Kalman Filtering

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1 Introduction

The Kalman filter (also known as a Linear Quadratic Estimator), is a state estimation algorithm that uses noisy measurements observed over time. The algorithm produces estimates of the state that tend to be more accurate that those based on a single measurement alone (as this observation is corrupted by noise). The algorithm estimates a *joint probability distribution* over the variables at each timestep. The estimation of the joint probability distribution only relies on the information in the previous timestep (i.e. Markov property). The use of the Kalman filter is motivated by an example of estimating the uncertain states of a ballistic missile.

2 Ballistic Model Dynamics

In this section, a derivation of the dynamics of a ballistic missile in discrete time is presented. The derivation follows what is presented in Wikipedia [1]. A **Ballistic Missile** is a missile with a high, arching trajectory, which is initially powered and guided but falls under gravity on to its target. Ballistic missile defense systems may employ the use of a smaller guided missile to intercept the ballistic missile in flight to target. In order to intercept an incoming missile, the interceptor needs to "learn" the state of the missile so it knows where to go!

A pioneer ballistic missile was the V2 (Fig. 1), developed in Nazi Germany in the 1930s and 1940s under the direction of Wernher von Braun. By the end of World War II in Europe in May 1945, over 3,000 V-2s had been launched. An estimated 2,754 civilians were killed in London by V-2 attacks with another 6,523 injured (`https://en.wikipedia.org/wiki/V-2_rocket`).

Using Newton's second law under constant acceleration, one can derive the equations of motion in the horizontal (x) and vertical (y) directions. An example of a ballistic trajectory in 2 dimensions is shown in Fig. 2. In the $[0, t]$ time interval, the equations of motion in the x direction can be expressed as:

German V2 Rocket (replica)

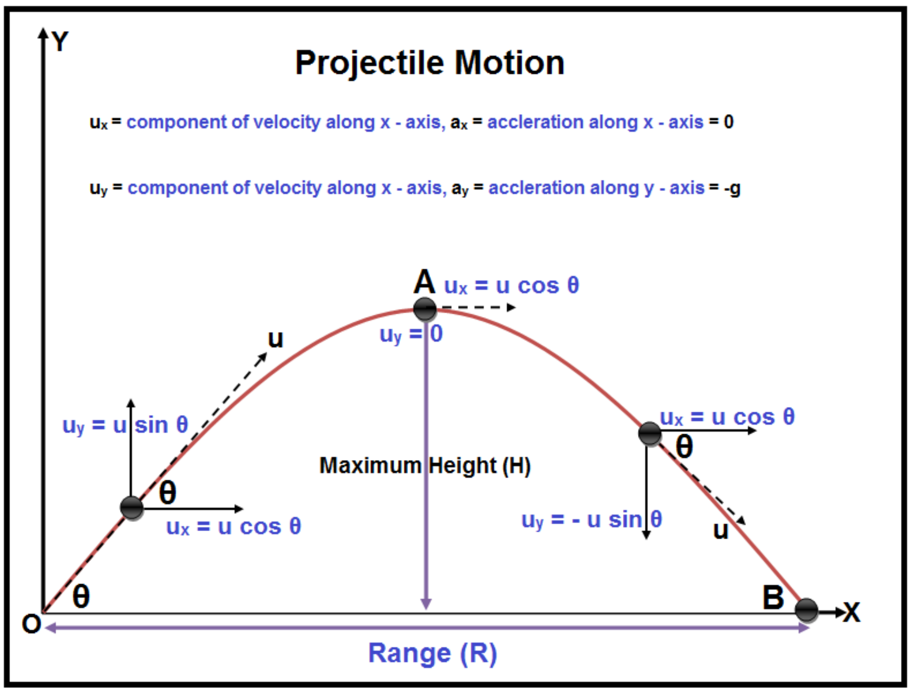

Figure 2: Ballistic trajectory. u_x and u_y denote the x and y direction velocities (respectively)

$$
\begin{aligned}
a_x(t) &= a, \quad \text{constant} && (1)\\
u_x(t) &= \int_0^t a_x(\tau)d\tau + u_x(0) && (2)\\
&= at && (3)\\
x(t) &= \int_0^t \Big(u_x(\tau)\Big)d\tau + x(0) && (4)\\
&= u_x(0)t + \frac{1}{2}at^2 && (5)
\end{aligned}
$$

