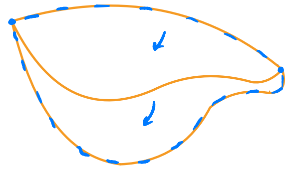
Exam Exercices

① Prove that Homotopy relation is an equivalence relation.

An homotopy is any family of maps $f_t : X \to Y$, $t \in I$, such that the associated map $F: X \times I \to Y$, $F(x,t) = f_t(x)$ is continuous. On the other hand, $f_0, f_1 : X \to Y$ are homotopic if there exists a homotopy connecting them, i.e., if $I = [0,1]$ $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$, $\forall x \in X$.



- Reflexivity: $f \simeq f$
  $I = [0,1]$, $f_t(x) = f(x)$, $\forall t \in [0,1]$
  $F(x,t) = f_t(x) = f(x)$, $\forall t \in [0,1]$ and $\begin{matrix} F(x,0) = f(x) \\ F(x,1) = f(x) \end{matrix}$ so $f \simeq f$

- Symmetry: If $f_0 \simeq f_1$, then $\exists F: X \times [0,1] \to Y$, $F(x,t) = f_t(x)$ continuous s.t. $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.
  If we consider $G: X \times [0,1] \to Y$ s.t. $G(x,t) = F(x,1-t)$, then
  $\begin{matrix} G(x,0) = F(x,1) = f_1(x) \\ G(x,1) = F(x,0) = f_0(x) \end{matrix}$ so $f_1 \simeq f_0$

- Transitivity: If $f_0 \simeq f_1 \Rightarrow \exists F_1 : X \times [0,1] \to Y$, $F_1(x,t) = f_t^1(x)$ s.t. $\begin{matrix} F_1(x,0) = f_0(x) \\ F_1(x,1) = f_1(x) \end{matrix}$
  and
  $f_1 \simeq f_2 \Rightarrow \exists F_2 : X \times [0,1] \to Y$, $F_2(x,t) = f_t^2(x)$ s.t. $\begin{matrix} F_2(x,0) = f_1(x) \\ F_2(x,1) = f_2(x) \end{matrix}$

If we consider $G: X \times [0,1] \to Y$ s.t. $G(x,t) = \begin{cases} F_1(x,2t), & 0 \le t < \frac{1}{2} \\ F_2(x,2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$, then

$\left.\begin{matrix} G(x,0) = F_1(x,0) = f_0(x) \\ G(x,1) = F_2(x,1) = f_2(x) \end{matrix}\right\} \Rightarrow f_0 \simeq f_2$

S^n is the unit sphere in $\mathbb{R}^{n+1}$ — Frontera del disco de dimensión n

② Construct an explicit deformation retraction of $\mathbb{R}^n \setminus \{0\}$ onto S^{n-1}.

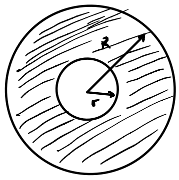
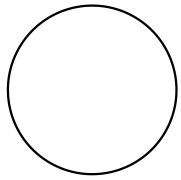
A deformation retraction of a space X onto a subspace A is family of maps $f_t : X \to X$, $t \in I$, such that $f_0 = \mathbb{1}_X$, $f_1(X) = A$ and $f_t|_A = \mathbb{1}_A$, $\forall t \in I$.

Let $X = \mathbb{R}^n \setminus \{0\}$, $f_t : X \to X$ s.t. $f_t(x) = (1-t)x + t \cdot \frac{x}{\|x\|}$, so we have:

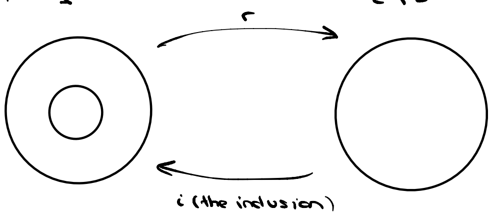
- $f_0(x) = x \cdot \mathbb{1}_{\mathbb{R}^n \setminus \{0\}}$
- $x \in \mathbb{R}^n \setminus \{0\}$, $f_1(x) = \frac{x}{\|x\|}$ and $\|f_1(x)\| = \left\| \frac{x}{\|x\|} \right\| = \frac{1}{\|x\|} \cdot \|x\| = 1$, so $f_1(x) \in S^{n-1}$ $\forall x \in \mathbb{R}^n \setminus \{0\}$
- if $x \in S^{n-1}$ $(\|x\| = 1)$, $f_t(x) = (1-t)x + t \frac{x}{\|x\|}$ and $\|f_t(x)\| = \|(1-t)x + t\frac{x}{1}\| =$
  $= \|x - tx + tx\| = \|x\| = 1 \Rightarrow \|f_t(x)\| = 1 \Rightarrow f_t(S^{n-1}) = S^{n-1}$

③ Show that ring (space) is homotopy equivalent to S^1.

Two topological spaces X, Y are homotopy equivalent if there are continuous maps
$\begin{matrix} f: X \to Y \\ g: Y \to X \end{matrix}$ s.t. $\begin{matrix} f \circ g \simeq \mathbb{1}_Y \\ g \circ f \simeq \mathbb{1}_X \end{matrix}$

$R > r > 0$, $R = \{x \in \mathbb{R}^2 \mid r < \|x\| < R\}$  , $S^1 = \{x \in \mathbb{R}^2 / \|x\| = 1\}$ 

S^1 is a retraction of R so there exists a deformation retraction $f_t : R \to S^1$ s.t. $f_0 = \mathbb{1}_R$, $f_1(R) = S^1$ and $f_t|_{S^1} = \mathbb{1}_{S^1}$



so $r \circ i \simeq \mathbb{1}$ $\quad r : R \to S^1$

$i \circ r \simeq \mathbb{1}$ $\quad i : S^1 \to R$

4 Show if $g_0, g_1 : Y \to Z$ are homotopic and $f_0, f_1 : X \to Y$ are homotopic, then the composition $g_0 \circ f_0$ is homotopic with $g_1 \circ f_1$.

$g_0 \simeq g_1 \Rightarrow \exists G : Y \times I \to Z$ (continuous map) s.t. $G(y,0) = g_0(y)$ and $G(y,1) = g_1(y)$

$f_0 \simeq f_1 \Rightarrow \exists F : X \times I \to Y$ (continuous map) s.t. $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$

└→ homotopies

Then the homotopy between $g_0 \circ f_0$ and $g_1 \circ f_1$ is $H(x,t) = (g_t \circ f_t)(x)$. To see that this defines an homotopy, note that H can be written as a composition of continuous maps:

$$\begin{tikzcd}
X \times I \arrow[r, "1_X \times \Delta_I"'] \arrow[rrr, bend left, "H"] & X \times I \times I \arrow[r, "F \times 1_I"'] & Y \times I \arrow[r, "G"'] & Z
\end{tikzcd}$$

$(x,t) \longmapsto (x,t,t)$

$(x,t,s) \longmapsto (F(x,t),s)$

$(y,s) \longmapsto G(y,s)$

$(x,t) \longmapsto G(F(x,t),s)$

5 Which letters are homotopy equivalent?

- $\{A, R, D, O, P, Q\}$ are homotopy equivalent to S^1.
- $\{B\}$ none
- $\{C, E, F, \dots\}$ are contractible (the identity map is homotopy eq. to constant map)

A map is null-homotopic if it's homotopic to a constant map.

6 Show that homeomorphic spaces are homotopy equivalent.

We know that 2 topological spaces are homotopy equivalent if there exists continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to $\mathbb{1}_Y$ and $g \circ f$ is homotopic to $\mathbb{1}_X$. Each map is called homotopy equivalence and g is said to be a homotopy inverse to f (and viceversa). On the other hand, two spaces are homeomorphic if $\exists f : X \to Y$, a bijective and continuous map, with continuous inverse ($f^{-1} : Y \to X$).

