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Semester: 2
Academic Year: 2022/2023

Faculty of Applied Physics and Mathematics
Exercises
Differential Equations II

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1. **Using the definition examine the stability of solutions of the initial value problem: $\frac{dx}{dt} = -2x,\ x(0) = 1$.**

   We have the differential equation $\frac{dx}{dt} = -2x$, that is a first-order linear homogeneous differential equation.

   For solving it, we can use the method of variables' separation. So we can go as follows:

   $$\frac{dx}{dt} = -2x \implies \frac{dx}{x} = -2dt \implies \int \frac{dx}{x} = \int -2dt \implies ln|x| = -2t + C$$

   Solving x we obtain:

   $$x(t) = Ce^{-2t}$$

   If we consider $t = 0$, we obtain that $x(0) = C$. To examine the stability of the solution, we need to look at the behavior of the solutions for different initial conditions. We know that A solution $x(t)$ of a differential equation is said to be asymptotically stable if it approaches a fixed value (called an equilibrium point) as t goes to infinity, and all solutions sufficiently close to it also approach the same equilibrium point as t goes to infinity.

   Let's consider two cases:

   - **Case 1:** $C > 0$
     If $C > 0$, then the solution is of the form $x(t) = C^{-2t}$, which is a decreasing function of t. As t increases, the value of $x(t)$ approaches zero. Therefore, we say that the solution is asymptotically stable.
   - **Case 2:** $C < 0$
     If $C < 0$, then the solution is of the form $x(t) = C^{-2t}$, which is a increasing function of t. As t increases, the value of $x(t)$ approaches negative infinity. Therefore, we say that the solution is unstable.

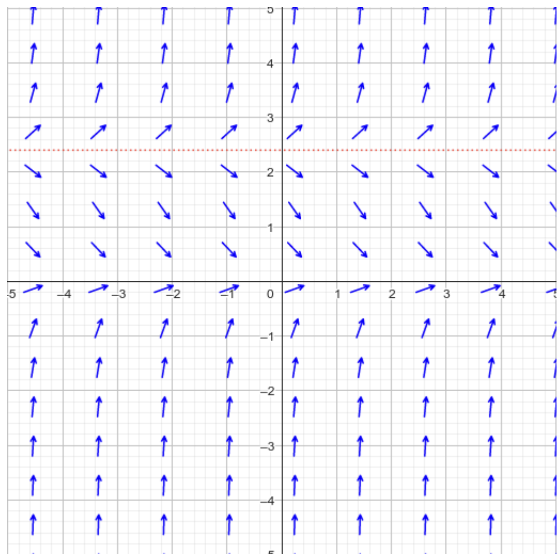
   Since $x(t) = 1 = C$, we get that the solution is asymptotically stable.

2. **Sketch representative phase-time plots and discuss stability properties of critical points for various of the number β for the following equation:** $y'(t) = y(y - \beta)$

   The equation $y'(t) = y(y - \beta)$ has to critical points: $y = 0$ and $y = \beta$. To examine the stability properties of these points we sketch the phase-time plots for different values of β.

- **Case 1:** $\beta > 0$

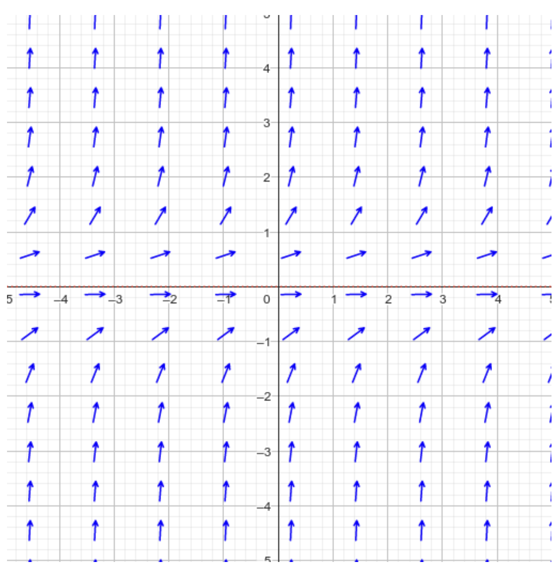
  When β is positive, the sign of $y'(t)$ depends on the sign of $y(y-\beta)$. When $y < 0$ or $y > \beta$ we have that $y'(t) > 0$ and, if $0 < y < \beta$, we have that $y'(t) < 0$. Therefore, since the derivative represents the slope of the tangent line to the graph of that function at a certain point:

