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Faculty of Applied Physics and Mathematics
Exercises II
Differential Equations II

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1. **Show that the following function**

$$G(t,s) = \begin{cases} ln\frac{2}{s}\ dla & 1 \leq t \leq s \leq 2 \\ ln\frac{2}{t}\ dla & 1 \leq s \leq t \leq 2 \end{cases}$$

is the Green function for the problem $-ty'' - y' = 0, y'(1) = y(2) = 0$**.**

Let L be a linear differential operator of the form

$$L = \frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}\right] + q(delta_0)$$

In our case,

$$L = \left[-t\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t}\right]$$

And let $\vec{D}$ be the vector-valued boundary conditions operator such that

$$\vec{D_u} = \begin{bmatrix} \alpha_1 u'(0) + \beta_1 u(0) \\ \alpha_2 u'(l) + \beta_2 u(l) \end{bmatrix}$$

In our case, $\alpha = y(1)$ and $\beta = y(2) = 0$.

Let now $f(x)$ be a continuous function in $[0,l]$ and $L_u = f$, $\vec{D_u} = \vec{0}$ is the solution for $f(x) = 0$, $\forall x\ s.t.\ u(x) = 0$.

There is one and only one solution $u(x)$ which satisfies $L_u = f$, $\vec{D_u} = \vec{0}$ and it is given by

$$h(x) = \int_0^l f(s)G(x,s)ds$$

where $G(x,s)$ is a Green function satisfying the following conditions:

- $G(x,s)$ is continuous in x and s.
- For $x \neq s$, $L(G(x,s)) = 0$.
- For $s \neq 0$, $\vec{D}G(x,s) = \vec{0}$.
- $G'(s_{0^+} + s) - G'(s_{0^-} - s) = \frac{1}{f(s_0)}$.
- $G(x,s) = G(s,x)$

In this case we have an homogeneous linear second order differential equation. The general solution for this type of equations can be written as

$$a(x)y'' + b(x)y' + c(x)y = 0$$

So according to the previous definitions, our $u(x)$ would be of the form:

$$u(x) = \int_0^l G(t,s)f(s)ds + r(t)$$

Then, we only have to check if the given Green's function satisfies the necessary conditions, that is:

- $G(t,s)$ is continuous in t and s. ✓
- For $t \neq s$, $L(G(t,s)) = 0$, so

$$L(G(t,s)) = \left[-t\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} \right] G(t,s)$$

$$\frac{\partial}{\partial t}G(t,s) = \left\{ \begin{array}{ll} 0 & 1 \leq t \leq s \leq 2 \\ \frac{-1}{t} & 1 \leq s \leq t \leq 2 \end{array} \right. \quad \frac{\partial^2}{\partial t^2}G(t,s) = \left\{ \begin{array}{ll} 0 & 1 \leq t \leq s \leq 2 \\ \frac{1}{t^2} & 1 \leq s \leq t \leq 2 \end{array} \right.$$

$$L(G(t,s)) = \left\{ \begin{array}{ll} 0 & 1 \leq t \leq s \leq 2 \\ 0 & 1 \leq s \leq t \leq 2 \end{array} \right.$$

  So second condition ✓.
- For $s \neq 0$, $\vec{D}G(t,s) = \vec{0}$ ✓

$$G'(0,s) = 0 \;\; G'(2,s) = 0 \;\; G(0,s) + G'(0,s) = 0$$

- $G(t,s) = G(s,t)$ ✓
- $G'(s_{0^+} + s) - G'(s_{0^-} - s) = \frac{1}{f(s)}$ ✓

So, the function is the Green's function for the problem.

