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An Investigation into Topology in the Quantum Hall Effect and Non-Abelian Anyons

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1 An Introduction to the Hall Effect

$$\rho = R\frac{A}{l} \tag{1.1}$$

is defined as the resistance of a wire of length l and cross-sectional area A with units $[\rho] = \Omega m$ in our day-to-day world. But we define it in the more general $\mathbb{R}^n$ to have units of Ωm^{n-2}. We can rewrite Eq. 1.1 to see that the resistance of a hypercube of length l in d dimensions is given by

$$R = \rho l^{2-d} \tag{1.2}$$

Thus, we have that in $2D$ the resistivity and resistance are equal. Furthermore, we have by dimensional analysis that $[\frac{e^2}{h}R] = \frac{C^2}{J\cdot s}\frac{J\cdot s}{C^2} = 1$, is a dimensionless quantity. We will see that this term multiplying the resistance ($\frac{e^2}{h}$ is the quantum of conductance).

It's often the case that in crystal structures, the outermost (and hence most loosely-bound) valence electrons are free to move throughout the structure. We can model this as what's known as an electron gas. That is to say, we ignore all potentials except those representing the boundaries. In this regime, the electrons effectively become particles in a box.

Localisation is the lack of diffusion. For example, imagine a cluster of a certain gas A is released in a chamber full of gas B. Gas A particles will diffuse and eventually become indistinguishable from gas B. Localisation is the phenomenon of gas A not diffusing and remaining in a relatively small cluster within the chamber. This logic can be applied to the electron gas found in a crystal structure. One very common form of this is weak localisation, a topic very relevant to condensed matter physics and is briefly discussed in A.

The Ordered Electron Gas in the Presence of Magnetic Field

We've seen the quantum interference caused in the $2D$ regime under no magnetic field and non-interacting electrons. Now let us turn our focus to the case of a magnetic field with a pure electron gas. That is to say an electron gas with the only potential being the periodic potential found due to the crystal lattice. This, however, can be dealt with through the use of the effective mass.

In doing this, we have achieved full translational symmetry. Hence, there is no preferred reference frame and we are free to make use of the Lorentz Transformation.

In the lab frame, we say that the electrons are stationary. We then boost to a reference frame moving with velocity $-\mathbf{v}$ with respect to the lab frame. Thus, the electrons are moving at velocity $\mathbf{v}$ in our frame with current density

$$\mathbf{J} = -ne\mathbf{v}, \quad \text{Where } n \text{ is the electron density.} \tag{1.3}$$

Since the electrons are stationary, we can infer that there is no electric field in the lab frame. The experimental setup provides the lone electromagnetic field in the form of a $\hat{z}$ pointing magnetic field.

$$\begin{aligned} \mathbf{E} &= \mathbf{0} \\ \mathbf{B} &= B\hat{z} \end{aligned} \tag{1.4}$$

To find expressions for the electromagnetic fields in our boosted frame, let's first rotate in xy plane such that $\mathbf{v} \parallel \hat{x}$. We can do this because alongside translational symmetry, our electron gas is isotropic in the ϕ contained to the plane. This greatly eases the computation of the Lorentz Transformation.

Recall the electromagnetic field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{1.5}$$

Notice that we've set $c = 1$ for clarity. This factor is unimportant since $\mathbf{E} = 0$ anyways, that being said - we shall leave the electric field terms in for the time being for completeness. Furthermore, recall the form of the Lorentz Transformation with $\beta = -|\mathbf{v}|/c$ and $\gamma = [1-\beta^2]^{-1/2}$

$$\Lambda^{\nu}_{\ \mu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.6}$$

$$F'^{\mu\nu} = \Lambda^{\mu}_{\sigma}\Lambda^{\nu}_{\tau}F^{\sigma\tau} \tag{1.7}$$

$$= \begin{pmatrix} 0 & -E_x\gamma^2(1-\beta^2) & -\gamma(E_y - \beta B_z) & -\gamma(E_z + \beta B_y) \\ E_x\gamma^2(1-\beta^2) & 0 & -\gamma(B_z - \beta E_y) & \gamma(B_y + \beta E_z) \\ \gamma(E_y - \beta B_z) & \gamma(B_z - \beta E_y) & 0 & -B_x \\ \gamma(E_z + \beta B_y) & -\gamma(B_y + \beta E_z) & B_x & 0 \end{pmatrix} \tag{1.8}$$

We note here that the electromagnetic field tensor has kept its algebraic structure. That is to say, it is still antisymmetric. Let's take a linear order approximation in β and make sure to enforce the conditions stated in Eq. 1.4.

$$\gamma = \frac{\beta}{\sqrt{1-\beta^2}} \approx \beta \tag{1.9}$$

$$F'^{\mu\nu} = \begin{pmatrix} 0 & 0 & \beta B & 0 \\ 0 & 0 & -B & 0 \\ -\beta B & B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.10}$$

We can use the structure seen in $F^{\mu\nu}$ to read off the electromagnetic fields in the boosted frames

$$\begin{aligned} \mathbf{E}' &= -\mathbf{v} \times \mathbf{B}' \\ \mathbf{B}' &= B\hat{z} = \mathbf{B} \end{aligned} \tag{1.11}$$

We can perform a sanity-check based on the fact that the electrons are moving at constant velocity both in magnitude and direction i.e. they have zero acceleration. In turn, this means that the Lorentz Force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ must be identically zero.

Combining Eq. 1.3 and Eq. 1.11 we can represent the Electric field as

$$\begin{aligned} \mathbf{E}' &= \frac{B}{ne}\mathbf{J} \times \hat{z} \\ E'_i &= \rho_{ij}J^j = \frac{B}{ne}\epsilon_{ijk}J^j\hat{z}^k \\ \implies &\begin{cases} \rho_{ij} = \frac{B}{ne}\epsilon_{ijk}\hat{z}^k = \frac{B}{ne}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \sigma_{ij} = \rho_{ij}^{-1} = \frac{ne}{B}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{cases} \end{aligned} \tag{1.12}$$

We employ dimensional analysis to check that our expression is correct. $[\rho] = \frac{[B]}{[ne]} = \frac{N \cdot s}{C \cdot m^{\frac{1}{m^2}} C} = \frac{J \cdot s}{C^2} = \Omega$ as expected since we're working in $2D$ where $R = \rho$. In the case where $d \neq 2$, one need only change the dimensions of n to suit and recover the units for Eq. 1.2.

Note that if a material were to be a perfect conductor then Eq. 1.12 would read $\rho_{ij} = 0$ which is clearly not the case. The same applies if the material were to be a perfect insulator. We see now that the resistivity is simply a linear function in the magnetic field.

Recall, however, our assumption that there were no impurities. Our whole argument was based on the logical steps that no impurities (and non-interacting electrons) $\implies$ translation invariance $\implies$ the validity of the Lorentz transformation, which gave us our linear relationship.

Classical Approximation

Let's classically analyse the motion of an electric charge in $2D$ under the influence of an orthogonal magnetic field. We'll describe the particle as an electron but it's important to note that the derivation is done in terms of a positive point charge. For calculations then, one must switch $e \to -e$. We write the Hamiltonian

$$\begin{aligned} H = \frac{p^2}{2m} - \frac{e}{m}\mathbf{p}\cdot\mathbf{A} + \frac{e^2}{2m}A^2, \quad \mathbf{A} &\coloneqq \frac{1}{2}\mathbf{B}\times\mathbf{r} \\ = \frac{1}{2}m\mathbf{v}^2, \qquad \mathbf{v} &\coloneqq \frac{1}{m}\Big(\mathbf{p} - e\mathbf{A}\Big) \end{aligned} \tag{1.13}$$

It's important to note here that $\mathbf{v}$ is the mechanical velocity of the system and is related to the actual motion of the electron in configuration space. Hence, $\mathbf{p}_m = m\mathbf{v}$ is the mechanical momentum of the system. Contrary to the canonical momentum which is a mathematical quantity i.e. not an observable. Let's consider a canonical transformation from $(\mathbf{r},\mathbf{p}) \to (\mathbf{Q},\mathbf{P})$. That is a transformation from a certain basis into the another basis of the phase space which preserves the Poisson Bracket

$$\begin{aligned} \{\cdot,\star\} = \sum_i \frac{\partial\cdot}{\partial r^i}\frac{\partial\star}{\partial p^i} - \frac{\partial\star}{\partial r^i}\frac{\partial\cdot}{\partial p^i} \\ \{r^i, p^j\} = \delta^{ij} \end{aligned} \tag{1.14}$$

Since the electron is contained to the $2D$ plane, we know that $\mathbf{r}$ and $\mathbf{p}$ only have x and y components. Calculating all the Poisson brackets $\{p_i, A_j\}$

$$\{p_x, A_x\} = \left\{p_x, -\frac{B}{2}y\right\} = -\frac{B}{2}\{y, p_x\} = 0 \tag{1.15}$$

$$\{p_x, A_y\} = \left\{p_x, \frac{B}{2}x\right\} = -\frac{B}{2}\{x, p_x\} = -\frac{B}{2} \tag{1.16}$$

$$\{p_y, A_x\} = \left\{p_y, -\frac{B}{2}y\right\} = \frac{B}{2}\{x, p_y\} = \frac{B}{2} \tag{1.17}$$

$$\{p_y, A_y\} = \left\{p_y, \frac{B}{2}x\right\} = -\frac{B}{2}\{y, p_y\} = 0 \tag{1.18}$$

We'll now calculate the Poisson bracket between components of the mechanical velocity $\mathbf{v}$ as defined in Eq. 1.13.

$$\{v_x, v_y\} = \frac{1}{m^2}\{p_x - eA_x, p_y - eA_y\} \tag{1.19}$$

$$= \frac{1}{m^2}\Big[\{p_x, p_y\} - e\{p_x, A_y\} - e\{A_x, p_y\} + e^2\{A_x, A_y\}\Big] \tag{1.20}$$

$$\{A_x, A_y\} \sim \{x, y\} = 0 \implies \{v_x, v_y\} = \frac{e}{m^2}\Big[0 + \{p_y, A_x\} - \{p_x, A_y\} + 0\Big] \tag{1.21}$$

$$\text{From above we have,} \quad \{v_x, v_y\} = \frac{e}{m^2}\left(\frac{B}{2} + \frac{B}{2}\right) = \frac{eB}{m^2} \tag{1.22}$$

We now consider the Canonical Transformation defined by $\mathbf{P} = \alpha(\hat{z}\times\mathbf{v}$ and $\mathbf{Q}$ another linear combination of $\mathbf{p}$ and $e\mathbf{A}$. We enforce the condition that the only canonical pairs, i.e. pairs with unit Poisson bracket, are $\{Q_x, Q_y\}$ and $\{P_x, P_y\}$. We note that the cross product in $\mathbf{P}$ manifests as a switch in the components of $\mathbf{v}$ together with a change of sign on the P_y component. Combining this fact with the antisymmetry of the Poisson bracket (i.e. $\{a,b\} = -\{b,a\}$), we see that

$$\{P_x, P_y\} = \left(\frac{\alpha}{m}\right)^2\{v_x, v_y\} = \left(\frac{\alpha}{m}\right)^2\left(\frac{eB}{m^2}\right) = 1 \implies \alpha = \pm\frac{1}{\sqrt{eB}} \tag{1.23}$$

We choose the principal value and have a well defined $\mathbf{P} \sim \mathbf{p} - e\mathbf{A}$. Now, to create our basis we just need $\mathbf{Q}$ to be linearly independent. So we change their relative sign. We choose to include an overall minus sign such that $\{Q_x, Q_y\} = 1$.

$$\begin{aligned} i\mathbf{P} &= \frac{1}{\sqrt{eB}}\hat{z}\times(\mathbf{p} - e\mathbf{A}) \\ \mathbf{Q} &= -\frac{1}{\sqrt{eB}}\hat{z}\times(\mathbf{p} + e\mathbf{A}) \end{aligned} \tag{1.24}$$

We can now rewrite Eq. 1.13 as

$$H = \frac{eB}{2m}P^2 \tag{1.25}$$

A reminder of the relevant Hamiltonian formalism can be found in B. Returning now to our Hamiltonian. We see that it can be expanded to include the P_x and P_y canonical pairs explicitly as $H = \frac{eB}{2m}\Big(P_x^2 + P_y^2\Big)$. It's clear now that since $\frac{\partial H}{\partial Q_i} = 0$, the vector $\mathbf{Q}$ is conserved. We want to perform yet another canonical transformation to solve for P_x and P_y now. Let's look at using generating functions as a means of executing these transformations.

We can now utilise Eq. B.13 with $p = P_x$, $q = P_y$ and $Q = \Phi$, $P = \Gamma$. Suppose that Φ and Γ are given by

$$P_x = f(\Gamma)\cos\Phi \tag{1.26}$$

$$P_y = f(\Gamma)\sin\Phi \tag{1.27}$$

$$P_x = \frac{\partial F}{\partial P_y} = P_y\frac{P_x}{P_y} = P_y\cot\Phi \tag{1.28}$$

$$\text{Integrating yields,} \qquad F(P_y,\Phi) = \frac{1}{2}P_y^2\cot\Phi + C(\Phi) \tag{1.29}$$

$$\Gamma = -\frac{\partial F}{\partial \Phi} = \frac{P_y^2}{2}\csc^2\Phi + C'(\Phi) = \frac{f(\Gamma)^2}{2} \implies C(\Phi) = 0 \tag{1.30}$$

$$f(\Gamma) = \sqrt{2\Gamma} \implies \begin{cases} P_x = \sqrt{2\Gamma}\cos\Phi \\ P_y = \sqrt{2\Gamma}\sin\Phi \\ H' = \frac{eB}{m}\Gamma \end{cases} \tag{1.31}$$

Furthermore, we see that $\{\Phi, H'\} = \frac{eB}{m} = \dot{\Phi} = \omega_c$ the angular frequency which is known as the classical cyclotron frequency. Similarly, $\dot{\Gamma} = 0$ and is hence constant. Using the fact that Γ is constant we can write the motion of P_x and P_y through phase space as

$$\begin{aligned} P_x &= \sqrt{2\Gamma}\cos\Big(\frac{eB}{m}t + \Phi_0\Big) \\ P_y &= \sqrt{2\Gamma}\sin\Big(\frac{eB}{m}t + \Phi_0\Big) \end{aligned} \tag{1.32}$$

$$\mathbf{P} = \frac{1}{\sqrt{eB}}\hat{z}\times\Big(\mathbf{p} - \frac{e}{2}\mathbf{B}\times\mathbf{r}\Big) = \frac{1}{\sqrt{eB}}\Big(\hat{z}\times\mathbf{p} - \frac{e}{2}\hat{z}\times\Big(\mathbf{B}\times\mathbf{r}\Big)\Big) \tag{1.33}$$

$$= \frac{1}{\sqrt{eB}}\Big(\hat{z}\times\mathbf{p} - \frac{e}{2}\Big((\hat{z}\cdot\mathbf{r})\overset{0}{\cancel{\mathbf{B}}} - (\hat{z}\cdot\mathbf{B})\mathbf{r}\Big)\Big) \tag{1.34}$$

$$= \frac{1}{\sqrt{eB}}\Big(\hat{z}\times\mathbf{p} + \frac{eB}{2}\mathbf{r}\Big) \tag{1.35}$$

Similarly, for $\mathbf{Q}$ we have

$$\mathbf{Q} = \frac{1}{\sqrt{eB}}\Big(-\hat{z}\times\mathbf{p} + \frac{eB}{2}\mathbf{r}\Big) \tag{1.36}$$

$$\implies \mathbf{r} = \frac{\sqrt{eB}}{eB}\Big(\mathbf{P} + \mathbf{Q}\Big) = \frac{1}{\sqrt{eB}}\Big(\mathbf{P} + \mathbf{Q}\Big) \tag{1.37}$$

Therefore, we have that the centre of the circular motion is at $\frac{1}{\sqrt{eB}}\mathbf{Q}$, and the radius of the motion is simply $\frac{1}{\sqrt{eB}}|\mathbf{P}|$. Then the physical meaning of $\mathbf{Q}$ is the effective "centre of mass" of the system. Obviously mass is not the factor here but the particle's charge and interaction with the magnetic field, but it's a worthwhile comparison to make. Similarly, the meaning of $\mathbf{P}$ stems from the relative motion in the system about the "centre of mass".

The most important takeaway here is the connection between the motion of the electron in the Hall setup and the harmonic oscillator of which has a similar form as the expanded Hamiltonian in Eq. 1.25.

Semi-Classical Approximation

We've already seen how our system seems to veer towards that of a harmonic oscillator. As we venture further into the Hall effect, and hence into quantum mechanics, we see this connection manifest more thoroughly. The characteristic energy scale of the harmonic oscillator is given by $\hbar\omega_c$, the ground state energy. Similarly, we can find the characteristic length as $\sqrt{\langle x^2\rangle - \langle x\rangle^2}$ in the ground state.

$$l \coloneqq \sqrt{\langle x^2\rangle - \langle \cancelto{0}{x}\rangle^2}, \quad \langle x^2\rangle = \langle 0|\, \hat{x}^2\, |0\rangle \tag{1.38}$$

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega_c}}\Big(m\omega_c\hat{x} \mp i\hat{p}\Big) \implies \hat{x} = \sqrt{\frac{\hbar}{2m\omega_c}}\Big(\hat{a}_+ + \hat{a}_-\Big) \tag{1.39}$$

Recall the definition of the commutator $[\cdot,\star] = \cdot\star - \star\cdot$

$$\hat{a}_-\hat{a}_+ = \frac{1}{2\hbar m\omega_c}\Big(\underbrace{\hat{p}^2 + (m\omega_c x)^2}_{2m\hat{H}}\Big) - \frac{i}{2\hbar}\underbrace{[\hat{x},\hat{p}]}_{i\hbar} = \frac{\hat{H}}{\hbar\omega_c} + \frac{1}{2} \tag{1.40}$$

$$\text{The same argument yields} \quad \hat{a}_+\hat{a}_- = \frac{\hat{H}}{\hbar\omega_c} - \frac{1}{2} \tag{1.41}$$

$$\hat{x}^2 = \frac{\hbar}{2m\omega_c}\Big(\hat{a}_+^2 + \hat{a}_-^2 + \hat{a}_-\hat{a}_+ + \hat{a}_+\hat{a}_-\Big) \tag{1.42}$$

$$\langle 0|\, \hat{x}^2\, |0\rangle = \frac{\hbar}{2m\omega_c}\left(\langle 0|\, \hat{a}_+^2\, |0\rangle + \langle 0|\, \hat{a}_-^2\, |0\rangle + \frac{2}{\hbar\omega_c}\langle 0|\, \hat{H}\, |0\rangle\right) \tag{1.43}$$

$$= \frac{\hbar}{2m\omega_c}\left(\langle 0|2\rangle + \langle 2|0\rangle + 2\frac{\hbar\omega_c}{\hbar\omega_c}\right) = \frac{\hbar}{2m\omega_c} \tag{1.44}$$

$$l = \sqrt{\frac{\hbar}{2m\omega_c}} \tag{1.45}$$

In fact, we drop the $\sqrt{2}$ factor to define the characteristic length $l \coloneqq \sqrt{\frac{\hbar}{eB}}$. This is done so as to match the physical interpretation - this being that the area $2\pi l^2$ contains exactly one quantum of magnetic flux Φ_0 defined to be.

$$\begin{aligned} \Phi_0 &= \frac{h}{e} \\ B &= \frac{\Phi_0}{2\pi l^2} \end{aligned} \tag{1.46}$$

In this regime we call this characteristic length - the magnetic length. We require the condition $k_F l \gg 1$, where k_F is the Fermi wavevector, as this turns out to be an equivalent statement as $\hbar\omega_c \ll \epsilon_F$, the Fermi energy. This is done as for high magnetic fields, the energy becomes quite high and inversely the radius of the motion becomes very small. When the radius gets small enough where we cannot neglect the size of the electron's wave-packet the semi-classical approximation fails.

Consider the Bohr-Sommerfeld quantisation which states that the circumferences of the orbits of motion are of the form

$$2\pi r_n = \lambda n$$

where λ is the de Broglie wavelength. Recall that Eq. 1.13 tells us that the canonical momentum is expressed as the following (remember that since we're going to be dealing with an electron explicitly here we make the substitution $e \to -e$).

$$\mathbf{p} = m\mathbf{v} - e\mathbf{A} \tag{1.47}$$

It's this canonical momentum that determines the de Broglie wavelength.Since we have the freedom that comes with the Gauge potential - we're free to fix this as the symmetric gauge $\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}$ since $\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B}$. Remember that in the presence of a positively pointing magnetic field in the $\hat{z}$ direction, an electron will move in a counter-clockwise fashion.

Similarly, our Gauge potential will also point in the counter-clockwise direction.

$$p = \sqrt{(mv)^2 + (\frac{eB}{2}r_n)^2 - emvr_nB}, \quad v = r_n\omega_c \tag{1.48}$$

$$= \sqrt{(m\omega_c r_n)^2 + (\frac{eB}{2}r_n)^2 - em\omega_c B r_n^2} \tag{1.49}$$

$$= \sqrt{(m\omega_c r_n - \frac{eB}{2}r_n)^2} = m\omega_c r_n - \frac{eB}{2}r_n \tag{1.50}$$

$$p = \frac{h}{\lambda} = \frac{h}{2\pi r_n}n \tag{1.51}$$

$$\implies m\omega_c r_n - \frac{eB}{2}r_n = (eB - \frac{eB}{2})r_n = \frac{eB}{2}r_n = \frac{h}{2\pi r_n}n \tag{1.52}$$

Rearranging this we get

$$r_n = \sqrt{\frac{hn}{\pi eB}} = \sqrt{\frac{\Phi_0}{\pi B}n} \tag{1.53}$$

Finding the area contained by these orbits we retrieve the remarkable equation

$$\Phi_n = B\pi r_n^2 = \Phi_0 n \implies \Delta\Phi = \Phi_0 \tag{1.54}$$

This tells us that time we move to the next allowed radius we in fact add one more quantum of magnetic flux to the interior of the orbit.

Quantum Dynamics in Strong Magnetic Fields

We chose the symmetric gauge - $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ in our classical approximation. Now, we look at the Landau gauge

$$\mathbf{A} = xB\hat{y} \tag{1.55}$$

$\mathbf{A}$ is now explicitly a function of x which means we lose translation invariance in the x direction. Since the Hamiltonian is a function of $\mathbf{A}$ it's a worthwhile question to ask if we the physics changes too? The Hamiltonian given in Eq. 1.13 can be rewritten as

$$H = \frac{1}{2m}\Big(p_x^2 + (p_y + exB)^2\Big), \quad \text{Making our transformation } e \to -e \tag{1.56}$$

$$\dot{x} = \{x, H\} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \tag{1.57}$$

$$\dot{p}_x = \{p_x, H\} = -\frac{\partial H}{\partial x} = -\frac{eB}{m}(p_y + exB) \tag{1.58}$$

$$m\ddot{x} = \dot{p}_x = -\frac{eB}{m}(p_y + exB) \tag{1.59}$$

$$\dot{y} = \frac{p_y + xeB}{m} \tag{1.60}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = 0 \tag{1.61}$$

Combining these results we arrive at

$$\begin{aligned} m\ddot{x} &= -\frac{eB}{m}\dot{y} \\ m\ddot{y} &= \frac{eB}{m}\dot{x} \end{aligned} \tag{1.62}$$

Which are the equations for the needed circular motion. Thus, the physics itself has not changed and we're completely valid in choosing this gauge. In fact we can choose any gauge and the physics shall not change.

Claim: The equations of motion are invariant under gauge transformation.
Proof:
We use Einstein Summation Convention here - that is to say repeated indices are summed over.

