observed from both the frames. The frames may be moving with respect to each other. The position vector of the particle P with respect to the frame S is $\bar{r}\ P_1S = \overrightarrow{OP}$. The position vector of the particle with respect to the frame S´ is $\bar{r}_{P_1S^1} = \overline{O'P}$. The position of the frame S´ (the origin of frame S´ in fact) with respect to the frame S is OO´.

Now, $\quad \overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}$



$$\boxed{\vec{r}_{ps} = \vec{r}_{ps'} + \vec{r}_{s's}}$$

$$\vec{r}_{ps'} = \vec{r}_{ps} - \vec{r}_{s's}$$

Differentiate w.r.t. time

$$\vec{v}_{ps'} = \vec{v}_{ps} - \vec{v}_{s's}$$

Differentiate again

$$\vec{a}_{ps'} = \vec{a}_{ps} - \vec{a}_{s's}$$

In general

$$\boxed{\vec{v}_{12} = \vec{v}_{1s} - \vec{v}_{2s}}$$

or $\vec{V}_1 = \vec{V}_{12} + \vec{V}_2$

or $\vec{V}_{12} = \vec{V}_1 - \vec{V}_2$

$$\vec{V}_{A_1D} = \vec{V}_{AB} + \vec{V}_{BC} + \vec{V}_{CD} \quad \text{...........} \ (3)$$

Similarly in cartesian form

$V_1(x) = V_{1,2}(x) + V_2(x)$
$V_2(y) = V_{1,2}(y) + v_2(y)$

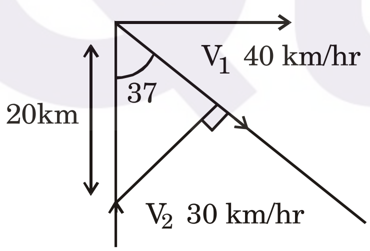
Also, $\dfrac{d\vec{r}_{12}}{dt} = \vec{V}_{12}, \ \dfrac{d\vec{v}_{12}}{dt} = \vec{a}_{12}$

1.7.1 Shortest distance of approach

Example 34

I starts from origin in x direction with v = 40 km/hr 2 is initially 20 km from origin and starts in y direction as shown in fig. Find out shortest distance and time takes to reach.

Solution.



$$\vec{V}_{12} = \vec{V}_1 - \vec{V}_2 = 40\tilde{i} - \left(30\tilde{j}\right)$$

$$= \left(40\tilde{i} - 30\tilde{j}\right)$$

$$\tan\phi = \left|\frac{\vec{v}_1}{\vec{v}_2}\right| = \frac{30}{40} = \frac{3}{4}, \quad \phi = 37°$$

$$\left|\vec{V}_{12}\right| = \sqrt{40^2 + 30^2} = 50 \text{ km/hr}$$

Now, once we have studied the motion of 1 with respect to 2, we can consider 2, to be stationary and then

Anslyse the motion as if we are sitting on 2

$$\therefore \quad \text{time} = \frac{l\sin\theta}{\vec{v}_{12}} = \left(\frac{12}{50}\right) = 0.245$$

2nd Method : (Not preferable)

Perform the analysis from ground

$$S = \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2 + (z_2 - z_1)^2}$$

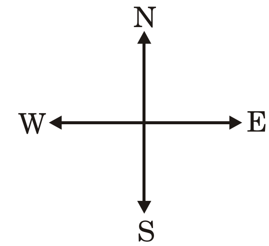
and put $\left(\dfrac{ds}{dt}\right) = 0$

Remember

Motion of one projectile with respect to anoter projectile is always rectilinear.

Example 35

Car A has an acceleration of 2 m/s² due east and car B, 4 m/s² due north. What is the acceleration of car B with respect to car A ?



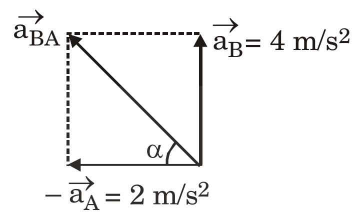
Solution. It is a two dimensional motion. Therefore,

$\vec{a}_{BA}$ = acceleration of car B with respect to car A. = $\vec{a}_B - \vec{a}_A$

Here, $\vec{a}_B$ = acceleration of car B = 4 m/s² (due north)

and $\vec{a}_A$ = acceleration of car A = 2 m/s² (due east)

$$\left|\vec{a}_{BA}\right| = \sqrt{(4)^2 + (2)^2} = 2\sqrt{5}\,\text{m/s}^2$$



and $\alpha = \tan^{-1}\left(\frac{4}{2}\right) = \tan^{-1}(2)$

Thus, $\vec{a}_{BA}$ is $2\sqrt{5}$ m/s² at an angle of $\alpha = \tan^{-1}$ (2) from west towards north.

1.7.2 Minimum Distance between Two Bodies in Motion

When two bodies are in motion, the minimum distance between them or the time when one body overtakes the other can be solved easily by the principle of relative motion. In these type of problems one body is assumed to be at rest and the relative motion of the other body is considered. By assuming so two body problem is converted into one body problem and the solution becomes easy. Following example will illustrate the statement.

Example 36

An open lift is moving upward with velocity 10 m/s. It has an upward acceleration of 2 m/s². A ball is projected upwards with velocity 20 m/s relative to ground. Find :

(a) time when ball again meets the lift.

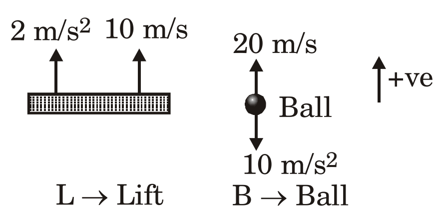
(b) displacement of lift and ball at that instant.

(c) distance travelled by the ball upto that instant. Take g = 10 m/s²

Solution. (a) At the time when ball again meets the lift.

$$S_L = S_B$$

$$\therefore \quad 10t + \frac{1}{2} \times 2 \times t^2 = 20t - \frac{1}{2} \times 10t^2$$



Solving this equation we get,

t = 0 and $t = \frac{5}{3}$ seconds

$\therefore$ Ball will again meet the lift after $\frac{5}{3}$ second.

(b) At this instant $S_L = S_B = 10 \times \frac{5}{3} + \frac{1}{2} \times 2 \times \left(\frac{5}{3}\right)^2 = \frac{175}{9}\text{m} = 19.4 \text{ m}$

(c) For the ball $u \uparrow \downarrow a$. Therefore we will first find t_0, the time when its velocity becomes zero.

$$t_0 = \left|\frac{u}{a}\right| = \frac{20}{10} = 2 \text{ second}$$

As $t\left(\frac{5}{3}\text{second}\right) < t_0$, distance and displacement are equal or

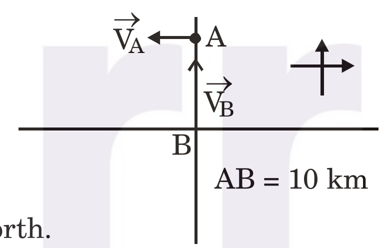
d = 19.4 m.

Example 37

Two ships A and B are 10 km apart on a line running south to north. Ship A farther north is streaming west at 20 km/hr and ship B is streaming north at 20 km/hr. What is their distance of closest approach and how long do they take to reach it ?

Solution : Ships A and B are moving with same speed 20 km/hr in the directions shown in figure. It is a two dimensional, two body problem with zero acceleration. Let us find $\vec{v}_{BA}$

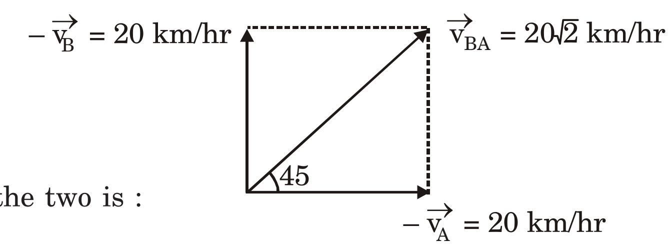
$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$



Here, $\left|\vec{v}_{BA}\right| = \sqrt{(20)^2 + (20)^2} = 20\sqrt{2}$ km/hr

i.e., $\vec{v}_{BA}$ is $10\sqrt{2}$ km/hr at an angle of 45 from east towards north.

Thus, the given problem can be simplified as :

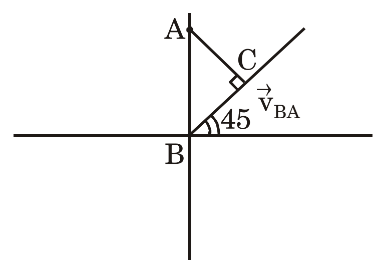


A is at rest and B is moving with $\vec{v}_{BA}$ in the direction shown in Fig.

