Complementary Problems

① $X \sim N(0,1)$

W with distribution $U(\{-1,1\})$, t.i. $P[W=-1] = P[W=1] = \frac{1}{2}$

independents

$Y = WX$. Start calculating it's distribution function:

$F_Y(y) = P[Y \leq y] = P[WX \leq y] = P[X \leq y]P[W=1] +$

$+ P[-X \leq y]P[W=-1] = \frac{1}{2}P[X \leq y] + \frac{1}{2}P[X \geq -y] =$

$= \frac{1}{2}P[X \leq y] + \frac{1}{2}P[X \leq y] = P[X \leq y] = F_X(y)$

↳ $X \sim N(0,1)$ $P[X \leq y] = P[X \geq -y]$ is symmetric

Then, $Y \sim N(0,1)$, since $X \sim N(0,1)$.

Now, we want to see if $Cov(X,Y) = 0$.

$Cov(X,Y) = E[XY] = E[X^2 W] \underset{\text{independents}}{=} E[X^2]E[W] =$

$= E[X^2]\cdot\left(-1\cdot\frac{1}{2} + 1\cdot\frac{1}{2}\right) = 0 \Rightarrow X, Y$ are uncorrelated.

Let's see that they are not independent.

Define $g(X) = \begin{cases} X & \text{if } W=1 \\ -X & \text{if } W=-1 \end{cases}$. It's a measureable function.

It's easy to see that $Y = g(X) \Rightarrow$ They are not independent.

Another way to see it is that, if they are independents,

$Y/_{X=x}$ should be $Y \sim N(0,1)$, but $Y_{X=x}(y) \in \{-x, x\}$!!

② $X \sim N_p(\mu, \Sigma)$, $\Sigma > 0$, $f_X(x) = K$

Since X has a MND, we know that

$$f_X(x) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right\} = K$$

Because f_X is a distribution function, we know that

$f_X(x) \geq 0$, so, if $K \leq 0 \Rightarrow$ The geometrical focus is ϕ.

Suppose $K > 0$, then:

$$\exp\left\{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right\} = K(2\pi)^{p/2}|\Sigma|^{1/2} \Rightarrow$$

$$\Rightarrow -\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu) = \ln\left(K(2\pi)^{p/2}|\Sigma|^{1/2}\right) \Rightarrow$$

$$\Rightarrow (x-\mu)'\Sigma^{-1}(x-\mu) = -2\ln\left(K(2\pi)^{p/2}|\Sigma|^{1/2}\right) = \delta$$

$0 < |\Sigma| = \prod_{i=1}^{p} \lambda_i$, λ_i the eigenvalues of $\Sigma \Rightarrow |\Sigma|^{1/2} = \prod_{i=1}^{p} \lambda_i^{1/2}$

Then,

$$\mathcal{I} = -2\ln\Big(\kappa \underbrace{(2\pi)^{p/2}}_{\vee \atop 0} \cdot \underbrace{\prod_{i=1}^{p} \lambda_i^{1/2}}_{\vee \atop 0}\Big)$$. Therefore, for $\kappa > 0$

We have that $(x-\mu)'(\mathcal{I}\Sigma)^{-1}(x-\mu) = 1$. This is a hyperellypsoid centered in μ and with principal axis determinated by λ_i and lenght $\frac{1}{\sqrt{\mathcal{I}\lambda_i}}$.

Now, $\sqrt{\mathcal{I}} \in \mathbb{R} \Leftrightarrow \mathcal{I} > 0 \Leftrightarrow \ln(\kappa(2\pi)^{p/2}|\Sigma|^{1/2}) < 0 \Leftrightarrow$

$\Leftrightarrow 0 < \kappa(2\pi)^{p/2}|\Sigma|^{1/2} \Leftrightarrow 0 < \kappa < \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}}$

Therefore, $\kappa \in I = \left(0, \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}}\right) \Rightarrow f_X(x) = \kappa$ is an hyperellypsoid. Otherwise, ϕ.

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Let's consider the characteristic function of Y:

$$\phi_Y(t) = E[e^{itY}] = E[e^{it(\alpha + DX + Z)}] = E[e^{it(\alpha+DX)} e^{itZ}]$$

Because X and Y are independent, we can go as follows:

$$\phi_Y(t) = E[e^{it'(\alpha+DX)}]E[e^{it'Z}] =$$

$$= E[e^{it'\alpha}]E[e^{it'DX}]E[e^{it'Z}] = e^{it'\alpha}\phi_X(D't)\phi_Z(t)$$

