Convergence of a symmetric MPFA Method on quadrilateral grids

Universität Stuttgart

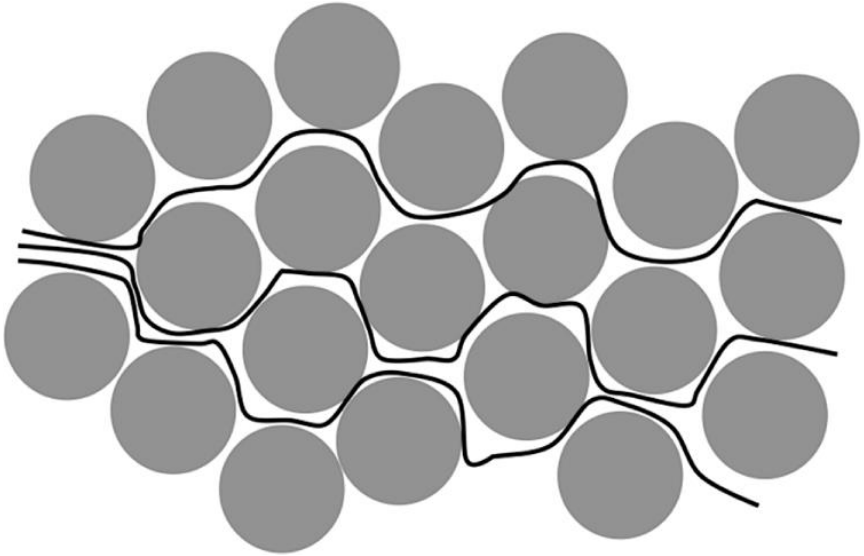
SimTech

Paper from Aavatsmark, et al.

Presented by Daniel Fink at the M.Sc. SimTech-Seminar

Dezember 3, 2019

porous media



Model equation

$$-div(\boldsymbol{K}\, grad\, p) = g \quad on \quad \Omega$$

with

- $\boldsymbol{K}$ - permeability tensor (symmetric and positive definite)
- p - pressure (potential)
- g - source term

→ Can be solved with MPFA method (control volume discretization)

Motivation

- MPFA method

→ Stability, monotonicity, convergency

- Matrix of coefficients are nonsymmetric

- Symmetric MPFA method

→ Transformation to orthogonal reference space

- No convergency theory available for general MPFA

→ Reformulation as mixed finite element (MFE) problem

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Overview

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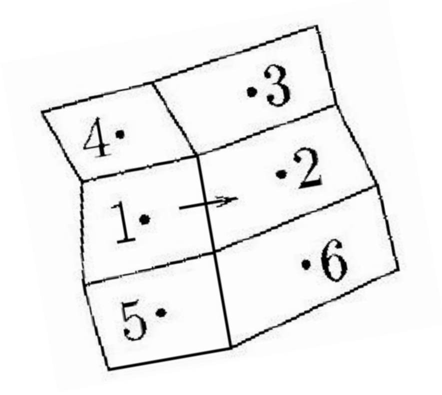


Short repetition of MPFA (2D)

↓

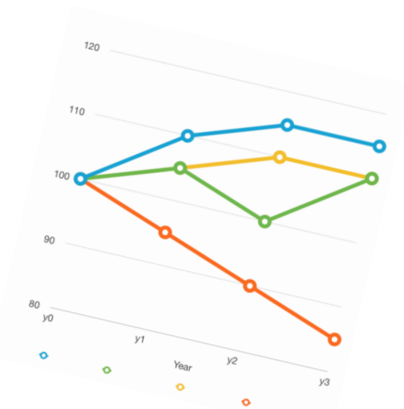
Introduction into symmetric MPFA

↓

Reformulation as MFE

↓

Numerical experiments



MPFA Method (2D)

MPFA Method

- Integrate model equation $-div(\boldsymbol{K}\,grad\,p) = g$

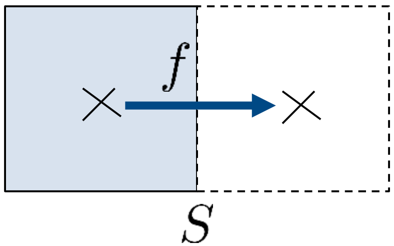
easy part

$$-\int_{\Omega_i} div(\boldsymbol{K}\,grad\,p)\,d\tau = -\int_{\partial\Omega_i} (\boldsymbol{K}\,grad\,p)\cdot\boldsymbol{n}\,d\sigma = \boxed{-\int_{\Omega_i} g\,d\tau}$$

- Formulation is locally conserved if fluxes are equal on the cell surface

hard part

$$\boxed{f = -\int_S (\boldsymbol{K}\,grad\,p)\cdot\boldsymbol{n}\,d\sigma}$$



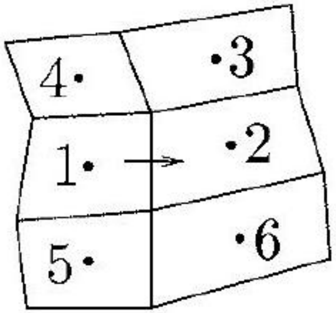
→ Until now, this is just the control volume approach

MPFA Method

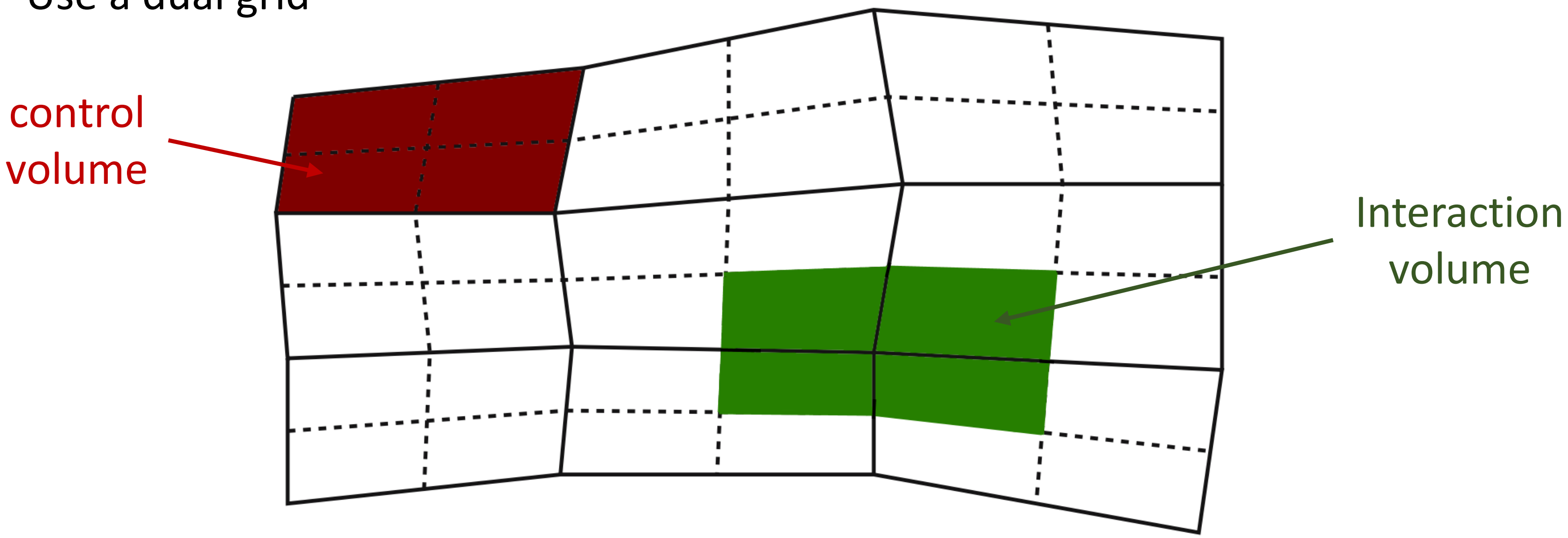
- **Multi point** flux approximation

→ Principal directions of ***K*** are not aligned with the grid

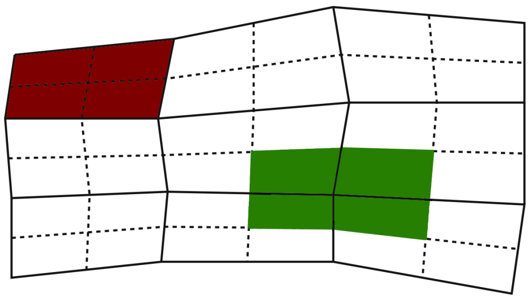
→ Need more information from neighboring cells