2. **Determinate Green's function for the following problems:**

   (a) $ty''(t) + y'(t) = t$, $y(1) = y(e) = 0$
   First we have to solve the homogeneous equation associated to our problem, $ty''(t) + y'(t) = 0$. It's a second order Euler homogeneous DE.
   We can make the change $y = t^r$

$$t(t^r)'' + (t^r)' = 0 \implies r^2 r^{r-1} - rt^{r-1} + rt^{r-1} = r^2 t^{r-1} = 0 \implies r = 0$$
$$y(t) = c_1 t^0 + c_2 ln(t)t^0 \implies \underline{y(t) = c_1 + c_2 ln(t)}$$

   Now we set the boundary conditions to that solution, so

$$y(1) = 0 \implies c_1 = 0 \implies \underline{y_1 = ln(t)}$$
$$y(e) = 0 \implies c_1 + c_2 = 0 \implies \underline{y_2 = 1 - ln(t)}$$

   Now, we calculate the Wronskian

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} ln(t) & 1 - ln(t) \\ \frac{1}{t} & -\frac{1}{t} \end{vmatrix} = -\frac{ln(t)}{t} - \frac{1 - ln(t)}{t} = -\frac{1}{t}$$

Then, Green's function is

$$G(t,s) = \begin{cases} \frac{y_1(s)y_2(t)}{W(y_1,y_2)(s)} & a \le s \le t \le b \\ \frac{y_1(t)y_2(s)}{W(y_1,y_2)(s)} & a \le t \le s \le b \end{cases} \implies \boxed{G(t,s) = \begin{cases} -sln(s)(1 - ln(t)) & 1 \le s \le t \le e \\ -sln(t)(1 - ln(s)) & 1 \le t \le s \le e \end{cases}}$$

(b) $y'' - y = 0,\ y(0) = y'(0),\ y(a) + \lambda y(a) = 0,\ a > 0$

We have again a second order Euler homogeneous ODE, so let's take $y = r$

$$r^2 - r^0 = 0 \implies r^2 = 1 \implies r = \pm 1 \implies \underline{y(t) = c_1 e^t + c_2 e^{-t}}$$

Now, we set the boundary conditions to that solution:

$$y(0) = y'(0),\ y'(t) = c_1 e^t - c_2 e^{-t} \implies c_1 + c_2 = c_1 - c_2 = 0 \implies c_2 = 0 \implies \underline{y_1 = e^t}$$

$$y(a) + \lambda y(a) = 0, a > 0 \implies c_1 e^a + c_2 e^{-a} + \lambda c_1 e^a + \lambda c_2 e^{-a} = 0 \implies (1+\lambda) c_1 e^a + (1-\lambda) c_2 e^{-a} = 0 \implies$$

$$\implies c_1 e^a + \frac{c_2}{e^a} = 0 \implies c_1 e^{2a} = -c_2 \implies \underline{y_2} = e^t + e^{2a} e^{-t} = \underline{e^t + e^{2a-t}}$$

Now, we calculate the Wronskian

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & e^t + e^{2a-t} \\ e^t & e^t - e^{2a-t} \end{vmatrix} = -e^{2t} - e^{2a} - (e^{2t} + e^{2a}) = -2e^{2a}$$

Then, Green's function is

$$G(t,s) = \begin{cases} \frac{y_1(s)y_2(t)}{W(y_1,y_2)(s)} & a \le s \le t \le b \\ \frac{y_1(t)y_2(s)}{W(y_1,y_2)(s)} & a \le t \le s \le b \end{cases} \implies \boxed{G(t,s) = \begin{cases} -\frac{e^t + e^{2a-t}}{2e^s} & 0 \le s \le t \le a \\ -\frac{e^t(e^s + e^{2a})}{2e^{3s}} & 0 \le t \le s \le a \end{cases}}$$

(c) $y'' + 4y = cos(t),\ y(0) = 0, y'(\pi) = 0$

Let's solve the homogeneous equation $y'' + 4y = 0$. We can make the change $y = e^{\alpha t}$:

$$(e^{\alpha t})'' + 4e^{\alpha t} = 0 \implies \alpha^2 e^{\alpha t} + 4e^{\alpha t} = 0 \implies e^{\alpha t}(\alpha^2 + 4) = 0 \implies \alpha = \pm 2i$$

