1. General aspects about random vectors

The <u>mean vector</u> of X is $\mu_X := E[X] = \begin{pmatrix} E[X_1] \\ \vdots \\ E[X_p] \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}$

<u>Linearity Property</u>: $X = (X_1, \dots, X_p)'$ r.v. and define

$$Y = BX + b, \quad \text{with } B \text{ a } q \times p \text{ constant matrix}, \; b \text{ a } q \times 1 \text{ constant vector}$$

Then, we have that $\mu_Y = B\mu_X + b$

<u>Dem</u>: $\mu_Y = E[Y] = E[BX + b] = E\left[\begin{pmatrix} \langle b_1 X \rangle + b_1 \\ \vdots \\ \langle b_q X \rangle + b_q \end{pmatrix}\right] =$

$$\downarrow$$

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ \vdots & & \ddots & \vdots \\ b_{q1} & \cdots & \cdots & b_{qp} \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_p \end{pmatrix} = \begin{pmatrix} b_{11}X_1 + b_{12}X_2 \cdots \\ \vdots \\ b_{q1}X_1 + b_{q2}X_2 \cdots \end{pmatrix}$$

$$= \begin{pmatrix} E[\langle b_1 X \rangle + b_1] \\ \vdots \\ E[\langle b_q X \rangle + b_q] \end{pmatrix} = \begin{pmatrix} E[\langle b_1 X \rangle] + E[b_1] \\ \vdots \\ E[\langle b_q X \rangle] + E[b_q] \end{pmatrix} =$$

$$= \begin{pmatrix} E[\langle b_1 X \rangle] + b_1 \\ \vdots \\ E[\langle b_q X \rangle] + b_q \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{p} b_{1j} E[X_j] \\ \vdots \\ \sum_{j=1}^{p} b_{qj} E[X_j] \end{pmatrix} + b =$$

$$= \begin{pmatrix} b_1' \cdot E[X] \\ \vdots \\ b_q' \cdot E[X] \end{pmatrix} + b = BE[X] + b = B\mu_X + b \quad \blacksquare$$

Idea a seguir: Aplicar def, producto de matrices y desarrollar siguiendo las propiedades de la esperanza.

The <u>covariance</u> matrix of X is defined as

$$\Sigma_X = \mathrm{Cov}(X) := E[(X - \mu_X)(X - \mu_X)'] = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1p} \\ \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdots & \sigma_{pp} \end{pmatrix}$$

$$\sigma_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)]$$

Note that $\sigma_{ii} = E[(X_i - \mu_i)^2] = \mathrm{var}(X_i)$ and $\sigma_{ij} = \sigma_{ji}$

The class of covariance matrices of dim $p\times p$ coincides with the class of symmetric non-negative definite matrices of dim $p\times p$.

That's means Σ symmetric and corr. matrix $\Leftrightarrow$ Σ non-negative definite.

Dem: $\Rightarrow$| Σ symmetric and is the cov. matrix of a r.v. $\underline{X}$ with mean vector μ. Let $\alpha \in \mathbb{R}^p$, we have then

$0 \leq \mathrm{Var}(\alpha'\underline{X}) = E[(\alpha'\underline{X} - \alpha'\mu)^2] = E[(\alpha'\underline{X} - \alpha'\mu)(\alpha'\underline{X} - \alpha'\mu)'] =$

$= \alpha' E[(\underline{X}-\mu)(\underline{X}-\mu)']\alpha = \alpha'\Sigma\alpha \Rightarrow \alpha'\Sigma\alpha \geq 0 \Rightarrow$ Non-negative

$\Leftarrow$| Σ symmetric and non-negative. Take $r = \underline{\mathrm{rank}(\Sigma)}$, $r \leq p$. We can write $\underline{\Sigma = CC'}$ for some C of dim. $p\times 1$. Let $\underline{Y} = (Y_1, \dots, Y_r)'$ with $\mu_{\underline{Y}} = 0$ and $\Sigma_{\underline{Y}} = I_r$, $\underline{Y \sim N_r(0, I_r)}$ and let $\underline{\underline{X} = C\underline{Y}}$. We have then:

$E[\underline{X}] = E[C\underline{Y}] = CE[\underline{Y}] = C\cdot 0 = 0$

$\mathrm{Cov}(\underline{X}) = E[(\underline{X}-\mu_{\underline{X}})(\underline{X}-\mu_{\underline{X}})'] = E[\underline{X}\underline{X}'] = E[(C\underline{Y})(C\underline{Y})'] =$

$= E[C\underline{Y}\underline{Y}'C'] = CE[\underline{Y}\underline{Y}']C' = C I_r C' = CC' = \Sigma$ ∎

Idea: $\Rightarrow$| Symmetric and cov. matrix. Use $\mathrm{var}(\alpha'\underline{X})$ to see it's non negative. $\Leftarrow$| Symmetric and non-negative. $\Sigma = CC'$ of dim $p\times 1$ and $Y = (Y_1 \dots, Y_r)'$, $r = \mathrm{rank}(\Sigma)$, $Y \sim N_r(0, I_r)$ and take $\underline{X} = C\underline{Y}$. Use $\mathrm{Cov}(\underline{X})$ to see it's Σ.

For any covariance matrix, we can distinguish the following cases:

$\underline{\Sigma \text{ positive-definite } (\Sigma > 0)}$

Σ non-singular, $|\Sigma| > 0$, $\exists\, \Sigma^{-1}$.

In this case, we can consider a "$\underline{\text{normalization}}$" of $\underline{X} \sim (\mu, \Sigma)$ as $Z = C^{-1}(\underline{X}-\mu) \sim (0_p, I_{p\times p})$, where C is a matrix of dim $p\times p$ s.t. $\Sigma = CC'$

Dem: $E[Z] = E[C^{-1}(\underline{X}-\mu)] = C^{-1}(E[\underline{X}-\mu]) =$

$= C^{-1}(E[\underline{X}] - \mu) = C^{-1}(\mu - \mu) = 0_p$

The equation $\Delta(\underline{x},\mu) = K$, $K \geq 0$ and $x \in \mathbb{R}^p$, defines a hyperellipsoid in $\mathbb{R}^p$ in such a way that the points transformed by normalization correspond to a p-dimensional Euclidean sphere of radius K with center at the origin O.

$\underline{\text{Dem}}$: $\Delta(\underline{x},\mu) = K \Rightarrow \Delta^2(\underline{x},\mu) = K^2 \Rightarrow (\underline{x}-\mu)'\Sigma^{-1}(\underline{x}-\mu) = K^2 \overset{K \neq 0}{\Rightarrow}$

$\Rightarrow (\underline{x}-\mu)'(K^2\Sigma)^{-1}(\underline{x}-\mu) = 1$ is an hyperellipsoid because

$\Sigma > 0 \Rightarrow \Sigma^{-1} > 0 \Rightarrow (K^2\Sigma)^{-1} > 0$.