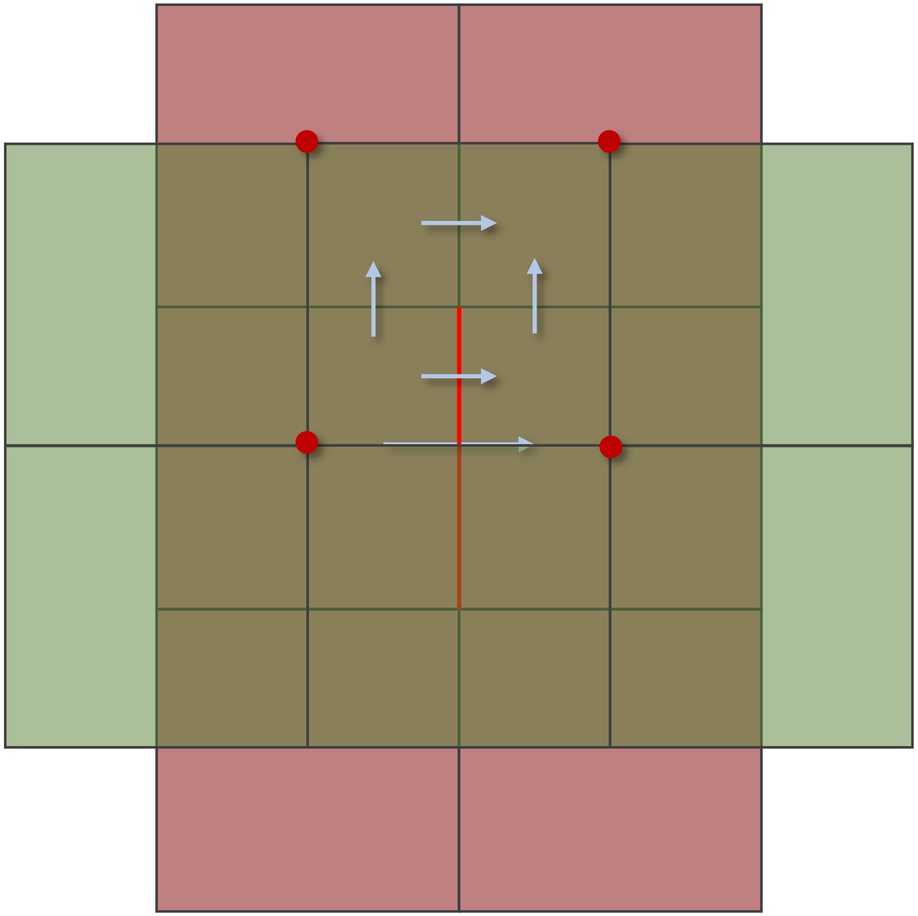


- Use a dual grid



MPFA Method





Assuming (O-Method):

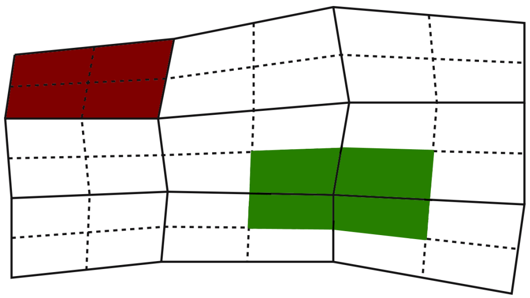
- continuity of flux on each half edge
- continuity of pressure on each center of half edge

→ 4x4 system of linear equations

$$f = \boldsymbol{T}p$$

$$f_e = \sum_{i \in I} t_{e,i} p_i$$

MPFA Method

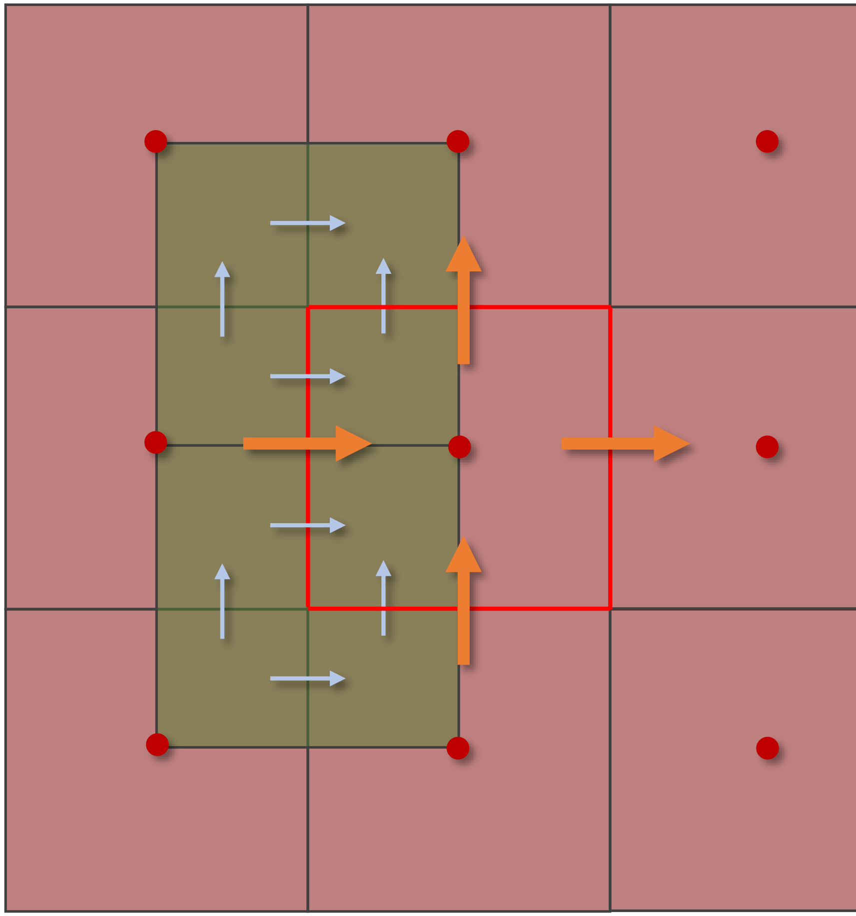


- Do the same procedure for the other interaction volume
- Add the two half edge fluxes