We have 2 complex roots, so the general solution has the form $y(t) = e^{\alpha t}(c_1 cos(\beta t) + c_2 sin(\beta t))$, so we have

$$y(t) = c_1 cos(2t) + c_2 sin(2t)$$

Now we set the boundary conditions to obtain y_1 and y_2:

$$y(0) = 0 \implies c_1 = 0 \implies \underline{y_1 = sin(2t)}$$

$$y'(\pi) = 0,\ y'(t) = -2c_1 sin(2t) + 2c_2 cos(2t) \implies 0 = y'(\pi) = 2c_2 \implies \underline{y_2 = 2cos(2t)}$$

Now, we calculate the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} sin(2t) & 2cos(2t) \\ 2cos(2t) & -4sin(2t) \end{vmatrix} = -4sin^2(2t) - 4cos^2(2t) = -4$$

Then, Green's function is

$$G(t,s) = \begin{cases} \frac{y_1(s)y_2(t)}{W(y_1,y_2)(s)} & a \le s \le t \le b \\ \frac{y_1(t)y_2(s)}{W(y_1,y_2)(s)} & a \le t \le s \le b \end{cases} \implies \boxed{G(t,s) = \begin{cases} \frac{2sin(2s)cos(2t)}{-4} & 0 \le s \le t \le \pi \\ \frac{2sin(2t)cos(2s)}{-4} & 0 \le t \le s \le \pi \end{cases}}$$

(d) $y'' + k^2 y = 0$, $y(0) = y'(1) = 0$, $k \neq 0$

We have a lineal homogeneous ODE of second order. We can make the change $y = e^{\alpha t}$:

$$(e^{\alpha t})'' + k^2(e^{\alpha t}) = 0 \implies \alpha^2 e^{\alpha t} + k^2 e^{\alpha t} = 0 \implies e^{\alpha t}(\alpha^2 + k^2) = 0 \implies \alpha = \pm ik$$

As in the previous exercise, the general solution would be:

$$y(t) = c_1 cos(kt) + c_2 sin(kt)$$

Now we set the boundary conditions:

$$y(0) = 0 \implies c_1 = 0 \implies \underline{y_1 = sin(kt)}$$

$$y'(1) = 0,\ y'(t) = -kc_1 sin(kt) + kc_2 cos(kt) \implies 0 = y'(1) = -kc_1 sin(k) + kc_2 cos(k) \implies$$
$$\implies k(c_2 cos(k) - c_1 sin(k)) = 0 \implies \underline{y_2} = cos(k)cos(kt) + sin(k)sin(kt) = \underline{cos(kt-k)}$$

Now, we calculate the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} sin(kt) & cos(kt-k) \\ kcos(kt) & -ksin(kt-k) \end{vmatrix} =$$

$$= -ksin(kt)sin(kt-k) - kcos(kt)cos(kt-k) = -kcos(k)$$

Then, Green's function is

$$G(t,s) = \begin{cases} \frac{y_1(s)y_2(t)}{W(y_1,y_2)(s)} & a \leq s \leq t \leq b \\ \frac{y_1(t)y_2(s)}{W(y_1,y_2)(s)} & a \leq t \leq s \leq b \end{cases} \implies \boxed{G(t,s) = \begin{cases} -\frac{sin(ks)cos(kt-k)}{kcos(k)} & 0 \leq s \leq t \leq 1 \\ -\frac{sin(kt)cos(ks-k)}{kcos(k)} & 0 \leq t \leq s \leq 1 \end{cases}}$$

3. **Find (using the Green function) the solution to the boundary value problem for the following DE:**

(a) $y'' = sin(x)$, $y(0) = y(1) = 0$

First, we have to find the solution to the associated homogeneous equation $y'' = 0$.
If $f'(x) = g(x) \implies f(x) = \int g(x)dx \implies y' = \int 0dx \implies y' = c_1$. If we repeat it again, we will obtain that

$$y = \int c_1 dt \implies y(t) = c_1 x + c_2$$

Imposing boundary conditions:

$$y(0) = 0 \implies c_2 = 0 \implies \underline{y_1 = x}\ \ y(1) = 0 \implies c_1 = -c_2 \implies \underline{y_2 = x - 1}$$