Therefore, the minimum distance between the two is :

$$S_{min} = AC = AB \sin 45° = 10\left(\frac{1}{\sqrt{2}}\right)\text{km}$$

$= 5\sqrt{2}$ km **Ans.**



and the desired time is

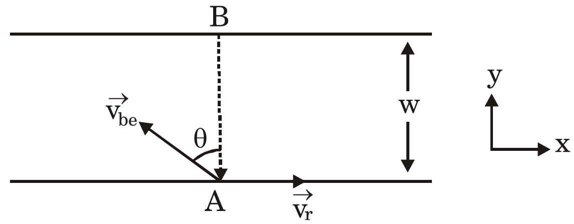
$$t = \frac{BC}{\left|\vec{v}_{BA}\right|} = \frac{5\sqrt{2}}{20\sqrt{2}} \quad (BC = AC = 5\sqrt{2} \text{ km})$$

$= \frac{1}{4}$ hr = 15 minutes. **Ans.**

1.7.3 River-boat Problems

In river boat problems we come across the following three terms :

$\vec{v}_r$ = absolute velocity of river



$\vec{v}_{br}$ = velocity of boatman with respect to river or velocity of boatman in still water

and $\vec{v}_b$ = absolute velocity of boatman

Here, it is important to note that $\vec{v}_{br}$ is the velocity of boatman with which he steers and $\vec{v}_b$ is the actual velocity of boatman relative to ground.

Further, $\vec{v}_b = \vec{v}_{br} = \vec{v}_r$

Now, let us derive some standard results and their special cases.

A boatman starts from point A on one bank of a river with velocity $\vec{v}_{br}$ in the direction shown in Fig. River is flowing along positive x-direction with velocity $\vec{v}_r$. Width ofthe river is w. Then

$$\vec{v}_b = \vec{v}_r + \vec{v}_{br}$$

Therefore, $v_{bx} = v_{rx} + v_{brx} = v_r - v_{br}\sin\theta$

and $v_{by} = v_{ry} + v_{bry} = 0 + v_{br}\cos\theta = v_{br}\cos\theta$

Now, time taken by the boatman to cross the river is :

$$t = \frac{w}{v_{by}} = \frac{w}{v_{br}\cos\theta}$$

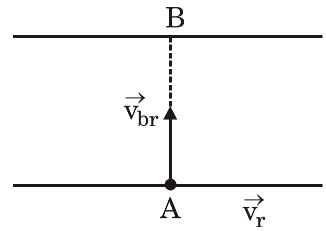
or $$\boxed{t = \frac{w}{v_{br}\cos\theta}} \quad \text{.... (i)}$$

Further, displacement along x-axis when he reaches on the other bank (also called drift) is

$$x = v_{bx}\, t = \left(v_r - v_{br}\sin\theta\right)\frac{w}{v_{br}\cos\theta}$$

or $$\boxed{x = \left(v_r - v_{br}\sin\theta\right)\frac{w}{v_{br}\cos\theta}} \quad \text{... (ii)}$$

Two special cases are ;

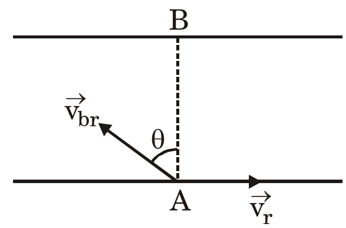


(i) Condition when the boatman crosses the river in shortest interval of time

From equation (i) we can see that time (t) will be minimum when $\theta = 0$, i.e. the boatman should steer his boat perpendicular to the river current.

Also, $t_{min} = \dfrac{w}{v_{br}}$ as cos θ = 1

(ii) Condition when the boatman wants to reach point B, i.e., at a point just opposite from where he started



In this case, the drift (x) should be zero

$\therefore \quad x = 0$

or $\quad (v_r - v_{br}\sin\theta)\dfrac{w}{v_{br}\cos\theta} = 0$ or $v_r = v_{br} \sin\theta$

or $\quad \boxed{\sin\theta = \dfrac{v_r}{v_{br}} \text{ or } \theta = \sin^{-1}\left(\dfrac{v_r}{v_{br}}\right)}$

Hence, to reach point B the boatman should row at an angle $\theta = \sin^{-1}\left(\dfrac{v_r}{v_{br}}\right)$ upstream from AB.

Further, since sin θ > 1

So, if $v_r \geq v_{br}$, the boatman can never reach at point B. Because if $v_r \geq v_{br}$, sin θ = 90 and it is just impossible to reach at B if θ = 90 . Moreover it can be seen that $v_b = 0$ if $v_r = v_{br}$ and θ = 90 . Similarly, if $v_r > v_{br}$, sin θ > 1, i.e., no such angle exists. Pratically it can be realized in this manner that it is not possible to reach at B if river velocity (v_r) is too high.

(iii) Shortest path

Path length travelled by the boatman when he reaches the opposite shore is

$$s = \sqrt{w^2 + x^2}$$

When $v_r < v_{br}$. In this case x = 0, when $\theta = \sin^{-1}\left(\dfrac{v_r}{v_{br}}\right)$

or $\quad \boxed{s_{min} = w \text{ at } \sin^{-1}\left(\dfrac{v_r}{v_{br}}\right)}$

When $v_r > v_{br}$: In this case x is minimum, where $\dfrac{dx}{d\theta} = 0$

or $\quad \dfrac{d}{d\theta}\left\{\dfrac{w}{v_{br} - \cos\theta}(v_r - v_{br}\sin\theta)\right\} = 0$

or $\quad -v_{br}\cos^2\theta - (v_r - v_{br}\sin\theta)(-\sin\theta) = 0$

or $\quad -v_{br} + v_r \sin\theta = 0$

or $\quad \theta = \sin^{-1}\left(\dfrac{v_{br}}{v_r}\right)$

Now, at this angle we can find x_{min} and then s_{min} which comes out to be

$$\boxed{s_{min} = w\left(\dfrac{v_r}{v_{br}}\right) \text{ at } \theta = \sin^{-1}\left(\dfrac{v_{br}}{v_r}\right)}$$

Example 38

A man can row a boat with 4 km/hr in still water. If he is crossing a river where the current is 2 km/hr.

(a) In what direction will his boat be headed if he wants to reach a point on the other bank, directly opposite to starting point ?

(b) If width of the river is 4 km, how long will the man take to cross the river, with the condition in part (a) ?

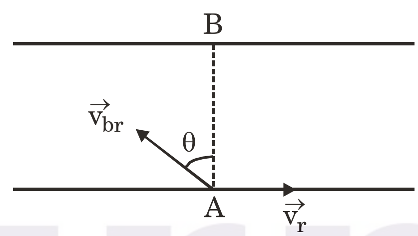
(c) In what direction should he head the boat if he wants to cross the river in shortest time and what was is this minimum time ?

(d) How long will it take him to row 2 kms up the stream and then back to his starting point ?

Solution. (a) Given, that v_{br} = 4 km/hr and v_r = 2 km/hr

$$\therefore \quad \theta = \sin^{-1}\left(\frac{v_r}{v_{br}}\right) = \sin^{-1}\left(\frac{2}{4}\right) = \sin^{-1}\left(\frac{1}{2}\right) = 30^\circ$$

Hence, to reach the point directly opposite to starting point he should head the boat at an angle of 30 With AB or 90 + 30 = 120 with the river flow.



(b) Time taken by the boatman to cross the river

w = width of river = 4 km

v_{br} = 4 km/hr and θ = 30

$$t = \frac{4}{4\cos 30^\circ} = \frac{2}{\sqrt{3}} \text{ hr.}$$

(c) For shortest time θ = 0

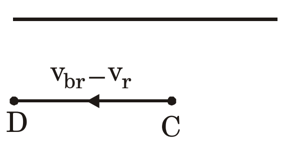
and $\quad t_{min} = \dfrac{w}{v_{br}\cos 0^\circ} = \dfrac{4}{4} = 1$ hr

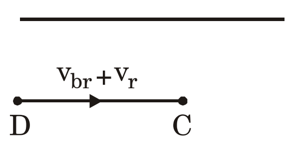
Hence, he should head his boat perpendicular to the river current for crossing the river in shortest time and this shortest time is 1 hr.