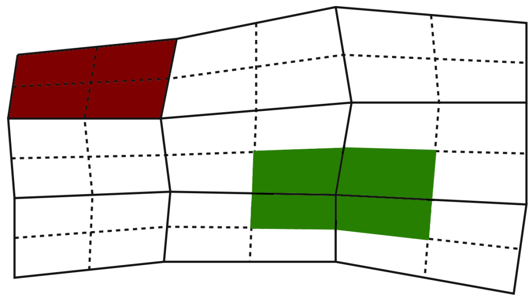
→ $f_a + f_b = f$

- Do this for all edges of a cell

→ $f_1 + f_2 - f_3 - f_4 = \int_{\Omega_i} g \, d\tau$



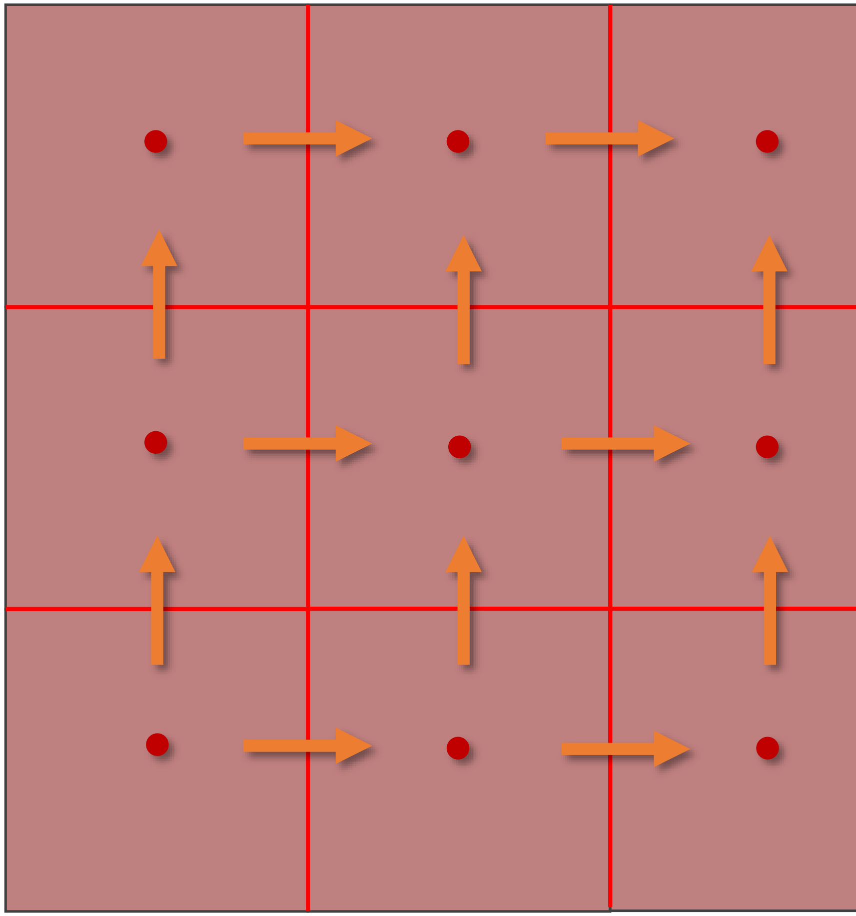
MPFA Method



- Assembling the entire grid

→ $\boldsymbol{M}p = r$

with $\boldsymbol{M}$ being nonsymmetric



Symmetric MPFA Method

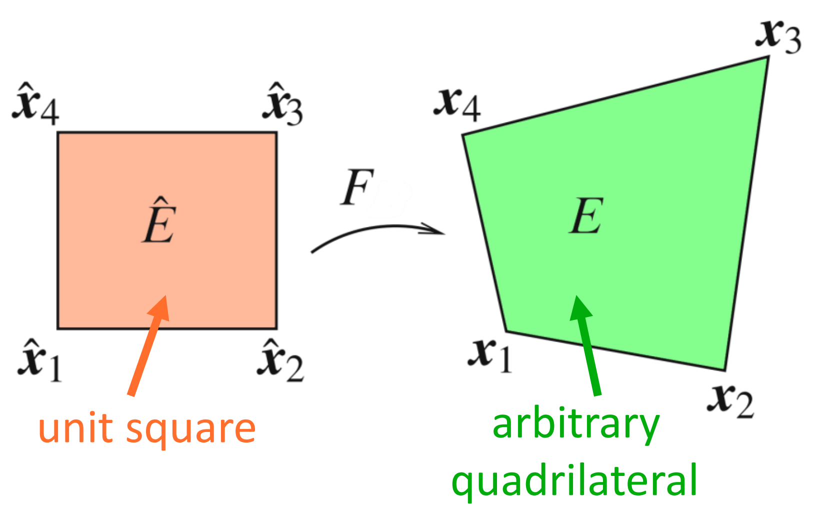
Symmetric MPFA Method

What is the idea behind symmetric MPFA?

- Instead of physical space, use an orthogonal reference space
- Apply MPFA O-Method there
- $F\colon \hat{E} \to E$ is a diffeomorphism
- Jacobian Matrix $\boldsymbol{D} \neq \text{const}$

→ Evaluation needed

→ Different symmetric MPFA versions



Symmetric MPFA Method

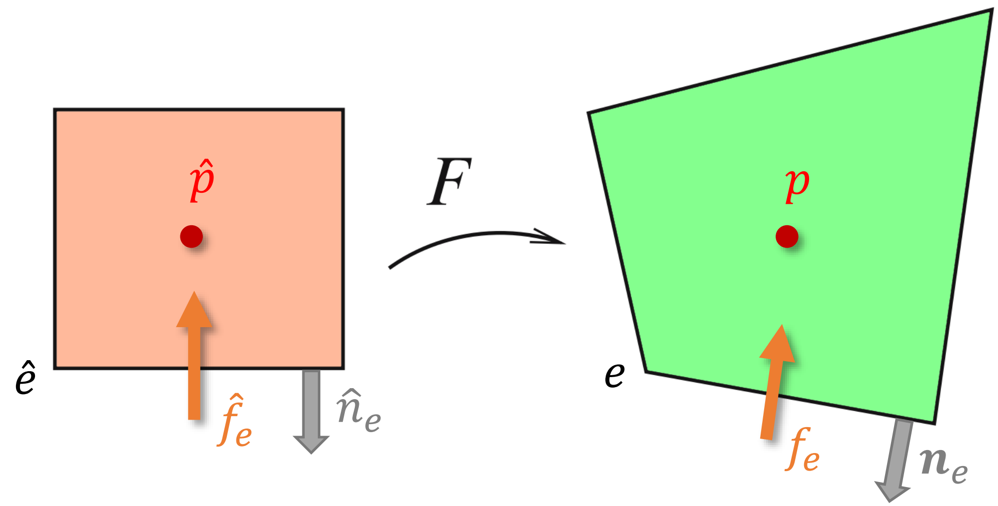
How are the quantities transformed?

- Permeability

embodies permeability and geometry → not cell-wise constant

→ $\widehat{\boldsymbol{K}} = \det \boldsymbol{D} \cdot \boldsymbol{D}^{-1}\boldsymbol{K}\boldsymbol{D}^{-T}$

→ Symmetric and positive definite



- Pressure

→ $\hat{p} = p \circ F(\hat{x})$

remain unchanged

- Flux

→ $f_e = -\int_e \boldsymbol{K}\, grad\, p \cdot \boldsymbol{n}_e\, ds = -\int_{\hat{e}} \widehat{\boldsymbol{K}}\, grad\, \hat{p} \cdot \hat{n}_e\, d\hat{s} = \hat{f}_e$

Symmetric MPFA Method

How does it become symmetric?