From Hamilton's equations we have the change of the momentum vector is given as

$$p_i = -\partial_i H = -\frac{1}{2m}\partial_i\Big((p_j + eA_j)(p^j + eA^j)\Big) = -\frac{1}{2m}\Big(p_j p^j + 2ep_j A^j + e^2 A_j A^j\Big) \tag{1.63}$$

$$= -\frac{e}{m}\Big(p^j\partial_i A_j + eA^j\partial_i A_j\Big) = -ev^j\partial_i A_j \tag{1.64}$$

$$\dot{r}_i = \frac{\partial H}{\partial p^i} = \frac{1}{m}(p_i + eA_i) \tag{1.65}$$

$$m\ddot{r}_i = \dot{p}_i + e\dot{A}_i = \dot{p}_i + e\partial_j A_i v^j \quad \text{From the multivariable chain rule. } D_t\mathbf{A} = \sum_j \frac{\partial \mathbf{A}}{\partial x^j}\dot{x}^j \tag{1.66}$$

Combining results we have:

$$m\ddot{r}_i = ev^j\Big(\partial_j A_i - \partial_i A_j\Big) = -ev^j\epsilon_{ijk}\Big(\boldsymbol{\nabla}\times\mathbf{A}\Big)^k = -e\epsilon_{ijk}v^j B^k = -e\Big(\mathbf{v}\times\mathbf{B}\Big)_i \tag{1.67}$$

We have retrieved the Lorentz force for a general gauge $\mathbf{A}$ only with the condition that the magnetic field is uniform. Therefore, the equations of motion are invariant under these gauge transformations.

■

The Hamiltonian using the Landau gauge can be written as

$$\hat{H} = \frac{1}{2m}\Big(\hat{p}_x^2 + (\hat{p}_y + eB\hat{x})^2\Big) \tag{1.68}$$

Notice here that we have the same translation symmetry in the y direction that we were talking about before. Since if we disregard the $x - p_x$ canonical pair, this is just a free particle and hence has the plane wave as its solution, we attempt the separation of variables

$$\psi_k(x,y) = e^{iky}\phi_k(x) \tag{1.69}$$

Looking at the y momentum operator we see

$$\hat{p}_y\psi_k = -i\hbar\partial_y e^{iky}\phi_k(x) = \hbar k\psi_k \tag{1.70}$$

Great! Our wavefunction is an eigenstate of $\hat{p}_y$ and so we can make the replacement $\hat{p}_y \to \hbar k$ in the Hamiltonian. Now, we have the Schrodinger Equation,

$$\hat{H}_k \cancel{e^{iky}}\phi_(x) = \epsilon_k \cancel{e^{iky}}\phi_k(x) \implies \hat{H}_k\phi_(x) = \epsilon_k\phi_k(x) \tag{1.71}$$

$$\hat{H}_k \coloneqq \frac{1}{2m}\Big(\hat{p}_x^2 + (\hbar k + eBx)^2\Big) = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2}m\omega_c^2(x + kl^2)^2$$

$\hat{H}_k$ is simply the $1D$ harmonic oscillator Hamiltonian. This is precisely the underlying connection we eluded to earlier. Albeit, now in only one dimension as the other degree of freedom is contained in the y momentum (and hence k) in this gauge. The central position of this Hamiltonian is given as $X_k = -kl^2$, which interestingly enough depends on the y momentum. Since the y plane wave can be made redundant in the Schrodinger Equation as shown in Eq. 1.71, we have an entire family of allowed energies for every y plane wave

$$\epsilon_{kn} = (n + \frac{1}{2})\hbar\omega_c \tag{1.72}$$

Thus, the wave function is given by combining the plane wave and harmonic oscillator wave function

$$\psi_{kn}(x,y) = \frac{1}{\sqrt{L}}e^{iky}H_n(\frac{x + kl^2}{l})\exp\Big\{-\frac{(x + kl^2)^2}{2l^2}\Big\} \tag{1.73}$$

In this context, the harmonic oscillator levels are called *Landau levels*. Let's choose to work with a rectangular sample of horizontal with L_x and vertical height L_y. In particular, let the right hand endge of our sample be placed at $x = 0$ and hence the left hand edge is at $x = -L_x$. Let's also suppose that the boundary conditions are periodic in the y direction. Although our lack of translation symmetry in the x direction prevents us from having periodic boundaries in the x direction. Although since our wave function is of the form $\psi(x, 0) \sim \exp\{-(x + kl^2)^2/(2l^2)\}$, it will vanish quite quickly away from the centre position $X_k = -kl^2$. Since the sample's x values range from 0 to $-L_x$, the k values for which the centre position is within these limits (and hence the wave packet is mainly within the sample) are elements of the interval $(0, L_x/l^2)$. It's clear to see that the states at the left and right boundaries are quite different and hence there cannot be boundary conditions that are periodic in x.

Our plane wave solution under periodic boundary conditions for the y direction gives us the very simple solution to the wave length in the basis state $\lambda = L_y$ which gives us $k = 2\pi/L_y$. But if we say there are N states for every Landau level, we know that k can in fact be any integer multiple ranging up to $k = 2\pi N/L_y$. We can also count the amount of allowed states by utilising our allowed range for k values discussed above which means that at a maximum N is given by

$$\frac{L_x}{l^2} = \frac{2\pi N}{L_y} \implies N = \frac{L_x L_y}{2\pi l^2} = \frac{L_x L_Y}{\Phi_0} B \tag{1.74}$$

This tells us that the number of states in each Landau level given by N is simply the number of magnetic flux quanta within the sample. Thus, there is one state in each Landau level per flux quantum. This is precisely the result we found in the semi-classical approximation, in particular Eq. 1.54.

Notice that even though the allowed wavevectors are indexed by a one-dimensional quantity (the x position range) the degeneracy associated with each Landau level is quite large in the $2D$ area. The reason being - our periodic boundary conditions give us the fact that $k_n = 2\pi n/L_y$ so as $\exp ik_n(0) = \exp ik_n L_y \ \forall n$. That tells us that the spacing between allowed k vectors is $\Delta_k = 2\pi/L_y$. The range, however, is of the size $(0, L_x)$. Thus, an increase in overall scale of the sample, L, increases the amount of allowed vectors whilst simultaneously decreasing the spacing between them.

Semi-classical interlude

Returning for a moment to the semi-classical approximation.

Claim: Using the fact that the energy for the nth harmonic oscillator state is $(n + 1)\hbar\omega_c$, under the semi-classical approximation we can retrieve the same result that the width of the support of the wavefunction scales as $\sqrt{n}l$.
Proof: The support of the wavefunction is merely the region of space where $\psi \neq 0$. This translates to the locations where we can find the particle, under the semi-classical regime, i.e. the allowed radii r_n.

Recall Eq. 1.13, where $\mathbf{v}$ is the mechanical velocity of the electron. But the magnitude of the mechanical velocity is nothing more than $v = r\omega$. This tells us that the Hamiltonian, and hence the energy, is given by

$$H = \epsilon_n = \frac{1}{2} m r_n^2 \omega_c^2 \tag{1.75}$$

But from the condition in the claim we have that the nth energy is given by

$$\epsilon_n = (n + \frac{1}{2})\hbar\omega_c \tag{1.76}$$

$$\implies (n + \frac{1}{2})\hbar\omega_c = m\Big(\frac{r_n}{\sqrt{2}}\Big)^2 \omega_c^2 \tag{1.77}$$

$$(n + \frac{1}{2})\frac{\hbar}{m\omega_c} = (w_n)^2, \tag{1.78}$$

$$w_n \coloneqq \frac{r_n}{\sqrt{2}} \text{ is approximately the width of the support} \tag{1.79}$$

$$\Delta w_n \coloneqq \sqrt{(w_{n+1})^2 - (w_n)^2} \implies \Delta w_n = \sqrt{n}\sqrt{\frac{\hbar}{m\omega_c}} = \sqrt{n}l \tag{1.80}$$

Hence, the semi-classical equivalent of the width of the support scales as $\sqrt{n}l$ as claimed. ■

Now that we have the eigenfunctions for the electron in a strong magnetic field, we aim to relate them back to the more physically intuitive, semi-classical case. Let's consider a wave packet located at some position with definite momentum, $p = \hbar K$. Say, the momentum is large enough i.e. $\frac{(\hbar K)^2}{2m} \gg \hbar\omega_c$ (the cyclotron energy). Since the wave packet is quite energetic, and hence of large enough scale to be modeled semi-classically, we know that the wave packet will consist of different Landau level states. Furthermore, these states will be centred about $\bar{n} = \frac{(\hbar K)^2}{2m\hbar\omega_c}$ Our total steady state wave function is going to be a sum over Landau levels obviously, but what precisely is being summed? We're going to have the harmonic oscillator esque wave function from Eq 1.71. We're going to need to weight it by some factor a_n that can be tuned to fit the initial conditions. But this factor should be a function of k since we're going to need control over which states within each Landau level the wave packet will be comprised of. Finally, we also need to include a factor of N, the number of states within a Landau level, as a state is more likely to fall into a more potentially populous Landau level. Since $N = \frac{L_y}{2\pi}k$ we have that the small change is N, $\mathrm{d}N = \frac{L_y}{2\pi}\,\mathrm{d}k$. Now, we can construct our wave function. To make this a time-dependent wave function we need only apply the time-evolution operator, $\exp\{-i\frac{E_n}{\hbar}t\}$.

$$\Psi(\mathbf{r},t) = \sum_n \int \frac{L_y}{2\pi}\,\mathrm{d}k\, a_n(k)\psi_{nk}(\mathbf{r})\exp\left\{-i(n+\frac{1}{2})\omega_c t\right\} \tag{1.81}$$

To show that this indeed corresponds to circular motion we should consider $\langle r^2\rangle$.

Let's imagine we have a very strong magnetic field such that we can ignore the higher Landau levels. Since we have degenerate states, made of combinations of stationary basis states, our state will also be stationary. Stationary state $\implies$ doesn't change with time $\implies$ the total current is going to be zero. However, let's look at the current density $\mathbf{J}$. For this, we're going to look at the mechanical velocity multiplied by the electron charge applied to the lowest Landau level.

$$\psi_k(\mathbf{r}) = \frac{1}{\sqrt{\pi^{1/2}L_y l}}e^{iky}\exp\left\{-\frac{1}{2l^2}(x+kl^2)^2\right\} \tag{1.82}$$

$$\langle\mathbf{J}\rangle = -\frac{e}{m}\langle\Psi_k|\,(\mathbf{p}+e\mathbf{A})\,|\Psi_k\rangle \tag{1.83}$$

$$\langle J_y\rangle = -\frac{e}{m\sqrt{\pi}lL_y}\iint \mathrm{d}x\,\mathrm{d}y\, e^{iky-iky}(\hbar k+eBx)\exp\left\{-\frac{1}{l^2}(x+kl^2)^2\right\} \tag{1.84}$$

$$= -\frac{e\omega_c \cancel{L_y}}{\sqrt{\pi}l\cancel{L_y}}\int \mathrm{d}x\ (x+kl^2)\exp\left\{-\frac{1}{l^2}(x+kl^2)^2\right\} \tag{1.85}$$

$$= -\frac{el\omega_c}{2\sqrt{\pi}}\int \mathrm{d}u\exp\{-u\} = \frac{el\omega_c}{2\sqrt{\pi}}\exp\left\{-\frac{1}{l^2}(x+kl^2)^2\right\}\Big|_{-2kl^2}^{0} \tag{1.86}$$

We choose these bounds as the wave packets are contained within the x coordinates $-2kl^2$ and 0.

$$\implies \langle J_y\rangle \propto \exp\{-k^2l^2\} - \exp\{-k^2l^2\} = 0 \tag{1.87}$$

Which has units of $\frac{C\,m}{s}$ as expected, given our definition of $\mathbf{J}$. [1]

Let's add in a potential term. Specifically, $V(\mathbf{r}) = eEx$, an electric field pointing in the $-x$ direction. Since we still

[1] If it were the case that $kl^2 = L_x/2$ i.e. the centre point of the wave packet is the midpoint of the horizontal length. In this case, when we integrate over x we're integrating an odd function over a symmetric region and is 0. This also leads to the circular motion.

retain y translation symmetry, and so choose the Landau gauge. Then the Hamiltonian becomes

$$H_k = \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_c^2(x + kl^2)^2 + eEx \tag{1.88}$$

$$= \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_c^2\left(x^2 + 2kl^2x + 2\frac{eE}{m\omega_c^2}x + k^2l^2\right) \tag{1.89}$$

$$= \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_c^2\left((x + kl^2 + \frac{eE}{m\omega_c^2})^2 - 2\frac{eEkl^2}{m\omega_c^2} - (\frac{eE}{m\omega_c^2})^2\right) \tag{1.90}$$

$$= \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_c^2(x - X_k)^2 - eEkl^2 - \frac{1}{2}\frac{e^2E^2}{m\omega_c^2}, \quad X_k \coloneqq -kl^2 - \frac{eE}{m\omega_c^2} \tag{1.91}$$

$$= \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_c^2(x - X_k)^2 + eE(X_k + \frac{eE}{m\omega_c^2}) - \frac{e^2E^2}{2m\omega_c^2} \tag{1.92}$$

$$= \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_c^2(x - X_k)^2 + eEX_k + \frac{e^2E^2}{2m\omega_c^2} \tag{1.93}$$

$$= \underbrace{\left(\frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_c(x - X_k)^2\right)}_{\text{Our normal Hamiltonian (Eq. 1.71)}} + \left(eEX_k + \frac{1}{2}m(\frac{E}{B})^2\right) \tag{1.94}$$

The new Hamiltonian is the sum of the old Hamiltonian and a new term relating to the **E** field. This tells us that the energy eigenvalue should also follow this same pattern. Again, as we're working with $n = 0$, the lowest Landau level, we have

$$\epsilon_k = \frac{1}{2}\hbar\omega_c + eEX_k + \frac{1}{2}m\bar{v}^2, \quad \bar{v} \coloneqq -\frac{E}{B} \tag{1.95}$$

The group velocity is given by $\frac{\partial\omega_k}{\partial k} = \frac{1}{\hbar}\frac{\partial\epsilon_k}{\partial k} = \frac{eE}{\hbar}\frac{\partial X_k}{\partial k} = -\frac{eEl^2}{\hbar} = \bar{v}$. Since the **E** field is constant, we have a linear potential and hence the wave packet propagates as expected. This new Hamiltonian breaks the x translational symmetry. This means that the symmetry we used to calculate $\langle J_y\rangle = 0$ doesn't exist any more.

We see from Eq. 1.95 that the energy is a linear function of position in X_k. This means that the Landau levels now are effectively tilted. Now if we draw a line at a certain energy $\tilde{E}$, we can hit multiple Landau levels - that is to say, there exist degeneracies between Landau Levels.

A Gauge Independent Quantization of the Hamiltonian

Credit to Dr. Chaolun Wu for a far cleaner derivation of the energy levels of the quantum Hall system without the need for gauge fixing. We begin with the classical Hamiltonian Eq. 1.25 then go about the Canonical quantization[2]

$$H = -\frac{1}{2m}\left(\partial_i - ieA_i\right)^2 \tag{1.96}$$

Then, we notice that this can be rewritten in terms of the covariant derivative

$$H = -\frac{1}{2m}D_i^2, \qquad D_i \coloneqq \partial_i - ieA_i \tag{1.97}$$

Furthermore, the Field Strength Tensor

$$F_{ij} = \partial_{[i}A_{j]} = \partial_iA_j - \partial_jA_i \tag{1.98}$$

$$= -\frac{1}{ie}\left(\partial_i\partial_j - \partial_j\partial_i - ie\partial_iA_j + ie\partial_jA_i - e^2A_iA_j + e^2A_jA_i\right) \tag{1.99}$$

$$= \frac{i}{e}[D_i, D_j] \tag{1.100}$$

But since F is antisymmetric 2D tensor then it only has one independent componenet namely $F_{xy} = \frac{i}{e}[D_x, D_y]$ which we know from the structure of F to just be B. Furthermore, being restricted to $d = 2$ means that we can complexify the problem to attain complex version of each vectors - the holomorphic and anti-holomorphic coordinates i.e.

$$\mathbf{X} \mapsto (z, \bar{z}) = (x + iy,\ x - iy), (D_x, D_y) \mapsto (D, \bar{D}) = (D_x + iD_y, D_x - iD_y) \tag{1.101}$$

[2]We set $\hbar = 1$ and the electric charge to be dimensionless.

Note that since $\mathrm{Im}\{\mathbf{D}\} \neq 0$ then $\bar{D} \neq D^*$. This gives us four independent derivatives

$$\begin{aligned} D &= D_x + iD_y = \partial_x + i\partial_y - ie\Big(A_x + iA_y\Big) \\ \bar{D} &= D_x - iD_y = \partial_x - i\partial_y - ie\Big(A_x - iA_y\Big) \\ D^* &= D_x^* - iD_y^* = \partial_x - i\partial_y + ie\Big(A_x - iA_y\Big) \\ \bar{D}^* &= D_x^* + iD_y^* = \partial_x + i\partial_y + ie\Big(A_x + iA_y\Big) \end{aligned} \tag{1.102}$$

But then we shall look at computing the commutators between these operators,

$$\big[D, \bar{D}\big] = [D_x + iD_y, D_x - iD_y] = -i[D_x, D_y] + i[D_y, D_x] = -2i[D_x, D_y] = -2i\frac{eB}{i} = -2eB \tag{1.103}$$

But since this commutator is real then

$$\big[D^*, \bar{D}^*\big] = -2eB \tag{1.104}$$

Any other combination can, however, be attacked via

$$\begin{aligned} \big[D, \bar{D}^*\big] = \big[D + D^* - D^*, \bar{D}^* + \bar{D} - \bar{D}\big] = \big[D, \bar{D}\big] - \big(\big[D, \bar{D}\big]\big)^* - \big[D^*, \bar{D}\big] \\ \implies \mathrm{Re}\big\{\big[D, \bar{D}^*\big]\big\} \sim \mathrm{Im}\big\{\big[D, \bar{D}\big]\big\} = 0 \end{aligned} \tag{1.105}$$

Similarly for the real part of $\big[D, \bar{D}^*\big]$ and hence for $\big[D^*, \bar{D}\big]$ via complex conjugation. After normalisation by a factor which we notice is precisely $-1/l^2$ - where l is the magnetic length. Then by relabelling using $a \sim D, a^\dagger \sim \bar{D}$ and likewise with b and D^*,

$$\begin{aligned} a &= -\frac{il}{\sqrt{2}}D, \qquad a^\dagger = -\frac{il}{\sqrt{2}}\bar{D} \\ b &= -\frac{il}{\sqrt{2}}D^*, \qquad b^\dagger = -\frac{il}{\sqrt{2}}\bar{D}^* \end{aligned} \tag{1.106}$$

We then get the commutation relations

$$\big[a, a^\dagger\big] = \big[b, b^\dagger\big] = 1 \tag{1.107}$$

With all other combinations null. This clearly resembles the harmonic oscillator. Let's investigate this hunch further by looking at the Hamiltonian. Indeed, we know that H is constructed from a Kronecker delta, in $d = 2$ though $\exists$ one other independent $2-$tensor namely the Levi-Civita tensor. We know that δ_{ij} being symmetric implies the Hermitian status of H but if we're to add the antisymmetric ϵ_{ij} then we must also make it Hermitian by adding a factor of i. That is to say,

$$H = -\frac{1}{2m}\Big(D_x^2 + D_y^2\Big) = -\frac{1}{2m}\underbrace{\Big(\delta^{ij}D_iD_j + i\epsilon^{lk}D_lD_k\Big)}_{=D_x^2+D_y^2+iD_xD_y-iD_yD_x} + \frac{i}{2m}\overbrace{\epsilon^{mn}D_mD_n}^{[D_x,D_y]} \tag{1.108}$$

Looking at the first term we see that

$$D_x^2 + D_y^2 + iD_xD_y - iD_yD_x = \Big(D_x - iD_y\Big)\Big(D_x + iD_y\Big) = \bar{D}D \tag{1.109}$$

which we can evaluate in term of the creation-annihilation operators giving $H_{\mathrm{ex}} = -\frac{1}{2m}(\sqrt{2}ia^\dagger/l)(\sqrt{2}ia/l)$. From the latter we recall the value of that commutator from above. Putting this together we see

$$H = \frac{1}{ml^2}a^\dagger a + \frac{eB}{m}$$

Going a step further by recalling the definition of the magnetic length as well as the cyclotron frequency and find

$$H = \omega_c\Big(a^\dagger a + \frac{1}{2}\Big) \tag{1.110}$$

Recalling the classical case above we see that the $a, a^\dagger$ pair feature in the Hamiltonian they're responsible for the motion about the quasi-centre of mass, or at least their classical counterparts are, and the degeneracy in b and $b^\dagger$ tells us we can connect this with the motion of the guiding centre. Hence we can write

$$\begin{aligned} p &\sim a^\dagger + b, \qquad & \bar{p} &\sim a + b^\dagger \\ z &\sim a - b^\dagger, \qquad & \bar{z} &\sim a^\dagger - b \end{aligned} \tag{1.111}$$

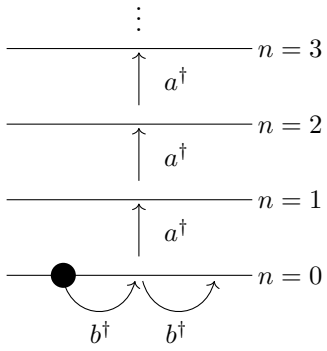
2 Integer Quantum Hall

Landau Levels and the Single-Particle Wavefunction

Since we know that the Hamiltonian looks similar to that of a harmonic oscillator, we can build up the eigenstates in a similar way by using the previously defined creation and annihalation operators [22]. The wavefunction will hence look like the following:

$$|nm\rangle = \frac{1}{\sqrt{n!m!}}(a^\dagger)^n(b^\dagger)^m|00\rangle \tag{2.1}$$

In this wavefunction, the first index is changed by the a and $a^\dagger$ operators, and the second index by b and $b^\dagger$ operators. Since only a and $a^\dagger$ appear in the Hamiltonian, these are the operators that move between energy levels of the system, while b and $b^\dagger$ move between degenerate states within these energy levels. These energy levels are known as Landau levels, and are each separated by an energy of $\hbar\omega_c$.