We know that $X \sim N_p(\mu_X, \Sigma_X)$, $\Sigma_X \geq 0 \iff$ It's characteristic function is of the form

$$\phi_X(t) = \exp\{it'\mu_X - \frac{1}{2}t'\Sigma t\}$$

Since $X \sim N_r(0, \Sigma_X)$ and $Z \sim N_p(0, \sigma^2 I_p)$, we have that:

$$\left.\begin{array}{l} \phi_X(D't) = \exp\{\cancel{it'0} - \frac{1}{2}t'D\Sigma_X D't\} \\ \phi_Z(t) = \exp\{\cancel{it'0} - \frac{1}{2}t'\sigma^2 I_p t\} \end{array}\right\} \Rightarrow \text{Using uniqess of c.f of a r.v.}$$

$$\Rightarrow \Phi_Y(t) = e^{it'\alpha} e^{-\frac{1}{2}t'D\Sigma_X D't} e^{-\frac{1}{2}t'\sigma^2 I_p t} =$$

$$= \exp\{it'\alpha - \frac{1}{2}t'(D\Sigma_X D' + \sigma^2 I_p)t\} \Rightarrow \boxed{Y \sim N(\alpha, D\Sigma_X D' + \sigma^2 I_p)}$$

Now, let's calculate the distribution of $\begin{pmatrix} Y \\ X \end{pmatrix}$:

$$\begin{pmatrix} Y \\ X \end{pmatrix} = \begin{pmatrix} \alpha + DX + Z \\ X \end{pmatrix} = \begin{pmatrix} \alpha \\ 0_r \end{pmatrix} + \begin{pmatrix} D \\ I_r \end{pmatrix} X + \begin{pmatrix} I_p \\ 0_r \end{pmatrix} Z$$

Let $X \sim N_p(\mu_X, \Sigma_X)$, $\Sigma_X > 0$

$Y = BX + b$, B a $p\times p$ constant matrix, b a $p\times 1$ constant vector $\Rightarrow Y \sim N_p(B\mu_X + b, B\Sigma_X B')$

$$\underbrace{\begin{pmatrix} \alpha \\ 0_r \end{pmatrix}}_{b} + \underbrace{\begin{pmatrix} D \\ I_r \end{pmatrix}}_{B} X \sim N_p\left(\cancel{\begin{pmatrix} D \\ I_r \end{pmatrix} \cdot 0} + \begin{pmatrix} \alpha \\ 0_r \end{pmatrix}, \begin{pmatrix} D \\ I_r \end{pmatrix} \Sigma_X \begin{pmatrix} D \\ I_r \end{pmatrix}'\right)$$

$$\underbrace{\begin{pmatrix} I_p \\ 0_r \end{pmatrix}}_{B} Z + \underbrace{0}_{b} \sim N_p\left(\cancel{\begin{pmatrix} I_p \\ 0_r \end{pmatrix} \cdot 0} + 0, \begin{pmatrix} I_p \\ 0_r \end{pmatrix} \sigma^2 I_p \begin{pmatrix} I_p \\ 0_r \end{pmatrix}'\right)$$

Result: Let X_k, $k=1,\dots,m$ be a p-dimensional r.v. with MND $X_k \sim N_p(\mu_k, \Sigma_k)$ being independent. Then, for every set of matrices A_k, $k=1,\dots,m$ (constant) of dimension $q\times p$, we have that:

$$Y := \sum_{k=1}^{m} A_k X_k \sim N_q\left(\sum_{k=1}^{m} A_k\mu_k, \sum_{k=1}^{m} A_k\Sigma_k A_k'\right)$$

As a consequence, if X_1, X_2 are p-dimensional independent r.v., then $\begin{matrix} X_1 \sim N_p(\mu_1,\Sigma_1) \\ X_2 \sim N_p(\mu_2,\Sigma_2)\end{matrix} \iff X_1+X_2 \sim N_p(\mu_1+\mu_2, \Sigma_1+\Sigma_2)$

(the linear combinations are also independ.)

Because of independent of X and ε, we have that:

$$\boxed{\begin{pmatrix} Y \\ X \end{pmatrix} \sim N_{p+r}\left(\begin{pmatrix} \alpha \\ 0_r \end{pmatrix}, \begin{pmatrix} D \\ I_r \end{pmatrix}\Sigma_X\begin{pmatrix} D \\ I_r \end{pmatrix}' + \begin{pmatrix} I_p \\ 0_r \end{pmatrix}\sigma^2 I_p\begin{pmatrix} I_p \\ 0_r \end{pmatrix}'\right)}$$

Now, we have to prove that $E[X|Y] = \Sigma_X D'\Sigma_Y^{-1}(Y-\alpha)$.
First, ~~[illegible]~~ we see if it's non-singular or not.

$$\det\left(\begin{pmatrix} D \\ I_r \end{pmatrix}\Sigma_X\begin{pmatrix} D \\ I_r \end{pmatrix}' + \begin{pmatrix} I_p \\ 0_r \end{pmatrix}\sigma^2 I_p\begin{pmatrix} I_p \\ 0_r \end{pmatrix}'\right) =$$

$$= \det\left(\begin{pmatrix} D\Sigma_X D' + \sigma^2 I_p & \boxed{D\Sigma_X} \\ \Sigma_X D' & \Sigma_X \end{pmatrix}\right) > 0 \ ??$$

