Game Theory

We call a zero-sum-game, a game that represents a pure conflict between our two players, that's if one gains, the other lose, so is a pure conflict situation.

Dominance: suppose a zero-sum matrix game. Each player has just one choice to make, so we define a pure strategy for Rose to be a choice of a row and for Colin to be the choice of a column.

$$\text{Rose} \quad \begin{array}{c} \text{Colin} \\ \begin{bmatrix} \boxed{2 \quad 1} \\ 3 \quad 0 \\ \boxed{-1 \quad 0} \end{bmatrix} \end{array}$$

No matters what Colin does, Rose's payoff is always higher when she plays row 1 instead of row 3

Row i dominates row i' if every entry in row i is greater or equal to the entry of row i' (because Rose wants the payoff to be as high as possible).
Column j dominates column j' if every entry in column j is less than or equal to entry of column j' (because Colin wants the payoff to be as low as possible).

Theorem: Let A be a zero-sum matrix game, if iterated deletion of dominated strategies reduced A to 1×1 matrix consisting of the entry in position (i,j) of the original matrix $\Rightarrow$ A has a saddle point at (i,j).

Also, a saddle point in a matrix game is an outcome where is less or equal to any entry in its row and greater or equal to any entry in its column.

Colin

Rose	A	B	C	D	Row min	
A	4	3	2	5	2	← maximin
B	-10	2	0	-1	-10	
C	7	5	2	3	2	← maximin
D	0	8	-4	-5	-5	
Column max.	7	8	2	5		
			↑ minimax			

Colin

Rose	A	B		
A	2	-3	-3	
B	0	2	0	← maximin
C	-5	10	-5	
	2	10		
	↑ minimax			

If minimax = maximin $\Rightarrow$ Saddle point strategies
if minimax $\neq$ maximin $\Rightarrow$ No saddle points, Rose assure that she wins at least maximin and Colin assure that Rose wins no more than 2.

Theorem: Any two saddle points in a matrix game have the same value. Furthermore, if Rose and Colin both play strategies containing a saddle point outcome, the result will be a saddle point.

If there is a number v such that Rose has strategy which guarantees she will win at least v and Colin strategy which guarantees Rose will win no more than v, then v is the value of the game

① In the following game, find all cases of dominance among Rose strategies and among Colin strategies

Colin

Rose	A	B	C	D
A	3	-6	2	-4
B	2	1	0	1
C	-4	3	-5	4

M

Initially no row dominance. On the other hand column B dominates D and column C dominates A

② The Dominance Principle can be extended to the Principle of Higher Order Dominance. The idea is that we first cross out any dominated strategies for Rose and for Colin. In the resulting smaller game, some strategies may be dominated, even though they weren't in the original game. Cross them out. Look at the new smaller game and continue until no new dominance appears. Try in the following game:

Colin

Rose	A	B	C	D	E
A	1	1	1	2	2
B	2	1	1	1	2
C	2	2	1	1	1
D	2	2	2	1	0

C C

" Si tu usando A ganas más que usando B, nunca vas a usar B, independientemente de si el resto la dominan "

Colin

Rose	C	D	E
A	1	2	2
~~B~~	1	1	2
~~C~~	1	1	1
D	2	1	0

A
A

→

Colin

Rose	C	D	E
A	1	2	2
D	2	1	0

E

→

Rose	C	E
A	1	2
D	2	0

③ Find all saddle points in the following games. Draw the movement diagrams for the games b) and c)

a)

	A	B	C	D
A	3	2	4	2
B	2	1	3	0
C	2	2	2	2

b)

	A	B	C
A	-2	0	4
B	2	1	3
C	3	-1	-2

c)

	A	B	C
A	4	3	8
B	9	5	1
C	2	7	6

Colin

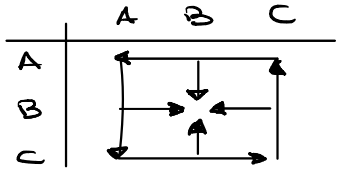
Rose	A	B	C	D	Row min	
A	3	2	4	2	2	← maximin
B	2	1	3	0	0	
C	2	2	2	2	2	← maximin
Col max	3	2	4	2		
		↑ minimax		↑ minimax		

⇒ 2 is a saddle point

1 is a saddle point

		Colin			
		A	B	C	Row min
Rose	A	-2	0	4	-2
	B	2	1	3	1 ← maximin
	C	3	-1	-2	-2
col max		3	1	4	
			↑ minimax		

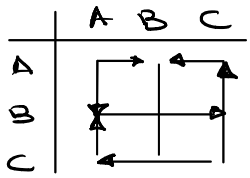
⇒



No saddle point

		Colin			
		A	B	C	Row min
Rose	A	4	3	8	3 ← maximin
	B	9	5	1	1
	C	2	7	6	2
Col max		9	7	8	
			↑ minimax		

⇒



For drawing the movement diagram, in each row we draw an arrow from each entry to the smallest to the smallest entry in that row. In each column we draw arrows to the largest entry in that column.

6 a) In Game 2.1, suppose you were Rose and you know that Colin was playing his strategies in the proportions given in my class results, and that what you did wouldn't change this. Which strategy would be best for you to play? Why? How would you calculate this?

	A	B	C	D		
A	12	-1	1	0	31%	0
B	5	1	7	-20	10%	-20
C	3	2	4	3	49%	2 ← maximin
D	-16	0	0	16	10%	-16
	20%	51%	2%	27%		
	12	2	7	16		
		↑ minimax				

The strategy most used by Colin is B. Also the most requrrent strategy for Rose is C. That's because there is a saddle point at outcome 2, so the movement diagram tends to it. So it is an equilibrium outcome.

The mixed strategies uses fixed probabilities. The expected value of getting payoffs $a_1, \dots, a_k$ with probabilities $p_1, \dots, p_k$ is $p_1 a_1 + \dots + p_k a_k$. If you know that your opponent is playing a given mixed strategy and will continue to play it regardless what you do, you should always play your strategy which has the largest expected value.

		Colin A	Colin B
Rose	A	2	-3
	B	0	3

Rose A: $(2)x - 3(1-x) = -3+5x$

Rose B: $(0)x + 3(1-x) = 3-3x$

$-3+5x = 3-3x \Leftrightarrow x = 3/4$

So if Colin uses $\frac{3}{4}A + \frac{1}{4}B$, he can assure that Rose wins, on average, no more than $\frac{3}{4}$ unit/game.

Colin A: $2(x) + 0(1-x) = 2x$

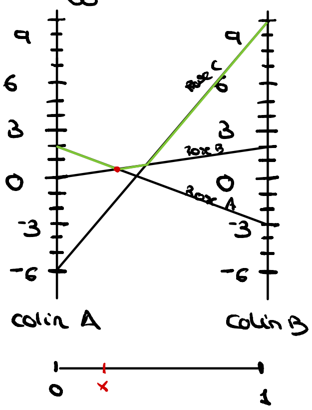
Colin B: $-3(x) + 3(1-x) = 3-6x$

$2x = 3-6x \Leftrightarrow x = 3/8$

So if Rose uses $\frac{3}{8}A + \frac{5}{8}B$, she assures of winning, on average, at least $\frac{3}{4}$ unit/game

So $3/4$ is the value of the game and $\frac{3}{4}A + \frac{1}{4}B$ Colin's optimal strategy and $\frac{3}{8}A + \frac{5}{8}B$ Rose's optimal strategy.

