WEEK 1: ORBITS

GOAL: The basic object in space flight is the elliptical orbit. Your goal is to get familiar with orbits, their shapes, and the dynamics within them

Kepler's Laws

While in space, and while getting to space, a rocket is mainly subjected to a single force: Gravity. If you want to get from Point A to Point B while in space, then you can't just point yourself at Point B and go since gravity will bend and curve where you go. Working against gravity is not possible, and so the objective of spaceflight is to find the best ways to work *with* gravity. That is, space flight is about falling well.

Many people and civilizations have contributed to our understanding of what gravity actually does - Ptolemy, Copernicus, Galileo, Kepler, Newton, Gauss, Einstein, Hawking and many astronomers whose names have been lost to history. The astronomy Johannes Kepler created a formulation that is powerful and accurate while relying on simple mathematical tools. Kepler gave us three main laws:

(1) The orbit of a planet is an ellipse with the Sun at one of the two foci.
(2) A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
(3) The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.

Each of these can be used to understand the nature of orbits better. And each of these can be dug into more. Answer the following questions. You can work together and do some online research.

Kepler Problems: Level 1

Kepler's First Law

(1) What is an ellipse? How can we think of an ellipse as being a more general version of a circle?
(2) What are the foci of an ellipse? How do they relate to the center of a circle?
(3) Draw an orbit around a planet. That is, draw a planet and an ellipse with it as a foci. Draw a space ship at a random point on the orbit. Label the following: Apoapsis, Periapse, Semi-Major Axis, Semi-Minor Axis, Altitude, Radius Vector.
(4) Explain how the following features of an ellipse - the orbit you take - are related: Semi-Major Axis, Periapse and Apoapsis. How do these relate to the diameter and radius of a circle?

Kepler's Second Law If you are orbiting the Earth, then your orbit does not have to be circular, it can be highly eccentric and stretched out. This will affect how you move around the planet. Kepler's Second Law gives us a rule for this: Let's say you start at Point A and draw a line to the center of the Earth, and then move for 10 minutes, ending at Point B. At Point B you draw another line to the center of the Earth. The two lines will cut a slice out of the ellipse that you're moving around. Kepler's Second Law says that as long as you travel 10 minutes, the area of this slice will be the same no matter where you start. Of course, the specific time of 10 minutes does not matter, any slice made by travelling a fixed amount of time will have the same area.

(1) Let's say that you're orbiting the Earth in a very elliptic orbit, and every 15 minutes you track how far you have travelled in that 15 minutes. You find that you travel a whole lot further when you are close to the Perigee than you do when you are close to the Apogee. Use Kepler's Second Law to explain why. Drawing a picture will help!
(2) You change your orbit so that you are now travelling in a perfect circle and continue your experiment of measuring distance travelled every 15 minutes. But now you find that you are travelling the same distance regardless of where you're at in the orbit! Explain how Kepler's Second Law applied to a circle means that your rotational speed is constant. (Hint: What is the formula for a sector of a circle?)

Kepler's Third Law Kepler's Third Law says that for any object orbiting a celestial body, if T is the time it takes for one orbit to complete and if A is the length of the Semi-Major Axis, then the value T^2/A^3 will be the same regardless of what is orbiting the celestial body and regardless of what the orbit looks like. Kepler called this the "Music of the Spheres" because it was so amazing to him.

(1) The orbit of the Earth is very circular with radius equal to 150,000,000 kilometers. The orbital period of the Earth is 365.25 days. We know that the orbital period of Mars is 687 days. What is the Semi-Major Axis of Mars's orbit?

Kepler Problems: Level 2

Kepler's First Law Kepler's First Law can be extended to say that any orbit takes the shape of a Conic Section - a circle, ellipse, parabola, or hyperbola. A circle and ellipse are "closed orbits" because they never escape the object they are orbiting. Parabolas and hyperbolas are "open orbits" because they have enough energy to escape the grasp of gravity. Mathematically, this means that the orbit "reaches infinity". A parabolic orbit has exactly the amount of energy needed to escape, and a hyperbolic orbit has more than enough energy. If you are chilling a distance R from a celestial body - like sitting on the surface of a planet with radius R, then the Escape Velocity is the minimum amount of speed needed to escape on an open orbit if you were to just accelerate in the direction exactly opposite the body. If the object has mass M, then the escape velocity is

$$v_e = \sqrt{\frac{2GM}{R}}$$

where $G = 6.67 \times 10^{-11} m^3/s^2 kg$ is the "Gravitational Constant".

(1) Earth's radius is about 4000 miles and its mass is about $6 \times 10^2 4kg$. Compute its escape velocity in meters per second.
(2) If you keep the same mass, but squeeze it into a smaller radius then what will happen to the escape velocity? If the radius of Earth was halved, then what would its new escape velocity be?
(3) The speed of light is approximately $3 \times 10^8 m/s$. How small would Earth's radius have to be for its escape velocity to be equal to the speed of light? What does this represent?

Kepler's Second Law Angular momentum is an important concept in the motion of objects. Like energy, it is something that does not change in scenario without outside influences. In an orbit, when you are close to the Apogee or Perigee, then the angular momentum is approximately equal to

$$L = rmv$$

where r is the distance you are from the center of the celestial body, m is the mass of your spacecraft, and v is your velocity. If the (fixed) time and distances travelled are small enough, then the path you travel is approximately a straight line. Use all of this and Kepler's Second Law to show that angular momentum near the Apogee of an orbit is equal to the angular momentum near the Perigee of the orbit.

