(3) Let X be a random variable with distribution $N(0,1)$. Let W be a random variable with distribution $U(\{-1,1\})$, that is, $P[W=-1] = P[W=1] = 1/2$, with X and W being independent. Let $Y = WX$. Prove that:

a) Y has distribution $N(0,1)$.

b) X and Y are uncorrelated.

c) X and Y are not independent.

For knowing the distribution of Y, we calculate the distribution function:

$$\varphi_Y(y) = P[Y < y] = \frac{1}{2} P[X < y] + \frac{1}{2} P[-X < y] = \frac{1}{2}\varphi_X(y) + \frac{1}{2}(1-\varphi_X(y)) = \varphi_X(y)$$

where $\varphi_X(x)$ is the distribution function of a standard normal $\Rightarrow Y$ is a standard normal

On the other hand, let's see they are uncorrelated:

$$Cov(X,Y) = E[XY] - E[X]E[Y] = E[X^2W] = E[X^2]E[W] = 0$$

Finally, let's see they are not independents. Suppose X and Y independents. That implies that $Y/X = x$ should be a normal distribution, but, $Y_{X=x}(y) \subset \{-x, x\}$ is not a normal distribution. This is a contradiction $\Rightarrow X$ and Y are not independents.

(2) Let X be a p-dimensional random vector with distribution $N_p(\mu, \Sigma)$, Σ non-singular, and let f_X be a corresponding density function. From the spectral decomposition (into eigenvalues, eigenvectors) of the matrix Σ, describe the geometric locus of the points determined by the equation $f_X(x) = K$, with $K \in \mathbb{R}$ (distinguish cases, according to the value of K).

We start with the first case $K < 0$. In this case, the equality does not hold for any value, so it is the null-set. On the other hand, for $K > 0$

$$f_X(x) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\{(x-\mu)'\Sigma^{-1}(x-\mu)\} = K$$

Using the logarithmic expression:

$$\ln(K(2\pi)^{p/2}|\Sigma|^{1/2}) = (x-\mu)'\Sigma^{-1}(x-\mu)$$

Descomposing the matrix Σ as $H'DH$, we have that:

$$\ln(K(2\pi)^{p/2}|\Sigma|^{1/2}) = (x-\mu)'(H'DH)^{-1}(x-\mu)$$

$$\ln(K(2\pi)^{p/2}|\Sigma|^{1/2}) = (x-\mu)'H'D^{-1}H(x-\mu)$$

So we have the following cases:

- $\ln(K(2\pi)^{p/2}|\Sigma|^{1/2}) \neq 0$. In this case we can divide all by the logarithm.

$$(x-\mu)'H' \frac{D^{-1}}{\ln(K(2\pi)^{p/2}|\Sigma|^{1/2})} H(x-\mu) = 1$$

In this case, the geometric locus of the points is an ellipse with semi-axes determinated by the eigenvectors of Σ^{-1} and, on the other hand, the length of the semi-axes is given by the associated eigenvalues λ_i, as

$$\frac{1}{a_i^2} = \frac{1}{\lambda_i \ln(K(2\pi)^{p/2}|\Sigma|^{1/2})} \Rightarrow a_i = \left(\lambda_i \ln(K(2\pi)^{p/2}|\Sigma|^{1/2})\right)^{1/2}$$

- $\ln(K(2\pi)^{p/2}|\Sigma|^{1/2}) = 0$.

$$0 = (x-\mu)'H'D^{-1}H(x-\mu)$$

We have an ellipse degenerated on one point, which coincides with the means' vector.