Idea: Take the expresion powered by 2.

Change of variables (linear case)

$\underline{x}$ p-dimensional r.vec

$\underline{y}$ p-dimensional r.vec.

$\underline{y} = g(\underline{x}) = B\underline{x} + b$, with B $p \times p$ constant matrix, non-singular

b $p \times 1$ constant vector

Then, $\underline{x} = B^{-1}(\underline{y}-b)$

$J_{g^{-1}}(\cdot) = |B|^{-1}$

$f_{\underline{y}}(y) = f_{\underline{x}}(B^{-1}(y-b))\,\text{abs}(|B|^{-1})$

$$\Sigma_Z = E[(Z-\mu_Z)(Z-\mu_Z)'] \overset{\mu_Z = 0_p}{=} E[ZZ'] = E[C^{-1}(X-\mu)(X-\mu)'(C^{-1})'] =$$

$$= C^{-1}E[(X-\mu)(X-\mu)'](C^{-1})' = C^{-1}\Sigma_X (C')^{-1} = C^{-1}CC'(C')^{-1} =$$

$$= I_{p\times p}$$

Idea: Use the definition of $E[Z]$ and $\mathrm{cov}(Z)$ and calculate.

In this case, we can also define the Mahalanobis distance of $X \sim (\mu, \Sigma)$ with respect to its mean vector μ as

$$\Delta(X,\mu) := \{(X-\mu)'\Sigma^{-1}(X-\mu)\}^{1/2}$$

Remarks:

- $\Delta(X,\mu) = \|Z\|_p$, Z normalization of X.

Dem: Take $Z = C^{-1}(X-\mu)$, $\Sigma = CC'$ and $\Delta(X,\mu)$

$$\Delta(X,\mu) = \{(X-\mu)'\Sigma^{-1}(X-\mu)\}^{1/2} = \{(X-\mu)'(CC')^{-1}(X-\mu)\}^{1/2} =$$

$$= \{(X-\mu)'(C^{-1})'C^{-1}(X-\mu)\}^{1/2} = \{Z'Z\}^{1/2} = \|Z\|_p$$

Idea: $\Sigma = CC'$ for taking Z and substitute in $\Delta(X,\mu)$

- $\Delta(X,\mu)$ is a r.v. with $E[\Delta^2(X,\mu)] = p$.

Dem: $E[\Delta^2(X,\mu)] = E[\|Z\|_p^2] = E[Z'Z] = E[\sum_{j=1}^{p} z_j^2] =$

$$= \underbrace{\sum_{j=1}^{p} E[z_j^2]}_{E[z_j]=0} = \sum_{j=1}^{p} \mathrm{var}(z_j) = \sum_{j=1}^{p} 1 = p$$

Idea: Change $\Delta^2(X,\mu) = \|Z\|_p^2$ and remember $\mu_Z = 0$ and $\Sigma_Z = 1$.

Σ positive-semidefinite ($\Sigma \geq 0$)

Σ singular, $|\Sigma| = 0$, $\nexists\, \Sigma^{-1}$.

As a consequence, we cannot perform normalization or Mahalanobis distance.

Remarks:

- $\mathrm{rank}(\Sigma) = r < p$, so $\Sigma = CC'$ with C a $p\times r$ matrix of rank r.
- At least, satisfies $\alpha' X = K$, $\alpha \neq 0$.

Dem: Let $Y = \alpha' X$ s.t. $\alpha'\Sigma_X\alpha = 0$, because $|\Sigma_X| = 0$,

then $E[Y] = E[\alpha' X] = \alpha' E[X] = \alpha'\mu = K$.

$$\mathrm{var}(Y) = E[(Y-\mu_Y)^2] = E[(Y-\mu_Y)(Y-\mu_Y)'] =$$

$$= E[(\alpha' X - K)(\alpha' X - K)'] \underset{K=\alpha'\mu_X}{=} \alpha' E[(X-\mu_X)(X-\mu_X)']\alpha = \alpha'\Sigma_X\alpha \underset{|\Sigma_X|=0}{=} 0$$

Idea: Take $Y = \alpha' X$, $\alpha \neq 0$ and remember $\alpha'\Sigma_X\alpha = 0$ because $|\Sigma_X| = 0$

Linear transformation: $X \sim (\mu_X, \Sigma_X)$ p-dimensional

$Y = BX + b$, with B a $q \times p$ constant matrix
b a $q \times 1$ constant vector

Then, $Y \sim (B\mu_X + b, B\Sigma_X B')$

Dem: First part already demostrated.

$$\Sigma_Y = E[(Y-\mu_Y)(Y-\mu_Y)'] = E[(BX+b-(B\mu_X+b))(BX+b-(B\mu_X+b))'] =$$
$$= E[B(X+\cancel{b}-\mu_X-\cancel{b})(X+\cancel{b}-\mu_X-\cancel{b})B'] = BE[(X-\mu_X)(X-\mu_X)']B' =$$
$$= B\Sigma_X B' \quad \boxtimes$$

Idea: Definition of Σ_Y and $Y = BX + b$.

Theorem: Let $X = (X_1, \dots, X_p)'$ be a r. vec. Then, the multivariate distribution of X is univocally determinated by the set of all the univariate distributions of the r. var. of the form $\alpha' X$, $\forall \alpha' \in \mathbb{R}^p$.

Dem: $\phi_X(t) = E[e^{it'X}]$ and let $Z_\alpha = \alpha' X$, $\alpha' \in \mathbb{R}^p$

$$Y_\alpha = \alpha' X = \sum_{j=1}^{p} \alpha_j X_j \Rightarrow \Phi_{Y_\alpha}(t) = E[e^{it'(\alpha' X)}]$$

- $t = 1 \Rightarrow \phi_{Z_\alpha}(1) = E[e^{i\alpha' X}] = \phi_X(\alpha)$
- $\alpha = t \Rightarrow \phi_{Z_t}(1) = \phi_X(t)$
$$\downarrow$$
$$Z_t = tX$$

Idea: $Y_\alpha = \alpha' X$, $\alpha' \in \mathbb{R}^p$
Consider $\phi_X(t)$ and $\phi_{Y_\alpha}(t)$ and follow. $\boxtimes$

" De este tema queda también los cambios de variable, pero si entran ~~o~~ en ejercicio, así que acabo aquí ".

