University of Stuttgart
Institute for Computational Physics

SimTech

ICP Group Seminar

Robustness analysis of quantum algorithms against coherent control errors

January 30th, 2023

Daniel Fink

Background

- Collaboration with Julian Berberich
- Jointly working in SimTech PN8
- Paper under current development

Robustness analysis of quantum algorithms against coherent control errors

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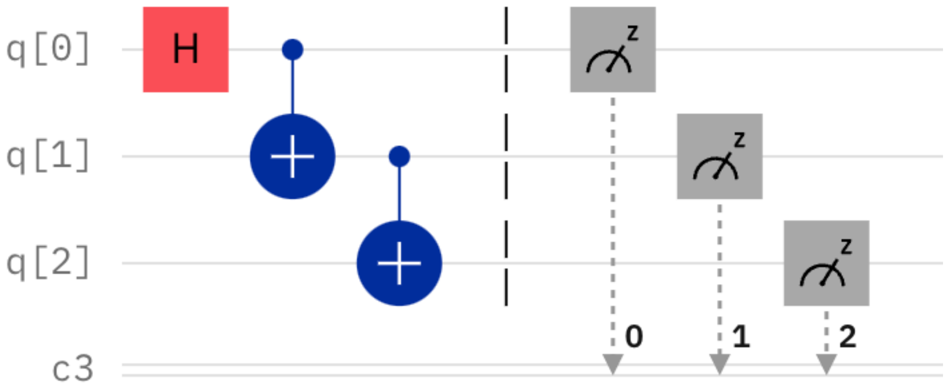
Noise poses a major obstacle in current quantum devices. Several recent studies have shown that coherent control errors, for which an ideal gate e^{-iH} is perturbed by an additional error gate

1 Introduction

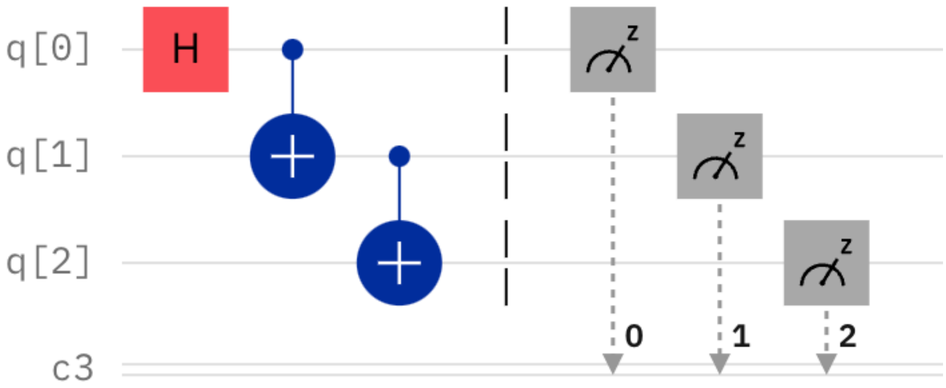
Quantum computing has emerged as a powerful tool to overcome limitations of classical computing and solve problems that were previously

What's the Problem?



What's the Problem?

3-GHZ State Circuit



$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

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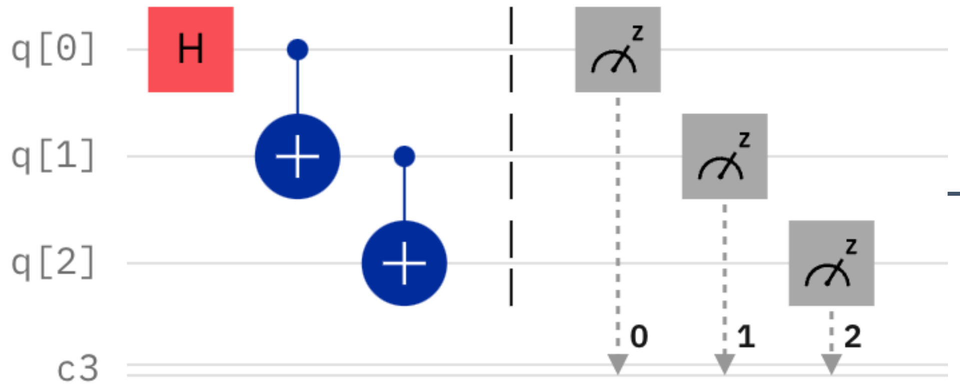

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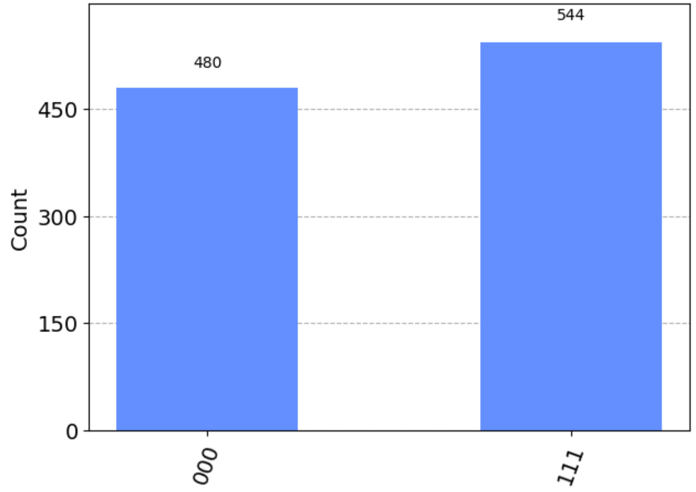
ideal