(d) $t = t_{CD} + t_{DC}$

or $\quad t = \dfrac{CD}{v_{br} - v_r} + \dfrac{DC}{v_{br} + v_r}$

$$= \frac{2}{4-2} + \frac{2}{4+2} = 1 + \frac{1}{3} = \frac{4}{3} \text{ hr.}$$





1.7.4 Aircraft Wind Problems

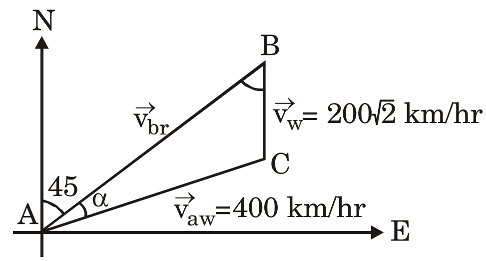
This is similar to river boat problem. The only difference is that $\vec{v}_{br}$ is replaced by $\vec{v}_{aw}$ (velocity of aircraft with respect to wind or velocity or aircraft in still air), $\vec{v}_r$ is replaced by $\vec{v}_w$ (velocity of wind) and $\vec{v}_b$ is replaced by $\vec{v}_a$ (absolute velocity of aircraft). Further, $\vec{v}_a = \vec{v}_{aw}$. The following example will illustrate the theory.

Example 39

An aircraft flies at 400 km/hr in still air. A wind of $200\sqrt{2}$ km/hr is blowing from the south. The pilot wishes to travel from A to a point B north east of A. Find the direction he must steer and time of his journey if AB = 1000 km.

Solution. Given that $\vec{v}_w = 200\sqrt{2}$ km/hr

$\vec{v}_{aw}$ = 400 km/hr and $\vec{v}_a$ should be along AB or in north-east direction. Thus, the direction of $\vec{v}_{aw}$ should be such as the resultant of $\vec{v}_w$ and $\vec{v}_{aw}$ is along AB or in north-east direction.



Let $\vec{v}_{aw}$ makes an angle α with AB as shown in Fig. 2.34. Applying sine law in triangle ABC, we get

$$\frac{AC}{\sin 45^\circ} = \frac{BC}{\sin\alpha}$$

or $$\sin\alpha\left(\frac{BC}{AC}\right)\sin 45^\circ = \left(\frac{200\sqrt{2}}{400}\right)\frac{1}{\sqrt{2}} = \frac{1}{2}$$

∴ α = 30

Therefore, the pilot should steer in a direction at an angle of (45 + α) or 75 from north towards east.

Further, $$\frac{\left|\vec{v}_a\right|}{\sin(180^\circ - 45^\circ - 30^\circ)} = \frac{400}{\sin 45^\circ}$$

or $$\left|\vec{v}_a\right| = \frac{\sin 105^\circ}{\sin 45^\circ} \times (400)\frac{km}{hr}$$

$$= \left(\frac{\cos 15^\circ}{\sin 45^\circ}\right)(400)\frac{km}{hr} = \left(\frac{0.9659}{0.707}\right)(400)\frac{km}{hr}$$

$$= 546.47\frac{km}{hr}$$

The time of journey from A to B is

$$t = \frac{AB}{\left|\vec{v}_a\right|} = \frac{1000}{546.47}hr$$

$$t = 1.83 \text{ hr.}$$

1.7.5 Rain Problems

In these type of problems we again come across three terms $\vec{v}_r, \vec{v}_m$ and $\vec{v}_{rm}$. Here

$\vec{v}$ = velocity of rain

$\vec{v}_r$ = velocity of man (it may be velocity of cyclist or velocity of motorist also)

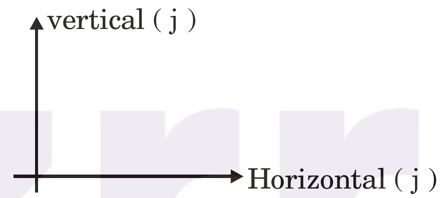
and $\vec{v}_m$ = velocity of rain with respect to man

Here $\vec{v}_{rm}$ is the velocity of rain which appears to the man. Now, let us take one example of this.

Example 40

To a man walking at the rate of 3 km/hr the rain appears to fall vertically. When he increases his speed to 6 km/hr it appears to meet him at an angle of 45 with vertical. Find the speed of rain.

Solution : Let $\tilde{i}$ and $\tilde{j}$ be the unit vectors in horizontal and vertical directions respectively.



Let velocity of rain be $\vec{v}_r = a\tilde{i} + b\tilde{j}$

The speed of rain will be $\left|\vec{v}_r\right| = \sqrt{a^2 + b^2}$ .... (i)

In the first case $\vec{v}_m$ = velocity of man $= 3\tilde{i}$

$\therefore \quad \vec{v}_{rm} = \vec{v}_r - \vec{v}_m = (a-3)\tilde{i} + b\tilde{j}$

It seems to be in vertical direction. Hence,

a – 3 = 0 or a = 3

In the second case $\vec{v}_m = 6\tilde{i}$

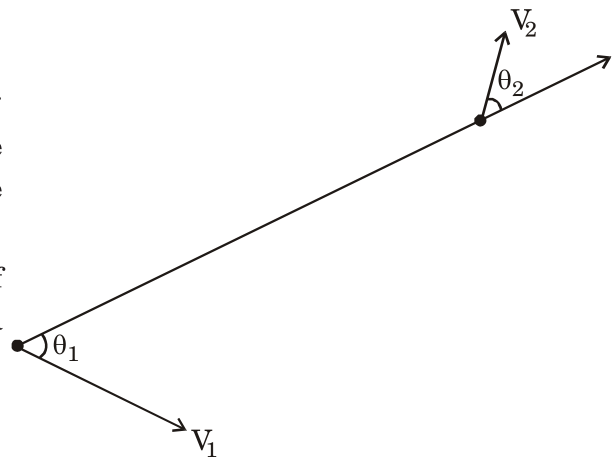
$\therefore \quad \vec{v}_{rm} = (a-6)\tilde{i} + b\tilde{j} = -3\tilde{i} = b\tilde{j}$

This seems to be at 45 with vertical.

Hence, $|b| = 3$

Therefore, from Eq. (ii) speed of rain is $\left|\vec{v}_r\right| = \sqrt{(3)^2 + (3)^2} = 3\sqrt{2}\,\dfrac{km}{hr}$

1.7.6 Velocity of Approach



$\frac{-ds}{dt} = (V_{rear} - V_{front})$ along the line joining the particles.

Velocity of approach can be found by taking the component of relative velocity along the line joining the particles.

While deciding the front and rear object, the direction of arrow of the line joining the particles can be taken as a guide.

Velocity of approach = $\left(\frac{-ds}{dt}\right) = \vec{V}_{rear} - \vec{V}_{front}$

$$\left(\frac{-ds}{dt}\right) = V_1 \cos\theta_1 - (V_2 \cos\theta_2)$$

Velocity of separation = $V_{front} - V_{rear}$

$$\frac{+ds}{dt} = (V_2 \cos\theta_2 - V_1 \cos\theta_1)$$

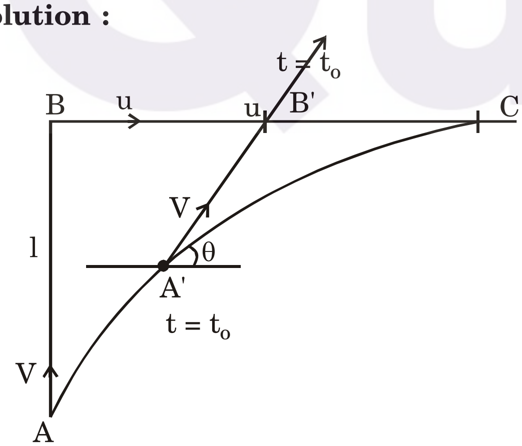
Example 41

A particle A moves with uniform (const.) speed V, continually aimed at B, which moves rectilinearly and uniformly with speed u (<v). Initially, $\vec{u} \perp \vec{v}$ and separation AB = l.

Find : (i) How soon the particles converge ?

(ii) Distance covered by each particle till they corverge.

Solution :



Let the particles meet at time = τ.

At any instant let the particles be at A´ and B´ respectively as shown.