		Colin A	Colin B
Rose	A	2	-3
	B	0	2
	C	-5	10



The vertical coordinate of the line above any point x gives Rose's expected payoff if Colin plays the mixed strategy $(1-x)A, xB$. Colin would want to choose the x to make the corresponding payoff to Rose as small as possible. That's the red point, that is the intersection of Rose A and B so the appropiate subgame to solve is

	A	B	Row diff	Rose odds	Prob
A	2	-3	5	$\lvert -2 \rvert = 2$	2/7
B	0	2	-2	$\lvert 5 \rvert = 5$	5/7
column diff	2	-5		+ 7	
Colin odds	5	2			
Prob	5/7	2/7			

		Colin A	Colin B	Rose prob.
Rose	A	2	-3	2/7
	B	0	2	5/7
Colin Prob.		5/7	2/7	

Rose expectations if Colin play $(5/7)A, (2/7)B$

Rose A: $\frac{5}{7}(2) + \frac{2}{7}(-3) = \frac{4}{7}$

Rose B: $\frac{5}{7}(0) + \frac{2}{7}(2) = \frac{4}{7}$

Rose C: $\frac{5}{7}(-5) + \frac{2}{7}(10) = -\frac{5}{7}$

Colin expectations if Rose play $(2/7)A, (5/7)B, (0)C$

Colin A: $\frac{2}{7}(2) + \frac{5}{7}(0) + 0(-5) = \frac{4}{7}$

Colin B: $\frac{2}{7}(-3) + \frac{5}{7}(2) + 0(0) = \frac{4}{7}$

Hacemos lo mismo si la matriz es $2 \times n$, pero tomando x más alto de la intersección de líneas inferior

<u>Minimax Theorem</u> : Every $m \times n$ matrix game has a solution. That's, $\exists\, v$, called the value of the game, and there are optimal (pure or mixed) strategies for Rose and C such that if Rose plays her optimal strategy, Rose's expected payoff will be $\geq v$, no matter what Colin does and if Colin plays his optimal strategy, Rose's expected payoff will be $\leq v$, no matter what Rose does. Futhermore, the solution can always be found as the solution of $k \times k$ subgame.

(3) Solve the following games :

a)

	A	B
A	-3	5
B	-1	3
C	2	-2
D	3	-6

b)

	A	B
A	-2	5
B	1	2
C	0	-2
D	0	4

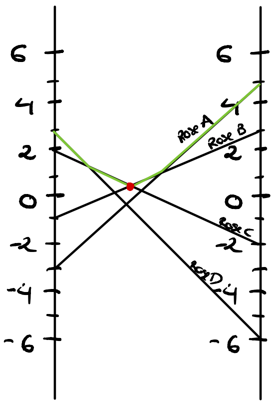
c)

	A	B	C	D	E
A	-4	2	0	3	-2
B	4	-1	0	-3	1

	A	B	
A	-3	5	-3
B	-1	3	-1 ← maximin
C	2	-2	-2
D	3	-6	-6
	3	5	

↑ minimax

No saddle-point



Red point is the intersection between Rose C and Rose B

	A	B	Diff	odds	Prob
B	-1	3	-4	4	1/2
C	2	-2	4	4	1/2
Diff	-3	5			
odds	5	3			
Prob	5/8	3/8			

- Value of the game = $\frac{1}{2}$
- Colin's optimy strategy: $(5/8)$A, $(3/8)$B
- Rose's optimy strategy: (0)A, $(1/2)$B, $(1/2)$C, (0)D

	A	B	
A	-2	5	-2
B	1	2	1
~~C~~	~~0~~	~~-2~~	~~-2~~ D
D	0	4	0
	1	5	

maximin

↑ minimax

1 is a saddle-point

→

	A	B
A	-2	5
B	1	2
D	0	4

A

→

	A
~~A~~	~~-2~~ → B
B	1
~~D~~	~~0~~ → B

→

	A
B	1

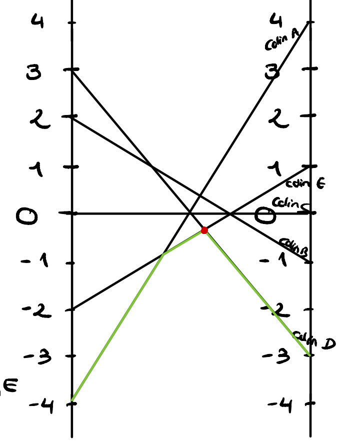
- Value of the game = 1
- Colin's optimy strategy : A
- Rose's optimy strategy: B

	A	B	C	D	E	
A	-4	2	0	3	-2	-4
B	4	-1	0	-3	1	-3 ← maximin
	4	2	0	3	1	

↑ minimax

No saddle-point

- Value of the game = $-\frac{3}{9}$
- Colin's optimy strategy: (0)A, (0)B, (0)C, $(3/9)$D $(6/9)$E
- Rose's optimy strategy: $(4/9)$A, $(5/9)$B



Intersection Colin D and Colin E

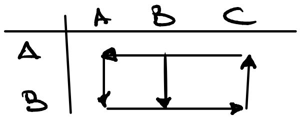
	D	E	
A	3	-2	4/9
B	-3	1	5/9
	3/9	6/9	

④ Some games have more than one solution. The value of the game is fixed, but the players may have several differents strategies which ensure this value.

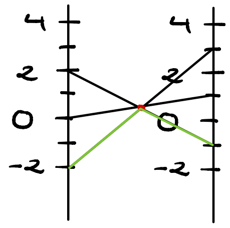
a) Draw the graph of the following game. What happens?

Colin

Rose	A	B	C
A	-2	0	2
B	3	1	-1



2 circles



b) Show that there are two differents optimal strategies for Colin, corresponding to the solutions for two different 2×2 subgames. The third 2×2 subgame does not yield a solution. In the graph, what is different about that subgame?

	A	B	
A	-2	0	-2
B	3	1	1 ← maximin
	3	1 ↑ minimax	

⇒ There is a saddle-point at 1, so Rose's and Colin's optymal play is B.

	B	C	
A	0	2	0 ← maximin
B	1	-1	-1
	1 ↑ minimax	2	

⇒ There is no saddle-point

- value of the game = $\frac{1}{2}$
- Colin's optymal strategy $(3/4)$ B, $(1/4)$ C, (0) A
- Rose's optymal strategy $(1/2)$ A, $(1/2)$ B

	B	C			
A	0	2	-2	2	1/2
B	1	-1	2	2	1/2
	-1	3			
	3/4	1/4			

	A	C	
A	-2	2	-2
B	3	1	1 ← maximin
	3	2 ↑ minimax	

⇒ There is no saddle-point

- Value of the game = $\frac{1}{2}$
- Colin's optymal strategy $(3/8)$ A, $(5/8)$ C, (0) B
- Rose's optymal strategy: $(1/2)$ A, $(1/2)$ B

	A	C	
A	-2	2	1/2
B	3	1	1/2
	3/8	5/8	

⑤ Solve the following games

a)

	A	B	C
A	3	0	1
B	-1	2	2
C	1	0	-1

b)

	A	B	C
A	5	2	1
B	4	1	3
C	3	4	3
D	1	6	2

c)

	A	B	C	D
A	4	-3	2	-4
B	4	-4	4	-2
C	0	1	-3	2
D	-5	2	-7	2
E	3	-2	2	-2

	A	B	C	
A	3	0	1	0 ← maximin
B	-1	2	2	-1
C	1	0	-1	-1
	3	2 ↑	2 ↑	

minimax

⇒ There is no saddle-point. Suppose Colin plays Colin A, B, C with probabilities $(x, y, 1-x-y)$

Rose A: $x(3) + y(0) + (1-x-y)(1) = 2x - y + 1$

Rose B: $x(-1) + y(2) + (1-x-y)(2) = -3x + 2$

Rose C: $x(1) + y(0) + (1-x-y)(-1) = 2x + y - 1$

$$\left.\begin{array}{l} 2x - y + 1 = -3x + 2 \Rightarrow 5x - y = 1 \\ 2x + y - 1 = -3x + 2 \Rightarrow 5x + y = 3 \end{array}\right\} \Rightarrow x = \frac{1}{2},\ y = 1$$

No sale porque hay dominancia!!

