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Exercises
Mathematical Approach to Symmetrical Phenomenon

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1. **Describe all representations $\varphi : \mathbb{Z}_n \to \mathbb{C}^*$. Does $\mathbb{Z}_n$ have irreducible representations $\rho : \mathbb{Z}_n \to GL(n, \mathbb{C})$, for $n > 1$? Please justify your answer.**

   We know that a representation of a group is a way of representing the elements of the group as matrices or linear transformations. A representation of a group G on a vector space V is a homomorphism from G to the group of invertible linear transformations of V.

   In the case of the group $\mathbb{Z}_n$ and the vector space $\mathbb{C}^*$, a representation $\varphi : \mathbb{Z}_n \to \mathbb{C}^*$ would be a homomorphism from the group of integers modulo n to the group of non-zero complex numbers. Essentially, it maps elements of the group $\mathbb{Z}_n$ to non-zero complex numbers in a way that preserves the group structure.

   There are many possible representations of $\mathbb{Z}_n$ on $\mathbb{C}^*$, including the following:

   - The trivial representation, where all elements of $\mathbb{Z}_n$ are mapped to the identity element of $\mathbb{C}^*$ ($\varphi(a) = 1, \forall a \in \mathbb{Z}_n$).
   - The regular representation, where elements of $\mathbb{Z}_n$ are mapped to complex roots of unity ($\varphi(a) = e^{2\pi iak/n}\ for\ k = 1, 2, ..., n-1$ where e is the base of the natural logarithm, i is the imaginary unit and a is a element of $\mathbb{Z}_n$). These representation are also referred to as the "cyclic representations of $\mathbb{Z}_n$". They are not irreducible and can be decomposed in one dimensional representation.
   - Any character of $\mathbb{Z}_n$, which is a homomorphism from $\mathbb{Z}_n$ to the unit circle in the complex plane.
   - Direct sum of any number of the above representations.

   It's worth noting that the above representations are not unique and there are infinitely many representations of $\mathbb{Z}_n$ on $\mathbb{C}^*$.

   On the other hand, $\mathbb{Z}_n$ does not have, in general, an irreducible representation on $GL(n, \mathbb{C})$ for $n > 1$. The reason is that all representations of $\mathbb{Z}_n$ on $GL(n, \mathbb{C})$ are equivalent to direct sums of the one-dimensional representations given by the homomorphism $\varphi(a) = e^{2\pi iak/n}\ for\ k = 1, 2, ..., n-1$. These one-dimensional representations are not irreducible and can be further decomposed into one dimensional representations.

   In addition, the group $GL(n, \mathbb{C})$ is not abelian and $\mathbb{Z}_n$ is abelian, so there's no homomorphism between them. Only if n is prime we have just one irreducible representation of dimension 1, which is the trivial representation.

   It's worth noting that there's a way to represent $mathbbZ_n$ with $GL(n, \mathbb{C})$ through projective representations, but they are not irreducible neither.

2. **Let $G = \{1, -1, i, -i\}$ (with multiplication of complex-numbers). Define $\varphi, \rho : G \to \mathbb{C}^*$ as follows: $\varphi_g = g$ and $\rho_i = -i$. Check if $\varphi \oplus \rho,\ \psi : G \to GL(2, \mathbb{C})$, where**

$$\psi_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

are equivalent. List all the matrices of the above representations.