From the equations:

$$
\begin{aligned}
u_x(t) - u_x(0) &= at && (6)\\
x(t) - x(0) &= u_x(0)t + \frac{1}{2}at^2 && (7)
\end{aligned}
$$

We substitute $\dot{x} \rightarrow u_x$, resulting in

$$
\begin{aligned}
\dot{x}(t) &= at + \dot{x}(0) && (8)\\
x(t) &= \dot{x}(0)t + \frac{1}{2}at^2 + x(0) && (9)
\end{aligned}
$$

Instead of modeling the whole trajectory (x and $\dot{x}$) continuously, we "discretize" the system (i.e. break up time into chunks), and model the trajectory over each chunk of time. Assuming all chunks of time are uniform in length of Δt, we consider the trajectory over an arbitrary interval between timesteps k and $k+1$. We now have:

$$
\begin{aligned}
x(k+1) &= x(k) + \dot{x}(k)\Delta t + \frac{1}{2}a\Delta t^2 && (10)\\
\dot{x}(k+1) &= \dot{x}(k) + a\Delta t && (11)
\end{aligned}
$$

Let $\boldsymbol{x}_k = [x(k), \dot{x}(k)]^T$. We can now rewrite the expression as:

$$
\boldsymbol{x}_{k+1} = \boldsymbol{F}\boldsymbol{x}_k + \boldsymbol{G}a_k \tag{12}
$$

with

$$
\boldsymbol{F} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{G} = \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} \tag{13}
$$

and $a_k = a$.

Using the same equations to model the y-direction, we can extend the previous dynamic equation to model both the x and y components of the ballistic missile trajectory. Let $\boldsymbol{x}_k = [x(k), \dot{x}(k), y(k), \dot{y}(k)]^T$. The equations of motion for the missile now become:

$$\boldsymbol{x}_{k+1} = \boldsymbol{F}\boldsymbol{x}_k + \boldsymbol{G}\boldsymbol{a}_k \tag{14}$$

with

$$\boldsymbol{F} = \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{G} = \begin{bmatrix} \frac{1}{2}\Delta t^2 & 0 \\ \Delta t & 0 \\ 0 & \frac{1}{2}\Delta t^2 \\ 0 & \Delta t \end{bmatrix} \tag{15}$$

and $\boldsymbol{a}_k = [0, -g]^T$.

Now let's assume that a disturbance imparts a constant acceleration on the missile at each timestep. The acceleration forces are random between timesteps, but follow a probability distribution. We express this as:

$$\boldsymbol{x}_{k+1} = \boldsymbol{F}\boldsymbol{x}_k + \boldsymbol{G}\boldsymbol{a}_k \tag{16}$$

with

$$\boldsymbol{F} = \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{G} = \begin{bmatrix} \frac{1}{2}\Delta t^2 & 0 \\ \Delta t & 0 \\ 0 & \frac{1}{2}\Delta t^2 \\ 0 & \Delta t \end{bmatrix} \tag{17}$$

and $\boldsymbol{a}_k = [0 + w_x(k), -g + w_y(k)]^T$.

In addition to our equations of motion, we also need to model how we "observe" the missile in motion. Suppose we can measure the x and y coordinates of the missile, with some uncertainty. We express this as:

$$\boldsymbol{z}_k = \boldsymbol{H}\boldsymbol{x}_k + \boldsymbol{v}_k \tag{18}$$

with

$$\boldsymbol{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{v}_k = \begin{bmatrix} v_x(k) \\ v_y(k) \end{bmatrix} \tag{19}$$

Putting the dynamics and the measurement together, this system can be expressed as:

$$\boldsymbol{x}_{k+1} = \boldsymbol{F}\boldsymbol{x}_k + \boldsymbol{G}\boldsymbol{a}_k \tag{20}$$
$$\boldsymbol{z}_k = \boldsymbol{H}\boldsymbol{x}_k + \boldsymbol{v}_k \tag{21}$$

where $\boldsymbol{x}_k$ and $\boldsymbol{z}_k$ are random variables. We've now developed a linear state space representation of the dynamics of the ballistic missile. We will next provide a very brief overview of random variables.