	A	B	C
A	3	0	1
B	-1	2	2
C	1	0	-1

→ A

$\longrightarrow$

	A	B	C
A	3	0	1
B	-1	2	2

C → B

$\longrightarrow$

	A	B	
A	3	0	1/2
B	-1	2	1/2
	1/3	2/3	

- Value of the game = 1
- Colin's optimy strategy: (1/3) A, (2/3) B, (0) C
- Rose's optimy strategy: (1/2) A, (1/2) B

	A	B	C	
A	5	2	1	1
B	4	1	3	1
C	3	4	3	3 ← maximin
D	1	6	2	1
	5	6	3 ↑ minimax	

⇒ There is a saddle-point at 3, so the optimal play for both players is C, but there is no dominance.

	A	B	C	D	
A	4	-3	2	-4	-4
B	4	-4	4	-2	-4
C	0	1	-3	2	-3
D	-5	2	-7	2	-7
E	3	-2	2	-2	-2 ← maximin
	4	2	4	2	

A → C; minimax ↑ under B and D

⇒ There is no saddle-point

$\longrightarrow$

	B	C	D
A	-3	2	-4
B	-4	4	-2
C	1	-3	2
D	2	-7	2
E	-2	2	-2

A → E

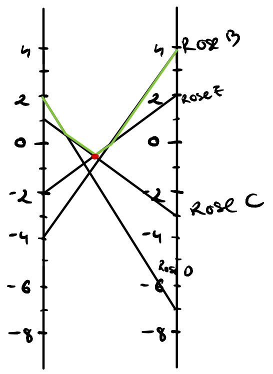
$\longrightarrow$

	B	C	D
B	-4	4	-2
C	1	-3	2
D	2	-7	2
E	-2	2	-2

D → B

$\longrightarrow$

	B	C
B	-4	4
C	1	-3
D	2	-7
E	-2	2



intersection Rose's E and Rose's C

	B	C	
C	1	-3	1/2
E	-2	2	1/2
	5/8	3/8	

- Value of the game = $-\frac{1}{2}$
- Rose's optimy strategy: (0) A, (0) B, (1/2) C, (0) D, (1/2) E
- Colin's optimy strategy: (0) A, (5/8) B, (3/8) C, (0) D

⑥ Solve the following game   No saddle-point

	A	B	C	D	E	F	
A	4	-4	3	2	-3	3	-4
B	-1	-1	-2	0	0	4	-2 ← maximin
C	-1	2	1	-1	2	-3	-3
	4	2	3	2	2	4	

E → B; minimax ↑ under B, D, E

	B	D	F
A	-4	2	3
B	-1	0	4
C	2	-1	-3

Note: Rose ABC vs Colin BDF

Rose A: $x(-4) + y(2) + (1-x-y)(3) = -7x - y + 3$

Rose B: $x(-1) + y(0) + (1-x-y)(4) = -5x - 4y + 4$

Rose C: $x(2) + y(-1) + (1-x-y)(-3) = 5x + 2y - 3$

$$\left.\begin{array}{l} -7x - y + 3 = -5x - 4y + 4 \Rightarrow 2x + 3y = 1 \\ 5x + 2y - 3 = -5x - 4y + 4 \Rightarrow 10x + 6y = 7 \end{array}\right\} \Rightarrow$$

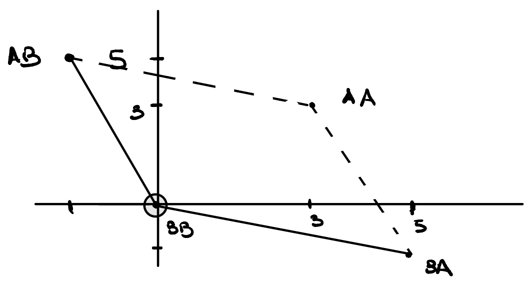
$\Rightarrow \boxed{x = \frac{5}{6},\ y = -\frac{2}{9}\ ??}$ No sale la cuenta

Nash Equilibria

An outcome of a game is non-Pareto-optimal if there is another outcome which would give both players higher payoffs, or would give one player the same payoff but the other player a higher payoff. The Pareto Principle says that to be acceptable as a solution to a game, an outcome should be Pareto optimal.

It is easy to see which outcomes of a game are Pareto optimal if we plot the outcomes in a coordinate plane. After we have plotted the points representing the pure strategy outcomes, mixed strategy outcomes are represented by points in the convex polygon enclosing the pure-strategy points. Pareto optimal outcomes are exactly those which lie on the "northeast" boundary of the payoff polygon.

	A	B
A	(3,3) →	(-1,5)
B	(5,-1) →	(0,0)



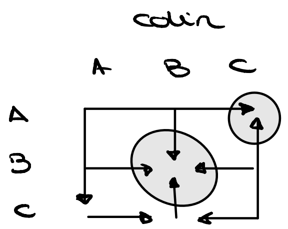
○ Equilibrium outcome

- - - Pareto optimal outcome

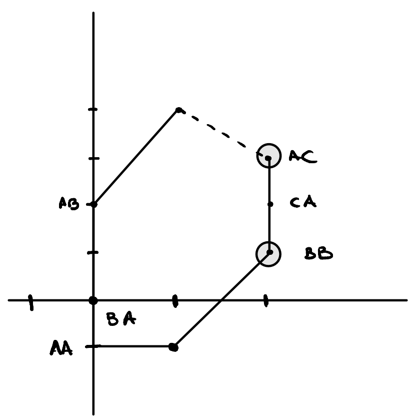
In a non-zero sum game, Rose's optimal strategy in Rose's game is called Rose's prudential strategy and the value of Rose's game is call Rose's security level. Analogy for Colin. Also, a player's counter-prudential strategy is his optimal response to his opponent's prudential strategy.

A two person game is solvable in the strict sense (SSS) if there is at least one equilibrium outcome which is Pareto optimal and, if there is more than one Pareto optimal equilibrium, all of them are equivalent and interchangeable.

		Colin A	Colin B	Colin C
Rose	A	(0,-1)	(0,2)	(2,3)
	B	(0,0)	(2,1)	(1,-1)
	C	(2,2)	(1,4)	(1,-1)



There are two equilibria at BB and AC. The payoff polygon shows that BB is not Pareto optimal, so AC is the unique Pareto optimal equilibrium.



The game is SSS and we would prescribe that Rose should play Rose A and Colin should play Colin C.

1. Find a Nash equilibrium in the next game and say if it is Pareto Optimal.

		A	B
Rose	A	(3,2)	(2,1)
	B	(4,3)	(1,4)

$$R = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

Rose wants to equate Colin's payoff.