$(12) \to \begin{pmatrix} D \\ I_r \end{pmatrix}\Sigma_X\begin{pmatrix} D \\ I_r \end{pmatrix}' \to D\Sigma_X + 0 \to D\Sigma_X$; $\begin{pmatrix} I_p \\ 0_r \end{pmatrix}\sigma^2 I_p\begin{pmatrix} I_p \\ 0_r \end{pmatrix}'$

$$\det\left(\begin{pmatrix} D\Sigma_X D' + \sigma^2 I_p & D\Sigma_X \\ \Sigma_X D' & \Sigma_X \end{pmatrix}\right) =$$

$|A| = \begin{vmatrix} B & C \\ D & E \end{vmatrix}$, B and E squared $\Rightarrow$

$\Rightarrow |A| = |E|\,|B - CE^{-1}D|$

$$= \det(\Sigma_X)\det(D\Sigma_X D' + \sigma^2 I_p - D\cancel{\Sigma_X}\cancel{\Sigma_X^{-1}}\Sigma_X D') =$$

$$= \det(\Sigma_X)\det(\cancel{D\Sigma_X D'} + \sigma^2 I_p - \cancel{D\Sigma_X D'}) = \det(\Sigma_X)\det(\sigma^2 I_p) > 0$$

$\hookrightarrow \Sigma_X > 0 \Rightarrow \det(\Sigma_X) > 0$

$\sigma^2 > 0 \Rightarrow \sigma^2 I_p > 0$

So, we have that the distribution is non-singular.

Result 3 (non-singular): Let X be a r.v. with $X \sim N_p(\mu, \Sigma)$ $(\Sigma > 0)$, and we have $X = \begin{pmatrix} X_{(1)} \\ X_{(2)} \end{pmatrix}$, $\mu = \begin{pmatrix} \mu_{(1)} \\ \mu_{(2)} \end{pmatrix}$, $\Sigma = \begin{pmatrix} \Sigma_{(11)} & \Sigma_{(12)} \\ \Sigma_{(21)} & \Sigma_{(22)} \end{pmatrix}$

Then, we have that:

1) $X_{(1)}$ and $X_{(2)} - \Sigma_{(21)}\Sigma^{-1}_{(11)} X_{(1)}$ are independent

2) $X_{(1)} \sim N_q(\mu_{(1)}, \Sigma_{(11)})$

$$X_{(2)} - \Sigma_{(21)}\Sigma^{-1}_{(11)} X_1 \sim N_{p-q}\left(\mu_{(2)} - \Sigma_{(21)}\Sigma^{-1}_{(11)}\mu_{(1)},\ \Sigma_{(22)} - \Sigma_{(21)}\Sigma^{-1}_{(11)}\Sigma_{(12)}\right)$$

3) $X_{(2)} / X_{(1)} \overset{x_{(1)}}{=}$ is a MND with

$$N_{p-q}\left(\mu_{(2)} + \Sigma_{(21)}\Sigma_{(11)}^{-1}(x_{(1)} - \mu_{(1)}),\ \Sigma_{(22)} - \Sigma_{(21)}\Sigma^{-1}_{(11)}\Sigma_{(12)}\right)$$

Applying the result, we have that:

$$\underset{\mu_{X/Y}}{\overset{\shortparallel}{E[X/Y]}} = \Sigma_X D' \underbrace{(D\Sigma_X D' + \sigma^2 I_p)}_{\Sigma_{(11)} = \Sigma_Y}{}^{-1} (Y - \alpha) = \Sigma_X D' \Sigma_Y^{-1}(Y - \alpha)$$

Finally, for calculating the distribution of Y/X, we can use the same result:

$X_{(1)} = X$ $\quad \mu_{(1)} = 0$ $\quad \Sigma = \begin{pmatrix} \Sigma_X & \Sigma_X D' \\ D\Sigma_X & \Sigma_Y \end{pmatrix}$

$X_{(2)} = Y$ $\quad \mu_{(2)} = \alpha$

$$Y/X \sim N_p\left(\alpha + D\cancel{\Sigma_X \Sigma_X^{-1}}(X - 0),\ \Sigma_Y - D\Sigma_X \cancel{\Sigma_X^{-1}\Sigma_X} D'\right) \sim$$

$$\sim N_p(\alpha + DX, \Sigma_Y - D\Sigma_X D') \sim N_p(\alpha + DX, \cancel{D\Sigma_X D'} + \sigma^2 I_p - \cancel{D\Sigma_X D'})$$

$$\boxed{\sim N_p(\alpha + DX, \sigma^2 I_p)}$$

② For the first and second task, we haven't used that $\Sigma_X > 0$, so they are still valid. Now, for (3) and (4), we have to re-do the demostration.

