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Faculty of Applied Physics and Mathematics
Exercises II
Differential Equations III

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1. **Find the Laplace transform for the following functions**

   (a) $f(t) = e^{3t}sin(t)$

   To complete the Laplace transform, we need our function $f(t)$ to satisfy the following conditions:

   - $f(t)$ is defined $\forall x \in [0, \infty)$
   - $f(t)$ is continuous for every interval $0 < x < b$
   - $f(t)$ is α-exponential, that is, $\forall x > x_0$, $|f(x)| < Ae^{\alpha x}$

   The Laplace transform of a function $f(t)$ with the above mentioned condition is defined as:

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt \tag{1}$$

   In our case, all functions in the exercise satisfy the conditions to compute their Laplace transform.

   In this first function we will have:

$$1 = \int_0^\infty e^{3t}sin(t)e^{-st}dt = \int_0^\infty sin(t)e^{t(3-s)}dt =$$
$$= \begin{bmatrix} u = sin(t) & du = cos(t) \\ dv = e^{(3-s)t} & v = \frac{(3-s)e^{(3-s)t}}{s^2-6s+9} \end{bmatrix} = \frac{(3-s)e^{(3-s)t}sin(t)}{s^2-6s+9} - \int \frac{(3-s)e^{(3-s)t}cos(t)}{s^2-6s+9}dt =$$
$$= \begin{bmatrix} u = cos(t) & du = -sin(t) \\ dv = \frac{(3-s)e^{(3-s)t}}{s^2-6s+9} & v = \frac{e^{(3-s)t}}{s^2-6s+9} \end{bmatrix} = \frac{(3-s)e^{(3-s)t}sin(t)}{s^2-6s+9} -$$
$$-\left(\frac{e^{(3-s)t}cos(t)}{s^2-6s+9} - \int -\frac{e^{(3-s)t}sin(t)}{s^2-6s+9}dt\right) = \left.\frac{(3-s)e^{(3-s)t}sin(t) - e^{(3-s)t}cos(t)}{s^2-6s+10}\right]_0^\infty = \frac{1}{s^2-6s+10}$$

   Therefore,

$$\boxed{\begin{array}{ll} if\ s-3<0 & \nexists L\{f(t)\} \\ if\ s-3>0 & L\left\{e^{3t}sin(t)\right\} = \frac{1}{s^2-6s+10} \end{array}}$$

   (b) $f(t) = \int_0^t \frac{sin(4t)}{t}dt$

   The function $f(t) = \int_0^t \frac{sin(4t)}{t}dt = sin(4t)$ is known as the sine integral. It's known $L\left\{\frac{sin(t)}{t}\right\} = arctan(\frac{1}{s})$. Then,

$$L\left\{\frac{sin(at)}{at}\right\} = \frac{1}{a}\left\{\frac{sin(at)}{t}\right\} = \frac{1}{a}arctan(\frac{1}{\frac{s}{a}}) - \frac{1}{a}arctan(\frac{a}{s})$$

so, $L\left\{\frac{sin(at)}{t}\right\} = arctan(\frac{a}{s})$.

Theorem: If $L\{f(t)\} = F(s)$, then $L\left\{\int_0^t f(t)dt\right\} = F(s)/s$.

Hence, as $L\left\{\frac{sin(t)}{t}\right\} = arctan(\frac{4}{s})$

$$\boxed{L\{f(t)\} = F(s) = \frac{1}{s}arctan(\frac{4}{s})}$$

(c) $f(t) = sin(t)cos(t)$

We know that $sin(t)cos(t) = \frac{1}{2}sin(2t)$. Therefore,

$$L(f(t)) = F(s) = \int_0^\infty \frac{1}{2}sin(2t)e^{-st}dt = \frac{1}{2}\int_0^\infty sin(2t)e^{-st}dt =$$

$$= \left[\begin{array}{cc} u = sin(2t) & du = 2cos(2t) \\ dv = e^{-st} & v = -\frac{e^{-st}}{s} \end{array}\right] = \frac{1}{2}\left[-\frac{sin(2t)e^{-st}}{s} - \int -\frac{2cos(2t)e^{-st}}{s}\right] =$$

$$= \left[\begin{array}{cc} u = 2cos(2t) & du = -4sin(2t) \\ dv = -\frac{e^{-st}}{s} & v = \frac{e^{-st}}{s^2} \end{array}\right] = \frac{1}{2}\left[-\frac{sin(2t)e^{-st}}{s} - \frac{2cos(2t)e^{-st}}{s^2} - \int \frac{4sin(2t)e^{-st}}{s^2}dt\right] =$$

$$= -\frac{sin(2t)e^{-st}}{s} - \frac{2cos(2t)e^{-st}}{s^2} - \frac{4}{s^2}\int sin(2t)e^{-st}dt$$

As $\int sin(2t)e^{-st}$ appears twice, we can solve it as follows:

$$\int sin(2t)e^{-st} = \frac{-sin(2t)e^{-st} - 2cos(2t)e^{-st}}{s^2+4}$$

$$\frac{1}{2}\int sin(2t)e^{-st} = \frac{-sin(2t)se^{-st} - 2cos(2t)e^{-st}}{s^2+4}$$

Between 0 and ∞:

$$\boxed{L\{f(t)\} = F(s) = \frac{1}{s^2+4}}$$

When $s > 0$. If $s < 0$ the Laplace transform does not exist.

