① Let $A$ be a random matrix with distribution $W_p(n, I_p)$. Prove that $tr(A)$ has distribution $\chi^2_{np}$.

From the result about marginalization by diagonalization, we know that, if $A \sim W_p(n, I_p) \Rightarrow a_{ii} \sim \chi_n^2$, $i = 1 \dots p$.

On the other hand, $tr(A) = \sum_{i=1}^{p} a_{ii}$, so, we want to prove the independence of $a_{ii}$, in order to use reproductive's results.

From definition, $A \overset{d}{=} \sum_{\alpha=1}^{n} Z_\alpha Z_\alpha'$, with $Z_\alpha \sim N_p(0, \underset{I_p}{\underset{\shortparallel}{\Sigma}})$, independents, $n \geq p$.

For each $\alpha$, we can write $Z_\alpha = \begin{pmatrix} Z_{\alpha 1} \\ \vdots \\ Z_{\alpha p} \end{pmatrix}$ and, because $Z_\alpha \sim N_p(0, I_p)$, we have that $Z_{\alpha 1}, \dots, Z_{\alpha p} \sim N_1(0, 1)$ and are independents, $\forall \alpha = 1, \dots, n$.

Now, we can write $a_{ii} \overset{d}{=} \sum_{\alpha=1}^{n} Z_{\alpha i}^2$, $i = 1, \dots, p$ and it is clear that they are independents $\Rightarrow tr(A) = \sum_{i=1}^{p} a_{ii} \sim \chi^2_{np}$ $\quad \left( np = \sum_{i=1}^{p} n \right)$

② Let $A$ be a random matrix with distribution $W_p(n, \Sigma)$ $(\Sigma > 0)$. Prove that, for any vectors $a, b \in \mathbb{R}^p$, the random variables $a'Aa$ and $b'Ab$ are independents if and only if $a'\Sigma b = 0$.

If one (or both) of the vectors is null, we have that (suppose is $a$) $a'Aa = 0$, it is a degenerated variable, so it is independent from any other variable and it's trivial that $a'\Sigma b = 0$ (the same if you take $b = 0$ instead of $a$).

Now, consider $a, b \in \mathbb{R}^p \setminus \{0\}$. We have then 2 cases:

1) $\exists \lambda \in \mathbb{R}$ such that $a = \lambda b$:

$a'Aa = \lambda b' A \lambda b = \lambda^2 b'Ab \Rightarrow a'Aa$ and $b'Ab$ are not independents

Therefore, $a'\Sigma b = \underset{\underset{0}{\neq}}{\lambda} \underset{\underset{0}{\vee}}{b'\Sigma b} \neq 0$ $\quad (\Sigma > 0)$

2) The vectors $a, b$ are independents:

In this case, consider $M = (a\ b)_{p \times 2}$ of rank $2 \leq p$ (because $a, b$ independents)

Therefore, we have that:

$$M'AM = \begin{pmatrix} a' \\ b' \end{pmatrix}_{2\times p} A_{p\times p} \, (a \; b)_{p\times 2} = \begin{pmatrix} a'Aa & a'Ab \\ b'Aa & b'Ab \end{pmatrix}_{2\times 2}$$

Use the property of reproductive by rectangular linear transformations:

$$M'AM \sim W_2(n, M'\Sigma M) = W_2\left(n, \begin{pmatrix} a'\Sigma a & a'\Sigma b \\ b'\Sigma a & b'\Sigma b \end{pmatrix}\right)$$

$\boxed{\Rightarrow}$ If $a'Aa$ and $b'Ab$ are independents. Then,

$$0 = \underset{\substack{\downarrow \\ 2^o \text{ order moment}}}{Cov}(a'Aa, b'Ab) = n\left((a'\Sigma b)^2 + (a'\Sigma b)^2\right) \Rightarrow a'\Sigma b = 0$$

$$Cov(a_{11}, a_{22}) = n(\sigma_{12}\sigma_{12} + \sigma_{12}\sigma_{12})$$

$\boxed{\Leftarrow}$ If $a'\Sigma b = 0$, using marginalization by blocks' result, we have that $a'Aa$ and $b'Ab$ are independents.

