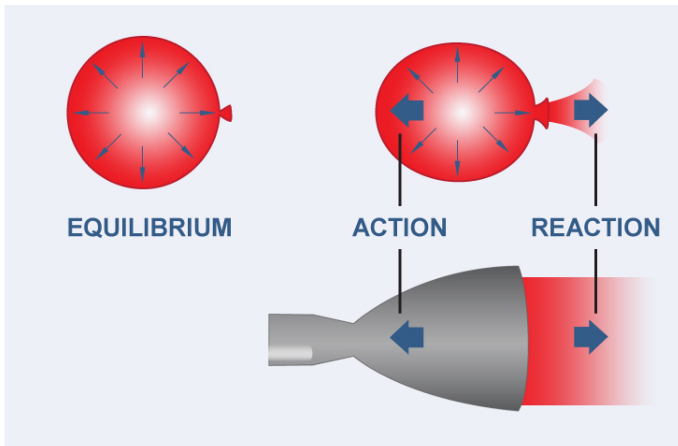
WEEK 2: THE ROCKET EQUATION

GOAL: Understand the role of the Rocket Equation in Rocket Design.

The Rocket Equation

Rocket engines work by creating enormous pressure in a small area and directing out in one direction through the nozzle. Thanks to Newton's Third Law which says that "For every action there is an equal and opposite reaction", the high pressure and extreme fast moving exhaust pushes the rocket in the opposite direction.



Some engines produce a lot of force by expelling a lot of fuel. Some engines produce little thrust, but use the fuel more efficiently. And other engines lie in the middle or prioritize other things. There are two main ways that we use to evaluate an engine:

(1) **Thrust -** This is the raw force that an engine creates.
(2) **Specific Impulse -** This is a measure of how efficiently the engine uses its fuel to create thrust.

When you are launching, you have to work against the force of gravity to get your rocket into orbit and so you need your thrust to be greater than the force of gravity on the rocket. The ratio between Thrust and Weight is the **Thrust-to-Weight** ratio (TWR). You often want to bring as little into space as you possibly can, and so what fuel you bring should be used well and so in-space engines generally look for a higher Specific Impulse.

The main goal of an engine is to change the speed of a rocket. A rocket engine works by burning fuel, and so how much speed it can create depends on how much fuel it can burn along with the particular properties of the engine. The equation that relates "Fuel Burnt" to "Speed Produced" is known as the **Rocket Equation** and it is:

$$\Delta v = I_{sp} g \ln\left(\frac{m_1}{m_2}\right)$$

This has some new symbols in it:

(1) Δv is the total change in velocity that has been made over a burn. Said aloud, this is "Delta V".
(2) I_{sp} is the Specific Impulse of the engine
(3) $g = 9.8\text{m/s}^2$ is the acceleration due to gravity
(4) m_1 is the mass that the rocket begins the burn with
(5) m_2 is the mass that the rocket ends the burn with

The ratio m_1/m_2 should be thought of as measuring how much fuel was burnt. So this formula uses the engine (through I_{sp}) to convert the change in fuel (via m_1/m_2) into a change in speed (as Δv). In a way, this equation is a rocket! In the end, the Rocket Equation allows us to begin to make smart choices about how to design our rockets based on what we have and what we need. While Orbits tell us how rockets move, the Rocket Equation tells us what rockets can do.

The way that it is written, you can use the "ln" button on your calculator to get Delta-V if you know the masses involved. If you know the Delta-V and need to figure out the masses needed, then you will need to solve a logarithmic equation - which many of you might not be able to do yet. But the equation can be rewritten as

$$\frac{m_1}{m_2} = \exp\left(\frac{\Delta v}{I_{sp} g}\right)$$

Here $\exp(X)$ is just a clean way to write e^X and $e \approx 2.718$. This can be used to get mass if you know the Delta-V.

INSTRUCTIONS: Complete either Levels 1+2, or Level 3, or the KSP Experimentation option.

The Rocket Equation: Level 1

(1) The units of Δv is speed (m/s usually), and the logarithm never has a unit. With this in mind, what are the units of I_{sp}? Why?
(2) Your rocket has a mass of 90,000kg, and a thrust of 1,500,000 Newtons (standard units of force). What is the TWR? Will this rocket be able to launch? Why or why not?
(3) You are in space and have an engine with specific impulse equal to $I_{sp} = 200$s. You start with a mass of $10,000$kg and you burn away 9500kg of fuel. How much did you accelerate?
(4) Your friend is in space next to you, but they have a ship with $I_{sp} = 300$. If they start with a mass of $10,000$kg as well, how much fuel will they have to burn to accelerate as much as you did?
(5) The Earth is in orbit around the Sun in a roughly circular orbit with lateral speed of about 30km/s. This lateral speed effectively makes Earth "miss" the sun as it falls around it. If you wanted to fly trash into the Sun - for permanent, safe disposal - then you would need to cancel out all of that lateral velocity so that it could just fall right in. The rocket engine with the highest specific impulse ever had $I_{sp} = 542$s. Assuming that only the trash remained after burning the fuel, compute how much mass you would need to burn if you wanted to dispose of one ton of trash this way.