The quantum Hall effect happens when the magnetic field B is very large, as this means there is very little tunnelling of electrons between different Landau levels. A large part of the discussion in quantum Hall is done in the lowest Landau level, as many of the effects seen here are the same in higher Landau levels, althoguh not all of them. The lowest Landau level is any state denoted by $|0m\rangle$ and with energy $\frac{1}{2}\hbar\omega_c$. It should also be noted that this is all independent of what gauge we choose. Let's now take the symmetric gauge and see what our wavefunction looks like. The creation and annihalation operators now look like the following:

$$a^\dagger \sim \partial_z - \frac{B}{4}\bar{z} \qquad b^\dagger \sim \partial_{\bar{z}} - \frac{B}{4}z \tag{2.2}$$

$$a \sim \partial_{\bar{z}} + \frac{B}{4}z \qquad b \sim \partial_z + \frac{B}{4}\bar{z} \tag{2.3}$$

We can use the fact that $a|0m\rangle = 0$ to get the wavefunction:

$$\left(\partial_{\bar{z}} + \frac{B}{4}z\right)\varphi_{0m}(z,\bar{z}) = 0 \tag{2.4}$$

solving this gives:

$$\varphi(z,\bar{z}) = f(z)e^{-\frac{B}{4}z\bar{z}} \tag{2.5}$$

Now we can apply a similar method using $b^\dagger$:

$$b^\dagger|00\rangle = 0 \implies \left(\partial_z + \frac{B}{4}\bar{z}\right)f(z)e^{-\frac{B}{4}z\bar{z}} \tag{2.6}$$

$$\implies f'(z) = 0 \implies \varphi_{00}(z,\bar{z}) = e^{-\frac{B}{4}z\bar{z}} = e^{-\frac{z\bar{z}}{4\ell^2}}, \quad \text{where } \ell \text{ is the magnetic length} \tag{2.7}$$

Now let's see how the $b^\dagger$ operator acts changes the wavefunction in the lowest Landau level.

$$b^\dagger\varphi_{0m}(z,\bar{z}) = \left(\partial_{\bar{z}} - \frac{B}{4}z\right)f(z)e^{-\frac{B}{4}z\bar{z}} = \left(-\frac{B}{2}z\right)\varphi_{0m}(z,\bar{z}) \tag{2.8}$$

$$\implies \varphi_{0m}(z,\bar{z}) \sim z^m e^{-\frac{B}{4}z\bar{z}} \tag{2.9}$$

Now let's see what higher Landau level wavefunctions look like.

$$\varphi_{n0}(z,\bar{z}) = (a^\dagger)^n e^{-\frac{B}{4}z\bar{z}} \tag{2.10}$$

It can be shown that $(a^\dagger)^n \sim e^{\frac{B}{4}z\bar{z}}\partial_z^n e^{-\frac{B}{4}z\bar{z}}$, which is like Rodrigues' formula for Laguerre polynomials:

$$L_n^\alpha(\xi) = \frac{1}{n!}\xi^{-\alpha}e^\xi\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^n\left(\xi^{n+\alpha}e^{-\xi}\right) \tag{2.11}$$

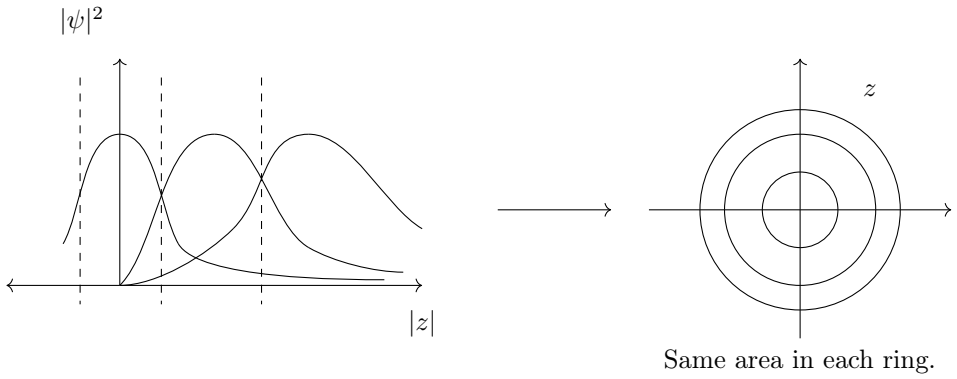
Hence, the general wavefunction goes like the following:

$$\varphi_{nm}(z,\bar{z}) = z^{m-n}L_n^{m-n}\left(-\frac{B}{2}z\bar{z}\right)e^{-\frac{B}{4}z\bar{z}} \tag{2.12}$$

Now let's look at the geometry of the lowest Landau level. The probability density is given by:

$$|\varphi_{0m}(z,\bar{z})|^2 = |z|^{2m}e^{-\frac{B}{2}|z|^2} \tag{2.13}$$

For $m = 0$, this is just a Gaussian, and each additional m adds a factor of $|z|^2$ to the probability density, which pushes the probability density further out from the origin.



Above this is plotted in both the radial and 2D picture, and one can see that the probability density is concentrated in a ring around the origin, with the radius of this ring increasing with m. Roughly speaking each ring covers the same area, and you can calculate the area of this by taking the expectation value of $|z|^2$:

$$\langle 00|\,\mathbf{x}^2\,|00\rangle = 2\pi\ell^2 = \frac{2\pi}{B} \tag{2.14}$$

The distance between nearby rings then gets closer as m increases, as they each cover the same area. Below is a density plot of the first few wavefunctions.

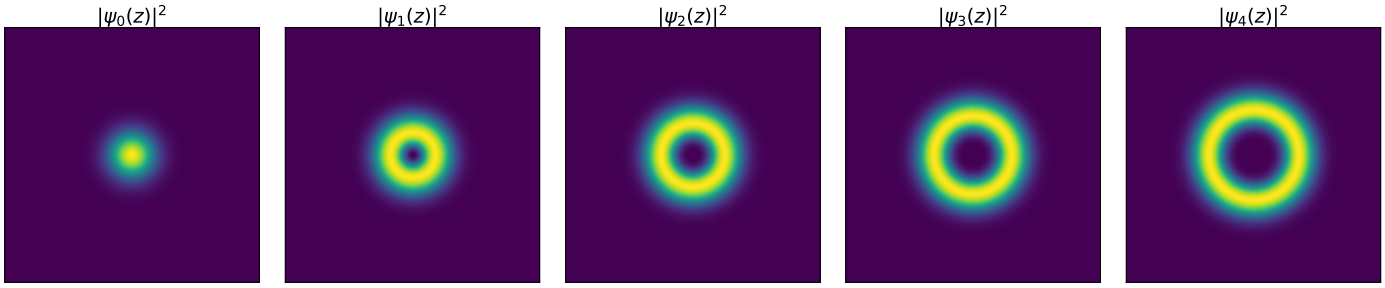
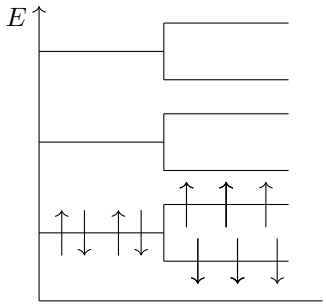


Figure 1: The probability density of the lowest Landau level wavefunctions.

You can count the degeneracy of the Landau level using this idea. Now we can think of a Zeeman splitting occuring, which means that each of these rings now contains one electron.



Hence, if there are N electrons in the Landau level, then the maximum value m can take is given by $m_{\max} = N - 1$. To make visualising this picture easier, one can think of adding a little potential $V(r)$ such that $m = 0$ is the ground state, and the filling of electrons is like pouring water onto a plate. Now what area is covered by N electrons?

$$A = 2\pi\ell^2 N = \frac{2\pi N}{B} \tag{2.15}$$

Rewriting this, we can find the degeneracy of the Landau level:

$$N = \frac{BA}{2\pi} \tag{2.16}$$

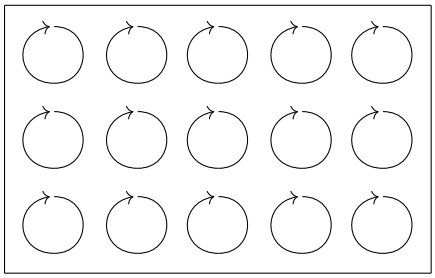
Restoring units, we get:

$$N = \frac{\Phi_{mag}}{h/e} = \frac{\Phi_{mag}}{\Phi_0} \tag{2.17}$$

Where Φ_{mag} is the magnetic flux through the area A, and $\Phi_0 = h/e$ is the magnetic flux quantum. This means that the degeneracy of the Landau level is quantised in units of Φ_0. Rearranging this, we can get that

$$\frac{1}{2\pi} = \frac{N/A}{B} = \frac{\rho}{B}, \tag{2.18}$$

where $\rho = N/A$ is the electron number density. In general, $\frac{\rho}{B} = \frac{\nu}{2\pi}$, where ν is the filling fraction. Since this is a fully filled Landau level, we can say that $\nu = 1$. We will see later that the fractional quantum Hall effect happens when ν is given by a fractional number. This ring structure is just one representation of the quantum Hall fluid. The classical picture is more like a liquid of electrons performing a sort of cycloton motion, which is obtained by using the coherent state basis.



The coherent state is just an eigenstate of b:

$$b\,|n\lambda\rangle = \lambda\,|n\lambda\rangle\,, \tag{2.19}$$

which is similar to the coherent state of a harmonic oscillator $a\,|\lambda\rangle = \lambda\,|\lambda\rangle$, where the solution is $e^{\lambda a^\dagger}$. Taking inspiration from this, we write the coherent state of the quantum Hall fluid as:

$$|n\lambda\rangle = e^{\lambda b^\dagger - \lambda^* b}\,|n0\rangle \tag{2.20}$$

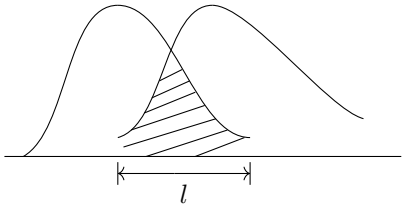
The reason we want to add the second part of the exponent is to ensure that it is Hermitian. Using the BCH formula, we can rewrite this as follows:

$$|n\lambda\rangle = e^{-\frac{|\lambda|^2}{2}} e^{\lambda b^\dagger} |n0\rangle \tag{2.21}$$

The overlap of different states is given by:

$$\langle n\lambda_1 | n\lambda_2 \rangle = e^{-\frac{1}{2}(|\lambda_1|^2 + |\lambda_2|^2)} e^{\lambda_1^* \lambda_2} \tag{2.22}$$

So, their overlap exponentially decays as the distance between them increases, as shown below.



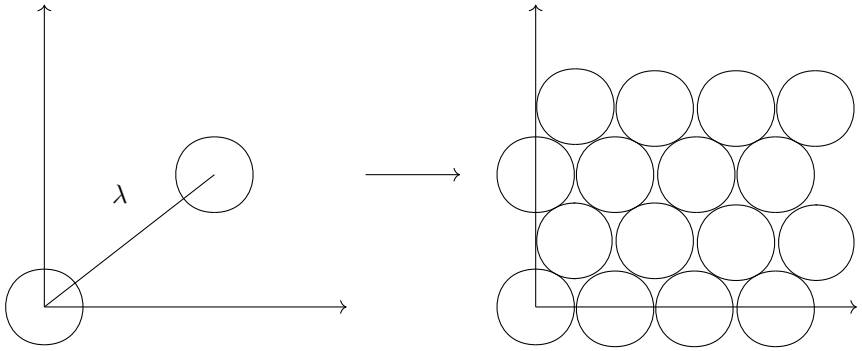
The completeness relation for this is:

$$\sum_n \int \frac{d\lambda d\lambda^*}{2\pi} |n\lambda\rangle \langle n\lambda| = 1 \tag{2.23}$$

The wavefunction in the coherent state basis looks like the following:

$$\varphi_{n\lambda}(z, \bar{z}) \sim e^{-\frac{|\lambda|^2}{2}} \left(\partial_z + \frac{i\bar{z}}{2\ell^2}\right)^n e^{-\frac{z\bar{z}}{4\ell^2} + \frac{i}{2\ell} z\lambda} \tag{2.24}$$

For $n = 0$, this is just a Gaussian shifted by a constant, where λ can be thought of as the displacement of the Gaussian from the origin. This is known as a magnetic translation in the literature. So each electron roughly lies in a circle radius ℓ, which gives each an area of $\sim \pi\ell^2$.



Total area $= 2\pi l^2$.

The factor of $e^{\lambda b^\dagger - \lambda^* b}$ is basically $e^{i\mathbf{k}\cdot\mathbf{x}}$. Let's try to see this.

$$e^{i\mathbf{k}\cdot\mathbf{x}} = e^{i(\bar{k}z + k\bar{z})} = e^{\bar{k}\ell(a^\dagger - b) - k\ell(a - b^\dagger)}, \text{ where we used } z = i\ell(a - b^\dagger), \quad \bar{z} = -i\ell(a^\dagger - b) \tag{2.25}$$

Using the BCH formula, we can rewrite this as:

$$e^{i\mathbf{k}\cdot\mathbf{x}} \sim e^{-kla^\dagger} e^{-\bar{k}\ell b^\dagger + k\ell b} e^{k\ell a} \tag{2.26}$$

Comparing this to the coherent state, we can see that $\lambda = -\ell\bar{k}$.

Lowest Landau Level Projection

The lowest Landau level projection can sometimes be very ugly, which is a technical challenge for the field as we often have to work in the lowest Landau level. The projection operator can be written in the following equivalent ways:

$$\sum_m |0m\rangle \langle 0m| = \mathcal{P}_{LLL} \tag{2.27}$$

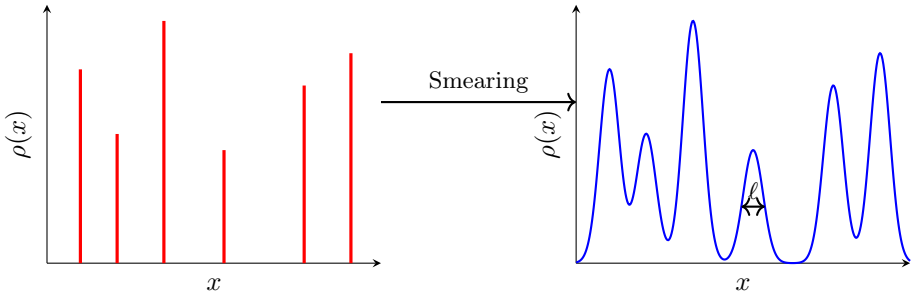
$$\int \frac{d\lambda d\lambda^*}{2\pi} |0\lambda\rangle \langle 0\lambda| = \mathcal{P}_{LLL} \tag{2.28}$$

Now say you have some quantum state that you want to project to the lowest Landau level, you can do so by the following:

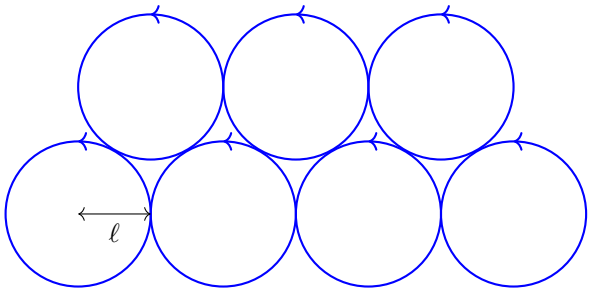
$$\psi_{LLL}(\mathbf{x}) = \hat{\mathcal{P}}_{LLL}\psi(\mathbf{y}) = \int d^2\mathbf{y}\mathcal{P}_{LLL}(\mathbf{x}-\mathbf{y})\psi(\mathbf{y}) \tag{2.29}$$

$$\mathcal{P}_{LLL}(\mathbf{x},\mathbf{y}) = \langle \mathbf{x}| \mathcal{P}_{LLL} |\mathbf{y}\rangle = \frac{1}{2\pi\ell^2} e^{\frac{(\mathbf{x}-\mathbf{y})^2}{4\ell^2} - \frac{i}{2\ell^2}\mathbf{x}\times\mathbf{y}} \tag{2.30}$$

So the projection operator acts in a similar way to a propagator, and essentially tends to smear out a wavefunction by a Gaussian [5]. For example, if the wavefunction was simply a Dirac delta function, it projects down to a Gaussian, which is precisely what we observe the wavefunctions to look like in the lowest Landau level. Physically, this can be thought



of as a manifestation of the resolution we have with these quantum hall states. We cannot "zoom in" to a length scale smaller than that of the magnetic length ℓ. Thinking back to the coherent state or classical picture, you cannot shrink the resolution to smaller than the area in which the particles are rotating.



Cyclotron Orbits with Magnetic Length ℓ

The other way one can think of this in the second quantized formula is that the anti-commutation relation of

$$\{\psi(\mathbf{x}), \psi(\mathbf{y})\} = \delta^{(2)}(\mathbf{x}-\mathbf{y}), \tag{2.31}$$

where the particles that are created are point-like gets projected to the following relation:

$$\{\psi_{LLL}(\mathbf{x}), \psi_{LLL}(\mathbf{y})\} \sim e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{4\ell^2}} \tag{2.32}$$

So, in the lowest Landau level the particles created are Gaussian-like instead or point-like. The intuition behind all of this is that in the lowest Landau level everything is controlled by this magnetic length ℓ.

Hall conductivity and the Many-Body Wavefunction

In 1980 people discovered the quantum Hall effect. Classically, the Hall conductivity for a sample of area A with number of electrons N_{el} is given by

$$\sigma_{xy} = \frac{N_{el}}{A}\frac{e}{B} \tag{2.33}$$

From this, you would expect a straight line with slope $\frac{A}{eN_{el}}$ when plotting resistivity (σ_{xy}^{-1}) against B, but this is not what we see in experiments. Instead, we see plateaus at certain values of the Hall conductivity, which are given by:

$$\sigma_H = \frac{e^2}{h}\nu \tag{2.34}$$

Where ν is the filling fraction. The plateaus are quantised, and the value of ν can be any rational number, which is why this is known as the quantum Hall effect. The plateaus are also characterised by regions of very low ohmic resistivity (ρ_{xx}), and the edges of these plateaus can be identified by spikes in ρ_{xx}, as shown below. above

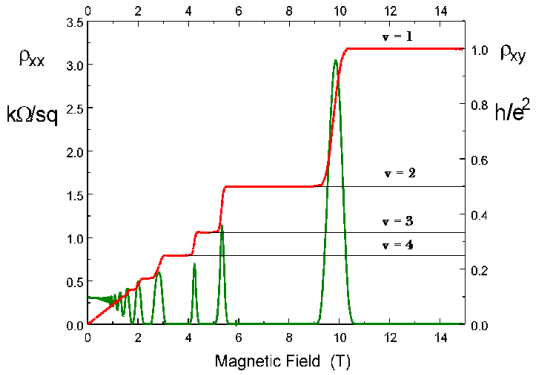
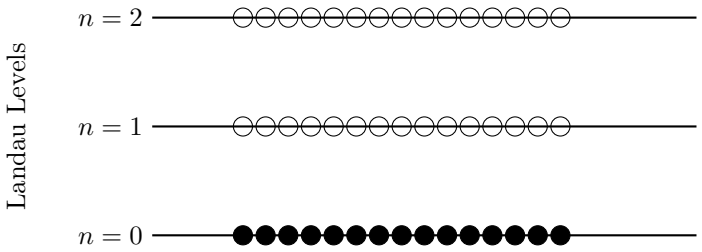
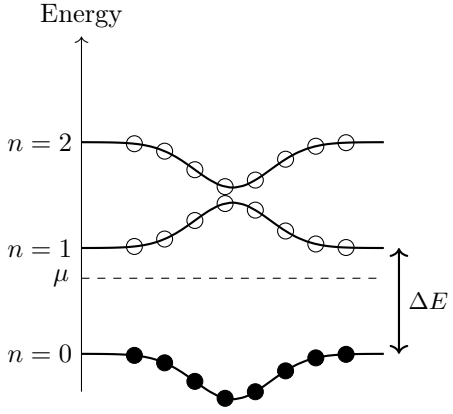


Figure 2: Plot of resistivity against magnetic field strength [21]

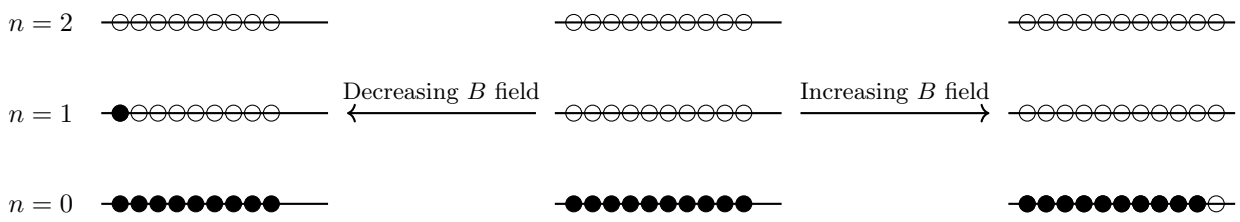
In the Landau level picture, $\sigma_{xx} = 0$ corresponds to fully filled Landau levels, as there are no states for the electrons to scatter to.



This is similar to the band theory picture, where gapped and gapless states are determined by the value of the chemical potential μ. In the picture below, σ_{xx} would also be zero as there is a gap there. Landau levels can therefore be thought of as flat bands.



As mentioned earlier, the filling fraction $\nu = 1$ corresponds to 1 fully filled Landau level, $\nu = 2$ corresponds to 2 fully filled Landau levels and so on. Each of these filling fractions have an associated plateau as shown above. But one would assume that this value should only correspond to a particular B-field, and that changing the B-field would deviate the Hall conductivity from this specific value. In addition to this, changing the B-field is mathematically equivalent to changing the electron density and hence the number of electrons (for a fixed area of a sample), which means that after a change in B you would expect your Landau level to no longer be filled, hence allowing for there to be a non-zero value for σ_{xx}. This is a result of the localisation of these extra states around impurities in the sample.



Now let's look at the fully filled lowest Landau level. We have the following single-particle wavefunctions in the symmetric gauge [6]:

$$\varphi_{00}(z) = e^{-\frac{z\bar{z}}{4\ell^2}} \tag{2.35}$$

$$\varphi_{01}(z) = ze^{-\frac{z\bar{z}}{4\ell^2}} \tag{2.36}$$

$$\varphi_{02}(z) = z^2 e^{-\frac{z\bar{z}}{4\ell^2}} \tag{2.37}$$

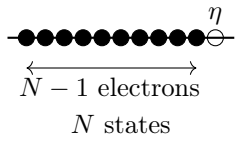
$$\vdots$$

$$\varphi_{0N-1}(z) = z^{N-1} e^{-\frac{z\bar{z}}{4\ell^2}} \tag{2.38}$$

We use what is known as the Slater determinant to antisymmetrize over all of these to obtain the many-body wavefunction. The many-body wavefunction of the Integer quantum Hall fluid in the lowest Landau Level is given by the following:

$$\Psi_{\nu=1}(z_1, \ldots, z_N) = \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_{N-1} \\ z_1^2 & z_2^2 & \ldots & z_{N-1}^2 \\ \vdots & \vdots & & \vdots \\ z_1^{N-1} & z_2^{N-1} & \ldots & z_{N-1}^{N-1} \end{pmatrix} e^{-\frac{1}{4\ell^2}\sum_i |z_i|^2} = \prod_{i<j}^{N-1} (z_i - z_j) e^{-\frac{1}{4\ell^2}\sum_i |z_i|^2} \tag{2.39}$$

It should be noted that this wavefunction is also translationally invariant. Now we want to consider what the wavefunction looks like if we introduce an empty state η like below.



η is called a hole here, like in band theory. It acts like a particle, and hence the wavefunction looks like the following:

$$\Psi_\eta(z_1, \ldots, z_N) = \left[\prod_{i=0}^{N-1} (z_i - \eta)\right] \left[\prod_{i<j}^{N-1} (z_i - z_j)\right] e^{-\frac{1}{4\ell^2}\sum_i |z_i|^2} \tag{2.40}$$

Going back to $\Psi_{\nu=1}$ state, we wish to find the density of this wavefunction:

$$\langle \Psi_{\nu=1} | \rho(\mathbf{x}) | \Psi_{\nu=1} \rangle \tag{2.41}$$

The easiest way to get this is by looking at the second quantisation picture:

$$\rho(\mathbf{x}) |\Psi_{\nu=1}\rangle = \psi^\dagger(\mathbf{x})\psi(\mathbf{x}) |\Psi_{\nu=1}\rangle = \psi^\dagger(\mathbf{x})\psi(\mathbf{x}) \int d\mathbf{x}_1 \cdots d\mathbf{x}_N \; \Psi_{\nu=1}(\mathbf{x}_1, \ldots, \mathbf{x}_N)\psi^\dagger(\mathbf{x}_1) \cdots \psi^\dagger(\mathbf{x}_N) |0\rangle \tag{2.42}$$

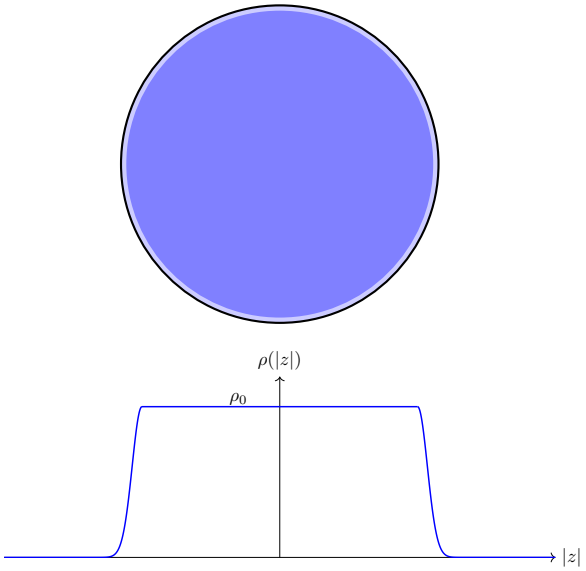
Using the anticommutation relations for the fermionic creation and annihilation operators,

$$\{\psi(x), \psi^\dagger(x')\} = \delta(x - x'), \tag{2.43}$$

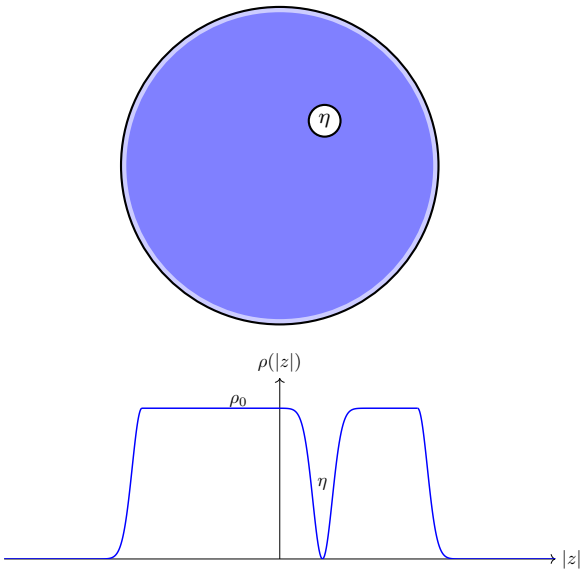
we can move the annihilation operator through the creation operators within the integral, each time picking up a delta-function term, until it annihilates the Fock vacuum at the end of the expression. Due to the total antisymmetrization of the wavefunction, this reduces down to the following (up to a factor):

$$\langle \Psi_{\nu=1} | \, \rho(\mathbf{x}) \, | \Psi_{\nu=1} \rangle \sim \int d\mathbf{x_2} \cdots d\mathbf{x_N} \, |\Psi(\mathbf{x}, \mathbf{x}_2, \ldots, \mathbf{x}_N)|^2 = \text{const.} \tag{2.44}$$

Since this is separable, the integrals over the other variables are just 1, and one is left with a constant density $\rho = \frac{N_{el}}{A}$, i.e. it is a uniform liquid. This makes sense given that the system is translationally invariant, so from a symmetry argument it must be of constant density. Thinking about the fully filled Landau level, it's just a uniform liquid disc.



