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Faculty of Applied Physics and Mathematics
Project
Mathematical Method of Physics

Contents

1 Elaborated Topic: Damped and Forced Oscillations

When you pluck a guitar string, the resulting sound has a steady tone and lasts a long time. The string vibrates around an equilibrium position, and one oscillation is completed when the string starts from the initial position, travels to one of the extreme positions, then to the other extreme position, and returns to its initial position. We define **periodic motion** to be any motion that repeats itself at regular time intervals, such as exhibited by the guitar string or by a child swinging on a swing.

- In the absence of friction, the time to complete one oscillation remains constant and is called the **period (T)**. Its units are usually seconds, but may be any convenient unit of time.
- A concept closely related to period is the frequency of an event. **Frequency (f)** is defined to be the number of events per unit time.

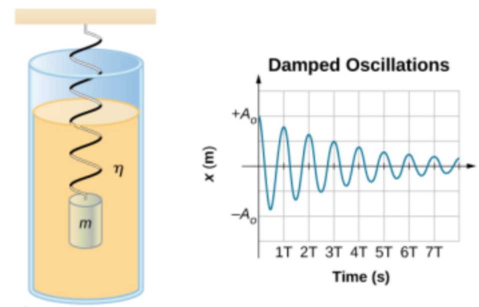
The most simple systems are simple **harmonic motion (SHM)**. A system that oscillates with SHM is called a simple **harmonic oscillator**. In this systems, the acceleration, and, therefore the net force, is proportional to displacement and acts in the opposite direction of the displacement.

1.1 Damped Oscillations

In the real world, oscillations seldom follow true SHM. Friction of some sort usually acts to dampen the motion so it dies away, or needs more force to continue. In this section, we examine some examples of damped harmonic motion and see how to modify the equations of motion to describe this more general case.

A guitar string stops oscillating a few seconds after being plucked. To keep swinging on a playground swing, you must keep pushing. Although we can often make friction and other non conservative forces small or negligible, completely undamped motion is rare. In fact, we may even want to damp oscillations, such as with car shock absorbers.

The figure shows a mass m attached to a spring with a force constant k. The mass is raised to a position A_0, the initial amplitude, and then released. The mass oscillates around the equilibrium position in a fluid with viscosity but the amplitude decreases for each oscillation. For a system that has a small amount of damping, the period and frequency are constant and are nearly the same as for SHM, but the amplitude gradually decreases as shown. This occurs because the non-conservative damping force removes energy from the system, usually in the form of thermal energy.



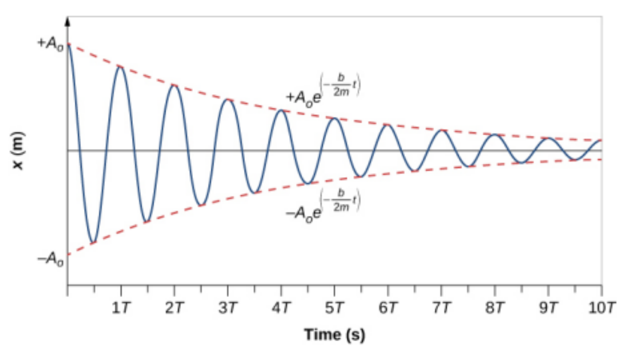
Consider the forces acting on the mass. Note that the only contribution of the weight is to change the equilibrium position, as discussed earlier in the chapter. Therefore, the net force is equal to the force of the spring and the damping force (F_D). If the magnitude of the velocity is small, meaning the mass oscillates slowly, the damping force is proportional to the velocity and acts against the direction of motion ($F_D = -bv$). The net force on the mass is therefore

$$ma = -bv - kx$$

Writing this as a differential equation in x we obtain

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{1}$$

To determine the solution to this equation, consider the following plot of position versus time.