Velocity of approach = v – (u cosθ)

$$\Rightarrow \left(\frac{-ds}{dt}\right) = v - (4\cos\theta)$$

$$\int_l^0 -ds = \int_0^\tau (v - u\cos\theta)\,dt$$

$$\boxed{l = v\tau - u\int_0^\tau \cos\theta\,dt} \quad \ldots (1)$$

For 2nd eqn.,

$$\Delta x_A = \int_0^\tau (v\cos\theta)\,dt = v\int_0^\tau \cos\theta\,dt$$

$$\Delta x_B = \Delta x_A$$

$$\Rightarrow (u\tau) = (\Delta x_A)$$

$$\boxed{u\tau = v\int_0^\tau \cos\theta\,dt.} \quad \ldots (2)$$

Solving (1) and (2)

$$l = v\tau - u\left(\frac{u\tau}{v}\right)$$

$$l = \left(v\tau - \frac{u^2\tau}{v}\right)$$

$$\therefore l = \frac{(v^2 - u^2)\tau}{v}$$

$$\tau = \frac{vl}{v^2 - u^2}$$

$$\therefore \text{ Time of meeting } = \frac{vl}{v^2 - u^2}$$

$$\text{Distance covered by B} = u\tau = \left(\frac{uvl}{v^2 - u^2}\right)$$

$$\text{Distance covered by A} = v\tau = \left(\frac{v^2 l}{v^2 - u^2}\right)$$

Example 42

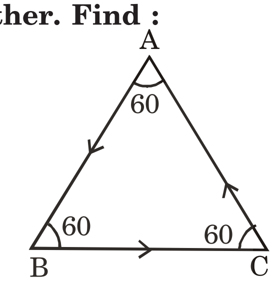
3 Particles are located at the corner of an equilateral triangle of side a. They all start moving simultaneously with constant speed v, with each heading towards the other. Find :

(a) the time after which the particle converge (τ)

(b) the distance covered by each particle they coverage

(c) Avg. angular acceleration of each particle about centroid from t = (τ/4) to (τ/2)

(d) Find inst. acceleration at t = 0, τ/4.



Solution.

(a) $$\frac{-ds}{dt} = v - (-v\cos 60)$$

$$\Rightarrow \frac{-ds}{dt} = \left(\frac{3v}{2}\right)$$

$$\therefore \int_a^0 -ds = \frac{3v}{2}\int_0^\tau dt$$

$$a = \left(\frac{3v}{2}\tau\right) \qquad \Rightarrow = \frac{2a}{3v}$$

(b) distance covered $= v\tau = \left(\frac{2a}{3}\right)$

(c) Avg. angular acceleration, Relative angular velocity about centroid O

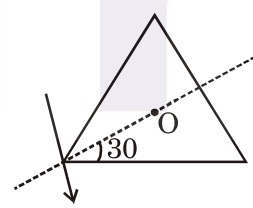
$$\omega = \frac{v\sin 30°}{R}$$

$$\omega = \frac{v\sin 30°}{\frac{a}{2}\sec 30°} = \frac{v\sin 30°\sqrt{3}}{a} = \left(\frac{\sqrt{3}\,v}{2a}\right)$$

$\therefore$ At t = τ/4

$$\int_a^s \frac{ds}{dt} = v - (-v\cos 60°)$$

$$a - s = \frac{3v}{2}t$$



$$\therefore s = a - \frac{3v}{2}t$$

$$c\left(\frac{2a}{3v}\right) = a - \frac{3v}{2}\left(\frac{2a}{3v}\right)\frac{1}{4} = \left(\frac{3a}{4}\right)$$

$$\therefore \omega_1 = \frac{\sqrt{3}\,v\;4}{2(3a)} = \left(\frac{2\sqrt{3}v}{3a}\right)$$

At t = τ/2

$$s = a - \frac{3v}{2}t_2$$

$$= a - \frac{3v}{2}\left(\frac{2a}{3v}\right)\frac{1}{2} = \left(\frac{a}{2}\right)$$

$$\therefore\ \omega_2 = \frac{2\sqrt{3}v}{2a} = \left(\frac{\sqrt{3}\,v}{a}\right)$$

Hence, $(\alpha)_{avg}$

$$= \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\dfrac{\sqrt{3}\,v}{a} - \dfrac{2\sqrt{3}}{39}}{\tau/4} = \frac{2\sqrt{3}V^2}{a^2}$$

(d) Acceleration

$$\Rightarrow \frac{d\vec{v}}{dt} = (v\omega)\tilde{n}$$

$$\vec{a} = \frac{v^2 \sin 30^\circ \sqrt{3}}{s}$$

$$\vec{a} = \frac{v^2 \sin 30^\circ \sqrt{3}}{\left[a - \left(\dfrac{3v}{2}\right)t\right]}$$

At $t = \tau/4$ $\quad a = \dfrac{\sqrt{3}v^2\,4}{2(3a)} = \left(\dfrac{2\sqrt{3}v^2}{3a}\right)$

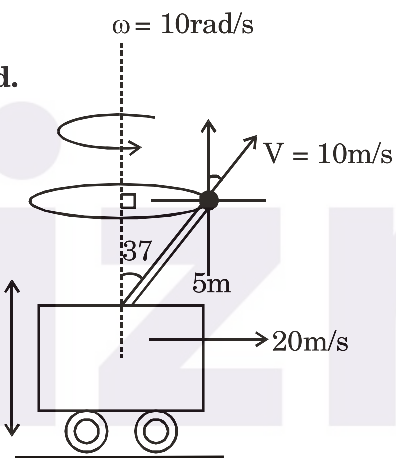
At $t = 0$, $\quad \boxed{a = \left(\dfrac{\sqrt{3}v^2}{2a}\right)}$

Example 43

A gun is fitted with trolley moving with v = 20 m/s. The gun rotates freely about vertical axis with ω = 10 rad/s. Length of gun = 5m, Height of trolley = 5m.

Given, muzzle speed = 10 m/s. Find :

Horizontal range before its strikes the ground.



Solution.

$$\vec{V}\text{shot} = \vec{V}\text{shot, gun} + \vec{V}\text{gun, trolley} + \vec{V}\text{trolley}$$

$$= [(V\cos 37^\circ)\,j + (V\sin 37^\circ)\,i] + [(5\sin 37^\circ)\,co]\,(-\tilde{k})$$

$$+ [20i]$$

$$= (26\tilde{i}) + 8\tilde{j} - 30\tilde{k}$$

$$\therefore \vec{V}\text{shot, ground} = (26\tilde{i} + 8\tilde{j} - 3\tilde{k})\text{ m/s}$$

$$\vec{a}x = 0,\ \vec{a}z = 0,\ \vec{a}y = (-g)\text{ m/s}^2$$

$$y = y_0 + \left[uyt + \frac{1}{2}ayt^2\right]$$

also, $y = 0$, $y_0 = 5 + 5\cos 37 = 9$m

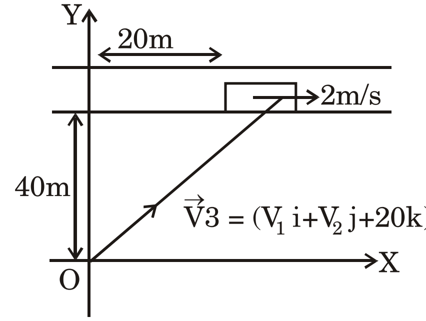
$$0 - 9 = 8t - \frac{1}{2}gt^2$$

$$\therefore\ 5t^2 - 8t - 9 = 0$$

$$\therefore\ \boxed{t = \left(8 \pm \sqrt{64 + 180}\right)}$$

Example 44

A block is floating in a river with a velocity v = 2m/s. a stone is projected as shown. The stone strikes the block. Find v_1 **and** v_2.



Solution.