Certainly, any homeomorphism $f : X \to I$ is a homotopy equivalence with homotopy inverse f^{-1}.

(7) Let X be a space and $f: S^1 \to X$ a continuous map. Show that f is null-homotopic (homotopic to a constant map) if and only if there is a continuous map $g: D^2 \to X$ with $g|_{S^1} = f$.

Más adelante

The fundamental group

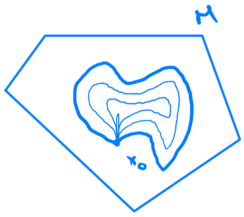
Loop: $f:[0,1] \to X$ continuous s.t. $f(0) = f(1) = x_0 \longrightarrow$ basepoint

Two $\boxed{\text{paths}}$ $f, g: [0,1] \to X$ s.t. $f(1) = g(0)$ we define $f \circ g = \begin{cases} f(2t) & \text{if } 0 \le t \le 1/2 \\ f(2t-1) & \text{if } 1/2 < t \le 1 \end{cases}$

continuous function from $[0,1] \to X$ $\Rightarrow$ A loop is a path from x to x.

The fundamental group is the set of all homotopy classes of loops $f:[0,1] \to X$ at basepoint x_0 and denoted by $\pi_1(X, x_0)$

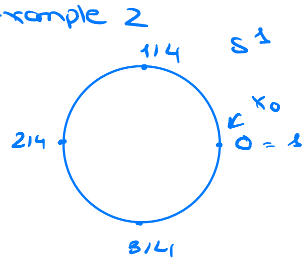
Example 1



All loops at x_0 are null-homotopic (homotopic to the constant loop $\alpha_0: [0,1] \to M$, $\alpha(t) = \alpha_0(0)$, $\forall t$

$\pi_1(M) = \pi_1(\text{disk}) = \pi_1(\text{polygon}) = \{[\alpha_0]\}$

Example 2



α_0 constant loop

$\alpha_1(t) = t$ $\qquad$ $\alpha_{-1}(t) = -t \bmod 1$

$\alpha_2(t) = 2t \bmod 1$ $\qquad$ $\alpha_{-2}(t) = -2t \bmod 1$

$\alpha_3(t) = 3t \bmod 1$

In general $[\alpha_m] \cdot [\alpha_n] = [\alpha_{m+n}]$

$\pi_1(S^1) = \{[\alpha_n] : n \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$

(8) Let X be a convex set, show that $\pi_1(X, x_0) = 0$

X is a convex set if given any 2 points of X, the set contains the whole line segment that joins them.

If X is a convex set and $\gamma: [0,1] \to X$ is a loop with $x_0 \in X$ as a basepoint, then γ is null-homotopic (homotopic to the constant loop $\alpha_0: [0,1] \to X$, $\alpha(t) = \alpha_0(0) = x_0$)

because we define the homotopy:

$f_t(x) = tx_0 + (1-t)\gamma(x)$, $F: X \times I \to X$, $F(x,0) = \gamma(x)$

$F(x,t) = f_t(x)$ $\quad F(x,1) = x_0 = \alpha_0(0)$

(It's well defined because X is convex)

As γ is any loop on X we conclude that $[\gamma] = [\alpha_0]$, $\forall \gamma$ loop and $\pi_1(X)$ has only one element (It's a trivial group).

(9) Let X be a path connected space, show that the fundamental group does not depend on the choice of the basepoint ($\pi_1(X, x_0) = \pi_1(X)$)

X is a <u>path connected space</u> if X is a topological space in which any two points can be joined by a continuous image of a simple arc, that is, a space X for any two points x_0 and x_1 on which there is a continuous mapping $f: I \to X$, $I = [0,1]$, s.t. $f(0) = x_0$ and $f(1) = x_1$.

Let X be a path connected set and take $x, y \in X$. Because it's path connected, exists f connecting x and y.

We will show that $\Phi: \pi_1(X, x) \to \pi_1(X, y)$ is an isomorphism.

$$[g] \longmapsto [f^{-1} g f]$$

- It's an homomorphism: $\Phi([gh]) = [f^{-1} gh f] = [f^{-1} g f f^{-1} h f] = [f^{-1} g f][f^{-1} h f] = \Phi(g)\Phi(f)$

- It's bijective: $\Phi^{-1}: \pi_1(X, y) \to \pi_1(X, x)$ $\quad \Phi \circ \Phi^{-1} = \mathbb{1}$

$$[g] \longmapsto [f g f^{-1}] \qquad \Phi^{-1} \circ \Phi = \mathbb{1}$$

So if X is a path connected set, the choice of the basepoint is not important.

(10) Let X, Y be a path connected spaces, then $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$

<u>Property</u>: A map $f: I \to X \times Y$ is continuous $\Leftrightarrow$ $f_1: I \to X$
$f_2: I \to Y$ are continuous.

$f(x) = (f_1(x), f_2(x))$

A loop $f: I \to X \times Y$ with basepoint (x_0, y_0) is equivalent to two loops $f_1: I \to X$ with basepoint x_0
$f_2: I \to Y$ with basepoint y_0

Similarly, a homotopy f_t of a loop in $X \times Y$ is equivalent to two homotopies f_t^1 and f_t^2 of the corresponding loop in X and Y respectively.

isomorphism between groups

Thus we have a correspondence $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$

$$[f] \mapsto ([f_1], [f_2])$$

(11) $T = S^1 \times S^1$, find $\pi_1(T)$

By exercise 10: $\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$

⑫ Please compute Euler characteristic χ for S^1, S^2, $S^1 \vee S^1$, $S^2 \vee S^2$, $S^1 \times S^1$ ...

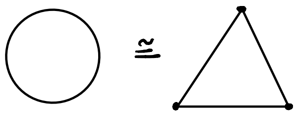
The Euler characteristic is a topological invariant, a number that describes a topological space's shape regardless the way it's bent.

space constructed by cell - complex

For a finite CW complex Σ, the Euler characteristic $\chi(\Sigma)$ is defined to be $\sum_n (-1)^n c_n$, where c_n is the numbers of n-cells of Σ, generalizing the familiar formula vertices - edges + faces for 2-dimensional complexes.

<u>Euler Characteristic via triangulations</u>: Let K be a simplicial complex of dim $K = n$. Let α_i the number of i-simplices of K, then, $\chi(K) = \sum_{i=0}^{n} (-1)^n \alpha_i$

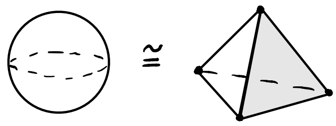
S^1



dim $K = 2$

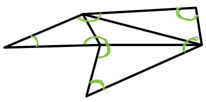
$\alpha_0 = 3$
$\alpha_1 = 3$ $\quad \chi(K) = 3 - 3 + 0 = 0$
$\alpha_2 = 0$