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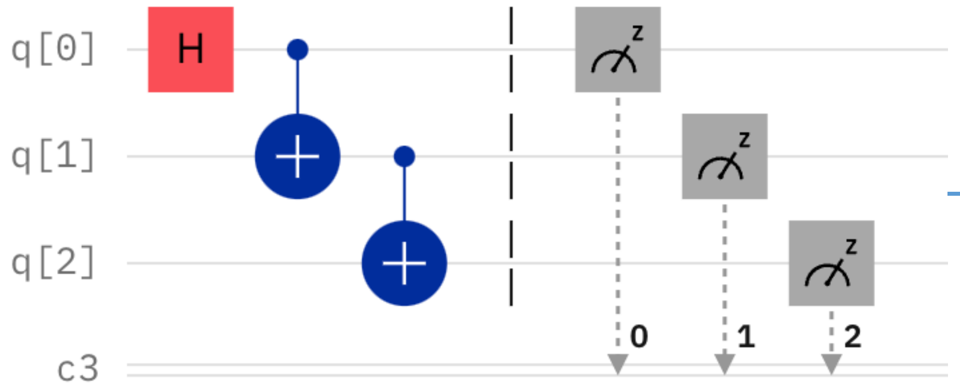

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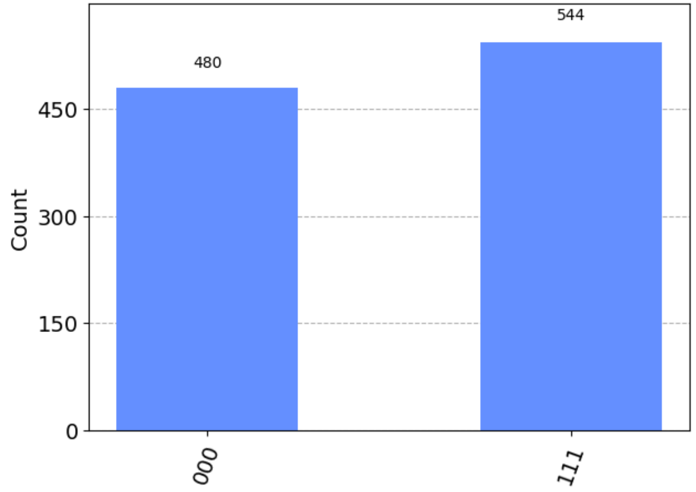
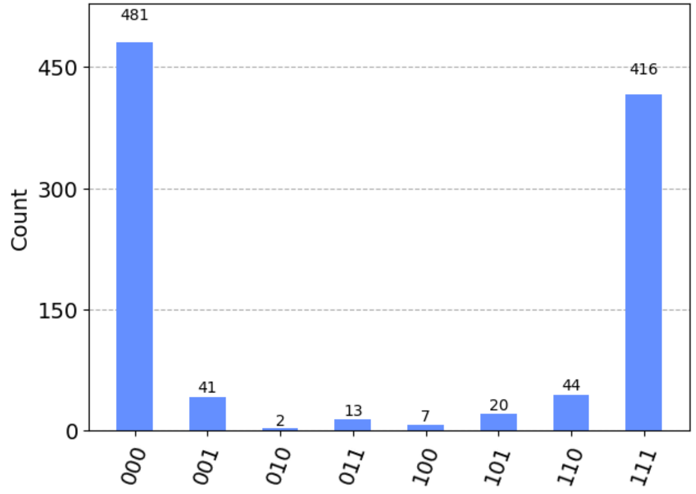
ideal

errors





What's the Problem?

SimTech

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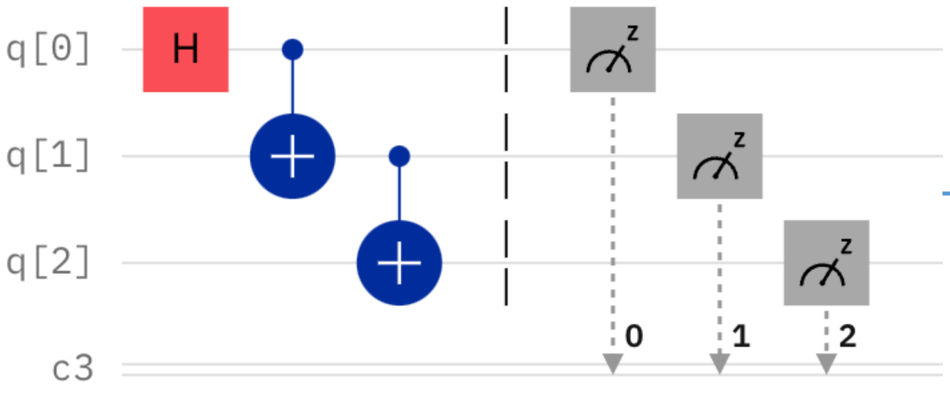