Now if we have a hole like before, one can visualise this as having a constant density away from the hole, with a dip of width $\sim \ell$ at the position of the hole η. This is an example of a soliton or a vortex in the liquid. This hole can also move around the liquid freely.



Like was mentioned earlier, one way of creating a hole state is by threading one magnetic flux quantum through the sample, since $N = \frac{\Phi_{mag}}{\Phi_0}$ this will create one more empty state.

Disorder in the sample

QHE experiments are often performed using semiconductors such as GaAs$-$Ga$_x$ Al$_{1-x}$ As[23]. Contrary to possibly pre-held rationale regarding the ideal levels of impurities in materials found after undergoing a canonical condensed matter course or working in high temperature superconductivity, for example. In particular, the notion that experimentalists should endeavour to generate as pristine sample as is possible, though with some notably interesting physics coming from the inclusion of defects in these samples[3]. However, in the context of the quantum Hall effect we, in fact, require a relatively large number of impurities in our sample to be able to measure plateaux at all. We can model these impurities through some potential $V(z)$. A natural example to use could be a set of repulsive (attractive) delta functions $V(z) = \pm \sum_i v_i \delta(z - z_i)$. In a similar picture to the Zeeman splitting of the Hydrogen, we want to induce this symmetry breaking to split the *very* degenerate states found in each Landau level. But we don't want these splittings to cause mixing of the Landau levels since the premise of these plateuax to begin with require a large gap between levels. Looking at (1.110) we note the difference between Landau levels to be ω_c and hence we want $V \ll \omega_C$.

The density of states for the system with no disorder should look like a collection of straight lines horizontally with vertical gap ω_c in the bulk but will close together in the edge. This will spread out the density of states into more Gaussian-esque shapes. The states available on either side of these peaks will allow for the plateaux to be of finite width. We can vary the flux being threaded through the sample, say by decreasing $\mathbf{B}$ - or we could equivalently increase the chemical potential. Thereby reducing the number of free states for electrons to go into or increasing the number of electrons respectively. If there were no disorder, then the next available states for these electrons would be the next Landau level. However, the existence of impurities acting as attractive potentials gives the electrons some place to exist - pinned down to these impurities - in such a way as to not upset the dynamics of the rest of the quantum Hall fluid.

Indeed, Girvin uses an intuitive analogy to sea level[7] where we view the system as a basin which would be flat if not for any impurities. The introduction of impurities, both attractive and repulsive, that are more smooth than the earlier discussed delta functions can be viewed as mountains and chasms in this basin. Recalling that the chemical potential can be effectively thought of as the voltage across our sample, we can raise and lower the chemical potential at will. This sea in this analogy represents electrons. When the sea level is such that there is no "water" that exists from edge-to-edge i.e. all of the water is localised to the troughs then all of our electrons are localised to the defects. Since the water is contained and unable to reach from one side to the other, the electron gas is unable to pass a current and is in an insulating state. But for every trough $\exists$ a sea level - chemical potential - for which the shoreline doesn't hold in the water like a lake any more but switches to the edges of the sample. Once this happens and the water can carry a current (both in the sense of waterflow and of particle/hole excitations) and we've reached a plateau transition where the conductivity σ_{xx} is non-zero - called an extended state. As we increase this further, though, we return to only adding to the localised states until we once again phase transition from a relatively insulating phase to a very brief conducting phase and then back to insulating.

This picture, detailing the addition of *electrons* to localised states that don't effect the conductivity is the basis of the plateaux seen in the *Integer* quantum Hall effect. A very similar picture is painted in the case of the *Fractional* quantum Hall effect but this will be discussed later. We also mention the duality between the edge states and the localised states here too which shall also be further explained when discussing edge states.

The Plasma Analogue

Let's consider the quantum Hall liquid in the bulk and away from any impurities - which are very small when compared to overall scale of the sample of $N \gg 1$ electrons. It is clear then that we have a translation invariance and so we should expect a uniform density. But, to be rigourous, we shall investigate this further. Defining density operator,

$$\rho(z) = \overbrace{\psi^\dagger(z)\psi(z)}^{\text{second quantized}} \equiv \overbrace{\sum_{i=1}^{N} \delta^{(2)}(z - z_i)}^{\text{first quantized}} \tag{2.45}$$

Then, using non-normalised wavefunction since we have been pretty lax with ensuring normalisation, we can write the expectation value as

$$\langle \rho \rangle = \frac{\langle \psi | \rho | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\int \mathrm{d}\underline{z}\, |\psi(\{z_i\}|\rho(z)}{\int \mathrm{d}\underline{z}\, |\psi(\{z_i\}|} \tag{2.46}$$

[3]The likes of p- or n-type semiconductors being an example.

Where the integration measure $\mathrm{d}\underline{z} \coloneqq \prod_i \mathrm{d}^2 z_i$. As theorists, our first instinct may be to use the delta functions to integrate out all but one of the integration variables and then perform the integration analytically. But given the complexity of the Laughlin wavefunction (3.3) we wish you good luck in that endeavour. So, as computational theoretical physicists our next instinct may be to numerically perform the integral. For a dozen of particles you may be able to get a reasonable answer in reasonable time but given that a finite sample as a lot more particles than a dozen particles then there's little hope in completing a computation in your lifetime. So that leaves us the only option of being quite clever to figure it out. Indeed, Laughlin was that clever. We notice that this has the form of a quantity in statistical physics if we enforce

$$\begin{aligned} \mathcal{Z} &= \int \mathrm{d}\underline{z}\, e^{-\beta \mathcal{U}} \\ -\beta \mathcal{U} &\coloneqq \ln |\psi|^2 \end{aligned} \tag{2.47}$$

Then recalling the precise form of the Laughin wavefunction we get that

$$-\beta \mathcal{U} = 2p \sum_{i<j} \ln |z_i - z_j| - \frac{1}{2l^2} \sum_i |z_i|^2 \tag{2.48}$$

Which, if we identify $\beta \coloneqq 2/p$ - for a reason we shall come back to - we can rewrite as

$$\mathcal{U} = -p^2 \sum_{i<j} \ln |z_i - z_j| + \frac{p}{4l^2} \sum_i |z_i|^2 \tag{2.49}$$

Now, this very well remind you of something. Especially if you're fresh on your electrostatics. If not, we briefly discuss in C.

Now, using (C.4) we can see that the first term in the energy $\mathcal{U}$ carries precisely the same form we would expect from a $2d$ system of point charges with charge p. This should give us some indication that we can draw an analogy between the Laughlin wavefunction and the sphere of physics including electromagnetism. Indeed, we find this in the form of the *plasma analogue*. Here, we map the Laughlin wavefunction to this $\beta\mathcal{U}$ pair and hence build a fictitious plasmatic system governed by this energy. Now the question is how we should interpret the second term. This is fairly simple - it's just the energy due to a constant background charge. We can see that by yet again taking the laplacian and getting that

$$-\partial^2 \left(\frac{|z_i|^2}{4l^2} \right) = -1/l^2, \qquad -\partial^2 \phi = 2\pi\rho \tag{2.50}$$

$$\rho = -\frac{1}{2\pi l^2} \tag{2.51}$$

Where ρ is the charge that any of the fictitious electrons feel due to the background. We see now that our identification of β is precisely what allows us to attribute relative positive and negative charges. Over a large scale the point charge contributions will smear out and result in an overall neutral homogeneous plasma. Then, since the overall charge must be null and each fictitious electron has charge p in this construction then if we replace ρ by an equivalent np - where n is the number density - then we retrieve

$$n = \frac{1}{2\pi l^2\, p} \tag{2.52}$$

Which is precisely the density in the $\nu = 1/p$ state. It's also interesting to note that this system with the fictitious electron charge of $p \in \mathbb{Z}^+$ is effectively the same as saying that each electron takes up not just one state but p states. This notion is something we will cover later and is known as a *Composite Fermion* and will be more thoroughly discussed in 3.5.

3 Fractional Quantum Hall

Historically, the first observed fractional quantum Hall state is the $\nu = \frac{1}{3}$ state, which has the longest plateau and is the most stable among all of the fractional quantum Hall states also. Other stable states that were observed early on looked like the following:

$$\nu = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \ldots \to \frac{1}{2} \tag{3.1}$$

This series can be characterised by $\nu = \frac{p}{2p+1}$ [22]. There is also always a correpsonding series on the other side of $\nu = \frac{1}{2}$ which goes as:

$$\nu = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \ldots \to \frac{1}{2} \tag{3.2}$$

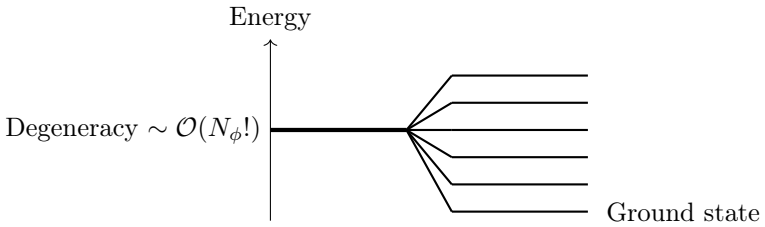
Now, if one wants to see what happens in higher Landau levels, one need just to add an integer, i.e. the corresponding state for $\nu = \frac{1}{3}$ on the first Landau level is $\nu = 1 + \frac{1}{3} = \frac{4}{3}$. So, in some sense all of these higher states can be mapped back to the lowest Landau level, and indeed many of these states share properties across different Landau levels. However, some states like $\nu = \frac{5}{2}$ or $\nu = \frac{12}{5}$ are not quite the same as the lowest Landau level states. To study the fractional quantum Hall effect, we will start with the simplest type of states, the $\frac{1}{p}$ (for odd p) state, also known as the Laughlin state.

The Laughlin Wavefunction

Let's take the $\nu = \frac{1}{3}$ state as an example, the Landau level will look something like the following:

$N_\phi = 15$, $N = 5$ filled states

From looking at this picture, it seems as though there is a large degeneracy of states with the same energy, which is a problem because any state can scatter to many other states. This would imply that $\sigma_{xx} \neq 0$, and the state is gapless or metallic in nature. A gapped state is equivalent to having a ground state with no degeneracy. Experiments tell us that this state is gapped, so how is the degeneracy lifted? In fact, it is lifted by the electron-electron interaction. In the lowest Landau level, this interaction is repulsive, so naturally the electrons will wish to spread out as much as possible. This is then essentially a 2D Fermi liquid now.



Now it remains to find out what this ground state or so called Laughlin wavefunction actually looks like. It turns out that the wavefunction of the IQH fluid is actually a special case of the Laughlin wavefunction. The Laughlin wavefunction [17] is given by:

$$\Psi_L(z_1, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^p e^{-\frac{1}{4\ell^2} \sum_i |z_i|^2} \tag{3.3}$$

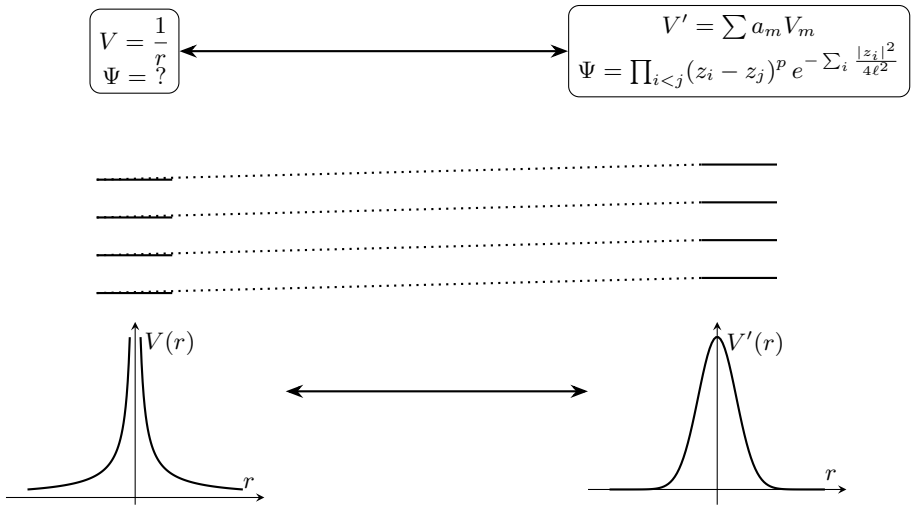
Where p is a positive odd integer. The IQH fluid is just the case where $p = 1$. In actual fact, this is not analytically the exact ground state, but it is a very good approximation and has a high overlap with the numerical solution for the low energy case. Since the Hamiltonian for the lowest Landau level is just the interaction Hamiltonian (up to a constant), we can say that the Hamiltonian here is given by

$$H_{LLL} = \sum_{i<j} \frac{1}{|z_i - z_j|} = V \tag{3.4}$$

The Laughlin wavefunction as mentioned is not the true ground state solution for this Hamiltonian. However, Haldane came up with a way of finding a series of Hamiltonians for which this is the ground state.

$$V' = \sum_{m=0} a_m \langle 0m| V |0m\rangle = \sum_{m=0} a_m V_m \tag{3.5}$$

This kind of technique is known as Haldane pseudopotential [10]. The justification for using this as the ground state of the Coulomb potential is a notion known as adiabaticity. This is the notion that if one can continuously deform one potential into the other, then one wavefunction should continuously deform into the other one. Here, one must consider the entire spectrum of the Hamiltonians, ensuring that there is not a "level-crossing" for energy levels. The idea here is that if the pseudopotential has the same qualitative properties as the actual potential, then the solutions to it should also have the same qualitative properties as the analytical solution. This justification enables us to study the Laughlin wavefunction.



The highest power of z_i for the Laughlin state will be $pN - 1$. Physically, what does this mean? Consider what happens as one electron approaches the other. For $\nu = 1$, the wavefunction will go to zero linearly, since there are factors of $(z_i - z_j)$ in the wavefunction. However, for $\nu = \frac{1}{3}$, the factors are $(z_i - z_j)^3$, which means that the wavefunction goes to zero in a cubic nature. Going back to the symmetric gauge picture, this also relates to electrons taking up only every third ring in the 2D diagram. This matches the intuition from classical knowledge of the Couloumb potential, as electrons will want to spread out as much as possible, so occupying every third state allows them to maximise their collective distance from each other and hence minimise the force they feel from the Coulomb potential.

Charged Excitations

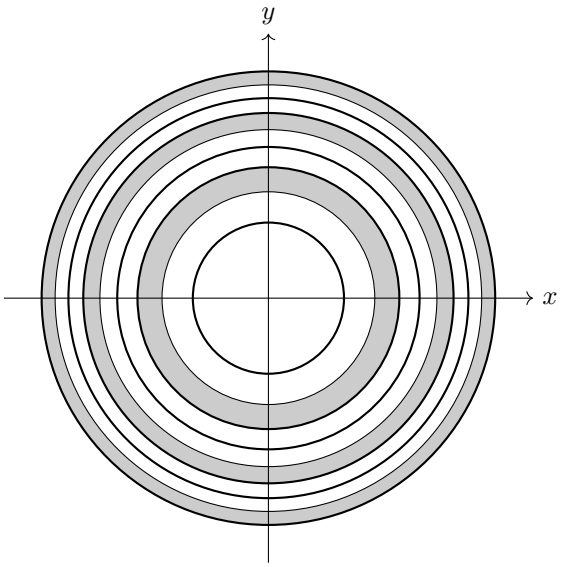
We can now look towards building up this notion of composite fermions. Firstly, let's just look at the notion of charge excitations on the ground state of the system. For a $\nu = 1/p$ state, let's define the ground state $\left|\Psi_{1/p}\right\rangle$ such that

$$\Psi_{1/p}(\{z_k\}) = \left\langle \{z_k\} \middle| \Psi_{1/p} \right\rangle = \prod_{i<j} (z_i - z_j)^p \exp\left\{ -\sum_{i=1}^{N} \frac{|z_i|^2}{4l^2} \right\} \tag{3.6}$$

Looking at the formalism of second quantization to elicit changes in excitations. In particular, we can view a simple example of a neutral excitation in the form of the density operator $\rho(\mathbf{z}) = \psi^\dagger(\mathbf{z})\psi(\mathbf{z})$ in the position basis or, via Fourier Transform, in the momentum basis

$$\rho_k = \overbrace{\psi^\dagger_{k+p}}^{\text{particle}} \underbrace{\psi_p}_{\text{hole}} \tag{3.7}$$

In this context, particles and holes are both charged but oppositely. For the most part, the charged excitations are the most interesting ones in the context of the QHE. Then the question becomes what is the relevant operator associated with an excitation we wish to study.



By threading one flux quantum through the centre of our system then i.e. $N_\Phi \mapsto N_\Phi + 1$, then we add another empty state in the middle and push the entire fluid out a bit more. But since our state will spread out one electron charge amongst p states - in this case $p = 3$ - then this addition of a state won't carry with it the lack of one charge but rather the lack of $1/3$ charges. Recall though that in this case we really consider a hole to have positive charge since we have our quantum Hall fluid as a baseline. So what we've really done here is add in a *quasi-hole* with *fractional charge* which we'll see leads to *fractional statistics* and hence anyons. We can continue to thread more fluxes and the holes can appear in separate places each carrying with them a fractional charge. But if we place them all together then they almost "merge" into an electron hole with positive unit charge. This is not dissimilar to the fractional charge associated with quarks and hence the hadronisation seen in joining multiple valence quarks into a composite particle with integer charge[9].

In looking for the operator associated with these quasi-holes then we can return to the Laughlin wavefunction. We know that the wavefunction has to go to zero as any z_i approaches the location of a quasi-hole, say η. Since a quasi-hole should be thought of on a similar footing as a real electron we can make a decent guess that if we just add in a factor of $(z_- - \eta)\,\forall\, z_i$ we'd get an accurate wavefunction which we do. That is to say, we have the new wavefunction

$$\Psi_\eta(\{z_k\}) = \prod_{l=1}^{N}(z_l - \eta)\prod_{i<j}(z_i - z_j)^p \exp\left\{-\sum_{i=1}^{N}\frac{|z_i|^2}{4l^2}\right\} \tag{3.8}$$

Notice the power difference in the products demonstrating the fractional charge associated with the quasi-hole. The additional power of each z_i can also be seen as spreading out the overall wavefunction which, again, makes sense in the physical picture we're dealing with. Then we can clearly see that the operator associated with the creation of a quasi-hole

$$|qh(\eta)\rangle = \prod_{i=1}^{N}(z_i - \eta)\,|\Psi_{1/p}\rangle \tag{3.9}$$

One can also see the fractional charge associated with the quasi-hole through the power difference in the context of the Plasma Analogue 2.5 where an additional logarithmic term is added with dependence on η which is explained in more detail in [22].

An alternative, and slightly hand-waving, way of looking at the conductivity and fractional charge goes as follows. We can argue based on the inability of electrons to scatter and produce a current parallel to an electric field that the only contribution to the current density in the x direction will be the scaled y component of the electric field i.e $\mathbf{J}^x = \sigma_{xy}\mathbf{E}^y$. By integrating we can convert the current density to current and electric field to a potential and get that $I = \sigma_{xy}\varepsilon$. Using Faraday's law of induction coupled with the definition of the current as $I = \frac{\mathrm{d}q}{\mathrm{d}t}$ we get the relation that $\mathrm{d}q = -\sigma_{xy}\,\mathrm{d}\Phi$. But in our language then the change in flux is, up to a constant multiplying, just the change in the number of fluxes

and hence $\mathrm{d}q \propto -\sigma_{xy}\,\mathrm{d}N_\Phi$. This tells us two things. The first is that by threading a magnetic flux the charge associated will go down - this is equivalent to the charges spreading out in the picture. The second is that, when normalised by an IQHE state for example, we can directly see that since the lack of charge associated is only $1/p$ then $\sigma_{xy} \propto 1/p$ just as expected.

Now that we've aptly covered quasi-holes, we can look at quasi-particles. The central difference in this case is that we remove a quantum of flux and hence we get a surplus charge. So if we're just conjugating charge by changing from quasi-hole to quasi-particles then it's a good guess that we can just complex cojugate the quasi-hole wavefunction[4]. In which case we get a guess that goes as

$$|qp(\eta)\rangle = \prod_i (\bar{z}_i - \bar{\eta})\,|\Psi_{1/p}\rangle \tag{3.10}$$

However, as we can see from (1.111), having this $\bar{z}$ term will send the state to the first Landau level instead of the lowest Landau level we want to work in for a $\nu = 1/p$ state. Therefore, we want to project onto the lowest Landau level. This is done by just removing the $a^\dagger$ term thereby effectively setting it to zero. Doing this will we can see that

$$0 = -i\sqrt{2}\left(l\frac{\partial}{\partial z} - \frac{\bar{z}}{4l}\right) \implies \bar{z} = 4l^2\frac{\partial}{\partial z} \tag{3.11}$$

We then split up the normal $\bar{z}_k$ into $\bar{z}_k/2 + 2l^2\partial_k$.

$$\prod_k \left(\left(2l^2\frac{\partial}{\partial z_k} + \frac{\bar{z}_k}{2}\right) - \bar{\eta}\right)\prod_{i<j}(z_i - z_j)^p \exp\left\{-\sum_{i=1}^N \frac{|z_i|^2}{4l^2}\right\} \tag{3.12}$$

If we split each ∂_k derivative up into its effect on the polynomial and exponential then we can see that the exponential will bring down precisely a factor of $-\frac{\bar{z}_k}{4l^2}$ which will beautifully cancel the extra term we have. Therefore the quasi-particle wavefunction is simply

$$\Psi_\eta^{qp}(\{z_k\}) = \langle\{z_k\}|qp(\eta)\rangle = \left[\prod_{l=1}^N 2l^2\left(\frac{\partial}{\partial z_k} - \bar{\eta}\right)\prod_{i<j}(z_i - z_j)^p\right]\exp\left\{-\sum_{i=1}^N \frac{|z_i|^2}{4l^2}\right\} \tag{3.13}$$

Where the derivative only acts within the square bracket.