Rose plays $p = (x, 1-x)$

$$pC = (x, 1-x)\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = (2x+3-3x,\ x+4-4x) = (-x+3, -3x+4) \Rightarrow$$

$$\Rightarrow -x+3 = -3x+4 \qquad x = \frac{1}{2} \qquad \Rightarrow p = (1/2, 1/2) \qquad pC = (5/2, 5/2)$$

Colin's expected payoff of 5/2

Colin plays $q = (y, 1-y)$ to equate Rose's payoff

$$Rq = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = (y+2,\ 3y+1) \Rightarrow y+2 = 3y+1 \quad y = \frac{1}{2} \Rightarrow q = (1/2, 1/2) \quad Rq = (5/2, 5/2)$$

Rose's expected payoff of 5/2

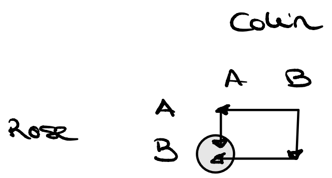
Nash equilibrium based in mixed strategy:

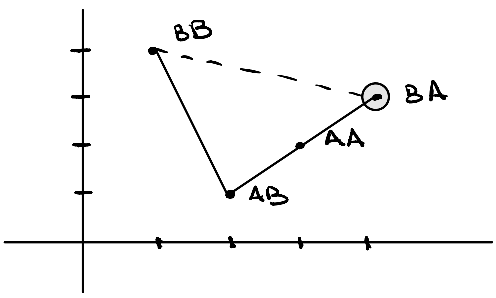
Rose: $\frac{1}{2}A$, $\frac{1}{2}B$

Colin: $\frac{1}{2}A$, $\frac{1}{2}B$

$\Rightarrow$ Expected payoff $(5/2, 5/2)$. It's not Pareto optimal because the outcome (4,3) is higher for both players.

If we use simple - pure strategy:





It is Pareto optimal, so the game is SSS and Rose's optimal strategy is Rose B and Colin's optimal strategy is Colin A.

② For each of the following games, draw the movement diagram and the payoff polygon, and identify all (pure-strategy) equilibria and Pareto optimal outcomes. Determinate if it is SSS or not.

a)

Rose \ Colin	A	B
A	(2,2)	(4,3)
B	(3,4)	(1,1)

b)

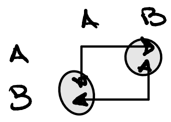
Rose \ Colin	A	B
A	(2,2)	(4,1)
B	(1,3)	(3,4)

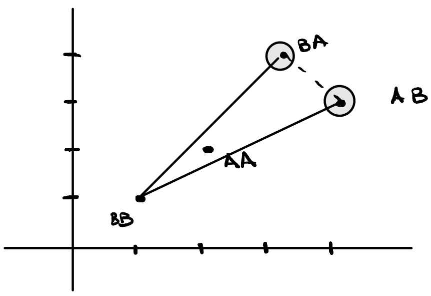
c)

Rose \ Colin	A	B	C
A	(3,0)	(5,2)	(0,4)
B	(2,2)	(1,1)	(3,3)
C	(4,1)	(4,0)	(1,0)

d) Game c) with AC payoff changed to (0,5)

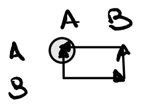
	A	B
A	(2,2)	(4,3)
B	(3,4)	(1,1)

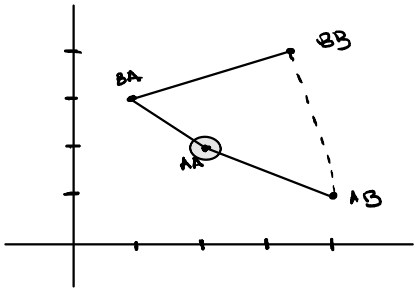




There are two pure-strategy equilibrium in (3,4) and (4,3), and they aren't interchangable and equivalent, so the game is not SSS.

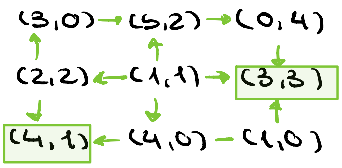
	A	B
A	(2,2)	(4,1)
B	(1,3)	(3,4)

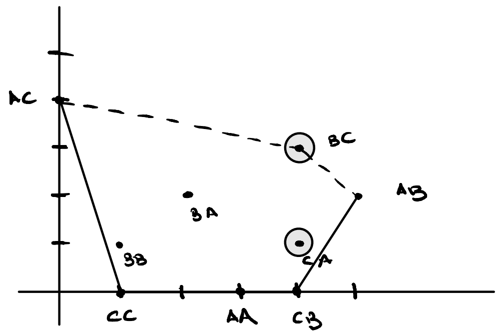




There is an only pure-strategy equilibrium and but it's not Pareto optimal, so the game is not

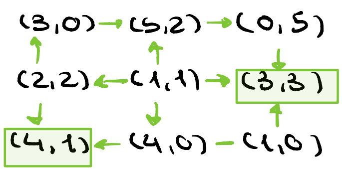
	A	B	C
A	(3,0)	(5,2)	(0,4)
B	(2,2)	(1,1)	(3,3)
C	(4,1)	(4,0)	(1,0)

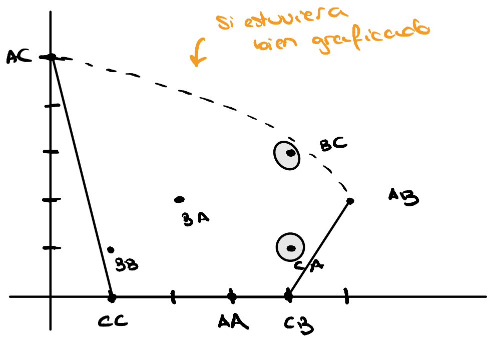




There are two pure-strategy equilibrium in (3,3) and (4,1), but the last one is not Pareto optimal, so (3,3) is the only Pareto optimal equilibrium ⇒ ⇒ the game is SSS and the solution is BC.

	A	B	C
A	(3,0)	(5,2)	(0,5)
B	(2,2)	(1,1)	(3,3)
C	(4,1)	(4,0)	(1,0)





Now (3,3) is not Pareto optimal ⇒ The game is not SSS.

③ For the following game

	A	B	C
A	(0,-1)	(0,2)	(2,3)
B	(0,0)	(2,1)	(1,-1)
C	(2,2)	(1,4)	(1,-1)

a) verify Rose's prudential strategy is $(\frac{1}{7}A, \frac{2}{7}B, \frac{4}{7}C)$ and say the security level.

b) Colin's prudential strategy

c) Expected playoff if both play prudentially

d) counter - prudential play

We know that Rose's prudential strategy is Rose's optimal play in Rose game:

$$R = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \qquad \eta R = (2-2x-2y,\ -x+y+1,\ x+1) \Rightarrow \begin{cases} -x-3y = -1 \\ 2x - y = 0 \end{cases} \Rightarrow$$

$$\eta = (x, y, 1-x-y) \qquad \Rightarrow x = 1/7,\ y = 2/7$$

$\Rightarrow \eta = (1/7, 2/7, 4/7)$, so Rose's prudential strategy is $\frac{1}{7}A, \frac{2}{7}B, \frac{4}{7}C$ and the security level is $\eta R = (8/7, 8/7, 8/7) \rightarrow$ (security level $8/7$).

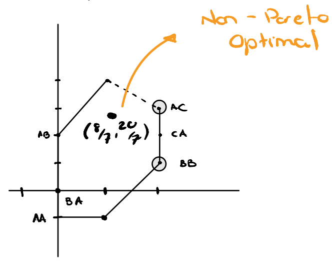
Now, we go with Colin. If we do the same:

$$C = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & -1 \\ 2 & 4 & -1 \end{bmatrix} \qquad q = (x, y, 1-x-y)$$

$Cq^t \Rightarrow$ The resulted system has no value solution. That's because we know that Rose's wants Colin to loose as much as possible, so Rose won't play the Rose C when he know that with Rose B, Colin loose more

$$\begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$ It has a saddle-point at 2 so Colin's prudential strategy is Colin B and the security level is 2.