S^2

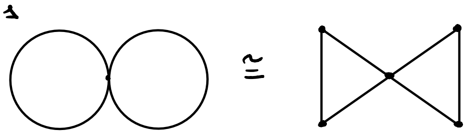


dim $K = 3$

$\alpha_0 = 4$
$\alpha_1 = 6$ $\quad \chi(K) = 4 - 6 + 4 = 2$
$\alpha_2 = 4$



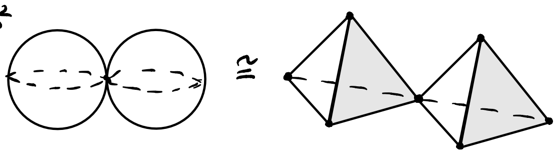
$S^1 \vee S^1$



dim $K = 2$

$\alpha_0 = 5$
$\alpha_1 = 6$ $\quad \chi(K) = 5 - 6 + 0 = -1$
$\alpha_2 = 0$

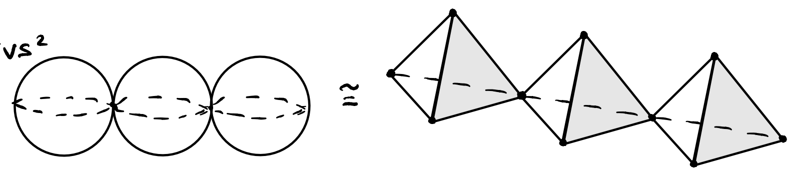
$S^2 \vee S^2$



dim $K = 3$

$\alpha_0 = 7$
$\alpha_1 = 12$ $\quad \chi(K) = 7 - 12 + 8 = 3$
$\alpha_3 = 8$

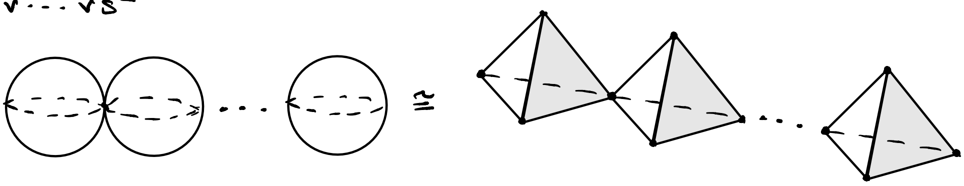
$S^2 \vee S^2 \vee S^2$



$\alpha_0 = 10$
$\alpha_1 = 18$ $\quad \chi(K) = 10 - 18 + 12 = 4$
$\alpha_3 = 12$

⋮

$S^2 \vee \overset{n)}{\cdots} \vee S^2$



$\alpha_0 = 3n + 1$
$\alpha_1 = 6n$ $\quad \chi(K) = 3n+1-6n+4n =$
$\alpha_2 = 4n$ $\quad = n+1$

<u>Euler characteristic via Homology groups</u>: $\chi(\Sigma) = \sum_n (-1)^n \operatorname{Rank} H_n(\Sigma)$

$H_n(\Sigma) = \mathbb{Z}^r \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_p}$

$\operatorname{Rank} H_n(\Sigma) = r$

Le quitas un abierto a cada conjunto y los unes por ese abierto

Unión conexa

(13) Using the property from exercise 20.15 - Greenberg, please show that:

1) N - whole torus (connected sum of n torus), $\chi(T \# T \# T \cdots \# T) = 2-2n$

2) N - whole projective space, $\chi(\mathbb{RP}^2 \# \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2) = 2-n$

<u>Property</u>: $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$

$$\chi(T \# \overset{n)}{\cdots} \# T) = \chi(T \# (T \# \overset{n-1)}{\cdots} \# T)) = \overset{\overset{0}{\shortparallel}}{\chi(T)} + \chi(T \# \overset{n-1)}{\cdots} \#) - 2 =$$
$$= \chi(T \# (T \# \overset{n-2}{\cdots} \# T)) - 2 = \overset{\overset{0}{\shortparallel}}{\chi(T)} + \chi(T \# \overset{n-2}{\cdots} \# T) - 2 - 2 = \cdots$$
$$\cdots = \overset{\overset{0}{\shortparallel}}{\chi(T)} + \chi(T \# T) - (n-2)2 =$$
$$= \overset{\overset{0}{\shortparallel}}{\chi(T)} + \overset{\overset{0}{\shortparallel}}{\chi(T)} - 2 - 2n + 4 = \underline{2-2n}$$

$$\chi(S^n) = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even} \end{cases}, \quad \chi(S^1 \times \overset{n)}{\cdots} \times S^1) = \chi(S^1) \cdot \chi(S^1) \cdots = 0$$

$$\chi(\mathbb{RP}^n) = \frac{1}{2}\chi(S^n) \Rightarrow \chi(\mathbb{RP}^2) = \frac{1}{2}\chi(S^2) = 1, \text{ so:}$$

$$\chi(\mathbb{RP}^2 \# \overset{n)}{\cdots} \# \mathbb{RP}^2) = \chi(\mathbb{RP}^2 \# (\mathbb{RP}^2 \# \overset{n-1)}{\cdots} \# \mathbb{RP}^2)) = \overset{\overset{1}{\shortparallel}}{\chi(\mathbb{RP}^2)} + \chi(\mathbb{RP}^2 \# \overset{n-1)}{\cdots} \# \mathbb{RP}^2) - 2 =$$
$$= \chi(\mathbb{RP}^2 \# (\mathbb{RP}^2 \# \overset{n-2)}{\cdots} \# \mathbb{RP}^2)) - 1 = \overset{\overset{1}{\shortparallel}}{\chi(\mathbb{RP}^2)} + \chi(\mathbb{RP}^2 \# \overset{n-2)}{\cdots} \# \mathbb{RP}^2) - 2 - 1 = \cdots$$
$$\cdots = \chi(\mathbb{RP}^2 \# \mathbb{RP}^2) - 1 \cdot (n-2) = 1 + 1 - \cancel{2} - n + \cancel{2} = \underline{2-n}$$

<u>Notes</u>

<u>Def</u>: A <u>simplicial complex</u> K on a subset of vertices V_K is a family of simplicies S_K with the following properties:

1. Each vertex of K is a simplex, that is, K contains all elements of V_K.
2. Every non-empty subset of S_K belongs to K.

A simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions.

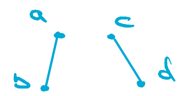
- 0 - dimensional → A point
- 1 - dimensional → Line - segment
- 2 - dimensional → Triangle
- 3 - dimensional → Tetrahedron
- 4 - dimensional → 5 - cell
- ⋮

K - simplex is a K - dimensional polytope which is the <u>convex hull</u> of its K+1 vertices.

↳ Smallest convex set that contains it

<u>Example</u>

$K = \{\langle a,b\rangle, \langle c,d\rangle, \langle a\rangle, \langle b\rangle, \langle c\rangle, \langle d\rangle\}$ is a simplicial complex on vertices $\{a,b,c,d\}$



<u>Def</u>: If s is a simplex with $(n+1)$ elements, we will say that s has dimension n. A simplicial complex K has dimension n if K has, at least, one n-simplex, but does not contain a simplex of higher dimension.

<u>Def</u>: Consider K a simplicial complex. Given $s, s' \in K$, two simplicies, we say that s' is a <u>face</u> of s if $s' \subseteq s$ (proper face if $s' \neq s$)

Def: Given a simplex s, we define the <u>clausure</u> of s as the simplicial complex made of all faces of s and we define the <u>skeleton</u> of s, $\dot{s}$, as the simplicial complex made of the proper faces of s.

<u>Example</u>: Consider the simplex $s = \langle a,b,c \rangle$

$cl(s) = \{ \langle a,b,c \rangle, \langle a,b \rangle, \langle b,c \rangle, \langle a,c \rangle, \langle a \rangle, \langle b \rangle, \langle c \rangle \}$

$\dot{s} = \{ \langle a,b \rangle, \langle b,c \rangle, \langle a,c \rangle, \langle a \rangle, \langle b \rangle, \langle c \rangle \}$

Def: Let K be a simplicial complex of dimension n and consider $p \in \mathbb{N}$, $0 \leq p \leq n$. The <u>p-skeleton</u> of K is the simplicial complex made of the simplicies of K with dimension less or equal to p.

The 1-skeleton of $cl(s)$ is $\dot{s}$

Triangulable spaces and Polyhedra

Def: A collection of points $\{a_0, \dots, a_k\}$ in $\mathbb{R}^n$ are linealy independent if the vectors $\overrightarrow{a_0a_1}, \dots, \overrightarrow{a_0a_k}$ are linearly independent.