  The critical point $y = 0$ is unstable and any solution that start close to it will move away from it as time goes on. The critical point $y = \beta$ is semi-stable. If solution starts close to $y = \beta$ and has initial value $y(0) < \beta$, it will move towards $y = \beta$ as time goes on. On the other hand, if solution starts close to $y = \beta$ and has initial value $y(0) > \beta$ it will move away $y = \beta$ as time goes on.

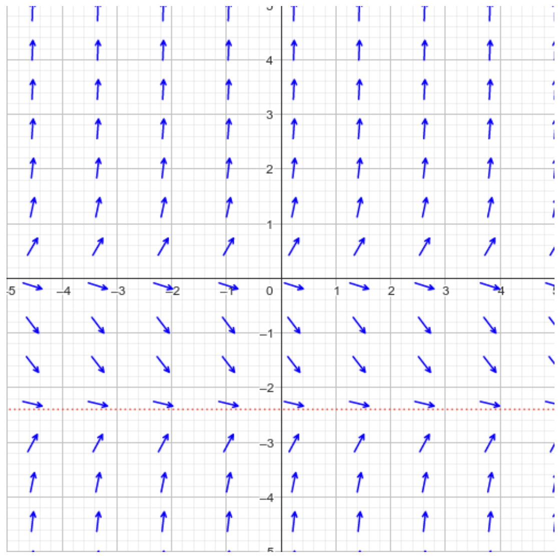
- **Case 2:** $\beta = 0$

  In this case, we have that $y'(t) = y^2$, so we have that $y'(t) \geq 0 \; \forall y$.

  In this case we only have the critical point $y = 0$ that is unstable.

- **Case 3:** $\beta < 0$
  When β is negative, the sign of $y'(t)$ depends on the sign of $y(y-\beta)$. When $y > 0$ or $y < \beta$ we have that $y'(t) > 0$ and, if $0 > y > \beta$, we have that $y'(t) < 0$.

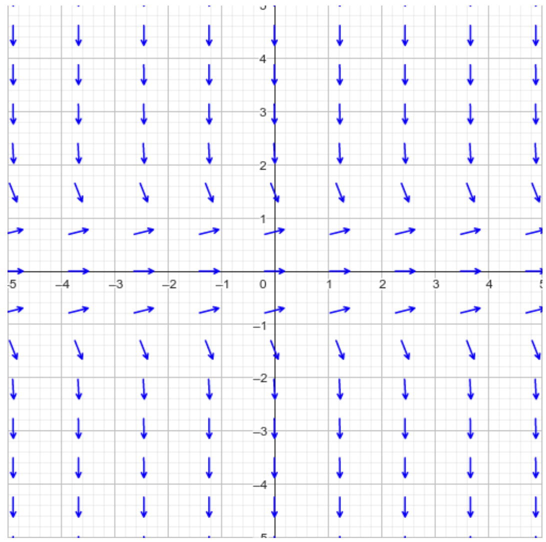
  

  In this case, the critical point $y = 0$ is unstable and any solution that start close to it will move away from it as time goes on. The critical point $y = \beta$ is semi-stable. If solution starts close to $y = \beta$ and has initial value $y(0) > \beta$, it will move towards $y = \beta$ as time goes on. On the other hand, if solution starts close to $y = \beta$ and has initial value $y(0) < \beta$ it will move away $y = \beta$ as time goes on.