Kepler's Third Law In a special mission, three manned space ships were all equidistributed in geosynchronous orbits around the Earth. But a failure on the craft behind of you means that you need to try to rendezvous with it to rescue the other pilot. To do this, you drop your Perigee by some amount to speed up your orbital period so that when you return to the Apogee, you will meet the other ship. Use Kepler's Third Law to find how close you should drop your Perigee.

Kepler Hands-On: Anatomy of an Orbit Scientific Poster

There are some really cool scientific posters which look awesome but are a great resources for important facts. See the following links for some examples:

(1) The Standard Model of Fundamental Particles and Interactions
(2) The Solar System
(3) The Periodic Table of Elements
(4) The History and Fate of the Universe
(5) The Scale of the Universe

Kepler's Laws give us vital information about orbits and there's a lot of vocabulary and facts that are useful to keep in mind and know about orbits from them. Especially for spaceflight. This task is to create a poster which beautifully displays and communicates all this formation, including all the vocabulary that we will be wanting to know. What are orbits? What are their shapes? What features and parameters do they have? Etc.

This poster can either be done by hand and in-person using art materials at the school. This can be done digitally as well, and if you are able to show up in person then we can work on printing it, laminating it, and hanging it in our classroom.

Kepler Hands-On: Computing Mars's Orbit

Finding the path that planets take in the sky is the main motivation for most of historical astronomy. Kepler's Laws are not an exception. In this challenge, you will use the historical computations that Kepler used in order to discover the orbit of Mars. These computations were used specifically to formulate Kepler's Third Law.

The underlying math for this is relatively simple - Using the Tangent Function and solving Systems of Equations - but the physics are built on Kepler's formulations. These can help develop your intuition about orbits and see how even simple laws like this can result in powerful computations.The computations are built on a couple of assumptions. First, that the Earth's

orbit is circular, not entirely correct but it is circular enough for this to be relatively accurate. Second, that we know Mars's orbital period to be 697 days, which is also reasonably accurate.

To do this triangulation, we place the solar system on a Cartesian Plane with the sun at the origin. The Earth then travels in a circle around the origin, and Mars travels on an ellipse with the origin as one of the foci. Since we know the distance of the Earth from the sun, the only thing we need to know in order to find Earth's location is its angle on the plane. We can't measure Mars's location directly, but we can measure the angle of the line drawn from Earth to Mars. These historical measurements then come as pairs of the form

$$[\text{Earth's Angle on the Plane, The angle of the line from Earth to Mars}]$$

The first in this pair will let us find the point that Earth is located on the plane. We can use the second data point to find the slope of a line through that point. Mars is somewhere on this line, but it could be anywhere on this line. There is then another partner pair of such measurements that were made exactly 697 days apart, which ensures that Mars is actually located at the same point on the plane. We can then use both of these pairs of measurements to find to lines that pass through the same point, and this point happens to be where Mars was when each of those measurements were made. Systems of Linear Equations will then let us find that point. Doing this for many different places will then give the location for Mars at many different points in time.

(1) Use the attached data sheet of measurements (Link) to find as many Mars locations as possible.
(2) Use these locations to show that the orbit of Mars is not circular.
(3) The equation of an ellipse which has the origin as a foci is

$$r = \frac{P}{1 + E\cos(\theta - \omega)}$$

where you can adjust P, E, ω to get different ellipses. This is also written in Polar Coordinates, rather than Cartesian Coordinates. Plot the points where Mars was on Desmos, then use this formula to find the P, E, ω that best fit all of these points.
(4) The value of E that you got is the **Eccentricity** of the orbit, which measures how elongated it is. Look up the actual eccentricity of Mars's orbit and compare it to your own. How might you get a better measurement?

Beyond Kepler

If you complete Level 2, and feel like you can handle more, then you can try these

(1) A more accurate definition of angular momentum is

$$L = rmv_\perp$$

where $v_\perp$ is the component of the velocity vector which is perpendicular to the radius vector (the vector between the two objects). Generalize the method in Level 2 to use Kepler's Second Law to prove that angular momentum is the same everywhere on an orbit. This will use a mix of geometry and limits.
(2) If there is an object of mass m orbiting an object of mass M with fixed angular momentum L, then the formula for the total energy of the system is:

$$E = -G\frac{mM}{r} + \frac{1}{2}m|\mathbf{v}|^2$$

Here G is the Gravitational Constant, r is the distance between the two objects, and $|\mathbf{v}|$ is the speed of the orbiting object. This formula can be interpreted as:

$$\text{Total Energy} = -\,(\text{Gravitational Binding Energy}) + (\text{Kinetic Energy})$$

Interpret what is happening in the following scenarios: $E < 0$, $E = 0$, $E > 0$. Use these explanations to derive the formula for Escape Velocity (probably better named "Escape Speed"), and use it to interpret the escape velocity formula.
(3) The velocity vector can be written as $\mathbf{v} = [v_\parallel, v_\perp]$, where $v_\parallel$ is the component of velocity parallel to the radius vector and $v_\perp$ is the component perpendicular to the radius vector. The speed $|\mathbf{v}|$ is the length of the velocity vector. Use this to show that we can write

$$E = -G\frac{mM}{r} + \frac{L^2}{2mr^2} + \frac{1}{2}mv_\parallel^2$$

Interpret this formula in a way analogous to the previous interpretation. What is $v_\parallel$ at the apoapsis and periapsis? Use this to find a formula for these two lengths. Interpret what happens to these points when $E = 0$.
(4) You are in an elliptic orbit around a planet with a limited amount of fuel, and you need to escape it. Find the strategy which gets you into an escape trajectory and uses the least amount of fuel. This means finding the most optimal *place* in the orbit and *direction* to burn your engines. Can you prove that this is the most optimal strategy?