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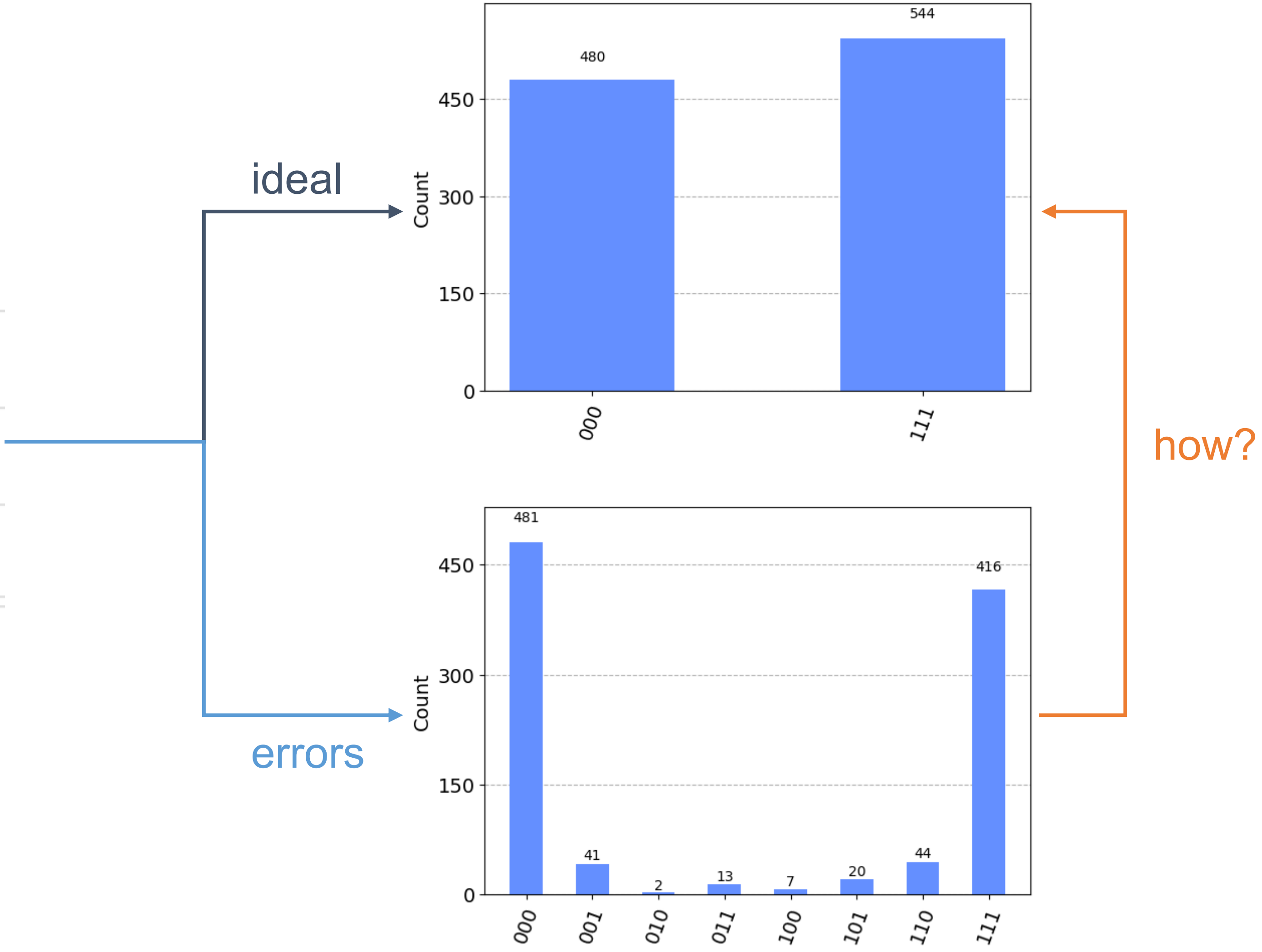


Agenda

- Quantum Computing
- Handling Errors: QEC & QEM
- Our approach: Lipschitz bounds
- Experiments
- Conclusion & Outlook

Quantum Computing

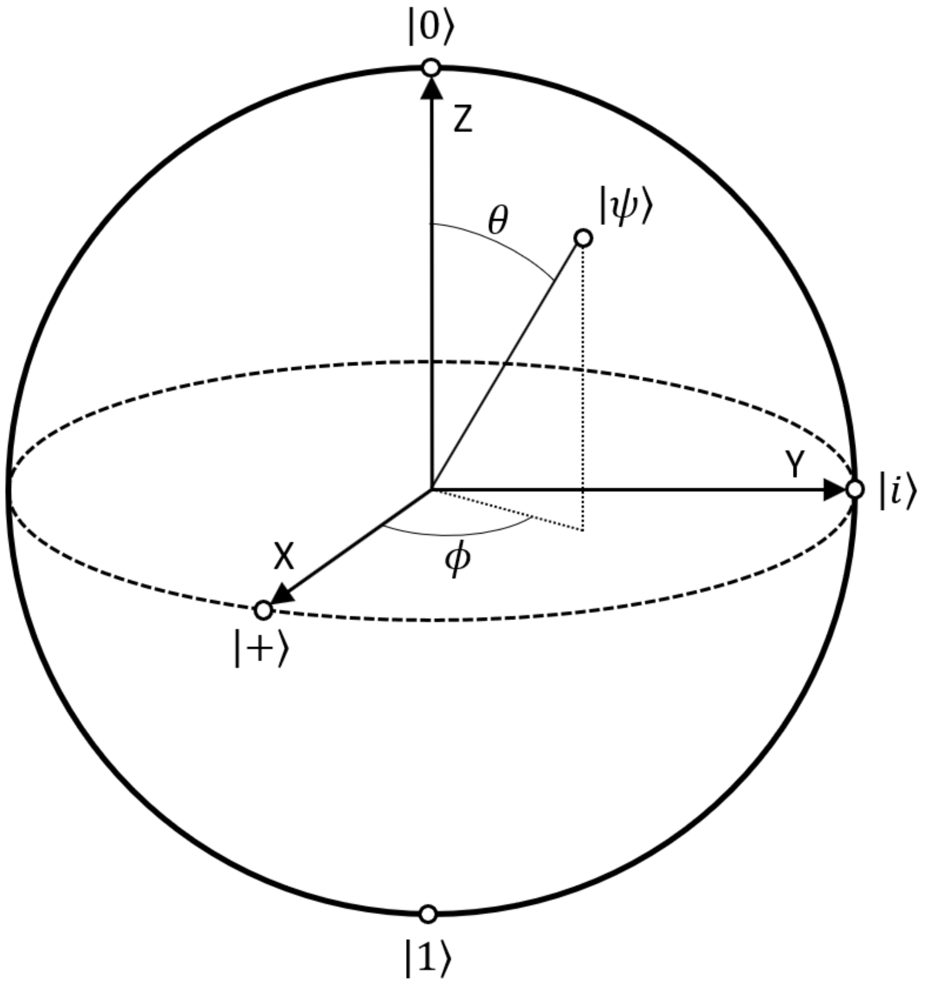
Qubits

- General single qubit state: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$

→ $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$, with $\theta, \phi \in \mathbb{R}$ (up to a global phase)