Doing a mixed strategy $\frac{1}{7}AB + \frac{2}{7}BB + \frac{4}{7}CB = (8/7, 20/7)$ Rose gets his security level and Colin's better outcome that the security level



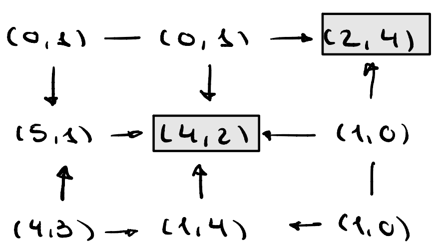
④ Consider the following game:

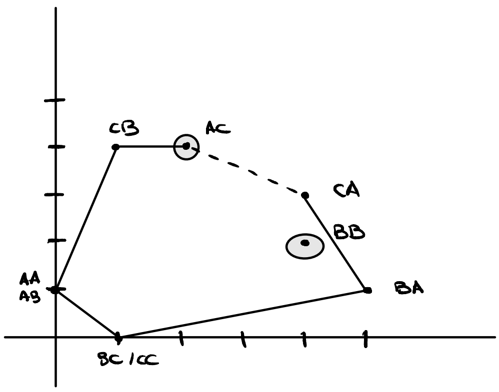
	A	B	C
A	(0,1)	(0,1)	(2,4)
B	(5,1)	(4,2)	(1,0)
C	(4,3)	(1,4)	(1,0)

a) Draw movement diagram and payoff polygon

b) Suppose dominated strategies. What happens?

c) Would you feel comfortable with the « solution »?





There are two pure - strategy equilibrium, but only one is Pareto Optimal, so the game is SSS and the solution is AC (2,4).

Now, we consider dominance:

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} (0,1) & (0,1) & (2,4) \\ (5,1) & (4,2) & (1,0) \\ (4,3) & (1,4) & (1,0) \end{bmatrix} \end{array} \longrightarrow \begin{bmatrix} (0,1) & (0,1) & (2,4) \\ (5,1) & (4,2) & (1,0) \end{bmatrix} \longrightarrow \begin{bmatrix} (0,1) & (2,4) \\ (4,2) & (1,0) \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} (0,1) & (2,4) \\ (4,2) & (1,0) \\ (1,4) & (1,0) \end{bmatrix} \longrightarrow \begin{bmatrix} (0,1) & (2,4) \\ (4,2) & (1,0) \end{bmatrix}$$

We obtain a 2×2 game, with rows and columns that contain the 2 equilibrium outcomes.

The Prisoner's Dilemma

	A	B
A	(0,0)	(-2,1)
B	(1,-2)	(-1,-1)

We see that strategy B is dominant for both players, but is non-Pareto-optimal. In fact, if both plays B, is the worst case possible.

Suppose that Colin will make his choice contingent on what he thinks Rose will do, and Rose will make her choice contingent on what contingent policy she thinks Colin is following. In this complicated scenario, might cooperation arise? Howard formalized this idea as what he called metagame. Consider that Colin's choice be contingent on Rose's choice, so that Colin has 4 strategies:

I: Choose A regardless of what he thinks Rose will do.
II: Choose the same alternative he thinks Rose will choose,
III: Choose the opposite alternative he thinks Rose will choose,
IV: Choose B regardless of what he thinks Rose will do.

The resulting game is the first level of metagame

	I : AA	II : AB	III : BA	IV : BB
A	(0,0)	(0,0)	(-2,1)	(-2,1)
B	(1,-2)	(-1,-1)	(1,-2)	(-1,-1)

Si repetimos con Rose nos da 16 opciones y el equilibrio en cooperar, pero esto involucra la suposición de leer la mente del otro

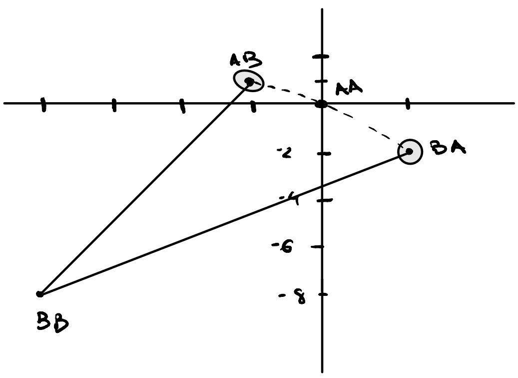
... Hubo una competición de programas de ordenador y blablabla ...

TIT FOR TAT won. The strategy should be:
- Nice → Start cooperating and never be first to defect.
- Retaliatory → It should reliably punish defection by its opponent
- Forgiving → Having punished defection, it should be willing to try cooperating again
- Clear → Its pattern of play should be consistent and easy to predict.

④ Find the pure strategy Nash equilibrium in this game. Are they Pareto optimal? Find also the mixed Nash strategy. Finally, how would you play the game?

	A	B
A	(0,0)	(-2,1)
B	(1,-2)	(-8,-8)

$(0,0) \rightarrow \boxed{(-2,1)}$
$\downarrow \qquad\qquad \uparrow$
$\boxed{(1,-2)} \leftarrow (-8,-8)$



There are two pure-strategy equilibriums that are Pareto optimal and they are no interchangeable and equivalent, so the game is not SSS.

Let's calculate now the mixed strategy:
Rose play $p = (x, 1-x)$ so

$$C = \begin{bmatrix} 0 & -1 \\ -2 & -8 \end{bmatrix} \Rightarrow pC = (2x-2 \quad , 9x-8) \Rightarrow -2+2x = 9x-8$$
$$x = \frac{6}{7}$$

So Rose's prudential play is $\frac{6}{7}A, \frac{1}{7}B$. On the other hand, Colin's play $q = (y, 1-y)$

$$R = \begin{bmatrix} 0 & -2 \\ 1 & -8 \end{bmatrix} \Rightarrow Rq^t = (-2+2y, 9y-8) \Rightarrow 2y-2 = 9y-8 \Rightarrow \text{Colin's prudential}$$
$$y = \frac{6}{7}$$

play is $\frac{6}{7}A, \frac{1}{7}B$

$$pRq = (6/7, 1/7)\begin{bmatrix} 0 & -2 \\ 1 & -8 \end{bmatrix}\begin{pmatrix} 6/7 \\ 1/7 \end{pmatrix} = -2$$

$$\Rightarrow (-2,-2) \text{ is not Pareto Optimal}$$

$$pCq = (6/7, 1/7)\begin{bmatrix} 0 & -1 \\ -2 & -8 \end{bmatrix}\begin{pmatrix} 6/7 \\ 1/7 \end{pmatrix} = -2$$

I will play using the TIT FOR TAT philosophy. That's (if I'm Rose) play A while Colin doesn't threat me (while he plays A) and if one round he plays B, I start playing B until he plays A again.

Strategic Moves

Up to this point, we have considered both players choose their plays without no comunication and simultaneously. Now, we are going to consider that one player moves first and make his move known to other player or players can talk each other before they move. So now, commitments, threats and promises become possible.

① The following games all have outcome which do not give Rose her maximal payoff. For each one, say whether Rose could benefit by any of the following strategic moves.

a) Colin

Rose	A	B
A	(3,4)	(4,3)
B	(2,2)	(1,1)

Rose threatens if "Colin choose A, then I choose B"

b)

	A	B
A	(2,4)	(3,3)
B	(1,2)	(4,1)

Rose threatens "A then B" and promise if "B then A"

c)

	A	B
A	(3,2)	(1,1)
B	(2,4)	(4,3)

Make Colin choose first.

Games Against Nature

- **Laplace**: Choose the row with the highest average entry. Equivalently, choose the row with the highest row sum.

