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Faculty of Applied Physics and Mathematics
Exercises
Partial Differential Equations

Contents

1 Elaborated Topic: Method of characteristics for solving first order PDEs

A first order PDE is an equation which contains $u_x(x,t),\ u_t(x,t)$ and $u(x,t)$. In order to obtain a unique solution we must impose an additional condition, e.g., the values of $u(x,t)$ on a certain line.

1.1 Linear 1st order PDE

A linear 1st order PDE is of the form

$$\vec{a}(x,t)u_x + \vec{b}(x,t)u_t + \vec{c}(x,t)u = \vec{g}(x,t)$$

We assume that $\vec{c}(x,t) \neq 0$, then we can divide by $\vec{c}(x,t)$ and obtain the PDE in the form

$$u_t(x,t) + c(x,t)u_x(x,t) + a(x,t)u(x,t) = g(x,t) \tag{1}$$

We want to consider t as "time", and prescribe for $t = 0$ the initial condition

$$u(x,0) = u_0(x) \tag{2}$$

with a given function $u_0(x)$. The problem consisting of the PDE 1 and the initial condition 2 is called an **initial value problem**.

A **(classical) solution** of the initial value problem is a function $u(x,t)$ such that the derivatives $u_t(x,t),\ u_x(x,t)$ exist, and $u(x,t),\ u_t(x,t),\ u_x(x,t)$ are all continuous. Furthermore the PDE 1 is satisfied for all points (x,t), and the initial condition 2 is satisfied for all x.

1.2 Characteristics

We observe that $u_t(x,t) + c(x,t)u_x(x,t)$ is a directional derivative in the direction of the vector $(c(x,t),1)$ in the (x,t) plane. If we plot all these direction vectors in the (x,t) plane we obtain a direction field. We can find curves $x = X(t)$ fitting into this direction field by solving the ODE

$$X'(t) = c(X(t),t) \tag{3}$$

These curves are called **characteristic curves** or **characteristics**. The characteristic passing through the point $(x_0, 0)$ on the x-axis satisfies the initial condition

$$X(0) = x_0$$

We now consider a solution $u(x,t)$ of the initial value problem determined by 1 and 2, and the characteristic curve $x = X(t)$ with $X(0) = x_0$. Restricting $u(x,t)$ to the characteristic curve defines the function

$$v(t) := u\left(X(t),t\right)$$

We can find $v'(t)$ by using the chain rule

$$\begin{aligned} v'(t) &= u_x\left(X(t),t\right)X'(t) + u_t\left(X(t),t\right) = \\ &= u_x\left(X(t),t\right)c\left(X(t),t\right) + u_t\left(X(t),t\right) \end{aligned}$$

Using the ODE 3. Since $u(x,t)$ satisfies the PDE 1 the function $v(t)$ satisfies the ODE

$$v'(t) + a\left(X(t),t\right)v(t) = g\left(X(t),t\right) \tag{4}$$

and initial condition

$$v(0) = u_0(x_0) \tag{5}$$

The result is that we can solve the PDE by solving a family of 1st order ODEs. For a given point (x,t) we first have to find x_0 so that the corresponding characteristic $X(t)$ passes through (x,t). We then solve the initial value problem 4,5 for the solution v. Then $v(t)$ is the solution $u(x,t)$ in the point (x,t).

1.3 Recipe for solving linear 1st order PDE

We obtain the following recipe for finding the solution $u(x,t)$ of the initial value problem 4,5:

1. Find the characteristics: Solve the initial value problem

   $$X'(t) = c\left(X(t),t\right), \quad X(0) = x_0$$

   Since the solution depends on x_0, we will denote it by $X_{(x_0)}(t)$.

2. For a given point (x,t) find the starting point x_0 of the characteristic through (x,t): For a given x,t solve the equation

   $$X_{(x_0)}(t) = x$$

   for x_0, yielding an expression $x_0 = p(x,t)$.

3. With the function $X(t) = X_{(x_0)}(t)$ from step 1 solve the initial value problem

   $$v'(t) + a\left(X(t),t\right)v(t) = g\left(X(t),t\right), \quad v(0) = u_0(x_0)$$

   Since the solution depends on x_0, we will denote it by $v_{(x_0)}(t)$.