(d) $f(t) = cos^3(t)$

$cos^3(t) = \frac{1}{4}(3cos(t) + cos(3t))$

$L(f(t)) = F(s) = \int_0^\infty \frac{e^{-st}}{4}(3cos(t) + cos(3t))dt = \frac{3}{4}\int_0^\infty e^{-st}cos(t)dt + \frac{1}{4}\int_0^\infty cos(3t)e^{-st}dt$

Let's solve each integral:

$$\int_0^\infty e^{-st}cos(t)dt = \left[\begin{array}{cc} u = cos(t) & du = -sin(t) \\ dv = e^{-st} & v = -\frac{e^{-st}}{s} \end{array}\right] = -\frac{e^{-st}}{s}cos(t) - \int \frac{sin(t)e^{-st}}{s}ds =$$

$$= \left[\begin{array}{cc} u = sin(t) & du = cos(t) \\ dv = -\frac{e^{-st}}{s} & v = \frac{e^{-st}}{s^2} \end{array}\right] = -\frac{cos(t)e^{-st}}{s} - \frac{sin(t)e^{-st}}{s^2} - \frac{1}{s^2}\int cos(t)e^{-st}ds \quad (2)$$

$$\int e^{-st}cos(t)dt\left(1 + \frac{1}{s^2}\right) = -\frac{cos(t)e^{-st}}{s} - \frac{sin(t)e^{-st}}{s^2} \quad (3)$$

Replace 3 into 2:

$$\int_0^\infty e^{-st}cos(t)dt = \left.\frac{sin(t)e^{-st} - cos(t)se^{-st}}{1+s^2}\right]_0^\infty = \frac{s}{s^2+1} \tag{4}$$

Let's go with the second integral:

$$\int_0^\infty cos(3t)e^{-st}dt = \left[\begin{array}{cc} u = cos(3t) & du = -3sin(3t) \\ dv = e^{-st} & v = -\frac{e^{st}}{s} \end{array}\right] = -\frac{e^{-st}cos(3t)}{s} - \int \frac{3sin(3t)e^{-st}}{s}dt =$$

$$= \left[\begin{array}{cc} u = -3sin(3t) & du = -9cos(3t) \\ dv = -\frac{e^{-st}}{s} & v = \frac{e^{-st}}{s^2} \end{array}\right] = -\frac{cos(3t)e^{-st}}{s} + \frac{3sin(3t)e^{-st}}{s^2} + \int \frac{9cos(3t)e^{-st}}{s^2}dt =$$

$$= \left.\frac{3sin(3t)e^{-st} + cos(3t)se^{-st}}{s^2+9}\right]_0^\infty = \frac{s}{s^2+9} \tag{5}$$

Hence, using 4 and 5 we have that:

$$F(s) = \frac{3}{4}\frac{s}{s^2+1} + \frac{1}{4}\frac{s}{s^2+9} = \boxed{\frac{s^3+7s}{s^4+10s+9} \quad for\ s > 0}$$

As before, if $s < 0$, $\nexists L(f(t))$.

2. **Find the inverse Laplace transform of the following functions:**

(a) $F(s) = \frac{s-1}{s^2-2s-3}$

$$\frac{s-1}{s^2-2s-3} = \frac{s-1}{(s+1)(s-3)} = \frac{a}{s+1} + \frac{b}{s-3}$$

$$s-1 = a(s-3) + b(s+1)$$
$$s = 1 \implies -2 = -4a \implies a = \frac{1}{2}$$
$$s = 3 \implies 2 = 4b \implies b = \frac{1}{2}$$

$$\frac{1/2}{s+1} + \frac{1/2}{s-3} \implies \frac{s-1}{s^2-2s-3} = \frac{1}{2(s+1)} + \frac{1}{2(s-3)}$$

So, we have that:

$$L^{-1}(f(t)) = L^{-1}\left\{\frac{1}{2(s+1)}\right\} + L^{-1}\left\{\frac{1}{2(s-3)}\right\}$$