Sz = 0 {displacement of stone in vertical direction = 0]

$$0 = (u_z t) - \frac{1}{2} g t^2$$

$$\therefore t = \left(\frac{2u_z}{g}\right) = \frac{40}{10} = 4s$$

Sx = 20 m Sy = 40 m

$\vec{V}_{stone,\ boat} = \vec{V}_{stone} - \vec{V}_{boat}$

$= (v_1 - 2)\ i + v_2\ j + 20k$

$(V_1 - 2)\ t = Sx$

$\Rightarrow (V_1 - 2)\ 4 = 20$

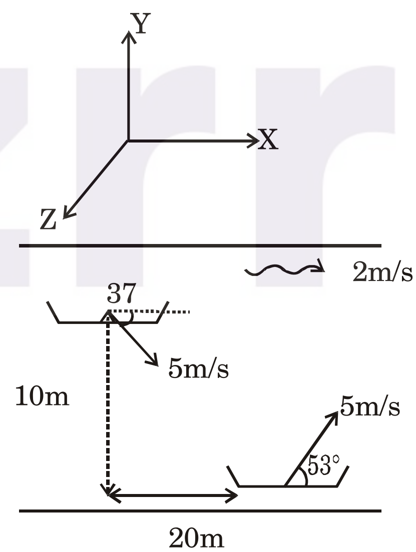
$\Rightarrow V_1 = 7$ m/s

also, $V_2\ t = Sy$

$$\Rightarrow v_2 = \frac{40}{4} = 10 \text{ m/s.}$$

Example 45

2 boats are moving relative to water as shown. Now, a bomb is projected from 1st boat with velocity $\vec{v} = (v_1 i + v_2 j + 20k)$ **and it explodes the 2nd boat. Find** $\vec{v}_1$ and $\vec{v}_2$ **.**



Solution.

$$\vec{V}_{bomb,\ ground} = \vec{V}_{bomb,\ boat_1} + \vec{V}_{boat_1,\ river} + \vec{V}_{river,\ ground}$$

$$\Rightarrow v_1 i + v_2 j + 20k = \vec{v}\,\text{bomb, boat 1} + (4i + 3j) + 2i$$

$$\vec{v}\,\text{bomb, boat 1} = (v_1 - 6)\ i + (v_2 - 3)\ j\ 20k$$

$$\vec{v}\,\text{bomb, boat 2} = \vec{v}\,\text{bomb, boat 1} + \vec{v}\,\text{boat 1, boat 2}$$

$$= [(v_1 - 6)i + (v_2 - 3)\ j + 20K] + [\vec{V}\,\text{boat1, river} - \vec{V}\,\text{boat2, river}]$$

$$= [(v_1 - 6)i + (v_2 - 3)\ j + 20K] + [(4i + 3j) - (3i + 4j)]$$

$$= (v_1 - 5)\ i + (v_2 - 4)\ j + 20\ k$$

Now time of flight $= \left(\frac{2uz}{g}\right) = \frac{2 \times 20}{10} = 4s$

also,

$S_y = (v_2 - 4)\ 4 = 10$

$\therefore v_2 = \frac{13}{2}$ m/s

$= 3.5$ m/s

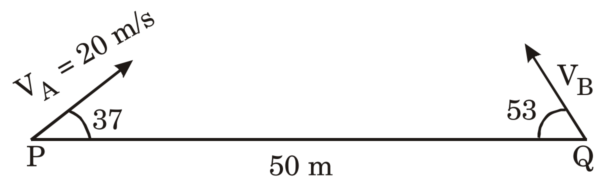
$S_x = (v_1 - 5)\ 4 = 20$

$v_1 = 10$ m/s

MISCELLANEOUS EXAMPLES

Example 1

Two particles are projected from ground as shown. If the 2 particles move in same plane and collide, find v_B and find the time and height at which collision takes place.



Solution.

Since the particle mut collide $\vec{v}_{AB}$ must be directed along PQ.

$\Rightarrow$ Normal component of $\vec{v}_{AB}(y)$ must be zero.

$V_{AB}(y) = V_A(y) - V_B(y) = 0$

$= 20 \sin 37 - V_B \sin 53 = 0$

$20 \sin 37 = V_B \sin 53$

$\therefore$ $V_B = 15$ m/s

$\vec{V}_{AB}(x) = \vec{V}_A(x) - \vec{V}_B(x)$

$= 20 \cos 37 - (-15 \sin 37)$

$= 20 \times \frac{4}{5} + 15 \times \frac{3}{5} = 25$ m/s

$\therefore$ time $= \left(\frac{50}{25}\right) = 2\,s$

$$H = uyt - \frac{1}{2}ayt^2$$

$$= 20 \times \frac{3}{5}(4) - \frac{1}{2} \times 10 \times 4 = 28\,m$$

Concept : The resultant Velocity (relative) of the two projectiles colliding with each other, must lie along the line joing them

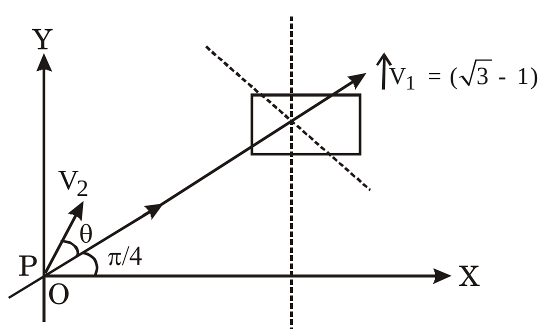
Example 2

A trolley A is moving along a straight line parallel to y-axis with v = $\left(\sqrt{3}-1\right)$ m/s in a smooth horizontal XY plane. A stone is projected along OA making an angle with + X axis, along the surface with speed V_2, when the trolley makes an angle $(\pi/_4)$ with x-axis.

the stone hits the trolley

(i) If motion of stone is observed from the trolley frame then find the angle θ made by its velocity vector along the axis.

(ii) If $\phi = 4\theta/_3$ find v_2?



Solution :

let, $\vec{V}_s$ be velocity of stone

$\vec{V}_t$ be velocity of trolley

and $\vec{V}_{st}$ be velocity of stone w.r.t. trolley

For collision to happen

$\vec{v}_{st}$ must be directed along OA.

$\vec{V}_{st} = \vec{V}_s - \vec{V}_t$

$= \vec{V}_{st} = V_s \sin(\phi - 45) - V_b \sin{}^{\pi}/_{4} = 0$

$V_s \sin(\phi - 45) = V_t \sin{}^{\pi}/_{4}$

$$V_s = \frac{(\sqrt{3}-1)}{\sqrt{2}\sin(\phi - 45)}$$

Now, in the reference from of trolley the angle made by the velocity vector of stone with X-axis $= {}^{\pi}/_{4}$ (as it is directed along OA)

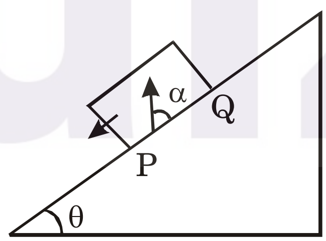
$$\therefore \quad \phi = \frac{4\theta}{3} = \left({}^{\pi}/_{3}\right)$$

Example 3

A large heavy box is sliding without friction down a smooth place of indination θ. From a point P on the bottom of the box, a particle is projected inside the box, The initial speed of the particle with respect to the box is u **and the directions of projection makes an angle α, with the bottom as shown in the figure.**

(a) **find the distance along the bottom PQ.**

(b) **if the horizontal displacement of the particle is zero, as seen by ground, find the speed of the box w.r.t. ground at the instant when the particle was projected.** (JEE 1998)



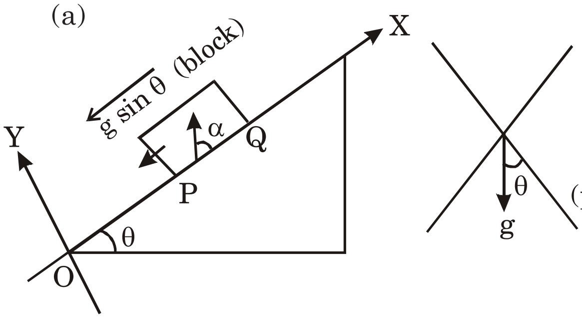
Solution.

Let $\vec{a}_p$ be acceleration of particle

$\vec{a}_b$ the acceleration of block

then $\vec{a}_{pb}$ is the acceleration of particle w.r.t. block

(a)



Normal component of acceleration should be zero

$\Rightarrow \quad (\vec{a}_{pb})_y = (\vec{a}_p)_y - (\vec{a}_b)_y$

$= (g \sin\theta - (+ g \sin\theta) = 0$

$(\vec{a}\, pb)x = (\vec{a}\, p)x - (\vec{a}\, b)x$

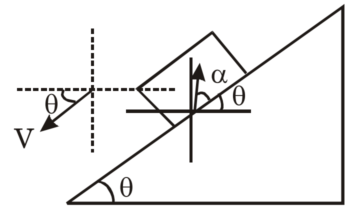
$= g \cos\theta - 0 = g \cos\theta$

Range PQ $= \left(\dfrac{u^2 \sin 2\theta}{g \cos\theta}\right)$

(b) Horizontal displacement = 0

Horizontal comp. of velocity of particle as observed from ground = 0

i.e.



$(\vec{v}\, p)_x = 0$

$\Rightarrow (\vec{v}\, p)_x = (\vec{v}\, pb)_x + (\vec{v}\, b)_x$

$\Rightarrow 0 = u \cos(\alpha + \theta) + (- v \cos\theta)$

$\therefore v = \dfrac{u \cos(\alpha + \theta)}{\cos\theta}$

Example 4

Two trains are approaching each other on a long straight track with constant speed of v km/hr each. When the trains are l km apart, a bird just in front of one train flies at a speed ω km/hr (ω > v) towards the other train. When it arrives just in front of that train, it turns and flies back towards the firs train. In this way, it flies back and forth between the two trains until the final moment when it is sandwiched between the trains.