  If you have no reason to assume one event is more likely than another, you can do no better than to assume that they are equally likely, so choose the expected value principal.

- **Wald**: Write down the minimum entry in each row. Choose the row with the largest minimum.

  You might want to be cautious, considering the worst that could happen to them and making that worst as good as possible.

  It choose the saddle-point strategy if the game has it or choose a pure strategy (the maximin strategy) which will not be the mixed strategy game theoric solution. This is reasonable when the choice is to be made just one, however if the game is gonna be played many times choosing the optimal game theoric mixed strategy might be the appropiate way to implement Wald's philosophy.

- **Maximin principle** assumes that the worst will happen.
- **Maximax principle** assumes that the best will happen.
- **Hurwicz**: Choose a "coefficient of optimism" α between 0 and 1. For each row, compute α(row maximum) + $(1-\alpha)$(row minimum) and choose the row with the highest average.
- **Savage**: For the regret matrix, write down the largest entry in each row. Choose the row for which this largest entry is smallest.

  Savage proposed minimize the maximum regret.

  Start from the original matrix and write down the regret matrix in which entry is the difference between the corresponding entry in the original matrix and the largest entry in its column.

- Axiom 1 (Symmetry): Rearranging the rows or columns should not affect which strategy is recommended as best (Reordenar no afecta)
- Axiom 2 (Strong Domination): If every entry in row X is larger than the corresponding entry in row Y, a method should not recommend method Y.
- Axiom 3 (Linearity): The recomend strategy should not change if you sum a constant or you multiply by a positive constant to all entries in the matrix.
- Axiom 4 (Column Duplication): If we add a duplicated column, strategy shouldn't change.
- Axiom 5 (Bonus Invariance): The recommended strategy shouldn't change if a constant is added to every entry in some column.
- Axiom 6 (Row Adjunction): Suppose a method chooses row X as the best strategy to follow in the game against nature, and the new row Z is added to the matrix. The method should choose either X or Z, but not some other row.

1. Find what each decision method (Laplace, Wald, Hurwicz with a coefficient of optimism suitable for you and savage) would tell a company manager to do in the following decision situation. The manager has no information about what the economy will be like three years from now when the payoff will come

		Way up	Slightly up	Slightly down	Way down
Manager	Hold steady	3	2	2	0
	Expand slightly	4	2	0	0
	Expand greatly	6	2	0	-2
	Diversify	1	1	2	2

Laplace → (7, 6, 6, 6) . Choose Row 1 (Hold steady)

Wald → (0, 0, −2, 1) . Choose Row 4 (Diversify)

Hurwicz → $(3\alpha, 4\alpha, 8\alpha - 2, \alpha + 1) \rightarrow \alpha = \frac{3}{4} \rightarrow (9/4, 12/4, \frac{24}{4} - 2, \frac{3}{4} + 1)$

Choose Row 3 (Expand greatly)

Savage :

$$\begin{bmatrix} 3 & 0 & 0 & 2 \\ 2 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 5 & 1 & 0 & 0 \end{bmatrix} \begin{matrix} 3 \\ 2 \\ 4 \\ 5 \end{matrix} \leftarrow$$

Choose Row 2 (Expand slightly)

2. Show that Wald method violates the Bonus invariance axiom by considering the effect of adding 2 to every entry in column C, and 1 to every entry in column D, in the example matrix of this chapter.

Nature

You	A	B	C	D
A	2	2	0	1
B	1	1	1	1
C	0	4	0	0
D	1	3	0	0

$\xrightarrow{C+2,\ D+1}$

	A	B	C	D
A	2	2	2	2
B	1	1	3	2
C	0	4	2	1
D	1	3	2	1

(0, 1, 0, 0) → Choose B $\neq$ (2, 1, 0, 1) → Choose A

AS (Bonus invariance) !!

Show that Hurwicz method violates Bonus invariance by considering the effect of adding a suitable bonus to some column of the matrix ($\alpha = 3/4$)

	A	B	C	D
A	2	2	0	1
B	1	1	1	1
C	0	4	0	0
D	1	3	0	0

$\xrightarrow{C+4}$

	A	B	C	D
A	2	2	4	1
B	1	1	5	1
C	0	4	4	0
D	1	3	4	0

(6/4, 1, 3, 9/4) → Row C

(13/4, 16/4, 12/4, 12/4) → Row B

(3) Show that the Savage method violates Row adjudication axiom by considering the effect of adjoining Row E: 0 0 0 3 to the example matrix

	A	B	C	D
A	2	2	0	1
B	1	1	1	1
C	0	4	0	0
D	1	3	0	0

$\longrightarrow$

	A	B	C	D
A	2	2	0	1
B	1	1	1	1
C	0	4	0	0
D	1	3	0	0
E	0	0	0	3

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} 2 \\ 3 \\ 2 \\ 1 \end{matrix} \leftarrow \text{Row D}$$

$$\begin{bmatrix} 0 & 2 & 1 & 2 \\ 1 & 3 & 0 & 2 \\ 2 & 0 & 1 & 3 \\ 1 & 1 & 1 & 3 \\ 2 & 4 & 1 & 0 \end{bmatrix} \begin{matrix} 2 \\ 3 \\ 3 \\ 3 \\ 4 \end{matrix} \begin{matrix} \leftarrow \text{Row A} \\ \\ \\ \\ \\ \end{matrix}$$

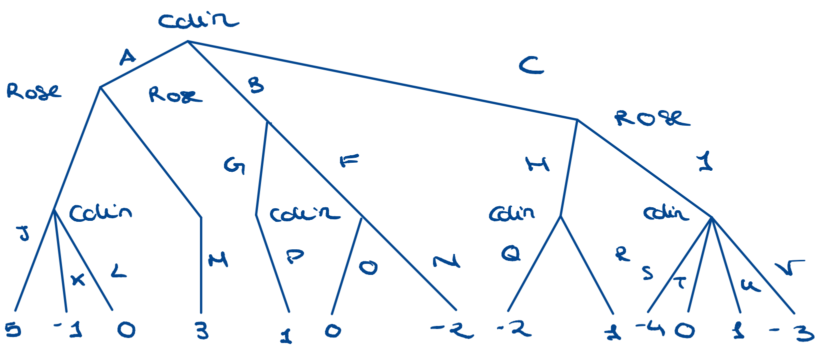
solo podría dar E o D

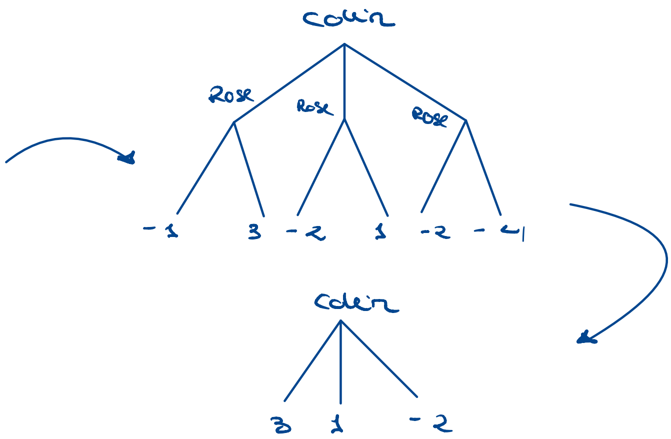
Game Trees

A game is said to be perfect information in the game tree if
- no nodes are labeled by CHANCE, and
- all information sets consist of a single node.

Chance plays no part in the game, and each player always knows all previous moves which have been made in the game.