Using the information given in the exercise,

$$E[X/Y] = \Sigma_X D' \Sigma_Y^{-}(Y - \alpha) = \Sigma_X D'(D\Sigma_X D' + \sigma^2 I_p)^{-}(Y - \alpha)$$

We need to prove that $\Sigma_Y^- = \Sigma_Y^{-1}$.

The existence of Σ_Y^{-1} depends on Σ_Y being invertible $\Rightarrow$

$\Rightarrow \Sigma_Y$ must be positive-definite $(x'\Sigma_Y x > 0, \ \forall x \in \mathbb{R}^p \setminus \{0\})$

- If $\Sigma_Z > 0$ is trivial
- otherwise, $x'\Sigma_Y x = x'(D\Sigma_Z D' + \sigma^2 I_p)x =$

$$= \underbrace{x'D\Sigma_Z D'x}_{\substack{\geq \\ 0}} + \underbrace{x'\sigma^2 I_p x}_{\substack{= \\ \|x\|\sigma^2 > 0 \\ \downarrow \\ x \neq 0}} > 0 \Rightarrow \Sigma_Y \text{ is positive-definite} \Rightarrow$$

$$\Rightarrow \Sigma_Y \Sigma_Y^- \Sigma_Y = \Sigma_Y \Rightarrow \Sigma_Y^- = \Sigma_Y^{-1} \Rightarrow E[Z/Y] = \Sigma_Z D'\Sigma_Y^{-1}(Y - \alpha)$$

For (d), we use the same result, but generalizing using the information given in the task.

$$Y/X \sim N_p(\alpha + D\Sigma_Z \Sigma_Z^- X, \ D\Sigma_Z D' + \sigma^2 I_p - D\underbrace{\Sigma_Z\Sigma_Z^-\Sigma_Z}_{\Sigma_Z} D') \sim$$

$$\sim N_p(\alpha + D\Sigma_Z\Sigma_Z^- X, \ \sigma^2 I_p)$$

because $\sigma^2 = \alpha'\Sigma\alpha$, so $\sigma = (\alpha'\Sigma\alpha)^{1/2}$

③ Let's consider $(X - \mu)$. Since $X \sim N_p(\mu, \Sigma) \Rightarrow (X-\mu) \sim N_p(0, \Sigma)$

$\forall \alpha \in \mathbb{R}^p$, $\alpha'(X - \mu) \sim N_p(0, \alpha'\Sigma\alpha)$ $\quad (\alpha'\Sigma\alpha \geq 0$, since $\Sigma \geq 0)$

Using the auxiliary result, we have that $E[\alpha'(X-\mu)^k] = \begin{cases} (\alpha'\Sigma\alpha)^{k/2}(k-1)!! & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases} =$

$$= \begin{cases} (\alpha'\Sigma\alpha)^m \cdot (2m-1)!! & \text{if } k = 2m \text{ (even)} \\ 0 & \text{if } k = 2m-1 \text{ (odd)} \end{cases}$$

Let's see that $(2m-1)!! = \dfrac{(2m)!}{2^m \cdot m!}$ for $m \in \mathbb{N}$, by induction:

- $m = 1 \rightarrow 1!! = \prod_{i=1}^{1}(2i-1) = 1 = 1 = \dfrac{2!}{2 \cdot 1!}$
- $m = 2 \rightarrow 2!! = \prod_{i=1}^{2}(2i-1) = 3 \cdot 1 = 3 = 3 = \dfrac{4!}{4 \cdot 2!} = \dfrac{\cancel{4} \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{4} \cdot \cancel{2} \cdot \cancel{1}}$

$\vdots$

$$(2(m+1)-1)!! = (2m+1)!! = \prod_{i=1}^{m+1}(2i-1) = \left(\prod_{i=1}^{m}(2i-1)\right)(2m+1) \underset{\text{by induction}}{=}$$

$$= \frac{(2m)!}{2^m \cdot m!}(2m+1) = \frac{(2m+1)!}{2^m \cdot m!} \cdot \frac{m+1}{m+1} = (m+1) \cdot \frac{(2m+1)!}{2^m(m+1)!} \cdot \frac{2}{2} =$$

$$= (2m+2)\cdot\frac{(2m+1)!}{2^{m+1}(m+1)!} = \frac{(2(m+1))!}{2^{m+1}(m+1)!} \Rightarrow$$

$$\Rightarrow E[\alpha'(X-\mu)^K] = \begin{cases} (\alpha'\Sigma\alpha)^m \cdot \dfrac{(2m)!}{2^m\cdot m!} & \text{if } K=2m \text{ (even)} \\ 0 & \text{if } K=2m-1 \text{ (odd)} \end{cases}$$

④ $\operatorname{rank}(\Sigma)=K$, $\Sigma = H\Lambda H'$

$H=(H_1|H_2)$, $\Lambda = \left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right)$, $D=\operatorname{diag}(\lambda_1,\dots,\lambda_K)$, $\lambda_1,\dots,\lambda_K>0$