(a) **Show that the bird makes infinite trips and the time interval between these trips form a geometrical series with the common ratio** $\dfrac{(\omega - v)}{(\omega + v)}$.

(b) **Find the total distance travelled by the bird.**

(c) **Taking l = 20 km, v = 50 km/hr, ω = 70 km/hr, draw the v–t and x – t graphs for the problem.**

Solution.

(a) Let t_1, t_2, t_3, ... tn be the time intervals for the first, second, third, ... trips respectively. Let us calculate the time tn for the nth trip. Before this trip, the bird and the trains have already been in motion for

$S_{n-1} (= t_1 + t_2 + t_3 + ... \quad .... + t_{n-1})$ seconds. Hence the separation between the trains at this instant is $l - S_{n-1}(2v)$, where 2v is the velocity of approach for the trains. The velocity of approach between the bird and a train is (v + ω).

Time for the nth trip :

$$t_n = \frac{\text{seperation}}{v+\omega} = \frac{l - 2vS_{n-1}}{v+\omega} \quad \text{........ (i)}$$

$\Rightarrow (v + \omega)\, t_n + l - S_{n-1}(2v)$

replacing n by n – 1, we have :

$(v + \omega)\, t_{n-1} = l - S_{n-2}(2v)$ .... (ii)

subtracting the equations (i) & (ii), we get,

$(v + \omega)(t_n - t_{n-1}) = -2v(S_{n-1} - S_{n-2})$

$\Rightarrow (v + \omega)(t_n - t_{n-1}) = -2vt_{n-1}$

$$\Rightarrow \frac{t_n}{t_{n-1}} = 1 - \frac{2v}{\omega + v} = \frac{\omega - v}{w + v}$$

Thus we conclude that the time intervals for the successive trips are in geometric progression with common ratio $\dfrac{(\omega - v)}{(\omega + v)}$.The bird makes infinite trips of decreasing intervals.

Example 5

A rubber ball is released from a height of 4.90 m above the floor. It bounces repeatedly, always rising to 81/100 of the height through which it falls.

(a) **Ignoring the practical fact that the ball has a finite size (in other words, treating the ball as point mass that bounces an infinite number of times), show that its total distance of travel is 46.7 m.**

(b) **Determine the time required for the infinite number of bounces.**

(c) **Determine the average speed.**

Solution.

Let' h = 4.9 m

(a) distance travelled

$$= h+2\left[\frac{81}{100}h+\left(\frac{81}{100}\right)^2 h+\ldots\ldots\ldots\ldots\ldots\right]$$

$$= h+2h\left(\frac{0.81}{1-0.81}\right)$$

$$= 4.9+\frac{9.8\times 0.81}{0.19} = 46.7 \text{ m}$$

(b) time required to fall through infinite bounces

time required to fall through height h = $\sqrt{\frac{2h}{g}}$

⇒ total time

$$= \sqrt{\frac{2h}{g}}+2\left(\sqrt{\frac{2nh}{g}}+\sqrt{\frac{2n^2h}{g}}+\ldots\ldots\ldots\right)$$

where $n = \frac{81}{100}$

⇒ total time = $\sqrt{\frac{2h}{g}}+2\sqrt{\frac{2h}{g}}\left(\frac{\sqrt{n}}{1-\sqrt{n}}\right)$

$$= \sqrt{\frac{2h}{g}}(1+18) = 19\,\text{sec.}$$

(c) average speed

$$\text{Average Speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{46.7}{19} = 2.46 \text{ m/s}$$

Example 6

Consider a collection of a large number of particles each with speed v. The direction of velocity is randomly distributed in the collection. Show that the magnitude of the relative velocity between a pair of particles averaged over all the pairs in the collection is greater than v.

Solution :

As shown in Fig. let $\vec{v}_1$ and $\vec{v}_2$ be the velocities of any two particles and θ be the angle between them. As each particle has speed v, so

$$\left|\vec{v}_1\right| = \left|\vec{v}_2\right| = v$$

The magnitude of relative velocity $\vec{v}_{21}$ of particle 2 with respect to 1 is given by

$$v_{21} = \sqrt{\left|-\vec{v}_1\right|^2+\left|\vec{v}_2\right|^2-\left|\vec{v}_1\right|\left|\vec{v}_2\right|\cos\left(180^\circ-\theta\right)}$$

$$= \sqrt{v^2\ v^2 - 2v\,v\cos\theta} = \sqrt{2v^2(1-\cos\theta)}$$

$$= \sqrt{2v^2 \times 2\sin^2\frac{\theta}{2}} = 2v\sin\frac{\theta}{2} \qquad \left[1-\cos 2\theta = 2\sin^2\theta \therefore 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

As the velocities of the particles are randomly distributed, so θ can vary from 0 to 2π. If (v_{21}) is the magnitude of the average velocity when averaged over all pairs, then

$$(v_{21}) = \frac{\int_0^{2\pi} 2v\sin\frac{\theta}{2}\,d\theta}{\int_0^{2\pi} d\theta} = \frac{2v\left[\frac{-\cos\frac{\theta}{2}}{1/2}\right]_0^{2\pi}}{[\theta]_0^{2\pi}}$$

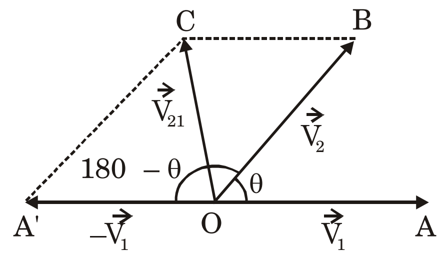


Fig. 2.69.

$$= \frac{-4v\left[\cos\frac{\theta}{2}\right]_0^{2\pi}}{2\pi - 0} = \frac{2v}{\pi}[\cos\pi - \cos 0] = \frac{2v}{\pi}[-1-1] = \frac{4v}{\pi} = 1.273\,v$$

Clearly, $(v_{21}) > v$

Example 7

Fig. 2.70 shows a rod of length l resting on a wall and the floor. Its lower end A is pulled towards left with a constant velocity v. Find the velocity of the other end B downward when the rod makes an angle θ with the horizontal.

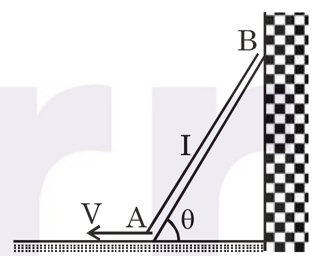


Fig. 2.70

Solution.

In such type of problems, when velocity of one part of a body is given and that of other is required, we first find the relation between the two displacements, then differentiate them with respect to time. Here θ the distance from the corner to the point A is x and that up to B is y. Then

$$v = \frac{dx}{dt}$$

and $\qquad v_B = \frac{dv}{dt}\ v_B = -\frac{dv}{dt}$ (– sign denotes that v is decreasing)

Further $\qquad x^2 + y^2 = l^2$

Differentiating with respect to time t $\qquad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$$xv = yv_B$$

$$v_B = \frac{x}{y}v = v\cot\theta$$

Example 8

In a car race, car A takes time t less than car B and passes the finishing point with a velocity v more than the velocity with which car B passes the point. Assuming that the cars start from rest and travel with constant accelerations a_1 and a_2, show that $\frac{v}{t} = \sqrt{a_1 a_2}$.

Solution.

Let s be the distance covered by each car. Let the times taken by the two cars to complete the journey be t_1 and and t_2, their velocities at the finishing point be v_1 and v_2 respectively. According to the problem,

$$v_1 - v_2 = v \text{ and } t_2 - t_1 = t$$

Now,
$$\frac{v}{t} = \frac{v_1 - v_2}{t_2 - t_1}$$

$$= \frac{\sqrt{2a_1 s} - \sqrt{2a_2 s}}{\sqrt{\frac{2s}{a_2}} - \sqrt{\frac{2s}{a_1}}} = \frac{\sqrt{a_1} - \sqrt{a_2}}{\sqrt{\frac{1}{a_2}} - \sqrt{\frac{1}{a_1}}}$$

$$= \sqrt{a_1 a_2}$$ **Hence proved.**

Example 9

A particle moves in the plane according to the law $x = k\,t$, $y = kt\,(1 - at)$, when k and a are positive constants, and t is the time. Find :

(a) the equation of the particle's trajectory $y(x)$

(b) the velocity v and the acceleration a of the point as a function of time.