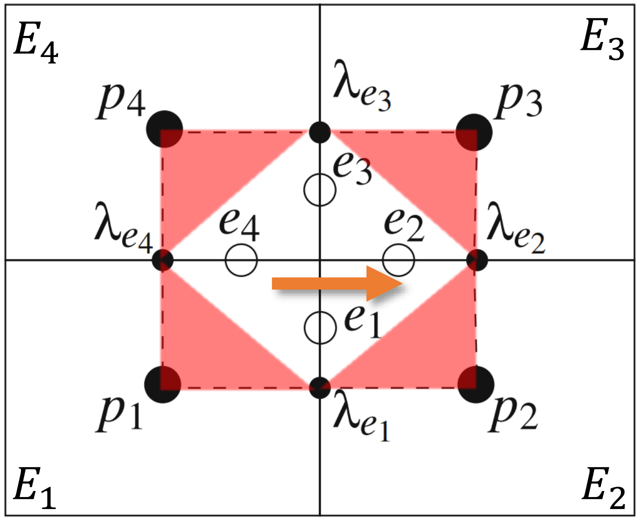
- Flux through e_1:

→ $$f_{e_1} = -\int_{\hat{e}} \widehat{\boldsymbol{K}}\, grad\, \hat{p} \cdot \widehat{\boldsymbol{n}}_{\boldsymbol{e}}\, d\hat{s}$$
$$\approx -\widehat{\boldsymbol{K}}_{\boldsymbol{E_1}}\, grad\, \hat{p} \cdot \widehat{\boldsymbol{n}}_{\boldsymbol{e}}/2$$

with $\widehat{\boldsymbol{K}}_{\boldsymbol{E_1}}$ being the evaluation of $\widehat{\boldsymbol{K}}$ at some point

→ $\widehat{\boldsymbol{K}}_{\boldsymbol{E_1}} = const\ on\ E_1$

reference space



- Assuming $\hat{p}$ linear on each subcell

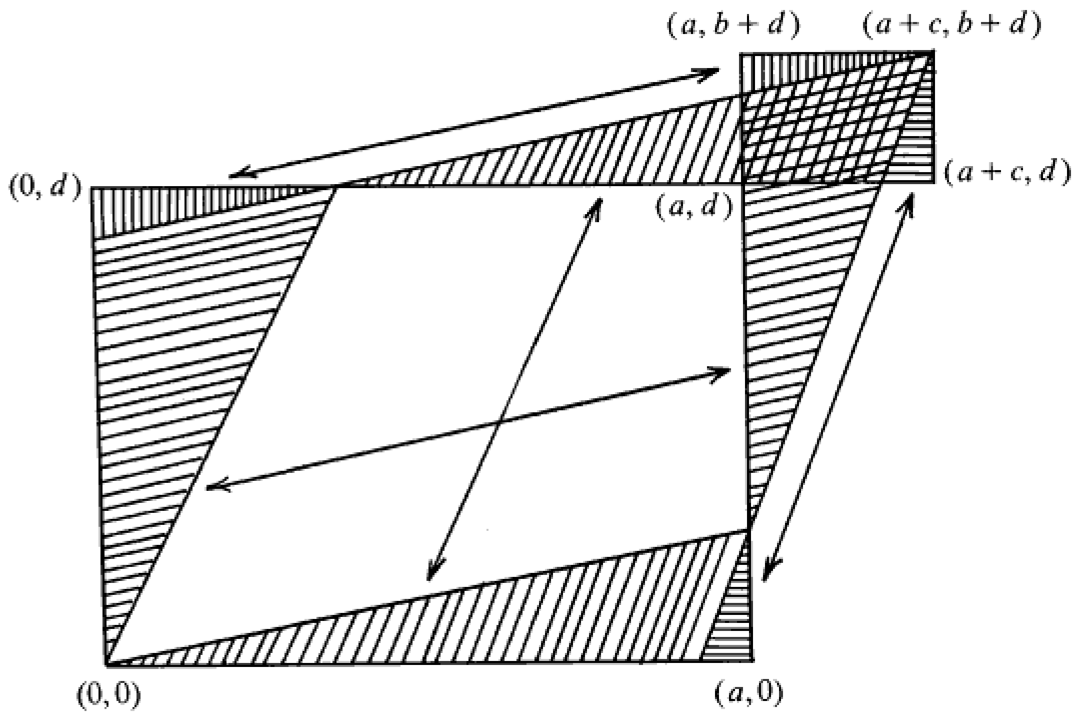
→ $grad\ \hat{p} = const$

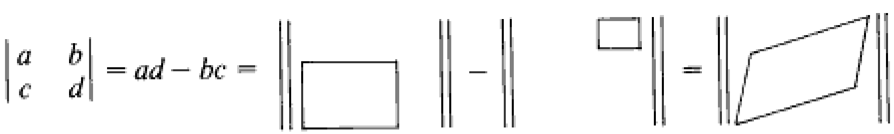
- Do the same procedure for the other subcells and eliminate λ_i

→ $\boldsymbol{A}\boldsymbol{f} = \boldsymbol{p}$, with $\boldsymbol{A}$ being symmetric and positive definite

Symmetric MPFA Method

Where should the Jacobian be evaluated?

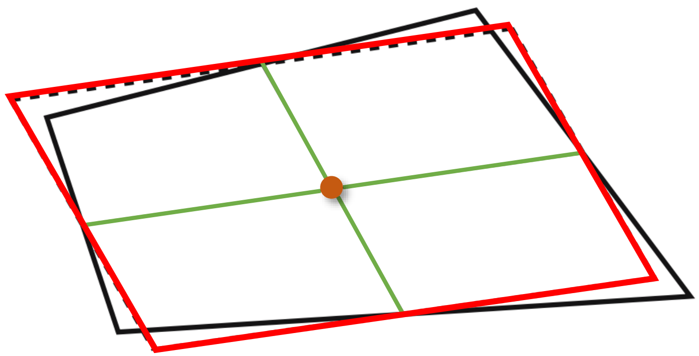




Symmetric MPFA Method

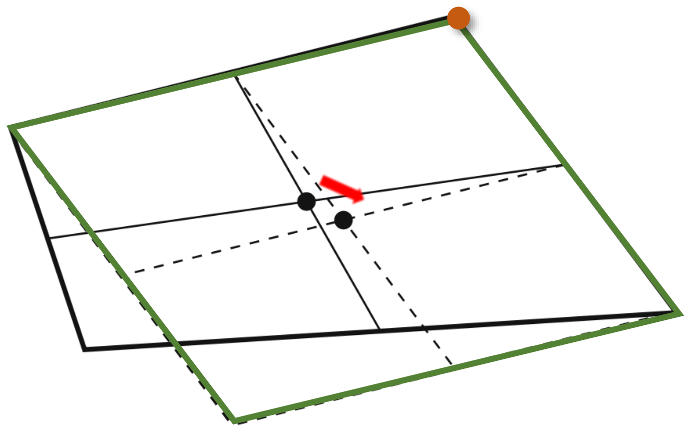
Where should the Jacobian be evaluated?

Cell center



- Correct distances from cell center to edge midpoints
- Wrong length and orientation of edges

Corner



- Correct length and orientation of edges
- Cell center moved

Reformulation as MFE

Reformulation as MFE

$-div(\boldsymbol{K}\, grad\, p) = g$ ➡ $\boldsymbol{u} = -\boldsymbol{K}\, grad\, p$ and $div(\boldsymbol{u}) = g$

velocity

- Assuming $p = 0\ on\ \partial\Omega$

→ Weak formulation: Find $(\boldsymbol{u}, p) \in H(div) \times L^2$ such that

$$\left(\boldsymbol{K}^{-1}\boldsymbol{u}, \boldsymbol{v}\right) - \left(p, div(\boldsymbol{v})\right) = 0 \qquad \forall \boldsymbol{v} \in H(div)$$

$$(div(\boldsymbol{u}), q) = (g, q) \qquad \forall q \in L^2$$

→MFE (discrete): Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$\left(\boldsymbol{K}^{-1}\boldsymbol{u}_h, \boldsymbol{v}\right) - \left(p_h, div(\boldsymbol{v})\right) = 0 \qquad \forall \boldsymbol{v} \in V_h$$

$$(div(\boldsymbol{u}_h), q) = (g, q) \qquad \forall q \in Q_h$$

Reformulation as MFE

What finite element spaces can be used?

velocity

Broken Raviart-Thomas	**Brezzi-Douglas-Marini**
$V_h = \widehat{RT_0}^{1/2}$	$V_h = \widehat{BDM_1}$
→ MPFA midpoint evaluation	→ MPFA corner point evaluation

pressure → Piecewise constants in each cell

Reformulation as MFE

How does the MPFA become an MFE?