4. Now the solution of the initial value problem 4,5 is obtained by setting $x_0 = p(x,t)$ with the function $p(x,t)$ from step 2:
$$u(x,t) = v_{(x_0)}(t)\big|_{x_0=p(x,t)}$$

Example: Solve the initial value problem

$$u_t + xu_x + u = 3x, \quad u(x_0) = tan^{-1}(x)$$

Here $c(x,t) = x$ and $u_0(x) = tan^{-1}(x)$.

1. Find the characteristics: Solve the initial value problem

$$X'(t) = X(t), \quad X(0) = x_0$$

The general solution of the ODE is $X(t) = Ce^t$, and the initial condition gives

$$X_{(x_0)}(t) = x_0 e^t$$

2. Fora given point (x,t) find the starting point x_0 of the characteristic through (x,t). For given x,t solve the equation $X_{(x_0)}(t) = x$, i.e.
$$x_0 e^t = x$$
for x_0. This gives $x_0 = xe^{-t}$, hence
$$p(x,t) = xe^{-t}$$

3. With $X(t) = x_0 e^t$ solve the initial value problem $v'(t) + a\left(X(t),t\right)v(t) = g\left(X(t),t\right), v(0) = u_0(x_0)$:

$$v'(t) + v(t) = 3x_0 e^t, \quad v(0) = tan^{-1}(x_0)$$

This is a linear in-homogeneous ODE. The solution of the homogeneous ODE is Ce^{-t}. The particular solution is of the form ae^t,plugging this into the ODE yields $ae^t + ae^t = 3x_0 e^t$, hence $a = \frac{3}{2}x_0$. The general solution of the ODE is $v(t) = Ce^{-t} + \frac{3}{2}x_0 e^t$. The initial condition gives $C + \frac{3}{2}x_0 = tan^{-1}(x_0)$ or $C = tan^{-1}(x_0) - \frac{3}{2}x_0$. Therefore we obtain

$$v_{(x_0)}(t) = \left[tan^{-1}(x_0) - \frac{3}{2}x_0\right]e^{-t} + \frac{3}{2}x_0 e^t$$

4. Now we insert $x_0 = p(x,t) = xe^{-t}$ into this expression to obtain the solution:

$$u(x,t) = v_{(x_0)}(t)\big|_{x_0=xe^{-t}} = \left[tan^{-1}(xe^{-t}) - \frac{3}{2}(xe^{-t})\right]e^{-t} + \frac{3}{2}(xe^{-t})e^t =$$
$$= tan^{-1}(xe^{-t})e^{-t} - \frac{3}{2}xe^{-2t} + \frac{3}{2}x$$

1.4 Difficulties

We have reduced the solution of the PDE to the solution of two families of IVPs for ODEs. Does this method always give us a (classical) solution $u(x,t)$? What could possibly go wrong?

1.4.1 Interval where X(t) exists

An ODE initial value problem $v'(t) = f(t, v(t)), v(0) = v_0$ has unique solution $v(t)$ if the function f is sufficiently smooth (*e.g., $f(t, v)$ and $f_v(t, v)$ are continuous*). However, it can happen that the solution only exists on an interval $t \in [0, t_*)$ and it does not exists for $t \geq t_*$.

A simple example is the IVP

$$v'(t) = v(t)^2, \quad v(0) = v_0$$

with $v_0 > 0$. In this case the solution is $v(t) = \frac{1}{a^{-1}-t}$, and the solution exists only for $t < a^{-1}$.

For a given point (x_1, t_1) we need to follow the characteristic backwards to $t = 0$ in order to find $x_0 = p(x_1, t_1)$. This means we are solving the following ODE problem for decreasing t:

$$X'(t) = c\left(X(t), t\right), \quad X(t_1) = x_1$$

For the method of characteristics to work we need that the solution of this IVP exists for $t \in [0, t_1]$. If this is the case for all points (x_1, t_1), we are able to find the characteristic curve $(X(t), t)$ from (x_1, t_1) back to $(x_0, 0)$. We then need to solve the IVP for $v(t) := u\left(X(t), t\right)$. Note that this is a linear ODE, so the solution is guaranteed to exists for all time.