As conclusion, we have that $y = 0$ is always unstable and $y = \beta$ is semi-stable.

3. **Sketch phase-time plots for the following equations. Indicate critical points where solutions increase and decrease, and where solutions change concavity. (a)** $y'(t) = y^2 - y^4$**, (b)** $y'(t) = cost$

   - **(a)** $y'(t) = y^2 - y^4$
     To find the critical points, we need to solve $y'(t) = y^2 - y^4 = 0 \implies y^2(1-y^2) = 0$. So, critical points are $y = -1$, $y = 0$, $y = 1$.
     To determinate where solutions increase and decrease,we can look at the sign of the derivative for different values of y:
     - If $y < -1$ then $y'(t) < 0$, so solutions will be decreasing.
     - If $-1 < y < 0$ then $y'(t) > 0$, so solutions will be increasing.
     - If $0 < y < 1$ then $y'(t) > 0$, so solutions will be increasing.
     - If $y > 1$ then $y'(t) < 0$, so solutions will be decreasing.



To determine where solutions change concavity, we need to look at the second derivative of y:

$$y''(t) = 2yy'(t) - 4y^3y'(t) = 2y(y^2 - y^4) - 4y^3(y^2 - y^4) = 2y^3 - 2y^5 - 4^5 + 4y^7 = 2y^3 - 6y^5 + 4y^7 =$$

$$= y^3(2 - 6y^2 + 4y^4) = 0 \implies y = 0,\ y = 1,\ y = -1,\ y = \frac{\sqrt{2}}{2},\ y = -\frac{\sqrt{2}}{2}$$

Then:

- If $y < -1$ we have $y''(t) < 0$. Then, solutions are concave down.
- If $-1 < y < -\frac{\sqrt{2}}{2}$ we have $y''(t) > 0$. Then, solutions are concave up.
- If $-\frac{\sqrt{2}}{2} < y < 0$ we have $y''(t) < 0$. Then solutions are concave down.
- If $0 < y < \frac{\sqrt{2}}{2}$ we have $y''(t) > 0$. Then solutions are concave up.
- If $\frac{\sqrt{2}}{2} < y < 1$ we have $y''(t) < 0$. Then solutions are concave down.
- If $y > 1$ we have $y''(t) > 0$. Then solutions are concave up.

- **(b)** $y'(t) = cos(t)$

  To determine where solutions increase and decrease,we can look at the sign of the derivative for different values of t:

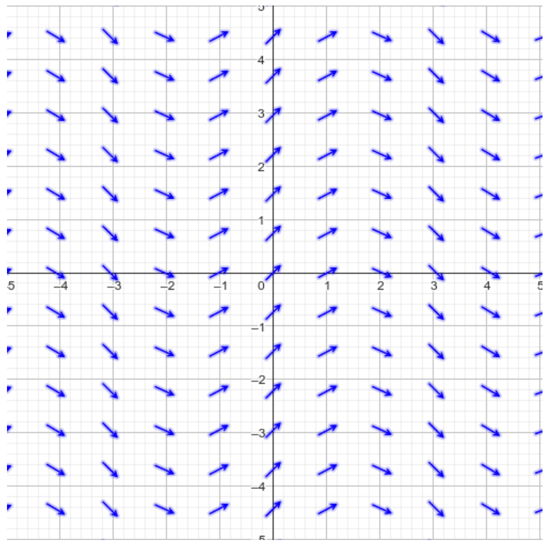
  - If $t \in \{x | 0 < x < \frac{\pi}{2}\} \cup \{x | \frac{3\pi}{2} < x < 2\pi\}$ then $cos(t) > 0$.
  - If $t \in \{x | \frac{\pi}{2} < x < \pi\} \cup \{x | \pi < x < \frac{3\pi}{2}\}$ then $cos(t) < 0$.

  To determine where solutions change concavity, we need to look at the second derivative of y:

$$y''(t) = -sen(t) = 0 \implies t = 0,\ t = \pi$$

  Then:

  - If $0 < t < \pi$ we have $y''(t) < 0$. Then, solutions are concave down.
  - If $0\pi < t < 2\pi$ we have $y''(t) > 0$. Then, solutions are concave up.