$$(\boldsymbol{K}^{-1}\boldsymbol{u}_h, \boldsymbol{v}) - \left(p_h, div(\boldsymbol{u}_h)\right) = 0 \qquad \forall v \in V_h$$

$$(div(\boldsymbol{u}_h), q) = (g, q) \quad \forall q \in Q_h$$

replace this by a quadrature formula

$\hat{a}_h(\widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{v}})$

Approximate $\widehat{\boldsymbol{K}}$

→ Midpoint vs. corner point evaluation

Apply trapezoidal rule

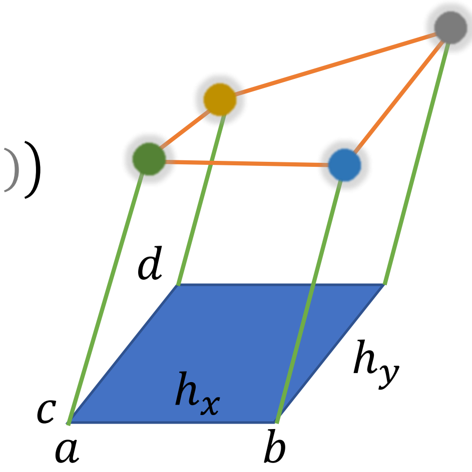
Note: $\left(\boldsymbol{K}^{-1}\boldsymbol{u}, \boldsymbol{v}\right)_E = \left(\widehat{\boldsymbol{K}}^{-1}\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{v}}\right)_{\hat{E}}$

$a_E(\boldsymbol{u}, \boldsymbol{v}) = a_{\hat{E}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{v}})$

Reformulation as MFE

- Trapezoidal rule:

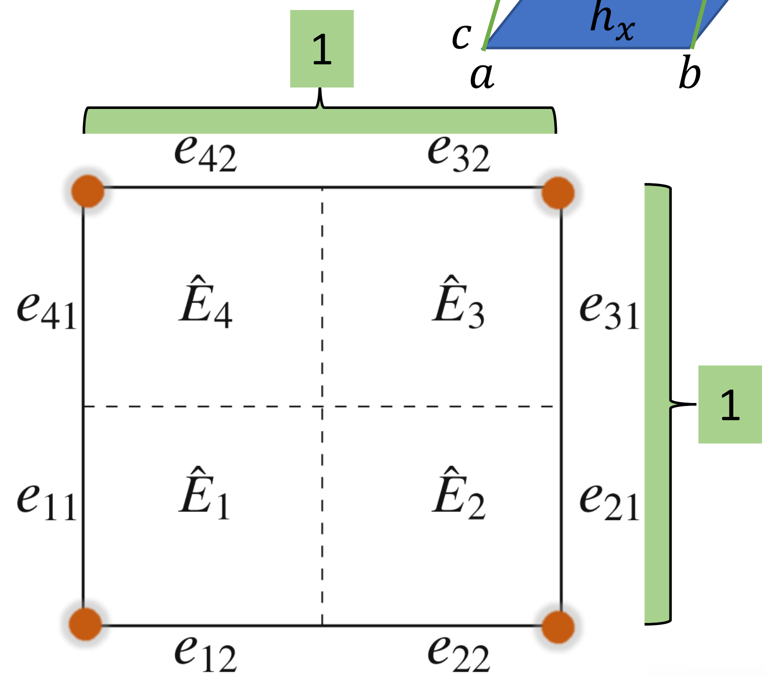
$$\int_a^b \int_c^d f(x,y)\,dx\,dy \approx \frac{1}{4} h_x h_y \big(f(a,c) + f(b,c) + f(a,d) + f(b,d) \big)$$



$$\rightarrow \left(\widehat{\boldsymbol{K}}^{-1}\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{v}}\right)_{\hat{E}} = \int_{\hat{E}} \widehat{\boldsymbol{K}}^{-1}\,\widehat{\boldsymbol{u}} \cdot \widehat{\boldsymbol{v}}\,d\tau$$

$$\approx \frac{1}{4}\sum_{i=1}^{4} \widehat{\boldsymbol{K}}_{\widehat{E}_i}^{-1}\,\widehat{\boldsymbol{u}}(\widehat{\boldsymbol{x}}_i) \cdot \widehat{\boldsymbol{v}}(\widehat{\boldsymbol{x}}_i)$$

$$= \hat{a}_{\hat{E}}(\widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{v}})$$



- Do this for all cells

$$\rightarrow a_h(\boldsymbol{u}, \boldsymbol{v}) = \sum_{\hat{E}} \hat{a}_{\hat{E}}(\widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{v}})$$

Reformulation as MFE

- The MPFA as an MFE formulation:

Find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}) - \big(p_h, div(\boldsymbol{v})\big) = 0 \qquad \forall \boldsymbol{v} \in V_h$$

$$(div(\boldsymbol{u}_h), q) = (g, q) \qquad \forall q \in Q_h$$

→ Has a unique solution (with $V_h = {RT_0}^{1/2}$ or $V_h = BDM_1$)

→ Methods are equivalent with the MPFA method (midpoint or corner point)

→ Local 4×4 velocity system

→ Global pressure system

symmetric and positive definite

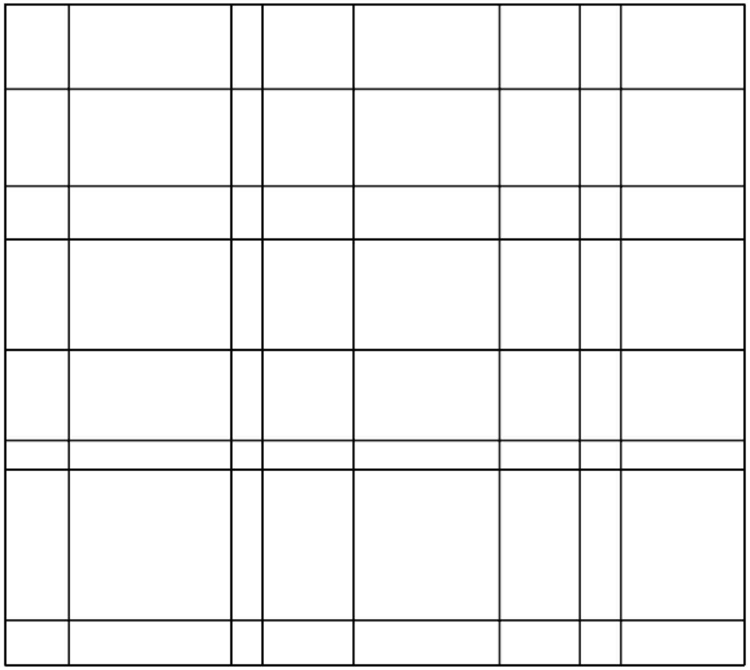
Numerical Experiments

Numerical Experiments

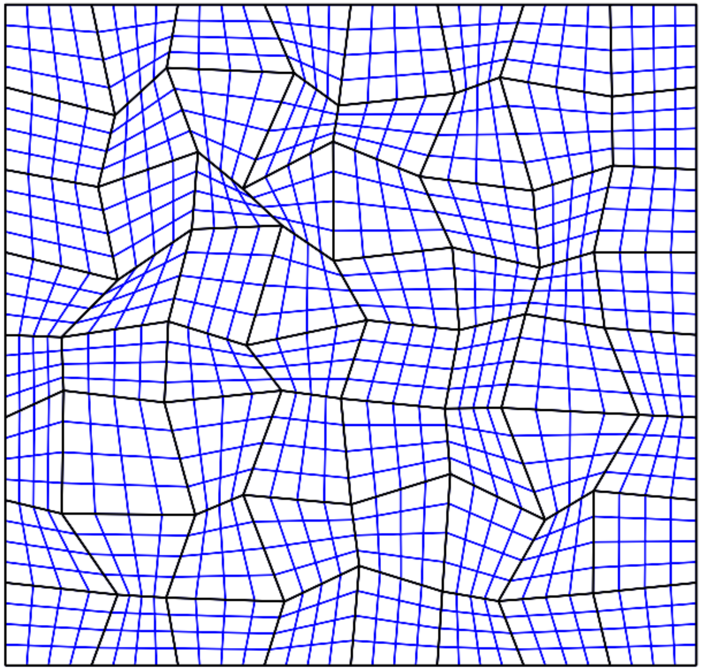
Use homogeneous medium and smooth solutions:

$p(x, y) = \cos(2\pi\, x) \cos(2\pi\, y)$ and $\boldsymbol{K} = const$

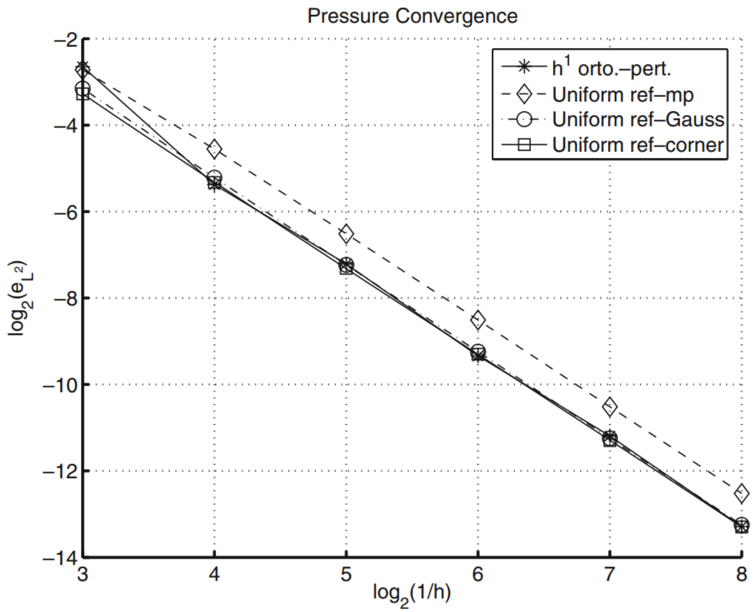
Orthogonal grid

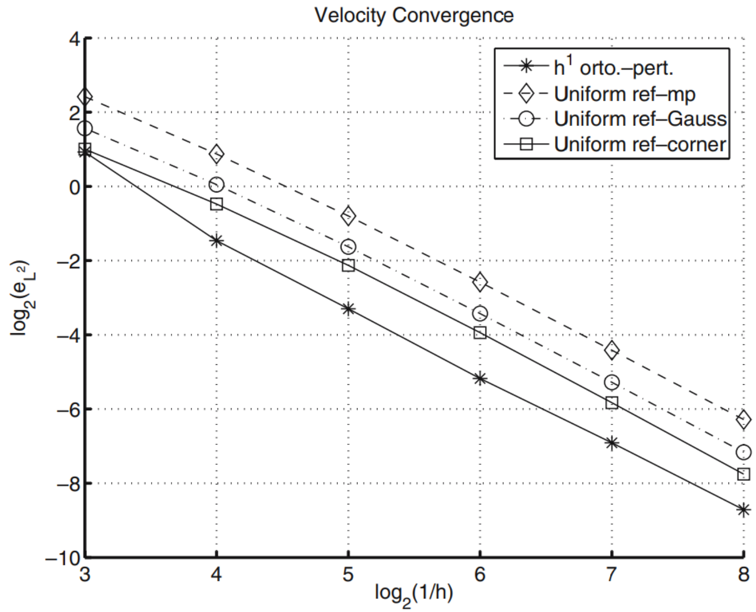


Uniform refined grid



Numerical Experiments

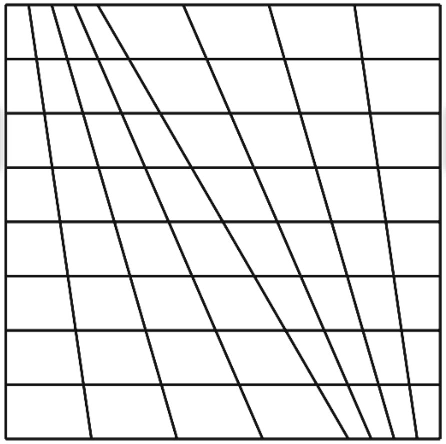




→ Second order convergency for all test cases

→ Corner point evaluation performs best

Numerical Experiments



perturb the corners

$$\tilde{x}_i = x_i + R_x h^\gamma$$

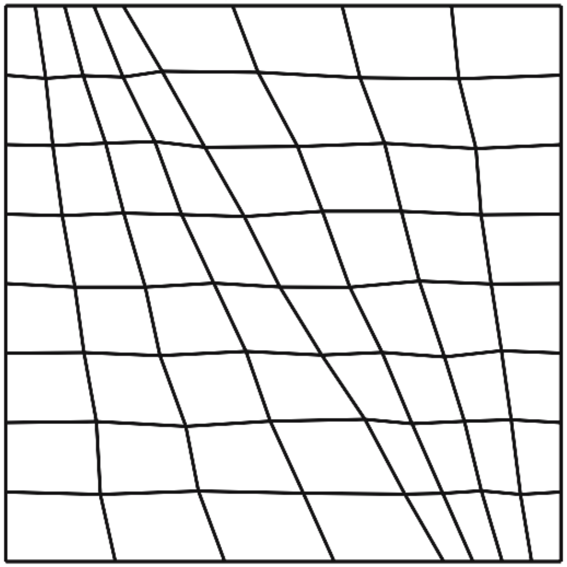
$$\tilde{y}_i = y_i + R_y h^\gamma$$

with

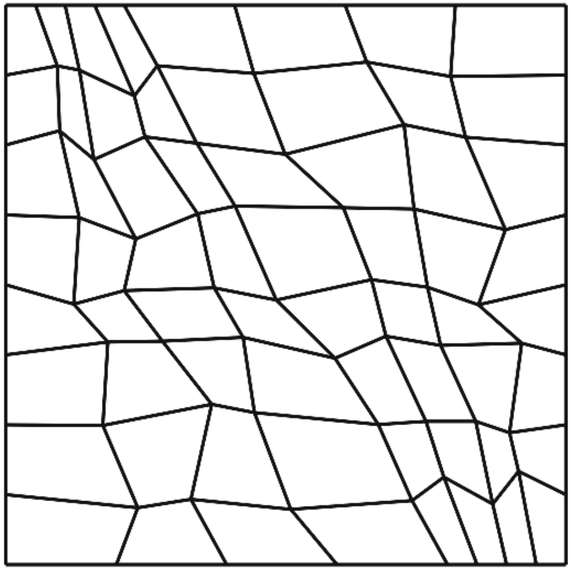
$$R_x \in [-0.5, 0.5]$$

$$R_y \in [-0.5, 0.5]$$

randomly



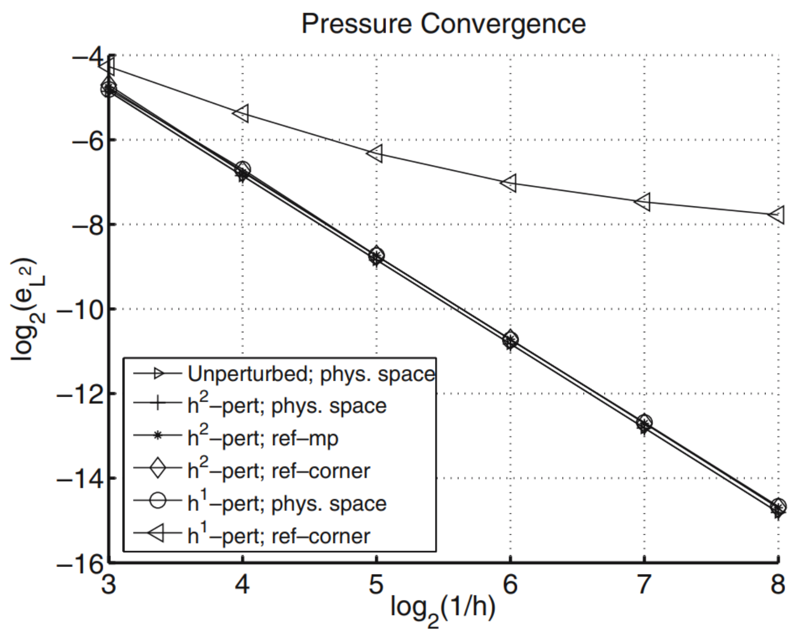
h^2- perturbed grid

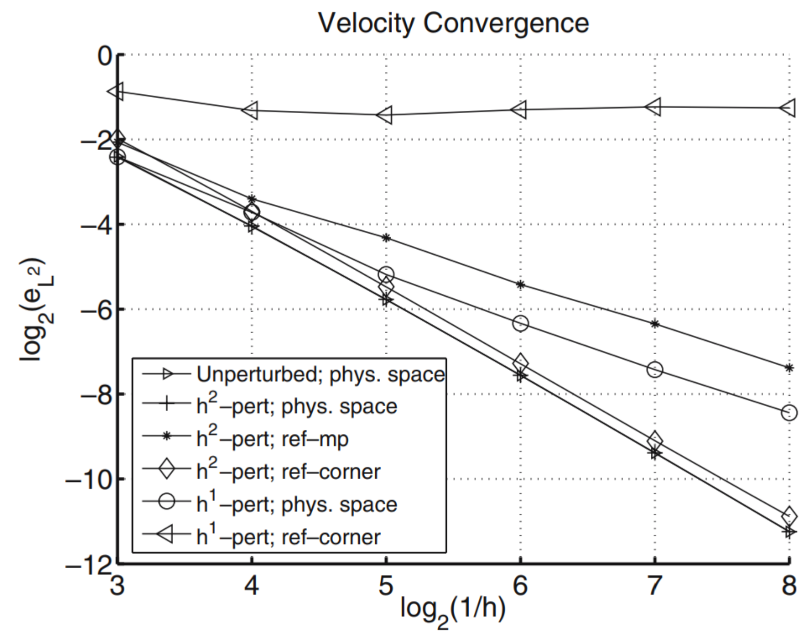


h^1- perturbed grid

→ Perturb in each refinement step

Numerical Experiments





→ $O(h^2)$ convergency rate for h^2-perturbed grids

→ No convergency for h^1-perturbed grids in reference space

→ $O(h^1)$ convergency rate for midpoint evaluation in reference space

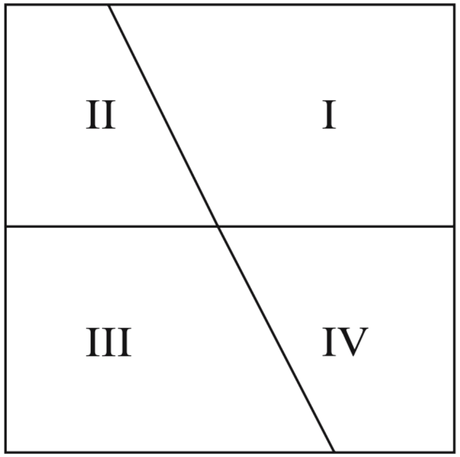
→ $O(h^2)$ convergency rate for corner point evaluation in reference space