$\Sigma^+ = H_1D^{-1}H_1'$ (Moore - Penrose inverse matrix of Σ)

a) $\Sigma\Sigma^+\Sigma=\Sigma$

$$\Sigma = H\Lambda H' = (H_1|H_2)\left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right)\begin{pmatrix} H_1' \\ H_2' \end{pmatrix} = H_1DH_1'$$

Then,

$$\Sigma\Sigma^+\Sigma = H_1D\cancel{H_1'}\,\cancel{H_1}D^{-1}\cancel{H_1'}\,\cancel{H}DH_1' = H_1D\,\cancel{D^{-1}}\,\cancel{D}H_1' = H_1DH_1' = \Sigma$$

b) $\Sigma^+\Sigma\Sigma^+ = \Sigma^+$

$$H_1D^{-1}\cancel{H_1'}\,\cancel{H_1}DH_1'\cancel{H_1}D^{-1}H_1' = H_1D^{-1}\cancel{DD^{-1}}H_1' = H_1D^{-1}H_1' = \Sigma^+$$

c) $(\Sigma^+\Sigma)' = \Sigma^+\Sigma$

$$(H_1D^{-1}\cancel{H_1'}\,\cancel{H_1}DH_1')' = (H_1\cancel{D^{-1}}DH_1')' = (\cancel{H_1}\cancel{H_1'})' = (Id)' = Id$$

$$\Sigma^+\Sigma = H_1D^{-1}\cancel{H_1'}\,\cancel{H_1}DH_1' = H_1\cancel{D^{-1}}DH_1' = \cancel{H_1}\cancel{H_1'} = Id \quad \Big\} \Rightarrow$$

$\Rightarrow (\Sigma^+\Sigma) = \Sigma^+\Sigma$

d) $(\Sigma\Sigma^+)' = \Sigma\Sigma^+$

$$(H_1D\cancel{H_1'}\,\cancel{H_1}D^{-1}H_1')' = (H_1\cancel{DD^{-1}}H_1')' = (\cancel{H_1}\cancel{H_1'})' = (id)' = id \quad \Big\} \Rightarrow (\Sigma\Sigma^+)' = \Sigma\Sigma^+$$

$\Sigma\Sigma^+ = id$

⑤ $\Sigma = (\sigma_{ij})$ a 3×3 symmetric matrix

$\sigma_{11}=\sigma_{22}=\sigma_{33}=1$, $\sigma_{12}=\sigma_{21}=0$

¿if $(\sigma_{13}+\sigma_{23}) > \frac{3}{2} \Rightarrow \Sigma$ is not - positive -definite?

$$\Sigma = \begin{pmatrix} 1 & 0 & \sigma_{13} \\ 0 & 1 & \sigma_{32} \\ \sigma_{13} & \sigma_{32} & 1 \end{pmatrix}, \ \det(\Sigma) = 1-\sigma_{13}^2-\sigma_{23}^2$$

Suppose that Σ is positive definite, $\Sigma > 0 \Rightarrow \det(\Sigma) > 0 \Rightarrow$

$\Rightarrow 1 - \tau_{13}^2 - \tau_{32}^2 > 0 \Rightarrow 1 - (\tau_{13}^2 + \tau_{32}^2) > 0 \Rightarrow$

$\Rightarrow \tau_{13}^2 + \tau_{32}^2 < 1$ and, as the tasks says, we are going to consider $\begin{cases} \tau_{13}^2 + \tau_{32}^2 < 1 \\ \tau_{13} + \tau_{32} > \frac{3}{2} \end{cases}$

- $\tau_{31} + \tau_{32} = \frac{3}{2} \Rightarrow \tau_{32} = -\tau_{31} + \frac{3}{2}$
- $\tau_{13}^2 + \tau_{32}^2 = 1 \Rightarrow \tau_{13}^2 + \left(-\tau_{31} + \frac{3}{2}\right)^2 = \tau_{13}^2 + \tau_{13}^2 + \frac{9}{4} - 3\tau_{31} = 1 \Rightarrow$

$\Rightarrow 2\tau_{13}^2 - 3\tau_{31} + \frac{5}{4} = 0 \Rightarrow \tau_{31} = \dfrac{-(-3) \pm \sqrt{(3)^2 - 4 \cdot 2 \cdot (5/4)}}{2 \cdot 2} =$

$= \dfrac{3 \pm \sqrt{9-10}}{4} = \dfrac{3 \pm \sqrt{-1}}{4} \notin \mathbb{R}$. The system has no solution $\Rightarrow$

$\Rightarrow \nexists\ \tau_{13}, \tau_{23}$ s.t. $\begin{cases} \tau_{13}^2 + \tau_{32}^2 < 1 \\ \tau_{13} + \tau_{32} > 3/2 \end{cases} \Rightarrow \Sigma$ is not positive definite, at least for $\tau_{13} + \tau_{32} > 3/2$ //

⑥ $Z \sim N_p(0, I_p)$

$Y_1 = C_1 Z$, C_1 a $k_1 \times p$ matrix, $k_1 \leq p$

$Y_2 = C_2 Z$, C_2 a $k_2 \times p$ matrix, $k_2 \leq p$

¿ condition of independence of Y_1 and Y_2?