The curve resembles a cosine curve oscillating in the envelope of an exponential function $A_0e^{-\alpha t}$ where $\alpha = \frac{b}{2m}$. So, the solution is

$$x(t) = A_0e^{-\frac{b}{2m}t}cos(\omega t + \Phi) \tag{2}$$

To prove that it is the right solution, take the first and second derivatives with respect to time and substitute them into 1. It is found that 2 is the solution if

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

Recall that the angular frequency of a mass undergoing SHM is equal to the square root of the force constant divided by the mass. This is often referred to as the **natural angular frequency**, which is represented as

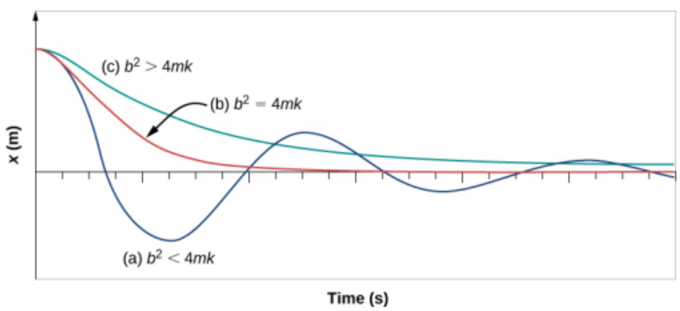
$$\omega_0 = \sqrt{\frac{k}{m}} \tag{3}$$

The angular frequency for damped harmonic motion becomes

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} \tag{4}$$

Recall that when we began this description of damped harmonic motion, we stated that the damping must be small. If you gradually increase the amount of damping in a system, the period and frequency begin to be affected, because damping opposes and hence slows the back and forth motion. (The net force is smaller in both directions.) If there is very large damping, the system does not even oscillate.

The angular frequency is $\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$. As b increases, $\frac{k}{m} - \left(\frac{b}{2m}\right)^2$ becomes smaller and, eventually, reaches zero when $b = \sqrt{4mk}$. It b becomes any larger, $\frac{k}{m} - \left(\frac{b}{2m}\right)^2$ becomes a negative number and $\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$ is a complex number.



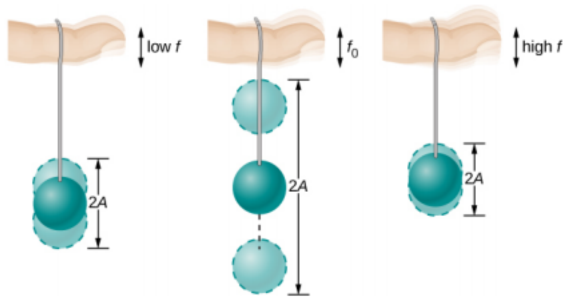
The previous figure shows the displacement of a harmonic oscillator for different amounts of damping. When the damping constant is small, $b < \sqrt{4mk}$ the system oscillates while the amplitude of the motion decays exponentially. This system is said to be **underdamped**, as in curve (a). Many systems are underdamped, and oscillate while the amplitude decreases exponentially, such as the mass oscillating on a spring. The damping may be quite small, but eventually the mass comes to rest. If the damping constant is $b = \sqrt{4mk}$ the system is said to be **critically damped**, as in curve (b). An example of a critically damped system is the shock absorbers in a car. It is advantageous to have the oscillations decay as fast as possible. Here, the system does not oscillate, but asymptotically approaches the equilibrium condition as quickly as possible. Curve (c) represents an **overdamped** system where $b > \sqrt{4mk}$. An overdamped system will approach equilibrium over a longer period of time.

Critical damping is often desired, because such a system returns to equilibrium rapidly and remains at equilibrium as well. In addition, a constant force applied to a critically damped system moves the system to a new equilibrium position in the shortest time possible without overshooting or oscillating about the new position.

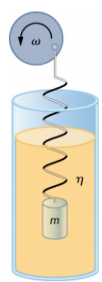
1.2 Forced Oscillations

Sit in front of a piano sometime and sing a loud brief note at it with the dampers off its strings.It will sing the same note back at you—the strings, having the same frequencies as your voice, are resonating in response to the forces from the sound waves that you sent to them. This is a good example of the fact that objects—in this case, piano strings—can be forced to oscillate, and oscillate most easily at their natural frequency. In this section, we briefly explore applying a periodic driving force acting on a simple harmonic oscillator. The driving force puts energy into the system at a certain frequency, not necessarily the same as the natural frequency of the system. Recall that the natural frequency is the frequency at which a system would oscillate if there were no driving and no damping force.

