(3) Let Y be a random vector defined by

$$Y = \alpha + DX + Z$$

where α is a $p\times 1$ vector, D is a $p\times r$ matrix, X is a random vector with $N(0, \Sigma_X)$

a) obtain the distribution of Y.

b) Obtain the distribution of the join vector $\binom{X}{Y}$

c) Prove that $E[X/Y] = \Sigma_X D' \Sigma_Y^{-1} (Y - \alpha)$

d) Prove that Y/X has the distribution $N_p(\alpha - DX, \sigma^2 I_p)$

Consider the characteristic function of Y and we calculate it from X and Z.

$$\Phi_Y(t) = E[e^{itY}] = E[e^{it(\alpha + DX + Z)}] = E[e^{it(\alpha+DX)} e^{itZ}]$$

Because X and Z are independent, we have that $\alpha + DX + Z$ are independent too.

$$\Phi_Y(t) = E[e^{it'(\alpha+DX+Z)}] = E[e^{it'(\alpha+DX)}]E[e^{itZ}] =$$

$$= e^{it'\alpha} E[e^{it'DX}]E[e^{it'Z}] = e^{it'\alpha}\Phi_X(D't)\Phi_Z(t)$$

We know that:

$$\Phi_X(D't) = e^{-\frac{1}{2}t'D\Sigma_X D't}$$

$$\Phi_Z(t) = e^{-1/2 t'\sigma^2 I_p t}$$

So, by unicity of the characteristic function:

$$\Phi_Y(t) = e^{it'\alpha - 1/2 t'(D\Sigma_X D' + \sigma^2 I_p)t'} \Rightarrow Y \sim N(\alpha, D\Sigma_X D' + \sigma^2 I_p)$$

Now, we calculate the distribution of the join vector:

$$\binom{Y}{X} = \binom{\alpha + DX + Z}{X} = \binom{\alpha}{0_r} + \binom{D}{I_r}X + \binom{I_p}{0_r}Z$$

We know that:

$$\binom{\alpha}{0_r} + \binom{D}{I_r}X \sim N_r\left(\binom{\alpha}{0_r}, \binom{D}{I_r}\Sigma_X\binom{D}{I_r}'\right)$$

$$\binom{I_p}{0_r}Z \sim N_p\left(0, \binom{I_p}{0_r}\sigma^2 I_p \binom{I_p}{0_r}'\right)$$

By independece of both distributions, we have that:

$$\binom{Y}{X} \sim N_{p+r}\left(\binom{\alpha}{0_r}, \binom{D}{I_r}\Sigma_X\binom{D}{I_r}' + \binom{I_p}{0_r}\sigma^2 I_p\binom{I_p}{0_r}'\right)$$

For (c) we can just apply a theory result and prove that the distribution is not singular, that's means,

$$\det\left(\binom{D}{I_r}\Sigma_X\binom{D}{I_r}' + \binom{I_p}{0_r}\sigma^2 I_p\binom{I_p}{0_r}'\right) > 0$$

$$\det\begin{pmatrix} D\Sigma_X D' + \sigma^2 I_p & D\Sigma_X \\ \Sigma_X D' & \Sigma_X \end{pmatrix} > 0$$

If a determinant can be divided in blocks of the form $|A| = \begin{vmatrix} B & C \\ D & E \end{vmatrix}$, with B and E squared, we can say that $|A| = |E||B - CE^{-1}D|$

$$\det(\mathrm{cov}\binom{X}{Y}) = \det(\Sigma_X)\det(D\Sigma_X D' + \sigma^2 I_p - D\Sigma_X D') = \det(\Sigma_X)\det(\sigma^2 I_p) > 0$$

Applying the theory result:

$$E[X/Y] = \Sigma_X D'(D\Sigma_X D' + \sigma^2 I_p)^{-1}(Y - \alpha) = \Sigma_X D'\Sigma_Y^{-1}(Y - \alpha)$$

Finally, applying the same result:

$$X/Y \sim N_p(\alpha + D\Sigma_X\Sigma_X(X - 0), D\Sigma_X D' + \sigma^2 I_p - D\Sigma_X\Sigma_X^{-1}\Sigma_X D') \equiv N_p(\alpha + DX, \sigma^2 I_p)$$

(2) In relation to the previous exercise, prove that the results obtained are still valid in the case where Σ_X is singular, with the exception that in the part (d) we will have that the distribution of Y/X is $N_p(\alpha + D\Sigma_X \Sigma_X^- X, \sigma^2 I_p)$, with Σ_X^{-1} being a generalized inverse of the matrix Σ_X (that's some matrix s.t. $\Sigma_X \Sigma_X^- \Sigma_X = \Sigma_X$).