3. Prove that if $X_1 \dots X_N$ constitute a simple random sample distribution $N_p(\mu, \Sigma)$ $(\Sigma > 0)$, and the mean vector parameter $\mu$ is known, then the maximum-likelihood estimator of $\Sigma$ is $\hat{\Sigma} := \frac{1}{N}\sum_{\alpha=1}^{N}(X_\alpha - \mu)(X_\alpha - \mu)'$. Check if it is unbiased or not.

Let's find the maximum of $\ln(L(\mu, \Sigma))$, respect $\Sigma$, knowing $\mu$:

$$\ln(L(\mu, \Sigma)) = \ln(L(\Sigma)) = -\frac{pN}{2}\ln(2\pi) - \frac{N}{2}\ln(|\Sigma|) - \frac{1}{2}\sum_{\alpha=1}^{N}(x_\alpha - \mu)'\Sigma^{-1}(x_\alpha - \mu)$$

Because $\sum_{\alpha=1}^{N}(x_\alpha - \mu)'\Sigma^{-1}(x_\alpha - \mu)$ is an scalar, we can consider its trace:

$$tr\left(\sum_{\alpha=1}^{N}(x_\alpha - \mu)'\Sigma^{-1}(x_\alpha - \mu)\right) = tr\left(\Sigma^{-1}\sum_{\alpha=1}^{N}(x_\alpha - \mu)(x_\alpha - \mu)'\right)$$

Therefore, we have that:

$$\ln(L(\Sigma)) = -\frac{pN}{2}\ln(2\pi) - \frac{N}{2}\ln(|\Sigma|) - \frac{1}{2}tr\left(\Sigma^{-1}\sum_{\alpha=1}^{N}(x_\alpha - \mu)(x_\alpha - \mu)'\right)$$

Take $f(\Sigma) = 2\left[\ln(L(\Sigma)) + \frac{pN}{2}\ln(2\pi)\right]$ and applying Watson's lemma, we have that it has only one maximum in $\frac{1}{N}\sum_{\alpha=1}^{N}(x_\alpha - \mu)(x_\alpha - \mu)'$. It is easy to see that $f(\Sigma)$ and $\ln(L(\Sigma))$ reaches the same maximum because is a ~~[illegible]~~ traslation in the plane $xy$.

Let's see it is unbiased:

$$E[\hat{\Sigma}] = \frac{1}{N} E\left[\sum_{\alpha=1}^{N} (x_\alpha-\mu)(x_\alpha-\mu)'\right] \underset{\text{linearity}}{=} \frac{1}{N} \sum_{\alpha=1}^{N} E[(x_\alpha-\mu)(x_\alpha-\mu)'] =$$

$$= \frac{1}{N} \sum_{\alpha=1}^{N} \Sigma = \Sigma \quad \Rightarrow \text{It's unbiased}$$

4 Let $A$ be a random matrix with distribution $W_p(n,\Sigma)$ $(\Sigma > 0)$.

Prove that, using wishart density function,

$$E[|A|^r] = |\Sigma|^r \cdot 2^{pr} \cdot \frac{\Gamma_p(\frac{1}{2}n+r)}{\Gamma_p(\frac{1}{2}n)}, \quad \forall r > 0$$

Let $\Delta_p$ be the set of real-symmetric-positive-definite matrices of dim $p$. By definition:

$$E[|A|^r] = \int_{\Delta_p} |A|^r \cdot \frac{|A|^{\frac{n-p-1}{2}} \exp\{-\frac{1}{2} tr(\Sigma^{-1}A)\}}{2^{\frac{np}{2}} |\Sigma|^{n/2} \Gamma_p(\frac{n}{2})} dA =$$

$$= \int_{\Delta_p} \frac{|A|^{\frac{(n+2r)-p-1}{2}} \exp\{-\frac{1}{2}tr(\Sigma^{-1}A)\}}{2^{\frac{np}{2}} |\Sigma|^{n/2} \Gamma_p(\frac{n}{2})} \cdot \frac{|\Sigma|^r 2^{pr} \Gamma_p(\frac{1}{2}n+r)}{|\Sigma|^r 2^{pr} \Gamma_p(\frac{1}{2}n+r)} dA =$$

$$= \int_{\Delta_p} \frac{|A|^{\frac{(n+2r)-p-1}{2}} \exp\{-\frac{1}{2}tr(\Sigma^{-1}A)\}}{2^{\frac{p(n+2r)}{2}} |\Sigma|^{\frac{n+2r}{2}} \Gamma_p(\frac{n+2r}{2})} \cdot \frac{2^{pr} |\Sigma|^r \Gamma_p(\frac{n+2r}{2})}{\Gamma_p(\frac{n}{2})} dA =$$