Most of us have played with toys involving an object supported on an elastic band, something like the paddle ball suspended from a finger like in the following figure



Imagine the finger in the figure is your finger. At first, you hold your finger steady, and the ball bounces up and down with a small amount of damping. If you move your finger up and down slowly, the ball follows along without bouncing much on its own. As you increase the frequency at which you move your finger up and down, the ball responds by oscillating with increasing amplitude. When you drive the ball at its natural frequency, the ball's oscillations increase in amplitude with each oscillation for as long as you drive it. The phenomenon of driving a system with a frequency equal to its natural frequency is called **resonance**. A system being driven at its natural frequency is said to **resonate**. As the driving frequency gets progressively higher than the resonant or natural frequency, the amplitude of the oscillations becomes smaller until the oscillations nearly disappear, and your finger simply moves up and down with little effect on the ball.



Consider a simple experiment. Attach a mass m to a spring in a viscous fluid, similar to the apparatus discussed in the damped harmonic oscillator. This time, instead of fixing the free end of the spring, attach the free end to a disk that is driven by a variable-speed motor. The motor turns with an angular driving frequency of ω. The rotating disk provides energy to the system by the work done by the driving force ($F_d = F_0 sin(\omega t)$)

Using Newton's second law ($\vec{F}_{net} = m\vec{a}$), we can analyze the motion of the mass. The resulting equation is similar to the force equation for the damped harmonic oscillator, with the addition of the driving force:

$$-kx - b\frac{dx}{dt} + F_0 sin(\omega t) = m\frac{d^2x}{dt^2} \tag{5}$$

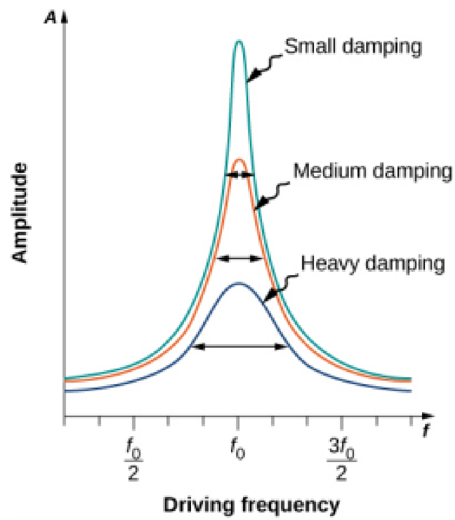
When an oscillator is forced with a periodic driving force, the motion may seem chaotic. The motions of the oscillator is known as transients. After the transients die out, the oscillator reaches a steady state, where the motion is periodic. After some time, the steady state solution to this differential equation is

$$x(t) = Acos(\omega t + \Phi) \tag{6}$$

Once again, it is left as an exercise to prove that this equation is a solution. Taking the first and second time derivative of $x(t)$ and substituting them into the force equation shows that $x(t) = Asin(\omega t + \Phi)$ is a solution as long as the amplitude is equal to

$$A = \frac{F_0}{\sqrt{m^2\left(\omega^2 - \omega_0^2\right)^2 + b^2\omega^2}} \tag{7}$$

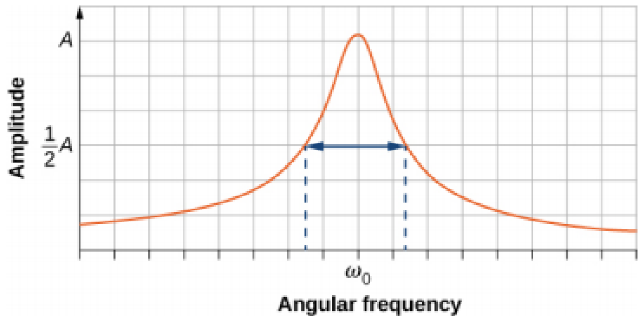
where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural angular frequency of the system of the mass and spring. Recall that the angular frequency, and therefore the frequency, of the motor can be adjusted. Looking at the denominator of the equation for the amplitude, when the driving frequency is much smaller, or much larger, than the natural frequency, the square of the difference of the two angular frequencies $\left(\omega^2 - \omega_0^2\right)^2$ is positive and large, making the denominator large, and the result is a small amplitude for the oscillations of the mass. As the frequency of the driving force approaches the natural frequency of the system, the denominator becomes small and the amplitude of the oscillations becomes large. The maximum amplitude results when the frequency of the driving force equals the natural frequency of the system $\left(A_{max} = \frac{F_0}{b\omega}\right)$



The previous figure shows a graph of the amplitude of a damped harmonic oscillator as a function of the frequency of the periodic force driving it. Each of the three curves on the graph represents a different amount of damping. All three curves peak at the point where the frequency of the driving force equals the natural frequency of the harmonic oscillator. The highest peak, or greatest response, is for the least amount of damping, because less energy is removed by the damping force. Note that since the amplitude grows as the damping decreases, taking this to the limit where there is no damping ($b = 0$), the amplitude becomes infinite.