For (1) and (2) I haven't use the condition that Σ_X is non-singular, so they are still correct. For (3) and (4) we have to re-do the demostration:

Consider the generalize result for (3):

$$X/Y \sim N_r\left(\Sigma_X D'(D\Sigma_X D' + \sigma^2 I_r)^- (Y-\alpha),\ \Sigma_X - \Sigma_X D'(D\Sigma_X D' + \sigma^2 I_p)^- D\Sigma_X\right)$$

So we can calculate:

$$E[X|Y] = \Sigma_X D'(D\Sigma_X D' + \sigma^2 I_p)^-(Y-\alpha) = \Sigma_X D' \Sigma_Y^- (Y-\alpha)$$

Finally, we just have to demostrate that Σ_Y is positive-defined ($\Sigma_Y^- = \Sigma_Y^{-1}$)

$$x'\Sigma_Y x = x'(D\Sigma_X D' + \sigma^2 I_p)x = x' D\Sigma_X D' x + x'\sigma^2 I_p x = x' D\Sigma_X D' x + x' + \sigma^2 \|x\|$$

Because Σ_X is non-negative-defined, we have that $x'D\Sigma_X D' x \geq 0$, so, we have $(x'D\Sigma_X D' x + x' + \sigma^2 \|x\|) > 0 \Rightarrow \Sigma_Y$ is positive-defined.

$$E[X|Y] = \Sigma_X D' \Sigma_Y^- (Y-\alpha)$$

For (4), we use the inverse of Σ_X

$$E[Y|X] = \alpha + D\Sigma_X \Sigma_X^- X$$

where,

$$\Sigma_{Y|X} = \Sigma_{(11)} - \Sigma_{(22)}\Sigma_{(22)} - \Sigma_{(21)} = D\Sigma_X D' - D\Sigma_X D' + \sigma^2 I_p = \sigma^2 I_p$$

(3) Let $X \sim N_p(\mu, \Sigma)$ and let $\alpha \in \mathbb{R}^p$. Prove that

$$E[(\alpha'(X-\mu))^k] = \begin{cases} \dfrac{(2m)!}{2^m m!}(\alpha'\Sigma\alpha)^m, & \text{if } k = 2m \text{ (even)} \\ 0, & \text{if } k = 2m-1 \text{ (odd)} \end{cases}$$

Consider $Y = \alpha'(X-\mu)$ and we will study the distribution:

$$E[Y] = E[\alpha'(X-\mu)] = \alpha'(E[X]-\mu) = 0$$

$$\operatorname{Var}(Y) = \operatorname{Var}(\alpha'(X-\mu)) = E[(\alpha'(X-\mu))'(\alpha'(X-\mu))] = E[(\alpha'(X-\mu))^2] - E[\alpha'(X-\mu)]^2 =$$

$$= E[(\alpha'(X-\mu))^2] = E[\alpha'(X-\mu)(X-\mu)'\alpha] = \alpha'\Sigma_X\alpha$$

Finally,

$$E[\alpha'(X-\mu)]^k = \begin{cases} (k-1)!!\,(\alpha'\Sigma\alpha)^{k/2} & \text{if } k=2m \\ 0 & \text{if } k = 2m-1 \end{cases} \Rightarrow E[\alpha'(X-\mu)]^k = \begin{cases} \dfrac{(2m)!}{2^m m!}(\alpha'\Sigma\alpha), & k=2m \\ 0, & k=2m-1 \end{cases}$$

(4) Let $X \sim N_p(0, \Sigma)$ with $\operatorname{rank}(\Sigma) = k$. Consider the spectral descomposition of Σ given by $\Sigma = H\Lambda H'$, with the partitioning $H = (H_1 | H_2)$ $\Lambda = \left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right)$, where $D = \operatorname{diag}(\lambda_1, \dots, \lambda_k)$, with $\lambda_1, \dots, \lambda_k > 0$.