2. MND. Case $\Sigma > 0$

Let $X = (X_1, \dots, X_p)'$ be a r. vec. $X \sim N_p(\mu, \Sigma)$ if

$$f_X(x) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right\}, \forall x \in \mathbb{R}^p$$

Dem: $f_X(x) \geq 0$, $\forall x \in \mathbb{R}^p$, because $|\Sigma|^{1/2} > 0$ and $\exp \geq 0$

$$\int_{\mathbb{R}^p} f_X(x) \underset{\substack{\downarrow \\ X = CZ+\mu}}{=} \int_{\mathbb{R}^p} \frac{|C|}{(2\pi)^{p/2} |C|^{1/2} |C|^{1/2}} \exp\left\{-\frac{1}{2} z'z\right\} dz =$$

$z = g(x) = C^{-1}(X-\mu)$ s.t. $\Sigma = CC'$, $|C| > 0$ $\quad \frac{dg_i}{dx_j} = c_{ij}^{-1}$

$$= \int_{\mathbb{R}^p} \frac{1}{(2\pi)^{p/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{p} z_i^2\right\} dz_1 \cdots dz_p = \prod_{i=1}^{p} \underbrace{\int_{\mathbb{R}^p} \frac{1}{\sqrt{2}} \exp\left\{-\frac{1}{2} z_i^2\right\} dz_i}_{\text{☒ } z_i \sim N(0,1)} =$$

$$= \prod_{i=1}^{p} 1 = 1$$

Idea: 1) trivial 2) take $X = CZ + \mu$

Characterization I: A p-dimensional r. vec. X has a non-singular MND, $X \sim N_p(\mu, \Sigma)$, $(\Sigma > 0)$, if and only if,

$X := AZ + \mu$, with A a $p \times p$ cte. matrix, non-singular, $\Sigma = AA'$

$Z \sim N_p(0, I_p)$

Dem: $\Rightarrow$) $X = AZ + \mu$ is a linear change, so, applying linearity we have $X \sim N_p(A\mu_Z + \mu, A\Sigma_Z A') = (\mu, AA') = (\mu, \Sigma)$

$\Leftarrow$) Take $Z = A^{-1}(X - \mu)$, $\Sigma = AA'$. Then, we know that

$Z \sim N_p(0, I_p)$ and $X = AZ + \mu$ ☒

Idea: $\Rightarrow$) Apply linearity

$\Leftarrow$) Apply normalitation

Result 1: Let $X = (X_1, \cdots, X_p)'$ be a r. vec. with non-singular MND, $X \sim N_p(\mu, \Sigma)$, $(\Sigma > 0)$. If Σ is diagonal $\Rightarrow$

$\Rightarrow X_i$, $i = 1, \cdots, p$ are independent each other and each one follows an UND, $X_i \sim N(\mu_i, \sigma_i^2)$. $i = 1, \cdots, p$

Dem: Take the distribution function of X.

$$f_X(x) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(X - \mu)' \Sigma^{-1} (X - \mu)\right\}$$

and suppose $\Sigma = \mathrm{diag}(\sigma_{11}^2 \cdots \sigma_{pp}^2)$, then, we have that $|\Sigma| = \prod_{i=1}^{p} \sigma_{ii}^2 \Rightarrow |\Sigma|^{1/2} = \prod_{i=1}^{p} \sigma_{ii}$ and $\Sigma^{-1} = \mathrm{diag}(\sigma_{11}^{-2} \cdots \sigma_{pp}^{-2})$

Then, $(X - \mu)' \Sigma^{-1} (X - \mu) = \sum_{i=1}^{p} \frac{(X_i - \mu_i)^2}{\sigma_{ii}^2}$. So, we get

that $f_X(x) = \frac{1}{(2\pi)^{p/2} \prod_{i=1}^{p} \sigma_{ii}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{p} \frac{(X_i - \mu_i)^2}{\sigma_{ii}^2}\right\} =$

$$= \prod_{i=1}^{p} \frac{1}{\sqrt{2\pi}\sigma_{ii}} \exp\left\{-\frac{1}{2} \frac{(X_i - \mu_i)}{\sigma_i^2}\right\} = \prod_{i=1}^{p} f_{X_i}(x_i) \Rightarrow \text{Independents}$$ ☒

Idea: Take the distribution func. and Σ diagonal and calculate to see that $f_X(x) = f_{X_1}(x_1) \cdots f_{X_p}(x_p)$

The converse of the result is true in the sense that if $X = (X_1, \dots, X_p)'$ is a r. vec. with mutually independent and each one $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, \dots, p$

$\sigma_i > 0$!!

then, $X \sim N_p(\mu, \Sigma)$ with $\Sigma = diag(\sigma_1^2 \dots \sigma_p^2)$ $(\Sigma > 0)$

Result 2: Let $X = (X_1, \dots, X_p)'$ be a r. vec. with non-singular MND $X \sim N_p(\mu, \Sigma)$, $(\Sigma > 0)$. Assume that the components of X are ordered in such a way that $X = \begin{pmatrix} X_{(1)} \\ X_{(2)} \end{pmatrix}$ with $X_{(1)} = (X_1, \dots, X_q)'$, $X_{(2)} = (X_{q+1}, \dots, X_p)'$ we have $\mu = \begin{pmatrix} \mu_{(1)} \\ \mu_{(2)} \end{pmatrix}$, $\Sigma = \begin{pmatrix} \Sigma_{(11)} & \Sigma_{(12)} \\ \Sigma_{(21)} & \Sigma_{(22)} \end{pmatrix}$.

Suppose $\Sigma_{(21)} = \Sigma_{(12)} = 0 \Rightarrow X_{(1)}$ and $X_{(2)}$ are independent and $X_{(1)} \sim N_q(\mu_{(1)}, \Sigma_{(11)})$, $X_{(2)} \sim N_{p-q}(\mu_{(2)}, \Sigma_{(22)})$

Dem: Take $\Sigma = \begin{pmatrix} \Sigma_{(11)} & 0 \\ 0 & \Sigma_{(22)} \end{pmatrix}$ and consider the distribution function $f_X(x) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(X-\mu)'\Sigma^{-1}(X-\mu)\right\}$

$\Sigma = diag(\Sigma_{(11)}, \Sigma_{(22)}) \Rightarrow \Sigma^{-1} = diag(\Sigma_{(11)}^{-1}\ \Sigma_{(22)}^{-1})$