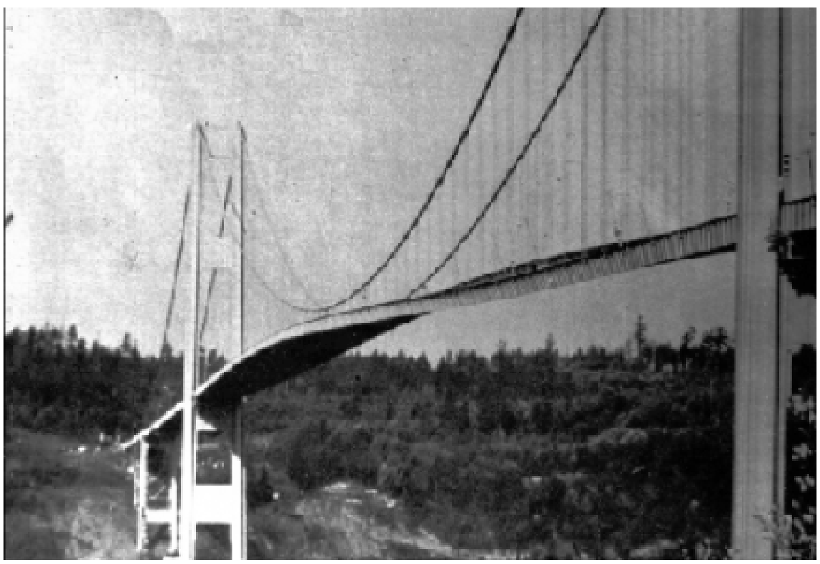
Note that a small-amplitude driving force can produce a large-amplitude response. This phenomenon is known as resonance. A common example of resonance is a parent pushing a small child on a swing. When the child wants to go higher, the parent does not move back and then, getting a running start, slam into the child, applying a great force in a short interval. Instead, the parent applies small pushes to the child at just the right frequency, and the amplitude of the child's swings increases.

It is interesting to note that the widths of the resonance curves shown depend on damping: the less the damping, the narrower the resonance. The consequence is that if you want a driven oscillator to resonate at a very specific frequency, you need as little damping as possible. For instance, a radio has a circuit that is used to choose a particular radio station. In this case, the forced damped oscillator consists of a resistor, capacitor, and inductor, which will be discussed later in this course. The circuit is "tuned" to pick a particular radio station. Here it is desirable to have the resonance curve be very narrow, to pick out the exact frequency of the radio station chosen. The narrowness of the graph, and the ability to pick out a certain frequency, is known as the quality of the system. The quality is defined as the spread of the angular frequency, or equivalently, the spread in the frequency, at half the maximum amplitude, divided by the natural frequency ($Q = \frac{\Delta\omega}{\omega_0}$) as shown in the following figure:



For a small damping, the quality is approximately equal to $Q \approx \frac{2b}{m}$.

These features of driven harmonic oscillators apply to a huge variety of systems. For instance, magnetic resonance imaging (MRI) is a widely used medical diagnostic tool in which atomic nuclei (mostly hydrogen nuclei or protons) are made to resonate by incoming radio waves (on the order of 100 MHz). In all of these cases, the efficiency of energy transfer from the driving force into the oscillator is best at resonance.



The picture shows a photograph of a famous example (the Tacoma Narrows bridge) of the destructive effects of a driven harmonic oscillation. The Millennium bridge in London was closed for a short period of time for the same reason while inspections were carried out. Observations lead to modifications being made to the bridge prior to the reopening.

2 Exercises

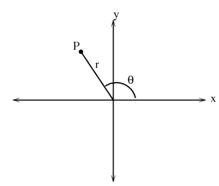
1. **Find the following integral**

$$\int_0^{+\infty} e^{-x^2}dx$$

Let's call $I = \int_0^{+\infty} e^{-x^2}dx$, so we can multiply it with itself

$$I^2 = I \cdot I = \left(\int_0^{+\infty} e^{-x^2}dx\right) \cdot \left(\int_0^{+\infty} e^{-y^2}dy\right) = \int_0^{+\infty}\int_0^{+\infty} e^{-\left(x^2+y^2\right)}dxdy \tag{8}$$

Now, we will make a change of variable to work with polar coordinates:



$$\begin{cases} r^2 = x^2+y^2 \\ dxdy = rdrd\theta \end{cases} \implies \begin{cases} r: 0 \to +\infty \\ \theta : 0 \to \frac{\pi}{2} \end{cases} \tag{9}$$

See that both variables go from $0 \to +\infty$, so when we change to polar coordinates, we are working in the first quater.