4. **Sketch a phase plot for each of the following systems. In each case describe what type of critical point the origin is and indicate whether or not the origin is asymptotically stable.**

- **(a)**

$$\left.\begin{aligned} x' &= y+1 \\ y' &= -4x+8 \end{aligned}\right\}$$

We start looking critical points, so we look points where $x' = 0$ and $y' = 0$. That implies that we are looking for points that are solution of the linear system:

$$\left.\begin{aligned} 0 &= y+1 \\ 0 &= -4x+8 \end{aligned}\right\}$$

Solving the system, we have a unique point that verifies it, that is $(x, y) = (2, -1)$.

Now we are going to look for eigenvalues. For doing it, we are going to express our system as follows:

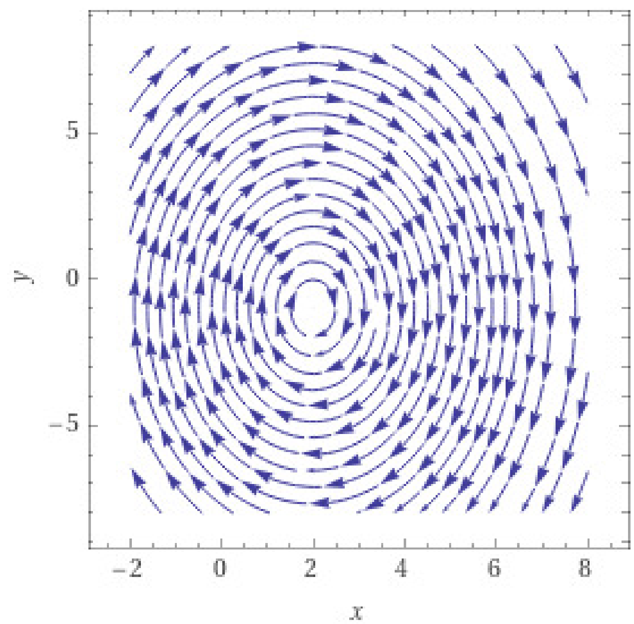
$$\vec{x}\,' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 1 \\ 8 \end{pmatrix} = A(t)\vec{x} + b(t)$$

Now we calculate the eigenvalues:

$$det(A(t) - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0 \implies \lambda = \pm\sqrt{-4} = \pm i\sqrt{4}$$

Because we got a complex solution, that indicates us that solutions are going to "rotate" around the center. If we take $x = 2$ and $y > -1$, then we have $x' > 0$, so that indicates us that the rotation is clockwise.

If we represent it, we see:



In this case, $(x, y) = (2, -1)$ is a stable point (center) since solutions just spiral around the center without decaying into or coming straight out it.

- **(b)**

$$\left.\begin{aligned} x' &= x + y \\ y' &= 2(x + y) \end{aligned}\right\}$$

We start looking critical points, so we look points where $x' = 0$ and $y' = 0$. That implies that we are looking for points that are solution of the linear system:

$$\left.\begin{aligned} 0 &= x + y \\ 0 &= 2x + 2y \end{aligned}\right\}$$

Solving the system, we have a infinite points that verifies it, that is $(x, y) = (\beta, -\beta)$. So all critical points are contained in the straight line $x + y = 0$.