Games of perfect information can be analyzed by a technique known as truncation or tree pruning . consider the next tree





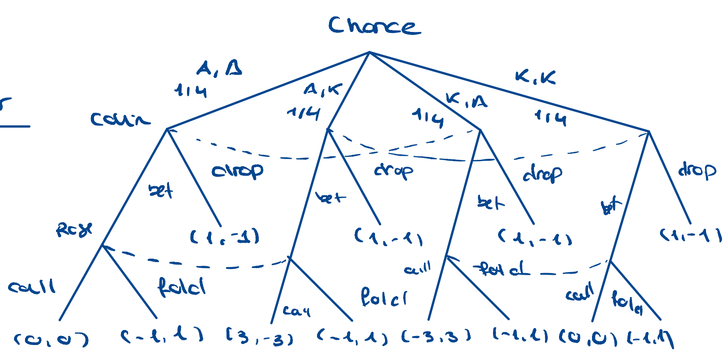
Consider Colin's last choice, Colin would always do his best to choose the branch leading to the smallest Rose payoff. so we can truncate the tree by removing its final branches and labeling each new final node by the smallest Rose payoff below that node. Now, Rose will do his best to choose the branch leading to the largest payoff, so we can truncate again. Finally, Colin will choose -2. We can follow the path.

If we wrote the matrix for this game and followed the truncation process that in each stage, strategies are eliminated by dominance, and it leaves a saddle-point in -2.

Since this process would work for any two-person zero-sum-game of perfect information, any such game will have a saddle-point (that are found by dominance). One interesting consequence of this analysis is : Consider a finite game of complete information between two players, white and Black, in which there are just two possible outcomes, "White wins" or "Black wins". Then, exactly one of the following statements must hold:
- White has a strategy which can force a win for white, no matter what Black does.
- Black has a strategy which can force a win for Black, no matter what white does.

1 check two of the payoffs Game :

Rose calls		Colin bets: always	A only	K only	never
	always	0	-1/4	$\boxed{5/4}$	1
	A only	1/4	1/4	1	1
	K only	-5/4	$\boxed{-1/2}$	1/4	1
	never	-1	0	0	1



a) 5/4

Probability	Rose hand, Colin hand	Outcome	Payoff Rose
1/4	A, A	Colin drops	+1
1/4	A, K	Colin bet, Rose calls	+3
1/4	K, A	Colin drops	+1
1/4	K, K	Colin bet, Rose calls	0

$(1/4)\cdot 1 + (1/4)\cdot 3 + (1/4)\cdot 1 + (1/4)\cdot 0 = 5/4$

b) - 1/2

Probability	Rose hand, Colin hand	Outcome	Payoff Rose
1/4	A, A	Colin bet, Rose folds	0
1/4	A, K	Colin drops	+1
1/4	K, A	Colin bet, Rose calls	-3
1/4	K, K	Colin drops	+1

$(1/4)\cdot 0 + (1/4)\cdot 1 + (1/4)\cdot(-3) + (1/4)\cdot 1 = -1/2$

2 Consider the same game of poker, except that the deck has now aces, kings and queens (Pista: Each player only bets or call with ace only or ace - king). Write and solve the 3×3 matrix. The solution should involve mixed strategies. Interpret the solution as advice to the players.

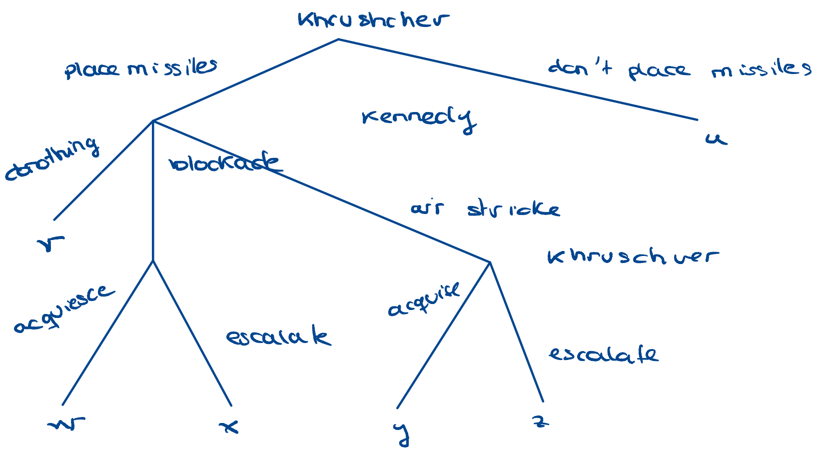
Idea: Hacer el árbol, y hacer como (1).

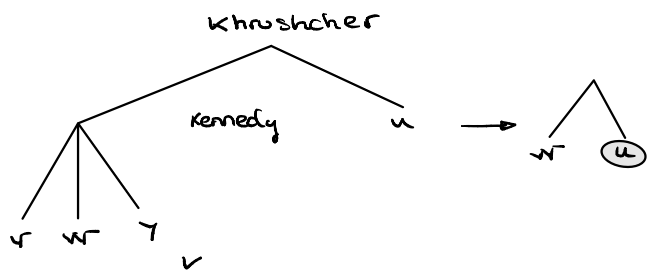
3 Suppose the players in the Cuban missile crisis had the following preference orderings for the outcomes

Kennedy : w, y, u, v, x, z

Khrushchev : v, u, w, y, z, x

Use truncation on the next tree to find the rational outcome on this game. That is not what happened. Can you think of some possible explanations for why not?





4 Explain why in figure 7.3. Rose has 8 strategies and Colin has 13.

Colin: A/JM, A/KM, A/LM, B/NP, B/OP, C/QS, C/QT, C/QU, C/QV, C/RS, C/RT.

5 For the following game

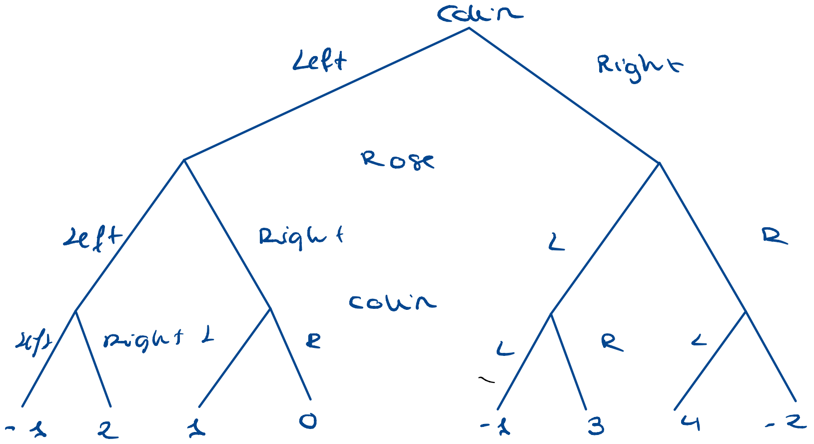
a) Find the solution by truncation.

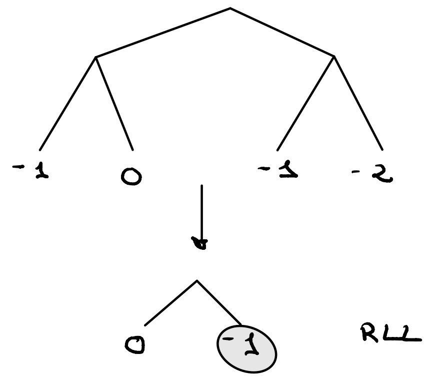
b) List all Colin strategies. Be sure to describe each one clearly

c) List all Rose strategies.

d) Write the matrix for the game

e) Solve the matrix and verify that the solution is the same by truncation





Rose: LL, LR, RL, RR

Colin: L/LL, L/LR, L/RL, L/RR, R/LL, R/LR, R/RL, R/RR

Colin

		L/LL	L/LR	L/RL	L/RR	R/LL	R/LR	R/RL	R/RR
Rose	LL	-1	-1	2	2	-1	-1	3	3
	LR	-1	-1	2	2	4	-2	4	-2
	RL	1	0	1	0	-1	-1	3	3
	RR	1	0	1	0	4	-2	4	-2

Primera opción. Si es L nos fijamos si es R ganará si L nos fijamos en si R nos fijamos

6 Suppose that the previous game is modified so that Colin does not know what choice Rose made before Colin has to make his second choice.