Prove that $\Sigma^+ = H_1 D^{-1} H_1'$ is the Moore-Penrose inverse matrix of Σ, that is, it satisfies:

a) $\Sigma\Sigma^+\Sigma = \Sigma$ b) $\Sigma^+\Sigma\Sigma^+ = \Sigma^+$ c) $(\Sigma^+\Sigma)' = \Sigma^+\Sigma$ d) $(\Sigma\Sigma^+)' = \Sigma\Sigma^+$

Consider Σ: $\Sigma = H\Lambda H' = (H_1 | H_2)\left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right)\begin{pmatrix} H_1' \\ H_2' \end{pmatrix} = H_1 D H_1'$

calculate $\Sigma\Sigma^+\Sigma$:

$$\Sigma\Sigma^+\Sigma = H_1 D H_1' H_1 D^{-1} H_1' H_1 D H_1' = \Im H_1 D H_1' = \Sigma$$

Where we are using that H_1 is an ortogonal matrix. The other 3 properties:

$$\Sigma^+\Sigma\Sigma^+ = H_1 D^{-1} H_1' H_1 D H_1' H_1 D^{-1} H_1' = H_1 D^{-1} H_1' = \Sigma^+$$

$$(\Sigma^+\Sigma)' = I' = I = \Sigma^+\Sigma \qquad (\Sigma\Sigma^+)' = I' = I = \Sigma\Sigma^+$$

5. Let $\Sigma = (\sigma_{ij})$ be a 3×3 symmetric matrix such that
$\sigma_{11} = \sigma_{22} = \sigma_{33} = 1, \quad \sigma_{12} = 0$
Prove that, at least for $(\sigma_{13} + \sigma_{23}) > \frac{3}{2}$, Σ is not a positive define matrix.

$$\det\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ x & y & 1 \end{pmatrix} = 1 - x^2 - y^2 > 0 \Rightarrow x^2 < 1 - y^2$$

By the other hand, applying that $x+y > \frac{3}{2}$ and taking squares:

$$x^2 > \frac{9}{4} + y^2 - 3y$$

Finally, joinning the two conditions, we solve the inecuation:

$$2y^2 - 3y + \frac{5}{4} < 0 \Rightarrow 2y^2 - 3y + \frac{5}{4} = 0 \text{ (Does not have sol. in } \mathbb{R})$$

So, the matrix is not positive - defined.

6. Let $Z \sim N_p(0, I_p)$. Let $Y_1 = C_1 Z$ and $Y_2 = C_2 Z$, with C_i a $K_i \times p$ matrix $K_i \leq p$ $(i = 1, 2)$. Find a necessary and sufficient condition for the independence of Y_1 and Y_2.

Consider $\Phi_{Y_1}(t_1) = e^{-1/2 t_1' C_1 C_1' t_1}$ and $\Phi_{Y_2}(t_2) = e^{-1/2 t_2' C_2 C_2' t_2}$ and we study if the characteristic function can be or not descomponed in the product of both.

$$Y := \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} Z \sim N\left(0, \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}(C_1' C_2')\right), \quad \Phi_Y(t) = e^{-1/2 t' \Gamma t}$$

$$\Gamma = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}(C_1' C_2')$$

We have to probe that $\Phi_Y(t) = \Phi_{Y_1}(t_1)\Phi_{Y_2}(t_2)$:

$$\Phi_{Y_1}(t_1)\Phi_{Y_2}(t_2) = e^{-1/2 t_1' C_1 C_1' t_1 + t_2' C_2 C_2' t_2}$$

$$(t_1\ t_2)\begin{pmatrix} C_1C_1' & C_1C_2' \\ C_2C_1' & C_2C_2' \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = t_1 C_1 C_1' t_1' + t_2 C_2 C_1' t_1' + t_1' C_1 C_2' t_2' + t_2 C_2 C_2' t_2'$$

For the conditions to be equal:

$$t_1 C_1 C_1' t_1' + t_2 C_2 C_1' t_1' + t_1' C_1 C_2' t_2' + t_2 C_2 C_2' t_2' = t_1' C_1 C_1' t_1 + t_2' C_2 C_2' t_2$$

We must have $C_2 C_1' = C_1 C_2'' = 0$.

