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Semester: 1
Academic Year: 2022/2023

Faculty of Applied Physics and Mathematics

Brouwer Theorem and Its Applications

Algebraic Topology

Abstract

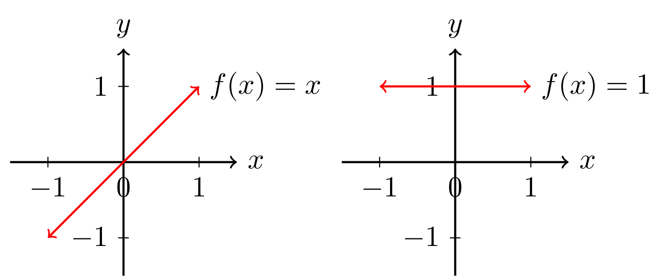
This assignment is about Brouwer's Theorem, also known as Brouwer fixed point theorem. We will see different concepts that are necessary to understand the theorem, as well as the different proofs that demonstrate its veracity. After all that previous work, we will see some examples and practical uses of the theorem.

1. Introduction. Fixed Points

Fixed points have many applications. One of their prime applications is in the mathematical field of game theory; here, they are involved in finding equilibrium. The existence and location of the fixed point(s) is important in determining the location of any equilibrium. They are then applied to some economics, and used to justify the existence of economic equilibrium in the market, as well as equilibrium in dynamical systems.

Definition. For a function $f : X \to X$, a fixed point $c \in X$ is any point where $f(c) = c$.

When a function has a fixed point, c, the point (c, c) is on its graph. Lets see some examples:



We can see that the first function is entirely fixed points, while the second has only one fixed point at 1.

Fixed points came into mathematical focus in the late 19^{th} century, when the mathematician Henri Poincaré began using them in them in topological analysis of nonlinear problems. On the other hand, Luitzen Egbertus Jan Brouwer, of the University of Amsterdam, started working with them in algebraic topology.

In 1909, Jan Brouwer formulated his fixed-point theorem, which was first published relating only to the three-dimensional case in 1909 and then in 1910 presented the next theorem:

Brouwer Fixed-Point Theorem in $\mathbb{R}$. Given a set $K \subset \mathbb{R}^n$ compact and convex, and a continuous function $f : K \to K$, then exists some $c \in K$ such that $f(c) = c$; that is, c is a fixed point.

The original wording of the theorem gave this result for n-simplexex (*A specific class of compact and convex sets, that is the "simplest" polygon in n dimensions that has $n+1$ vertices*), but we are going to focus on unit intervals and disks instead.

2. **General Definitions**

Definition. ***Topological Space.*** A topological space is a set, X, equipped with a collection of its subsets, T, which verify the following properties:

(a) $\emptyset \in T$.

(b) For any arbitrary collection $U_\alpha \in T,\ \alpha \in \Delta$, the union $\bigcup_{\alpha\in\Delta} U_\alpha \in T$.

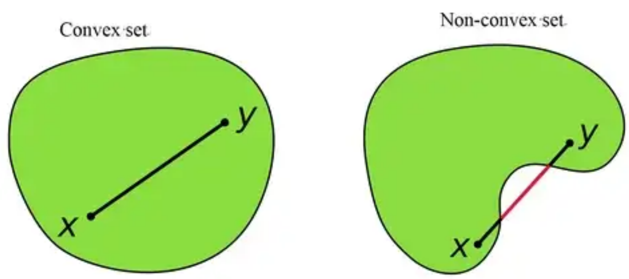
(c) For any $U_1, U_2 \in T$, the intersection $U_1 \cap U_2 \in T$.

Definition. ***Open Set.*** Any element of the topology is an open set. *In fact, the topology is the full collection of open sets, consisting of the basic sets, their infinite unions and finite intersections.*

In Euclidean metric space $\mathbb{R}^n$, tha basic open sets are open intervals, disks or balls, for n=1,2,3,..., respectively.

Definition. ***Closed Set.*** A set $F \subseteq \mathbb{R}^n$ is closed if its complement, F^C, is open. A set is also closed if it is the arbitrary intersection or infinite union of closed sets. In $\mathbb{R}$, closed intervals and singletons are closed.

Definition. ***Convex.*** A set $G \subseteq \mathbb{R}^n$ is said to be convex if. for any two points $g_1, g_2 \in G$, all points on the straight line segment connecting g_1 and g_2 are also in G.



Definition. ***Open Cover.*** A collection $\mathcal{A}$ of open sets in $\mathbb{R}^n$ is an open cover for set $\mathcal{A}$ if the union of all sets in $\mathcal{A}$ has A as a subset.