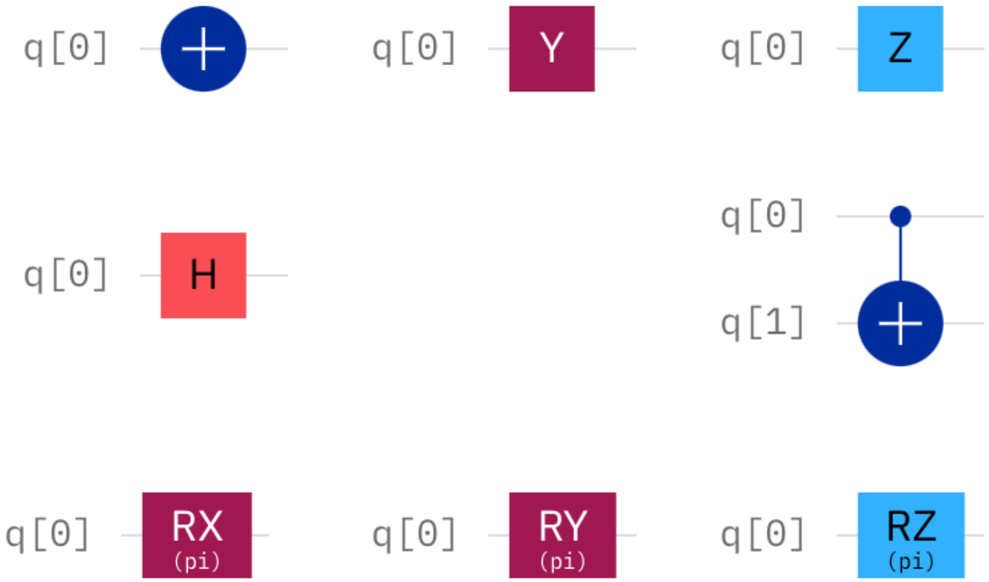
Bloch Sphere

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$



Quantum Gates

- Unitary operator $\mathrm{U}: H \to H, \quad \mathrm{U}|\psi_0\rangle = |\psi\rangle$
- Examples:



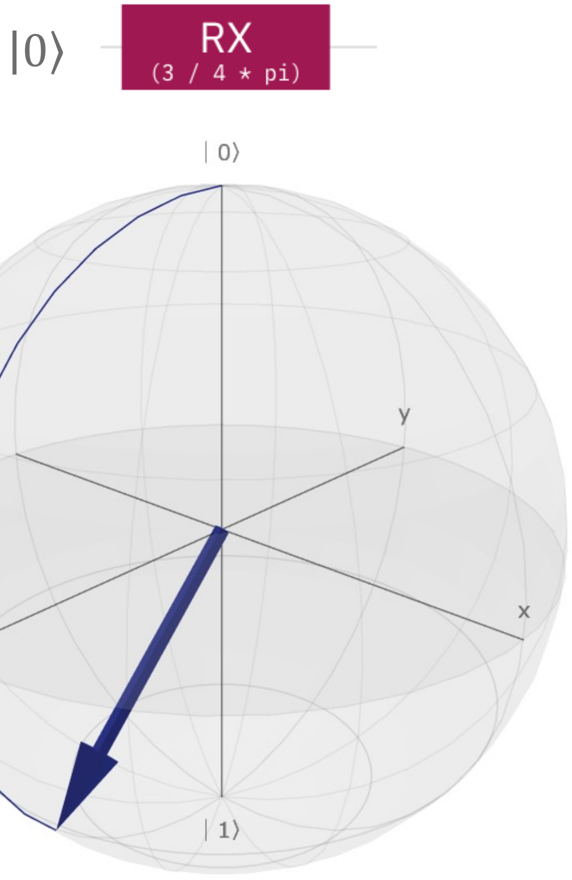
Pauli Gates

Hadamard

Controlled X (CX)

Pauli Rotation Gates

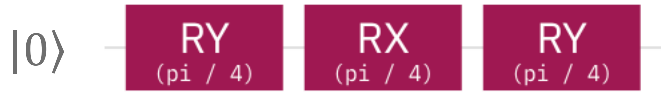
Quantum Gates

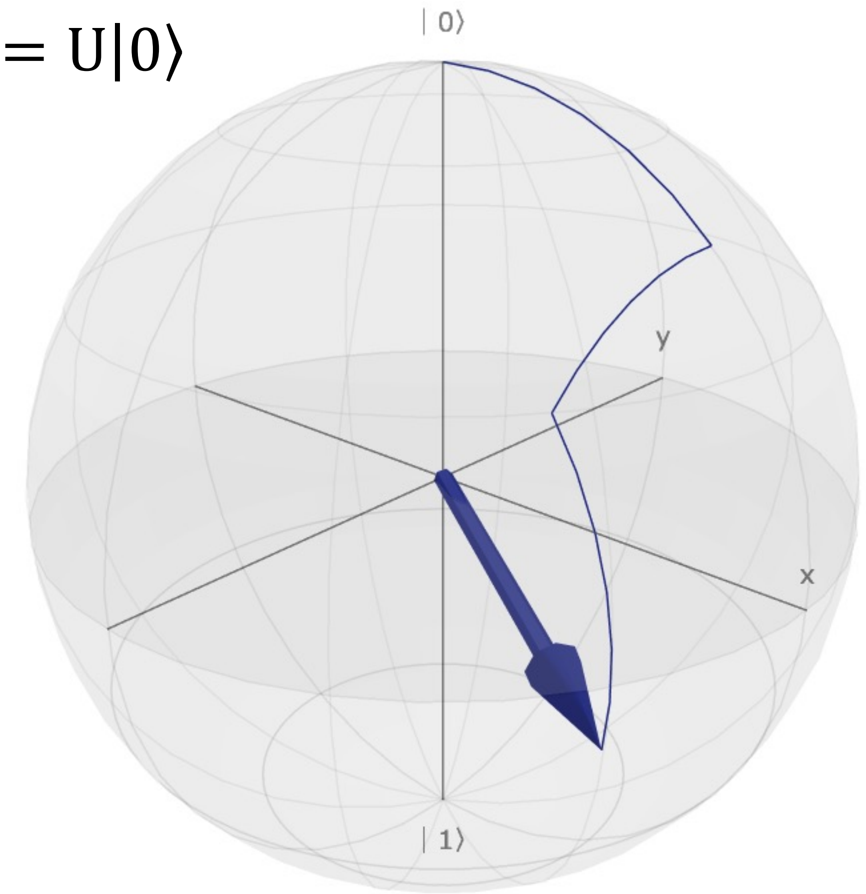


Quantum Circuits

- Gates together form a circuit: $|\psi\rangle = \mathrm{U_N} \dots \mathrm{U_1}|0\rangle = \mathrm{U}|0\rangle$