Let $X \sim N_p(\mu_X, \Sigma_X)$, $\Sigma_X > 0$

$Y = BX + b \Rightarrow Y \sim N_p(B\mu_X + b, B\Sigma_X B')$

We know that $Y_1 \sim N_p(0, C_1 \Sigma_Z C_1') \sim N_p(0, C_1 I_p C_1')$

$Y_2 \sim N_p(0, C_2 \Sigma_Z C_2') \sim N_p(0, C_2 I_p C_2')$

We can write Y as $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} Z$. Applying the general result, we have that

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N_{k_1+k_2}\left(\begin{pmatrix} 0_{k_1 \times 1} \\ 0_{k_2 \times 1} \end{pmatrix}, \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} I_p (C_1' \ C_2')\right) \sim$$

$$\sim N_{k_1+k_2}\left(\begin{pmatrix} 0_{k_1 \times 1} \\ 0_{k_2 \times 1} \end{pmatrix}, \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (C_1' \ C_2')\right)$$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N_{k_1+k_2}\left(\begin{pmatrix} 0_{k_1 \times 1} \\ 0_{k_2 \times 1} \end{pmatrix}, \underbrace{\begin{pmatrix} C_1 C_1' & C_1 C_2' \\ C_2 C_1' & C_2 C_2' \end{pmatrix}}_{\Sigma_Y}\right)$$

Let $X = (X_1, \dots, X_p)'$ be a random vector with non-singular MND, $X \sim N_p(\mu, \Sigma)$ $(\Sigma > 0)$

Assume that the components of X are ordered in such a way that

$$X = \begin{pmatrix} X_{(1)} \\ X_{(2)} \end{pmatrix}, \quad \text{with } X_{(1)} = (X_1, \dots, X_q)', \ X_{(2)} = (X_{q+1}, \dots, X_p)'$$

we have

$$\mu = \begin{pmatrix} \mu_{(1)} \\ \mu_{(2)} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{(11)} & \Sigma_{(12)} \\ \Sigma_{(21)} & \Sigma_{(22)} \end{pmatrix} = \begin{pmatrix} \Sigma_{(11)} & 0 \\ 0 & \Sigma_{(22)} \end{pmatrix}$$

$\Rightarrow$ $X_{(1)}$ and $X_{(2)}$ independent and $X_{(1)} \sim N_q(\mu_{(1)}, \Sigma_{(11)})$

$X_{(2)} \sim N_{p-q}(\mu_{(2)}, \Sigma_{(22)})$

(No podemos usar este resultado como hace wolch $(\Leftarrow) \Rightarrow$)

Consider $\phi_Y(t) = \exp\left\{ it'\mu_Y - \frac{1}{2} t' \Sigma_Y t \right\} =$

$$= \exp\left\{ \cancel{it' \begin{pmatrix} 0_{q \times 1} \\ 0_{q \times 1} \end{pmatrix}}^{\,0} - \frac{1}{2} t' \begin{pmatrix} C_1 C_1' & C_1 C_2' \\ C_2 C_1' & C_2 C_2' \end{pmatrix} t \right\} =$$

$$= \exp\left\{ -\frac{1}{2} t' \begin{pmatrix} C_1 C_1' & C_1 C_2' \\ C_2 C_1' & C_2 C_2' \end{pmatrix} t \right\}$$

Suppose Y_1 and Y_2 independent $\Rightarrow \phi_Y(t) = \phi_{Y_1}(t_1)\,\phi_{Y_2}(t_2)$

$$\phi_{Y_1}(t_1) = \exp\left\{ -\frac{1}{2} t_1' C_1 I_p C_1' t_1 \right\}$$

$$\phi_{Y_2}(t_2) = \exp\left\{ -\frac{1}{2} t_2' C_2 I_p C_2' t_2 \right\}$$

$$\phi_{Y_1}(t_1)\,\phi_{Y_2}(t_2) = \exp\left\{ -\frac{1}{2} t_1' C_1 C_1' t_1 - \frac{1}{2} t_2' C_2 C_2' t_2 \right\} =$$

$$= \exp\left\{ -\frac{1}{2} t' \begin{pmatrix} C_1 C_1' & C_1 C_2' \\ C_2 C_1' & C_2 C_2' \end{pmatrix} t \right\} = \phi_Y(t) \Leftrightarrow$$