Solution.

(a) $x = kt,\ y = kt\,(1 - at)$

Equation of trajectory, $y = kt - kat^2 = x - ka\frac{x^2}{k^2}$

or $y = x - a\frac{x^2}{k}$ (parabola)

(b) $v_x = \frac{dx}{dt} = k;\ v_y = \frac{dy}{dt} = k - 2at = k(1 - 2at)$

$\therefore$ $\vec{v} = \tilde{I}v_x + \tilde{J}v_y = k\tilde{I} + k(1 - 2at)\tilde{J}$

$\therefore$ $v = \sqrt{k^2 + k^2(1 - 2at)^2} = k\sqrt{1 + (1 - 2at)^2}$

Acceleration $\vec{a} = \frac{d\vec{v}}{dt} = -2ak\tilde{J} = \text{constant}$

Example 10

A point moves rectilinearly with deceleration whose modulus depends on the velocity v of the particle as $\alpha = k\sqrt{v}$ where k is a positive constant. At the initial moment the velocity of the point is equal to v_0. What distance will it traverse before it stops? What time will it take to cover that distance?

Solution.

Let t_0 be the time in which it comes to a stop

Given that $-\dfrac{dv}{dt} = k\sqrt{v}$

$$\int_0^0 k\,dt = \int_{v_0}^0 -\frac{dv}{\sqrt{v}} = \int_0^{v_0} \frac{dv}{\sqrt{v}}$$

$$kt_0 = \left[2\sqrt{v}\right]_0^{v_0} = 2\sqrt{v_0}$$

$$t_0 = \frac{2}{k}\sqrt{v_0}$$

To find the distance covered before stopping,

We have, $\dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = v\dfrac{dv}{ds}$ as $v = \dfrac{ds}{dt}$

Now, $\dfrac{dv}{dt} = -k\sqrt{v}$

Therefore, $v\dfrac{dv}{ds} = -k\sqrt{v} \quad \Rightarrow \quad \sqrt{v}\,dv = -k\,ds$

$\therefore \quad \displaystyle\int_{v_0}^0 \sqrt{v}\,dv = -\int_0^v k\,ds \quad \Rightarrow \quad s = \frac{2}{3k} v_0^{3/2}$ **Ans.**

Example 11

A man can row a boat in still water at 3 km/h. He can walk at a speed of 5 km/h on the shore. The water in the river flows at 2 km/h. If the man rows across the river and walks along the shore to reach the opposite point on the river bank, find the direction in which he should row the boat so that he could reach the opposite shore in the least possible time. Also calculate this time. The width of the river is 500 m.

SOLUTION : Suppose the boatman rows with velocity $\vec{V}_{br}$ in the direction shown in Fig. 2.82.

Given, $\vec{v}_{br}$ = 3 km/hr, v_r = 2 km/hr,

and v_w = walking speed = 5 km/hr

$$\vec{V}_b = \vec{V}_{br} + \vec{V}_r$$

$\therefore \quad v_{bx} = v_r - v_{br} \sin\theta = 2 - 3\sin\theta$

and $\quad v_{bx} = v_{br}\cos\theta = 3\cos\theta$

Time taken to reach the other side

$$t_1 = \frac{v}{v_{bv}} = \frac{0.5}{3\cos\theta} = \frac{1}{6\cos\theta} \qquad \text{....(i)}$$

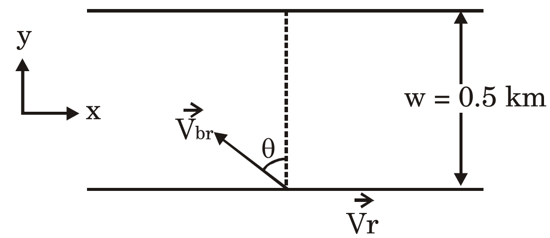


Fig. 2.82

horizontal drift $\quad x = v_{bx}\, t_1 = (2 - 3\sin\theta)\dfrac{1}{6\cos\theta} = \dfrac{1}{3\cos\theta} - \dfrac{\tan\theta}{2}$

Time to travel this distance by walking $\quad t_2 = \dfrac{x}{v_w} = \dfrac{1}{15\cos\theta} - \dfrac{\tan\theta}{10} \qquad \text{... (ii)}$

Total time $\quad t = t_1 + t_2 = \dfrac{1}{10}\left(\dfrac{7}{3\cos\theta} - \tan\theta\right) \qquad \text{...(iii)}$

For time to be minimum $\dfrac{dt}{d\theta} = 0$

or $\quad \dfrac{7}{3}\sec\theta\tan\theta - \sec^2\theta = 0$

or $\quad \sin\theta = \dfrac{3}{7}$ **Ans.**

For Eq. (iii) $\quad t_{min} = \dfrac{1}{10}\left(\dfrac{7}{3} \times \dfrac{7}{\sqrt{40}} - \dfrac{3}{\sqrt{40}}\right)$

$= 0.21$ hr. **Ans.**

Example 12

A river of width 'a' with straight parallel banks flows due north with speed u. The points O and A are on opposite banks and A is due east of O. Coordinate axes Ox and Oy are taken in the east and north directions respectively. A boat, whose speed is v relative to water, starts from O and crosses the river. If the boat is steered due east and u varies with x as $u = x(a - x)v/a^2$. Find

(a) equation of trajectory of the boat

(b) time taken to cross the river

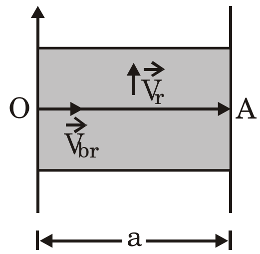
(c) absolute velocity of boatman when he reaches the opposite bank

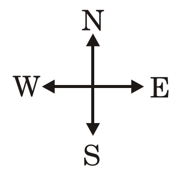
(d) the displacement of boatman when he reaches the opposite bank from the initial position.

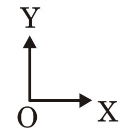
Solution.

(a) Let $\vec{V}_{br}$ be the velocity of boatman relative to river, $\vec{V}_r$ the velocity of river and $\vec{V}_b$ is the absolute velocity of boatman. Then

$$\vec{V}_b = \vec{V}_{br} + \vec{V}_r$$







Given; $\left|\vec{V}_{br}\right| = v$

and $\left|\vec{V}_r\right| = v$

Now, $$u = v_y = \frac{dy}{dt} = x(a - x)\frac{v}{a^2} \quad \text{...(i)}$$

and $$v = v_x = \frac{dx}{dt} = v \quad \text{...(ii)}$$

Dividing Eqs. (i) by (ii), we get

$$\frac{dy}{dx} = \frac{x(a - x)}{a^2} \quad \text{or} \quad dy = \frac{x(a - x)}{a^2}dx$$

or $$\int_0^y dy = \int_0^x \frac{x(a - x)}{a^2}dx$$

or $$y = \frac{x^2}{2a} - \frac{x^3}{3a^2} \quad \text{...(iii)}$$

This is the desired equation of trajectory.

(b) Time taken to cross the river is $t = \dfrac{a}{v_x} = \dfrac{a}{v}$ **Ans.**

(c) When the boatman reaches the opposite side, x = a or $v_y = 0$

Hence, resultant velocity of boatman is v along positive x-axis or due east

(d) from Eq. (iii)

$$y = \frac{a^2}{2a} - \frac{a^3}{3a^2} = \frac{a}{6} \quad \text{at } x = a \qquad \text{(at opposite bank)}$$

Hence displacement of boatman will be

$$\vec{s} = x\hat{i} + y\hat{j} \quad \text{or} \quad \vec{s} = a\hat{i} = a\hat{i} + \frac{a}{b}\hat{j}$$

Example 13

A particle is thrown over a triangle from one end of a horizontal base and after grazing the vertex falls on the other end of the base. If α and β be the base angles and θ the angle of projection, prove that $\tan\theta = \tan\alpha + \tan\beta$.