Definition. ***Compact.*** Let (X, T) be a topological space; if every open covering $\mathcal{A}$ of A contains a finite sub-covering (*a finite sub-collection of* $\mathcal{A}$ *that is still an open cover for* A) then, A is compact.

In the most familiar of cases, the real numbers with the usual topology, a set must simply be closed and bounded in order to be compact, as shown be the **Heine-Borel Theorem**. In particular, this is true for $\mathbb{R}^n$ with the usual topology.

Definition. ***Continuous.*** Let (X, T_X) and (Y, T_Y) be topological spaces. A function $f : X \to Y$ is continuous if, for V an open subset of Y, $f^{-1}(V)$ is open in X. The open sets in X and Y are the member sets of T_X and T_Y, respectively.

Definition. ***Open Map.*** Let (X, T_X) and (Y, T_Y) be topological spaces. A function $f : X \to Y$ is an open map if, for U an open subset of X, $f(U)$ is open in Y.

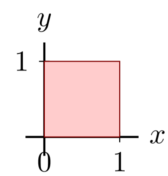
Definition. ***Bijection.*** A function $f : X \to Y$ is a bijection if it is injective and surjective; that is to say, f is a bijection if $\forall y \in Y \implies \exists x \in X$ such that $f(x) = y$ and if $f(x_1) = f(x_2) \implies x_1 = x_2$. A function that is bijective will have a well-defined inverse; that is, its inverse will be a function.

Definition. ***Homeomorphism.*** A homeomorphism is a function that is continuous, an open map, and bijective. It is clear in this context, then, how being an open map relates to it having a continuous inverse, and how all of this relates to structures defined through open sets. The existence of a homeomorphism between two sets is sufficient to show that the two sets are homeomorphic.

If two sets are homeomorphic, then they are topologically equivalent. Thus, topological properties that hold for one set will hold for any set homeomorphic to it (*in fact, it is this quality that makes a property topological*).

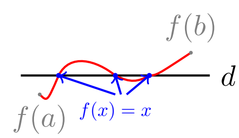
3. **Dimension 1 Case**

The most simple case to consider the fixed point theorem is when the set $K \subset \mathbb{R}$ has $\mathbb{R}$ only having 1 dimension, and is in fact the unit 'square' $I = [0, 1]$. For a continuous function $f : [0, 1] \to [0, 1]$ to have fixed points, it must be so that there is point $c \in X$ where $f(c) = c$. While K is one-dimensional, however, the actual work will be done in $[0, 1]^2$, which has two dimensions.



The proof relies on the Intermediate Value Theorem:

Intermediate Value Theorem. Let $X, Y \subseteq \mathbb{R}$. Given a function $f : X \to Y$ that is continuous on $[a, b] \subseteq X$, there exists for every $d \in (f(a), f(b)) \subseteq Y$ (assuming, without loss of generality, that $f(a) \leq f(b)$) some $c \in (a, b)$ such that $f(c) = d$.



When dealing with one dimension, any closed and convex subset of $\mathbb{R}$ is homeomorphic to $[0, 1]$. We can then show that any one-dimensional case for the Brouwer Fixed Point Theorem is equivalent to the case in $[0, 1]$, and thus, the Theorem applies there.

(a) **Basic Proof of the Brouwer Fixed Point Theorem on Set** $[0, 1]$

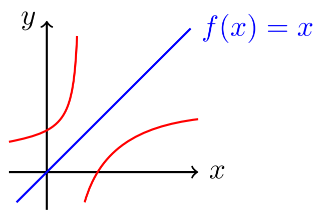
Given that set K is compact and convex, and that function $f : K \to K$ is continuous, then there exists some $c \in K$ such that $f(c) = c$;that is, c is a fixed point.

Proof: Let $f : [0, 1] \to [0, 1]$ be continuous function on the unit square. The function $g(x) = f(x) - x$ is also continuous on the unit square as well, as it is difference of continuous f and the identity function $i(x) = x$ which is also continuous. When $x = 0$, $f(0) \geq 0$ and $g(0) = f(0) - 0$, so $g(0)$ is either positive or 0. Now when $x = 1$, $f(1) \leq 1$, and $g(1) = f(1) - 1$. Similarly, $g(1)$ is either negative or 0.

Since g is a continuous function on closed set, the Intermediate Value Theorem applies. Then we have $g(0) \geq 0$ and $g(1) \leq 0$, and it must be so that there is a $c \in [0, 1]$ such that $g(c) = d$, for any d, but in particular when $d = 0$. Thus, we have a point c where $g(c) = f(c) - c = 0$, thus; $f(c) = c$, and therefore c is our fixed point.