The Rocket Equation: Level 2

(1) You are delivering a 0.45 ton satellite into orbit. The engine that you can use to circularize your orbit after launch has a specific impulse of 320s and has mass 1.5 tons. The Δv required to circularize your orbit is 1200m/s. Assuming that the mass of the fuel tank is 1/8th the mass of the fuel itself, find the least amount of fuel that you will have to bring after launch to get your satellite into orbit. What is the total mass that you need to bring with you through the initial launch burn?
(2) You fly the mission in the previous problem but with a less powerful engine, in terms of thrust, but has a specific impulse of 345s and weighs only 0.5 tons. Redo the previous problem with this engine instead. Make comparisons.
(3) In KSP you have designed a nice launch module that you like, and you're able to copy/paste it to use it different rockets. Each copy of the module has initial mass of 90 tons, a final mass of 20 tons, and it has a specific impulse of 235s. Compute:
  (a) The Delta-V possible if your payload has mass 5tons.
  (b) The Delta-V possible if your payload has mass 10tons.
  (c) The Delta-V possible if your payload has mass 5tons but you attach two of these modules. Note: The mass doubles, but the specific impulse remains the same.
  (d) The Delta-V possible if your payload has mass 10tons but you attach two of these modules.
  (e) Use the results to examine how well rockets work as the mass scales up.
(4) Use all of these problems to discuss the rule-of-thumb in rocket design: **Bigger is not always better.**

The Rocket Equation: Level 3 (Requires Trigonometry)

During launch, the two main forces on the rocket are Thrust and Gravity. Gravity works against thrust to steal away some Δv, and since the acceleration due to gravity is $g = 9.8\text{m/s}^2$ the total Delta-V has the form

$$\Delta v = I_{sp} g \ln\left(\frac{m_1}{m_2}\right) - gt$$

where t is the time of the launch burn.
NOTE: The starred questions require Calculus. These are optional, but encouraged if you have the toolset.

(1) You have a rocket whose engine has a specific impulse of 285s and burns 72 tons of fuel over 146 seconds. If a total Delta-V of 1200m/s is needed to reach an altitude of 100km, then what is the largest mass that this rocket can deliver to that altitude?
(2) This formula is, technically, only viable for when the rocket is going directly vertical. Get more exact with the formula by including angle of attack. Modify the equation for ΔV by drawing a Force Diagram of the rocket when its angle away from vertical is θ. If we use numbers from the unmodified equation, are we playing it safe or are we underestimating what we can actually do? Why?
(3) To orbit Kerbin at 100km, you need to be moving at a speed of about 2000m/s in the direction parallel to the surface. Say that you launch exactly vertically and reach a velocity of 2000m/s at 100km going straight up. How much will you still have to accelerate if you want to get into orbit and in what direction? What would be an optimal strategy to do this acceleration that takes advantage of gravity to do this?
(4) A **Gravity Turn** during launch is a turn away from a vertical direction during launch. Identify two advantages that a Gravity Turn gives and use these problems and math to explain your thoughts.

(5) How might atmosphere be a reason to avoid doing a Gravity Turn too early? Describe what an optimal launch from the Mun would look like. Compare it to an optimal launch from Kerbin.
(6) If your engine burns fuel at a rate of r, then find a formula for the mass that a rocket with initial mass m would have after burning for t seconds. Use this to find a formula for $\Delta v(t)$, the total Delta-V after burning for t seconds (include the gravitational term!).
(7) (*) For launch to actually happen at $T = 0$, then the derivative of $\Delta v(t)$ at $t = 0$ must be positive. If your engine has specific impulse I_{sp} and burns fuel at a rate of r, then find the largest mass that this engine can lift.
(8) (*) The force of Thrust must exceed the force of gravity. The force of gravity is mg. With this in mind, and the inequality from the previous problem, make conjecture for a formula that relates Thrust, Specific Impulse, and rate of fuel consumption.

The Rocket Equation: KSP Experimentation (Requires MechJeb2)

The goal of a launch is to get as close to an orbital trajectory as possible using the least amount of fuel. During launch, there are three main forces that act on a rocket: Thrust, Gravity, Aerodynamic Drag. Each of these contribute to total Delta-V in different ways:

$$\Delta v_{\text{Total}} = \Delta v_{\text{Thrust}} - \Delta v_{\text{Gravity}} - \Delta v_{\text{Drag}}$$

The Rocket Equation gives us a pretty clear way to analyze the first of these, and Newton's Laws of force and gravitation help us with the second one, but the component due to drag is often much more difficult to analyze. The goal of this section is to use KSP mods to investigate how all of these things work during launch and to use data to find an optimal launch trajectory.

To do this, you will have to install the KSP mod "MechJeb2", instructions for which are on the website. This will give you access to a part called "MechJeb 2" that communicates live information from the rocket, including graphs which analyze these different components of Delta-V and information about Launch Trajectory.

(1) Create a rocket that you can consistently use to get into a 100km orbit around Kerbin. Attach the MechJeb 2 part to it somewhere.
(2) On the launchpad, click the "MechJeb" tab at the top right and select the "Flight Recorder" option. This will bring up a graph with different selections of data to display.
(3) Select to display information about "Pitch", "ΔV", "Gravity Loss", and "Drag Loss". This gives information about the Trajectory, Total Delta-V, Delta-V lost due to gravity, and Delta-V lost due to drag respectively.
(4) Formulate at least 3 different launch trajectories which achieve orbit to analyze. Run each of these three scenarios with the flight log active, and record the data after getting into orbit. A screenshot will work, but if you want to do some real data analysis then you can export it as a CSV file.
(5) Write an analysis of what an efficient launch looks like using the data to support your conclusions. Formulate and describe a fourth launch strategy based off of the analysis which optimizes the fuel needed.
(6) Execute your fourth launch strategy using the flight recorder to record its performance. Write a final analysis that discusses this fourth launch.