Finally, we can ask the question "Well what if I have more than one quasi-particle or quasi-hole? What will my wavefunction look like then?" to which we reply that for a system with N electrons where we add M quasi-holes and then K quasi-particles then the wavefunction is simply

$$\Psi\Big(\{\eta_a\},\{\zeta_b\};\{z_c\}\Big) = \left[\prod_{i=1}^N\prod_{j=1}^K\left(2l^2\frac{\partial}{\partial z_i} - \bar{\zeta}_j\right)\prod_{k=1}^M(z_i - \eta_k)\right]\prod_{i<j}(z_i - z_j)^p\exp\left\{-\sum_{i=1}^N\frac{|z_i|^2}{4l^2}\right\} \tag{3.14}$$

But What Are Vortices?

Simply, vortices are objects that behave in particular ways when we rotate a particle - whose position is denoted by z - around the vortex at the origin then it picks up a phase factor of $e^{i\arg z}$. But the arg function is not determined at $z = 0$ then we need the wavefunction to really go as $f(|z|)e^{i\arg z}$ such that $f(0) = 0$. This is precisely what we see in the Laughlin wavefunction. However, when we do the same thing dragging one vortex around another we find that the phase factor goes as $i\arg z/2p$ leading to fractional statistics.

Fractional Statistics

Fermions and Bosons

In our navigation towards the understanding of anyons via fractional statistics, a good place to set sail is the statistics associated with Fermions and Bosons. Let's consider a system of two *identical* particles with its state denoted by [5]

$$|ab\rangle = |a\rangle \otimes |b\rangle \tag{3.15}$$

[4] The justification for this can easily be seen through covariant derivative that would appear in the action: $D_\mu = \partial_\mu - ieA_\mu$. Here complex conjugation and charge conjugation are identical operations.

[5] We can also construct the same system in the second quantized formalism by replacing states by the operators creating a particle in different states.

Where one particle is in the state $|a\rangle$ and the other is in the state $|b\rangle$. Then we define the permuting operator $\mathcal{P}$ which swaps the particles between states i.e.

$$|ba\rangle = \mathcal{P}\,|ab\rangle \tag{3.16}$$

This can be seen visually in the space-time diagram.

Which means that if we interchange particles twice we find that

$$|ab\rangle = \mathcal{P}^2\,|ab\rangle \tag{3.17}$$

This tells us that the eigenvalue of the $\mathcal{P}^2$ operator is just one. So if $\Lambda = 1$ corresponds to $\mathcal{P}^2$, then λ corresponding to $\mathcal{P}$ is ± 1. So we can construct two linearly independent eigenvectors

$$|\pm\rangle = |ab\rangle \pm |ba\rangle \implies \mathcal{P}\,|\pm\rangle = |ba\rangle \pm |ab\rangle = \pm\,|\pm\rangle \tag{3.18}$$

We associate the $-$ states with Fermions which are anti-symmetric and the $+$ states with Bosons which are symmetric. In general, we build this state of two identical particles as

$$|\psi\rangle = |ab\rangle + e^{i\phi}\,|ba\rangle \tag{3.19}$$

where ϕ is the statistical angle of the system. Which for Fermions $\phi = \pi \mod 2\pi$ and Bosons $\phi = 0 \mod 2\pi$. We'll later see that for the relevant anyons we find $\phi = \pi/p$ but this will be discussed in full further on.

The Berry Phase

One of the most interesting applications of quantum mechanics to magnetic field dynamics is the *AharonovâĂŞBohm effect*. This tells us that when one takes a charged particle along a closed path C in a region with zero magnetic field yet non-zero vector potential $\mathbf{A}$, we pick up a phase proportional to the enclosed magnetic flux. However, in this setup the vector potential is really just some parameter of the system. So then we can attempt to generalise this notion to other parameters too. Let's consider a normalised state parameterised by a set $\{\eta_i\}$ defined by $|\eta\rangle$ with $\langle\eta|\eta\rangle = 1$. Given that we're still looking at the $U(1)$ group of transformation since we just want to look at phase differences in the change of parameters, we should see if we can draw some more concrete connections to electromagnetism. In particular, let's suppose that the phase that would normally be associated with a gauge transformation $U = e^{i\phi}$ is now associated with the state itself $|\eta\rangle$. In the electromagnetism context we know that the vector potential is found by

$$\mathbf{A} = -i\,U^\dagger \partial\, U = -iU^\dagger U\,(i\partial\phi) = \partial\phi \tag{3.20}$$

Then this would become

$$\mathcal{A}_i = -i\,\langle\eta|\frac{\partial}{\partial\eta^i}|\eta\rangle \tag{3.21}$$

This is known as the **Berry connection**. We can also construct a relevant "magnetic field" as

$$\mathcal{B}_i = \epsilon_{ijk}\frac{\partial}{\partial\eta^j}\mathcal{A}_k \tag{3.22}$$

Though it's very important to note that this is not a physical magnetic field. It's just a mathematical construction named because of its duality to electromagnetism. Nothing more. Now we can further define the **Berry curvature** which has the same form as the familiar field-strength tensor.

$$\mathcal{F}_{ij} = \frac{\partial}{\partial\eta^{[i}}\mathcal{A}_{j]} \tag{3.23}$$

Notice that we've *cleverly* chosen for our parameters to be the set of η_i. Let's now consider these as two parameters that locate the position of the quasi-hole. But since we're working in $d = 2$ then there's only one independent parameter since F_{ij} has four components but is anti-symmetric.

$$\mathcal{F}_{xy} = \frac{\partial\mathcal{A}_y}{\partial\eta^x} - \frac{\partial\mathcal{A}_x}{\partial\eta^y} = i\left[\frac{\partial}{\partial\eta^y}\left\langle\eta\middle|\frac{\partial\eta}{\partial\eta^x}\right\rangle - \frac{\partial}{\partial\eta^x}\left\langle\eta\middle|\frac{\partial\eta}{\partial\eta_y}\right\rangle\right] \tag{3.24}$$

We could also consider the parameter to be in momentum space. This forms the basis of the Anomalous quantum Hall effect which occurs in the momentum space and without an external magnetic field.

Sticking to our parameters being the location of the quasi-hole we find that the associated phase change i.e. the statistical phase, is given - as in electromagnetism - by

$$\theta = \oint_C \mathcal{A}_i \, \mathrm{d}\eta^i \tag{3.25}$$

As one could imagine, actually going about calculating the Berry connection can be a quite involved procedure. So we should try our best to simplify it as much as possible. Recall above where I said that the ket in the calculation must be normalised - well the Laughlin wavefunction $\langle z_i | \eta_i \rangle$ we use is very clearly not normalised itself. So we shall then define a new ket which is normalised

$$|\psi\rangle = \frac{1}{\sqrt{Z}} |\eta_i\rangle \tag{3.26}$$

$$1 = \langle \psi | \psi \rangle = \frac{1}{Z} \langle \eta | \eta \rangle \implies Z = \langle \eta | \eta \rangle \tag{3.27}$$

$$Z = \int \prod_i \mathrm{d}^2 z_i \exp\left\{ \sum_{j,k} \ln |z_j - \eta_k|^2 + p \sum_{l<m} \ln |z_l - z_m|^2 - \frac{1}{2l^2} \sum_n |z_n|^2 \right\} \tag{3.28}$$

This looks quite a bit like the partition function[6] we encountered in the plasma analogy. Except for a few physical points we'd need to consider. First and foremost is that in this setup the holes are to be seen as a charge impurity.[7] So we should treat the impurities with positions η_i in dual to the electrons with position z_j with charges of unity and p respectively. Leveraging the plasma analogy more explicitly in a fluid state, then as long as impurities are sufficiently far apart then electrons will swarm around them. The large distance between impurities justifies our want to look at an impurity in isolation with a number of electrons. These electrons - having opposite charge to the impurities - will swarm around the impurities. So now we have a positive charge surrounded by a roughly equal amount of negative charges[8]. If we take a step back then it starts to become a little bit difficult to distinguish an electron and an impurity. Taking a few more steps back we wouldn't be able to see the difference even if we squinted. So we take out our trusty *electric-field-ometer* and see that the field generated by this multi-electron-impurity system matches an electric field of an overall weaker, even nigh neutral, charge. This is a concept known as *screening*. This entire setup will have energy along the lines of

$$U(z_i; \eta_j) = -\sum_{a<b} \ln \left| \frac{\eta_a - \eta_b}{l} \right|^2 - p \sum_{j,k} \ln \left| \frac{z_j - \eta_k}{l} \right|^2 - p^2 \sum_{l<m} \ln \left| \frac{z_l - z_m}{l} \right|^2 + \frac{p}{4l^2} \sum_n |z_n|^2 + \frac{1}{4l^2} \sum_q |\eta_q|^2 \tag{3.29}$$

Then we can define the *plasma partition function* as

$$\Xi = \int \prod_i \mathrm{d}^2 z_i \, e^{-\beta U} = \exp\left\{ \frac{1}{p} \sum_{a<b} \ln \left| \frac{\eta_a - \eta_b}{l} \right|^2 - \frac{1}{2pl^2} \sum_i |\eta_i|^2 \right\} Z \tag{3.30}$$

Through our screening argument, we see that each of the relatively few impurities (compared to the number of electrons) will be effectively nullified. Implying that, up to leading order, they won't feature heavily in our system-wide analysis. In particular, if we then compute the partition function of the entire fictitious system as seen in (3.30) then we won't have any reliance on the positions of the impurities whatsoever since they've all been counteracted by the electrons. You might ask, however, about the very *slight* caveat mentioned in the above paragraph about the plasma system being in a fluid state. Well what if it's not? That's an excellent question but only pertinent to filling factors $p \gtrsim 70$ [2] and so not entirely relevant for physical discussions.

Now we've argued that Ξ is a constant at least up to leading order let's replace it with Ξ_0 to indicate this. Then we can say that

$$Z = \Xi_0 \exp\left\{ -\frac{1}{p} \sum_{a<b} \ln \left| \frac{\eta_a - \eta_b}{l} \right|^2 + \frac{1}{2pl^2} \sum_i |\eta_i|^2 \right\} \tag{3.31}$$

[6] Technically we should have constant dividing within each log term for dimensional reasons but this constant we will leave implicit here.

[7] This is suboptimal language since impurities in the normal quantum Hall context often refers to static differences in the material itself. Which is the type of impurity that leads to the plateaux that we discussed previously.

[8] We should expect approximately equal amount since the system should be in equilibrium.

Moving back we can see that the Berry connection (3.21) for the impurity location - treated as a parameter - η can be rewritten in a simpler form thusly. Recall that $|\eta\rangle$ is entirely holomorphic in η_i[9] and hence $\langle\eta|$ is wholely anti-holomorphic. A neat implication here is then that

$$\langle\eta|\frac{\partial}{\partial\eta^i} \equiv \frac{\partial}{\partial\eta^i}\langle\eta| \tag{3.32}$$

Then we can change the order of operators and see that

$$\mathcal{A}_i = -i\langle\psi|\frac{\partial}{\partial\eta^i}|\psi\rangle = -i\frac{1}{\sqrt{Z}}\langle\eta|\frac{\partial}{\partial\eta^i}\frac{1}{\sqrt{Z}}|\eta\rangle = -i\frac{Z}{\sqrt{Z}}\frac{\partial}{\partial\eta^i}\frac{1}{\sqrt{Z}} - \frac{i}{Z}\langle\eta|\frac{\partial}{\partial\eta^i}|\eta\rangle \tag{3.33}$$

$$\mathcal{A}_i = \frac{i}{2Z}\frac{\partial}{\partial\eta^i}Z - \frac{i}{Z}\frac{\partial}{\partial\eta^i}Z = -\frac{i}{2}\frac{\partial}{\partial\eta^i}\ln Z \tag{3.34}$$

$$= -\frac{i}{2}\frac{\partial}{\partial\eta^i}\left[-\frac{1}{p}\sum_{a<b}\ln\left|\frac{\eta_a-\eta_b}{l}\right|^2 + \frac{1}{2pl^2}\sum_j|\eta_j|^2\right] \tag{3.35}$$

$$\implies \mathcal{A}_i = \frac{i}{2p}\sum_{a\neq i}\frac{1}{\eta_a-\eta_i} - \frac{i}{4pl^2}\bar{\eta}_i \tag{3.36}$$

Aside: Fractional Charge

Now that we've built up all the formalism of the Berry connection, we might as well use it for something a little off topic before moving onto the main event. Let's take our system and move one impurity in a closed loop C that does not enclose any other impurities. Well then by Cauchy's theorem the first term in $\mathcal{A}_i$ will be null and so we can just focus on the second. We could also have found the anti-holomorphic part but it should be clear by symmetry - except for the fact that the derivative only acts on the $1/\sqrt{Z}$ term - that we can skip this step here and notice that $\bar{\mathcal{A}}_i = i\eta/4pl^2$ and so the Berry phase for a single particle moving - hence dropping the subscript - is given by

$$\gamma \coloneqq \oint_C \mathcal{A}\,\mathrm{d}\bar{\eta} + \bar{\mathcal{A}}\,\mathrm{d}\eta = -\frac{i}{4pl^2}\oint_C \bar{\eta}\,\mathrm{d}\eta - \eta\,\mathrm{d}\bar{\eta} \tag{3.37}$$

Let's move from $\eta, \bar{\eta}$ to x, y coordinates briefly.

$$\bar{\eta}\,\mathrm{d}\eta - h.c. = 2i\,\mathrm{Im}\{\bar{\eta}\,\mathrm{d}\eta\} = 2i(x\,\mathrm{d}y - y\,\mathrm{d}x) \tag{3.38}$$

We can calculate this integral simply by defining the one form $\omega \coloneqq x\,\mathrm{d}y - y\,\mathrm{d}x$ with $C \equiv \partial\Omega$ and then seeing

$$\gamma = \frac{1}{2pl^2}\oint_C \omega = \frac{i}{2pl^2}\int_\Omega \mathrm{d}\omega \tag{3.39}$$

$$\mathrm{d}\omega = \mathrm{d}x\wedge\mathrm{d}y - \mathrm{d}y\wedge\mathrm{d}x = 2\,\mathrm{d}x\wedge\mathrm{d}y \tag{3.40}$$

$$\implies \gamma = \frac{1}{pl^2}\int_\Omega \mathrm{d}x\wedge\mathrm{d}y = \frac{A}{pl^2} = \frac{e}{p}\frac{BA}{Be\cdot l^2} = \frac{e}{p}\Phi \tag{3.41}$$

Which is precisely the phase factor picked up due to Aharonov-Bohm effect but for a particle with charge e/p. Thus, the impurities we're working with here do indeed have fractional charge.

Impurities, quasi-holes, and anyons

Now that we can attribute the phase due to the second term in the Berry connection to the Aharonov-Bohm effect then we can focus on the first term which only contributes if we enclose a quasi-hole (impurity) in the path of another. Suppose we have N of these quasi-holes with which we exchange one quasi-hole, the i^{th}. The phase factor for a full rotation is as follows

$$i\gamma = -\frac{1}{2p}\sum_{j\neq i}\oint_C \underbrace{\frac{1}{\eta_j-\eta_i}}_{\coloneqq f_j}\mathrm{d}\eta_i + h.c. = -\frac{\pi i}{p}\sum_{j\neq i}(\mathrm{Res}(f_j) + h.c.) = -\frac{2\pi i}{p}N \tag{3.42}$$

[9] Don't get the Laughlin wavefunction $|\eta\rangle$ confused with any of the Plasma analogy terms since these are most definitely not holomorphic in η_i.

So if moving about by 2π yields this then an exchange - which is motion by π gives us

$$e^{i\gamma} = e^{-i\frac{\pi}{p}N} \tag{3.43}$$

Now we can see plain as day that quasi-holes in the fractional quantum Hall effect obey fractional statistics and hence are indeed anyons.

The Moore-Read State

As we saw previously, the Laughlin wavefunction is given by:

$$\Psi_L(z_1, \ldots, z_N) = \prod_{i<j}(z_i - z_j)^p e^{-\frac{1}{4\ell^2}\sum_i |z_i|^2} \tag{3.44}$$

Here p is a positive odd integer. Now, what if we want to describe a system with filling fraction $\nu = \frac{1}{2}$? Here, an issue is encountered with the fact that a fermionic wavefunction must be antisymmetric under the exchange of two particles, which means that the wavefunction must change sign when two particles are exchanged. However, this wavefunction is symmetric for $p = 2$, as it is a product of two symmetric functions. Hence, we must edit this equation in some way so that it becomes antisymmetric and still preserves essential information about the state such as translational invariance. It turns out that the simplest way to do this is using the Pfaffian, which is a function that is antisymmetric under the exchange of two particles. The Pfaffian is defined as follows:

$$\text{pf}(A) = \frac{1}{2^{N/2}(N/2)!}\epsilon^{i_1 \ldots i_n} A_{i_1 i_2} A_{i_3 i_4} \ldots A_{i_{n-1} i_n} \tag{3.45}$$

For an antisymmatric matrix A, the Pfaffian is the square root of the determinant. A more familiar example of a Pfaffian in physics is the Pfaffian of the electromagnatic field strength tensor $F_{\mu\nu}$, which is given by:

$$\text{pf}(F_{\mu\nu}) = F_{01}F_{23} - F_{02}F_{13} + F_{03}F_{12} = \mathbf{E} \cdot \mathbf{B} \tag{3.46}$$

Using this, we can write the wavefunction for the $\nu = \frac{1}{2}$ state as follows:

$$\Psi_{\nu=\frac{1}{2}}(z_1, \ldots, z_N) = \text{pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^2 e^{-\frac{1}{4\ell^2}\sum_i |z_i|^2} \tag{3.47}$$

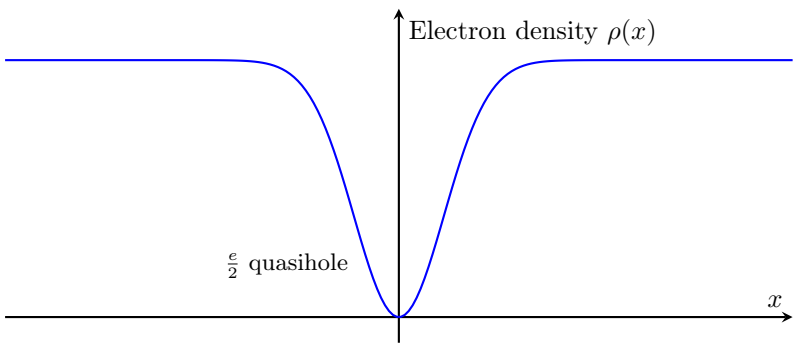
This is known as the Moore-Read wavefunction [19] and is a good description of the $\nu = \frac{1}{2}$ fractional quantum Hall state. In the thermodynamic limit, this Pfaffian factor does not change the filling fraction. This is not the true ground state of the Coulomb interaction, but is the true ground state of the following Hamiltonian:

$$H = \sum_{i<j<k} \delta^{(2)}(z_i - z_j)\delta^{(2)}(z_j - z_k) \tag{3.48}$$

The really interesting properties of this state are those of the quasihole and the quasiparticle. The quasihole state is given by

$$|\eta\rangle = \prod_{i=1}^{N}(z_i - \eta)\Psi_{MR}, \tag{3.49}$$

where η is the position of the quasihole. This punches a hole like below:



It turns out that for this state you can actually write a different operator that punches two holes. These two holes each have a charge of $\frac{e}{4}$, and an $\frac{e}{2}$ hole is the fusion of two of these holes. The operator for two holes of size $\frac{e}{4}$ is given by:

$$|\eta_1, \eta_2\rangle = \text{pf}\left(\frac{(z_i - \eta_1)(z_j - \eta_2) + (z_j - \eta_1)(z_i - \eta_2)}{z_i - z_j}\right)|\Psi_{MR}\rangle \tag{3.50}$$

One can see that for $\eta_1 = \eta_2$, this gives the same operator as before. These $\frac{e}{4}$ holes can only show up in pairs, there does not exist a wavefunction that has a single $\frac{e}{4}$ hole. This is because there is no way you can write down a Pfaffian that is well defined for just one of these $\frac{e}{4}$ holes, hence why you need two holes. Now consider a state with 4 of these holes. The wavefunction will look like the following:

$$|\eta_1, \eta_2, \eta_3, \eta_4\rangle_{(12)(34)} = \text{pf}\left(\frac{(z_i - \eta_1)(z_i - \eta_2)(z_j - \eta_3)(z_j - \eta_4)}{z_i - z_j}\right)|\Psi_{MR}\rangle \tag{3.51}$$

Notice that we could have equivalently swapped some of the indices around, which would give different states such as the ones below:

$$|\eta_1, \eta_2, \eta_3, \eta_4\rangle_{(13)(24)} = \text{pf}\left(\frac{(z_i - \eta_1)(z_i - \eta_3)(z_j - \eta_2)(z_j - \eta_4)}{z_i - z_j}\right)|\Psi_{MR}\rangle \tag{3.52}$$

$$|\eta_1, \eta_2, \eta_3, \eta_4\rangle_{(14)(23)} = \text{pf}\left(\frac{(z_i - \eta_1)(z_i - \eta_4)(z_j - \eta_2)(z_j - \eta_3)}{z_i - z_j}\right)|\Psi_{MR}\rangle \tag{3.53}$$

These all seem like different states, however, only two of them are linearly independent. Their relation is given by the following identity:

$$\text{pf}_{(12)(34)} - \text{pf}_{(13)(24)} = \frac{\eta_{14}\eta_{23}}{\eta_{13}\eta_{24}}\left(\text{pf}_{(12)(34)} - \text{pf}_{(14)(23)}\right), \qquad \text{where } \eta_{ij} = \eta_i - \eta_j \tag{3.54}$$

The density plots of these states look identical, but they are actually different states. For $2n$ vortices, there is a 2^{n-1} degenerate states. Hence, this is like the dimension of this Hilbert space of vortices.

$$\dim(V^{2n}) = 2^{n-1} \tag{3.55}$$

This dimension is similar to that of a spinor system, and that's because these are actually Majorana spinors. This is related to the superconductor viewpoint. Now what happens if we have one of these degenerate states and wind one anyon around the other? The electron density will remain the same, but the state will change to a linear combination of the two degenarate states. In this way, these are non-abelian anyons. Going further beyond this, let's try to generalise this Moore-Read state. Earlier it was mentioned that this state is the exact ground state with a Hamiltonian that looks like:

$$H = \sum_{i<j<k} \delta^{(2)}(z_i - z_j)\delta^{(2)}(z_i - z_k) \tag{3.56}$$

Now we want to generalise this to the exact ground state of not a three body but an n-body interaction:

$$H = H = \sum_{i<j<k<\ldots} \delta^{(2)}(z_i - z_j)\delta^{(2)}(z_j - z_k)\delta^{(2)}(z_k - z_l)\cdots \tag{3.57}$$

It turns out that you can rewrite the Moore-Read state in the following equivalent way:

$$z_i, \quad i = 1, \ldots, N \text{ (even)} \tag{3.58}$$

$$v_i \in \{z_1, \ldots, z_{N/2}\}, \quad w_i \in \{z_{N/2+1}, \ldots, z_N\} \tag{3.59}$$

$$\Psi_{MR} = \mathcal{S}\left[\prod_{i<j}^{N/2}(v_i - v_j)(w_i - w_j)\right]\prod_{i<j}^{N}(z_i - z_j)^{m-1} \tag{3.60}$$

The generalisation of this is that if you have three delta functions in the Hamiltonian, then you divide it into 3 groups, if you have 4 then you divide it into 4 groups, and so on. So, in general it will look like:

$$\Psi = \mathcal{S}\left[\prod_{i<j}^{N/p}\underbrace{(v_i^{(1)} - v_j^{(1)})(v_i^{(2)} - v_j^{(2)})(v_i^{(3)} - v_j^{(3)})\ldots}_{p \text{ groups}}\right]\prod_{i<j}^{N}(z_i - z_j)^{m-1}, \tag{3.61}$$

where the filling fraction is:

$$\nu = \frac{p}{p(m-1)+2} \tag{3.62}$$

Another thing to note here is that even m corresponds to a fermionic state, while odd m corresponds to a bosonic state. $m = 2$ and $p = 2$ corresponds to the Moore-Read state. Changing to $p = 3$ gives us the $\nu = \frac{3}{5}$ state, which has a particle-hole conjugate state of $\nu = \frac{2}{5}$ related to it. This can be imagined by replacing holes with particles and particles with holes. Interestingly, at the real lowest Landau level, the Moore-Read state is not realised. It turns out, this is actually observed at the $\nu = \frac{5}{2}$ state, which is the corresponding state at the 1st Landau level (as the lowest Landau level has two spins). Similarly, the $\nu = \frac{3}{5}$ state is not observed to behave like this due to being in the lowest Landau level. This is because the state illustrated here is actually a superconductor, and a superconductor requires attractive interaction, which cannot be realised in the lowest Landau level. This attractive interaction in higher Landau levels is made possible by the nodes of the wavefunctions given by the Laguerre polynomials. Similar to the Moore-Read state, this occurs in the second Landau level or the $\nu = \frac{13}{5}$ and $\nu = \frac{12}{5}$ states. This is known as the Read-Rezayi state [20]. For these states, we know that if you do the braiding of the anyons it is non-abelian. It turns out that these are actually Ising anyons. Considering the two degenerate states as $|I = 1\rangle$ and $|I = 2\rangle$, we can relate this to the Berry connection in the following way:

$$A_{IJ} = -i \langle \psi_I | \partial_\eta | \psi_J \rangle \tag{3.63}$$

$$\theta_{IJ} = \oint_C \mathbf{A} \cdot d\mathbf{l} = \oint_C \langle \psi_I | \partial_\eta | \psi_J \rangle \, d\eta \tag{3.64}$$

$$U = e^{i\theta_{IJ}} \in SU(2) \tag{3.65}$$

This matrix is a non-abelian gauge transformation matrix. θ_{IJ} is called a holonomy. These angles are no longer just scalars, they are instead matrices.