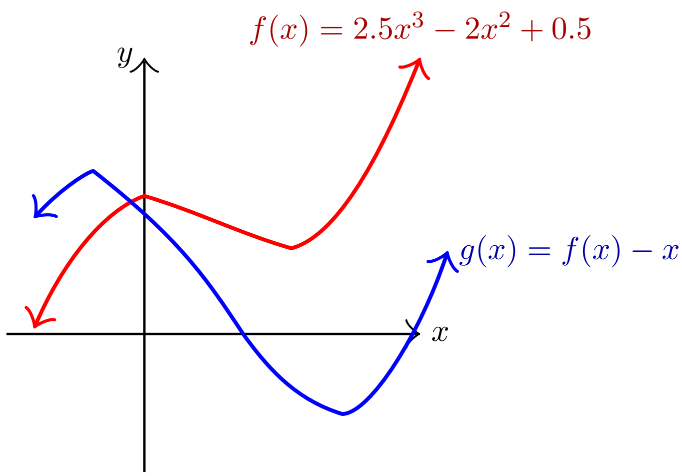
□

To consider it through explanation, note a fixed point requires passing through the line $f(x) = x$. Thus, a function without a fixed point cannot intersect this line. That, however, leaves something such as the figure below, which is not continuous.



It is impossible for a continuous function to not intersect the line $i(x) = x$; however, to intersect that line is to have a fixed point, as all point on $i(x) = x$ are in fact fixed points.

For $g(x) = f(x) - x$, instead of trying to intersect $i(x) = x$, we are trying to not intersect the zero line $h(x) = 0$. It is easier to show, using the Intermediate Value Theorem, that g intersects the constant function h that it is show that f intersects i.



(b) **Extension to Homeomorphic Sets**

In order to determine whether a fixed point is guaranteed for some other compact convex interval K, then one must determinate whether or not homeomorphism can be found between K and $[0, 1]$. In fact is, then K also has a fixed point for any continuous functions from K into itself.

In higher dimensions, we can show that $f : K \to K$ has a fixed point under the same conditions: compactness and convexness of the set K, and continuity of the function f.

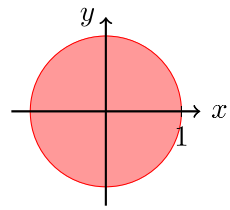
4. **Proof of the Brouwer Fixed Point Theorem for disk in 2D**

Definition. ***Closure.*** Let (X, T) be a topological space, and let $G \subseteq X$. The Closure of G, written $\overline{G}$, is the intersection of all closed sets that fully contain G. The closure of a set will always be closed.

Definition. ***Retraction.*** Let set $S \subseteq \mathbb{R}^2$ with $B \subseteq S$. We call $r : S \to B$ a retraction if it is continuous and $r(b) = b$, $\forall b \in B$.

We will consider this with S being a disk, and B being the 'surface' or boundary of that disk. Or, rather, we will consider the lack of existence of such a retraction.

In $\mathbb{R}^2$, the unit disk can be defined by $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \mid ||(x, y)|| \leq 1\}$ and the unit circle as $C = \{(x, y) \in \mathbb{R}^2 ||| (x, y)|| = 1\}$.



No Retraction Theorem. There does not exist any retraction from an closed unit disk $\mathbb{D}$ to its boundary, C.

We will need the No Retraction Theorem in order to classify a function without a fixed point as a retraction that violates the above theorem. Because so much the proof of the Bouwer Fixed Point Theorem rest on the No Retraction Theorem, we also present its proof here for $\mathbb{D} \subset \mathbb{R}^2$.

Proof: Let $r : \mathbb{D} \to C$ be a retraction from the unit disk $\mathbb{D}$ to its boundary, C. Consider $a, b \in C$; by removing these from C, we create two disjoint open arcs that compose $C - \{a, b\}$. Now let $A = r^{-1}(a)$, and $B = r^{-1}(b)$. Since r is a retraction, $a \in A$ and $b \in B$, and so A and B intersect C. Since r is continuous, and $\{a\}$ and $\{b\}$ are closed, A and B must also be closed. Furthermore, a and b can be the only points where A and B, respectively, can intersect C, as they are the only elements of A and B that are in C. Note that $\overline{(C - \{a, b\})} = C$. We can, then, find a subset of $\mathbb{D} \backslash (A \cup B)$ whose closure will contain C. Let us call this set P. We can choose it so that it is the open and path-connected P.