Given $\{a_0, \dots, a_k\}$ a collection of independent points, the $\boxed{k\text{-simplex}}$ generated by them $\underbrace{\langle a_0, \dots, a_k \rangle}_{\text{closed subset of } \mathbb{R}^n} = \{ x \in \mathbb{R}^n \mid x = \sum_{i=0}^{k} \lambda_i a_i,\ \lambda_i \geq 0,\ \sum_{i=0}^{k} \lambda_i = 1 \}$

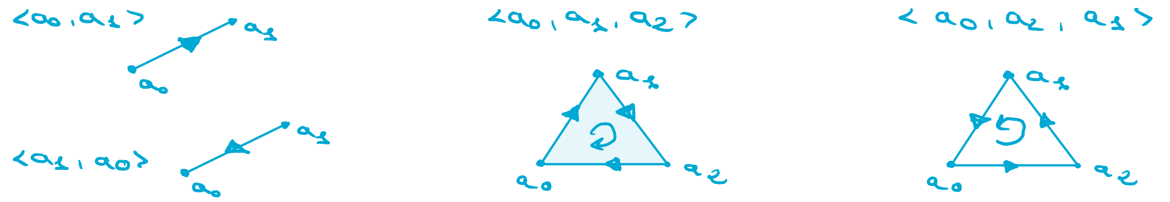
Geometrical

Def: A <u>geometrical simplicial complex</u> K is a simplicial complex so that the vertices are in an Euclidean space $\mathbb{R}^n$.

The union of simplicies of a geometrical simplicial complex K with usual topology in $\mathbb{R}^n$ is denoted by $|K|$. It's a bounded and closed set $\Rightarrow |K|$ is compact.

Def: We say that a topological space (X, τ) is <u>triangulable</u> if there are a geometrical simplicial complex K and a homeomorphism $f: |K| \to X$. In this context, (K, f) is a triangulation of the space (X, τ). We sometimes say that a triangulable space is a polyhedron.

Def: An <u>orientated p-simplex</u> is an ordered collection of $(p+1)$ points. Notice that for each p-simplex (without order), there are $(p+1)!$ different oriented p-simplicies.



Def: Given σ_1 and σ_2 two oriented p-simplicies on the same set of vertices, we will say that $\langle a_0, \dots, a_p \rangle = \sigma_1 \underset{\text{equivalent}}{\sim} \sigma_2 = \langle b_0, \dots, b_p \rangle$ if there is an <u>even permutation</u> ρ so that $b_i = a_{\rho(i)}$

Número par de trasposiciones

Def: A simplicial complex K is oriented if its simplicies are oriented.

p- Chain Groups of a simplicial complex

Consider K an oriented simplicial complex. We define the <u>space of the p-chains</u> as

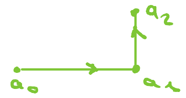
$$C_p(K) = \left\{ \sum_{i=1}^{N_p} m_i \cdot \sigma_i^p \ : \ \sigma_i^p \text{ is an oriented } p\text{-simplex of } K,\ m_i \in \mathbb{Z} \right\}$$

with N_p the number of p-simplices of K. In $C_p(K)$ we introduce the operation given by

$$\left(\sum_{i=1}^{N_p} m_i \cdot \sigma_i^p \right) + \left(\sum_{i=1}^{N_p} \tilde{m}_i \cdot \sigma_i^p \right) = \sum_{i=1}^{N_p} (m_i + \tilde{m}_i)\, \sigma_i^p$$

The geometrical meaning of the elements of $C_p(K)$ is as follows:

- $m\langle a_0, \dots, a_p \rangle$ with $m > 0$ means that you run m-times the complex $\langle a_0, \dots, a_p \rangle$
- $m\langle a_0, \dots, a_p \rangle$ with $m < 0$ means that you run m-times the complex $\langle a_p, \dots, a_0 \rangle$
- The general case $m_1 \langle a_0, \dots, a_p \rangle + m_2 \langle \tilde{a}_0, \dots, \tilde{a}_p \rangle$ means you concatenate both movements, i.e., $\langle a_0, a_1 \rangle - \langle a_2, a_1 \rangle$



<u>Def</u>: Consider K an orientated simplicial complex. Given σ^p and $\sigma^{p-1} \in K$, we define the number $[\sigma^p, \sigma^{p-1}]$ as follows:

- If σ^{p-1} has a vertex that is not vertex of σ^p $\Rightarrow [\sigma^p, \sigma^{p-1}] = 0$
- If all vertices of σ^{p-1} are vertices of σ^p $\Rightarrow \exists!$ vertex $v \in \sigma^p$ s.t. $v \notin \sigma^{p-1}$. We have that $[\sigma^p, \sigma^{p-1}] = 1$ if $\sigma^{p-1} = \langle a_0, \dots, a_{p-1} \rangle$ and $\underline{\sigma^p = \langle v, a_0, \dots, a_{p-1} \rangle}$, otherwise $[\sigma^p, \sigma^{p-1}] = -1$

Orientated (Position counts). Also see permutations.

Homology Groups

<u>Boundary Map</u> → consider K an oriented simplicial complex. Let $\{\sigma_1^p, \dots, \sigma_{n_p}^p\}$ and $\{\sigma_1^{p-1}, \dots, \sigma_{n_{p-1}}^{p-1}\}$ be the collection of p-simplices and $(p-1)$-simplices of K. Then, we define the p-boundary map as $\mathbb{Z}$-linear map

$$\partial_p : C_p(K) \longrightarrow C_{p-1}(K) \text{ so that}$$
$$\partial_p(\sigma_j^p) \longmapsto \sum_{k=1}^{N_{p-1}} [\sigma_j^p, \sigma_k^{p-1}]\, \sigma_k^{p-1}$$

$$\partial_m : C_m(K) \longrightarrow C_{m-1}(K)$$
$$\partial_{m-1} : C_{m-1}(K) \longrightarrow C_{m-2}(K)$$
$$\vdots$$
$$\partial_1 : C_1(K) \longrightarrow C_0(K)$$

<u>Theorem</u>: Consider K an orientated simplicial complex and $\sigma^p, \sigma^{p-2} \in K$ $(2 \leq p \leq \dim K)$ so that σ^{p-2} is a face of σ^p, then:

$$\sum_{\sigma_j^{p-1} \in K} [\sigma^p, \sigma_j^{p-1}][\sigma_j^{p-1}, \sigma^{p-2}] = 0$$

In summary, imagine that K is a orientated simplicial complex with $\dim K = m$, then we have that:

$$\partial_{m-1} \circ \partial_m : C_m(K) \longrightarrow C_{m-2}(K), \quad \partial_{m-1} \circ \partial_m = 0 \qquad (\partial^2 = 0)$$

$$\boxed{\operatorname{Im} \partial_p \subset \operatorname{Ker} \partial_{p-1}}$$

Def: A p-chain z_p with $z_p \in \ker(\partial_p)$ is a <u>p-cycle</u>. We denote by $Z_p(K)$ the set of all p-cycles of K. For $p=0$, $Z_0(K) = C_0(K)$

Def: A p-chain $b_p \in C_p(K)$ so that there is $c_{p+1} \in C_{p+1}(K)$ with $b_p = \partial_{p+1}(c_{p+1})$ is called <u>p-boundary</u>. The set of all p-boundaries of K is denoted by $B_p(K)$
If $p = \dim K \Rightarrow B_p(K) = \{0\}$.

By the previous comments, $B_p(K) \subset Z_p(K)$.