1.4.2 Smoothness of given function $u_0(x)$

For a classical solution we need that the resulting solution $u(x, t)$ has continuous derivatives $u_x(x, t)$ and $u_t(x, t)$. This means that in particular $u_x(x, 0) = u_0'(x)$ must be continuous. Therefore, we need to assume that the given initial function $u_0(x)$ is continuous and has a continuous derivative. If this is the case, and if the above condition on the existence interval of the ODE for $X(t)$ is satisfied one can then show that $u(x, t)$, $u_t(x, t)$ and $u_x(x, t)$ are continuous.

1.5 Method of Characteristics for PDE $u_t + c(x, t)u_x = f(x, t, u)$

Actually, the method of characteristics works in the same way for the more general case of the IVP

$$u_t + c(x, t)u_x = f(x, t, u), \quad u(x, 0) = u_0(x)$$

Note that the right hand side may contain nonlinear terms. Here is how we find one characteristic curve and the solution $u(x, t)$ on this characteristic curve:

Recipe: For a fixed value x_0 do the following:

1. Find the characteristic curve going through $(x_0, 0)$: Solve the IVP

$$X'(t) = c\left(X(t), t\right), \quad X(0) = x_0$$

2. Find the solution restricted to this characteristic curve: We can find the function $v(t) := u\left(X(t), t\right)$ by solving the IVP

$$v'(t) = f\left(X(t), t, v(t)\right), \quad v(0) = u_0(x)$$

Note that the ODE for $v(t)$ may now be nonlinear, so we need to assume that the solution exists for the times $t \in [0, T]$, we are interested in.

1.6 Method of Characteristics for quasi-linear PDE $u_t + c(x,t,u)u_x = f(x,t,u)$

It turns out that we can generalize the method of characteristics to the case of so-called **quasi-linear** 1st order PDEs:

$$u_t + c(x,t,u)u_x = f(x,t,u), \quad u(x,0) = u_0(x) \tag{6}$$

Note that now both the left hand side and the right hand side may contain nonlinear terms. Assume that $u(x,t)$ is a solution of the initial value problem 6. If we already have this solution we can define characteristics by solving the IVP

$$X'(t) = c\left(X(t), t, u\left(X(t), t\right)\right), \quad X(0) = x_0$$

We restrict the solution $u(x,t)$ to this characteristic curve and let $v(t) := u\left(X(t), t\right)$. Then we can write the IVP for $X(t)$ as

$$X'(t) = c\left(X(t), t, v(t)\right), \quad X(0) = x_0 \tag{7}$$

The function $v(t)$ satisfies

$$v'(t) = f\left(X(t), t, v(t)\right), \quad v(0) = u_0(x_0) \tag{8}$$

Note that we need $v(t)$ for 7, and we need $X(t)$ for 8, so the two ODEs are coupled and form a **system of 1st order ODEs**. But we now from the theory of ODEs that an initial value problem for a system of 1st order ODEs has a unique solution. It can also be solved numerically (*e.g., using Euler method or Runge-Kutta method, or ode45 in Matlab*). Here is how we find one characteristic curve and the solution $u(x,t)$ on this characteristic curve:

Recipe: For a fixed value x_0, do the following:
Find the functions $X(t), v(t)$ by solving the following IVP for a system of ODEs:

$$\begin{aligned} X'(t) &= c\left(X(t), t, v(t)\right) \quad X(0) = x_0 \\ v'(t) &= f\left(X(t), t, v(t)\right) \quad v(0) = u_0(x) \end{aligned} \tag{9}$$

This system is nonlinear, so we need to assume that the solution exists for the times $t \in [0,T]$ we are interested in.