3 Brief Overview of Random Variables

Consider a random variable (scalar) x:

- $E[x] = \bar{x}$ is the **expected value** of x, also known as the mean
- $\mathrm{Var}(x) = E[x - \bar{x}]^2$ is the **variance** of x, that represents the amount by which a random variable differs from the expected value

Consider two random variables (vector) $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$:

- $E[\boldsymbol{x}_1] = \bar{\boldsymbol{x}}_1$ is the expected value (mean) of $\boldsymbol{x}_1$
- $E[\boldsymbol{x}_2] = \bar{\boldsymbol{x}}_2$ is the expected value (mean) of $\boldsymbol{x}_2$
- $\mathrm{Cov}(\boldsymbol{x}_1, \boldsymbol{x}_2) = E[(\boldsymbol{x}_1 - \bar{\boldsymbol{x}}_1)(\boldsymbol{x}_2 - \bar{\boldsymbol{x}}_2)^T]$ is the **covariance** of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, that measures the joint variability of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$

The **variance** matrix is a special case of the covariance matrix. If $\boldsymbol{x} = [x_1, x_2]^T$, then:

$$\mathrm{Var}(\boldsymbol{x}) = E[(\boldsymbol{x} - \bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}})^T] \tag{22}$$
$$= \begin{bmatrix} (x_1 - \bar{x}_1)^2 & (x_1 - \bar{x}_1)(x_2 - \bar{x}_2) \\ (x_1 - \bar{x}_1)(x_2 - \bar{x}_2) & (x_2 - \bar{x}_2)^2 \end{bmatrix} \tag{23}$$

Notice that the *trace* of the variance matrix is:

$$\mathrm{tr}(\mathrm{Var}(\boldsymbol{x})) = (x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2 \tag{24}$$

This implies that minimizing the trace of the variance matrix drives $x_1 \to \bar{x}_1$ and $x_2 \to \bar{x}_2$.

4 Kalman Filter Overview

A Kalman Filter builds estimates of the covariance matrix of the *estimate* of the state $\boldsymbol{x}_k$. The algorithm is structured in such a way that the trace of the covariance matrix of the estimate is *minimized*. The algorithm is *recursive* in that the estimate at the present state depends only on information from the past timestep plus the present observation. In contrast to batch estimation techniques (like LSE - which we learned in class), the Kalman Filter doesn't require the history of estimates and observations

Let $\hat{\boldsymbol{x}}_{k+1|k}$ denote the estimate of the state at time $k+1$, made using information from step k. We call this the *a priori* estimate (sometimes also called a prediction) of $\boldsymbol{x}_{k+1}$. This estimate is made before the the system is observed at step k, before we've seen $\boldsymbol{z}_{k+1}$. Let $\boldsymbol{P}_{k+1|k}$ denote the covaraince matrix estimate associated with the error in the estimate of $\hat{\boldsymbol{x}}_{k+1|k}$. $\boldsymbol{P}_{k+1|k}$ captures the uncertainty associated with our estimate of the state $\hat{\boldsymbol{x}}_{k+1|k}$ given what we know at timestep k.

Now we observe the measurement: $\boldsymbol{z}_k$. Let $\hat{\boldsymbol{x}}_{k+1|k+1}$ denote the estimate of the state at time k, made using information from step $k+1$. We call this the *a posteriori* estimate of $\boldsymbol{x}_{k+1}$. This estimate is made after the the system is observed at step k+1, after we've seen $\boldsymbol{z}_{k+1}$!. Let $\boldsymbol{P}_{k+1|k+1}$ denote the covaraince matrix estimate associated with the error in the estimate of $\hat{\boldsymbol{x}}_{k+1|k+1}$. $\boldsymbol{P}_{k+1|k+1}$ captures the uncertainty associated with our estimate of the state $\hat{\boldsymbol{x}}_{k+1|k+1}$ given we have all of the information at timestep $k+1$. The prediction $\hat{\boldsymbol{x}}_{k|k-1}$ will be different from the estimate $\hat{\boldsymbol{x}}_{k|k}$!