<u>Very important</u>: We define the <u>homology group</u> of order p of K as the quotient group
$$Z_p(K)/B_p(K) = H_p(K).$$

<u>Remark</u>: By simple results in Algebra, a subgroup of a $\boxed{\text{free abelian group}}$ is always a free abelian group. Hence, $Z_p(K)$ and $B_p(K)$ are free abelian groups that are finitely generated. However, $H_p(K) = \frac{Z_p(K)}{B_p(K)}$ is an abelian group that is finitely generated but not always free, that is:
$$H_p(K) = G \oplus T_1 \oplus \cdots \oplus T_k, \text{ with } G \text{ a free abelian group}$$
and T_i a cyclic group of finite order.

G es un grupo libre si $\exists S \subseteq G$ tq todo elemento de G se puede expresar de forma única como producto finito de elementos de S y sus inversos.

<u>Example</u>: $\mathbb{Z}^2 \oplus \underset{\{0,1\}}{\mathbb{Z}_2}$

<u>Theorem</u>: Consider K a s.c. Let K_1, K_2 be two orientated simplicial complex obtained from K $\Rightarrow H_p(K_1) \approx H_p(K_2), \forall p$

Informally speaking, to define the homological group of a simplicial complex K, we have to introduce an orientation in all simplices of K. The previous result says that the homological group does not depend on the orientation we choose.

(14) Homology group computations.

<u>Apuntes de los videos</u>

<u>Computing homology groups</u>

X space
$$\cdots\ C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$
3-dim chains, 2-dim chains, 1-dim chains, 0-dim chains

$\boxed{\partial^2 = 0}$

If you $\partial_{n-1} \circ \partial_n = 0$
$C_n \to C_{n-1} \to C_{n-2}$

$H_1 = Z_1/B_1$

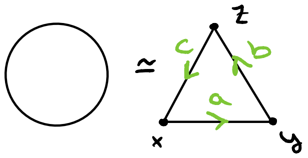
$\ker(\partial_1) = Z_1$ group of cycles $\subseteq C_1$
$\operatorname{Img}(\partial_2) = B_1$ group of boundaries $\subseteq C_1$

Same stage $H_n = Z_n/B_n$

Note that $B_1 \subseteq Z_1$, because you take an element from $C_2 \xrightarrow{\partial_2}$ you get a boundarie $\xrightarrow[\partial^2 = 0]{\partial_1}$ you get 0 $\Rightarrow$ It's a cycle

Some examples

Ex 1 $X = S^1$



skeleton of a Δ

3 line segments

vertices: $\{x, y, z\}$

We have 3 1-simplices (segments)

Let's provide them an orientation

$a = \langle x, y\rangle \quad b = \langle y, z\rangle \quad c = \langle z, x\rangle$

So X as an orientaded simplicial complex is $X = \{\langle x,y\rangle, \langle y,z\rangle, \langle z,x\rangle, \langle x\rangle, \langle y\rangle, \langle z\rangle\}$

Let's look up for $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

- $C_0 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ (because the bases consists of the 3 vertices)
  $\quad x \quad y \quad z$ A tipycall element in C_0 is $\alpha x + \beta y + \gamma z$, $\alpha, \beta, \gamma \in \mathbb{Z}$
- $C_1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ (because the bases are a, b, c)
  $\quad a \quad b \quad c$ A tipycall element in C_1 is $ma + nb + \ell c$, $m, n, \ell \in \mathbb{Z}$
- $C_2 = 0$ (because we are not considering the solid triangle)
- $\partial_0 = 0$ (because it sends all to 0)
  $x \mapsto 0$
  $y \mapsto 0$
  $z \mapsto 0$
- ∂_1: $a \mapsto y - x$
  $b \mapsto z - y$
  $c \mapsto x - z$
- ∂_2: $0 \mapsto 0$

Now, we have to compute in each stage Z_p and B_p:

- $Z_0 = \operatorname{Ker}(\partial_0) = C_0 = \langle x, y, z\rangle$ (Everything is sent to 0)
- $B_0 = \operatorname{Im}(\partial_1) = \langle y - x, z - y, x - z\rangle$
- what is $H_0 = \dfrac{Z_0}{B_0} = \dfrac{\langle x, y, z\rangle}{\langle y-x, z-y, x-z\rangle}$

Quotient by $B_0 \approx$ setting elements of B_0 to 0.

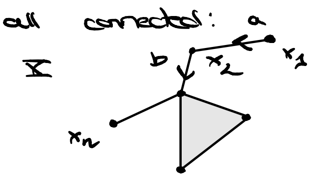
$$\left.\begin{array}{l} y - x = 0 \Rightarrow x = y \\ z - y = 0 \Rightarrow z = y \end{array}\right\} \Rightarrow x = y = z \Rightarrow H_0 \simeq \mathbb{Z}$$ (because every element is of form $nx + B_0$)

- $Z_1 = \operatorname{Ker}(\partial_1)$
  $\partial(\ell a + mb + nc) = \ell(y - x) + m(z - y) + n(x - z) = (n - \ell)x + (\ell - m)y + (m - n)z = 0$
  $\Rightarrow n = \ell,\ \ell = m,\ m = n \Rightarrow \ell = m = n \Rightarrow \underline{\operatorname{Ker}(\partial_1) = \langle a + b + c\rangle}$
  $\hookrightarrow$ Múltiplos de $a+b+c$ (Significa que los ciclos son recorrer el perímetro del triángulo)
- $B_1 = \operatorname{Im}(\partial_2) = 0$
- $H_1 \cong Z_1 / B_1 \cong Z_1 / 0 \cong Z_1 \cong \mathbb{Z}$ (because elements are $n(a+b+c)$)

Conclusion: $H_0(S^1) = H_1(S^1) = \mathbb{Z}$, $\quad H_n(S^1) = 0, \ \forall n \geq 2$

Question: What does $H_0(X)$ measure?

The number of conected components. Suppose a more complicated space, but all conected:



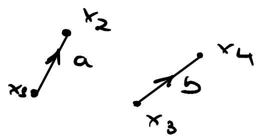
X

$C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

$Z_0 = C_0 = \ker(\partial_0)$

$B_0 = \operatorname{Im}(\partial_1) = \langle x_2 - x_1, x_3 - x_2, \dots\rangle$

Equating all elements of $B_0 \equiv 0 \Rightarrow$ Setting all vertices equal to each other $\Rightarrow x_1 = x_2 = \dots = x_n. \Rightarrow$ quotient is nx that is $\mathbb{Z}$.

If $\mathcal{X}$ 

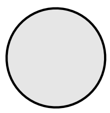
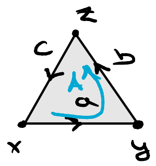
$$Im(\partial_1) = \langle x_2 - x_1, x_4 - x_3 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$Ker(\partial_0) = \langle x_1, x_2, x_3, x_4 \rangle \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$H_0 = \frac{Ker(\partial_0)}{Im(\partial_1)} \cong \boxed{\mathbb{Z} \oplus \mathbb{Z}}$$

→ one for each connected component

Ex 2 $\mathcal{X} = D$, disk

 $\cong$ 

2 - simplex

It's also the same but changing C_2: $0 = C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

Because we have a 2-cell (A) ← $\mathbb{Z}$ $\qquad \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \qquad \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

$A = \langle x, y, z \rangle$

H_0 is the same that before ($H_0 \cong \mathbb{Z}$)

- ▶ $Z_1 = Ker(\partial_1) = \langle a+b+c \rangle$
- ▶ $B_1 = Im(\partial_2) = \langle a+b+c \rangle$

$$\partial_2(A) = \langle yz \rangle - \langle xz \rangle + \langle xy \rangle = b - (-c) + a = a+b+c$$

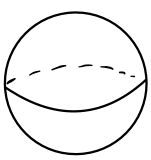
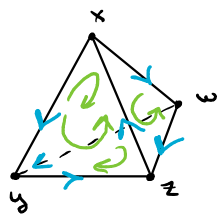
"Quitas la primera componente" menos "quitas la segunda componente" más "quitas la tercera componente".