We can write the ODE system 9 using the vector notation. Let

$$\vec{w}(t) := \begin{bmatrix} X(t) \\ v(t) \end{bmatrix}$$

so that $X(t) = w_1(t),\ v(t) = w_2(t)$. Then, the system can be written in the form

$$\vec{w}'(t) = \vec{G}\left(t, \vec{w}(t)\right). \quad \vec{w}(0) = \vec{w}^{(0)} \tag{10}$$

where we define

$$\vec{G}(t, \vec{w}) = \begin{bmatrix} c(w_1, t, w_2) \\ f(w_1, t, w_2) \end{bmatrix}, \quad \vec{w}^{(0)} = \begin{bmatrix} x_0 \\ u_0(x) \end{bmatrix} \tag{11}$$

We will consider some examples where this problem is easy to solve. But in general it will not be possible to find a solution with pencil and paper, and we should use a numerical method.

1.7 Solving the ODE system for quasi-linear PDE in Matlab

In Matlab we can use **ode45** to solve an initial value problem numerically.

We first should define the functions $c(x,t,u)$, $f(x,t,u)$ and $u_0(x)$. We use the @-syntax to define functions in Matlab. E.g., for $c(t,x,u) = 4-x$, $f(x,t,u) = u$ and $u_9(x) = e^{-x^2}$ we would use

```
c=@(x,t,u) 4-x;
f=@(x,t,u) u;
u0=@(x) exp(-x^2);
```

We then can define the vector-values function $\vec{G}(t,\vec{w})$ as in 11 by

```
G=@(t,w) [c(w(1),t,w(2));f(w(1),t,w(2))];
```

We pick a starting point x_0 and the times t_j for which I want to compute $X(t_j)$,$v(t_j)$. Let **tval** be the vector containing the values t_j. E.g.

```
x0=0.7;
tval=0:.1:3;
```

Now we can use **ode45** to solve the ODE system 10 by

```
[ts,ws]=ode45(G,tval,[x0;u0(x0)]);
```

In this case **ts** is the same as **tval**. The array **ws** has two columns, the first column contains the values of $X(t_j)$, the second column contains the values of $v(t_j)$:

```
X=ws(:,1);   v)ws(:,2);
```

We can now plot the characteristic consisting of the points $(X(t_j),t_j)$:

```
plot(X,tval,'.-')
```

The vector **v** now contains the solution values at those points, i.e., $v_j = u\,(X(t_j),t_j)$.

In order to find the solution at more points we just repeat this procedure for many starting points x_0.

Let's create a m-file called **quasilin** with the main code. In it we specify a vector **xval** of initial points x_0. We then have a loop iterating over those points. At the end we obtain an array **X** and an array **v** where each row corresponds to one value of x_0. Each column corresponds to one value of t_j. Finally, we plot all trajectories together, and we plot the graph of the solution u at time t_j by plotting the j-th column of **v** vs. the j-th column of **X**

```
function [X,v] = quasilin(c,f,u0,xval,tval)
% [X,v] = quasilin(c,f,u0,xval,tval)
% solve quasilinear 1st order PDE
%   u_t + c(x,t,u)*u_x = f(x,t,u),     u(0)=u0(x)
% using method of characteristics (solving ODEs with ode45)
%
% inputs:
%   c:     coefficient fct c, define by c = @(x,t,u) ...
%   f:     right hand side fct f, define by f = @(x,t,u) ...
%   u0:    initial condition fct u0, define by u0 = @(x) ...
%   xval:  values for x0, define by xval = xmin:step:xmax
%   tval:  values for t,  define by tval = tmin:step:tmax
% outputs:
%   X: x-values of characteristics
%   v: u-values of solution
% X,v are arrays where row index corresponds to x0-values xval
%                      column index corresponds to t-values tval
% at time tval(j) the x-values are X(:,j), the u-values are v(:,j)
% to plot the solution u vs. x at time tval(3) use  plot(X(:,3),v(:,3))
%
% Example: Burgers equation u_t + u*u_x = 0, u(x,0)=atan(x)
%   c = @(x,t,u) u;        % c(x,t,u)=u
%   f = @(x,t,u) 0;        % f(x,t,u)=0
```

```
%   u0 = @(x) atan(x);     % initial condition u0(x)=atan(x)
%  [X,v] = quasilin(c,f,u0,-5:.1:5,0:.1:3);

% This uses colorcurves.m
% Download colorcurves.m from the course web page!

Nx = length(xval);
Nt = length(tval);
X = zeros(Nx,Nt);
v = zeros(Nx,Nt);
%  need to solve ODE system  X'=c(X,t,v), v'=F(X,t,v)
%  let w = (X,v), i.e., w(1)=X, w(2)=v
%  then we can write ODE system as w'=G(t,w)
G = @(t,w) [ c(w(1),t,w(2)) ; f(w(1),t,w(2)) ]; % define fct G
for i=1:Nx
  x0 = xval(i);
  [ts,ws] = ode45(G,tval,[x0;u0(x0)]); % solve system of ODEs w'=G(t,w)
  X(i,:) = ws(:,1);   % 1st column of ws are values of X, store as row
  v(i,:) = ws(:,2);   % 2nd column of ws are values of v, store as row
end
% each row of array X contains solution for one value of x0
% each row of array v contains solution for one value of x0

% plot characteristics in (x,t) plane
figure(1)
plot(X',tval,'k')     % plot rows of X vs tval
xlabel('x'); ylabel('t')
title('characteristics')

% plot solutions u(x,t_j) vs x, for all times
figure(2)
colorcurves(X,tval,v)  % plot columns of v vs. columns of X
xlabel('x'); ylabel('t'); zlabel('u')
title('Rotate this graph with the mouse')
```