Consider a linear dynamic system of the form:

$$\boldsymbol{x}_{k+1} = \boldsymbol{F}_k \boldsymbol{x}_k + \boldsymbol{G}_k \boldsymbol{u}_k + \boldsymbol{w}_k, \quad \boldsymbol{w}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_k) \tag{25}$$

$$\boldsymbol{z}_k = \boldsymbol{H}_k \boldsymbol{x}_k + \boldsymbol{v}_k, \quad \boldsymbol{v}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}_k). \tag{26}$$

We define the following terms that will be needed during the derivation:

- $\hat{\boldsymbol{x}}_{k+1|k+1}$: a posteriori estimate
- $\hat{\boldsymbol{x}}_{k+1|k}$: a priori prediction
- $\tilde{\boldsymbol{x}}_{k|k} = (\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k})$: estimation error
- $\tilde{\boldsymbol{x}}_{k+1|k} = (\boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1|k})$: prediction error
- $\boldsymbol{P}_{k|k} = E\big[\tilde{\boldsymbol{x}}_{k|k}\ \tilde{\boldsymbol{x}}_{k|k}^T\big]$: estimation error covariance matrix
- $\boldsymbol{P}_{k+1|k} = E\big[\tilde{\boldsymbol{x}}_{k+1|k}\ \tilde{\boldsymbol{x}}_{k+1|k}^T\big]$: prediction error covariance matrix

Beginning with the *a posteriori* estimate of the state at the present timestep, k, we obtain the prediction of the state at the next timestep:

$$\hat{\boldsymbol{x}}_{k+1|k} = \boldsymbol{F}_k \hat{\boldsymbol{x}}_{k|k} + \boldsymbol{G}_k \boldsymbol{u}_k \tag{27}$$

The error in this prediction is:

$$
\begin{aligned}
\tilde{\boldsymbol{x}}_{k+1|k} &= \boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1|k} = \boldsymbol{F}_k \boldsymbol{x}_k + \boldsymbol{G}_k \boldsymbol{u}_k + \boldsymbol{w}_k - \boldsymbol{F}_k \hat{\boldsymbol{x}}_{k|k} - \boldsymbol{G}_k \boldsymbol{u}_k && (28)\\
&= \boldsymbol{F}_k \big(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k}\big) + \boldsymbol{w}_k && (29)\\
&= \boldsymbol{F}_k \tilde{\boldsymbol{x}}_{k|k} + \boldsymbol{w}_k && (30)
\end{aligned}
$$

$$
\boxed{\tilde{\boldsymbol{x}}_{k+1|k} = \boldsymbol{F}_k \tilde{\boldsymbol{x}}_{k|k} + \boldsymbol{w}_k}
$$

With the prediction error formula, we can determine the *a priori* covariance matrix at the next timestep using the *a posteriori* covariance at the present step

$$
\begin{aligned}
\boldsymbol{P}_{k+1|k} &= E\Big[\tilde{\boldsymbol{x}}_{k+1|k}\ \tilde{\boldsymbol{x}}_{k+1|k}^T\Big] && (31)\\
&= \Big[\big(\boldsymbol{F}_k \tilde{\boldsymbol{x}}_{k|k} + \boldsymbol{w}_k\big)\ \big(\boldsymbol{F}_k \tilde{\boldsymbol{x}}_{k|k} + \boldsymbol{w}_k\big)^T\Big] && (32)\\
&= E\Big[\boldsymbol{F}_k \tilde{\boldsymbol{x}}_{k|k}\ \tilde{\boldsymbol{x}}_{k|k}^T \boldsymbol{F}_k^T\Big] + E\Big[\boldsymbol{w}_k \boldsymbol{w}^T\Big] && (33)\\
&= \boldsymbol{F}_k \boldsymbol{P}_{k|k} \boldsymbol{F}_k^T + \boldsymbol{Q}_k && (34)
\end{aligned}
$$

$$
\boxed{\boldsymbol{P}_{k+1|k} = \boldsymbol{F}_k \boldsymbol{P}_{k|k} \boldsymbol{F}_k^T + \boldsymbol{Q}_k}
$$

We use the *a priori* state estimation used to compute the predicted measurement:

$$
\hat{\boldsymbol{z}}_{k+1|k} \triangleq \boldsymbol{H}_{k+1} \hat{\boldsymbol{x}}_{k+1|k} + \boldsymbol{v}_{k+1} \tag{35}
$$

We can define the measurement prediction error as:

$$
\tilde{\boldsymbol{z}}_{k+1|k} \triangleq \boldsymbol{z}_{k+1} - \hat{\boldsymbol{z}}_{k+1|k} \tag{36}
$$