$$= \frac{2^{pr} |\Sigma|^r \Gamma_p(\frac{n+2r}{2})}{\Gamma_p(\frac{n}{2})} \underbrace{\int_{\Delta_p} \frac{|A|^{\frac{(n+2r)-p-1}{2}} \exp\{-\frac{1}{2}tr(\Sigma^{-1}A)\} dA}{2^{\frac{p(n+2r)}{2}} |\Sigma|^{\frac{n+2r}{2}} \Gamma_p(\frac{n+2r}{2})}}_{1 \text{ because density function } W_p(n+2r,\Sigma)} =$$

$$= \frac{2^{pr} |\Sigma|^r \Gamma_p(\frac{n+2r}{2})}{\Gamma_p(\frac{n}{2})}$$

⑤ Let $X_1, \dots, X_N$ be a simple random sample of a distribution $N_p(\mu, \Sigma)$ $(\Sigma > 0)$. Let $A = \sum_{\alpha=1}^{N} (X_\alpha - \bar{X})(X_\alpha - \bar{X})'$ be the sample dispersion matrix, with $\bar{X} = \frac{1}{N}\sum_{\alpha=1}^{N} X_\alpha$, the sample mean vector.

a) Prove that the matrix $\mathrm{Cov}(\bar{X}, X_\alpha - \bar{X}) = 0$, for $\alpha = 1, \dots, N$. Deduce $\bar{X}$ and $A$ are independents.

Consider the vector $\begin{pmatrix} \bar{X} \\ X_\alpha - \bar{X} \end{pmatrix}_{2p\times 1}$ for each $\alpha = 1 \dots N$. Then,

$$\begin{pmatrix} \bar{X} \\ X_\alpha - \bar{X} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{N} I_p & \cdots & \frac{1}{N} I_p & \cdots & \frac{1}{N} I_p \\ -\frac{1}{N} I_p & \cdots & \underbrace{\frac{N-1}{N} I_p}_{\alpha\text{'s column}} & \cdots & -\frac{1}{N} I_p \end{pmatrix}_{(2p\times Np)}}_{= M_\alpha,\ \mathrm{rank}(M_\alpha) = 2p} \begin{pmatrix} X_1 \\ \vdots \\ X_\alpha \\ \vdots \\ X_p \end{pmatrix}_{(Np\times 1)}$$

Because independence of $X_\alpha \sim N_p(\mu, \Sigma)$, we have:

$$\begin{pmatrix} X_1 \\ \vdots \\ X_\alpha \\ \vdots \\ X_p \end{pmatrix} \sim N_{pN}(\tilde{\mu}, \tilde{\Sigma}), \quad \tilde{\mu} = \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix}_{(Np\times 1)}, \quad \tilde{\Sigma} = \mathrm{diag}(\Sigma \cdots \Sigma)_{(Np\times Np)}$$

So, using the result of linearity transformations for non-singular MND

$$\begin{pmatrix} \bar{X} \\ X_\alpha - \bar{X} \end{pmatrix} \sim N_{2p}(M_\alpha \tilde{\mu}, M_\alpha \tilde{\Sigma} M_\alpha'), \quad \forall \alpha = 1 \dots N$$

$$M_\alpha \tilde{\Sigma} M_\alpha' = \begin{pmatrix} \frac{1}{N} I_p & \cdots & \frac{1}{N} I_p & \cdots & \frac{1}{N} I_p \\ -\frac{1}{N} I_p & \cdots & \frac{N-1}{N} I_p & \cdots & -\frac{1}{N} I_p \end{pmatrix}_{(2p\times Np)} \begin{pmatrix} \Sigma & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & \Sigma & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & \Sigma \end{pmatrix}_{(Np\times Np)} \begin{pmatrix} \frac{1}{N} I_p & -\frac{1}{N} I_p \\ \vdots & \vdots \\ \frac{1}{N} I_p & \frac{N-1}{N} I_p \\ \vdots & \vdots \\ \frac{1}{N} I_p & -\frac{1}{N} I_p \end{pmatrix}_{(Np\times 2p)} =$$