Usage of quasilin:

```
[X,v]=quasilin(c,f,u0,xval,tval);
```

Here **c**, **f**, **u0** are functions which should be defined using **@**:

```
c=@(x,t,u)...; f=@(x,t,u)...; u0=@(x)...;
```

xval is the vector of x_0-values, **tval** is the vector of t-values.

X,v are the arrays containing the coordinates of the characteristics and the values of v as explained above. We can plot the solution at time t_j by plotting column j of array **v** vs. column j of array **X**. E,g,. to plot the solution at time tval(3) we use

```
plot(X(:,3),v(:,3)1@@);
```

Example of quasilin:

Consider

$$u_t + (4-x)u_x = u, \quad u(x,0) = e^{-x^2}$$

Here,

$$c(t,x,u) = 4-x, \quad f(t,x,u) = u, \quad u_0(x) = e^{-x^2}$$

```
c=@(x,t,u) 4-x;
f=@(x,t,u) u;
u0=@(x) exp(-x^2);
[X,v]=quasilin(c,f,u0,-5:.1:5,0:.1:1);
```

2 Exercises

1. **Solve the following equation** $xuu_X + yuu_y = x^2 + u^2$

   We have a non-homogeneous EDP of first order which will be solved by characteristic method. We can write our equation as

$$xu\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = x^2 + u^2$$

so it's in the form

$$P(x,y,z)\frac{\partial u}{\partial x} + Q(x,y,z)\frac{\partial u}{\partial y} + R(x,y,z)\frac{\partial u}{\partial z} = 0$$

Now, the first step of the method is to express it as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \implies \frac{dx}{xu} = \frac{du}{yu} = \frac{du}{x^2+u^2}$$

We can integrate each member of the first equality

$$\frac{dx}{xu} = \frac{dy}{uy} \implies \frac{1}{u}\int\frac{dx}{x} = \frac{1}{u}\int\frac{dy}{y} \implies e^{1/u}\left(\frac{x}{y}\right) = c_1$$

$$\begin{aligned}
uv &= x \\
dx &= duv + dvu \qquad & xu &= u^2 v \\
\frac{dx}{du} &= v + \frac{dv}{du}u \qquad & x^2 &= u^2v^2 \\
\frac{dx}{xu} &= \frac{du}{x^2+u^2} \qquad & \frac{dx}{du} &= v + \frac{dv}{du}u \\
\frac{v + \frac{dv}{du}u}{u^2 v} &= \frac{1}{u^2(1+v^2)} \\
1 + \frac{dv}{du}u &= \frac{1}{1+v^2} \\
\frac{dv}{du}u &= \frac{-v^2}{1+v^2} \\
\int \frac{1}{u}du &= -\int\frac{1+v^2}{v^2}dv \\
ln(u) &= v - \frac{2}{v^3} \\
c_2 &= \frac{2}{v^3} + ln(u) - v \\
v &= \frac{x}{u} \qquad & c_2 &= \frac{u^3}{x^3} + ln(u) - \frac{x}{u}
\end{aligned}$$