- Example:





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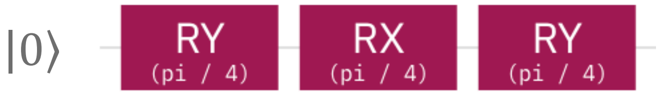
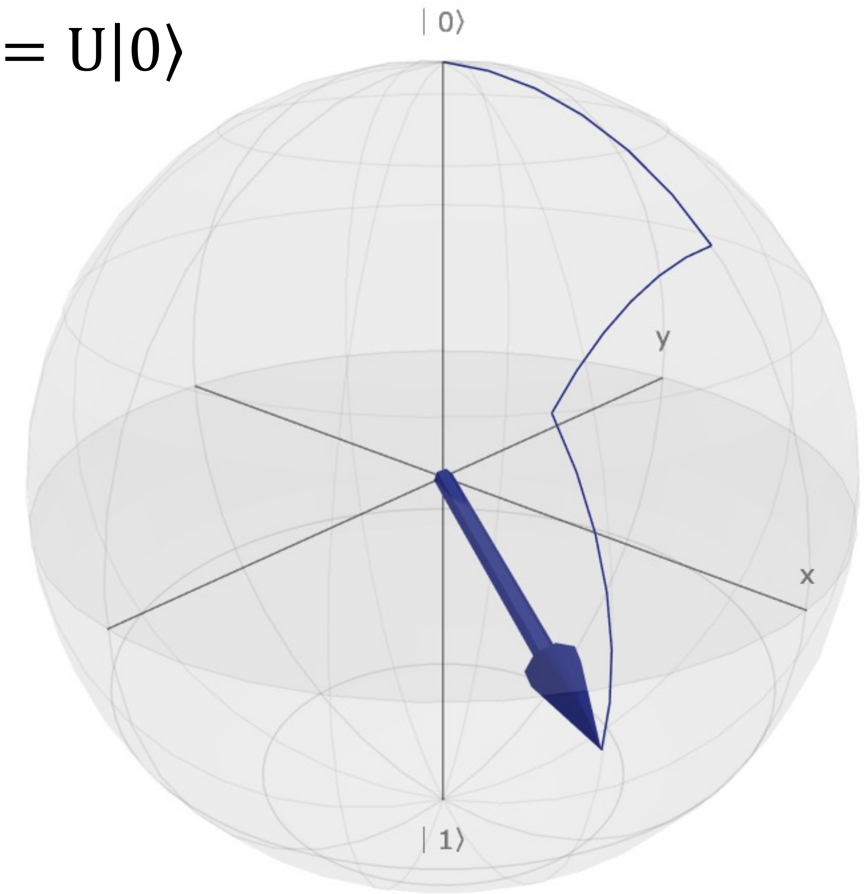
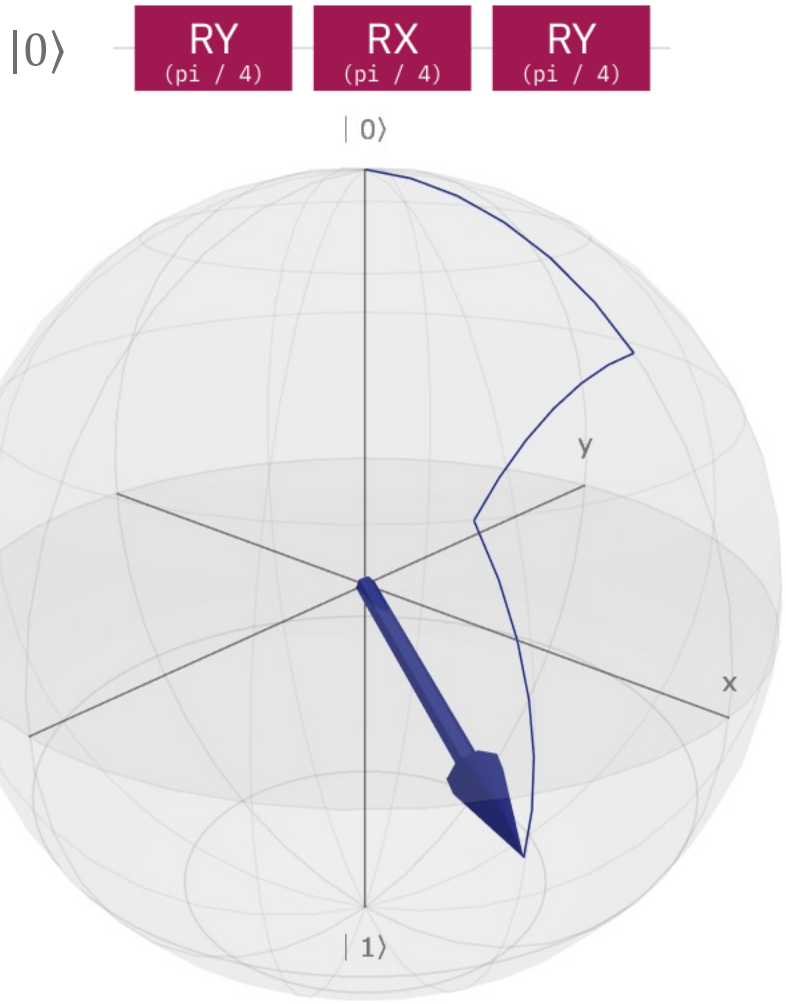