Composite Fermions

The remarkable success of the Laughlin wavefunction in describing certain fractional quantum Hall states raises a natural question: can we develop a unified framework that explains the broader hierarchy of observed fractional states? The answer lies in one of the most elegant theoretical constructions in condensed matter physics - the composite fermion picture, originally developed by Jain in 1989 [15].

Flux Attachment and the Effective Magnetic Field

The central insight of composite fermion theory is that electrons in strong magnetic fields can be viewed as binding with magnetic flux quanta to form new quasiparticles. This process, known as flux attachment, transforms the many-body problem of strongly interacting electrons into a more tractable problem of weakly interacting composite fermions [15, 16].

To understand this transformation, consider the Berry phase picked up by an electron as it moves along a closed path C enclosing area A. In the presence of a magnetic field B, this phase is:

$$\gamma_{\text{electron}} = -\oint_C \mathbf{A} \cdot d\mathbf{l} = -BA \tag{3.66}$$

Now consider attaching $2p$ flux quanta to each electron. Each flux quantum $\phi_0 = h/e$ contributes an additional Berry phase. If we have N electrons within the area A, and each carries $2p$ flux quanta, then a test electron moving around this region picks up an additional phase:

$$\gamma_{\text{flux}} = -2p \cdot N \cdot 2\pi \cdot \frac{h}{e} = -4\pi p N/e \tag{3.67}$$

The electron density within area A is $\rho = N/A$, so we can write:

$$\gamma_{\text{flux}} = -4\pi p \rho A/e \tag{3.68}$$

The total Berry phase becomes:

$$\gamma_{\text{total}} = -BA - 2p\rho A \cdot 2\pi/e = -(B - 2p\rho\Phi_0)A/e \tag{3.69}$$

This is precisely the Berry phase for a particle in an effective magnetic field:

$$B^* = B - 2p\rho\Phi_0 \tag{3.70}$$

For electrons at filling factor ν, the density is $\rho = \nu B/\phi_0$, giving:

$$B^* = B(1 - 2p\nu) \tag{3.71}$$

This is the fundamental result of composite fermion theory: by attaching $2p$ flux quanta to each electron, we transform the problem into one of composite fermions moving in a reduced effective magnetic field B^*.

Mean-Field Approximation

In the mean-field approximation, we treat the attached flux as producing a uniform effective magnetic field B^*, and the composite fermions as non-interacting particles in this field. The composite fermion filling factor ν^* is related to the electron filling factor by:

$$\nu^* = \frac{\rho\phi_0}{|B^*|} = \frac{\rho\phi_0}{|B - 2p\rho\phi_0|} \tag{3.72}$$

Using $\rho = \nu B/\phi_0$, we obtain:

$$\nu^* = \frac{\nu B/\phi_0 \cdot \phi_0}{|B(1 - 2p\nu)|} = \frac{\nu}{|1 - 2p\nu|} \tag{3.73}$$

For the case where $B^* > 0$ (i.e., $\nu < 1/(2p)$), this gives:

$$\nu^* = \frac{\nu}{1 - 2p\nu} \tag{3.74}$$

Inverting this relation:

$$(\nu^*)^{-1} = \nu^{-1} - 2p \tag{3.75}$$

When $\nu^* = n$ (an integer), the composite fermions completely fill n effective Landau levels, leading to incompressible states. From the relation above:

$$n^{-1} = \nu^{-1} - 2p \Rightarrow \nu = \frac{n}{2pn + 1} \tag{3.76}$$

This generates the primary Jain sequences observed experimentally and will be discussed more thoroughly in 3.6.q

Microscopic Wavefunction Construction

The microscopic approach constructs explicit many-body wavefunctions that capture the flux attachment through the Jastrow factor. Building on the Laughlin construction discussed earlier, the composite fermion wavefunction takes the form:

$$\Psi_\nu^{\text{CF}}(z_1, \ldots, z_N) = \prod_{i<j}(z_i - z_j)^{2p}\Psi_{\nu^*}^{\text{IQH}}(z_1, \ldots, z_N)\exp\left(-\frac{1}{4\ell_B^2}\sum_k |z_k|^2\right) \tag{3.77}$$

where $\Psi_{\nu^*}^{\text{IQH}}$ is the integer quantum Hall wavefunction at filling ν^*. The factor $\prod_{i<j}(z_i - z_j)^{2p}$ ensures that when two electrons approach each other ($z_i \to z_j$), the wavefunction vanishes with power $2p + 1$, reflecting the $2p$ attached flux quanta.

Connection to Laughlin States and the $m \to 2p + 1$ Substitution

To understand the deep connection between Laughlin states and composite fermions, we must carefully examine the relationship between the Laughlin parameter m and the composite fermion flux attachment parameter p. The Laughlin wavefunction at filling $\nu = 1/m$ (where m is odd) takes the form:

$$\Psi_{\text{Laughlin}} = \prod_{i<j}(z_i - z_j)^m \exp\left(-\frac{1}{4\ell_B^2}\sum_k |z_k|^2\right) \tag{3.78}$$

In the composite fermion framework, we decompose this power m into two physically distinct contributions:

$$m = 1 + 2p \quad \Rightarrow \quad p = \frac{m-1}{2} \tag{3.79}$$

This allows us to rewrite the Laughlin wavefunction as:

$$\Psi_{\text{Laughlin}} = \prod_{i<j}(z_i - z_j)^{2p} \cdot \prod_{i<j}(z_i - z_j)^1 \cdot \exp\left(-\frac{1}{4\ell_B^2}\sum_k |z_k|^2\right) \tag{3.80}$$

The physical meaning of this decomposition is interesting in two manners in particular. The first being that the factor $\prod_{i<j}(z_i - z_j)^{2p}$ represents the attachment of $2p$ flux quanta to each electron. These additional zeros ensure electrons avoid each other more strongly than Pauli exclusion alone would require, capturing the correlations that minimize Coulomb repulsion. The second being that the remaining part corresponds to the $\nu^* = 1$ integer quantum Hall state of composite fermions.

Second Quantization and the Chern-Simons Transformation

While the wavefunction approach provides intuitive insight, the full power of composite fermion theory emerges in the second quantized formulation. Here, the flux attachment is implemented through a unitary transformation that relates electron creation operators to composite fermion creation operators.

The key insight is that electron creation operators $c^\dagger(\mathbf{r})$ can be related to composite fermion creation operators $\psi^\dagger(\mathbf{r})$ through a unitary transformation U:

$$c^\dagger(\mathbf{r}) = U\psi^\dagger(\mathbf{r})U^\dagger \tag{3.81}$$

In the Chern-Simons formulation - which is discussed further in 3.7 - this transformation takes the specific form:

$$U = \exp\left(i\int d^2\mathbf{r}\int d^2\mathbf{r}'\rho(\mathbf{r})G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\right) \tag{3.82}$$

where $\rho(\mathbf{r}) = \psi^\dagger(\mathbf{r})\psi(\mathbf{r})$ is the composite fermion density operator, and $G(\mathbf{r}-\mathbf{r}')$ is chosen to attach $2p$ flux quanta to each particle. For $2p = 2$ (the most common case), we have:

$$G(\mathbf{r}-\mathbf{r}') = i\pi\,\text{sgn}(\mathbf{r}-\mathbf{r}')\theta(|\mathbf{r}-\mathbf{r}'| - a) \tag{3.83}$$

where a is a short-distance cutoff.

A crucial subtlety is that U is only approximately unitary. In the mean-field approximation, where we replace $\rho(\mathbf{r}) \to \langle\rho(\mathbf{r})\rangle = \bar{\rho}$ (constant density), the transformation becomes exactly unitary. This is the regime where composite fermions behave as free particles in an effective magnetic field. Beyond mean-field, the density fluctuations $\delta\rho(\mathbf{r}) = \rho(\mathbf{r}) - \bar{\rho}$ break the unitarity, corresponding to residual interactions between composite fermions.

To understand the composite fermion algebra, we compute anticommutators like $\{\psi^\dagger(\mathbf{r}_1), \psi^\dagger(\mathbf{r}_2)\}$ using the Baker-Campbell-Hausdorff (BCH) formula. Starting from the canonical fermionic anticommutation relations:

$$\{c^\dagger(\mathbf{r}_1), c^\dagger(\mathbf{r}_2)\} = 0, \quad \{c(\mathbf{r}_1), c^\dagger(\mathbf{r}_2)\} = \delta^2(\mathbf{r}_1 - \mathbf{r}_2) \tag{3.84}$$

Using $\psi^\dagger(\mathbf{r}_1) = U^\dagger c^\dagger(\mathbf{r}_1)U$, the anticommutator becomes:

$$\{\psi^\dagger(\mathbf{r}_1), \psi^\dagger(\mathbf{r}_2)\} = U^\dagger[c^\dagger(\mathbf{r}_1), c^\dagger(\mathbf{r}_2)]U \tag{3.85}$$

In the mean-field approximation where U is exactly unitary, this gives:

$$\{\psi^\dagger(\mathbf{r}_1), \psi^\dagger(\mathbf{r}_2)\} = 0 \tag{3.86}$$

so composite fermions behave like regular fermions. Beyond mean-field, the BCH formula generates correction terms from the non-commutativity of density operators:

$$\{\psi^\dagger(\mathbf{r}_1), \psi^\dagger(\mathbf{r}_2)\} = \sum_{n=1}^{\infty}\frac{B_n}{n!}\,\text{ad}_X^n(Y) \tag{3.87}$$

where X and Y involve gauge field fluctuations, and B_n are Bernoulli numbers.

This operator formalism reveals that composite fermions couple to an emergent Chern-Simons gauge field a_μ whose dynamics encode the flux attachment. The effective action becomes:

$$S = \int d^3x \left[\psi^\dagger (i\partial_0 + a_0)\psi - \frac{1}{2m^*}|(\nabla - ia)\psi|^2 + \frac{1}{4\pi}\epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \right] \tag{3.88}$$

The second quantized approach thus provides the rigorous foundation for understanding both the success of the mean-field composite fermion picture and the subtle quantum corrections that arise in the full microscopic theory [18, 11].

Jain States

For composite fermions there are two cases, $\rho = \pm\nu^* B^*$, where these correspond to different directions for this net magnetic field. This means that the filling fraction looks like:

$$\nu = \frac{\nu^*}{\nu^*(m-1) \pm 1} \tag{3.89}$$

Setting $m = 3$, we get two series of filling fractions:

$$\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \ldots \to \frac{1}{2} \quad \text{and} \quad 1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \ldots \to \frac{1}{2} \tag{3.90}$$

The wavefunction for this is then given by [15]:

$$\Psi(z_1, \ldots z_N) = \mathcal{P}_{LLL} \left[\prod_{i<j}^{N} (z_i - z_j)^{m-1} \Phi_{\nu^*}(z, \bar{z}) \right] \tag{3.91}$$

where $\Phi_{\nu^*}(z, \bar{z})$ is the lowest Landau level wavefunction. So $m-1$ is the number of fluxes that you attach to the composite particle. Even m corresponds to a composite fermion, and odd m corresponds to a composite boson. For composite boson, there is no Pauli exclusion principle, and the lowest Landau level is not gapped. Also, there will be no higher Landau levels, as they will all condense to the lowest Landau level. This is still a valid picture. Comparing the two pictures for $B = 3$, attaching two fluxes to a fermion results in $B^* = 3 - 2 = 1$, while attaching 3 for a boson gives $B^* = 3 - 3 = 0$. This means that for a boson there is no Landau level seperation any more, and Bose-Einstein condensation will occur at low temperatures. Topologically composite fermion and composite boson pictures are similar, but the dynamics of the theories are different. The composite boson picture was actually the first Chern-Simons effective theory for quantum Hall [25]. The fermionic picture then came in 1991 [18]. Looking at the $\nu = \frac{1}{2}$ state, $B^* = 2 - 2 = 0$ implies that there is no B^* for the state, and hence it behaves like a Fermi liquid. Fermi liquid is gapless, and is hence very conducting. This is what is seen for the $\nu = \frac{1}{2}$ state in experiments, i.e. there is no plateau. There is no net B-field, which means that in the lowest Landau level one can only have a Fermi liquid. In higher Landau levels the other possibility is a superconductor, which can only be achieved with an attractive electron-electron interaction. In the lowest Landau level this is not possible, as the projection of the Coulomb potential $\langle 0m| V |0m\rangle$ is always repulsive. However, at higher Landau levels, the zeros of the Laguerre polynomials can induce an attractive interaction and hence one can get a superconducting phase. Based on composite fermion scheme, the wavefunction becomes something like that of a Fermi Gas, where the different modes are producted up to the Fermi momentum. This is known as the HLR state [11], which is a gapless state.

$$\Psi(z_1, \ldots z_N) = \mathcal{P}_{LLL} \left[\prod_{i<j} (z_i - z_j)^2 \prod_{a=0}^{k_F} e^{i\mathbf{k}_a \cdot \mathbf{z}} \right] \tag{3.92}$$

Alternatively, if one has an attractive interaction, the Fermi Gas wavefunction must be replaced by the ground state of a superconductor, which is the Bardeen-Cooper-Schrieffer (BCS) wavefunction [1].

$$\Psi(z_1, \ldots z_N) = \mathcal{P}_{LLL} \left[\prod_{i<j} (z_i - z_j)^2 \Psi_{BCS}(\mathbf{z}_1, \ldots, \mathbf{z}_N) \right] \tag{3.93}$$

The composite fermion fills up to ν^* level, where

$$\nu = \frac{\nu^*}{\nu^*(m-1) \pm 1} \tag{3.94}$$

This is known as the Jain sequence [15]. For this, only abelian Chern-Simons is required, which means that the anyons are also abelian. Conversely non-abelian Chern-Simons is described by non-abelian anyons. But what is Chern-Simons and why is it important?

Chern-Simons Theories

We have looked at the quantum Hall effect in quite a bit of detail from a microscopic perspective, i.e. digging into the specifics of quantum Hall wavefunctions. Now we wish to zoom out and consider a more low energy or effective field theory, looking more at the coarse grain or topological nature of the quantum Hall fluid. The theories that describe this long-range behaviour are known as Chern-Simons theories [22]. For this effective field theory approach, we will treat A_μ as a background field, i.e. it is purely a parameter of the theory and is not a dynamical field, so equations of motion cannot be obtained from it. Also, it should be noted that this A_μ is not the same as that used to generate the original B-field for the quantum Hall effect to occur in the first place. Instead, this can be thought of as a perturbation of this original 4-potential. For a field theory, A_μ couples to the appropriate 4-current J^μ, and this shows up in the following form in the action:

$$S_A = \int d^3x' \; J^\mu A_\mu \tag{3.95}$$

Assuming that this J^μ obeys the continuity equation, we can also see that this action is gauge invariant up to a boundary term:

$$A_\mu \to A_\mu + \partial_\mu \chi \tag{3.96}$$

$$S_A \to S_A + \int d^3x' \; J^\mu \partial_\mu \chi \tag{3.97}$$

Integrating by part and using $\partial_\mu J^\mu = 0$ we have:

$$\int d^3x' \; J^\mu \; \partial_\mu \chi = \int d^3x' \; \underbrace{\partial_\mu (J^\mu \chi)}_{\text{Total derivative}} - \int d^3x' \; \chi \underbrace{\partial_\mu J^\mu}_{=0} = \text{Boundary term} \tag{3.98}$$

Since we're only interested in an effective field theory, we only need to find an "effective action." An effective action can be thought of as one without the dynamical fields, and can be found by looking at the partition funtion of the field theory, where "fields" means the dynamical fields of the theory:

$$\mathcal{Z}[A] = \int \mathcal{D}(\text{fields}) \; e^{iS[\text{fields};\, A]/\hbar} \equiv e^{iS_{\text{eff}}[A]/\hbar} \tag{3.99}$$

This effective action doesn't give equations of motion, but does give the 4-current's response to electromagnetic perturbations:

$$\frac{\delta S_{\text{eff}}[A]}{\delta A_\mu(x)} = \int d^3x' \; J^\mu(x') \underbrace{\frac{\delta A_\mu(x')}{\delta A_\mu(x)}}_{=\delta(x-x')} = \langle J^\mu(x) \rangle \tag{3.100}$$

Now, what will this effective action actually look like? To see this, we must consider the constraints we have on it:

1. Like before, this effective action must be invariant under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$.
2. The effective action must also be preserved under other symmetries the system may have. These symmetries may include classical examples such as rotational or translational invariance.
3. Since we are looking at the low energy, effective action, we only care about the leading order terms here. Since the effective action is purely a function of A_μ and its derivatives, both of which have units of inverse length, we want to find an action that has the fewest number of A_μ's and ∂_μ's as possible.

In 3+1 dimensions, the effective action is given by

$$S_{\text{eff}}[A] \sim \int d^4x' \; F_{\mu\nu}F^{\mu\nu}, \text{ where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.101}$$

However, the quantum Hall effect is unique to 2+1 dimensions for which there is an even lower order action:

$$S_{CS}[A] = \frac{k}{4\pi}\int d^3x' \; \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \tag{3.102}$$

This is lower order than the 3+1D effective action, now let's check that it satisfies Gauge invariance.

$$A_\mu \to A_\mu + \partial_\mu\chi \tag{3.103}$$

$$S_{CS}[A] \to S_{CS}[A] + \int d^3x' \; \epsilon^{\mu\nu\rho}\partial_\mu\chi\partial_\nu A_\rho + \int d^3x' \; \partial_\mu\chi \underbrace{\epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho\chi}_{=0} + \int d^3x' \; A_\mu \underbrace{\epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho\chi}_{=0} \tag{3.104}$$

Again, we can show that this is gauge invariant up to a boundary term by integrating the second term by parts:

$$\int d^3x' \; \epsilon^{\mu\nu\rho}\partial_\mu\chi\partial_\nu A_\rho = \int d^3x' \; \underbrace{\partial_\nu(\epsilon^{\mu\nu\rho}\partial_\mu\chi A_\rho)}_{\text{Total derivative}} - \int d^3x' A_\rho \underbrace{\epsilon^{\mu\nu\rho}\partial_\nu\partial_\mu\chi}_{=0} = \text{ Boundary term} \tag{3.105}$$

It can also be seen that this breaks parity and time reversal symmetry, which are known features of the quantum Hall effect. This can be seen by changing one of x_0, x_1 or x_2 to having a minus sign. The measure is invariant under this transformation, but for the change of some $x_\mu \to -x_\mu$, there is a corresponding change of $A_\mu \to -A_\mu$ and $\partial_\mu \to -\partial_\mu$, so the action changes by an overall minus sign. Now, let's compute the current density by taking the functional derivative of this action.

$$J^i(x) = \frac{\delta S_{CS}[A]}{\delta A_i(x)} = \frac{k}{2\pi}\int d^3x'\epsilon^{\mu\nu\rho} \underbrace{\frac{\delta A_\mu(x')}{\delta A_i(x)}}_{=\delta(x-x')\delta_{ij}} \partial_\nu A_\rho = \epsilon^{i\nu\rho}\partial_\nu A_\rho = \frac{k}{2\pi}\epsilon^{ij}E_j(x) \tag{3.106}$$

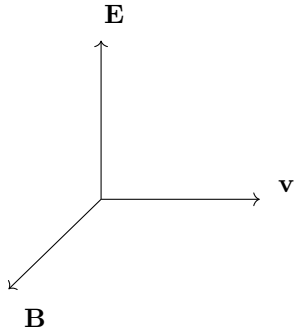
Hence, it can be said here that the Hall conductivity is given by $\sigma_{xy} = \frac{k}{2\pi}$. Comparng this to the Hall conductivity for ν filled Landau levels, it can be determined that $k = \frac{e^2\nu}{\hbar}$. We can also compute the charge density J^0 in a similar fashion:

$$J^0(x) = \frac{\delta S_{CS}[A]}{\delta A_0(x)} = \frac{k}{2\pi}B \tag{3.107}$$

Remembering that this B-field is from the perturbed 4-potential, this can again be identified with the earlier expression in section 2 when $k = \frac{e^2\nu}{\hbar}$. Going back to the classical picture, one gets:

$$J^i = \rho v^i \implies v^i = \epsilon^{ij}\frac{E_j}{B} \tag{3.108}$$

This is called the drift velocity, and is illustrated below.



4 Ising Anyons from $p_x + ip_y$ Superconductors and Moore-Read States

The Moore-Read quantum Hall state and p-wave BCS superconductors are proposed to be in the same topological class, and both have Majorana fermions as excitations to their ground states [14]. In this section, it is demonstrated that composite particles made up of these excitations follow Ising anyon fusion rules and the R-matrix and F-matrix of the Isiang anyon fusion category are derived from physical considerations of the system.

Ising Anyon Fusion Rules

Given Majorana fermions γ_i which satisfy the following relations,

$$\gamma_i^\dagger = \gamma_i, \tag{4.1}$$

$$\{\gamma_i, \gamma_j\} = 2\delta_{i,j}, \tag{4.2}$$

the Hilbert space for the ith region in the sample has a basis spanned by $|+\rangle_i$ and $|-\rangle_i$ which indicate even and odd numbers of Majoranas in the ith region. Equivalently, this may be viewed as the presence/absence of a Majorana in the region, as an even number of Majoranas will annihilate. Note that one cannot define states such that a vacuum state is annihilated by the annihilation operator, because the Majorana condition would then mean that a filled state cannot be created from the vaccum using a creation operator - similarly, there is no number operator on this space of states as there is an a regular fermionic Fock space.

The multiparticle Hilbert space may be attained by applying the usual tensor product structure with the required antisymmetry under exchanges in the order of the tensor product. This structure can be used to describe the presence/absence of Majoranas in n distinct regions. However, as Majorana fermions are Hermitian, their relationship to Dirac fermions may be viewed as analogous to the relationship between real and complex numbers. That is to say,

$$c_{ij}^\dagger = \frac{1}{2}(\gamma_i - i\gamma_j) \text{ and } c_{ij} = \frac{1}{2}(\gamma_i + i\gamma_j) \tag{4.3}$$

are creation and annihilation operators for Dirac fermions, just as $a \pm ib$ is a complex number for real a and b. This can be seen from the fact that c_{ij} and $c_{ij}^\dagger$ satisfy the following relations:

$$\{c_{ij}, c_{kl}^\dagger\} = \delta_{ij,kl},\ \{c_{ij}, c_{kl}\} = \{c_{ij}^\dagger, c_{kl}^\dagger\} = 0. \tag{4.4}$$

Such fermionic operators define a Hilbert space spanned by $|0\rangle_{ij}$ and $|1\rangle_{ij}$.