Now we are going to look for eigenvalues. For doing it, we are going to express our system as follows:

$$\vec{x}\,' = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \cdot \vec{x} = A(t)\vec{x}$$

Now we calculate the eigenvalues:

$$det(A(t) - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 2 = -3\lambda + \lambda^2 = \lambda(-3+\lambda) = 0 \implies$$

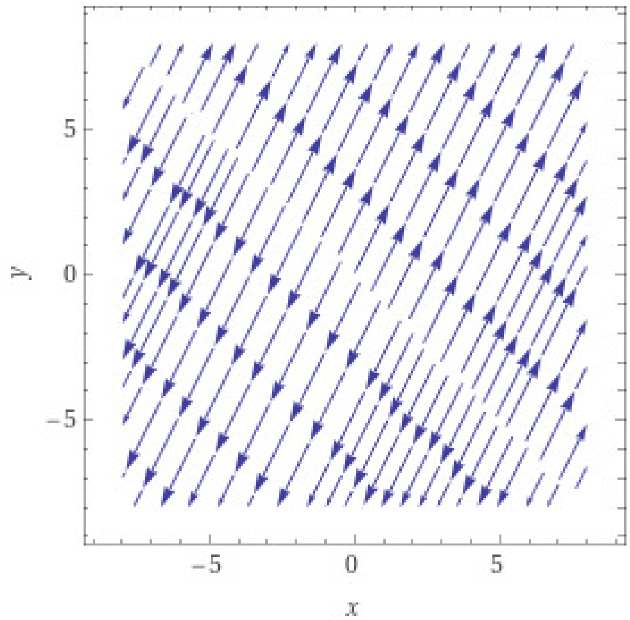
$$\implies \lambda = 0,\ \lambda = 3$$

We calculate now the associated eigenvectors:

$$V_0 = \{x \in \mathbb{R}^2 | A(t)x = 0\} = \{x \in \mathbb{R}^2 | x + y = 0\}$$

$$V_3 = \{x \in \mathbb{R}^2 | (A(t) - 3I)x = 0\} = \{(0, 0)\}$$

If we take $x = 1$ and $y < -1$, then we have that $x' < 0$. That indicates us that solutions below $x + y = 0$ decrease. On the other hand, if $x = 1$ and $y > -1$ then we have that $x' > 0$. That indicates us that solutions below $x + y = 0$ increase. If we represent it we have:



Since solutions go straight out from the critical points, the solutions are unstable.

5. **Using the Routh-Hurwitz criterion, test the stability conditions of the following equation**

$$y^4 + 3y''' + 3y'' + 3y' + 2y = 0$$

The Routh-Hurwitz criterion is based on the number of roots of the polynomial that lie in the right-half of the complex plane. If all the roots have negative real parts, then the system is stable. For doing it, we construct a table, called the Routh table, using the coefficients of the polynomial or, in this case, the characteristic equation of the differential equation. This table allows us to determinate the number of sign changes in the first column, which indicates the number of roots with positive real parts. If there are no sign changes, then all the roots have negative real parts and, following, the system is stable.

For the given differential equation, the characteristic equation is:

$$s^4 + 3s^3 + 3s^2 + 3s + 2 = 0 = a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

s^4	a_0	a_2	a_4
s^3	a_1	a_3	a_5
s^2	$b_1 = \frac{a_1 a_2 - a_3 a_0}{a_1}$	$b_2 = \frac{a_1 a_4 - a_5 a_0}{a_1}$	
s	$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1}$		
1	a_n		

s^4	1	3	2
s^3	3	3	0
s^2	2	2	
s	0		
1	2		

We can see that all elements of the first column of the Routh table are positive or 0, so there is no sign change in that column, and, then, the given differential equation is stable.

6. **Consider the following system of equations**

$$\left.\begin{aligned} y_1'(t) &= y_3 \\ y_2'(t) &= -3y_1 \\ y_3'(t) &= \alpha y_1 + 2y_2 - y_3 \end{aligned}\right\}$$

For which values of the parameter α the zero solution is asymptotically stable

To analyze the stability of the zero solution, we first consider the coefficient matrix:

$$A(t) = \begin{pmatrix} 0 & 0 & 1 \\ -3 & 0 & 0 \\ \alpha & 2 & -1 \end{pmatrix}$$

The eigenvalues of $A(t)$ will determinate the stability of the zero solution. If all eigenvalues have negative real parts, the zero solution is asymptotically stable. On the other hand, if at least one eigenvalue has a positive real part, the zero solution is unstable.