We now that $c_1 = \varphi_1,\ c_2 = \varphi_2$, so the solution is

$$\boxed{u(x,y,z) = F(\varphi_1,\varphi_2)} = F\left(e^{1/u}\left(\frac{x}{y}\right), \frac{u^3}{x^3} + ln(u) - \frac{x}{u}\right)$$

2. **Derive the solutions of the problem**

$$u_{xy} + tan(y)u_x = 2xtan(y);$$

$$u(x,0) = x^2 + e^{x^3}, u(0,y) = y^{10} + cos(y)$$

We can integrate both sides of the equation by x. This would give us:

$$\int u_{xy} + tan(y)u_x \, dx \ = u_y + tan(y)u + C_1(y)$$

$$\int 2xtan(y) \, dx \ = x^2 tan(y)$$

Going back to the equation we end up with

$$u_y + tan(y)u = x^2 tan(y) + C_1(y)$$

We will now transform the EDP into an ODE using a variable exchange:

$$u(x,y) = v(y)$$

$$u_y = v'(y)$$

This gives us the ODE:

$$v' = tan(y)(x^2 - v) => -\frac{v'}{v - x^2} = tan(y)$$

We can now integrate to solve which gives

$$ln|\frac{1}{v - x^2}| = ln|sec(y)| + C_2(x) => ln|\frac{cos(y)}{v - x^2}| = C_2(x)$$

Reordering the terms and elevating to the power of e we finally end up with

$$v = cos(y)C_2(x) + x^2$$

Now we reverse the variable exchange and so we have the general solution respect $C_1(y), C_2(x)$:

$$u(x,y) = cos(y)C_2(x) + x^2 + C_1(y)$$

We now apply the initial conditions

$$u(x,0) = x^2 + e^{x^3} = C_2(x) + x^2 + C_1(0) => e^{x^3} = C_2(x) + C_1(0)$$

$$C_2(x) = e^{x^3}; C_1(0) = 0$$

Same process with the other condition

$$u(0,y) = cos(y)C_2(0) + C_1(y) = y^{10} + cos(y)$$

$$C_2(0) = 1; C_2(y) = y^{10}$$

So the solution subjected to our initial conditions is

$$u(x,y) = cos(y)e^{x^3} + x^2 + y^{10}$$

3. **Solve the following problem**

$$u_{xx} + u_{xy} = 0, u(x,0) = x^3 . u(0,y) = y^5$$

We can firstly try integrating by x

$$\int u_{xx} + u_{xy}\, dx => u_x + u_y = C(y)$$

So we wan to solve by characteristic curves

$$\begin{aligned} x(s,t) &\to x_t = 1 \\ y(s,t) &\to y_t = 1 \end{aligned}$$

$$u_t = C(y) \to \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

This means that

$$\begin{aligned} x &= t + C_1(s) \\ y &= t + C_2(s) \\ u &= C_3(s) \end{aligned}$$

$$t = x - C_1 = y - C_2 \to y - x = C_4(s) \to s = C_4^{-1}(y-x)$$

This constant exists because $C_4 \in C^1$ Last but not least

$$u = C_3(C_4^{-1}(y-x)) \to u = C_5(y-x)$$

Now we substitute with our initial conditions

$$u(0,y) = C(y) = y^5 \to u(x,y) = (y-x)^5$$

4. **Find (graph) the shape of the string given by the equation $u_{tt} = u_{xx}$ of the following times $t = 0,\ t = \frac{1}{2a},\ t = \frac{1}{a}$, if the initial position is**

$$\varphi(x) = \frac{1}{4x^2+1}$$

and the initial velocity vector is zero.