So applying 9 into 8, we have:

$$I^2 = \int_0^{\pi/2} d\theta \int_0^{+\infty} e^{-r^2} rdr = \left[\begin{array}{c} u = r^2 \\ du = 2rdr \end{array}\right] = \int_0^{\pi/2} d\theta \int_0^{+\infty} e^{-u}\frac{du}{2} =$$

$$= \theta]_0^{\pi/2} \cdot -\frac{1}{2}e^{-u}\Big]_0^{+\infty} = \left(\frac{\pi}{2}-0\right)\left(-\frac{1}{2}\right)\left(e^{-\infty}-e^0\right) = \frac{\pi}{4} \tag{10}$$

So, from 10, we get that

$$I^2 = \frac{\pi}{4} \implies \boxed{I = \frac{\sqrt{\pi}}{2}}$$

2. **Find the center of mass of a thin plate D between $\frac{x^2}{9}+\frac{y^2}{4}=1$ and $y=0$ for $y \geq 0$, if the density is $\rho(x,y)=1$**

Let's isolate y from the given equation.

$$\frac{x^2}{9}+\frac{y^2}{4} = 1 \implies y^2 = 4\left(1-\frac{x^2}{9}\right) \implies y = \pm 2\sqrt{1-\frac{x^2}{9}}$$

Since we have that $y \geq 0$, then $y = 2\sqrt{1-\frac{x^2}{9}}$. If we represent the region between $y = 2\sqrt{1-\frac{x^2}{9}}$ and $y=0$, we can see that $x \in [-3,3]$.

We know that the center of mass of a thin plate D is the point $(x_c, y_c) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$, where

$$m = \int\int_D \rho(x,y)dxdy, \quad M_x = \int\int_D y\rho(x,y)dxdy, \quad M_y = \int\int_D x\rho(x,y)dxdy$$

So let's calculate all the necessary data.

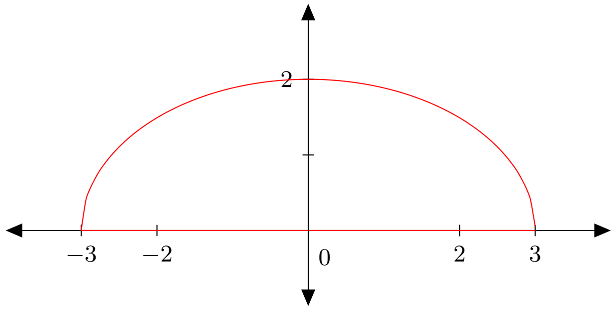


Figure 1: Region

$$m = \int_{-3}^{3} \left(\int_{0}^{2\sqrt{1-\frac{x^2}{9}}} dy \right) dx = \int_{-3}^{3} 2\sqrt{1-\frac{x^2}{9}} dx = 2\int_{-3}^{3} \sqrt{\frac{9-x^2}{9}} =$$
$$= \frac{2}{3}\int_{-3}^{3} \sqrt{9-x^2} = \left[\begin{array}{c} x = 3sin(t) \implies t = arcsin(\frac{x}{3}) \\ dx = 3cos(t)dt \end{array} \right] = \frac{2}{3}\left(\int 3cos(t)\sqrt{9-9sin^2(t)}\, dt \right) =$$
$$= \left[\begin{array}{c} sin^2(t) + cos^2(t) = 1,\ so \\ 9cos^2(t) = 9 - 9sin^2(t) \end{array} \right] = \frac{2}{3}\left(\int 3cos(t)\sqrt{9cos^2(t)}dt \right) = 6\int cos^2(t)dt =$$
$$= \left[cos^2(t) = \frac{cos(2t)+1}{2} \right] = 6\left(\frac{1}{2}\int cos(2t)dt + \frac{1}{2}\int dt \right) = 6\left(\frac{sin(2t)}{4} + \frac{t}{2} \right) =$$
$$= \left[t = arcsin(\frac{x}{3}) \right] = \frac{3}{2}sin\left(2arcsin(\frac{x}{3}) \right) + 3arcsin(\frac{x}{3}) \Bigg]_{-3}^{3} = \left[sin(2arcsin(\frac{x}{3}) = \frac{2x\sqrt{1-\frac{x^2}{9}}}{3} \right] =$$
$$\left[t = arcsin(\frac{x}{3}) \right] = \frac{3}{2}\frac{2x\sqrt{1-\frac{x^2}{9}}}{3} + 3arcsin(\frac{x}{3}) \Bigg]_{-3}^{3} = x\sqrt{1-\frac{x^2}{9}} + 3arcsin(\frac{x}{3}) \Bigg]_{-3}^{3} =$$
$$= 3\left(arcsin(1) - arcsin(-1) \right) = \boxed{3\pi} \quad (11)$$