Consider a closed arc of C, called C_a, that contains a. Let C_a have endpoints x_a, y_a. Both x_a and y_a will be in $\overline{P}$; thus, there exists a path that connects them. Furthermore, since we have defined P as a subset of $\mathbb{D} \backslash (A \cup B)$, this path cannot intersect A or B. However, unioning this path with $C \backslash \{a, b\}$ results in another connected set. This implies that the retraction image of that union of the path and $C \backslash \{a, b\}$ is $C \backslash \{a, b\}$, because the path avoided A and B. But the image of a connected set under a continuous function cannot be diskonnected, so we have a contradiction. Therefore, it must be that r, the retraction, cannot exist.

$\square$

The No Retraction Theorem proved above will be the cornerstone for the following proof for the Brouwer Fixed Point Theorem on $\mathbb{D}$.

Brouwer Fixed Point Theorem on $\mathbb{D} \subset \mathbb{R}^2$. Given that function $f : \mathbb{D} \to \mathbb{D}$ is continuous, then there exists some $c \in \mathbb{D}$ such that $f(c) = c$; that is, c is a fixed point.

Proof: Let $\mathbb{D}$ be the unit disk in $\mathbb{R}^2$. Let $f : \mathbb{D} \to \mathbb{D}$ be continuous, but suppose that it does not have a fixed point. Now, let $r : \mathbb{D} \to \mathbb{D}$ be another function that, for each $x \in \mathbb{D}$, assigns it to the tip of the ray that extends from the boundary of $\mathbb{D}$ (*the unit disk* C) and passes through $f(x)$, then x.

This will be well-defined since $f(x) \neq x$, $\forall x \in \mathbb{D}$. As r is defined in terms of f, and f is continuous, r will also be continuous.

However, consider x_0, a point which itself lies on C. In this situation, $r(x)$ must equal x, and thus, r is a retraction. But no such retraction can exist, due to the *No Retraction Theorem.* This contradicts that f can exist as it is, with no fixed points. Therefore, it must be so that any $f : \mathbb{D} \to \mathbb{D}$ must in fact have a fixed point.

$\square$

Again, this will also be true for any sets in $\mathbb{R}^2$ that are homeomorphic to $\mathbb{D}$ (*that is to say, compact convex sets*). Thus, this actually satisfies any possible case of compact convex set in $\mathbb{R}^2$.

5. General Proof

Now, we will move on to proving the Brouwer Fixed Point Theorem in any dimensional $\mathbb{R}^n$. First, however, a few things must be defined.

Definition. A C^1 function is continuous and has a continuous derivate.

Stone Weierstrass Theorem. Given a continuous function, it can be approximated to any degree with a sub-algebra which separate points. That is, one many get as close as one likes to the original function. A polynomial (*which is* C^1) is a point-separating sub-algebra.

We will only be using Stone Weierstrass Theorem to give us polynomials; thus, while Stone Weierstrass does allow for other functions to be used as approximations, we will not mind those.

Inverse Function Theorem. Let $X \subseteq R^n$ be open, and let function $f : X \to R^n$ be continuously differentiable and its derivative (*expressed as a matrix of partial derivatives*) has inverse at point $c \in X$, then it is also invertible in a neighborhood about c.

First, as it is again such an integral part, we will prove the general dimensional case of the No Retraction Theorem, before using it on the Brouwer Fixed Point Theorem. The addition of more dimensions requires changes to the theorem, as derivatives are now used to prove it and non-C^1 functions do not have usable derivatives. We proceed by assuming that such a retraction can exist, and then disproving its existence through contradiction.

No Retraction Theorem. There can be no C^1 retraction from the unit n-dimensional ball $\mathbb{D}^n$ to its boundary, the unit 'sphere' B^{n-1}.

Proof: Let $r : \mathbb{D}^n \to B^{n-1}$ be a C^1 retraction from the unit n-dimensional disk $\mathbb{D}^n$ to its boundary, B^{n-1}. Let $g(x) = r(x) - x$, let $t \in [0, 1]$ be fixed, and let $f_l(x) = x + lg(x) = x(1 - l) + lr(x)$. For $x \in \mathbb{D}^n$, note be the Triangle inequality that $||f_l(x)|| \leq ||x||(1 - l) + l||r(x)||$, because l and $1 - l$ both have magnitude less than 1. Furthermore, because x and $r(x)$ must also have magnitude less than 1, $||f_l(x)|| \leq (1 - l) + l = 1$; this makes f_l a function from $\mathbb{D}^n$ to $\mathbb{D}^n$. Furthermore, $f_l(x) = x(1 - l) + lr(x) = x(1 - l) + lx = x$ if $x \in B^{n-1}$, due to r being a retraction. This makes all point of in B^{n-1} fixed point of f_l.