$$= \begin{pmatrix} \frac{1}{N}\Sigma & \cdots & \frac{1}{N}\Sigma & \cdots & \frac{1}{N}\Sigma \\ -\frac{1}{N}\Sigma & \cdots & \frac{N-1}{N}\Sigma & \cdots & -\frac{1}{N}\Sigma \end{pmatrix}_{(2p\times Np)} \begin{pmatrix} \frac{1}{N} I_p & -\frac{1}{N} I_p \\ \vdots & \vdots \\ \frac{1}{N} I_p & \frac{N-1}{N} I_p \\ \vdots & \vdots \\ \frac{1}{N} I_p & -\frac{1}{N} I_p \end{pmatrix}_{(Np\times 2p)} =$$

$$= \begin{pmatrix} \frac{1}{N}\Sigma & 0_{p\times p} \\ 0_{p\times p} & \frac{N-1}{N}\Sigma \end{pmatrix}_{(2p\times 2p)} \Rightarrow \mathrm{Cov}(\bar{X}, X_\alpha - \bar{X}) = 0_{p\times p}, \ \forall \alpha = 1 \dots N.$$

Therefore, $\bar{X}$ and $X_\alpha - \bar{X}$ are independents $\Rightarrow$ $A$ and $\bar{X}$ independents.

b) Find the matrix $B$ such that $A = X'BX$ with $X = (X_1 \dots X_N)'$ ($(N \times p)$ - dimensional matrix).

$$A = \sum_{\alpha=1}^{N} (X_\alpha - \bar{X})(X_\alpha - \bar{X})' = (X_1 - \bar{X}, \dots, X_N - \bar{X}) \begin{pmatrix} X_1' - \bar{X}' \\ \vdots \\ X_N' - \bar{X}' \end{pmatrix} =$$

$$= (X_1 - \bar{X}, \dots, X_N - \bar{X}) \left[ \begin{pmatrix} X_1' \\ \vdots \\ X_N' \end{pmatrix} - \begin{pmatrix} \bar{X}' \\ \vdots \\ \bar{X}' \end{pmatrix} \right] =$$

$$= (X_1 - \bar{X}, \dots, X_N - \bar{X}) \left[ \begin{pmatrix} X_1' \\ \vdots \\ X_N' \end{pmatrix} - \frac{1}{N} \begin{pmatrix} (\sum_{\alpha=1}^{N} X_\alpha)' \\ \vdots \\ (\sum_{\alpha=1}^{N} X_\alpha)' \end{pmatrix} \right] =$$

$$= \left[ (X_1, \dots, X_N) - \frac{1}{N} \left( \sum_{\alpha=1}^{N} X_\alpha \cdots \sum_{\alpha=1}^{N} X_\alpha \right) \right] \left[ \begin{pmatrix} X_1' \\ \vdots \\ X_N' \end{pmatrix} - \frac{1}{N} \begin{pmatrix} (\sum_{\alpha=1}^{N} X_\alpha)' \\ \vdots \\ (\sum_{\alpha=1}^{N} X_\alpha)' \end{pmatrix} \right] =$$

$$= \left[ (X_1, \dots, X_N) \left( I_N - \frac{1}{N} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}_{N \times N} \right) \right] \left[ \left( I_N - \frac{1}{N} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}_{N \times N} \right) \begin{pmatrix} X_1' \\ \vdots \\ X_N' \end{pmatrix} \right] =$$

$$= X'_{p \times N} B_{N \times N} X_{N \times p}, \text{ where, } B = \left( I_N - \frac{1}{N} \mathbb{1}_{N \times N} \right) \left( I_N - \frac{1}{N} \mathbb{1}_{N \times N} \right).$$

c) Assuming that $N > p$, find some $(p \times p)$ - dimensional matrix $G$ such that $GAG' \sim W_p(N-1, I_p)$

We know from Fisher's theorem, that $A \sim W_p(N-1, \Sigma)$. Using reproductive result for 'rectangular' linear transformations, we have that $GAG' \sim W_p(N-1, G\Sigma G')$, dim of $G$ $p \times p$. We just need to find $G$ s.t. $G\Sigma G' = I_p$. Take a non - singular matrix $C$ of dim $p \times p$ s.t. $\Sigma = CC'$ ($C$ exists because $\Sigma > 0$). Let $G = C^{-1} \Rightarrow G\Sigma G' = C^{-1}CC'(C^{-1})' = I_p$