Having in mind that $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$, we got

$$\boxed{L^{-1}\{f(t)\} = \frac{1}{2}(e^{-t} + e^{3t})}$$

(b) $F(s) = \frac{5s}{(s^2+1)(s-1)}$

$$\frac{5s}{(s^2+1)(s-1)} = \frac{a+bs}{s^2+1} + \frac{c}{s-1}$$

$$5s = (a+bs)(s-1) + c(s^2+1)$$
$$s = 1 \implies 5 = 2c \implies c = \frac{5}{2}$$
$$5s = as - a + bs^2 - bs + \frac{5}{2}s^2 + \frac{5}{2}$$
$$5s = s(a-b) + s^2(b + \frac{5}{2}) + (\frac{5}{2} - a)$$
$$b + \frac{5}{2} = 1 \implies b = -\frac{5}{2}$$
$$\frac{5}{2} - a = 0 \implies a = \frac{5}{2}$$

$$\frac{5/2 - 5s/2}{s^2+1} + \frac{s/2}{s-1} = \frac{5(1-s)}{2(s^2+1)} + \frac{5}{2(s-1)} \implies$$
$$\implies \frac{5s}{(s^2+1)(s-1)} = \frac{5}{2(s-1)} + \frac{5(1-s)}{2(s^2+1)}$$

So we have that:

$$L^{-1}(F(s)) = \frac{5}{2}L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{5}{2}L^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{5}{2}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

We knows that

$$L\left\{sin(at)\right\} = \frac{a}{s^2+a^2} \implies L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = sin(at)$$

Therefore,

$$L^{-1}\left\{\frac{1}{s^2+1}\right\} = sin(t)$$

On the other hand, $L(cos(at)) = \frac{s}{s^2+a^2}$, so

$$L^{-1}\left\{\frac{s}{s^2+1}\right\} = cos(t)$$

And we already know that $L^{-1}\left\{\frac{1}{s-1}\right\} = e^t$
Joining all these results:

$$\boxed{L^{-1}(F(s)) = \frac{5}{2}(sin(t) - cos(t) + e^t)}$$

(c) $F(s) = \frac{1}{s+1}$

As $L(e^{at}) = \frac{1}{s-a} \implies L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$. For that reason, $\boxed{L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}}$, because $\frac{1}{s+1} = \frac{1}{s-(-1)}$

(d) $F(s) = \frac{2}{s^3-s^2+s-1}$

$$\frac{2}{s^3-s^2+s-1} = \frac{a}{s-1} + \frac{b}{s^2+1}$$

$$2 = a(s^2+1) + b(s-1)$$
$$s = 1 \implies 2 = 2a \implies a = 1$$
$$s = 0 \implies 2 = a - b \implies b = -1$$

$$\frac{2}{s^3-s^2+s-1} = \frac{1}{s-1} + \frac{-1}{s^2+1}$$

So we have that

$$L^{-1}(f(t)) = L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{-1}{s^2+1}\right\}$$

Having in mind that $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ and $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = sin(at)$, we get that

$$\boxed{L^{-1}\left\{f(t)\right\} = e^t + sin(-t)}$$

3. **Use the method of Laplace transform to solve the following initial value problems:**

(a) $y''(t) - 3y'(t) + 2y(t) = 8te^{-t},\ y(0) = y'(0) = 0,\ y''(0) = 1$
Let's call $L\left\{y(x)\right\} = F(s)$. We can express the transformation of the derivative as $L\left\{y'(x)\right\} = sF(s) - y(0)$ and we are going to call it $F_1(s)$. If we apply that formula in a recursive way:

$$L\left\{y^{n)}(x)\right\} = s^nF(s) - s^{n-1}y(0) - s^{n-2}y'(0) - ... - y^{n-1)}(0)$$

All these derivatives of $y(x)$ in the origin are the initial conditions. Once we have this, we can solve the exercise:

$$L\left\{y''(t)\right\} - 3L\left\{y'(t)\right\} + 2L\left\{y(t)\right\} = 8L\left\{te^{-t}\right\}$$

Then,

$$\begin{aligned}
s^3F(s) - s^2y(0) - sy'(0) - y''(0) - 3sF(s) - 3y(0) + 2F(s) &= \frac{1}{(s+1)^2} \\
L\left\{te^{-t}\right\} = \int_0^\infty te^{-t}e^{-st} &= \frac{1}{(s+1)^2} \\
s^3F(s) - 0s^2 - 0s - 1 - 3sF(s) - 0\cdot 3 - 2F(s) &= \frac{1}{(s+1)^2} \\
(s^3 - 3s + 2)F(s) = \frac{1}{(s+1)^2} + 1 \implies F(s) &= \frac{1 + (s+1)^2}{(s+1)^2(s^3 - 3s + 2)}
\end{aligned}$$