7. Let $Y \sim N_3(\mu, \Sigma)$, where, $\mu = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 6 & 1 & -2 \\ 1 & 13 & 4 \\ -2 & 4 & 4 \end{pmatrix}$.

a) Find the distribution of $Z = 2Y_1 - Y_2 + 3Y_3$.
b) Find the joint distribution of $Z_1 = Y_1 + Y_2 + Y_3$ and $Z_2 = Y_1 - Y_2 + 2Y_3$
c) Find the distribution of Y_2.
d) Find the distribution of Y_1 and Y_3.
e) Find the joint distribution of Y_1, Y_3 and $\frac{1}{2}(Y_1 + Y_2)$
f) Find the vector Z s.t. $Z = (T')^{-1}(Y - \mu) \sim N_3(0, I)$, where T is the matrix corresponding to Cholesky's factorization $\Sigma = T'T$.
g) Find the vector Z. s.t. $Z = \Sigma^{-1/2}(Y - \mu) \sim N_3(0, I)$, with $\Sigma^{-1/2}$ being the inverse of the matrix corresponding to the square-root factorization $\Sigma = \Sigma^{1/2}\Sigma^{1/2}$.

Start calculating the determinant of Σ for knowing the type of distribution

$$\det(\Sigma) = 144 > 0 \Rightarrow \text{Non-singular distribution.}$$

Now, we have to find the distribution of $Z = 2Z_1 - Z_2 + 3Z_3$:

$$Z = (2 \ -1 \ 3)\begin{pmatrix} Y_1 \\ Y_2 \\ Z_3 \end{pmatrix}$$

So we can find the distribution by:

$$Z \sim N\left((2 \ -1 \ 3)\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \ (2 \ -1 \ 3)\Sigma\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}\right) \equiv N(17, 21)$$

Now, we find the join distribution of $Z_1 = Z_1 + Z_2 + Z_3$ and $Z_2 = Y_1 - Y_2 + 2Z_3$

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} Y_1 \\ Z_2 \\ Z_3 \end{pmatrix}$$

So, I can find the distribution by:

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \ \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}\Sigma\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix}\right) \equiv N\left(\begin{pmatrix} 8 \\ 10 \end{pmatrix}, \begin{pmatrix} 29 & -1 \\ -1 & 9 \end{pmatrix}\right)$$

Find the distribution of Z_2:

$$Y_2 = (0 \ 1 \ 0)\begin{pmatrix} Z_1 \\ Y_2 \\ Z_3 \end{pmatrix} \Rightarrow Z_2 \sim N(1, 13)$$

And also the distribution of Z_1 e Z_3:

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Z_3 \end{pmatrix} \Rightarrow \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\Sigma\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

Find they join distribution of $Y_1, Y_3, \frac{1}{2}(Z_1+Z_2)$:

$$\begin{pmatrix} Y_1 \\ Y_2 \\ 1/2(Z_1+Z_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Z_3 \end{pmatrix} \Rightarrow \begin{pmatrix} Z_1 \\ Y_2 \\ 1/2(Z_1+Z_2) \end{pmatrix} \sim N\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}\Sigma\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}\right)$$

$$\begin{pmatrix} Y_1 \\ Y_3 \\ 1/2(Z_1+Z_2) \end{pmatrix} \sim N\left(\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 & -2 & 7/2 \\ -2 & 4 & 1 \\ 7/2 & 1 & 23/4 \end{pmatrix}\right)$$

Now, we have to find the vector Z s.t. $Z = (T')^{-1}(Z - \mu) \sim N_3(0, I)$, with T the correspondent matrix of Cholesky's factorization $\Sigma = T'T$.