Numerical Experiments

- Use subdomains with different permeabilities

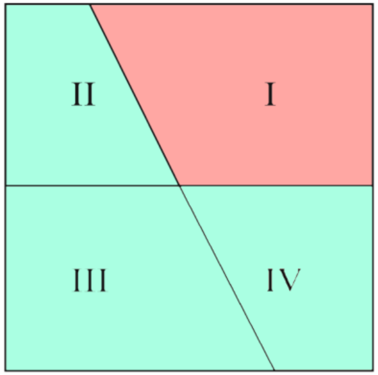
→ $\boldsymbol{K} \neq const$

→ Nonsmooth solutions occur

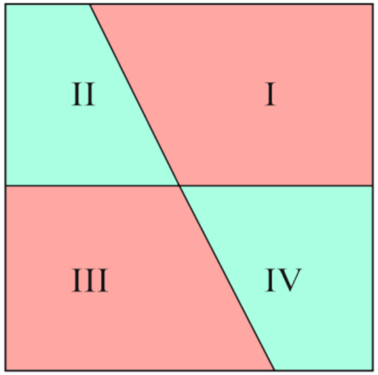


- Use interpolated Hilbert spaces $H^{1+\alpha}$ for pressure

→ i.e. assume $p \in H^{1+\alpha}$



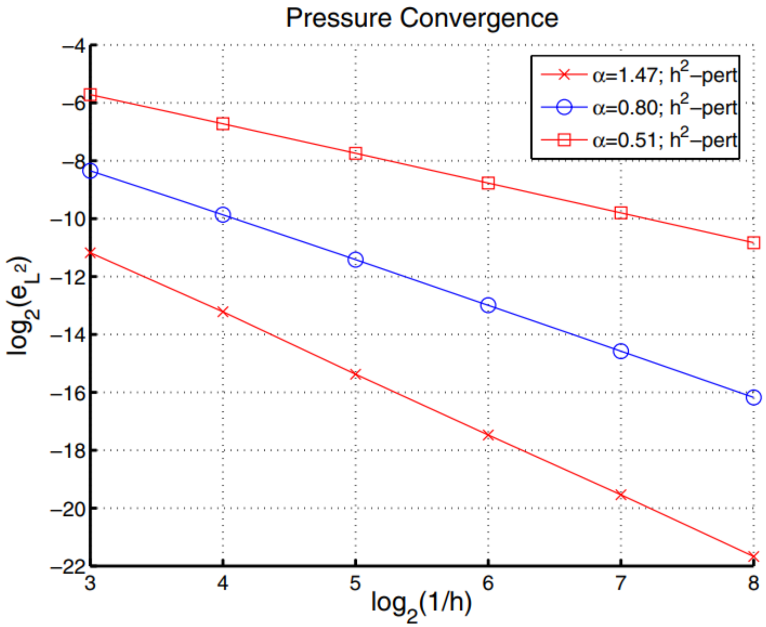
$\alpha = 1.47$



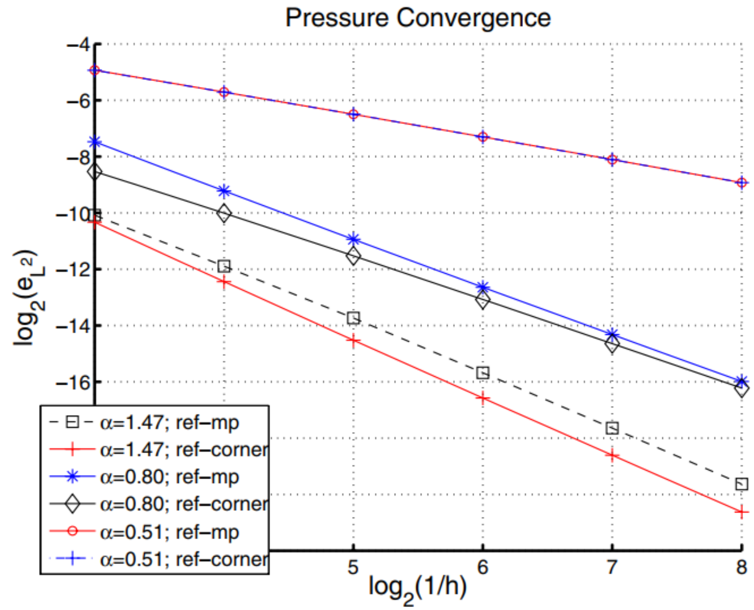
$\alpha = 0.80, \alpha = 0.51$

Numerical Experiments

h^2-perturbed physical space



h^2-perturbed reference space

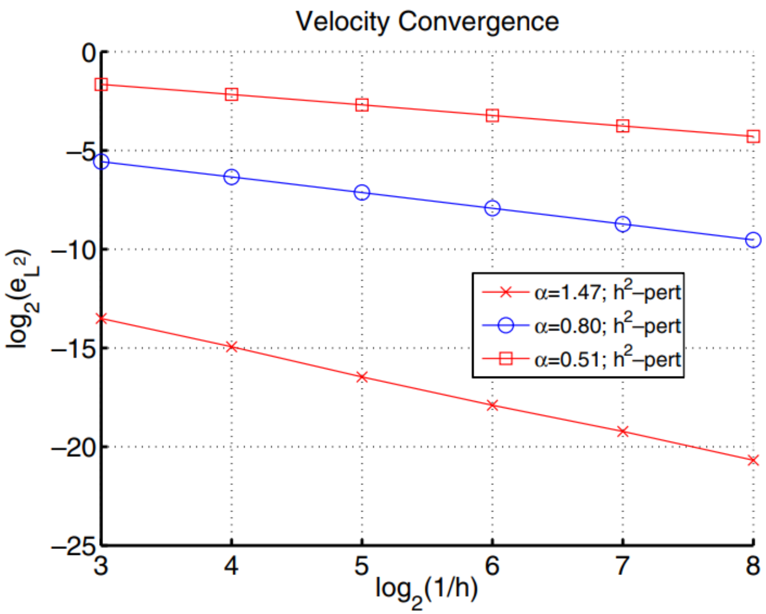


→ $\min(2, 2\alpha)$ order of convergency for all discretizations

→ Corner point evaluation performs better than midpoint

Numerical Experiments

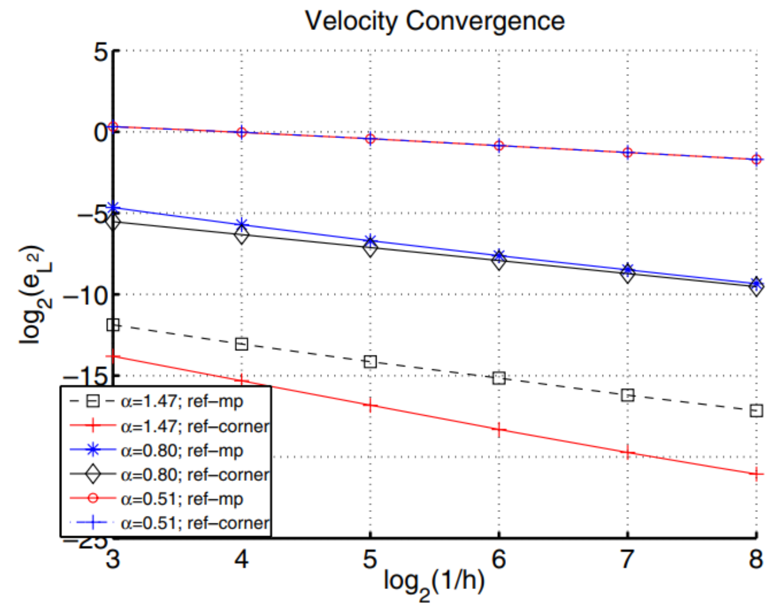
h^2-perturbed physical space



$\alpha = 1.47 \;\Rightarrow\; O(h^\alpha)$ convergency

$\alpha < 1 \;\Rightarrow\; O(h^\alpha)$ convergency

h^2-perturbed reference space

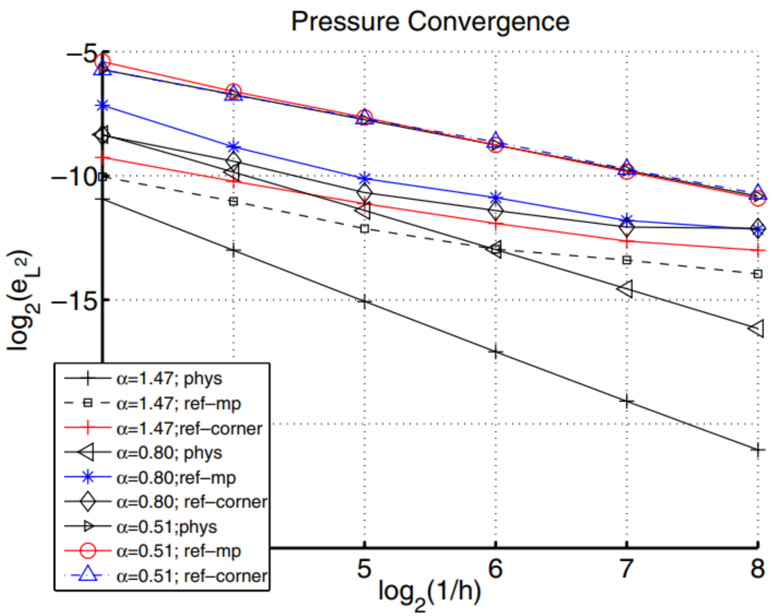


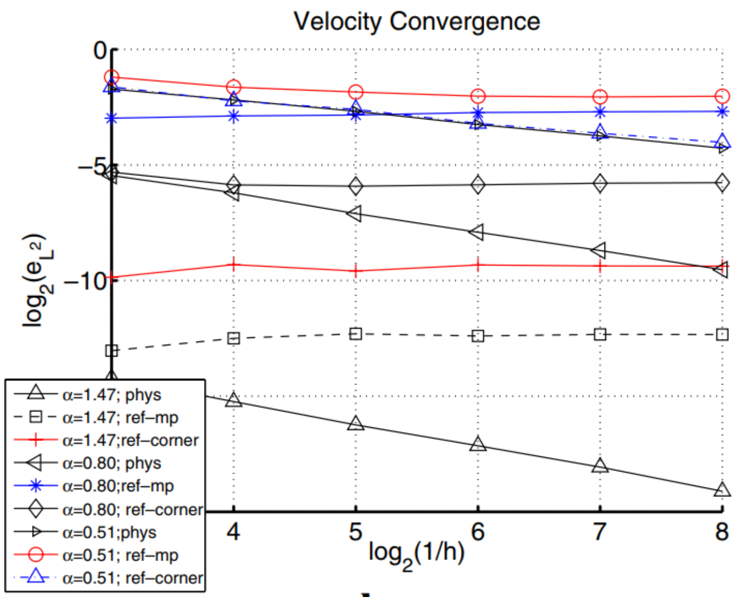
$$\alpha = 1.47 \;\Rightarrow\; \begin{cases} O(h^1) & \text{midpoint} \\ O(h^\alpha) & \text{corner point} \end{cases}$$

$\alpha < 1 \;\Rightarrow\; O(h^\alpha)$ convergency

Numerical Experiments

h^1-perturbed grids





$O\left(h^{\min(2,2\alpha)}\right)$ in physical space

In reference space: no convergence in the last refinement steps

$O\left(h^{\min(1,\alpha)}\right)$ in physical space

In reference space: no convergence at all

Conclusion

Conclusion

- MPFA method can be reformulated as MFE
- Trapezoidal quadrature rule → symmetric and positive definite cell centered pressure system
- Numerical Tests → Corner point evaluation more accurate than midpoint evaluation
- On h^2-perturbed mashes $\left\{ \begin{array}{l} O(h^2) \text{ convergence for corner point} \\ O(h^1) \text{ convergence for midpoint} \end{array} \right\}$ for velocities
- On h^1-perturbed mashes → both methods suffer reduction or loose convergency

Thank you for your attention.

→ Time for questions.