$\Rightarrow |\Sigma| = |\Sigma_{(11)}||\Sigma_{(22)}|$

$$(X-\mu)'\Sigma^{-1}(X-\mu) = ((X_1-\mu_1)'(X_2-\mu_2)')\begin{pmatrix} \Sigma_{(11)}^{-1} & 0 \\ 0 & \Sigma_{(22)}^{-1} \end{pmatrix}\begin{pmatrix} X_1-\mu_1 \\ X_2-\mu_2 \end{pmatrix} =$$

$$= ((X_1-\mu_1)'\Sigma_{(11)}^{-1}\ \ (X_2-\mu_2)'\Sigma_{(22)}^{-1})\begin{pmatrix} X_1-\mu_1 \\ X_2-\mu_2 \end{pmatrix} =$$

$$= (X_1-\mu_1)'\Sigma_{(11)}^{-1}(X_1-\mu_1) + (X_2-\mu_2)'\Sigma_{(22)}^{-1}(X_2-\mu_2)$$

Then, $f_X(x) = \frac{1}{(2\pi)^{p/2}|\Sigma_{(11)}|^{1/2}|\Sigma_{(22)}|^{1/2}} \cdot$

$$\exp\left\{-\frac{1}{2}(X_1-\mu_1)'\Sigma_{(11)}^{-1}(X_1-\mu_1) - \frac{1}{2}(X_2-\mu_2)'\Sigma_{(22)}^{-1}(X_2-\mu_2)\right\} =$$

$$= \frac{1}{(2\pi)^{q/2}|\Sigma_{(11)}|^{1/2}} \exp\left\{-\frac{1}{2}(X_1-\mu_1)'\Sigma_{(11)}^{-1}(X_1-\mu_1)\right\} \cdot \frac{1}{(2\pi)^{\frac{p-q}{2}}|\Sigma_{(22)}|^{1/2}} \exp\left\{-\frac{1}{2}(X_2-\mu_2)'\Sigma_{(22)}^{-1}(X_2-\mu_2)\right\}$$

$= f_{X_1}(x_1) f_{X_2}(x_2)$ ☒

Idea: Consider $\Sigma = \begin{pmatrix} \Sigma_{(11)} & 0 \\ 0 & \Sigma_{(22)} \end{pmatrix}$ and the distribution function and calculate.

Result 3: Let (as in 2) but $\Sigma = \begin{pmatrix} \Sigma_{(11)} & \Sigma_{(12)} \\ \Sigma_{(21)} & \Sigma_{(22)} \end{pmatrix}$. Then, we have that:

- $X_{(1)}$ and $X_{(2)} - \Sigma_{(21)}\Sigma_{(11)}^{-1}X_{(1)}$ are independent.
- $X_{(1)} \sim N_q(\mu_1, \Sigma_{(11)})$

  $X_{(2)} - \Sigma_{(21)}\Sigma_{(11)}^{-1}X_{(1)} \sim N_{p-q}\left(\mu_2 - \Sigma_{(21)}\Sigma_{(11)}^{-1}\mu_{(1)}, \Sigma_{(22)} - \Sigma_{(21)}\Sigma_{(11)}^{-1}\Sigma_{(12)}\right)$
- $\dfrac{X_{(2)}}{X_{(1)}} \sim N_{p-q}\left(\mu_2 + \Sigma_{(21)}\Sigma_{(11)}^{-1}(x_1 - \mu_1), \Sigma_{(22)} - \Sigma_{(21)}\Sigma_{(11)}^{-1}\Sigma_{(12)}\right)$

Dem: Descarted by teacher.

Characterization II: A p-dimensional r. vec. X has a non-singular MND, $X \sim N_p(\mu, \Sigma)$, if and only if, it's characteristic function is of the form

$$\phi_X(t) = \exp\left\{ it'\mu - \frac{1}{2}t'\Sigma t \right\}$$

Dem:

$$\phi_X(t) = E[e^{it'X}] = \int_{\mathbb{R}^p} e^{it'X} f_X(x)\,dx =$$

$$= \int \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{ it'X - \frac{1}{2}(X-\mu)'\Sigma^{-1}(X-\mu) \right\} dx$$

Consider the variable change $Y = C^{-1}(X-\mu)$, $\Sigma = CC'$. So we have that $X = CY + \mu$, $|\Sigma| = |C\,C'| = |C||C'| = |C|^2$

$\longrightarrow f_Y(y) = f_X(B^{-1}(Y-b))\,\mathrm{abs}(|B|^{-1})$

$$\Rightarrow \phi_X(t) = \int \frac{\boxed{\mathrm{abs}(|C|)}}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{ it'(CY+\mu) - \frac{1}{2}YY' \right\} dx =$$

$$= e^{it'\mu}\int \frac{1}{(2\pi)^{p/2}} \exp\left\{ it'CY - \frac{1}{2}YY' \right\} = \begin{bmatrix} t'C = \alpha' \\ \alpha_j = t'c_j \end{bmatrix} =$$

$$= \prod_{j=1}^{p} e^{it'_j\mu_j}\,\phi_{Y_j}(\alpha_j) = \exp\left\{ it'\mu - \frac{1}{2}t'\Sigma t \right\} \longleftarrow$$

$\sum_j t_j\mu_j = t'\mu$

$\sum_j \alpha_j^2 = \alpha'\alpha = t'c'ct = t'\Sigma t$

$$\phi_{Y_j}(\alpha_j) = E[e^{it'Y_j}] = f_Z(y_j)$$

$$Y \sim N(0, I)$$

$$Y_j \sim N(0,1) \Rightarrow \phi_{Y_j}(\alpha_j) = e^{-1/2\,\alpha_j^2}$$

∎

× Next Page

Linear change of variables: Let $\underline{X} \sim N_p(\mu_X, \Sigma_X)$, $\Sigma > 0$. Consider

$\underline{Y} = B\underline{X} + b$ with B a $p \times p$ constant matrix, non singular

b a $p \times 1$ constant vector

Then, $\underline{Y} \sim N_p(\mu_Y, \Sigma_Y) = N_p(B\mu_Y + b, B\Sigma_X B')$

Dem 1: Consider the distribution function of $\underline{X}$

$$f_X(x) = \frac{1}{(2\pi)^{p/2}|\Sigma_X|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{X}-\mu)'\Sigma_X^{-1}(\underline{X}-\mu)\right\} \text{ and}$$

apply the linear change $\underline{Y} = B\underline{X} + b \Rightarrow \underline{X} = B^{-1}(\underline{Y} - b)$

$$f_Y(g(x)) = f_Y(B^{-1}(\underline{Y}-b))\,\text{abs}|B^{-1}| =$$

$$= \frac{\text{abs}|B^{-1}|}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(B^{-1}(\underline{Y}-b)-\mu_X)'\Sigma_X^{-1}(B^{-1}(\underline{Y}-b)-\mu_X)\right\}$$

$$= \frac{1}{(2\pi)^{p/2}|\Sigma_Y|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{Y}'-b'-\mu_X'B')(B^{-1})'\Sigma_X^{-1}B^{-1}(\underline{Y}-b-B\mu_X)\right\}$$

$$|\Sigma_Y| = |B\Sigma_X B'| = |B^2||\Sigma_X| \Rightarrow |\Sigma_Y| = |B||\Sigma_X|^{1/2} \quad \left(\frac{\text{abs}|B^{-1}|}{|B|} = 1\right)$$

$$= \frac{1}{(2\pi)^{p/2}|\Sigma_Y|^{1/2}} \exp\left\{(\underline{Y}'-(B\mu+b))(B\Sigma B')^{-1}(\underline{Y}-(B\mu+b))\right\} \quad \blacksquare$$

Idea: Take the distribution function and apply the change, but we can demostrate this with the following result.