Since r is C^1, h must also be C^1, and there must then exist some C,, a constant, where $||g(x_2) - g(x_1)|| \leq C||x_2 - x_1||$.

Suppose that there are $x_1, x_2 \in \mathbb{D}^n$ with $x_1 \neq x_2$, but also with $f_l(x_1) = f_l(x_2)$. Using the definition of f_l, $f_l(x_1) = x_1 + lg(x_1) = x_2 + lg(x_2) = f_l(x_2)$, and from those we can derive $x_1 - x_2 = lg(x_2) - lg(x_1)$. Then, through, we have that $||x_1 - x_2|| = l||g(x_2) - g(x_1)|| \leq lC||x_1 - x_2||$ which means that $lC \geq 1$.

When $l < C^{-1}$, r_l must be injective, because in that case $lC < 1$ and $||x_1 - x_2|| \leq lC||x_1 - x_2||$ only if $x_1 - x_2 = 0$. Let $U_l = f_l[\mathbb{D}^n]$, and note that $f_l'(x) = (1, 1, 1, ..., 1) + lg'(x)$. We also know, due to g being C^1, that there exists some l_0 for which f_l' has a positive determinant when expressed as a matrix of partials for all $l \leq l_0$. This allows for the use of the inverse function theorem, so f_l is also invertible near that point. This allows for U_l to be open for sufficiently small l, as the continuity of f makes its inverse an open map. Let $l \in [0, l_0]$ be fixed yet arbitrary from here on. We now have a bijection.

However, suppose that $U_l = f_l(\mathbb{D}^n) \neq \mathbb{D}^n$. Clearly, $\mathbb{D}^n \subset U_l$ as f_l does not map outside of $\mathbb{D}^n$. It must then be so that the boundary of U_l will intersect the interior of $\mathbb{D}^n$ (*that is, the boundary of* U_l *must intersect a point that is not on the boundary of* $\mathbb{D}^n$. Let us call that point x_0. We have compactness, and moreover sequential compactness. Since y_0 is in the boundary of U_l, it is in the closure of U_l, and it is then a limit point. We can then find a sequence in U_l that converges to y; let us define this sequence in $\mathbb{D}^n$ as $(x_n) \subset \mathbb{D}^n$ for which $f(x_n) \to y_0$. But, as we have compactness, we can find a convergent sub-sequence of (x_n) as well. Suppose that $x_{n_m} \to x_0$; since f is continuous, this means that $f(x_{n_m}) \to f(x_0)$. However, $f(x_n) \to y_0$, and so $f(x_0) = y_0$. Yet, y_0 cannot be in U_l, as U_l is open and thus cannot contain its boundary. It must then be so that x_0 is in B^{n-1} the boundary of $\mathbb{D}^n$; otherwise it could not map to the boundary of U_l. But, as we have a retraction, $f(x_0) = x_0$; therefore, $x_0 = y_0$. This, however, would imply that $y_0 \in \mathbb{B}^{\ltimes - \nVdash}$, despite our initial condition that y_0 not be in the boundary of $\mathbb{D}^n$. Therefore, we have a contradiction, and so $f(\mathbb{D}^n) = U_l = \mathbb{D}^n$ for $l \in [0, l_0]$; that is, f_l is surjective. Thus, when $l \in [0, l_0]$ and $l < C^{-1}$, we have the f_l is both injective and surjective; so it is a bijection. From here on, we will only consider f_l where it is a bijection.

Because we have f_l continuous, we can have $F : [0, l] \to \mathbb{R}$ defined by $F(l) = \int_{\mathbb{D}^n} det\ f_l'(x)dx$. This is with $f_l' = (1, 1, 1, ..., 1) + lg'(x)$ being constructed as a (square) matrix. This will actually be n integrals, however, we will let dx serve as $dx_1dx_2...dx_n$ for these n dimensions. The determinant of a matrix can be written in the form of a polynomial. Note that F is a function of l (x *being completely removed during the integration process*), and so we can consider its determinant as a polynomial of l. But F is an integral of f_l, and it will grant the volume of $f_l(\mathbb{D}^n)$ (if $l < C^{-1}$). As $\mathbb{D}^n$ is a bijection, $f_l(\mathbb{D}^n) = \mathbb{D}^n$, and so this provides us a range for which the polynomial is constant. However, a polynomial that is constant on some interval is constant everywhere. We can now conclude that $F(l)$ gives the volume of $\mathbb{D}^n$ for all $l \in [0, 1]$.