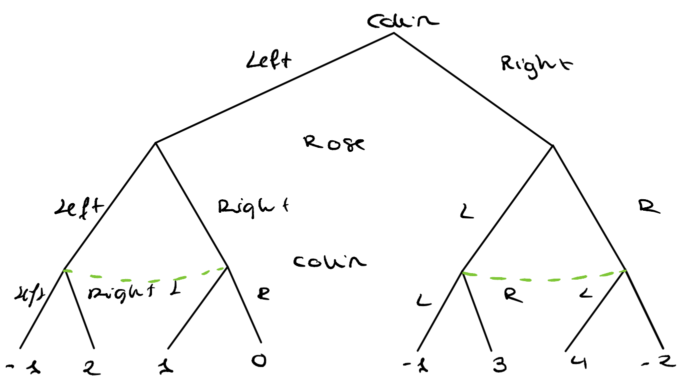
a) Show the information sets given by this condition

b) List all Colin strategies

c) Write the game matrix.

d) Solve the game the solution is 3x3, with a value of $\frac{1}{2}$. Has Colin's loss of information helped or hurt Colin?

Colin: L/LL, L/RR, R/LL, R/RR



Colin

		L/LL	L/RR	R/LL	R/RR
Rose	LL	-1	2	-1	3
	LR	-1	2	4	-2
	RL	1	0	-1	3
	RR	1	0	4	-2

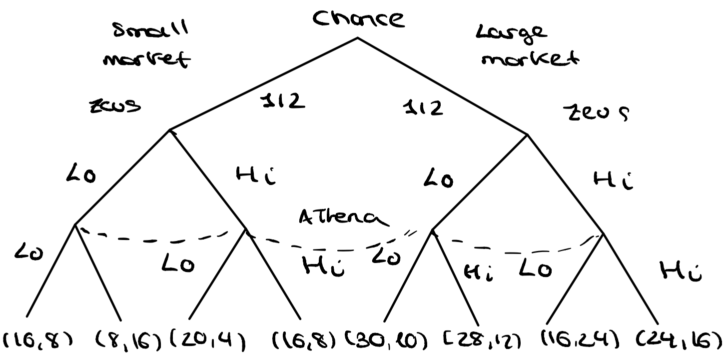
Competitive Decision Making

① Suppose that the companies move simultaneously, but that before they do, Zeus conducts a market survey. Athena knows the existence of the survey, but not its results.

a) What are the information sets for this game.

b) Write and solve the resulting 4×2 matrix game.

c) What effect would it have if Athena did not know the existence of the survey?



Zeus (small / large)

	Athena Lo	Athena Hi	
Lo/Lo	23	18	Hi/Lo
Lo/Hi	16	16	Hi/Lo
Hi/Lo	25	22	← saddle-point
Hi/Hi	18	20	Hi/Lo

$\frac{1}{2}(16,8) + \frac{1}{2}(30,10) = (8,4) + (15,5) = (23,9)$

$\frac{1}{2}(16,8) + \frac{1}{2}(16,24) = (8,4) + (8,12) = (16,8)$

$\frac{1}{2}(20,4) + \frac{1}{2}(30,10) = (10,2) + (15,5) = (25,7)$

...

Athena would play his optimal strategy in game

	Lo	Hi	
Lo	(23,9)	(18,14)	5/7
Hi	(18,14)	(20,12)	2/7
	2/7	5/7	

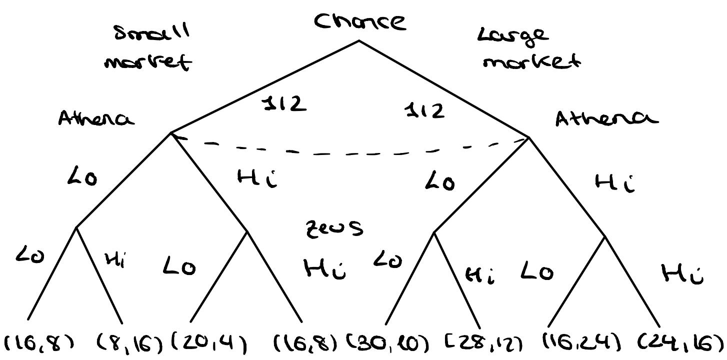
and Zeus Hi/Lo.

② Suppose that Athena must move first but that neither side knows CHANCE's move.

a) This game requires a different tree. Draw it, showing the information sets.

b) Write and solve the resulting 4×2 matrix game.

c) How much would it be worth to Zeus to have Athena move first, rather than having Zeus move first?



Zeus (Athena Lo / Athena Hi)

	Athena Lo	Athena Hi	
Lo/Lo	23	18	Lo/Hi
Lo/Hi	23	20	
Hi/Lo	18	18	Lo/Hi
Hi/Hi	18	20	Lo/Hi

The value of the game is 20 instead of 18 so Zeus would be 2 $ million better.

$(\frac{1}{2})(16,8) + \frac{1}{2}(30,10) = (23,9)$

$(\frac{1}{2})(16,8) + \frac{1}{2}(30,10) = (23,9)$

$(\frac{1}{2})(8,16) + (\frac{1}{2})(28,12) = (18, ..)$

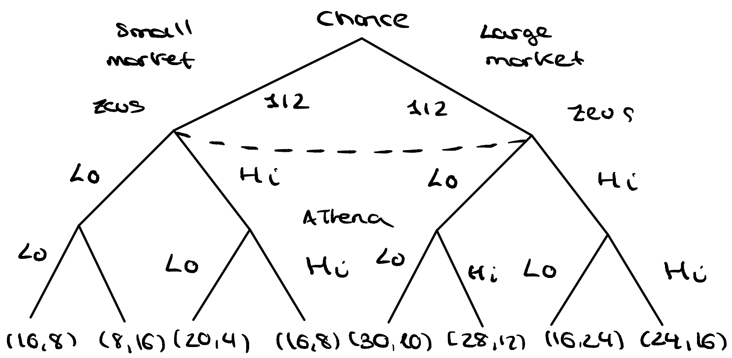
③ Suppose that Zeus move first, but doesn't do a market survey, and Athena does a market survey. Zeus know the existance of the survey but not its results.

a) what are the information sets for this game?

b) How many strategies do Athena and Zeus have?

c) Solve the game

d) what effect would it have if Zeus did not know the existance of the survey?



small market / large market
chose [ ] if Zeus [ ] / chose [ ] if Zeus [ ]

		LoLo/LoLo	LoHi/LoLo	HiLo/LoLo	HiHi/LoLo	...	$4^2 = 16$
Zeus	Lo						
	Hi						

Athenas will make best choose at each node. If Zeus choose Lo, its expectation is $(\frac{1}{2})(8,16) + \frac{1}{2}(28,12) = (18,14)$ → 18 payoff expected for Zeus. If Zeus choose Hi, its expectation is $(\frac{1}{2})(16,8) + (\frac{1}{2})(16,24) = (16,16)$ → 16 payoff expected for Zeus, so he should choose Lo.