- Quantum computers have a native gate set

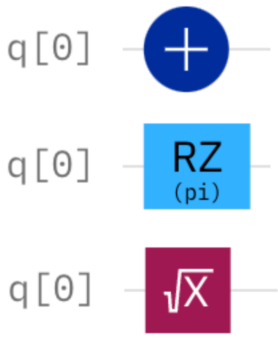
→ Transpilation is necessary

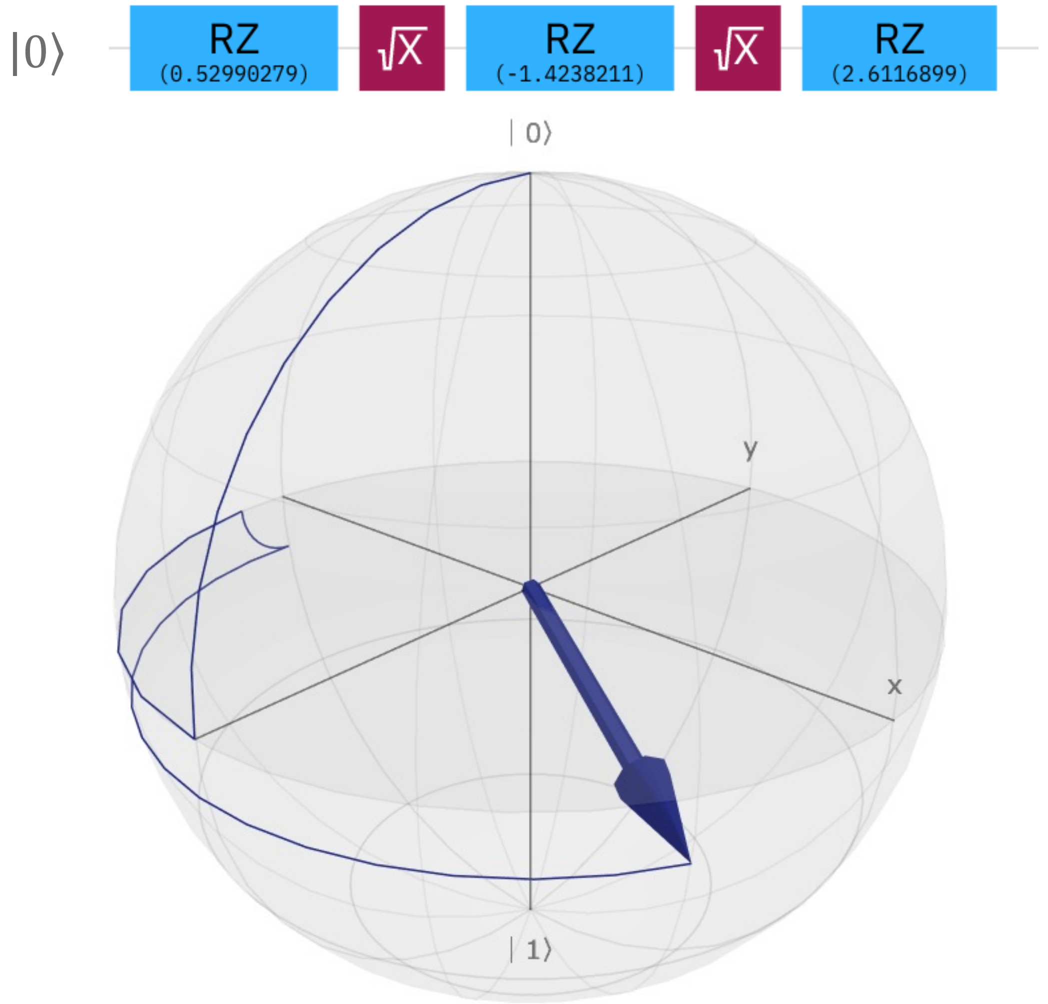


Transpilation



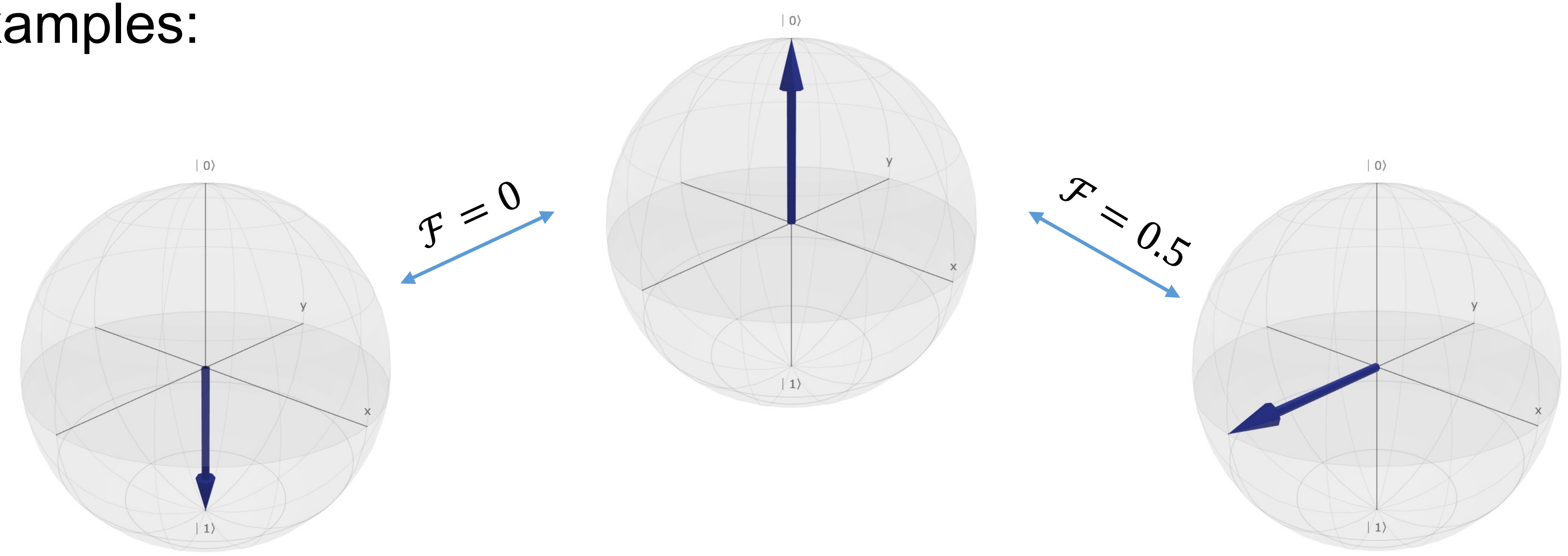
IBM Gate Set





Comparing Quantum States

- Goal: compare two states $|\psi\rangle \leftrightarrow |\phi\rangle$
- Define the Fidelity $\mathcal{F}(|\psi\rangle, |\phi\rangle) = |\langle\psi|\phi\rangle| \in [0,1]$
- Examples:



Errors in Quantum Computing

Quantum Errors

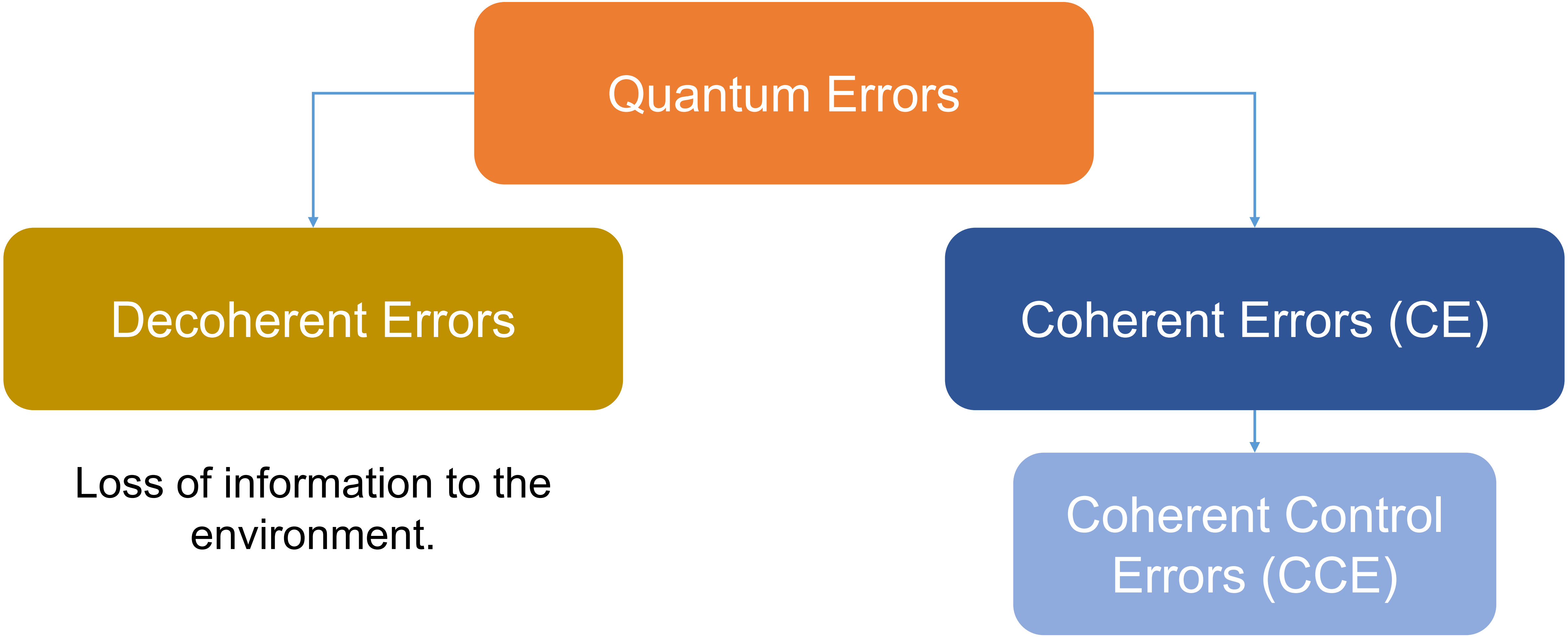
Decoherent Errors

Loss of information to the environment.

Coherent Errors (CE)

Information is preserved, but unintentional operations have been applied.

Errors in Quantum Computing



Coherent Errors

- Every Unitary can be written as $\mathrm{U} = e^{-i\mathrm{G}}$
- $\mathrm{G} = \mathrm{G}^{\dagger}$ is a Hermitian operator (generator)
- E.g.: $\mathrm{RZ}(\theta) = e^{-i\frac{\theta}{2}Z}$