Of particular note is that $F(1)$ gives us the volume, and that this volume will be greater than 0. However, consider the inner product (*sometimes called the dot product*) of f_l with itself, notated $\langle f_l, f_l \rangle$. Note that $f_l(x) = f(x)$ when in B^{n-1} for any x; hence $\langle f_l, f_l \rangle$ for $l = 1$ is simply $||f_1(x)|| = 1$. Consider any arbitrary vector $v \in \mathbb{R}^n$; the inner product of $vf_1'(x)$ and $f(x)$, is equal to the derivative, with respect to t, of the inner product of $\frac{1}{2}\langle f_1(xt + vt), f_1(xt + vt)\rangle$. However, this results in the derivative of $\frac{1}{2}(1)$, and the derivative of a constant is always 0. From this, we can see the determinant of f_1' itself will be 0, implying that $F(1) = 0$. However, that is in contradiction to the earlier claim that $F(1) > 0$. Therefore, it must be so that r, through which f is defined, cannot exist; there can be no C^1 retraction from the unit ball $\mathbb{D}^n$ to its boundary B^{n-1}.

$\square$

This is the proof outlined (*as a lemma for the Milnor-Rogers proof of the Brouwer Fixed Point Theorem*) for the general dimensional No Retraction Theorem. It is the Milnor-Rogers method that I will follow [1][2]. It is a topological method for proving the theorem; there any many others that are combinatorial. Let us state the theorem one more time:

Brouwer Fixed Point Theorem on $\mathbb{D}^n \subset \mathbb{R}^n$. Given that function $f : \mathbb{D}^n \to \mathbb{D}^n$ is continuous, then there exists some $c \in \mathbb{D}^n$ such that $f(c) = c$; that is, c is a fixed point.

Proof: Let $\frac{\epsilon}{2} > 0$ be fixed yet arbitrary, and let $f : \mathbb{D}^n \to \mathbb{D}^n$ be continuous; $\mathbb{D}^n$ is the n-dimensional unit ball, as before. The Stone-Weierstrass Theorem gives a sequence of C^1 functions $p_l : \mathbb{D}^n \to \mathbb{R}^n$ where $||p_l(x) - f(x)|| \leq \frac{1}{l}, \forall x \in \mathbb{D}^n$, with $l \in \mathbb{N}$. Let $q_l = (1 + \frac{1}{l})^{-1} p_l$ for each $l \in \mathbb{N}$. Then we have that $||q_l(x) - f(x)|| = ||(1 + \frac{1}{l})^{-1} p_l(x) - f(x)|| \leq 1 + \frac{1}{l}$ for any $x \in \mathbb{D}^n$, through substitution. We can choose, for $\frac{\epsilon}{2}$, an L_1 so that $||p_l(x) - f(x)|| \leq \frac{\epsilon}{3}$ for all $l \geq L_1$. Let $L_2 = L_1 + 1$, then $||p_l(x) - f(x)|| < \frac{\epsilon}{2}$ for all $l \geq L_2$, for any $x \in \mathbb{D}^n$. Thus, $q_l \to f$ uniformly. It is, then, also so that sub-sequence $q_{l_k} \to f$ uniformly; let $L_2 = K_1$.

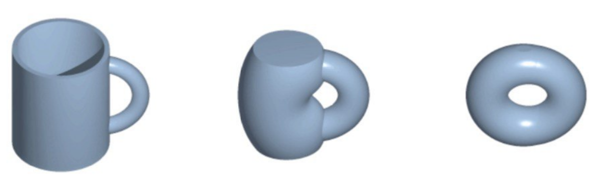
Let us define $h_l : \mathbb{D}^n \to B^{n-1}$ be the function that draws a straight line that touches $q_l(x)$, then x, and then returns the point where the line intersects $B^{n.1}$. For those h_l that have no fixed points, each is a C^1 map; it is derived from C^1 function q_l. However, it is also a retraction. It thus cannot exist, which means that q_l must have a fixed point, for all l, as otherwise h_l would be an impossible retraction.

Let $\{x_l\}_{l=1} \subset \mathbb{D}^n$ be the sequence of fixed points for q_l. Now, we are in sequentially compact space, and thus $\{x_l\}$ must have a convergent sub-sequence. Let x_{l_k} converge to $x_0 \in \mathbb{D}^n$; then for all $\frac{\epsilon}{2}$ there exists an $K_2 \in \mathbb{N}$ for which $||x_{l_k} - x|| < \frac{\epsilon}{2}$ for all $k \geq K_2$.

