An Investigation into Topology in the Quantum Hall Effect and Non-Abelian Anyons

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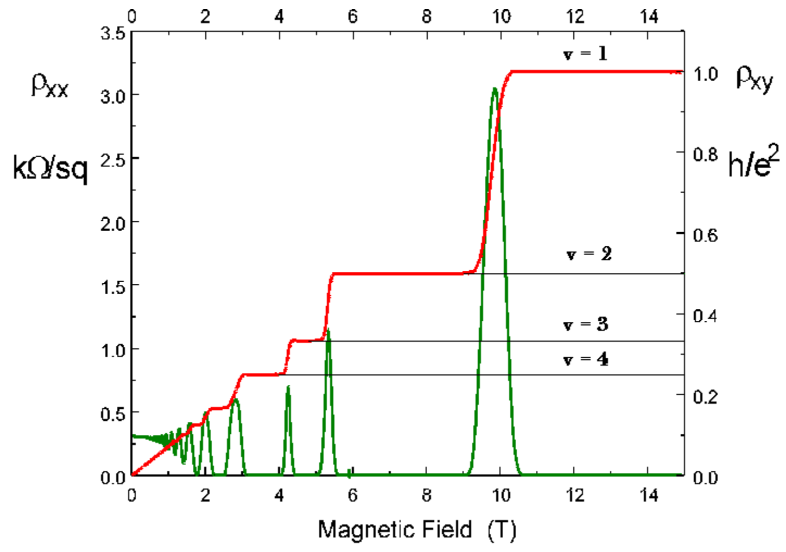
What is the Quantum Hall Effect?

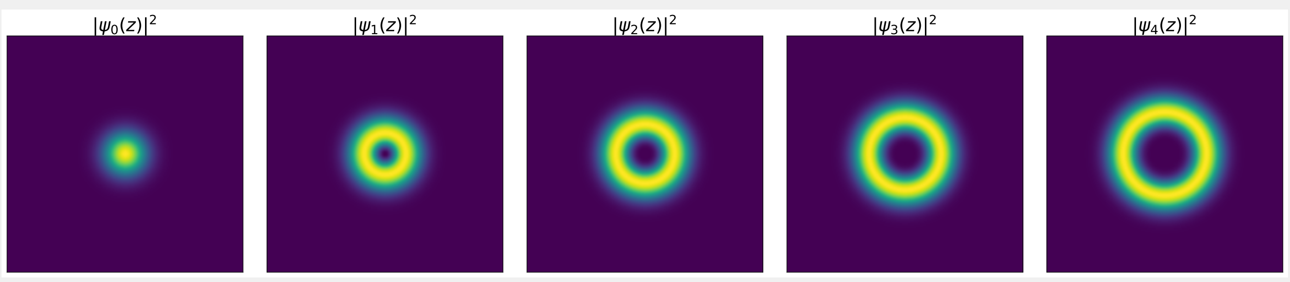
Classically, the Hall conductivity for a sample of area A with number of electrons N_{el} is given by $\sigma_{xy} = \frac{N_{el}}{A}\frac{e}{B}$. From this, you would expect a straight line with slope $\frac{A}{eN_{el}}$ when plotting resistivity (σ_{xy}^{-1}) against B, but this is not what we see in experiments. Instead, we see plateaus at certain values of the Hall conductivity, which are given by $\sigma_H = \frac{e^2}{h}\nu$, where ν is the filling fraction. The plateaus are quantised, and the value of ν can be any rational number, which is why this is known as the Quantum Hall effect. The plateaus are also characterised by regions of very low ohmic resistivity (ρ_{xx}), and the edges of these plateaus can be identified by spikes in ρ_{xx}, as shown below.



The Laughlin Wavefunction

The Hamiltonian of a Quantum Hall fluid is similar to that of a quantum harmonic oscillator. Using the rules for the creation and annihilation operators, the single-particle lowest Landau level wavefunction can be found to be given by the following.

$$H = \hbar\omega_c\left(a^\dagger a + \frac{1}{2}\right), \qquad \psi_m(z,\bar{z}) \propto z^m e^{-\frac{1}{4\ell^2}|z|^2}, \qquad \ell = \sqrt{\frac{\hbar}{eB}} \tag{1}$$