Result 4: Let $\underline{X} \sim N_p(\mu_X, \Sigma_X)$, $\Sigma_X > 0$, and consider

$\underline{Y} = B\underline{X} + b$, with B a $q \times p$ constant matrix of rank $q \leq p$

b a $q \times 1$ constant vector

Then, $\underline{Y} \sim N_q(\mu_Y, \Sigma_Y) = N_q(B\mu_X + b, B\Sigma_X B')$

Dem 2: Consider the c.f. of $\underline{Y}$:

$$\phi_Y(t) = \exp\left\{it'\mu_Y - \frac{1}{2}t'\Sigma_Y t\right\} = \exp\left\{it'(B\mu_X + b) - \frac{1}{2}t'(B\Sigma_X B')t\right\} \Rightarrow$$

$\Rightarrow \underline{Y} \sim N_q(B\mu_X + b, B\Sigma_X B')$ and the rank

of $B\Sigma_X B'$, $\text{rank}(B\Sigma_X B') = \min\{\text{rank}(B), \text{rank}(\Sigma_X), \text{rank}(B')\} = \min\{p, q\} = q$. $\blacksquare$

Idea: Directly with the c.f.

Dem:

$$\phi_{\mathbf{x}}(t) = E[e^{it'\mathbf{x}}] = \int_{\mathbb{R}^p} \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{ it'\mathbf{x} - \frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu) \right\} dx$$

$$\hookrightarrow \int_{\mathbb{R}^p} e^{it'\mathbf{x}} f_{\mathbf{x}}(x)dx$$

Consider the variable change $\mathbf{y} = C^{-1}(\mathbf{x}-\mu)$, $\Sigma = CC'$. So, we have that $|\Sigma| = |C||C'| = |C|^2$; $|C|^{-1} = |C|$

$$\Rightarrow \phi_{\mathbf{x}}(t) = \int_{\mathbb{R}^p} \frac{\text{abs}(|C|)}{(2\pi)^{p/2}|C|^2} \exp\left\{ it'(C\mathbf{y}+\mu) - \frac{1}{2}(C\mathbf{y}+\mu-\mu)'\overset{(CC')^{-1}}{\overset{\shortparallel}{\Sigma^{-1}}}(C\mathbf{y}+\mu-\mu) \right\} dy =$$

$$= e^{it'\mu} \int_{\mathbb{R}^p} \frac{1}{(2\pi)^{p/2}} \exp\left\{ it'C\mathbf{y} - \frac{1}{2}\mathbf{y}\mathbf{y}' \right\} dy = \left[\begin{matrix} \alpha_j = \sum_{i=1}^p t_i C_{ij} \\ t'C = \alpha' \end{matrix} \right] =$$

$$(\alpha_1 \cdots \alpha_p)\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = \alpha_1 y_1 + \cdots + \alpha_p y_p = \sum_{i=1}^p \alpha_i y_i$$

$$(t_1, \dots, t_p)\begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & & \vdots \\ c_{p1} & \cdots & c_{pp} \end{pmatrix} = \begin{pmatrix} t_1 c_{11} + \cdots + t_p c_{p1} \\ \vdots \\ t_p c_{1p} + \cdots + t_p c_{pp} \end{pmatrix}' =$$

$$\Rightarrow \alpha_j = \sum_{i=1}^p t_i C_{ij}$$

$$= e^{it'\mu} \int_{\mathbb{R}^p} \frac{1}{(2\pi)^{p/2}} \exp\left\{ i\sum_{i=1}^p \alpha_i y_i - \frac{1}{2} \sum_{i=1}^p y_i^2 \right\} dy =$$

$$(y_1 \cdots y_p)\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = y_1^2 + \cdots + y_p^2 = \sum_{i=1}^p y_p^2$$

$$= e^{it'\mu} \int_{\mathbb{R}^p} \frac{1}{(2\pi)^{p/2}} \exp\left\{ \sum_{i=1}^p i\alpha_i y_i - \frac{1}{2} y_i^2 \right\} dy_i =$$

$$= e^{it'\mu} \int_{\mathbb{R}^p} \frac{1}{(2\pi)^{p/2}} \prod_{i=1}^p \exp\left\{ i\alpha_i y_i - \frac{1}{2} y_i^2 \right\} dy_i =$$

$$\mathbf{y} \sim N(0, I) \Rightarrow \mathbf{y}_i \sim N(0,1)$$

$$= \prod_{i=1}^p e^{it'\mu} \int_{\mathbb{R}^p} \frac{1}{(2\pi)^{1/2}} \exp\left\{ i\alpha_i y_i - \frac{1}{2} y_i^2 \right\} dy_i =$$

$$= \prod_{i=1}^p e^{it'\mu} E[e^{i\alpha_j y_j}] = \prod_{i=1}^p e^{it'\mu} \phi_{y_j}(\alpha_j) = \exp\left\{ it'\mu - \frac{1}{2} t' \underbrace{CC'}_{\Sigma} t \right\}$$

∎

Result 5: Let $X = (X_1, \dots, X_p)' \sim N_p(\mu, \Sigma)$, $\Sigma > 0$.

Then, for every subvector $X_r = (X_{r_1}, \dots, X_{r_q})'$, $r = (r_1, \dots, r_q)'$, $q \leq p$.

we have that $X_r \sim N_q(\mu_r, \Sigma_r)$, $\Sigma_r > 0$, where

- μ_r is the subvector of μ corresponding to r.
- Σ_r is the submatrix of μ corresponding to r.