We can, then, combine these, and see that for any ϵ, there exists a $K = max\{K_1, K_2\}$ for which $||q_{l_k}(x_{l_k}) - f(x_0)|| < \epsilon$ for all $k \geq K$. However, $q_{l_k}(x_{l_k})$ is fixed, so $q_{l_k}(x_{l_k}) = x_{l_k}$ for each k. Thus, we have $||x_{l_k} - f(x_0)|| < \epsilon$. However, $x_{l_k} \to x_0$. Therefore, $x_0 = f(x_0)$, so f has a fixed point.

□

Now, we have only shown this result for one particular set, but, it is homeomorphic to any other compact and convex set; thus, on all compact and convex sets K, the Brouwer Fixed Point Theorem applies.



One of the most classic examples of two homeomorphic spaces, the coffee cup and the doughnut. Both are three-dimensional spaces with a hole in the middle. Neither is convex.

6. **Applications**

The Brouwer Fixed-Point Theorem is often used in the proving of the existence of Nash equilibriums. A Nash equilibrium occurs, in Game Theory, when the players know what strategies their opponents will use, know their strategies will not change, and also know that the current strategy they themselves are using is the best one to use. That is, both know what the other player is planning to do, and both know that their own current plans are the best strategy considering what their opponent is planning. They are incredibly important in Game Theory, being used to analyze problems or games where the different players act near-simultaneously. A specific example of this is modeling the market; Nash Equilibriums are used to predict and model actions taken during market crises.

Another application is in Dynamical Systems. Equilibriums, stable or unstable can be considered to be fixed points. Thus, in certaint spaces, one is guaranteed to have an equilibria.

A particular application of this it is to economics, this time more directly than through Nash Equilibriums. Fixed points are used to prove the existence of equilibria in the free market (for example, the meeting of supply and demand).

Other applications include coincidence theory and the Bass conjecture, and game theory in convex-valued multi-maps.

(a) **Interesting Application: Pure Exchange Economy and Equilibrium Prices**

We are looking at a system of m consumers, each of whom is initially endowed with fixed quantities of n different commodities. The consumers merely engaged in exchange.

Trade takes place, because each consumer wishes to acquire a bundle of commodities that is preferred to the initial endowment.

Assume a price vector $p = (p_1, ..., p_n)$ is announced. We see that one unit of commodity j has the same value as p_j/p_i units of commodity i. For any commodity bundle $c = (c_1, ..., c_n)$

$$p \cdot c = \sum_{i=1}^{n} p_i c_i = p_1 c_1 + ... + p_n c_n$$

is the market value of this bundle.

We use the following notation. For each consumer j and commodity i:

- w_i^j is $j's$ initial endowment of i.
- $x_i^j(p)$ is $j's$ final demand when the price vector is p.
- $w_i = \sum_{j=!}^{m} w_i^j$ is the total endowment for i.
- $x_i(p) = \sum_{j=1}^{m} x_i^j(p)$ is the aggregate demand for i
- If we add the demands of all consumers foe each commodity i and subtract the total initial endowment of i, then we get the so called **excess demand** $g_i(p) = x_i(p) - w_i$ of commodity i.

The total value of consumer $j's$ initial endowment at price p is

$$\sum_{i=1}^{m} p_i w_i^j$$

and the **budget constraint** for j

$$\sum_{i=1}^{m} p_i x_i^j(p) = \sum_{i=!}^{m} p_i w_i^j$$

Summing all budget constraints of the consumers we get **Walra's Law**:

$$\sum_{i=1}^{m} p_i x_i(p) = \sum_{i=1}^{m} p_i w_i \Longleftrightarrow \sum_{i=1}^{m} p_i (x_i(p) - w_i) = \sum_{i=1}^{m} p_i g_i(p) = 0$$

Question: Is it possible to find a price vector $p^* = (p_1^*, ..., p_n^*)$, a so called equilibrium price, which ensures that aggregate demand does not exceed the corresponding aggregate endowment:

$$\forall i = 1, ..., n : \quad x_i(p^*) \leq w_i \iff g_i(p^*) \leq 0?$$

The rule of free goods
In this case (*using Walra's Law)*) we see

$$\sum_{i=1}^{m} p_i^*(x_i(p^*) - w_i) = 0$$

where

- $p_i^* \geq 0$
- $x_i(p^*) - w_i \leq 0$

so $p_i^*(x_i(p^*) - w_i) \leq 0$ and

$$\forall i = 1, ..., n : \quad p_i^*(x_i(p^*) - w_i)$$

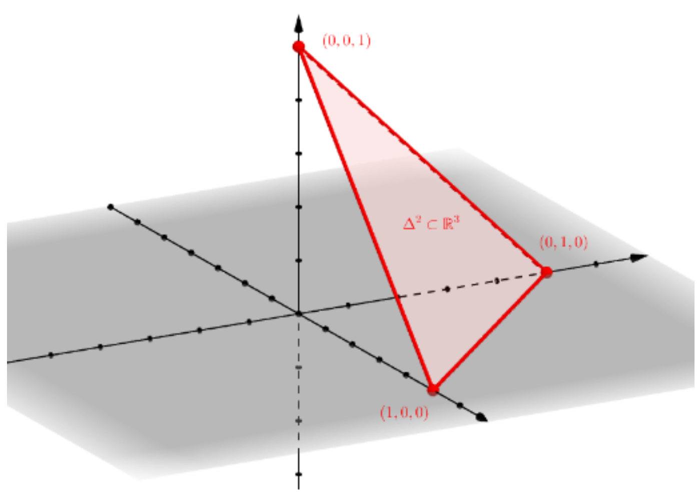
We have proved the rule of free goods: If any commodity is in excess supply in equilibrium, its price must be 0:

$$x_i(p^*) < w_i \implies p_i^* = 0$$

If there is a commodity for which market demand is strictly less than the total stock, then the equilibrium price must be 0.

Relative prices and Δ^{n-1}
Definition. We define the **standard unit simplex** $Delta^{n-1} \subset \mathbb{R}^n$ of dimension $n-1$ as

$$\Delta^{n-1} = \{x = (x_1, ..., x_n)^T \in \mathbb{R}^n : \forall x_i \geq 0, x_1 + ... + x_n = 1$$



The set Δ^{n-1} is $(n-1)$-dimensional convex and compact space, so if we try to apply Brouwer's fixed point theorem to a given continuous function F defined on Δ^{n-1}, it is only necessary to check that F is a self mapping. This means that you have to show that for all $(x_1, ..., x_n)$

with $x_1, ..., x_n \geq 0$ and $x_1 + ... + x_n = 1$ one has that $F_1(x_1, ..., x_n), ..., F_n(x_1, ..., x_n) \geq 0$ and $F_1(x_1, ..., x_n) + ... + F_n(x_1, ..., x_n) = 1$.

Coming back to our problem, it is rather obvious that only price ratios, or relative prices, matter in this economy. Hence we can restrict our attention to normalized prices p with $p_1 + ... + p_n = 1$ or shortly $p \in \Delta^{n-1}$. Suppose that the functions $g_1, ..., g_n$ are continuous on Δ^{n-1} and assume that for all $p \in \Delta^{n-1}$:

$$\sum_{i=1}^{n} p_i g_i(p) = 0$$

and the modified question is:

Is there a vector $p^* \in \Delta^{n-1}$ such that $g_i(p^*) \leq 0$ for all $i = 1, ..., n$?

We shall use Brouwer's fixed point theorem to prove the existence. to do so, we construct a continuous mapping from Δ^{n-1} into itself for which any fixed point gives equilibrium prices.

- Consider first the mapping $p' = F(p)$ defined component-wise by

$$p'_i = p_i + g_i(p)$$

  This simple price adjustment mechanism has a certain economic appeal. It maps the old price p_i to the new adjusted price $p_i + g_i(p)$ in such a way, that if the excess demand $g_i(p)$ is positive, so that the market demand exceeds the total available endowment, then the price will increase. But unfortunately, this mapping is not a self mapping of Δ^{n-1} because $p'_1 + ... + p'_n \neq 1$.

- Consider now the modified mapping $p' = F(p)$ defined component-wise by:

$$p'_i = \frac{1}{d(p)}(p_i + max\{0, g_i(p)\} \; with \; d(p) = 1 + \sum_{k=1}^{n} max\{0, g_i(p)\}$$

  This map F is continuous self mapping of Δ^{n-1} into itself. By Brouwer's fixed point theorem there is a fixed point $p^* \in \Delta^{n-1}$. This means that for all $i = 1, ..., n$ we have

$$p^*_i = \frac{1}{d(p^*)}(p^*_i + max\{0, g_i(p^*)\}$$

  or

$$p^*_i \cdot (d(p^*) - 1) = \{0, g_i(p^*)\}$$

  where $d(p^*) - 1) \geq 0$. Suppose that $d(p^*) > 1$. Then for all i with p^*_i we have $max\{0, g_i(p^*)\} > 0$ or $g_i(p^*) > 0$ and $\sum_{i=1}^{n} p^*_i g_i(p) > 0$. This is contradiction to Walra's Law.
  Hence $d(p^*) = 1$ and we see that $max\{0, g_i(p^*) = 0$ or $g_i(p^*) \leq 0$ for all $i = 1, ..., n$, so $p^* \in \Delta^{n-1}$ is an equilibrium price vector.

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