The matrices of the representation φ are given by $\varphi_1 = 1, \varphi_{-1} = -1, \varphi_i = i, \varphi_{-i} = -i$ and for ρ are given by $\rho_1 = -1, \rho_{-1} = 1, \rho_i = -i, \rho_{-i} = i$. So, the matrices of the representation $\varphi \oplus \rho$ are given by:

$$(\varphi \oplus \rho)_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ (\varphi \oplus \rho)_{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ (\varphi \oplus \rho)_i = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix},\ (\varphi \oplus \rho)_{-i} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

The matrices of the representation ψ are given by:

$$\psi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \psi_{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ \psi_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ \psi_{-i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note that the matrices for the elements 1 and -1 are identity matrices, and for i and $-i$ are not identity matrices, which are defined by the representation $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

To check if the representations $\varphi \oplus \rho$ and ψ are equivalent, we need to check if there exists a non-singular matrix P such that $P(\varphi \oplus \rho)P^{-1} = \psi, \forall g \in G$, but, since we have the matrices of both representations, we can see that they are not the same so the representations are not equivalent (*There is non-singular matrix P that satisfies the above equation for any $g \in G$*).

Note: For the exercise I have consider that ρ sends each element of the group to the negative corresponding element, but if with $\rho_i = -i$ we mean that ρ is only defined for $i, -i$, the exercise changes, and, in that case we will have $\rho_1 = \rho_{-1} = 1, \rho_i = -i, \rho_{-i} = i$ and, in that case, the matrices of $\varphi \oplus \rho$ and ψ are the sames, so the representations would be equivalent, because we just need to take P as the identity matrix.

3. **We write r for the rotation counterclockwise by $\pi/2$ and s for the reflection over the x-axis. Prove that the assigment**

$$\varphi(r^k) = \begin{bmatrix} i^k & 0 \\ 0 & (-i)^k \end{bmatrix}\ \varphi(sr^k) = \begin{bmatrix} 0 & (-i)^k \\ i^k & 0 \end{bmatrix}$$

defines a linear representation $\varphi : D_4 \to GL(k, \mathbb{C})$ of the dihedral group (symmetries of the square). Show that φ is irreducible.

The dihedral group D^4 is generated by two elements, r and s which correspond to a counterclockwise rotation by $\pi/2$ and a reflection over the x-axis, respectively. The group D^4 is defined as the set off all possible combinations of the form r^k and sr^k, where $0 \leq k \leq 3$.

First, we need to show that the assignment φ is well-defined, meaning that it assigns a unique matrix to each element of the group. Since the exponents k can only take the values $0, 1, 2, 3$ the matrices assigned to each element of the group are unique.

Next, we need to show that φ is linear, meaning that it satisfies the following properties:

- $\varphi(ab) = \varphi(a)\varphi(b)\ \forall a, b \in D^4$.
- $\varphi(1) = I,\ I$ is the identity matrix.

- $\varphi(a^{-1}) = \varphi(a)^{-1}, \forall a \in D^4$.

Let's consider the first property. For any a and b in D^4 we have:

$$\varphi(r^{k+m}) = \begin{bmatrix} i^{k+m} & 0 \\ 0 & (-i)^{k+m} \end{bmatrix} = \begin{bmatrix} i^k i^m & 0 \\ 0 & (-i)^k(-i)^m \end{bmatrix} = \begin{bmatrix} i^k & 0 \\ 0 & (-i)^k \end{bmatrix} \begin{bmatrix} i^m & 0 \\ 0 & (-i)^m \end{bmatrix} = \varphi(r^k)\varphi(r^m)$$

$$\varphi(sr^{k+m}) = \begin{bmatrix} 0 & (-i)^{k+m} \\ i^{k+m} & 0 \end{bmatrix} = \begin{bmatrix} 0 & (-i)^k(-i)^m \\ i^k i^m & 0 \end{bmatrix} = \begin{bmatrix} 0 & (-i)^k \\ i^k & 0 \end{bmatrix} \begin{bmatrix} 0 & (-i)^m \\ i^m & 0 \end{bmatrix} = \varphi(sr^k)\varphi(sr^m)$$

For the second property, it's easy to see when $k = 0$, $\varphi(r^0) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] = I$.