The measurement prediction error has an associated covariance matrix:

$$
\boldsymbol{S}_{k+1|k} = E\big[\tilde{\boldsymbol{z}}_{k+1|k}\ \tilde{\boldsymbol{z}}_{k+1|k}^T\big] \tag{37}
$$

Expanding the measurement *a priori* covariance matrix:

$$
\boxed{\boldsymbol{S}_{k+1|k} = \boldsymbol{H}_{k+1} \boldsymbol{P}_{k+1|k} \boldsymbol{H}_{k+1}^T + \boldsymbol{R}_{k+1}}
$$

We postulate a relationship between the *a posteriori* state estimate and the *a priori* state prediction at timestep $k+1$ of the form:

$$
\hat{\boldsymbol{x}}_{k+1|k+1} = \hat{\boldsymbol{x}}_{k+1|k} + \boldsymbol{K}_{k+1} \tilde{\boldsymbol{z}}_{k+1|k} \tag{38}
$$

Where $\boldsymbol{K}_{k+1}$ is a matrix that we will determine. Using this formula for the *a posteriori* state estimate, we we can derive the equation for the *a psoteriori* covariance matrix. To begin, we define the state estimation error:

$$\tilde{\boldsymbol{x}}_{k+1|k+1} = (\boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1|k+1}) \tag{39}$$

Define the *a posteriori* covaraince matrix as:

$$\begin{aligned}
\boldsymbol{P}_{k+1|k+1} &= \operatorname{cov}\Big[\tilde{\boldsymbol{x}}_{k+1|k+1}\Big] && (40)\\
&= \operatorname{cov}\Big[\big(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1}\big)\tilde{\boldsymbol{x}}_{k+1|k} - \boldsymbol{K}_{k+1}\boldsymbol{v}_{k+1}\Big] && (41)\\
&= \operatorname{cov}\Big[\big(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1}\big)\tilde{\boldsymbol{x}}_{k+1|k}\Big] - \operatorname{cov}\Big[\boldsymbol{K}_{k+1}\boldsymbol{v}_{k+1}\Big] && (42)\\
&= \big(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1}\big)\boldsymbol{P}_{k+1|k}\big(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1}\big)^T - \boldsymbol{K}_{k+1}\boldsymbol{R}_{k+1}\boldsymbol{K}_{k+1} && (43)
\end{aligned}$$

$$\boxed{\begin{aligned}\boldsymbol{P}_{k+1|k+1} = {} & \big(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1}\big)\boldsymbol{P}_{k+1|k}\big(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1}\big)^T \\ & - \boldsymbol{K}_{k+1}\boldsymbol{R}_{k+1}\boldsymbol{K}_{k+1}^T\end{aligned}}$$

The optimal Kalman Filter gain, $\boldsymbol{K}^*_{k+1}$ is determined by:

$$\boldsymbol{K}^*_{k+1} = \underset{\boldsymbol{K_{k+1}}}{\arg\min}\ \operatorname{tr}(\boldsymbol{P_{k+1|k+1}}) = \underset{\boldsymbol{K_{k+1}}}{\arg\min}\ E\Big[\big|\big|\boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1|k}\big|\big|_2^2\Big] \tag{45}$$

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{K}_{k+1}}\operatorname{tr}\big(\boldsymbol{P}_{k+1|k+1}\big) = {} & -2\big(\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k}\big)^T && (46)\\
& + 2\boldsymbol{K}_{k+1}\underbrace{\big(\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k}\boldsymbol{H}_{k+1}^T + \boldsymbol{R}_{k+1}\big)}_{\boldsymbol{S}_{k+1|k}} && (47)
\end{aligned}$$

Setting the above equation equal to 0 and solving yields:

$$\boxed{\boldsymbol{K}^*_{k+1} = \boldsymbol{P}_{k+1|k}\boldsymbol{H}_{k+1}^T\boldsymbol{S}_{k+1|k}^{-1}}$$

5 Experiment

Consider a 50 second simulation with timesteps of 0.1 seconds where a ballistic missile launched from the initial position of $(x_1, x_2) = (0, 300)$. The initial velocity of the missile is 500 meters per second with an angle (relative to x-axis of 75°, or $(\dot{x}_1, \dot{x}_2) = (500\cos(75°), 500\sin(75°))$. Random disturbances act on the missile and uncertainties are present in the measurement of the position of the missile. Both w_x and w_y are normally distributed with 0 mean and variance of 10, and w_x and w_y are normally distributed with 0 mean and variance of 750. The effectiveness of the Kalman filter in obtaining a good estimate of the position of the missile is shown in Figs. 3 - 6