The number operator in the fermionic Fock space may be written in terms of the Majorana operators as

$$c_{ij}^\dagger c_{ij} = \frac{1}{2}(1 + i\gamma_i\gamma_j). \tag{4.5}$$

Thus, in the Dirac fermion basis, the operator which creates a Majorana fermion in both the ith and jth region is

$$\gamma_i\gamma_j = -i(2c_{ij}^\dagger c_{ij} - 1) \tag{4.6}$$

$$= -i(\hat{p}_{|1\rangle_{ij}} - \hat{p}_{|0\rangle_{ij}}), \tag{4.7}$$

where $\hat{p}_{|a\rangle}$ is the projector onto $|a\rangle$. This demonstrates that the product of a pair of Majorana states is equivalent to some linear combination of filled and empty Dirac fermion states. This provides the first of the Ising anyon fusion rules:

$$\sigma \otimes \sigma = 1 \oplus \psi. \tag{4.8}$$

The fusion of these anyons can be viewed as a coarse graining procedure - we go from describing the states of sites i and j as a product of quantum states to describing their states together in one quantum state. This is effectively 'zooming out' our perception of the system, which involves some loss of information. Notice that there are four independent tensor products of states:

$$|+\rangle_i|+\rangle_j,\ |+\rangle_i|-\rangle_j,\ |-\rangle_i|+\rangle_j, \text{ and } |-\rangle_i|-\rangle_j. \tag{4.9}$$

Clearly, this basis of $\mathcal{H}_i \otimes \mathcal{H}_j$ is not isomorphic to basis of the fermionic Fock space $\mathcal{H}_{ij}$ produced in the coarse graining procedure. This shows that the fusion of MZMs is not described by a tensor product structure. This lack of a tensor

product decomposition of the fermionic Fock states is reminiscent of the definition of entangled quantum states - entangled states are states $|\Sigma\rangle_{ab} \in \mathcal{H}_i \otimes \mathcal{H}_j$ such that

$$|\Sigma\rangle_{ab} \neq |\phi\rangle_a |\psi\rangle_b \tag{4.10}$$

for any $|\phi\rangle_a$ and $|\psi\rangle_b$. The fermionic Fock states are such that

$$|0\rangle_{ij} \neq \sum |\phi\rangle_i |\psi\rangle_j \,, \text{ and } |1\rangle_{ij} \neq \sum |\phi\rangle_i |\psi\rangle_j \,, \text{ where } |\phi\rangle_i \in \{|+\rangle_i \,, |-\rangle_i\}, \;\; |\psi\rangle_j \in \{|+\rangle_j \,, |-\rangle_j\}. \tag{4.11}$$

In this sense, the basis vectors in the fermionic Fock space go beyond the definition of entanglement. Entanglement is the manifestation of non-locality of the quantum state of a composite system - hence, it is clear that the fusion of Majoranas into a Dirac fermion is non-local in nature. This non-locality demonstrates the topological nature of the excitations in the quantum hall state.

Equation (4.7) also shows that (up to a phase) the tensor product states $|+\rangle_i |+\rangle_j$ and $|-\rangle_i |-\rangle_j$ are equivalent in the coarse-grained picture, as are $|+\rangle_i |-\rangle_j$ and $|-\rangle_i |+\rangle_j$. This can be observed from the relations

$$\gamma_i\gamma_j |+\rangle_i |+\rangle_j \propto |-\rangle_i |-\rangle_j \,, \;\; \gamma_i\gamma_j |-\rangle_i |-\rangle_j \propto |+\rangle_i |+\rangle_j \,, \;\; \gamma_i\gamma_j |+\rangle_i |-\rangle_j \propto |-\rangle_i |+\rangle_j \,, \;\; \gamma_i\gamma_j |-\rangle_i |+\rangle_j \propto |+\rangle_i |-\rangle_j \,. \tag{4.12}$$

In conjunction with (4.7), these relations show that $|+\rangle_i |+\rangle_j$ and $|-\rangle_i |-\rangle_j$ are equivalent to one of $|0\rangle_{ij}$ or $|1\rangle_{ij}$ up to some phases, while $|+\rangle_i |-\rangle_j$ and $|-\rangle_i |+\rangle_j$ are equivalent to the other one one of $|0\rangle_{ij}$ or $|1\rangle_{ij}$ up to some phases. Because $|+\rangle$ and $|-\rangle$ are indistinguishable to the Majorana operators, we are free to choose which pair of tensor product states to associate with which fermionic Fock state. The details of the 2-1 map from tensor product states to fermionic Fock states by choosing

$$|+\rangle_i |+\rangle_j \cong |0\rangle_{ij} \text{ and } |+\rangle_i |-\rangle_j \cong |1\rangle_{ij} \,, \tag{4.13}$$

and

$$\gamma_i\gamma_j |+\rangle_i |+\rangle_j = |-\rangle_i |-\rangle_j \implies \gamma_i\gamma_j |-\rangle_i |-\rangle_j = |+\rangle_i |+\rangle_j \,. \tag{4.14}$$

These choices, along with (4.7) lead to the 2-1 map describing the fusion of Majorana states into fermionic Fock states

$$|+\rangle_i |+\rangle_j \mapsto |0\rangle_{ij} \,, \;\; -i\,|-\rangle_i |-\rangle_j \mapsto |0\rangle_{ij} \,, \;\; |+\rangle_i |-\rangle_j \mapsto |1\rangle_{ij} \,, \;\; i\,|-\rangle_i |+\rangle_j \mapsto |1\rangle_{ij} \,. \tag{4.15}$$

The fusion of the composite fermionic Fock states can be understood by considering the operator $i\gamma_i\gamma_j \otimes i\gamma_k\gamma_l$ on the composite Hilbert space $\mathcal{H}_{ij} \otimes \mathcal{H}_{kl}$:

$$i\gamma_i\gamma_j \otimes i\gamma_k\gamma_l = \hat{p}_{|1\rangle_{ij}} \hat{p}_{|1\rangle_{kl}} - \hat{p}_{|0\rangle_{ij}} \hat{p}_{|1\rangle_{kl}} - \hat{p}_{|1\rangle_{ij}} \hat{p}_{|0\rangle_{kl}} + \hat{p}_{|0\rangle_{ij}} \hat{p}_{|0\rangle_{kl}} . \tag{4.16}$$

Just as $i\gamma_i\gamma_j$ (which is often referred to as the 'parity operator') assigns a factor of $+1$ to $|1\rangle_{ij}$ and a factor of -1 to $|0\rangle_{ij}$, the 2-1 fusion map

$$|0\rangle_{ij} |0\rangle_{kl} \mapsto |0\rangle_{ij,kl} \,, \;\; |1\rangle_{ij} |1\rangle_{kl} \mapsto |0\rangle_{ij,kl} \,, \;\; |0\rangle_{ij} |1\rangle_{kl} \mapsto |1\rangle_{ij,kl} \,, \text{ and } |1\rangle_{ij} |0\rangle_{kl} \mapsto |1\rangle_{ij,kl} \,, \tag{4.17}$$

makes $i\gamma_i\gamma_j \otimes i\gamma_k\gamma_l$ have the same action on the new composite fermionic Fock states.

The fusion of a Majorana state with a fermionic Fock state results in a Majorana state. This can be viewed as a 2-1 map which simply projects out the fermionic degree of freedom.

These fusion rules are often written with $1 \equiv |0\rangle$, $\psi \equiv |1\rangle$, and $\sigma \equiv |+\rangle \,/\, |-\rangle$ as

	1	ψ	σ
1	1	ψ	σ
ψ	ψ	1	σ
σ	σ	σ	$1 \oplus \psi$

they are called 'Ising' fusion rules [24], as they emerge from the CFT arising from the critical Ising model. Given fusion rules, pentagon and hexagon equations from the theory of braided fusion categories present an avenue by which to arrive at unitary operators which correspond to changing the order of fusion in the chosen basis, and passing one anyon around another. However, these operators can be derived from the details presented above about the Majorana and fermionic states - in the following subsection, the process of doing so is described in full.

Ising Anyon R-Matrix

One piece of data characterising braided fusion categories (which describe anyons) is the R-matrix. This encodes the transformation of quantum states under anti-clockwise exchange of the anyons. A unitary operator performing this transformation must satisfy the following constraints:

$$\left(U_{ij}^{\dagger}\gamma_i U_{ij}\right)^2 = U_{ij}^{\dagger}\gamma_i^2 U_{ij} = 1, \tag{4.18}$$

and

$$U_{ij}^{\dagger} n_{ij} U_{ij} = n_{ij} \implies U_{ij}^{\dagger}\gamma_i\gamma_j U_{ij} = -\gamma_j\gamma_i. \tag{4.19}$$

This means that under the exchange of MZMs one of the two operators must pick up a factor of -1 and the other must pick up no phase, so either

$$\gamma_i \to \gamma_j \text{ and } \gamma_j \to -\gamma_i, \tag{4.20}$$

or

$$\gamma_i \to -\gamma_j \text{ and } \gamma_j \to \gamma_i. \tag{4.21}$$

Choosing the first option, the creation and annihilation operators in $\mathcal{H}_{ij}$,

$$c_{ij}^{\dagger} = \frac{1}{2}(\gamma_i - i\gamma_j), \text{ and } c_{ij} = \frac{1}{2}(\gamma_i + i\gamma_j) \tag{4.22}$$

are transformed into creation and annihilation operators in $\mathcal{H}_{ji}$

$$c_{ji}^{\dagger} = \frac{1}{2}(\gamma_j + i\gamma_i) = ic_{ij}^{\dagger}, \text{ and } c_{ji} = \frac{1}{2}(\gamma_j - i\gamma_i) = -ic_{ij}. \tag{4.23}$$

The phase difference picked up by the creation and annihilation operators is important - using the above relations, we may write the Majorana operators as a linear superposition of fermionic Fock operators either in $\mathcal{H}_{ij}$ or $\mathcal{H}_{ji}$:

$$\gamma_i = c_{ij} + c_{ij}^{\dagger} = i(c_{ji} - c_{ji}^{\dagger}), \text{ and } \gamma_j = i(c_{ij}^{\dagger} - c_{ij}) = c_{ji} + c_{ji}^{\dagger}. \tag{4.24}$$

From these relations, we obtain isomorphisms between the even and odd Majorana states and the vacuum and filled fermionic Fock states:

$$\begin{pmatrix} |+\rangle_i \\ |-\rangle_i \end{pmatrix} \cong \begin{pmatrix} |0\rangle_{ij} \\ |1\rangle_{ij} \end{pmatrix} \cong \begin{pmatrix} i\,|0\rangle_{ji} \\ |1\rangle_{ji} \end{pmatrix}, \tag{4.25}$$

$$\begin{pmatrix} |+\rangle_j \\ |-\rangle_j \end{pmatrix} \cong \begin{pmatrix} i\,|0\rangle_{ij} \\ -\,|1\rangle_{ij} \end{pmatrix} \cong \begin{pmatrix} |0\rangle_{ji} \\ |1\rangle_{ji} \end{pmatrix}. \tag{4.26}$$

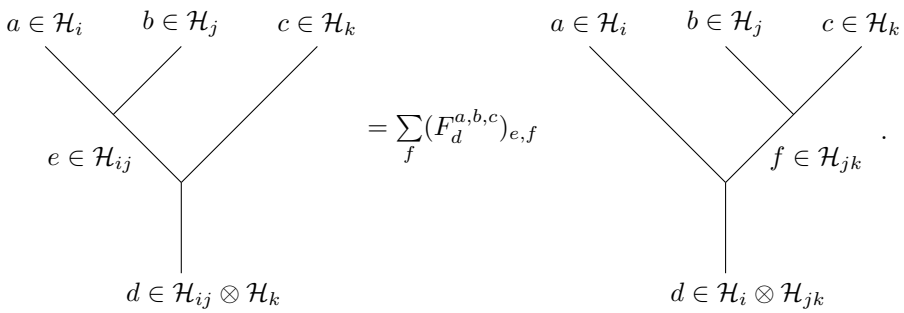
Both of these sets of isomorphisms imply

$$\begin{pmatrix} |0\rangle_{ji} \\ |1\rangle_{ji} \end{pmatrix} = e^{i\theta} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} |0\rangle_{ij} \\ |1\rangle_{ij} \end{pmatrix}. \tag{4.27}$$

With $\theta = -\pi/8$, this is one possible non-trivial R-matrix of an Ising braided fusion category - that is to say, it is a solution to the 'hexagon equations', which is a set of relations that ensure braided fusion categories are self-consistent objects. Up to an Abelian phase factor, we have derived this R-matrix from physical rather than mathematical considerations, linking the microscopic physical picture to the description of anyons using fusion categories.

Ising Anyon F-Symbols

In this section, we will further link the microscopic physical picture to the description of anyons using fusion categories by deriving the Ising fusion category F-symbols from the physics. F-symbols (or F-matrices) are essentially change of basis matrices which change the perceived order of fusion. Diagrammatically, with edges representing particles, causality flowing from top to bottom, and the joining of lines representing fusion of particles, the basis change that F-symbols carry out is defined as The basis on the left diagonalises $i\gamma_i\gamma_j$, while the one on the right diagonalises $i\gamma_j\gamma_k$ - to find the elements of $(F_{\sigma}^{\sigma,\sigma,\sigma})$, we will find a pair bases which diagonalise these operators and find a transformation between them.



While $\mathcal{H}_{ij} \otimes \mathcal{H}_k$ and $\mathcal{H}_i \otimes \mathcal{H}_{jk}$ are the Hilbert spaces that we are concerned with moving between, it is not clear how, for example, $|0\rangle_{ij} |-\rangle_k$ and $|+\rangle_i |1\rangle_{jk}$ are related from the information gathered thus far. However, we have established that $\mathcal{H}_{ij}$ is isomorphic to both $\mathcal{H}_i$ and $\mathcal{H}_j$. Thus, we are free to include a fourth, auxiliary Majorana and find a basis transformation that maps between a basis that diagonalises $i\gamma_j\gamma_k$ and $i\gamma_i\gamma_l$, and one which diagonalises $i\gamma_i\gamma_j$ and $i\gamma_l\gamma_k$. Then using our relations between $\mathcal{H}_{ij}$ and $\mathcal{H}_i$ or $\mathcal{H}_j$, we can attain the elements of $(F_\sigma^{\sigma,\sigma,\sigma})$.

Starting in the tensor product Hilbert space $\mathcal{H}_{il} \otimes \mathcal{H}_{jk}$, we want to get to the Hilbert space $\mathcal{H}_{ij} \otimes \mathcal{H}_{lk}$ - this is done by exchanging Majoranas l and j. We have established in the previous subsection that such an exchange must have the following effect:

$$\gamma_l \to \gamma_j, \text{ and } \gamma_j \to -\gamma_l. \tag{4.28}$$

The unitary operator which has this effect on the Majorana operators under its adjoint action is $U_{lj} = e^{i\theta}\frac{1}{\sqrt{2}}(1 + \gamma_l\gamma_j)$. Thus, (dropping the global phase factor and rewriting the Majorana operators as linear superpositions of fermionic Fock operators) the basis elements of $\mathcal{H}_{il} \otimes \mathcal{H}_{jk}$ are transformed by

$$U_{lj} = e^{i\theta}\frac{1}{\sqrt{2}}(1 - i(c_{12}^\dagger c_{34}^\dagger + c_{12}^\dagger c_{34} + c_{34}^\dagger c_{12} + c_{34}c_{12})). \tag{4.29}$$

This means the map takes the form [14]

$$\begin{pmatrix} |0\rangle_{il}|0\rangle_{jk} \\ |1\rangle_{il}|0\rangle_{jk} \\ |0\rangle_{il}|1\rangle_{jk} \\ |1\rangle_{il}|1\rangle_{jk} \end{pmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |0\rangle_{il}|0\rangle_{jk} \\ |1\rangle_{il}|0\rangle_{jk} \\ |0\rangle_{il}|1\rangle_{jk} \\ |1\rangle_{il}|1\rangle_{jk} \end{pmatrix} = \begin{pmatrix} |0\rangle_{ij}|0\rangle_{lk} \\ |1\rangle_{ij}|0\rangle_{lk} \\ |0\rangle_{ij}|1\rangle_{lk} \\ |1\rangle_{ij}|1\rangle_{lk} \end{pmatrix}. \tag{4.30}$$

Using the isomorphism between $\mathcal{H}_{il}$ and $\mathcal{H}_i$, this is

$$\begin{pmatrix} |0\rangle_{ij}|0\rangle_{lk} \\ |1\rangle_{ij}|0\rangle_{lk} \\ |0\rangle_{ij}|1\rangle_{lk} \\ |1\rangle_{ij}|1\rangle_{lk} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |+\rangle_i|0\rangle_{jk} \\ |-\rangle_i|0\rangle_{jk} \\ |+\rangle_i|1\rangle_{jk} \\ |-\rangle_i|1\rangle_{jk} \end{pmatrix}. \tag{4.31}$$

Now using the isomorphism between $\mathcal{H}_{lk}$ and $\mathcal{H}_k$,

$$\begin{pmatrix} i|0\rangle_{ij}|+\rangle_k \\ i|1\rangle_{ij}|+\rangle_k \\ -|0\rangle_{ij}|-\rangle_k \\ -|1\rangle_{ij}|-\rangle_k \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |+\rangle_i|0\rangle_{jk} \\ |-\rangle_i|0\rangle_{jk} \\ |+\rangle_i|1\rangle_{jk} \\ |-\rangle_i|1\rangle_{jk} \end{pmatrix}. \tag{4.32}$$

This leaves us with

$$\begin{pmatrix} |0\rangle_{ij}|+\rangle_k \\ |1\rangle_{ij}|+\rangle_k \\ |0\rangle_{ij}|-\rangle_k \\ |1\rangle_{ij}|-\rangle_k \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 0 & 0 & -1 \\ 0 & -i & -1 & 0 \\ 0 & i & -1 & 0 \\ i & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} |+\rangle_i|0\rangle_{jk} \\ |-\rangle_i|0\rangle_{jk} \\ |+\rangle_i|1\rangle_{jk} \\ |-\rangle_i|1\rangle_{jk} \end{pmatrix}. \tag{4.33}$$

As discussed when deriving the fusion rules, the fusion of fermionic Fock states with Majorana states is equivalent to projecting out the complex fermion. Therefore, we have two different versionms of the F-matrix $(F_\sigma^{\sigma,\sigma,\sigma})$, one which

applies when there was an even number of Majoranas before fusion:

$$\begin{pmatrix} (F_\sigma^{\sigma,\sigma,\sigma})_{1,1} & (F_\sigma^{\sigma,\sigma,\sigma})_{1,\psi} \\ (F_\sigma^{\sigma,\sigma,\sigma})_{\psi,1} & (F_\sigma^{\sigma,\sigma,\sigma})_{\psi,\psi} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ i & -1 \end{pmatrix}, \tag{4.34}$$

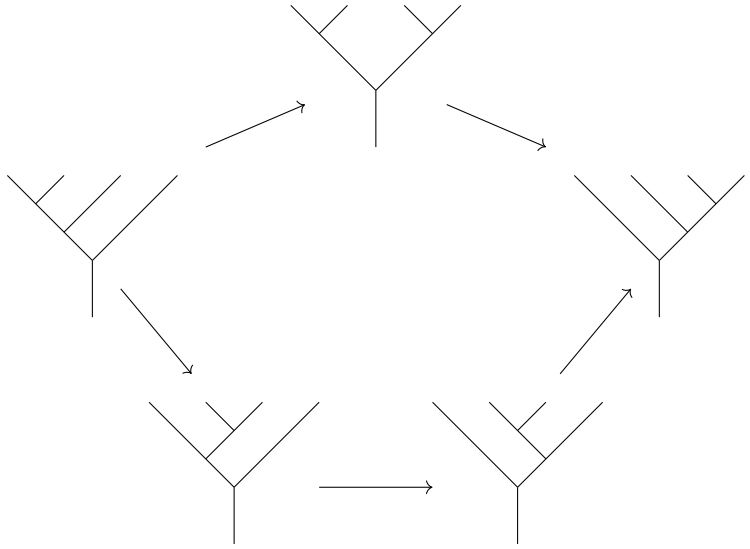
and one which applies when there was an odd number of Majoranas before fusion:

$$\begin{pmatrix} (F_\sigma^{\sigma,\sigma,\sigma})_{1,1} & (F_\sigma^{\sigma,\sigma,\sigma})_{1,\psi} \\ (F_\sigma^{\sigma,\sigma,\sigma})_{\psi,1} & (F_\sigma^{\sigma,\sigma,\sigma})_{\psi,\psi} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 \\ -i & -1 \end{pmatrix}. \tag{4.35}$$

Troublingly, neither of these F-matrices are of the form that is generally cited as the Ising anyon F-matrix,

$$\begin{pmatrix} (F_\sigma^{\sigma,\sigma,\sigma})_{1,1} & (F_\sigma^{\sigma,\sigma,\sigma})_{1,\psi} \\ (F_\sigma^{\sigma,\sigma,\sigma})_{\psi,1} & (F_\sigma^{\sigma,\sigma,\sigma})_{\psi,\psi} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{4.36}$$

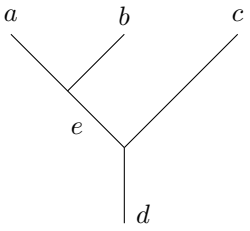
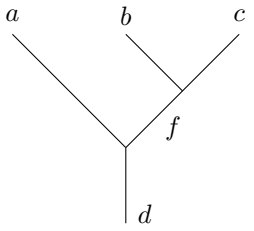
To understand how the F-matrices we have arrived at are related to this F-matrix, the way that F-matrices are defined in fusion categories must be understood. The 'pentagon axiom' places a constraint on the numerical values of F-symbols in the definition of fusion categories. The pentagon axiom states that the two alternative compositions of associators that completely change the basis of the four-particle fusion diagram must be equivalent. This can be visualised as follows (where each arrow indicates the application of an associator): The consistency equations imposed by the pentagon axiom



of monoidal categories is written explicitly as

$$(F_d^{e,g,h})_{i,c}(F_d^{a,b,c})_{e,f} = \sum_j (F_i^{a,b,g})_{e,j}(F_d^{a,j,h})_{i,f}(F_f^{b,g,h})_{j,c}. \tag{4.37}$$

Categorifying a set of fusion rules means finding a solution to these equations, the 'pentagon equations'. However, the solutions to the pentagon equations are non-unique. Given a solution to the pentagon equations, one can find infinitely many 'gauge equivalent' solutions since the $\mathbb{C}$-linear basis vectors $\mathrm{Hom}(a \otimes b, c)$ are defined up to a scalar, so pulling out a factor of $g_{a,b,c}$ from $\mathrm{Hom}(a \otimes b, c)$, one has so rescaling each F-symbol to cancel the ratio of gauge variables, one

$g_{a,b,e}g_{e,c,d}$  $= \sum_f (F_d^{a,b,c})_{e,f} g_{a,f,d}g_{b,c,f}$  ,

attains a new set of F-symbols,

$$(\mathcal{F}_d^{a,b,c})_{e,f} = \frac{g_{a,b,e}g_{e,c,d}}{g_{a,f,d}g_{b,c,f}} (F_d^{a,b,c})_{e,f}. \tag{4.38}$$

Plugging these factors into the pentagon equations, one finds the gauge variables cancel so the new set of F-symbols are also a solution. However, since the gauge variables define an isomorphism between two categorifications of the fusion rules, fusion categories defined by the gauge equivalent F-symbols are considered equivalent.

It is straightforward to check that the two versions of the F-matrix that we have derived by physical considerations are indeed related to the generally cited form by these gauge relations. Therefore, by physical considerations only, we have derived the F-matrix and the R-matrix of an Ising fusion category from excitations in Read-Reyazi quantum Hall states.