$$T = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} \qquad \Sigma = T'T \quad \Rightarrow \quad T = \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 1/\sqrt{6} & \sqrt{77/6} & 0 \\ -2/\sqrt{66} & 26/\sqrt{462} & 740/231 \end{pmatrix}$$

Finally, we have that

$$Z = \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 1/\sqrt{6} & \sqrt{77/6} & 0 \\ -2/\sqrt{66} & 26/\sqrt{462} & 740/231 \end{pmatrix}^{-1}\begin{pmatrix} Z_1 - 3 \\ Y_2 - 1 \\ Y_3 - 4 \end{pmatrix}$$

8 Let $Z \sim N_3(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \ \Sigma = \begin{pmatrix} 4 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Which of the following random variables and vectors are independent?

a) Y_1 and Y_2 b) Y_1 and Y_3 c) Y_2 and Y_3 d) (Y_1, Y_2) and Z_3 f) (Y_1, Y_3) and Z_2

Let's calculate $Z = (Y_1 \ Y_2)'$

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$$

$$Z \sim N_2\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \equiv N\left(\begin{pmatrix} 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 & -3 \\ -3 & 6 \end{pmatrix}\right)$$

They are not uncorrelated $\Rightarrow$ They are no independents

Let's calculate now $Z = (Z_1, Y_3)$:

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Z_3 \end{pmatrix}, \ Z \sim N\left(\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}\right)$$

Diagonal matriz $\Rightarrow$ They are independents

Now, we calculate $Z = (Z_2, Z_3)$

$$Z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \\ Y_3 \end{pmatrix}, \ Z \sim N\left(\begin{pmatrix} -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}\right)$$

Diagonal matrix $\Rightarrow$ They are independents.

We calculate (Z_1, Z_2) and Z_3:

$$Z = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\begin{pmatrix} Z_1 \\ Z_2 \\ Y_3 \end{pmatrix}, \ Z \sim N\left(\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \left(\begin{array}{cc|c} 4 & -3 & 0 \\ -3 & 6 & 0 \\ \hline 0 & 0 & 5 \end{array}\right)\right)$$

Diagonal matrix $\Rightarrow$ They are independents.

Finally, we have that:

$$z = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 1 & 0 \end{array}\right)\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \quad z \sim N\left(\begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}, \left(\begin{array}{cc|c} 4 & -3 & 0 \\ 0 & 0 & 5 \\ \hline -3 & 6 & 0 \end{array}\right)\right)$$

They are uncorrelated $\Rightarrow$ They are not independents.

(9) Assume that Y and X are subvectors of respective dimensions 2×1 and 3×1, with joint μ and Σ correspondingly partitioned as:

$$\mu = \left(\begin{array}{c} 3 \\ -2 \\ \hline 4 \\ -3 \\ 5 \end{array}\right), \quad \Sigma = \left(\begin{array}{cc|ccc} 14 & -8 & 15 & 0 & 3 \\ -8 & 18 & 8 & 6 & -2 \\ \hline 15 & 8 & 50 & 8 & 5 \\ 0 & 6 & 8 & 4 & 0 \\ 3 & -2 & 5 & 0 & 1 \end{array}\right)$$

Assume $\begin{pmatrix} Y \\ X \end{pmatrix} \sim N_5(\mu, \Sigma)$.

Find $E[Y/X]$ and $cov(Y/X)$.

We can use directly the result from theory and we get that:

$$E[Y/X] = \begin{pmatrix} 3X_3 - 12 \\ \frac{2}{3}X_1 + \frac{X^2}{6} - \frac{16}{3}X^3 - \frac{185}{6} \end{pmatrix} \qquad cov(Y/X) = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

(10) Assume that the random variables X and Y have joint distribution function

$$F(x,y) = \Phi(x)\Phi(y)[1 + \alpha(1-\Phi(x))(1-\Phi(y))]$$

with $|\alpha| \le 1$, where $\Phi(\cdot)$ denotes the standard normal distribution function. Prove that the marginal distributions corresponding to X and Y are standard normal.