Now we calculate the Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} x & x-1 \\ 1 & 1 \end{vmatrix} = 1$$

Consequently:

$$G(x,s) = \begin{cases} s(x-1) & 0 \leq s \leq x \leq 1 \\ x(s-1) & 0 \leq x \leq s \leq 1 \end{cases}$$

Now in order to find the solution $g(x)$, we have to compute the following integral:

$$y(x) = \int_0^1 G(x,s)f(s)ds, \quad f(s) = sin(s)$$

So, we have that:

$$\begin{aligned} y(x) &= \int_0^x s(x-1)sin(s)ds + \int_x^1 x(s-1)sin(s)ds = \\ &= (x-1)int_0^x ssin(s)ds + x\left[\int_x^1 ssin(s)ds - \int_x^1 sin(s)ds\right] = ... \end{aligned} \tag{1}$$

Solving $\int xsin(x)dx$:

$$\begin{aligned} \int xsin(x)dx = \begin{bmatrix} u = x & du = dx \\ dv = sin(x) & v = cos(x) \end{bmatrix} &= -xcos(x) - \int -cos(x)dx = \\ &= -xcos(x) + sin(x) + C \end{aligned} \tag{2}$$

Substituting 2 int 1:

$$\begin{aligned} ... \quad &= (x-1)\left[sin(s) - scos(s)\right]_0^x + x\left[sin(s) - scos(s)\right]_x^1 - x\left[cos(s)\right]_x^1 = \\ &= (x-1)(sin(x) - xcos(x)) + x(sin(1) - cos(1) - sin(x) + xcos(x)) - xcos(1) - xcos(x) = \\ &= xsin(x) - x^2cos(x) - sin(x) + xcos(x) + xsin(1) - \\ &- xcos(1) - xsin(x) + x^2cos(x) - xcos(1) - xcos(x) = ... \end{aligned}$$

So, the solution is:

$$... = \boxed{xsin(1) - sin(x) - 2xcos(1) = y(x)}$$

(b) $y'' - y = x$, $y(0) = y(1) = 0$

We know from exercise **2b** that the solution of the homogeneous equation is:

$$y(x) = c_1e^x + c_2e^{-x}$$

Adding the boundary conditions:

$$y(0) = 0 \implies c_1 + c_2 = 0 \implies \underline{y_1 = e^x - e^{-x}} y(1) = 0 \implies c_1e + c_2e^{-1} = 0 \implies \underline{y_2 = e^{x-1} - e^{1-x}}$$

$$W(y_1, y_2) = \begin{vmatrix} e^x - e^{-x} & e^{x-1} - e^{1-x} \\ e^x + e^{-x} & e^{x-1} + e^{1-x} \end{vmatrix} = 2e - 2e^{-1}$$

So the Green's function is:

$$G(x,s) = \begin{cases} \frac{(e^s - e^{-s})(e^{x-1} - e^{1-x})}{2(e - e^{-1})} & 0 \leq s \leq x \leq 1 \\ \frac{(e^x - e^{-x})(e^{s-1} - e^{1-s})}{2(e - e^{-1})} & 0 \leq x \leq s \leq 1 \end{cases}$$

To find the solution, we compute:

$$y(x) = \int_0^1 G(x,s)f(s)ds, \quad f(s) = s$$

So, we have:

$$y(x) = \int_0^x \frac{s(e^s - e^{-s})(e^{x-1} - e^{1-x})}{2(e - e^{-1})} ds + \int_x^1 \frac{s(e^x - e^{-x})(e^{s-1} - e^{1-s})}{2(e - e^{-1})} ds = \\ = \frac{e^{x-1} - e^{1-x}}{2(e - e^{-1})} \int_0^x s(e^s - e^{-s}) ds + \frac{e^x - e^{-x}}{2(e - e^{-1})} \int_x^1 s(e^{s-1} - e^{1-s}) ds = \quad ... \tag{3}$$