This ring structure is just one representation of the Quantum Hall fluid. The classical picture is more like a liquid of electrons performing a sort of cycloton motion, which is obtained by using the coherent state basis. The coherent state is just an eigenstate of b, $b|n\lambda\rangle = \lambda|n\lambda\rangle$, which is similar to the coherent state of a harmonic oscillator $a|\lambda\rangle = \lambda|\lambda\rangle$. It turns out that the many-body wavefunction of the Integer Quantum Hall fluid is actually a special case of a more general wavefunction, known as the Laughlin wavefunction. The Laughlin wavefunction is given by:

$$\Psi_L(z_1,\ldots,z_N) \propto \prod_{i<j}(z_i - z_j)^p e^{-\frac{1}{4\ell^2}\sum_i |z_i|^2} \tag{2}$$

Where p is a positive odd integer. The IQH fluid is just the case where $p = 1$. The Laughlin wavefunction is a many-body wavefunction that describes a fluid of electrons in the lowest Landau level, and it is known to be a good description of the Fractional Quantum Hall effect for $\nu = \frac{1}{p}$.

The Berry Phase in the Integer Quantum Hall Effect

Considering the ground state of a system, $|\psi(t)\rangle$, and some parameterised reference ground state $|n(\lambda(t)\rangle$ subject to $|\psi(t=0)\rangle = |n(\lambda(t=0))\rangle$ defined up to a unitary trasnformation $|\psi(t)\rangle = U(t)\,|n(\lambda(t))\rangle$. Moving the system on a closed path parameter-space then it shall return as a ground state up to a phase determined by two parts. (1) The expected dynamical phase picked up from the Schrödinger Equation and (2) an additional phase picked up through its path in parameter space known as the **Berry Phase**. We see this through the *Berry Connection* $\mathcal{A}_i$ such that $\dot{U} = -i\mathcal{A}_i\dot{\lambda}^i U$ and the subsequent phase:

$$e^{i\gamma} = \exp\left\{-i\oint_C \mathcal{A}_i(\lambda)\,\mathrm{d}\lambda^i\right\} \tag{3}$$

The curvature - or field strength - associated to the Berry connection is given by $\mathcal{F}_{ij} = \partial_{[i}A_{j]}$. Building the Quantum Hall system on the torus i.e. on a rectangle with periodic boundaries, and investigating the periodic parameters $\theta_{x,y} = 2\pi\Phi_{x,y}/\Phi_0$ - periodic since the Hamiltonian only depends on $\Phi_i \bmod \Phi_0$. Finding the Berry curvature of this system we find a notable duality

$$\mathcal{F}_{xy} = i\left[\frac{\partial}{\partial\theta_y}\langle\psi_0|\frac{\partial}{\partial\theta_x}|\psi_0\rangle - \frac{\partial}{\partial\theta_x}\langle\psi_0|\frac{\partial}{\partial\theta_y}|\psi_0\rangle\right] = -\frac{\hbar}{e^2}\sigma_{xy} \tag{4}$$

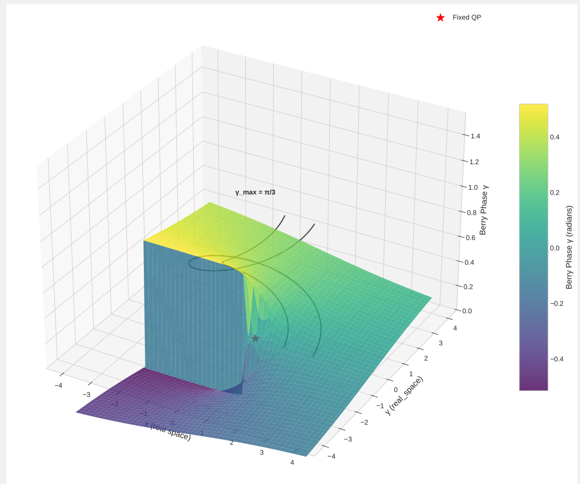
Where notice that in this space, the Berry curvature is identically just the **Kubo Formula** for the conductivity. Further integrating both sides of this over the torus in θ_i space and noting that integrating the curvature over the torus is nothing but the first Chern number $C \in \mathbb{Z}$ we see that the off-diagonal conductivity is quantized.

Anyons via Berry Phase in the Fractional Quantum Hall Effect

Fermions and Bosons, two of the most basic classifications of quantum mechanical particles, with characteristics such as their (anti-)symmetric properties. But parameterising quasi-particles such as vortices - or holes - by their positions η_k.

$$\Psi_{L,\{\eta_k\}}(z_1,\ldots,z_N) \propto \prod_{k,l}(\eta_k - z_l)\prod_{i<j}(z_i - z_j)^p e^{-\frac{1}{4\ell^2}\sum_i |z_i|^2} \tag{5}$$

Through an analogy to a plasmatic system's thermal state, the difference in power between the hole factors and the particle factors demonstrates a *fractional charge* associated with vortices.