We can take limits to infinity in x and y:

$$F(x) = \lim_{y\to\infty} F(x,y) = \Phi(x), \quad F(y) = \lim_{x\to\infty} F(x,y) = \Phi(y)$$

Important: $\lim_{x\to\infty} \Phi(x) = 1$, because it is a distribution function.

(11) Let $X_1, X_2, \dots$ be independent random vectors such that $X_i \sim N_m(\mu, \Sigma)$, $i = 1, 2, \dots$ and let $S_N = \sum_{i=1}^{N} X_i$. For $N_1 < N_2$:

a) Find the distribution of $(S'_{N_1}, S'_{N_2})'$

b) Find the conditional distribution of S'_{N_1} given S'_{N_2}.

For calculating the distribution, we use the characteristic function:

$$\Phi_Z(t) = E[e^{itZ}] = E[e^{it_1 S_{N_1} + it_2 S_{N_2}}] = E[e^{i(t_1+t_2)\sum_{i=1}^{N} X_i + it_2 \sum_{i=N_1+1}^{N_2} X_i}]$$

Applying that they are independents:

$$\Phi_Z(t) = E[e^{i(t_1+t_2)\sum_{i=1}^{N_1} X_i}]E[e^{it_2\sum_{i=N_1+1}^{N_2} X_i}] = \Phi_{\sum_{i=1}^{N} X_i}(t_1+t_2)\,\Phi_{\sum_{i=N_1+1}^{N_2} X_i}(t_2)$$

Using the reproductivity from normal, we have that:

$$\sum_{i=1}^{N_1} X_i \sim N(N_1\mu, N_1\Sigma), \quad \sum_{i=N_1+1}^{N_2} X_i \sim N((N_2-N_1)\mu, (N_2-N_1)\Sigma)$$

So, we affirm:

$$\Phi_{\sum_{i=1}^{N_1} X_i}(t_1) = e^{i(t_1+t_2)'N_1\mu - 1/2(t_1+t_2)'N_1\Sigma(t_1+t_2)}, \quad \Phi_{\sum_{i=N_1+1}^{N_2} X_i}(t_2) = e^{i(t_2)'(N_2-N_1)\mu - \frac{1}{2}(t_2)'(N_2-N_1)\Sigma(t_2)}$$

$$\Phi_Z(t) = e^{i(t_1', t_2')\begin{pmatrix} N_1\mu \\ N_2\mu \end{pmatrix} - \frac{1}{2}(t_1', t_2')\begin{pmatrix} N_1\Sigma & N_1\Sigma \\ N_1\Sigma & N_2\Sigma \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}} \Rightarrow Y \sim N_{2m}\left(\begin{pmatrix} N_1\mu \\ N_2\mu \end{pmatrix}, \begin{pmatrix} N_1\Sigma & N_1\Sigma \\ N_1\Sigma & N_2\Sigma \end{pmatrix}\right)$$

Finally, (from theory's results) knowing the distribution of Y, we can know the conditional dis.:

$$N_m(N_1\mu + N_1 N_2^{-1}(S'_{N_2} - N_2\mu), N_1\Sigma - N_1^2 N_2\Sigma)$$

(12) Assume that $X \sim N_3(0, \Sigma)$ with $\Sigma = \begin{pmatrix} 1 & p & 0 \\ 0 & 1 & p \\ p & p & 1 \end{pmatrix}$ Is there any value of p for which the variables $X_1 + X_2 + X_3$ and $X_1 - X_2 - X_3$ are independent?

Consider the vector $Z = \begin{pmatrix} X_1 + X_2 + X_3 \\ X_1 - X_2 - X_3 \end{pmatrix}$ and, applying theory's results, we have just to demostrate that components of Z are uncorrelated: they have to be uncorrelated

$$\Sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}\Sigma\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 4p+3 & -2p-1 \\ -2p-1 & 3 \end{pmatrix} \overset{?}{\Rightarrow} -2p-1 = 0 \Rightarrow p = -\frac{1}{2}$$

So, for $p = -\frac{1}{2}$, they are independents.