Notice that

$$\varphi'(x) = -\frac{8x}{(4x+1)^2} \quad \varphi''(x) = \frac{8(12x^2-1)}{(4x^2+1)^3}$$

so, $\varphi \in C^2(\mathbb{R})$ and $\psi(x) = 0 \in C^1(\mathbb{R}$. We are going to use the **d' Alambert Theorem:** Let $\varphi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$. Then, there exists a unique solution $u \in C^2\left(\mathbb{R} \times [0,+\infty)\right)$ of the initial value problem

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{12}$$

with $u(x,0)=\varphi(x),\ \frac{du}{dt}(x,0)=\psi(x),\ x\in\mathbb{R}$, which is given by

$$u(x,t)=\frac{1}{2}\left[\varphi(x-at)+\varphi(x+at)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(s)ds \tag{13}$$

If we express $u_{tt}=u_{xx}$ as 12 we have that

$$0=\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}$$

so, we can see that, for our problem $a=0,\ \psi(x)=0$. If we use the previous theorem, we have that the solution is of the form of 13, so:

$$u(x,t)=\frac{1}{2}\left[\varphi(x-t)+\varphi(x+t)\right]+\frac{1}{2}\int_{x-t}^{x+t}0ds=\frac{1}{2}\left[\varphi(x-t)+\varphi(x+t)\right] \tag{14}$$

Let's calculate it:

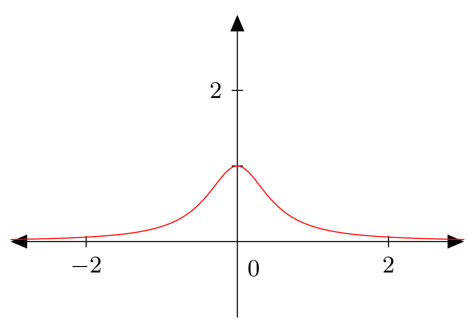
$$\varphi(x-t)=\frac{1}{4(x-t)^2+1}=\frac{1}{4x^2+4t^2-8xt+1},\quad \varphi(x+t)=\frac{1}{4(x+t)^2+1}=\frac{1}{4x^2+4t^2+8xt+1}$$

So, using 14,

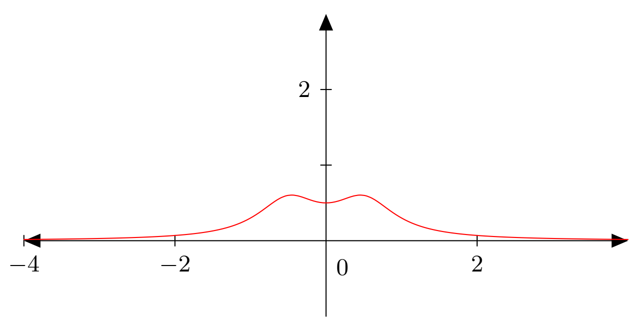
$$\boxed{u(x,t)=\frac{1}{2}\left[\frac{1}{4(x-t)^2+1}+\frac{1}{4(x+t)^2+1}\right]}=\frac{1}{2}\left[\frac{1}{4x^2+4t^2-8xt+1}+\frac{1}{4x^2+4t^2+8xt+1}\right]$$

We can check that it verifies that $u(x,0)=\varphi(x)$ and $u_t(x,0)=\psi(x)$. Let's draw now the graphs for each time:

- $\underline{t=0}$
  We know that, for $t=0$, $u(x,0)=\varphi(x)$

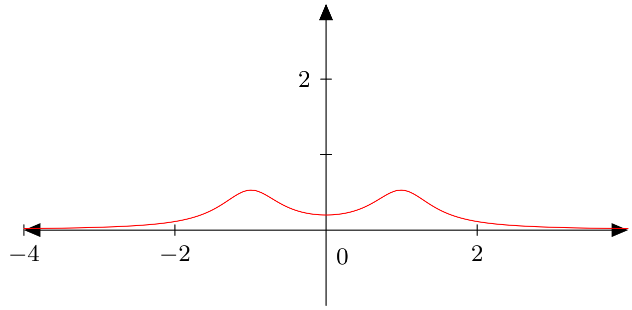


- $\underline{t=\frac{1}{2a}=\frac{1}{2}}$
  We have that $u(x,1/2)=\frac{1}{2}\left[\frac{1}{4(x-1/2)^2+1}+\frac{1}{4(x+1/2)^2+1}\right]$