$$\Leftrightarrow -\frac{1}{2} t_1' C_1 C_1' t_1 - \frac{1}{2} t_2' C_2 C_2' t_2 = -\frac{1}{2} (t_1' \ t_2') \begin{pmatrix} C_1 C_1' & C_1 C_2' \\ C_2 C_1' & C_2 C_2' \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \cancel{-\frac{1}{2}} \left(t_1' C_1 C_1' t_1 + t_2' C_2 C_2' t_2 \right) = \cancel{-\frac{1}{2}} \left(t_1' C_1 C_1' t_1 + t_1' C_1 C_2' t_2 + t_2' C_2 C_1' t_1 + t_2' C_2 C_2' t_2 \right)$$

$$\Leftrightarrow t_1' C_1 C_2' t_2 + t_2' C_2 C_1' t_1 = 0 \Leftrightarrow \boxed{C_1 C_2' = C_2 C_1' = 0}$$

⑦ $Y \sim N_3(\mu, \Sigma)$, $\mu = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 6 & 1 & -2 \\ 1 & 13 & 4 \\ -2 & 4 & 4 \end{pmatrix}$

$\det(\Sigma) = 144 > 0 \Rightarrow \Sigma$ positive define. We can write

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 & 1 & -2 \\ 1 & 13 & 4 \\ -2 & 4 & 4 \end{pmatrix} \right)$$

$$Z = 2Y_1 - Y_2 + 3Y_3 \Rightarrow Z = (2 \; -1 \; 3) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

Then, $Z \sim N\left((2 \; -1 \; 3) \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, (2 \; -1 \; 3) \begin{pmatrix} 6 & 1 & -2 \\ 1 & 13 & 4 \\ -2 & 4 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right) \sim$

$$\sim \boxed{N(17, 21)}$$

$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} Y_1 + Y_2 + Y_3 \\ Y_1 - Y_2 + 2Y_3 \end{pmatrix}$ (. . .) (Lo hago al final. Mucho cálculo y no entra cómputo)

(10) $F(x,y)=\Phi(x)\,\Phi(y)\,[1+\alpha(1-\Phi(x))(1-\Phi(y))]$, $|\alpha|\leq 1$

Let's calculate the distribution of the marginal variable X.

$$F_X(x)=\lim_{y\to+\infty} F(x,y) = \lim_{y\to\infty} \Phi(x)\,\Phi(y)\,[1+\alpha(1-\Phi(x))(1-\Phi(y))] =$$

$\lim_{x\to\infty} \Phi(x)=1$, since $\Phi(x)$ is a distribution function

$$= \Phi(x)\cdot 1\,[1+\overbrace{\cancel{\alpha(1-\Phi(x))(1-1)}}^{\to 0}] = \Phi(x)$$

. The same for $F_Y(y)=\lim_{x\to+\infty} F(x,y)=\Phi(y)$.

Therefore, since the distribution function of the marginal of X and Y are $\Phi(\cdot)$ that denotes the standard normal distribution function $\Rightarrow X, Y$ are standard normal.

(11) $X_1, X_2\dots$ independents , $X_i\sim N_m(\mu,\Sigma)$, $i=1,2,\dots$

$S_N=\sum_{i=1}^{N} X_i$, $N_1<N_2$

$d(S'_{N_1}, S'_{N_2})'$ distribution?

Let $X_i\sim N_p(\mu_i,\Sigma_i)$, $i=1,2,\dots,n$ and they are independents. Then, for every constant matrix $A_{k\;q\times p}$ we have that

$$Z:=\sum_{i=1}^{n} A_i X_i \sim N_q\left(\sum_{i=1}^{n} A_i\mu_i \,,\, \sum_{i=1}^{n} A_i\Sigma_i A_i'\right)$$

Let's see the distributions of S_{N_1} and S_{N_2}:

$$S_{N_1}=\sum_{i=1}^{N_1} X_i \sim N_m\left(\sum_{i=1}^{N_1}\mu \,,\, \sum_{i=1}^{N_1}\Sigma\right) \sim N_m(N_1\mu, N_1\Sigma)$$

$$S_{N_2}=\sum_{i=1}^{N_2} X_i \sim N_m(N_2\mu, N_2\Sigma)$$

Because $N_1<N_2$, we have that:

$$S_{N_2}=S_{N_1}+\sum_{i=N_1+1}^{N_2} X_i = S_{N_1}+\tilde{S}$$

We know that S_{N_1} , $\tilde{S}$ are independent, because they are linear combinations of independents vectors.