Then,

$$\boxed{L^{-1}\left\{F(s)\right\} = L^{-1}\left\{\frac{1 + (s+1)^2}{(s+1)^2(s^3 - 3s + 2)}\right\} = y(t)}$$

(b) $\left.\begin{array}{r} y'(t) - y(t) - z(t) = e^t \\ z'(t) + y(t) - z(t) = 0 \end{array}\right\} \; y(0) = 0, \; z(0) = 1$

We can express it as

$$\begin{bmatrix} y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

By the same reasoning in previous exercise, we can write:

$$\begin{bmatrix} sF(s) - y(0) \\ sG(s) - z(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} F(s) \\ G(s) \end{bmatrix} + \begin{bmatrix} L\left\{e^t\right\} \\ L\left\{0\right\} \end{bmatrix}$$

Our goal is to find $F(s)$ and $G(s)$:

$$\left.\begin{array}{r} sF(s) - 0 = F(s) + G(s) + L\left\{e^t\right\} \\ sG(s) - 1 = -F(s) + G(s) + L\left\{0\right\} \end{array}\right\} \implies \left.\begin{array}{c} sF(s) = F(s) + G(s) + \frac{1}{s-1} \\ sG(s) - 1 = G(s) - F(s) \end{array}\right\}$$

$$\begin{aligned}
F(s) &= (1-s)G(s) + 1 \\
s\left[(1-s)G(s) + 1\right] &= (1-s)G(s) + 1 + \frac{1}{s-1} \\
sG(s) - s^2G(s) + s &= G(s) - sG(s) + 1 + \frac{1}{s-1} \\
G(s)(s - s^2 - 2 + s) &= 1 + \frac{1}{s-1} - s \\
G(s) &= \frac{1}{2s - s^2 - 2}\,\frac{-s^2 + 2s}{s-1} \\
G(s) &= \frac{2s - s^2}{(s-1)(2s - s^2 - 2)}
\end{aligned}$$

$$F(s) = \frac{(1-s)(2s-s^2)}{(s-1)(2s-s^2-2)} + 1 = \frac{(1-s)(2s-s^2)+(s-1)(2s-s^2-2)}{(s-1)(2s-s^2-2)} =$$

$$= \frac{2s+s^3-s^2-2s^2+2s^2-s^3-2s-2s+s^2+2}{(s-1)(2s-s^2-2)} = \frac{2(1-s)}{(s-1)(2s-s^2-2)} = \frac{-2}{2s-s^2-2} =$$

$$= \frac{2}{s^2-2s+2}$$

The solution to our problem would be $y(t)$ and $z(t)$, where

$$y(t) = L^{-1}\left\{F(s)\right\}, \quad z(t) = L^{-1}\left\{G(s)\right\}$$

We have then:

$$L^{-1}\left\{F(s)\right\} = L^{-1}\left\{\frac{2}{s^2-2s+2}\right\} = L^{-1}\left\{\frac{2}{(s-1)^2+1}\right\} = 2L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} = 2e^t sin(t)$$

$$L^{-1}\left\{G(s)\right\} = L^{-1}\left\{\frac{2s-s^2}{(s-1)(2s-s^2-2)}\right\}$$

$$\frac{2s-s^2}{(s-1)(2s-s^2-2)} = \frac{a}{s-1} + \frac{b+cs}{s^2+2s+2}$$

$$2s-s^2 = -as^2+2as-2a-bs+b-cs^2+cs$$

$$2s-s^2 = s^2(-a-c)+s(2a-b+c)+(b.2a)$$

$$2s-s^2 = s^2(1-c)+s(c-b-2)+(b+2)$$

$$\left.\begin{array}{l} 1-c=-1 \\ c-b-2=2 \\ b+2=b \end{array}\right\} \implies c=2,\ b=-2$$

$$\frac{2s-s^2}{(s-1)(2s-s^2-2)} = -\frac{1}{s-1} + \frac{2s-2}{s^2-2s+2}$$

$$L^{-1}\left\{-\frac{1}{s-1} + \frac{2s-2}{s^2-2s+2}\right\} = -e^t + L{-}1\left\{frac2s-2s^2-2s+2\right\}$$

$$\frac{2s-2}{s^2-2s+2} = \frac{2(s-1)}{(s-1)^2+1} \implies L{-}1\left\{frac2s-2s^2-2s+2\right\} = 2e^t cos(t)$$

$$L^{-1}\left\{G(s)\right\} = -e^t + 2e^t cos(t)$$

So, we have that:

$$\boxed{\begin{array}{c} y(t) = 2e^t sin(t) \\ z(t) = 2e^t cos(t) - e^t \end{array}}$$