- $\underline{t = \frac{1}{a} = 1}$

  We have that $u(x,1) = \frac{1}{2}\left[\frac{1}{4(x-1)^2+1} + \frac{1}{4(x+1)^2+1}\right]$



5. **Solve the Cauchy Problem**

$$u_{tt} = a^2 u_{xx} + cos(x), u(x,0) = sin(x), u_t(x,0) = 1 + x.$$

Using Theorem 2 from W4notes we can prepare the following formula

$$u_{xx} - \frac{u_{tt}}{a^2} = -\frac{cos(x)}{a^2}.$$

We now express the solution as

$$u(x,t) = \frac{1}{2}(sin(x+at) + sin(x-at) + \frac{1}{2a}\int_{x-at}^{x+at} 1 + s\,ds \ + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} cos(s)\,ds\,d\tau$$

We will now develop each integral

(a)

$$\frac{1}{2a}\int_{x-at}^{x+at} 1 + s\,ds = \frac{1}{2a}[s + \frac{s^2}{2}]_{x-at}^{x+at} = \frac{1}{2a}(x + at + \frac{(x+at)^2}{2} - x + at - \frac{(x-at)^2}{2} =$$
$$= \frac{1}{2a}(2at + 2at) = 2t$$

(b)

$$\frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} cos(s)\,ds\,d\tau = \frac{1}{2a}\int_0^t [sin(s)]_{x-a(t-\tau)}^{x+a(t-\tau)}\,d\tau =$$
$$= \frac{1}{2a}\int sin(x+a(t+\tau)) - sen(x+a(t-\tau)\,d\tau = \frac{1}{2a}[-cos(x+a(t+\tau)) - cos(x+a(t-\tau))]_0^t =$$
$$= \frac{1}{2a}(-cos(x+2at) - cos(x) + 2cos(x+at)$$

Now we combine everything and thus we have

$$u(x,t) = \frac{1}{2}(sin(x+at) - sin(x-at)) + 2t + \frac{1}{2a}(2cos(x+at) - cos(x+2at) - cos(x)$$

6. **Find a function that is harmonic inside the unit circle $x^2 + y^2 < R$ and satisfies the boundary condition, i.e, $u_{|r=R} = cos\varphi$**

This is a Dirichlet problem with circular domain. We have that $r = R$ and $u(r,\varphi) = cos(\varphi) \implies f(\varphi) = cos(\varphi)$.

The Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x, y) \in \Omega$$

takes the following form in polar coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0, \quad 0 < r < R,\ 0 \leq \varphi \leq 2\pi$$
$$u(R, \varphi) f(\varphi) = cos(\varphi), \quad 0 \leq \varphi \leq 2\pi$$

The solution has the following form:

$$u(r, \varphi) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n cos(n\varphi) + B_n sin(n\varphi) r^n$$

where,

$$A_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\varphi) cos(n\varphi) d\varphi, \quad B_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\varphi) sin(n\varphi) d\varphi, \quad n = 1, 2...$$

Let's calculate it:

$$A_0 = \frac{1}{\pi R}\int_{-\pi}^{\pi} cos(\varphi) d\varphi = \left.\frac{sen(\varphi)}{\pi R}\right]_{-\pi}^{\pi} = 0$$

$$A_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\varphi) cos(n\varphi) d\varphi = \begin{cases} \frac{1}{\pi R^n} \left.\frac{cos(\varphi)sin(\varphi)+\varphi}{2}\right|_{-\pi}^{\pi} \\ \frac{1}{\pi R^n} \left.\frac{n cos(\varphi) sin(n\varphi) - sin(\varphi) cos(n\varphi)}{n^2-1}\right|_{-\pi}^{\pi} \end{cases}$$

So we have that $A_1 = \frac{1}{R}$ and $A_n = 0, \forall n \neq 1$. Also $B_n = 0, \forall n$ because we have a even function. So the solution would be:

$$\boxed{u(r\varphi) = \frac{r}{R} cos(\varphi)}$$