$$\int_0^x s(e^s - e^{-s}) ds = \begin{bmatrix} u = s & du = ds \\ dv = e^s - e^{-s} & v = e^s + e^{-s} \end{bmatrix} = s(e^s + e^{-s}) - int(e^s + e^{-s} ds]_0^x =$$

$$= s(e^s + e^{-s}) - e^s + e^{-s}]_0^x = x(e^x + e^{-x}) - e^x + e^{-x}$$

$$\int_x^1 s(e^{s-1} - e^{1-s}) ds = \begin{bmatrix} u = s & du = ds \\ dv = e^{s-1} - e^{1-s} & v = e^{s-1} + e^{1-s} \end{bmatrix} \\ = s(e^{s-1} + e^{1-s}) - \int e^{s-1} + e^{1-s} ds \Big]_x^1 = \\ = s(e^{s-1} + e^{1-s}) - e^{s-1} + e^{1-s}]_x^1 = 2 - x(e^{x-1} + e^{1-x}) + e^{x-1} - e^{1-x} \tag{4}$$

Hence, substituting 4 into 3, we have:

$$y(x) = \frac{(e^{x-1} - e^{1-x})\left[x(e^x + e^{-x}) - e^x + e^{-x}\right]}{2(e - e^{-1})} + \frac{(e^x - e^{-x})\left[2x(e^{x-1} + e^{1-x} + e^{x-1} - e^{1-x}\right]}{2(e - e^{-1})} = \\ = \frac{2xe^{-1} - 2xe + 2e^x - 2e^{-1}}{2(e - e^{-1}} = \boxed{\frac{x(e^{-1} - e) + e^x - e^{-x}}{e - e^{-1}} = y(x)}$$

(c) $xy'' + y' = x$, $y(1) = y(e) = 0$

From exercise **2a**, we know that

$$G(x, s) = \begin{cases} -sln(s)(1 - ln(x)) & 1 \leq s \leq x \leq e \\ -sln(x)(1 - ln(s)) & 1 \leq x \leq s \leq e \end{cases}$$

Therefore,

$$y(x) = \int_1^x -sln(s)(1 - ln(x)) ds + \int_x^1 -sln(x)(1 - ln(s)) ds = \\ = -\int_1^x sln(s) ds + ln(x) \int_1^x sln(s) ds + ln(x) \int_x^1 -s ds + ln(x) \int_x^1 sln(s) ds = ... \tag{5}$$

$$\int xln(x) dx = \begin{bmatrix} u = ln(x) & du = \frac{1}{x} dx \\ dv = x & v = \frac{x^2}{2} \end{bmatrix} = \frac{x^2}{2} ln(x) - \frac{1}{2} \int x dx = \frac{x^2}{2} ln(x) - \frac{x^2}{4} \tag{6}$$

Substituting 6 into 5:

$$... = (-1+ln(x))\left(\frac{x^2}{2}ln(x)-\frac{x^2}{4}-\frac{1}{2}ln(1)-\frac{1}{4}\right)-ln(x)\left(\frac{s^2}{2}\right]_x^l+$$

$$+ln(x)\left(\frac{e^2}{2}ln(e)-\frac{e^2}{4}-\frac{x^2}{2}ln(x)+\frac{x^2}{4}\right)=\frac{-x^2}{2}ln(x)+\frac{x^2}{4}+\frac{1}{4}+\frac{x^2}{2}ln^2(x)-\frac{x^2}{4}ln(x)-$$

$$-\frac{1}{4}ln(x)-ln(x)\frac{e^2}{2}+ln(x)\frac{x^2}{2}+ln(x)\frac{e^2}{2}ln(e)-\frac{e^2}{4}ln(x)-\frac{x^2}{2}ln^2(x)+ln(x)\frac{x^2}{4}=$$

$$=\boxed{\frac{x^2+1}{4}-\frac{e^2+1}{4}=y(x)}$$

(d) $y''-\pi^2y=cos(\pi x)$, $y(0)=y(1)$, $y'(0)=y'(1)$

First, we have to find the solution of the associated homogeneous equation:

$$y''-\pi^2y=0$$

Let's make the change $y=e^{\alpha t}$, so:

$$(e^{\alpha t})''-\pi^2e^{\alpha t}=0 \implies \alpha^2e^{\alpha t}=0 \implies \alpha=\pm\pi$$