- ▶ $H_1 = \frac{Z_1}{B_1} = \frac{\langle a+b+c \rangle}{\langle a+b+c \rangle} = 0$

- ▶ $Z_2 = Ker(\partial_2) = 0$ $\qquad (A \mapsto a+b+c)$
- ▶ $B_2 = Im(\partial_3) = 0$
- ▶ $H_2 = \frac{0}{0} = 0$

Conclusion: $H_0(D) = \mathbb{Z}$, $H_n(D) = 0$, $\forall n \geq 1$

Ex 3 S^2

 $\cong$ 

$$\mathcal{X} = \{ \overset{A}{\langle x, y, z \rangle}, \overset{B}{\langle x, z, w \rangle}, \overset{C}{\langle y, w, z \rangle}, \overset{D}{\langle x, w, y \rangle} \}$$

Those are the maximum simplicies. The space is determinated by them, but, $\mathcal{X}$ has more

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

- ▶ $C_0 = \mathbb{Z}^4 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ (Because we have 4 vertices)
  $\langle x, y, z, w \rangle$
- ▶ $C_1 = \mathbb{Z}^6 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ $\qquad a = \langle x, y \rangle$, $b = \langle y, z \rangle$, $c = \langle z, x \rangle$
  $\langle a, b, c, d, e, f \rangle$ $\qquad d = \langle x, w \rangle$, $e = \langle w, z \rangle$, $f = \langle w, y \rangle$
- ▶ $C_2 = \mathbb{Z}^4 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
  $\langle A, B, C, D \rangle$
- ▶ $C_3 = 0$ (Because it's not a solid tetrahedron)

- $Z_0 = \operatorname{Ker}(\partial_0) = C_0 = \langle x, y, z, w \rangle$
- $B_0 = \operatorname{Im}(\partial_1) = \langle y-x,\ z-y,\ x-z,\ w-x,\ z-w,\ y-w \rangle$

$$\partial_1 : C_1 \longrightarrow C_0$$

$$\begin{array}{llll} a \longmapsto y-x & & d \longmapsto w-x \\ b \longmapsto z-y & & e \longmapsto z-w \\ c \longmapsto x-z & & f \longmapsto y-w \end{array}$$

- $Z_0 / B_0 = \mathbb{Z} = \langle x + B_0 \rangle$

$$\left.\begin{array}{l} y-x=0 \Rightarrow x=y \\ z-y=0 \Rightarrow y=z \\ w-x=0 \Rightarrow x=w \end{array}\right\} \Rightarrow x=y=w=z$$

- $Z_1 = \operatorname{Ker}(\partial_1)$

$$\alpha(y-x) + \beta(z-y) + \gamma(x-z) + \delta(w-x) + \varepsilon(z-w) + \eta(y-w) = 0$$

$$(\gamma-\alpha-\delta)x + (\alpha-\beta+\eta)y + (\beta-\gamma+\varepsilon)z + (\delta-\varepsilon-\eta)w = 0$$

4 equations with 6 unknowns

Don't use fractions, you are in a group !!

$$\begin{array}{cccccc} \alpha & \beta & \gamma & \delta & \varepsilon & \eta \end{array}$$
$$\begin{pmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix} \underset{R_2 \to R_1+R_2}{\overset{R_1 \to -R_1}{\sim}} \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix} \sim$$

$$\underset{R_3 \to R_3+R_2}{\overset{R_2 \to -R_2}{\sim}} \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix} \underset{R_4 \to R_4+R_3}{\overset{R_3 \to -R_3}{\sim}} \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim$$

$$\underset{R_2 \to R_2-R_3}{\overset{R_1 \to R_1-R_3}{\sim}} \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \left.\begin{array}{l} \alpha = r-s-t \\ \beta = r-s \\ \gamma = r \\ \delta = s+t \\ \varepsilon = s \\ \eta = t \end{array}\right\} \Rightarrow$$

(with $\gamma \leftarrow r$, $\varepsilon \leftarrow s$, $\eta \leftarrow t$)

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \\ \eta \end{pmatrix} = r\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow$$

All solutions are multiples of cycles:

$$\alpha+\beta+\gamma$$
$$-\alpha-\beta+\delta+\varepsilon$$
$$-\alpha+\delta+\eta$$

$\Rightarrow$

$$\Rightarrow Z_1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^3$$ (We have 3 elements generating it)

Reescribirlo : $\langle \alpha+\beta+\gamma \, , -(\gamma+\delta+\varepsilon) \, , -\beta+\varepsilon-\eta \, , \delta+\eta-\alpha \rangle$

▸ $B_1 = \mathrm{Im}(\partial_2) = \langle a+b+c, \; -(e+d+c), \; e-b-f, \; f+d-a \rangle$

$\partial_2 : C_2 \longrightarrow C_1$

$$A \longmapsto \langle y,z\rangle - \langle x,z\rangle + \langle x,y\rangle = b-(-c)+a = a+b+c$$
$$B \longmapsto \langle z,w\rangle - \langle x,w\rangle + \langle x,z\rangle = -e-d-c$$
$$C \longmapsto \langle w,z\rangle - \langle y,z\rangle + \langle y,w\rangle = e-b-f$$
$$D \longmapsto \langle wy\rangle - \langle x,y\rangle + \langle x,w\rangle = f+d-a$$

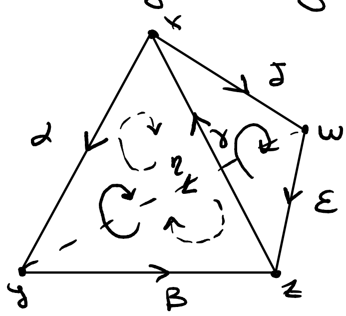
▸ $Z_1/B_1 = 0 \quad (Z_1 = B_1)$ (we know that $B_1 \subset Z_1$)

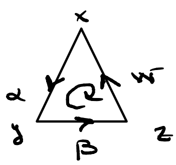
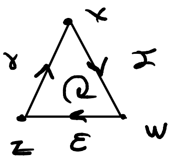
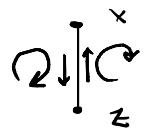
$$Z_1 = \langle \boxed{\alpha+\beta+\gamma}, \; -\alpha-\beta+\delta+\varepsilon, \; \boxed{-\alpha+\delta+\eta} \rangle \quad \Rightarrow B_1 = Z_1$$
$$B_1 = \langle \boxed{\alpha+\beta+\gamma}, \; \underline{-(\gamma+\delta+\varepsilon)}, \; \boxed{-\alpha+\delta+\eta}, \; -\beta+\varepsilon-\eta \rangle$$

$$(\alpha+\beta+\gamma)+(-\alpha-\beta+\delta+\varepsilon) = \underline{\gamma+\delta+\varepsilon}$$

▸ $Z_2 = \mathrm{Ker}(\partial_2)$

we what to find what combination of A, B, C, D will have bondary 0. Instead of doing it using algebra like the previous step, let's see it geometrically.