$$M_x = \int_{-3}^{3} \left(\int_{0}^{2\sqrt{1-\frac{x^2}{9}}} ydy \right) dx = \int_{-3}^{3} \left(\frac{y^2}{2} \right]_{0}^{2\sqrt{1-\frac{x^2}{9}}} dx = \int_{-3}^{3} 2\left(1 - \frac{x^2}{9} \right) dx =$$
$$= 2\left(\int_{-3}^{3} dx - \frac{1}{9}\int_{-3}^{3} x^2 dx \right) = 2\left(x - \frac{1}{9}\frac{x^3}{3} \right]_{-3}^{3} = \boxed{8} \quad (12)$$

$$M_y = \int_{-3}^{3} x\left(\int_{0}^{2\sqrt{1-\frac{x^2}{9}}} dy \right) dx = \int_{-3}^{3} x\left(2\sqrt{1-\frac{x^2}{9}} \right) dx = \int_{-3}^{3} 2x\sqrt{1-\frac{x^2}{9}} dx =$$
$$= \left[\begin{array}{c} t = 1 - \frac{x^2}{9} \\ dt = -\frac{2x}{9}dx \end{array} \right] = \int \sqrt{t}(-9)dt = -9\int \sqrt{t}dt = -9\frac{2}{3}t^{\frac{3}{2}} = -6\left(1 - \frac{x^2}{9} \right)^{\frac{3}{2}} \Bigg]_{-3}^{3} =$$
$$= \boxed{0} \quad (13)$$

Then, we have that:

$$\boxed{(x_c, y_c) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{0}{3\pi}, \frac{8}{3\pi}\right) = \left(0, \frac{8}{3}\pi\right)}$$

3. **Find the center of mass of a three dimensional region** $V: y = \sqrt{x},\ y = 2\sqrt{x},\ z = 0$ **and** $x + z = 6$**, if the density is** $\rho(x, y, z) = const$

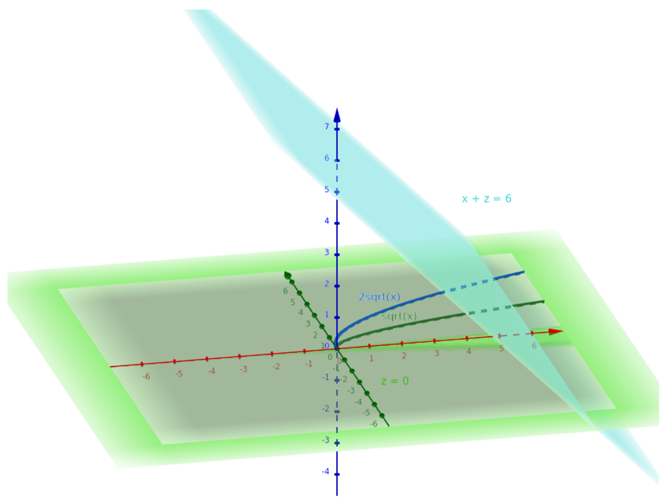
   We know that the center of mass of V is the point (x_c, y_c, z_c) such that:

$$x_c = \frac{M_{yz}}{m},\ y_c = \frac{M_{xz}}{m},\ z_c = \frac{M_{xy}}{m}$$

where

$$\begin{gathered} m = \int\int\int_V \rho(x, y, z)dxdydz \\ M_{xy} = \int\int\int_V z\rho(x, y, z)dxdydz \\ M_{xz} = \int\int\int_V y\rho(x, y, z)dxdydz \\ M_{yz} = \int\int\int_V x\rho(x, y, z)dxdydz \end{gathered}$$