5 Fibonacci Anyons from Composite Particle Construction

The composite particle construction of Fibonacci anyons [8] begins with considering a system of three layers of Dirac fermions in a background magnetic field with a Chern-Simons Lagrangian near a $\nu = 2 \to \nu = 1$ transition:

$$\mathcal{L}_{\text{micro}} = \sum_{n=1}^{3} \left(\bar{\Psi}_n (i\not{D} - m)\Psi_n - \frac{3}{2}\frac{1}{4\pi} A dA \right). \tag{5.1}$$

This Lagrangian contains a microscopic picture of Dirac fermions; integrating out the Dirac fermions provides an effective (low energy) Lagrangian. To do so, we define a partition function

$$Z = \int \mathcal{D}A \prod_{n=1}^{3} \left(\mathcal{D}\bar{\Psi}_n \mathcal{D}\Psi_n e^{i\int d^3x \left(\bar{\Psi}_n (i\not{D} - m)\Psi_n - \frac{3}{2}\frac{1}{4\pi} A dA\right)} \right). \tag{5.2}$$

Integrating the fermions out, we get

$$Z = \int \mathcal{D}A \prod_{n=1}^{3} \left(\text{Det}(i\not{D} - m) e^{-i\int d^3x \frac{3}{2}\frac{1}{4\pi} A dA} \right) = \int \mathcal{D}A \prod_{n=1}^{3} \left(e^{\text{Tr}\left(\log(i\not{D} - m)\right) - i\frac{3}{2}\frac{1}{4\pi}\int d^3x A dA} \right). \tag{5.3}$$

Clearly, the effective action for each layer is

$$S_{n,\text{eff}} = -i\text{Tr}\left(\log(i\not{D} - m)\right) - \frac{3}{2}\frac{1}{4\pi}\int d^3x A dA. \tag{5.4}$$

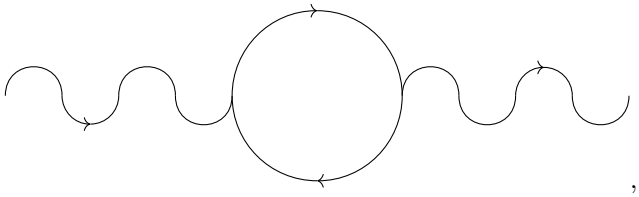
To evaluate the trace, we expand the logarithm for small gauge field (a valid thing to do because we are taking a low-energy limit):

$$\log(i\not{\partial} + \not{A} - m) = \log(i\not{\partial} - m) + \frac{1}{i\not{\partial} - m}\not{A} + \frac{1}{2}\frac{1}{i\not{\partial} - m}\not{A}\frac{1}{i\not{\partial} - m}\not{A} + \mathcal{O}(\not{A}^3). \tag{5.5}$$

The first term is free of A, so it can be ignored as an overall factor in Z. The second term in this expansion contributes a tadpole diagram, which is zero by Furry's theorem, to the effective action. Therefore, we must compute

$$\text{Tr}\left(\frac{1}{i\not{\partial} - m}\not{A}\frac{1}{i\not{\partial} - m}\not{A}\right) \tag{5.6}$$

to find lowest order contribution from the Dirac term to the effective action. This term corresponds to a one-loop Feynman diagram: and may be written using Dirac propagators as



,

$$\text{Tr}\left(\frac{1}{i\not{\partial} - m}\not{A}\frac{1}{i\not{\partial} - m}\not{A}\right) = \int \frac{d^3p}{(2\pi)^3} A_\mu(-p) \left[\int \frac{d^3k}{(2\pi)^3} \text{Tr}\left(\frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2}\gamma^\mu \frac{\not{k} + m}{k^2 - m^2}\gamma^\nu \right) \right] A_\nu(p). \tag{5.7}$$

Gamma matrices in $2+1$ dimensions may be represented in terms of scalar multiples Pauli matrices - choosing a mostly negative signature and

$$\gamma^0 = \sigma_x,\ \gamma^1 = i\sigma_y,\ \gamma^2 = -i\sigma_z, \tag{5.8}$$

it is clear that

$$\text{Tr}(\gamma^\mu) = 0,\ \text{Tr}(\gamma^\mu\gamma^\nu) = 2\eta^{\mu\nu},\ \text{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = 2i\epsilon^{\mu\nu\rho}, \text{ and } \text{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 0. \tag{5.9}$$

Therefore,

$$\int \frac{d^3k}{(2\pi)^3} \text{Tr}\left(\frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2}\gamma^\mu \frac{\not{k} + m}{k^2 - m^2}\gamma^\nu \right) = 2m(m\eta^{\mu\nu} + i\epsilon^{\mu\nu\rho}p_\rho) \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2 - m^2)\left((p+k)^2 - m^2\right)}. \tag{5.10}$$

Performing a Wick rotation on both p and k,

$$\int \frac{d^3k}{(2\pi)^3}\mathrm{Tr}\left(\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\gamma^\mu\frac{\not{k}+m}{k^2-m^2}\gamma^\nu\right) = -2mi(m\eta^{\mu\nu}+i\epsilon^{\mu\nu\rho}p_\rho)\int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+m^2)\left((p+k)^2+m^2\right)}$$
$$= -2mi(m\eta^{\mu\nu}+i\epsilon^{\mu\nu\rho}p_\rho)\frac{1}{4\pi}\arcsin\left(\frac{|p|}{\sqrt{p^2+4m^2}}\right)[4]. \quad (5.11)$$

Now taking the low energy limit of this result we obtain [4]

$$\int \frac{d^3k}{(2\pi)^3}\mathrm{Tr}\left(\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\gamma^\mu\frac{\not{k}+m}{k^2-m^2}\gamma^\nu\right) \approx -\frac{im}{4\pi|m|}(m\eta^{\mu\nu}+i\epsilon^{\mu\nu\rho}p_\rho). \quad (5.12)$$

Undoing the Wick rotation of p, we find

$$\mathrm{Tr}\left(\frac{1}{i\not{\partial}-m}\not{A}\frac{1}{i\not{\partial}-m}\not{A}\right) \approx \int\frac{d^3p}{(2\pi)^3}A_\mu(-p)\left[-\frac{im}{4\pi|m|}(m\eta^{\mu\nu}+i\epsilon^{\mu\nu\rho}p_\rho)\right]A_\nu(p). \quad (5.13)$$

Fourier transforming to position space and using $p_\mu = -i\partial_\mu$,

$$\mathrm{Tr}\left(\frac{1}{i\not{\partial}-m}\not{A}\frac{1}{i\not{\partial}-m}\not{A}\right) \approx i\frac{m}{|m|}\frac{1}{4\pi}\int d^3xAdA - i\frac{|m|}{4\pi}\int d^3xA^2, \quad (5.14)$$

so our effective action in each layer is

$$S_{n,\mathrm{eff}} = -\left(\frac{3}{2}-\frac{1}{2}\frac{m}{|m|}\right)\frac{1}{4\pi}\int d^3xAdA - \frac{|m|}{8\pi}\int d^3xA^2. \quad (5.15)$$

Only the first term is parity-odd, so it alone contributes to the Hall conductivity. It is apparent that the filling factor is

$$\frac{3}{2}-\frac{1}{2}\frac{m}{|m|}, \quad (5.16)$$

so for $m>0$ we have a $\nu=1$ integer quantum Hall state and for $m<0$ we have a $\nu=2$ integer quantum Hall state. It is at the quantum phase transition at $m=0$ that the composite particle construction of Fibonacci anyons may be carried out.

The reason for the importance of the $m=0$ point is that at this point, the theory described so far is proposed to satisfy a boson-fermion duality which relates the free Dirac fermions on each layer to bosons coupled to a $U(2)$ Chern-Simons gauge field. That is to say that $\mathcal{L}_{\mathrm{micro}}$ is dual to [8]

$$\mathcal{L}_{\mathrm{dual}} = \sum_{n=1}^{3}\left(|D\phi_n|^2 - r|\phi_n|^2 - |\phi_n|^4 + \frac{1}{4\pi}\mathrm{Tr}(a_n da_n - \frac{2i}{3}(a_n)^3) + \frac{1}{2\pi}Ad\mathrm{Tr}(a_1-a_2+a_3)\right), \quad (5.17)$$

with the quantum phase transition on this side of the duality being parameterised by r, with the bosons going from a gapped to ungapped phase at $r=0$. The choice of the signs of the charge of the gauge field in each layer are free to be determined as we like. The choice of signs presented above - with the middle layer's gauge field having opposite charge to the two outer layers' gauge fields - is crucial to obtaining the desired TQFT through flux attachment. First, an interlayer clustering term $\mathcal{L}_{\mathrm{cluster}}$ involving a Hubbard-Stratonovich field with order parameter $\phi_m^\dagger\phi_n$ is added to $\mathcal{L}_{\mathrm{dual}}$:

$$\mathcal{L}_{\mathrm{cluster}} = -\sum_{n,m}\phi_m^\dagger\Sigma_{m,n}\phi_n - V[\Sigma]. \quad (5.18)$$

By choice of potential $V[\Sigma]$, the gauge group on the three layers can be broken such that $U(2)\times U(2)\times U(2)\to U(2)$ and the a Higgs procedure may be applied, leaving only $a_1=a_2=a_3$ gauge field configurations. This results in the $U(2)_1$ Chern-Simons terms on the three levels to add and result in a $U(2)_3$ Chern-Simons term:

$$\sum_{n=1}^{3}\left(\frac{1}{4\pi}\mathrm{Tr}(a_n da_n - \frac{2i}{3}(a_n)^3) + \frac{1}{2\pi}Ad\mathrm{Tr}(a_1-a_2+a_3)\right) \to 3\frac{1}{4\pi}\mathrm{Tr}(ada - \frac{2i}{3}(a)^3) + \frac{1}{2\pi}Ad\mathrm{Tr}(a). \quad (5.19)$$

Two of the terms coupling the $U(2)$ gauge fields to the electromagnetic gauge field have cancelled due to the choice of signs of charges made earlier, so the remaining gauge field has the same charge as the original gauge fields on the first and third layers. To obtain the filling factor of the Hall state that this Lagrangian describes, we apply a level-rank duality [12] between TQFTs

$$SU(N)_k \leftrightarrow U(k)_{-N}. \tag{5.20}$$

With $k = 1$ and $N = 2$, this duality tells us that for each layer of our system, the Chern-Simons and E.M. coupling terms in $\mathcal{L}_{dual}$ are dual to a $\nu = 2$ integer quantum Hall state. In the case of our Lagrangian after including a clustering effect, the coupling to the E.M. gauge field is relatively only 1/3 as strong, so due to the choice of sign for the charge of the second layer's gauge field, the total filling factor after including a clustering is $\nu = 2/3$ rather than $\nu = 6$.

Now consider a single unit of flux being attached to each charge in our system. Because the charge carriers in $\mathcal{L}_{micro}$ are fermions, the attachment of flux to the charges results in composite bosonic charge carriers. This flux attachment is carried out by including an additional Abelian gauge field

$$3\frac{1}{4\pi}\mathrm{Tr}(ada - \frac{2i}{3}(a)^3) + \frac{1}{2\pi}Ad\mathrm{Tr}(a) \rightarrow 3\frac{1}{4\pi}\mathrm{Tr}(ada - \frac{2i}{3}(a)^3) + \frac{1}{2\pi}bd\mathrm{Tr}(a) + \frac{1}{4\pi}bdb + \frac{1}{2\pi}bdA + \frac{1}{4\pi}AdA. \tag{5.21}$$

Integrating out the gauge field b, we obtain

$$\mathcal{L}_{U(2)_{3,1}} = 2\frac{1}{4\pi}\mathrm{Tr}(ada) - \mathrm{Tr}(2i(a)^3) + \frac{1}{2\pi}Ad\mathrm{Tr}(a). \tag{5.22}$$

This is denoted as a $U(2)_{3,1}$ Chern-Simons theory where the two subscripts indicate the level of the Abelian and non-Abelian parts of the Chern-Simons action ($U(N)_{3,1}$ denotes $\frac{SU(N)_3 \times U(1)_N}{\mathbb{Z}}$). This Lagrangian describes a integer quantum Hall state with filling factor $\nu = 2$ [8].

It is the line content of TQFTs that correlates to the anyons in a fusion category. It has been shown [3] that $U(2)_{3,1}$ contains Fibonacci anyons and is dual to $(G_2)_1$, which is also known to contain Fibonacci anyons (and has been utilised in a theoretical construction of Fibonacci anyons in topological superconductors [13]). Therefore, by applying dualities of TQFTs and flux attachment to our quantum Hall system at a quantum phase transition, Fibonacci anyons have in theory been realised.

A Weak localisation

Let's ditch the gaseous continuum in favour of a lattice structure where an electron can move in a $1D$ random walk i.e. the electron moves from site to site in the lattice and can only move in the forward or backwards direction. In this regime, let's denote the following:

- l the length of a step.
- T the time taken for a step.
- $P_N(m)$ the probability the electron is found at site m at the Nth time step.

Suppose that there is an equal probability of an electron moving to the forwards or backwards. Therefore,

$$P_{N+1}(m) = \frac{1}{2}\Big[P_N(m-1) + P_N(m+1)\Big] \tag{A.1}$$

That is to say, the chance of finding the electron at site m is equal to $1/2$ the chance that the electron was just behind it $+$ $1/2$ the chance the electron just in front of it. Adding and subtracting a $P_N(m)$ term we have.

$$P_{N+1}(m) - P_N(m) = \frac{1}{2}\Big[P_N(m+1) - P_N(m) - \Big(P_N(m) - P_N(m-1)\Big)\Big] \tag{A.2}$$

$$T\frac{P_N(m+1) - P_N}{T} = l^2\frac{1}{2l^2}\Big[P_N(m+1) - P_N(m) - \Big(P_N(m) - P_N(m-1)\Big)\Big] \tag{A.3}$$

We see that in the limit, the left hand side is merely a time derivative and the right hand side, a second difference, becomes the Laplacian (although restricted to $1D$ for the moment).

$$\partial_t P(x,t) = D\,\partial_x^2 P(x,t), \qquad D \coloneqq \frac{l^2}{2T} \tag{A.4}$$

We see that Eq. A.4 is the diffusion equation. Let's apply the boundary conditions of $P(x,0) = \delta(x)$ and $P(\pm\infty, t) = 0$ and begin to solve. Using the Fourier transform (with implied bounds of $\pm\infty$), we have

$$\tilde{P}(k,t) = \int \mathrm{d}x\, e^{-ikx} P(x,t) \tag{A.5}$$

$$\tilde{P}(k,0) = \int \mathrm{d}x\, e^{-ikx}\delta(x) = 1 \tag{A.6}$$

$$P(x,t) = \frac{1}{2\pi}\int \mathrm{d}k\, e^{ikx}\tilde{P}(k,t), \tag{A.7}$$

We plug this into Eq. A.4 to find

$$\frac{1}{2\pi}\int \mathrm{d}k\, e^{ikx}\Big[\partial_t\tilde{P} + k^2 D\tilde{P}\Big] = 0 \tag{A.8}$$

$$\implies \partial_t\tilde{P} = -k^2 D\tilde{P} \implies \tilde{P} = \cancel{\tilde{P}(x,0)}^{1}\exp\big\{-k^2 Dt\big\} = \exp\big\{-k^2 Dt\big\} \tag{A.9}$$

$$P(x,t) = \frac{1}{2\pi}\int \mathrm{d}k \exp\Big(ikx - k^2 Dt\Big) = \frac{1}{2\pi}\exp\Big\{-\frac{x^2}{4Dt}\Big\}\underbrace{\int \mathrm{d}k\, e^{-(k\sqrt{Dt} - \frac{ix}{2\sqrt{Dt}})^2}}_{\sqrt{\pi}} \tag{A.10}$$

The probability is then given by $P(x,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\Big\{-\frac{x^2}{4Dt}\Big\}$. This can be expanded to $2D$, where we replace ∂_x^2 with $\nabla^2 = \partial_x^2 + \partial_y^2$. The solution then becomes

$$P(\mathbf{r},t) = \frac{1}{4\pi Dt}\exp\Big\{-\frac{r^2}{4Dt}\Big\} \tag{A.11}$$

This tells us the probability of the electron having made a loop, i.e. $\mathbf{r} = \mathbf{0}$ is given by $\frac{1}{4\pi Dt}$. Using this, we can analyse the relative difference in conductance, defined as $\sigma = \frac{1}{\rho}$, due to an electron looping around back to the origin. Intuitively,

we can see how the electron having net zero displacement will decrease the conductivity as less current is passing through the gas for constant voltage. The electron can be viewed as a wavepacket with most of its amplitude held in a length λ_F, the Fermi wavelength as found through its corresponding Fermi energy. The relative change in the conductivity is calculated by multiplying the probability of the electron returning the the origin by the area the electron maps out. It's crucial here to notice that the we're talking about the area the electron maps out between the average time it takes to scatter, τ_s, and the average time it takes to change phase, τ_ϕ, as after changing phase the electron's path won't be able to interfere as is needed. We have that the area is given by $\mathrm{d}A = \lambda_F \,\mathrm{d}x = \lambda_F v_F \,\mathrm{d}t$ where v_F is the Fermi velocity. Thus, the relative conductivity change is given by

$$\delta\sigma \approx -\sigma_0 \int_{\tau_s}^{\tau_\phi} \mathrm{d}t \, \frac{v_F \lambda_F}{4\pi D t} \propto -\frac{\lambda_F}{L} \ln \frac{\tau_\phi}{\tau_s} \tag{A.12}$$

Where $L = v_F \tau_s$ is the mean free path.

B Poisson Brackets, the Hamiltonian, and the Time Derivative

A staple result of Hamiltonian mechanics is that $\{\zeta, H\} = \dot{\zeta}$ for any ζ. This is perhaps the most useful expression of Hamilton's Equations. We now take the time to derive it from first principles.

Claim: $\dot{\zeta} = \{\zeta, H\}$ for any function $\zeta = \zeta(q, p)$ where $q = \{q^i\}$ and $p = \{p^i\}$ i.e. q represents all generalised position coordinates q^i's and likewise for the momenta represented as p.
Proof:

$$\{\zeta, H\} = \sum_i \frac{\partial \zeta}{\partial q^i}\frac{\partial H}{\partial p^i} - \frac{\partial H}{\partial q^i}\frac{\partial \zeta}{\partial q^i} \tag{B.1}$$

We define the action $S \coloneqq \int \mathrm{d}t \, L$ and the Hamiltonian $H \coloneqq \sum_i p^i \dot{q}^i$.

$$-L \tag{B.2}$$

$$S = \int_{t_0}^{t_1} \mathrm{d}t \sum_i p^i \dot{q}^i - H(p, q) \tag{B.3}$$

$$\delta S = \sum_i \int_{t_0}^{t_1} \mathrm{d}t \, \underbrace{p^i \delta \dot{q}^i}_{} + \dot{q}^i \delta p^i - \frac{\partial H}{\partial p^i}\delta p^i - \frac{\partial H}{\partial q^i}\delta q^i \tag{B.4}$$

$$\left\{ \int_{t_0}^{t_1} p^i \delta \dot{q}^i = p^i \delta q^i \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \mathrm{d}t \, \dot{p}^i \delta q^i \right. \tag{B.5}$$

$$= \sum_i p^i \delta q^i \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \Big(\dot{q}^i - \frac{\partial H}{\partial p^i}\Big)\delta p^i - \Big(\dot{p}^i + \frac{\partial H}{\partial q^i}\Big) \tag{B.6}$$

Now we use Hamilton's Principle, which states that the action is minimised along the physically taken trajectory together with the fact that the variation taken in q^i denoted by δq^i is to be null when at the boundaries.

$$\int_{t_0}^{t_1} \mathrm{d}t \, \Big(\dot{q}^i - \frac{\partial H}{\partial p^i}\Big)\delta p^i - \Big(\dot{p}^i + \frac{\partial H}{\partial q^i}\Big) = 0 \; \forall \text{ possible variations } \delta q^i \text{ and } \delta p^i \tag{B.7}$$

$$\implies \dot{p}^i = -\frac{\partial H}{\partial q^i}, \qquad \dot{q}^i = \frac{\partial H}{\partial p^i} \tag{B.8}$$

Using these relations in the definition of the Poisson Bracket given at the beginning of the proof we have

$$\{\zeta, H\} = \sum_i \frac{\partial \zeta}{\partial q^i}\frac{\mathrm{d}q^i}{\mathrm{d}t} + \frac{\partial \zeta}{\partial p^i}\frac{\mathrm{d}p^i}{\mathrm{d}t} = \frac{\mathrm{d}\zeta}{\mathrm{d}t} \tag{B.9}$$

■

Furthermore, we can also make the following claim:

Claim: There exists a function $F(q, Q, t)$ that generates the following transformation of coordinates $(q, p) \to (Q, P)$ such that the transformation preserves the Poisson bracket.

$$P^i = -\frac{\partial F}{\partial Q^i}, \qquad p^= \frac{\partial F}{\partial q^i}, \qquad H' = H + \frac{\partial F}{\partial t}$$

Proof: As we've seen, the Poisson bracket determines the time derivative of a quantity. In particular, $\{\mathbf{q}, H\}$ determines the equations of motion of the system. As seen in the proof of the previous claim, the minimisation of the action leads to this and hence is an equivalent statement. It's important to note though that the minimisation of the action, and hence the equations of motion, are invariant under the addition of a total derivative.

We define our generating function $F = F(q, Q, t)$. We then say that the action after the transformation

$$S' = S - \int \mathrm{d}F \tag{B.10}$$

$$\int \sum_i P^i \,\mathrm{d}Q^i - H' \,\mathrm{d}t = \int \sum_i p^i \,\mathrm{d}q^i - H \,\mathrm{d}t - \frac{\partial F}{\partial Q^i}\,\mathrm{d}Q^i - \frac{\partial F}{\partial q^i}\,\mathrm{d}q^i - \frac{\partial F}{\partial t}\,\mathrm{d}t \tag{B.11}$$

$$0 = \int \sum_i \Big(P^i + \frac{\partial F}{\partial Q^i}\Big)\,\mathrm{d}Q^i + \Big(\frac{\partial F}{\partial q^i} - p^i\Big)\,\mathrm{d}q^i + \Big(H + \frac{\partial F}{\partial t} - H'\Big)\,\mathrm{d}t \tag{B.12}$$

Using the same logic as before, each term must independently go to 0. Hence, we have

$$P^i = -\frac{\partial F}{\partial Q^i}, \qquad p^i = \frac{\partial F}{\partial q^i}, \qquad H' = H + \frac{\partial F}{\partial t} \tag{B.13}$$

■

C Electrostatics in N dimensions.

An often cited problem in electrostatics is the solution of the potential given a point charge. Let's consider this problem in $d = N > 1$. It can be shown from the Voss-Weyl formula and using the spherical symmetry to render the Laplace-Beltrami operator mute that the relevant Poisson Equation away from the origin is just

$$\frac{\partial}{\partial r}\Big(r^{N-1}\partial_r\phi\Big) = 0 \tag{C.1}$$

$$\partial_r\phi \equiv \boldsymbol{\nabla}\phi = \frac{C}{r^{N-1}} \implies \phi = -\frac{C}{(N-1)r^{N-2}} + D \tag{C.2}$$

Which obviously holds for $N > 2$. Now we enforce the condition that $\phi(r \to \infty) \to 0$ i.e. $D = 0$. Then we can deal with the delta function at the origin by integrating over a volume V such that the origin is in V. Then use the divergence theorem to turn the volume integral into a surface integral over ∂V and hence we get that

$$CA_{n-\text{ball}} = \rho \implies \phi(r) = -\frac{\Gamma(N/2)}{2\pi^{N/2}}\frac{\rho}{r^{N-2}} \tag{C.3}$$

However, for the case of $N = 2$ we get that $\phi \sim \ln(r)$ which we'll need to noramlise - in the unit sense - with some characteristic length of the system which we choose to be l_B. We can also redefine the density to include this area i.e. $\rho \to q/A$. In the end this gives us that

$$\phi = -q\ln\Big(\frac{r}{l_B}\Big), \qquad \partial^2\phi = -2\pi q\delta(r) \tag{C.4}$$

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