The characteristic polynomial of $A(t)$ is given by:

$$p(\lambda) = det(A(t) - \lambda I) = \begin{vmatrix} -\lambda & 0 & 1 \\ -3 & -\lambda & 0 \\ \alpha & 2 & -1-\lambda \end{vmatrix} = \lambda^2(-1-\lambda) - 6 + \lambda\alpha = -\lambda^3 - \lambda^2 + \alpha\lambda - 6 = 0$$

To find the values of α for which the zero solution is asymptotically stable, we need to determinate the values of α for which all the roots of $p(\lambda)$ has negative real parts.

One way to do this is using the Routh-Hurwitz criterion, which states that all roots of $p(\lambda)$ have negative real parts if and only if all the principal minors of the Routh array have the same sign. As we did in the previous exercise:

$$-s^3 - s^2 + \alpha s - 6 = 0 = a_0 s^3 + a_1 s^2 + a_2 s + a_3$$

s^3	a_0	a_2	a_4
s^2	a_1	a_3	a_5
s	$b_1 = \frac{a_1 a_2 - a_3 a_0}{a_1}$	$b_2 = \frac{a_1 a_4 - a_5 a_0}{a_1}$	
1	a_n		

s^3	-1	α	0
s^2	-1	-6	0
s	$\alpha + 6$	0	
1	-6		

So, we need $\alpha + 6 < 0$ for not having a sign change. So we can conclude that the values of α that makes the zero solution asymptotically stable are $\alpha < -6$.

7. **Find the Lyapunov function for the system**

$$\left.\begin{aligned} \frac{dx}{dt} &= -y^3 \\ \frac{dy}{dt} &= x^3 \end{aligned}\right\}$$

and discuss the stability of the zero solution.

To find the Lyapunov function for the system we can start by considering a function $V(x, Y)$ that is positive definite and has a derivative along the trajectories of the system that is negative definite. That is, we want to find a function $V(x, y)$ that satisfies the following conditions:

- $V(x,y) > 0, \forall (x,y) \neq (0,0)$
- $V(x,y) = 0$ only at $(0,0)$
- $\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} < 0,\ \forall (x,y) \neq (0,0)$

One possible Laypunov function for the given system is $V(x,y) = \frac{x^4}{4} + \frac{y^4}{4}$. To verify that this function satisfies the conditions above, we can compute its partial derivatives:

$$\frac{\partial V}{\partial x} = x^3 \quad \frac{\partial V}{\partial y} = y^3$$

Then, we can compute the derivative of V along the trajectories of the system:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = -x^3y^3 + y^3x^3 = 0$$

Since dV/dt is identically zero, we cannot use the direct Lyapunov method to conclude that the origin is stable. However, we can still use the indirect Lyapunov method that consists of finding a Lyapunov function V(x) for the system that satisfies the following conditions:

- $V(x)$ is positive definite, that is, $V(x) > 0,\ \forall x \neq 0$.
- $V(x)$ is radially unbounded, that is, $V(x) \to \infty$ as $||x|| \to \infty$
- $V(x)$ has a unique minimun at the origin, that is, $V(0) = 0$ and $V(x) > 0, \forall x \neq 0$

If such a Lyapunov function can be found, then the origin is stable in the sense of Lyapunov, and any nearby trajectory will approach the origin asymptotically as time goes to infinity.

So, by this method, we can conclude that he origin is asymptotically stable. This is because $V(x,y)$ is a positive definite function that is constant along the trajectories of the system, and it has a unique minimum at the origin. Therefore, any trajectory that starts sufficiently close to the origin will approach it asymptotically as $t \to \infty$.