If we represent V, we have that:



$$m = \int_0^6 \left(\int_0^6 \left(\int_{\sqrt{x}}^{2\sqrt{6}} cdy\right) dz\right) dx = \int_0^6 \left(\int_0^6 c\sqrt{x}dz\right) dx = 6c\int_0^6 \sqrt{x}dx = \boxed{24\sqrt{6}c}$$

$$M_{yz} = \int_0^6 \left(\int_0^6 \left(\int_{\sqrt{x}}^{2\sqrt{6}} xcdy\right) dz\right) dx = \int_0^6 (x\sqrt{x}cdz)\, dx = 6c\int_0^6 x\sqrt{x}dx = 6c\int_0^6 x^{3/2}dx =$$

$$= 6c\left(\frac{2x^{5/2}}{5}\right\Bigg|_0^6 = \boxed{\frac{6c72\sqrt{6}}{5}}$$

$$M_{xz} = \int_0^6 \left(\int_0^6 \left(\int_{\sqrt{x}}^{2\sqrt{6}} ycdy \right) dz \right) dx = c \int_0^6 \left(\int_0^6 2x - \frac{x}{2} dz \right) dx = 6c \int_0^6 2x - \frac{x}{2} dx =$$

$$= 6c \left(\int_0^6 2xdx - \int_0^6 \frac{x}{2} dx \right) = \boxed{276c}$$

$$M_{xy} = \int_0^6 \left(\int_0^6 \left(\int_{\sqrt{x}}^{2\sqrt{6}} zcdy \right) dz \right) dx = c \int_0^6 \left(\int_0^6 z\sqrt{x}dz \right) dx = 18c \int_0^6 \sqrt{x}dx = \boxed{72c\sqrt{6}}$$

So, the solution is:

$$\boxed{(x_c, y_c, z_c) = \left(\frac{M_{yz}}{m}, \frac{M_x z}{m}, \frac{M_x y}{m} \right) = (3.6, 2.76, 3)}$$

4. **Find the amplitude, period and initial position of particles that are in a simple harmonic motion of the following equation**

$$x(t) = -cos\left(3t + \frac{\pi}{4}\right)$$

- The **amplitude** is the maximum displacement from equilibrium and it's usually measured in meters.
- The **period** is the time it takes to complete one oscillation and it's usually measured in seconds.

We have a simple harmonic motion equation, whose general formula is given by the expression

$$x(t) = Acos(\omega t + \Phi)$$

where A is the amplitude, ω the angular frequency and Φ the phase shift. So, for our equation we have:

- **Amplitude**: $\boxed{A = 1m}$
- **Period**: $T = \frac{2\pi}{\omega} = \frac{2\pi}{3} \implies \boxed{T = \frac{2\pi}{3}s}$
- **Initial Position**: $x(0) = -cos(\frac{\pi}{4}) \implies \boxed{x(0) \approx -0.707m}$

5. **Expand the following function in Fourier series**

$$f(x) = \begin{cases} -(x+\pi) & dla \quad -\pi < x < -\frac{\pi}{2} \\ x & dla \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -(x-\pi) & dla \quad \frac{\pi}{2} < x \leq \pi \end{cases}$$

Let's call the Fourier Series Expansion of $f(x)$ as $s(x)$. We know that:

$$s(x) = a_0 + \sum_{n=1}^{\infty} a_n cos(nx) + b_n sin(nx) \tag{14}$$

where,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) cos(\frac{n\pi}{L}x)dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) sin(\frac{n\pi}{L}x)dx \tag{15}$$

In our case $L = \pi$ since we are in the interval $I = [-\pi, \pi]$. Let's calculate the coefficients:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{-\pi/2} -(x+\pi)dx + \int_{-\pi/2}^{\pi/2} xdx + \int_{\pi/2}^{\pi} -(x-\pi)dx\right) =$$
$$= \frac{1}{\pi}\left(-\left(\frac{x^2}{2} + \pi x\right]_{-\pi}^{-\pi/2} + \left(\frac{x^2}{2}\right]_{-\pi/2}^{\pi/2} - \left(\frac{x^2}{2} - \pi x\right]_{\pi/2}^{\pi}\right) =$$
$$= \frac{1}{\pi}\left(-\left(\frac{\pi^2}{8} - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \pi^2\right) + 0 - \left(-\frac{\pi^2}{8} + \frac{\pi^2}{2} + \frac{\pi^2}{2} - \pi^2\right)\right) = \frac{1}{\pi}\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) = 0$$