Dem: Take $B_{q\times p} = (b_{ij})$, $\begin{smallmatrix} i=1,\dots,q \\ j=1,\dots,p \end{smallmatrix}$ with $b_{ij} = \begin{cases} 1 & \text{if } j = r_i \\ 0 & \text{if } j \neq r_i \end{cases} \Rightarrow$

$\Rightarrow \text{rank}(B) = q$. Consider $X_r = BX + b \sim N_q(B\mu + b, B\Sigma B')$,

being $B\mu + b = \mu_r$ and $B\Sigma B' = \Sigma_r$ ■

Idea: Take $B = (b_{ij})$ $\begin{smallmatrix} i=1,\dots,q \\ j=1,\dots,p \end{smallmatrix}$ with $b_{ij} = \begin{cases} 1 & \text{if } j = r_i \\ 0 & \text{if } j \neq r_i \end{cases}$

and apply 4.

Characterization III: A p-dimensional random vector $X = (X_1, \dots, X_p)'$ has a non-singular MND if and only if every linear combination of the form $\alpha' X$, $\alpha \in \mathbb{R}^p \setminus \{0\}$ has a non-degenerate univariate ND.

Dem: $\boxed{\Rightarrow}$ Suppose $X \sim N_p(\mu, \Sigma)$, $\Sigma > 0$ and take $\alpha' = (\alpha_1, \dots, \alpha_p)'$
$\alpha' \in \mathbb{R}^p \setminus \{0\}$

and let $Y = \alpha' X$. $\hookrightarrow \alpha$ $(1\times p)$

Applying result 4 we have that $Y_\alpha \sim N_1(\alpha'\mu, \alpha'\Sigma\alpha)$, $\alpha'\Sigma\alpha > 0$ $q=1$

$\boxed{\Leftarrow}$ Suppose $\forall \alpha \in \mathbb{R}^p \setminus \{0\}$, we have that $Y_\alpha = \alpha' X \sim N_1(\mu_{Y_\alpha}, \sigma^2_{Y_\alpha})$ with $\sigma^2_{Y_\alpha} > 0$. It's neccessary that $\mu_{Y_\alpha} = \alpha\mu$ and $\sigma^2_{Y_\alpha} = \alpha'\Sigma\alpha \Rightarrow$

$\Rightarrow X \sim (\alpha\mu, \alpha'\Sigma\alpha)$ ■

Idea: $\Rightarrow)$ $\alpha' \in \mathbb{R}^p \setminus \{0\}$ (dim $1\times p$) apply linearity

$\Leftarrow)$ $\forall \alpha$ $Y_\alpha = \alpha' X$. Follow.

Let $X = (X_1, \dots, X_q)'$ a r.v. with $X \sim N_p(\mu, \Sigma)$, $\Sigma > 0$.

Consider $\Sigma = CC'$, $Z = C^{-1}(X - \mu) \sim N_p(0, I_p)$. Now,

define the r.v. $U = \frac{Z}{\|Z\|} = \frac{Z}{(Z'Z)^{1/2}}$. Since $\|U\| = \left\| \frac{Z}{\|Z\|} \right\| = \frac{\|Z\|}{\|Z\|} = 1$, we have that U is distributed on the unit sphere in $\mathbb{R}^p$, S_p, for every orthogonal matrix H, $U \stackrel{d}{=} HU$.

Also, we have that $U = \frac{Z}{(Z'Z)^{1/2}}$ and $R = (Z'Z)^{1/2}$ are independent.

No demostration.

3. Extension to the general case, $\Sigma \geq 0$

Let Σ be the covariance matrix of some p-dimensional r.v. Since $\Sigma \geq 0$ and is symmetric, it can be factorized in the form $\Sigma = H\Lambda H'$, with H a $p\times p$ orthogonal matrix ($H'H = HH' = I$)

Λ a $p\times p$ diagonal matrix (eigenvalues)

Let $r = \text{rank}(\Sigma)$, $r \leq p$, and, for convenience,

$\Lambda = \left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right)$, D a $r\times r$ non singular matrix

$H = (H_1 | H_2)$, H_1 a $p\times r$ matrix, H_2 a $p\times(p-r)$ matrix

Then,

$$\Sigma = (H_1 | H_2)\left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right)\begin{pmatrix} H_1' \\ H_2' \end{pmatrix} = H_1 D H_1'$$

C-II Extension: The c.f. is the same that for $\Sigma > 0$.

From the expectral descomposition and, taking into account that $HH' = (H_1\ H_2)\begin{pmatrix} H_1' \\ H_2' \end{pmatrix} = H_1H_1' + H_2H_2'$, we can write the c.f. of X as

$$\phi_X(t) = e^{it'\mu - \frac{1}{2}t'\Sigma t} \underset{HH' = I}{=} e^{it'HH'\mu - \frac{1}{2}t'H_1DH_1't} =$$

$$= e^{it'H_1H_1'\mu - \frac{1}{2}t'H_1DH_1't} \cdot e^{it'H_2H_2'\mu}$$

This expression suggests to consider the linear change of variables:

$$Y := H'X = \begin{pmatrix} H_1' \\ H_2' \end{pmatrix} X = \begin{pmatrix} H_1'X \\ H_2'X \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \Rightarrow X = HY$$

We have then

$$\mu_Y = H'\mu = \begin{pmatrix} H_1'\mu \\ H_2'\mu \end{pmatrix} = \begin{pmatrix} \mu_{Y_1} \\ \mu_{Y_2} \end{pmatrix}$$

$$\Sigma_Y = H'\Sigma H = H'H\Lambda H'H = \Lambda = \left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right) = \begin{pmatrix} \Sigma_{Y_1} & 0 \\ 0 & \Sigma_{Y_2} \end{pmatrix}$$

$\Rightarrow Y_1$ and Y_2 independents, $Y_1 \sim N_r(H_1'\mu, D)$, $Y_2 \sim N_{p-r}(H_2'\mu, 0)$

Therefore, we have that the c.f. of X can be interpreted as

$$\phi_X(t) = E[e^{it'X}] = E[e^{it'HY}] = \phi_Y(H't)$$

With the change variables

$$v = H't = \begin{pmatrix} H_1' \\ H_2' \end{pmatrix} t = \begin{pmatrix} H_1't \\ H_2't \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and we can write

$$\phi_Y(v) = \phi_X(t) = e^{it'H_1H_1'\mu - \frac{1}{2}t'H_1DH_1't}\, e^{it'H_2H_2'\mu}$$

That's, Y_1, Y_2 independents.