So, the solution is on the form of:

$$y=c_1e^{\pi x}+c_2e^{-\pi x}$$

Now, we set the boundary conditions:

$$y(0)=y(1) \implies c_1+c_2=c_1e^{\pi}+c_2e^{-\pi} \implies c_1=\frac{c_2}{e^{\pi}} \implies \underline{y_1=e^{\pi}e^{\pi x}+e^{-\pi x}}$$

$$y'(0)=y'(1),\ y'=c_1\pi e^{\pi x}-c_2\pi e^{-\pi x} \implies c_1\pi-c_2\pi=c_1\pi e^{\pi}-c_2\pi e^{-\pi} \implies$$

$$\implies c_1=\frac{e^{\pi}-1}{e^{\pi}(1-e^{\pi})}c_2 \implies \underline{y_2=\frac{e^{\pi}-1}{e^{\pi}(1-e^{\pi})}e^{\pi x}+e^{-\pi x}}$$

$$W(y_1,y_2)=\begin{vmatrix} e^{\pi}e^{\pi x}+e^{-\pi x} & \frac{e^{\pi}-1}{e^{\pi}(1-e^{\pi})}e^{\pi x}+e^{-\pi x} \\ \pi e^{\pi}e^{\pi x}-\pi e^{-\pi x} & \frac{\pi e^{\pi}-1}{e^{\pi}(1-e^{\pi})}e^{\pi x}-\pi e^{-\pi x}\end{vmatrix}=\frac{2\pi}{e(1-e^{\pi})}-2\pi e^{\pi}$$

So, the Green's function is:

$$G(x,s)=\begin{cases}\frac{(e^{\pi+\pi s}+e^{-\pi s})(-e^{\pi x-\pi}+e^{-\pi x})}{\frac{2\pi}{e(1-e^{\pi})}-2\pi e^{\pi}} & 0\le s\le x\le 1\\ \frac{(e^{\pi+\pi x}+e^{-\pi x})(-e^{\pi s-\pi}+e^{-\pi s})}{\frac{2\pi}{e(1-e^{\pi})}-2\pi e^{\pi}} & 0\le x\le s\le 1\end{cases}$$

Now, $y(x)=\int_0^1 G(x,s)f(s)ds$, with $f(s)=cos(\pi s)$

$$y(x)=\int_0^x G_1cos(\pi s)ds+\int_x^1 G_2cos(\pi s)ds=$$

$$=\frac{(e^{\pi}-1)(e^{2\pi x}-e^{\pi})(e^{\pi(2s+1)}-1)e^{1-\pi(s+k+1)}}{2\pi^2((e^{\pi}-1)e^{\pi+1}+1)}+\frac{(e^{\pi}-1)(e^{\pi(2x+1)}+1)(e^{2\pi s}+e^{\pi})e^{1-\pi(s+x+1)}}{2\pi^2((e^{\pi}-1)e^{\pi+1}+1)}=$$

$$=\boxed{\frac{(e^{\pi}-1)(e^{2\pi x}-e^{\pi})(e^{\pi(2s+1)}-1)e^{1-\pi(s+x+1)}+(e^{\pi}-1)(e^{\pi(2x+1)}+1)(e^{2\pi s}+e^{\pi})e^{1-\pi(s+x+1)}}{2\pi^2((e^{\pi}-1)e^{\pi+1}+1)}}$$

4. References

[1] Zengqin Zhao, *Solutions and Green's functions for some linear second-order three-point boundary value problems*

[2] LibreText, *Initial Value Green's Function*

[3] LibreText, *Boundary Value Green's Functions*