(Since the accounts are somewhat long and complex I have used an integral calculator to evaluate the values)

$$15 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(\frac{n\pi}{\pi}x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx =$$
$$= \frac{1}{\pi}\left(\int_{-\pi}^{-\pi/2} -(x+\pi)cos(nx)dx + \int_{-\pi/2}^{\pi/2} xcos(nx)dx + \int_{\pi/2}^{\pi} -(x-\pi)cos(nx)dx\right) =$$
$$= \frac{1}{\pi}\left(\frac{2cos(\pi n) + \pi n sin(\frac{\pi n}{2}) - 2cos(\frac{\pi n}{2})}{2n^2} + 0 - \frac{2cos(\pi n) + \pi n sin(\frac{\pi n}{2}) - 2cos(\frac{\pi n}{2})}{2n^2}\right) = 0$$

$$15 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx) =$$
$$= \frac{1}{\pi}\left(\int_{-\pi}^{-\pi/2} -(x+\pi)sin(nx)dx + \int_{-\pi/2}^{\pi/2} xsin(nx)dx + \int_{\pi/2}^{\pi} -(x-\pi)sin(nx)dx\right) =$$
$$= \frac{1}{\pi}\left(\frac{-2sin(\pi n) + 2sin(\frac{\pi n}{2}) + \pi n cos(\frac{\pi n}{2})}{2n^2} + \frac{2sin(\frac{\pi n}{2}) - \pi n cos(\frac{\pi n}{2})}{n^2} + \right.$$
$$\left. + \frac{-2sin(\pi n) + 2sin(\frac{\pi n}{2}) + \pi n cos(\frac{\pi n}{2})}{2n^2}\right) =$$
$$= \frac{1}{\pi}\left(\frac{-2sin(\pi n) + 2sin(\frac{\pi n}{2}) + \pi n cos(\frac{\pi n}{2})}{n^2} + \frac{2sin(\frac{\pi n}{2}) - \pi n cos(\frac{\pi n}{2})}{n^2}\right) = \frac{1}{\pi}\left(\frac{-2sin(\pi n) + 4sin(\frac{\pi n}{2})}{n^2}\right)$$

So, using 14, we have that:

$$\boxed{s(x) = \sum_{n=1}^{\infty}\left(\frac{-2sin(\pi n) + 4sin(\frac{\pi n}{2})}{\pi n^2}\right) sin(nx)}$$

6. **Determinate the Fourier transform of the following function**

$$f(x) = e^{-|x|}$$

We know that The Fourier Transformation is

$$F_{trans} = \int_{-\infty}^{+\infty} f(x)e^{-i\omega x}dx$$

So, taking into account that we can divide the absolute value:

$$
\begin{aligned}
F_{trans} &= \int_{-\infty}^{+\infty} e^{-|x|}e^{-i\omega x}dx = \int_{-\infty}^{0} e^{-|x|}e^{-i\omega x}dx + \int_{0}^{+\infty} e^{-|x|}e^{-i\omega x}dx = \\
&= \int_{-\infty}^{0} e^{x}e^{-i\omega x}dx + \int_{0}^{+\infty} e^{-x}e^{-i\omega x}dx = \int_{-\infty}^{0} e^{x-i\omega x}dx + \int_{0}^{+\infty} e^{-x-i\omega x}dx = \\
&= \int_{-\infty}^{0} e^{(1-i\omega)x}dx + \int_{0}^{+\infty} e^{-(1+i\omega)x}dx = \frac{1}{1-i\omega}e^{(1-i\omega)x}\Big]_{-\infty}^{0} - \frac{1}{1+i\omega}e^{-(1+i\omega)x}\Big]_{0}^{+\infty} = \\
&= \frac{1}{1-i\omega}e^{-i\omega x}e^{x}\Big]_{-\infty}^{0} - \frac{1}{1+i\omega}e^{-i\omega x}e^{-x}\Big]_{0}^{+\infty} = \frac{1}{1-i\omega}(1-0) - \frac{1}{1+i\omega}(0-1) = \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \\
&= \frac{1+i\omega+1-i\omega}{(1-i\omega)(1+i\omega)} = \boxed{\frac{2}{1+\omega^2}}
\end{aligned}
$$