Treating the position of these particles and vortices as different parameters then one can construct a Berry phase by moving the excitation through.

Particle	Fermion	Boson	$\nu = 1/p$ Anyon
Phase	π	2π	$2\pi/p$

Ising Anyons

The $\nu = 5/2$ Moore-Read state has Ising anyons as excitations. These anyons also appear as Majorana Zero Modes (MZMs) in $p_x + ip_y$ superconductors. The data defining the Ising fusion category can be extracted from the microscopic description of the system as follows.

Fusion Rules:
MZMs are represented by Majorana operators γ_i, wich satisfy the properties $\gamma_i^\dagger = \gamma_i$ and $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. The following linear superposition of two Majorana operators satisfies fermionic anti-commutation relations $\{c_{ij}^\dagger, c_{kl}\} = \delta_{ij,kl}$:

$$c_{ij} = \frac{1}{2}(\gamma_i + i\gamma_j). \tag{6}$$

Notice that this composition of the ith and jth Hilbert spaces is not done by a simple tensor product. This signifies the non-local, topological nature of the system. Creation of a jth MZM followed by an ith MZM in $\mathcal{H}_{ij}$ is equivalent to a sum of projectors onto $|0\rangle$ and $|1\rangle$:

$$\gamma_i\gamma_j = i(\hat{p}_{|0\rangle_{ij}} - \hat{p}_{|1\rangle_{ij}}). \tag{7}$$

This, along with the composition of fermions with other particles via tensor product of states gives Ising fusion rules:

$$\sigma\otimes\sigma = 1\oplus\psi, \psi\otimes\sigma = \sigma, \psi\otimes\psi = 1. \tag{8}$$

R-Matrix:
Exchanging MZMs i and j, one MZM must pick up a factor of -1 relative to the other. Therefore, we can choose

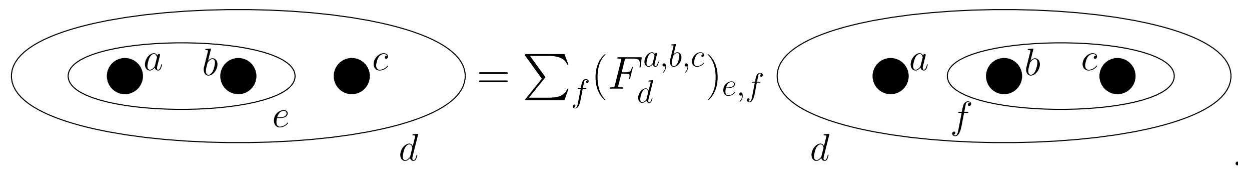
$$\gamma_i \to \gamma_j, \gamma_j \to -\gamma_i. \tag{9}$$

The coefficients picked up by $|0\rangle_{ij}$ and $|1\rangle_{ij}$ under exchange can be attained by noticing that these states are isomorphic to the states $|+\rangle_i$ and $|-\rangle_i$ defined by $\gamma_i|+\rangle_i = |-\rangle_i$, providing

$$\begin{pmatrix}|0\rangle_{ji}\\|1\rangle_{ji}\end{pmatrix} = e^{i\theta}\begin{pmatrix}1 & 0\\0 & i\end{pmatrix}\begin{pmatrix}|0\rangle_{ij}\\|1\rangle_{ij}\end{pmatrix}. \tag{10}$$

With $\theta = \pi/8$, this transformation matrix is the only non-trivial 'R-matrix' in the definition of Isiang anyons as objects in a fusion category.

F-Matrix:
F-matrices define basis transformations on systems of three particles, changing which pair are taken as composite first:



. (11)

The F-matrix for Ising anyons is found from the Majorana operators by including a fourth auxiliary MZM, pairing it with the c particle above, and finding the matrix form of the unitary transformation $\frac{1}{\sqrt{2}}(1 + \gamma_i\gamma_j)$ which swaps the auxiliary MZM with the central MZM. Composing this transformation with the R-matrix to move the auxiliary particle to the outside provides the Ising anyon F-matrix:

$$(F_\sigma^{\sigma,\sigma,\sigma}) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}. \tag{12}$$