Also, we know that $\tilde{S} \sim N_m((N_2 - N_1)\mu, (N_2 - N_1)\Sigma)$. Therefore,

$$\begin{pmatrix} S_{N_1} \\ S_{N_2} \end{pmatrix} = \begin{pmatrix} S_{N_1} \\ S_{N_1} + \tilde{S} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} S_{N_1} \\ \tilde{S} \end{pmatrix} =$$

$$= \begin{pmatrix} I_m \\ I_m \end{pmatrix} S_{N_1} + \begin{pmatrix} 0 \\ I_m \end{pmatrix} \tilde{S}$$

Because they are independent:

$$\begin{pmatrix} S_{N_1} \\ S_{N_2} \end{pmatrix} \sim N_{2m}\left(\begin{pmatrix} I_m \\ I_m \end{pmatrix} N_1\mu + \begin{pmatrix} 0 \\ I_m \end{pmatrix}(N_2 - N_1)\mu, \right.$$

$$\left. \begin{pmatrix} I_m \\ I_m \end{pmatrix} N_1 \Sigma \begin{pmatrix} I_m \\ I_m \end{pmatrix}' + \begin{pmatrix} 0 \\ I_m \end{pmatrix} (N_2 - N_1)\Sigma \begin{pmatrix} 0 \\ I_m \end{pmatrix}' \right) \sim$$

$$\sim N_{2m}\left(\begin{pmatrix} N_1\mu \\ N_1\mu + (N_2 - N_1)\mu \end{pmatrix}, \begin{pmatrix} N_1\Sigma & N_1\Sigma \\ N_1\Sigma & N_1\Sigma + (N_2 - N_1)\Sigma \end{pmatrix} \right) \sim$$

$$\boxed{\sim N_{2m}\left(\begin{pmatrix} N_1\mu \\ N_2\mu \end{pmatrix}, \begin{pmatrix} N_1\Sigma & N_1\Sigma \\ N_1\Sigma & N_2\Sigma \end{pmatrix} \right)}$$

¿ S_{N_1} / S_{N_2} ?

(Generalization: $\Sigma_x \Sigma_x^- \Sigma_x = \Sigma_x$)

If $\Sigma > 0$, we can just apply the result 3:

$$X_{(2)} / X_{(1)} \sim N_{p-q}\left(\mu_2 - \Sigma_{(21)}\Sigma_{(11)}^{-1} \cancel{\;\;}, \Sigma_{(22)} - \Sigma_{(21)}\Sigma_{(11)}^{-1}\Sigma_{(12)}\right)$$

↳ $(x_1 - \mu_1)$

Therefore, we will have,

$$S_{N_1} / S_{N_2} \sim N_m\left(N_1\mu - N_1\Sigma(N_2\Sigma)^{-1}(s_{N_2} - N_2\Sigma), N_1\Sigma - N_1\Sigma(N_2\Sigma)^{-1}N_1\Sigma\right)$$

If $\Sigma \geq 0$, we have to use the generalization $\Sigma\Sigma^-\Sigma = \Sigma$

$$S_{N_1} / S_{N_2} \sim N_m\left(N_1\mu - N_1\Sigma(N_2\Sigma)^-(s_{N_2} - N_2\Sigma), N_1\Sigma - N_1\Sigma(N_2\Sigma)^- N_1\Sigma\right) \sim$$

$$(N_2^{-1} = N_2^-) \sim N_m\left(N_1\mu - \frac{N_1}{N_2}\Sigma\Sigma^-(s_{N_2} - N_2\Sigma), N_1\Sigma - \frac{N_1^2}{N_2}\Sigma\Sigma^-\Sigma\right) \sim$$

$$\sim \boxed{N_m\left(N_1\mu - \frac{N_1}{N_2}\Sigma\Sigma^-(s_{N_2} - N_2\Sigma), N_1\Sigma - \frac{N_1^2}{N_2}\Sigma\right)}$$

12 $X \sim N_3(0, \Sigma)$, $\Sigma = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}$

¿ρ that makes $X_1 + X_2 + X_3$ and $X_1 - X_2 - X_3$ independent?

$$Z = \begin{pmatrix} X_1 + X_2 + X_3 \\ X_1 - X_2 - X_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = BX$$

$X \sim N_p(\mu, \Sigma)$ $\quad$ B $q \times p$

$Y = BX + b \Rightarrow Y \sim N_q(B\mu + b, B\Sigma B')$

So, we have that

$$Z \sim N_2 \left(\cancel{\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \cdot 0} \rightarrow 0, \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} \right) \sim$$

$$\sim N_2 \left(0, \begin{pmatrix} 1+\rho & 2\rho+1 & \rho+1 \\ 1-\rho & -1 & -\rho-1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} \right) \sim$$

$$\sim N_2 \left(0, \begin{pmatrix} 4\rho+3 & -2\rho-1 \\ -2\rho-1 & 3 \end{pmatrix} \right)$$

We want $X_{(1)}$ and $X_{(2)}$ to be independent. Using the second result:

Let $X = (X_1, \dots, X_p)'$ be a r.v. and suppose it's ordered in such a way that $X = \begin{pmatrix} X_{(1)} \\ X_{(2)} \end{pmatrix}$, with ...

$$\Sigma_{(21)} = \Sigma_{(12)} = 0 \iff -2\rho - 1 = 0 \iff \boxed{\rho = -\frac{1}{2}}$$