Now, since $Y_1 \sim N_r(H_1'\mu, D)$, with $D>0$, we have by C-I that

$$Y_1 \overset{d}{=} D^{1/2} Z + H_1'\mu \quad , \quad \text{with } Z \sim N_r(0, I_r)$$

$\hookrightarrow \Sigma = CC'$, but $D = D^{1/2} D^{1/2}$ because is diagonal.

and for $Z_2 = H_2'\mu \equiv 0Z + H_2'\mu$. Jointly, we can write

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \overset{d}{=} \begin{pmatrix} D^{1/2} Z + H_1'\mu \\ 0Z + H_2'\mu \end{pmatrix} = \begin{pmatrix} D^{1/2} \\ 0 \end{pmatrix} Z + H'\mu$$

So, $X = HY = H\begin{pmatrix} D^{1/2} \\ 0 \end{pmatrix} Z + HH'\mu = H_1 D^{1/2} Z + \mu$. Finally, denoting

$A = H_1 D^{1/2} \Rightarrow \underline{X \overset{d}{=} AZ + \mu}$, with A a $p \times r$ constant matrix of rank r
$Z \sim N_r(0, I_r)$

Taking into account the biunivocal correspondence between distributions and c.f., it must be $\phi_X = \phi_{AZ+\mu}$. That's, $\forall t \in \mathbb{R}^p$:

$$\phi_X(t) = \phi_{AZ+\mu}(t) = E[e^{it'(AZ+\mu)}] =$$
$$= e^{it'\mu} E[e^{it'AZ}] = e^{it'\mu} \phi_Z(A't)$$
$$= e^{it'\mu} e^{-\frac{1}{2}(t'A)(A't)} = e^{it'\mu - \frac{1}{2}t'(AA')t}$$

$\Rightarrow \underline{X \sim N_p(\mu, \Sigma), \ \Sigma = AA' \geq 0}$.

As a conclusion, we can state the following characterization.

C-I: General Case: A p-dimensional r.v. X has MNO

$X \sim N_p(\mu, \Sigma)$, $\Sigma \geq 0$, if and only if

$X \overset{d}{=} AZ + \mu$, with A a $p \times r$ matrix (constant), $\text{rank}(A) = r$
$Z \sim N_r(0, I_r)$ $\qquad \Sigma = AA'$

C-III: General Case: A p-dimensional r.v. $X = (X_1, \dots, X_n)'$ has a MNO if and only if $\alpha' X$, $\alpha \in \mathbb{R}^p$ has an univariate NO.

Dem: $\Rightarrow$) Suppose that $X \sim N_p(\mu, \Sigma)$, $\Sigma \geq 0$ and let $\alpha \in \mathbb{R}^p$, $Y_\alpha = \alpha' X$.

Take $\alpha' = B_{1 \times p}$, $0 = b_{1\times 1}$. Applying linear transformation,

we have $Y_\alpha \sim N_1(\alpha'\mu, \alpha'\Sigma\alpha)$, $\alpha'\Sigma\alpha \geq 0$

$\Leftarrow$) Suppose that $\forall \alpha \in \mathbb{R}^p$, $Y_\alpha = \alpha' X \sim N_1(\mu_{Y_\alpha}, \sigma^2_{Y_\alpha})$

Neccesary $\left.\begin{matrix} \mu_{Y_\alpha} = \alpha\mu \\ \sigma^2_{Y_\alpha} = \alpha'\Sigma\alpha \end{matrix}\right\}$, so $X \sim (\alpha\mu, \alpha'\Sigma\alpha)$

We know that the distribution of X is univocally defined by the univariate distributions set of the form αX, so $\phi_{Y_t}(1) = \phi_X(t)$, with $Y_t = t'X$.

$\phi_{Y_t}(s) = \exp\{ist'X - \frac{1}{2}s t'\Sigma t s\}$, $s = 1 \Rightarrow \phi_{Y_t}(t) = \exp\{it'X - \frac{1}{2}t'\Sigma t\} =$
$= \phi_X(t) \Rightarrow X \sim N_p(\mu, \Sigma)$ $\blacksquare$

Result: Let X_K, $K=1,\dots,m$ be a p-dimensional r.v. with MND
$X_K \sim N_p(\mu_K, \Sigma_K)$, respectivily
being independent. Then, for every set of matrices A_K, $K=1,\dots,m$ (constant) of dimension $q\times p$, we have that

$$Y := \sum_{K=1}^{m} A_K X_K \sim N_q\left(\sum_{K=1}^{m} A_K\mu_K, \sum_{K=1}^{m} A_K \Sigma_K A_K'\right)$$

Dem:

$$\phi_Y(t) = E[e^{it'Y}] = E[e^{it'(\sum_{K=1}^m A_K X_K)}] = E[e^{\sum_{K=1}^m i(A_K't)'X_K}] =$$

$$= \prod_{K=1}^{m} E[e^{i(A_K't)X_K}] = \prod_{K=1}^{m} \phi_{X_K}(A_K't) \underset{\uparrow}{=}$$

$X_K \sim N_p(\mu_K, \Sigma_K)$

$$= \prod_{K=1}^{m} \exp\left\{ i(A_K't)'\mu_K - \frac{1}{2}(A_K't)'\Sigma_K A_K't \right\} =$$

$$= \exp\left\{ it'\sum_{K=1}^{m} A_K\mu_K - \frac{1}{2}t'\sum_{K=1}^{m}(A_K\Sigma_K A_K')t \right\} \Rightarrow Y \sim N_q\left(\sum_{K=1}^{m} A_K\mu_K, \sum_{K=1}^{m} A_K\Sigma_K A_K'\right)$$

Idea: Use the change and the c.f.

4 Sphericall and elliptical distributions

A random vector $X = (X_1 \dots X_p)'$ is said to have spherical distribution if X and HX have the same distribution, $\forall H$ orthogonal matrix $(p\times p)$.

C-I: A r.v. $X = (X_1,\dots,X_p)'$ with a continuous type distribution has an spherical distribution if and only if it's density function can be expressed as $f(x) = g(x'x)$ for $g: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ (scalar func.)

C-II: ... if and only if it's characteristic function is of the form $\phi_X(t) = \xi(t't)$, $\xi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ (scalar func.)

Example 1: $Z \sim N_p(0, \sigma^2 I_p)$

$$f_Z(z) = \frac{1}{(2\pi)^{p/2}|\sigma^2 I_p|^{1/2}} \exp\left\{-\frac{1}{2} z'(\sigma^2 I_p)^{-1} z\right\} =$$

$\to \operatorname{diag}(\sigma^2 \dots \sigma^2) \Rightarrow$

$$= \frac{1}{(2\pi)^{p/2}\sigma^p} \exp\left\{-\frac{1}{2\sigma^2}\underbrace{zz'}_{s}\right\} \quad \Rightarrow |\sigma^2 I_p| = \prod_{i=1}^{p}\sigma^2 = (\sigma^2)^p$$

$$g(s) = \frac{1}{(2\pi)^{p/2}\sigma^p}\exp\left\{-\frac{1}{2\sigma^2}s\right\}, \quad g: \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$$

$g: \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $\xi: \mathbb{R}_0^+ \to \mathbb{R}$, (scalar func.)