Let's take a face  if I want to get 0 as a total sum I need the edge $\langle z,x\rangle$ to be conceled with something else and the only thing that can be conced with is when I continue the boundary of the other face that meet that edge γ  and if we see  they are going in the opposite way so that's means that the only way of that contribution to be 0 is that the coefficient of both faces are equal. We can repeat this in all edges and we will get that:

$$\partial_2(\langle x,y,z\rangle + \langle x,z,w\rangle + \langle y,w,z\rangle + \langle x,w,y\rangle) = 0 \Rightarrow Z_2 = \langle A+B+C+D\rangle = \mathbb{Z}$$

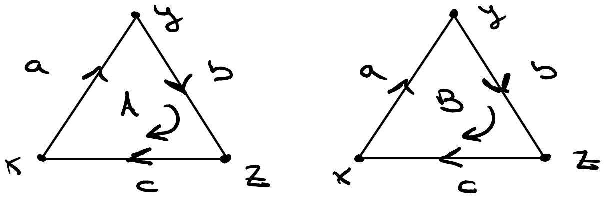
▸ $B_2 = \mathrm{Im}(\delta_3) = 0$

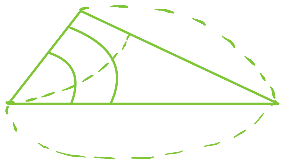
▸ $H_2 = \dfrac{Z_2}{B_2} = Z_2 = \mathbb{Z}$

Conclusion: $H_0(S^2) = \mathbb{Z}$, $H_1(S^2) = 0$, $H_2(S^2) = \mathbb{Z}$ and $H_n(S^2) = 0$, $\forall n \geq 3$

Another way: using Δ-complex or semi-simplicial complex. We allow ourselfs to think not necessary on a geometrical object in the space made of classical simplices but prepare our simplices to be curve and allow them to meet themselves at the boundaries.

Using this, another computation of S^2





Now our computation will be much easier because:

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$$\langle A, B\rangle \qquad \langle a,b,c\rangle \qquad \langle x,y,z\rangle$$

- $Z_1 = \ker(\partial_1) = \mathbb{Z}$

  $\alpha a + \beta b + \gamma c = \alpha(y-x) + \beta(z-y) + \gamma(x-z) = 0$

  $x(-\alpha+\gamma) + y(\alpha-\beta) + z(\beta-\gamma) = 0 \implies$

  $\implies \alpha = \beta = \gamma = Z_1 = \langle a+b+c\rangle$

- $B_1 = \operatorname{Im}(\partial_2) = \langle a+b+c\rangle = \mathbb{Z}$

  $\partial(A) = a+b+c$

  $\partial(B) = a+b+c$

- $H_1 = Z_1 / B_1 = 0 \qquad (Z_1 = B_1)$

- $Z_2 = \ker(\partial_2) = \langle A - B\rangle \simeq \mathbb{Z}$
- $B_2 = \operatorname{Im}(\partial_2) = 0$
- $H_2 = Z_2 / B_2 \simeq \mathbb{Z}$

Ex 4: D^3

All the same, except

$$0 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2$$

$$\mathbb{Z} \qquad \mathbb{Z}^4$$

We have a 3-cell $\langle x, y, z, w\rangle$

- $Z_3 = \ker(\partial_3) = 0 \qquad$ (3-cell $\mapsto A+B+C+D$)
- $B_3 = \operatorname{Im}(\partial_4) = 0$
- $H_3 = Z_3 / B_3 = 0$

- $Z_2 = \ker(\partial_2) = \mathbb{Z}$
- $B_2 = \operatorname{Im}(\partial_3) = \mathbb{Z}$
- $H_2 = 0$

Applications of homology

First application: We are going to use homology theory to prove that two topological spaces are not homeomorphic.

Recall: (X, T) is triangulable if there is a simplicial complex K and $\phi: |K| \to X$ a homeomorphism.

Theorem 1: Consider (X, T_1) and (Y, T_2) two t.s. and assume X and Y are triangulable with K_1 and K_2 associated triangulations. If there is $f: X \to Y$ a homeomorphism $\Rightarrow H_p(K_1) \cong H_p(K_2)$, $\forall p \in \mathbb{N}$.

Corollary: Consider B_N and B_M the Euclidean balls with center at the origin and radius 1 in $\mathbb{R}^N$ and $\mathbb{R}^M$. If $N \neq M \Rightarrow \nexists\, f: B_M \to B_N$ homeomorphism.

Def: Consider K a simplex complex with $\dim K = n$. Let $H_p(K) \cong \underbrace{\mathbb{Z}^{B_p}}_{\text{Free part}} \oplus \underbrace{\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_p}}_{\text{Torsion Part}}$

The exponent B_p is the p-th Betti's number of K.

Theorem (Euler-Poincaré): Assume K is a S.C. of $\dim K = n$ and let α_i be the number of i-simplices of K, then

$$\chi(K) = \sum_{i=0}^{n} (-1)^i \alpha_i = \sum_{i=0}^{n} (-1)^i \beta_i$$

Definition: A polyhedron in $\mathbb{R}^3$ is a compact, connected set limited by convex poligons and homeomorphic to S^2. We refer to these polygons as faces, the intersection of two faces is an edge and the intersection of two edges is a vertex. If a polyhedron has V vertices, E edges and F faces $\Rightarrow V - E + F = 2$.

7) Let X be a space and $f: S^1 \to X$ a continuous map. Show that f is null-homotopic (homotopic to a constant map) if and only if there is a continuous map $g: D^2 \to X$ with $g|_{S^1} = f$.

We are going to prove the next proposition instead:

Let X be a space. The following are equivalent:

1) Every map $S^1 \to X$ is homotopic to a constant map with image a point
2) Every map $S^1 \to X$ extends to a map $D^2 \to X$
3) $\pi_1(X, x_0) = 0$, $\forall x_0 \in X$

Proof

(1) ⇒ (2)| Let $f: S^1 \to X$ be a continuous map. We think of D^2 and S^1 as a subset of $\mathbb{R}^2$, and we will write points using polar coordenates (r, θ), $r \in [0,1]$.

If f is null-homotopic, there is a homotopy $f_t: S^1 \to X$ s.t. $f_1 = f$ and f_0 is the constant map $(r, \theta) \mapsto x_0$, $x_0 \in X$.

We can define $g: D^2 \to X$ as $g(r, \theta) = f_r(1, \theta)$. Let's see it's well-defined.

If $(r, \theta) = (r', \theta')$, with $r, r' > 0 \Rightarrow r = r'$ and $\theta' = \theta + 2\pi n$, for some $n \in \mathbb{Z}$.

Then $(1, \theta) = (1, \theta') \Rightarrow g(r', \theta') = f_r(1, \theta') = f_r(1, \theta) = g(r, \theta)$, so g is well-defined for $r > 0$. For $r = 0$, $g(0, \theta) = f_0(1, \theta) = x_0$, $\forall \theta$, so g is well-defined at $r = 0$.

The map g extends f because $g(1, \theta) = f_1(1, \theta) = f(1, \theta)$ ($\Rightarrow g|_{S^1} = f$, because if $x \in S^1 \Rightarrow x = (1, \theta)$)

Finally, let's see it's continuous. If $r > 0$ it's continuous by continuity of f_t.

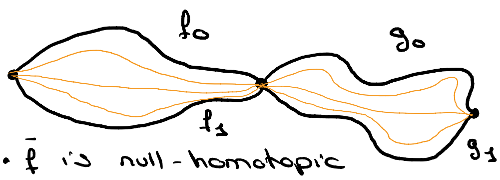
For $r = 0$, we take any sequence (r_n, θ_n) limiting to $(0, \theta)$ ($\lim_n r_n = 0$), so by continuity of f_t: $\lim_n g(r_n, \theta_n) = \lim_n f_{r_n}(1, \theta) = f_0(1, \theta) = g(0, \theta)$ hence g continuous at $(0, \theta)$.