For the third property, we know that $(r^k)^{-1} = r^{-k}$ and $(sr^k)^{-1} = sr^{-k}$ and, for any k, the matrix corresponding to the inverse element of a element $g \in D^4$ is the inverse of the matrix corresponding to g. Therefore, we have:

$$\varphi(r^{-k}) = \begin{bmatrix} i^{-k} & 0 \\ 0 & (-i)^{-k} \end{bmatrix} = \begin{bmatrix} i^k & 0 \\ 0 & (-i)^k \end{bmatrix}^{-1} = \varphi(r^k)^{-1}$$

$$\varphi(sr^{-k}) = \begin{bmatrix} 0 & (-i)^{-k} \\ i^k & 0 \end{bmatrix} = \begin{bmatrix} 0 & (-i)^k \\ i^{-k} & 0 \end{bmatrix}^{-1} = \varphi(sr^k)^{-1}$$

Therefore, the previous assignment defines a linear representation of the dihedral group D^4.

Now, to show that a representation is irreducible, we need to prove that there is no non-trivial subspace of the vector space on which the representation acts that is left invariant by the action of the group. In other words, we need to show that there is no non-trivial subspace of $\mathbb{C}^2$ on which the representation acts such that for any $g \in D^4$ and any vector v in the subspace, we have $\varphi(g)v$ still in that subspace.

One way to show this is to prove that the representation acts transitively on the vector space. This means that for any two non-zero vectors v and w in $\mathbb{C}^2$, there exists some $g \in D^4$ such that $\varphi(g)v = w$.

To prove this, we can take two non-zero vectors $v = [a, b]$ and $w = [c, d]$ in $\mathbb{C}^2$. We can then find integers k and m such that $i^k = a/c$ and $(-i)^k = b/d$, and we can set $g = r^k$. Then, we have $\varphi(g)v = \left[\begin{smallmatrix} i^k & 0 \\ 0 & i^{-k} \end{smallmatrix}\right]\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = [c, d] = w$. This shows that the representation acts transitively on the vector space $\mathbb{C}^2$ and, therefore, the representation is irreducible.

Alternatively, we can show that the representation is irreducible by showing that the only subspaces that are left invariant by the action of the group are the trivial subspaces, only the zero vector, or the whole space.

In this case, notice that for any non-zero vector $v \in \mathbb{C}^2$ the subspace generated by v is not left invariant by the action of the group. For example, for the vector $[1, 0]$, it's clear that the action of the group does not keep it inside the subspace generated by $[1, 0]$. Therefore, the representation is irreducible.

4. **Give an example of a finite group G and decomposable representation $\varphi : G \to GL(4, \mathbb{C})$** such that the $\varphi_g, g \in G$ do not have a common eigenvector.

   There are many possible examples:

- $G = C_4 = \{e, a, a^2, a^3\}$, the cyclic group of order 4 with the representation defined as follows:

$$\varphi(e) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \varphi(a) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \varphi(a^2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \varphi(a^3) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

  This representation is decomposable because it's equivalent to the direct sum of two 2-dimensional representations.

  It can be observed that each matrix in this representation is a permutation matrix, which means that the matrix doesn't have any non-zero eigenvalue. Thus, none of these matrices share a common eigenvector and the representation is not irreducible.

- $G = S^3$, the symmetric group on 3 elements, with representation defined as follows:

$$\varphi((12)) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \varphi((13)) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \varphi((23)) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  This representation is decomposable because it's equivalent to the direct sum of two 2-dimensional representations.

  It can be observed that in this representation, two of the matrices are permutation matrices and the other one is a diagonal matrix with eigenvalues 1, -1. Thus, these matrices don't share a common eigenvector, and the representation is not irreducible.

- $G = \mathbb{C}^2 \times \mathbb{C}^2$, the direct product of two cyclic group of order 2 with representation defined as follows:

$$\varphi((1,0)) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \varphi((0,1)) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad \varphi((1,1)) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

  This representation is decomposable because it's equivalent to the direct sum of two 2-dimensional representations.

  t can be observed that in this representation, all the matrices are diagonal matrices with eigenvalues 1,-1. Thus, these matrices don't share a common eigenvector, and the representation is not irreducible