Solution :

The situation is shown in Fig. 2.90

From figure, we have

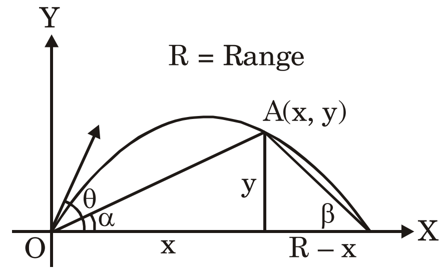


Fig. 2.90

$$\tan\alpha + \tan\beta = \frac{y}{x} + \frac{y}{R-x}$$

$$\tan\alpha + \tan\beta = \frac{yR}{x(R - x)} \quad \text{....(i)}$$

Equation of trajectory is $y = x\tan\theta\left[1 - \frac{x}{R}\right]$

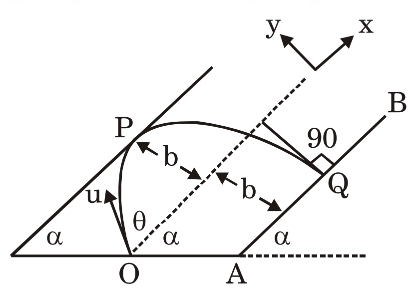
or, $$\tan\theta = \frac{yR}{x(R-x)} \quad \text{....(ii)}$$

From eqs. (i) and (ii) $\tan\theta = \tan\alpha + \tan\beta$

Example 14

Two parallel straight lines are inclined to the horizontal at an angle α. A particle is projected from a point mid way between them so as to graze one of the lines and strikes the other at right angles. Show that if θ is the angle between the direction of projection and either of lines, then

$$\tan\theta = \left(\sqrt{2}-1\right)\cot\alpha$$



Solution : Consider the motion of the particle from O to P.

The velocity v_y at P is zero.

$$v_y^2 = u_y^2 + 2a_ys_y$$

$\therefore$ $$0 = (u\sin\theta)^2 - 2(g\cos\alpha)b$$

or $$b = \frac{u^2\sin^2\theta}{2g\cos\alpha} \quad \text{...(i)}$$

Now, consider the motion of the particle from O to Q

The particle strikes the point Q at 90 to AB, i.e., its velocity along x-direction is zero.

Using $v_x = u_x + a_xt$, we have

$$0 = u\cos\theta - (g\sin\alpha)t$$

or $$t = \frac{u\cos\theta}{g\sin\alpha} \quad ...(ii)$$

For motion in y-direction, $$s_y = u_y t + \frac{1}{2}a_y t^2$$

or $$-b = u\sin\theta\left(\frac{u\cos\theta}{g\sin\alpha}\right) + \frac{1}{2}(-g\cos\alpha)\left(\frac{u\cos\theta}{g\sin\alpha}\right)^2 \quad ...(iii)$$

From Eqs. (i) and (iii)

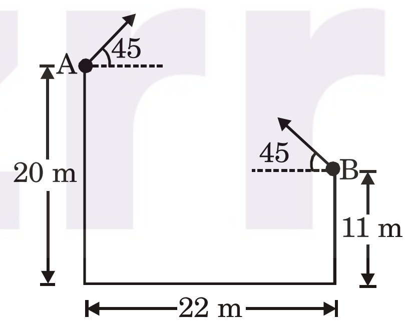
or $$-\frac{u^2\sin^2\theta}{2g\cos\alpha} = \frac{u^2\sin\theta\cos\theta}{g\sin\alpha} - \frac{gu^2\cos\alpha\cos^2\theta}{2g^2\sin^2\alpha}$$

or $$\frac{\sin^2\theta}{2\cos\alpha} = \frac{\sin\theta\cos\theta}{\sin\alpha} - \frac{\cos\alpha\cos^2\theta}{2\sin^2\alpha}$$

Solving we get $$\tan\theta = \left(\sqrt{2}-1\right)\cot\alpha$$

Example 15

Two particles are simultaneously thrown from the roofs of two high buildings as shown in figure. Their velocities are v_A = 2 m/s and v_B = 14 m/s respectively. Calculate the minimum distance between the particles in the process of their motion. Also find the time when they are at closest distance.



Solution :

Assuming A to be at rest

$$\vec{a}_{BA} = \vec{a}_B - \vec{a}_A = 0 \text{ as } \vec{a}_A = \vec{a}_B = g \text{ (downwards)}$$

Thus the relative motion between them is uniform.

Relative velocity of B with respect to A in vertical direction.

$$u_{BAV} = u_B\sin 45^\circ - u_A\sin 45^\circ = (14-2)\frac{1}{\sqrt{2}} = 6\sqrt{2}\text{ m/s}$$

Relative velocity of B with respect to A in horizontal direction

$$u_{BAH} = u_B\cos 45^\circ - (u_A\cos 45^\circ)$$

$$= (14+2)\frac{1}{\sqrt{2}} = 8\sqrt{2}\text{ m/s}$$

Horizontal distance between A and B after time t is

$$(u_{BAH})t = \left(22 - 8\sqrt{2}t\right)\text{m}$$

and vertical distance between A and B after time is

$$y = 9 - u_{BAV}\ t = \left(9 - 6\sqrt{2}t\right)\text{m}$$

Therefore, distance between them after time t is

$$S = \sqrt{x^2 + y^2}$$

or $$S^2 = x^2 + y^2 = \left(22 - 8\sqrt{2}t\right)^2 + \left(9 - 6\sqrt{2}t\right)^2$$

For S to be minimum $\frac{d}{dt}\left(S^2\right) = 0$

$\therefore$ $$2\left(22 - 8\sqrt{2}t\right)\left(-8\sqrt{2}\right) + 2\left(9 - 6\sqrt{2}t\right)\left(-6\sqrt{2}\right) = 0$$

or $$88 - 32\sqrt{2}t + 27 - 18\sqrt{2}t = 0$$

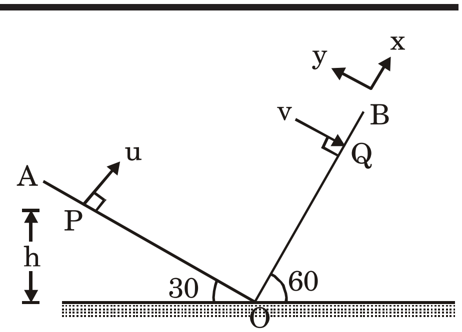
or $$t = \frac{23}{10\sqrt{2}} \text{ second}$$

$\therefore$ $$S_{min} = \sqrt{x^2 + y^2} \text{ at time } t = \frac{23}{10\sqrt{2}} \text{ second}$$

Substituting the values, we get $S_{min} = 6.0$ m

Example 16

Two inclined planes OA and OB having inclinations 30 and 60 with the horizontal respectively intersect each other at O, as shown in figure. A particle is projected from point P with velocity $u = 10\sqrt{3}$ m/s along a direction perpendicular to plane OA. If the particle strikes plane OB perpendicular at Q. Calculate



(a) time of flight

(b) velocity with which the particle strikes the plane OB

(c) height h of point P from point O

(d) distance PQ (Take g = 10 m/s²)

Solution.

Let us choose the x and y directions along OB and OA respectively. Then

$$u_x = u = 10\sqrt{3}\text{ m/s},\ u_y = 0$$

$$a_x = -\,g\sin 60^\circ = -\,5\sqrt{3}\text{ m/s}^2 \text{ and } a_y = -\,g\cos 60^\circ = -\,5\text{ m/s}^2$$

(a) At point Q, x-component of velocity is zero. Hence, substituting in

$$v_x = u_x + a_x t$$

$$0 = 10\sqrt{3} - 5\sqrt{3}t$$

or $$t = \frac{10\sqrt{3}}{5\sqrt{3}} = 2\text{s}$$

(b) At point Q $v = v_y = u_y + a_y t$

$\therefore$ $v = 0 - (5)(2) = -\,10$ m/s

Here, negative sign implies that velocity of particle at Q is along negative y direction

(c) Distance PO = | displacement of particle along y-direction | = $|S_y|$

Here, $$S_y = u_y t + \frac{1}{2}a_y t^2 = 0 - \frac{1}{2}(5)(2)^2 = -\,10\text{m}$$

$\therefore$ PO = 10 m

Therefore, $$h = PO\sin 30 = (10)\left(\frac{1}{2}\right)$$

or h = 5 m

(d) Distance OQ = displacement of particle along x-direction = s_x

Here, $$s_x = u_x t + \frac{1}{2}a_x\, t_2$$

$$= \left(10\sqrt{3}\right)(2) - \frac{1}{2}\left(5\sqrt{3}\right)(2)^2 = 10\sqrt{3}\text{ m}$$

or $$OQ = 10\sqrt{3}\text{ m}$$

$\therefore$ $$PQ = \sqrt{(PQ)^2 + (OQ)^2} = \sqrt{(10)^2 + \left(10\sqrt{3}\right)^2} = \sqrt{100+300} = \sqrt{400}$$

$\therefore$ PQ = 20 m