(2) ⇒ (3)| Let $x_0 \in X$ and let $[f] \in \pi_1(X, x_0)$. Choose a representative loop $f: S^1 \to X$ based at x_0. By hypothesis, f extends to a continuous map $\tilde{f}: D^2 \to X$. Intuitively speaking, we should be able to contract the loop f through the image of the disk in X. To make this precise, we think of D^2 as the unit disk in $\mathbb{R}^2$ and define $D^2 \to X$ by $f_t(x,y) = \tilde{f}((1-t)x + t, (1-t)y)$. Note that this is well defined since $((1-t)x + t, (1-t)y) \in D^2$. Then $f_0 = \tilde{f}$ and f_1 is the constant map sending evrything to $(1,0)$. Now consider $f_t|_{S^1}: S^1 \to X$. We have $f_0|_{S^1} = \tilde{f}|_{S^1} = f$, $f_1|_{S^1}(x,y) = (1,0)$, thus, f homotopic to a constant map, so $[f] = 0$, so $\pi_1(X, x_0) = 0$

(3) ⇒ (1)| Let $f: S^1 \to X$ be a continous map. Let x_0 be somepoint in the image of f. Then $[f] \in \pi_1(X, x_0)$, which is a trivial group by hypothesis, so f is nullhomotopic.

(15) Show that the composition of path satisfies the following cancellation property: if $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1 \Rightarrow f_0 \simeq f_1$.

$$\left.\begin{array}{l} f_0 \cdot g_0 \simeq f_1 \cdot g_1 \\ g_0 \simeq g_1 \end{array}\right| \Rightarrow f_0 \simeq f_1$$



If $\bar{f}$ is the inverse path ($\bar{f}(t) = f(1-t)$), we know $f \cdot \bar{f}$ is null-homotopic

$$f_0 \simeq f_0 \cdot (g_0 \cdot \bar{g}_0) \simeq (f_0 \cdot g_0) \cdot \bar{g}_0 \simeq (f_1 \cdot g_1) \cdot \bar{g}_0 \simeq f_1 (g_1 \cdot \bar{g}_0) \simeq f_1 (g_0 \cdot \bar{g}_0) \simeq f_1$$

Theorem 1.8: Every nonconstant polynomial with coefficients in $\mathbb{C}$ has root in $\mathbb{C}$.

We assume the polynomial is of the form $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. If $p(z)$ has no roots in $\mathbb{C}$, then for each number $r \geq 0$, the formula

$$f_r(s) = \frac{p(re^{2\pi i s}) / p(r)}{|p(re^{2\pi i s}) / p(r)|}$$

defines a loop in the unit circle $S^1 \subset \mathbb{C}$ based at 1. As r varies, f_r is a homotopy of loops based at 1. Since f_0 is the trivial loop $\Rightarrow [f_r] \in \pi_1(S^1)$ is zero for all r. Now fix a large value of r, bigger than $|a_1| + \cdots + |a_n|$ and bigger than 1. Then for $|z| = r$ we have

$$|z^n| > (|a_1| + \cdots + |a_n|)|z^{n-1}| > |a_1 z^{n-1}| + \cdots + |a_n| \geq |a_1 z^{n-1} + \cdots + a_n|$$

From the inequality $|z^n| > |a_1 z^{n-1} + \cdots + a_n|$ it follows that the polynomial $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$ has no roots on the circle $|z| = r$ when $0 \leq t \leq 1$. Replacing p by p_t in the formula for f_r above and letting t go from 1 to 0, we obtain a homology from the loop f_r to the loop $\omega_n(s) = e^{2\pi i n s}$. By

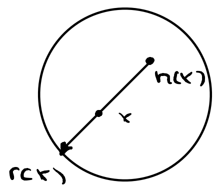
(Theorem 1.7: $\pi_1(S^1)$ is a infinite cyclic group generated by the homotopy class of the loop $\omega(s) = (\cos(2\pi s), \sin(2\pi s))$ based at $(1, 0)$)

theorem 1.7, ω_n represents n times a generator of the infinite cyclic group $\pi_1(S^1)$. Since we have shown that $[\omega_n] = [f_r] = 0 \Rightarrow n = 0$. Thus, the only polinomials without roots in $\mathbb{C}$ are constants

Theorem 1.9: Every continuous map $h: D^2 \to D^2$ has a fixed point, that is, a point $x \in D^2$ with $h(x) = x$.

Here we are using the standard notation D^n for the closed unit disk in $\mathbb{R}^n$, all vectors x of length $|x| \leq 1$. Thus, the boundary of D^n is the unit sphere S^{n-1}.

Suppose on the contrary that $h(x) \neq x$, $\forall x \in D^2$. Then, we can define a map $r: D^2 \to S^1$ by letting $r(x)$ be the points of S^1 where the ray in $\mathbb{R}^2$ starting at $h(x)$ and passing through x leaves D^2. Continuity of r is clear since small



perturbations of x produces small perturbations of $h(x)$, hence also small perturbations of the ray through these two points.

The crucial property of r, besides continuity, is that $r(x) = x$ if $x \in S^1$. Thus r is a retraction of D^2 onto S^1. We will show that no such retraction can exist.

Let f_0 be any loop in S^1. In D^2 there is a homotopy of f_0 to a constant loop, for example, $f_t(s) = (1-t) f_0(s) + t x_0$ where x_0 is the basepoint of f_0. Since the retraction r is the identity on S^1, the composition $r \circ f_t$ is then a homotopy in S^1 from $r f_0 = f_0$ to the constant loop at x_0. But this contradicts the fact that $\pi(S^1)$ is nonzero.

Theorem 1.10: For every continuous map $f: S^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$

The theorem says in particular that there is no one-to-one continuous map from S^2 to $\mathbb{R}^2$, so S^2 is not homeomorphic to a subspace of $\mathbb{R}^2$, an intuitively obvious fact that is not easy to prove directly.

If the conclusion is false for $f: S^2 \to \mathbb{R}^2$, we can define a map $g: S^2 \to S^1$ by $(f(x) - f(-x)) / |f(x) - f(-x)|$. Define a loop η circling the equator of $S^2 \subset \mathbb{R}^3$ by $\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0)$ and let $h: I \to S^1$ can be composed loop $g\eta$. Since $g(-x) = -g(x)$, we have the relation $h(s+1/2) = -h(s)$ $\forall s \in [0, 1/2]$.
The loop h can be lifted to a path $\tilde{h}: I \to \mathbb{R}$. The equation $h(s+1/2) = -h(s)$ implies that $\tilde{h}(s+1/2) = \tilde{h}(s) + q/2$ for some odd integer q that might conceivably depend on $s \in [0, 1/2]$. But in fact q is independent of s since by solving the equation $\tilde{h}(s+1/2) = \tilde{h}(s) + q/2$ for some q we see that q depends continuosly on $s \in [0, 1/2]$, so q must be a constant since it is constrained to integer values.
In particular, we have $\tilde{h}(1) = \tilde{h}(1/2) + q/2 = \tilde{h}(0) + q$. This means that h represents q times a generator of $\pi_1(S^1)$. Since q is odd, we conclude that h is not nullhomotopic. But h was the composition $g\eta: I \to S^2 \to S^1$ and η is nullhomotopic in S^2, so $g\eta$ is nullhomotopic in S^1 by composing a nullhomotopy of η with g. Thus we have arrived at a contradiction.

Proposition 1.12. $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$ if X and Y are path-connected
Same as exercise 10.

Theorem 2.26: If nonempty open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.
For $x \in U$, we have $H_k(U, U - \{x\}) \approx H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\})$ by excision. From the long exact sequence for the pair $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$ we get $H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \approx$
$\approx \tilde{H}_{k-1}(\mathbb{R}^m - \{x\})$. Since $\mathbb{R}^m - \{x\}$ deformation retracts onto a sphere S^{m-1}, we conclude that $H_k(U, U - \{x\})$ is $\mathbb{Z}$ for $k = m$ and 0 otherwise.
By the same reasoning, $H_k(V, V - \{y\})$ is $\mathbb{Z}$ for $k = n$ and 0 otherwise.
Since a homeomorphism $h: U \to V$ induces isomorphisms
$H_k(U, U - \{x\}) \to H_k(V, V - \{h(x)\})$, $\forall k$, we must have $m = n$.