<u>Example 2</u>: X r.v. with spherical distribution s.t. $P[X=0]=0$.

Define $U := \frac{X}{\|X\|}$, $X \neq 0$ and any assignment when $X=0$.

Then, U also has spherical distribution.

<u>Dem</u>: Let $X \stackrel{d}{=} HX$, H orthogonal matrix and X has spherical distribution. We can consider $g: \mathbb{R}^p \to S^p \subset \mathbb{R}^p$

$$y \mapsto g(y) := \frac{y}{\|y\|}$$

$$g(0) = (1, 0 \dots 0)$$

that's a measurable function.

$(X \stackrel{d}{=} Z \Rightarrow g(X) \stackrel{d}{=} g(Z))$

$$U = \frac{X}{\|X\|} \stackrel{\downarrow}{=} \frac{HX}{\|HX\|} = \frac{HX}{((HX)'HX)^{1/2}} = \frac{HX}{(X'H'HX)^{1/2}} = \underset{HU}{\underbrace{H\frac{X}{\|X\|}}}, \ \forall H \text{ orthogonal}$$

$\square$

$\Rightarrow U = HU$, $\forall U$ orthogonal

<u>Example 3</u>: More generally, if $X = (X_1, \dots, X_p)'$ is a p-dimensional r.v. with spherical distribution, we have then, given any 'isotropic' Borel-measurable radial transformation $h: \mathbb{R}^p \to \mathbb{R}^p$ of the form

$h(x) = g(\|x\|)x$, with $g: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ Borel-measurable

the transformed r.v. $h(X) = (h_1(X), \dots, h_p(X))'$ will have spherical distribution.

<u>Dem</u>:

$X \stackrel{d}{=} HX$, H orthogonal.

$Hh(X) = Hg(\|X\|)X = Hg(\|HX\|)X = g(\|HX\|)HX = h(HX) = h(X)$ $\square$

A random vector $X = (X_1, \dots, X_p)'$ is said to have <u>elliptical distribution</u> with parameters given by a p-dimensional vector μ and symmetric positive-definite $(p\times p)$-dimensional matrix V, if it can be expressed in the form:

$X = AZ + \mu$ with $V = AA'$, A $p\times p$ non-singular

$X \sim E_p(\mu, V)$ $\quad$ Z r.v. spherical distribution

<u>CI</u>: A r.v. X with continuous-type distribution has an elliptical distribution $\Longleftrightarrow$ $f_X(x) = |V|^{-1/2} g((x-\mu)'V^{-1}(x-\mu))$

or

$$\phi_X(t) = e^{it'\mu}\xi(t'Vt)$$

$g: \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $\xi: \mathbb{R}_0^+ \to \mathbb{R}$, (scalar func.)

Let $X = (X_1, \dots, X_p)'$ be a r.v. and let A be a $(p \times p)$ - dimensional matrix (constant). Consider the r.v. given by the quadratic form

$$X'AX$$

In particular, we consider the case where $X \sim N_p(\mu, \Sigma)$ which leads to non-centered χ^2 distribution.

" We don't have to study formulas, so I go directly with results and ~~demos proves~~ proofs."

Let $X \sim N_p(\mu, \Sigma)$, $(\Sigma > 0)$, and A a constant matrix ($p \times p$ symmetric)

Then,

$$E[X'AX] = tr(A\Sigma) + \mu' A \mu$$

$$Var(X'AX) = 2tr((A\Sigma)^2) + 4\mu' A \Sigma A \mu$$

Dem:

$$E[X'AX] = E[tr(X'AX)] \overset{tr(ab)=tr(ba)}{=} E[tr(AXX')] \overset{\text{linearity}}{=} tr(E[AXX']) =$$

$$= tr(AE[XX']) = tr(A(\Sigma + \mu\mu')) = tr(A\Sigma + A\mu\mu') =$$

$$\Sigma = E[(x-\mu)(x-\mu)'] = E[XX'] - \mu\mu' \Rightarrow E[XX'] = \Sigma + \mu\mu'$$

$$= tr(A\Sigma) + tr(A\mu\mu') = tr(A\Sigma) + tr(\mu' A \mu) = tr(A\Sigma) + \mu' A \mu$$

6. Quadratic forms based on r.v with MND

Let $X = (X_1, \dots, X_p)'$ be a r.v. and consider A a $p \times p$ constant matrix. The quadratic form is given by $X'AX$

If $X \sim N_p(\mu, \Sigma) \Rightarrow$ We have the non-centered χ^2 distribution.

Result: If Y_1 and Y_2 are independent, $Y_1 \sim \chi^2_{N_1}(\delta_1)$ $Y_2 \sim \chi^2_{N_2}(\delta_2)$, then $Y_1 + Y_2 \sim \chi^2_{N_1+N_2}(\delta_1+\delta_2)$

Use c.f. and separe it.

Let $X \sim N_p(\mu, \Sigma)$, $\Sigma > 0$

$$E[X'AX] = tr(A\Sigma) + \mu' A \mu$$

$$Var(X'AX) = 2tr((A\Sigma)^2) + 4\mu' A \Sigma A \mu$$

Use linearity, $tr(ab) = tr(ba)$ and remember $tr(A) = \sum_{i=1}^{p} A_{ii}$. If A is 1×1 then $tr(A) = A$

Result 1: Let $X \sim N_p(\mu, \Sigma)$, $\Sigma > 0$

$$(X-\mu)' \Sigma^{-1} (X-\mu) \sim \chi^2_p$$

$$x' \Sigma^{-1} x \sim \chi^2_p(\delta), \quad \mu \Sigma^{-1} \mu = \delta$$

1) $Y = C^{-1}(X-\mu)$, $Y \sim N_p(0, I_p)$, $\Sigma = CC'$

2) $V = C^{-1}X$, $V \sim N_p(C^{-1}\mu, (C^{-1})\Sigma((C^{-1})'))$

Result 4: Let $X \sim N_p(\mu, \Sigma)$, $\Sigma > 0$, B symmetric $p \times p$ $\Rightarrow$

$\Rightarrow$ ($X'BX \sim \chi^2_K(\delta)$, $K = rank(B)$, $\delta = \mu' B \mu$ $\Leftrightarrow$

$\Leftrightarrow$ $B\Sigma = (B\Sigma)^2$ or $B\Sigma B = B$ because symmetry)

$\Sigma = CC'$ non-singular $Y = C^{-1}X$, $X'BX = (CY)' B (CY)$

Apply result 2:

Result 2: $X'BX \sim \chi^2_k \Leftrightarrow B^2 = B$ (B idempotent) and, in that case $\delta = \mu' B \mu$ $rank(B) = tr(B)$