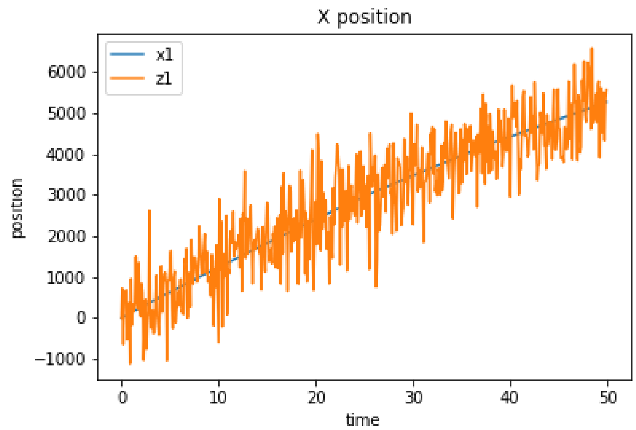

Figure 3: X position, state and measurement shown

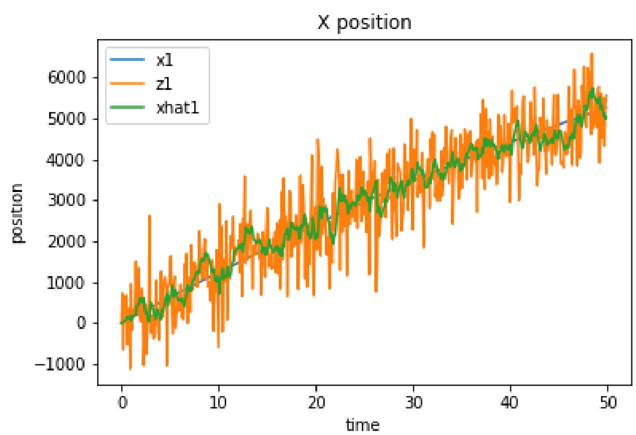

Figure 4: X position, state, measurement, and sate estimate shown

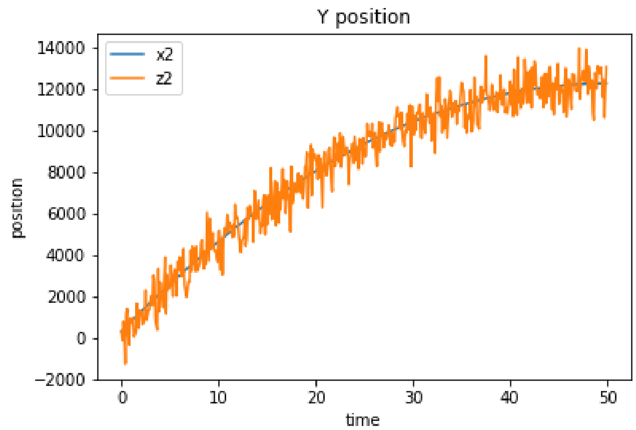

Figure 5: Y position, state and measurement shown

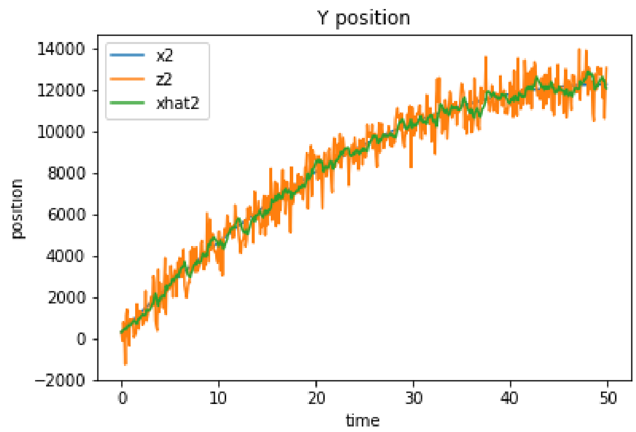

Figure 6: Y position, state, measurement, and sate estimate shown

References

[1] "Kalman filter," accessed July, 2017. [Online]. Available: https://en.wikipedia.org/wiki/Kalman_filter