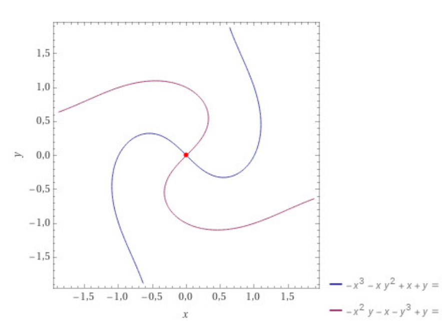
8. **Sketch phase-time plots for the following system:**

$$\left.\begin{array}{r} x'(t) = y(t) + x(t)\left[1 - y^2(t) - x^2(t)\right] \\ y'(t) = -x(t) + y(t)\left[1 - y^2(t) - x^2(t)\right] \end{array}\right\}$$

We start calculating critical points (points where $x'(t) = 0$ and $y'(t) = 0$). That means we have to solve system:

$$\left.\begin{array}{r} 0 = y(t) + x(t)\left[1 - y^2(t) - x^2(t)\right] \\ 0 = -x(t) + y(t)\left[1 - y^2(t) - x^2(t)\right] \end{array}\right\} \implies \left.\begin{array}{r} 0 = y + x - y^2x - x^3 \\ 0 = -x + y - y^3 - x^2y \end{array}\right\}$$

Solving the system (I solved it graphically), the only critical point is $(0,0)$.



To analyze the stability of the critical point, we need to compute the Jacobian matrix of the system:

$$J(x,y) = \begin{pmatrix} 1-2xy-3x^2 & 1-x^2 \\ -1 & 1-3y^2-x \end{pmatrix}$$

Now we evaluate it at the critical point $(0,0)$

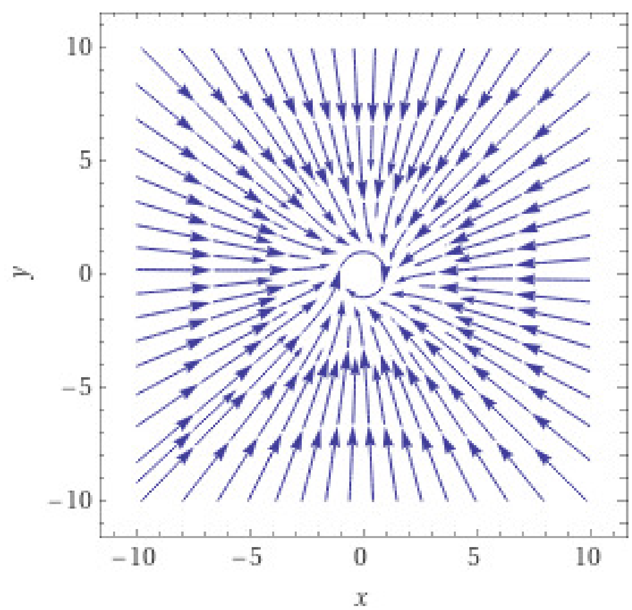
$$J(0,0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \implies p(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \implies \lambda^2 - 2\lambda + 2 = 0 \implies \lambda = 1 \pm i$$

So we know that the critical point is a center and trajectories of the system will form closed orbits around it.

For sketching the phase-time plot we are going to plot the nullclines, which are the curves where $x'(t) = 0$ and $y'(t) = 0$. These curves separate the phase plane into different regions where signs of $x'(t)$ and $y'(t)$ are different and, then, we can determinate the direction of the trajectories in each region by evaluating the signs of $x(t)$ and $y(t)$.

$$\left.\begin{array}{r} 0 = y(t) + x(t)\left[1 - y^2(t) - x^2(t)\right] \\ 0 = -x(t) + y(t)\left[1 - y^2(t) - x^2(t)\right] \end{array}\right\} \implies \left.\begin{array}{c} y = -x(1-x^2) \\ y = x/(1+x^2) \end{array}\right\}$$

Now we can evaluate the signs of $x'(t)$ and $y'(t)$ in each region. For example, in the region where $x < 0$ and $y > 0$, we have $x'(t) > 0$ and $y'(t) < 0$, which means that trajectories move down to the right. Repeating this process, we can get the next sketch:



Since trajectories are decaying and roating around center, the solutions are stable.