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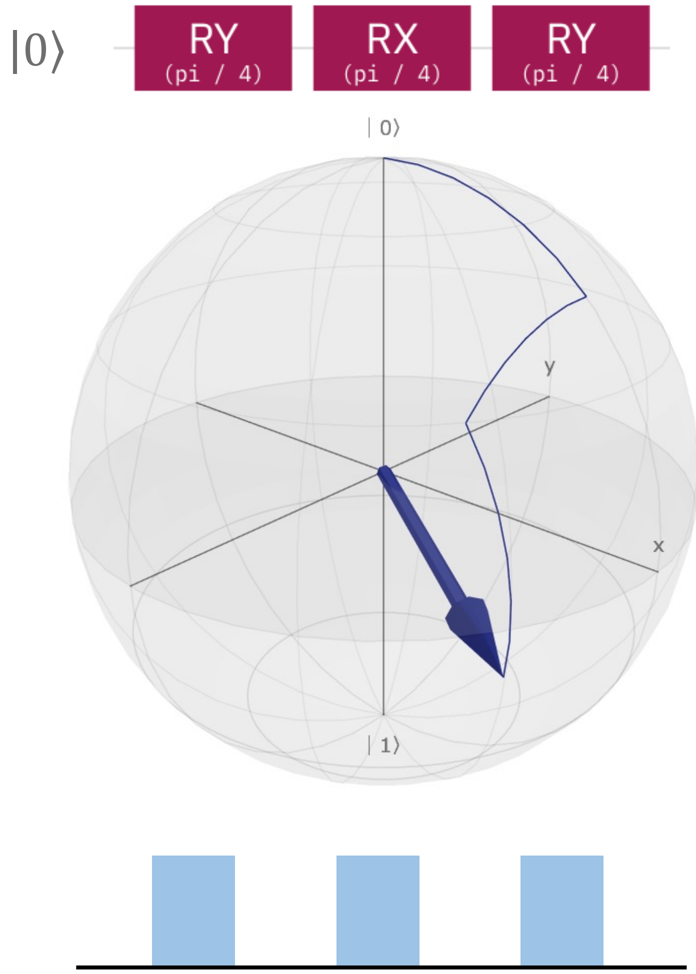
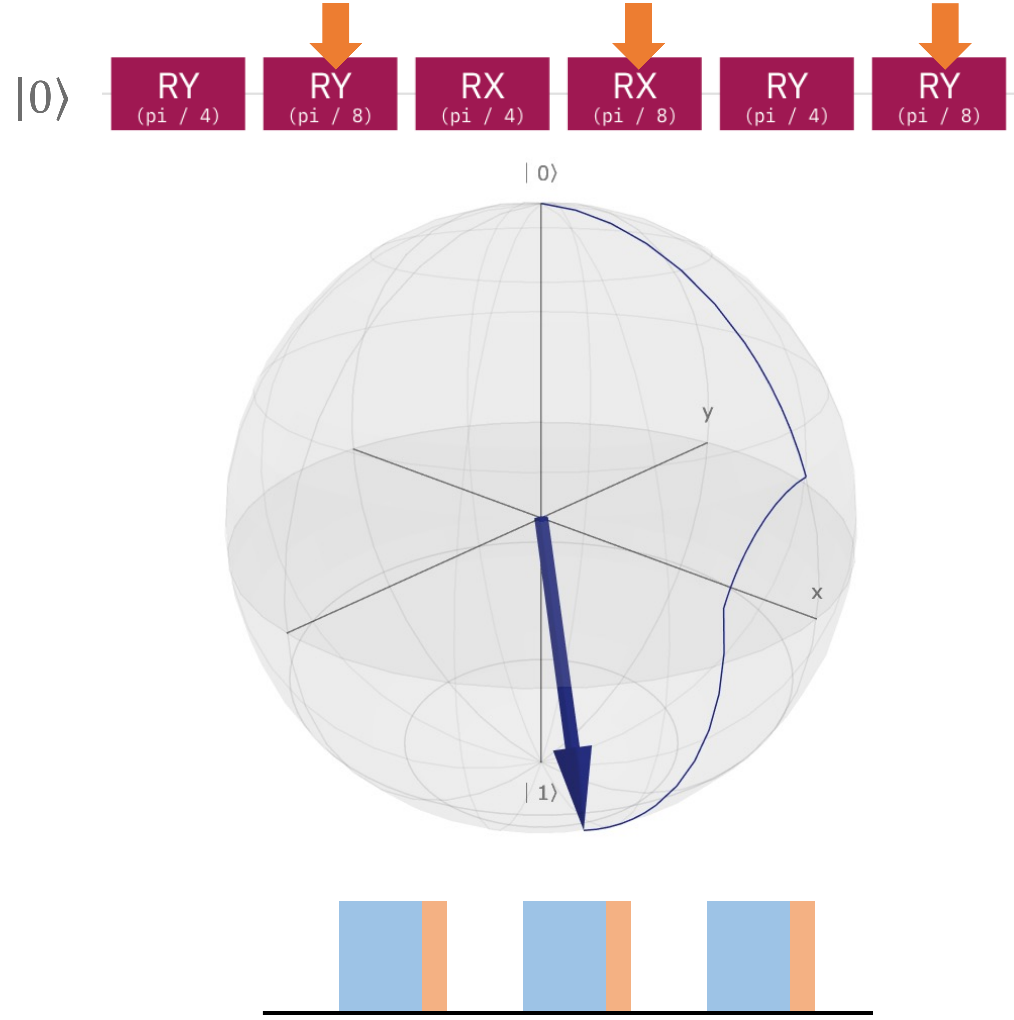
- Coherent error: $e^{-i\frac{\theta}{2}Z} \rightarrow e^{-i\frac{\theta}{2}Z} e^{-iG}$ any noise generator G
- Coherent control error: $e^{-i\frac{\theta}{2}Z} \rightarrow e^{-i(1+x)\frac{\theta}{2}Z}$ same generator

Coherent Control Errors





Handling Errors

Handling Errors

Procedures

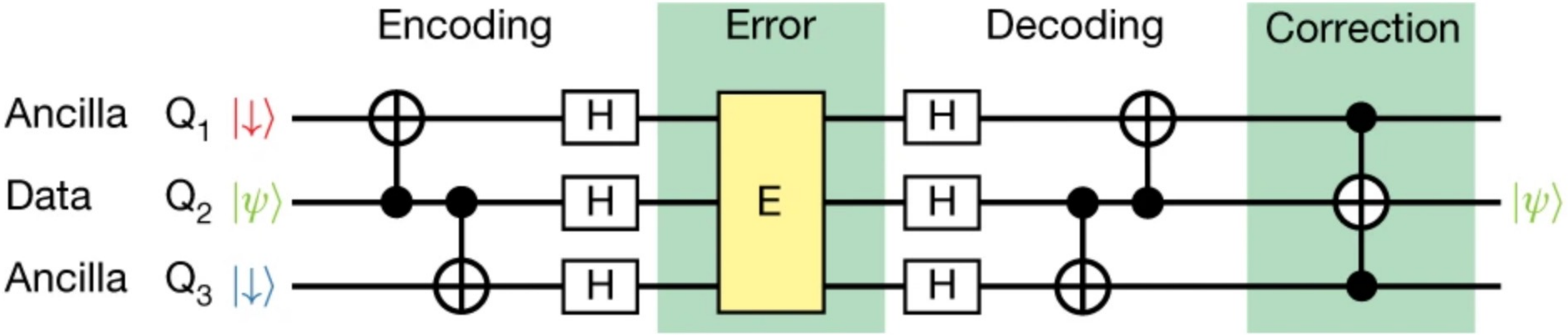
Quantum Error Correction (QEC)

Add additional gates to detect and compensate for errors.

Quantum Error Mitigation (QEM)

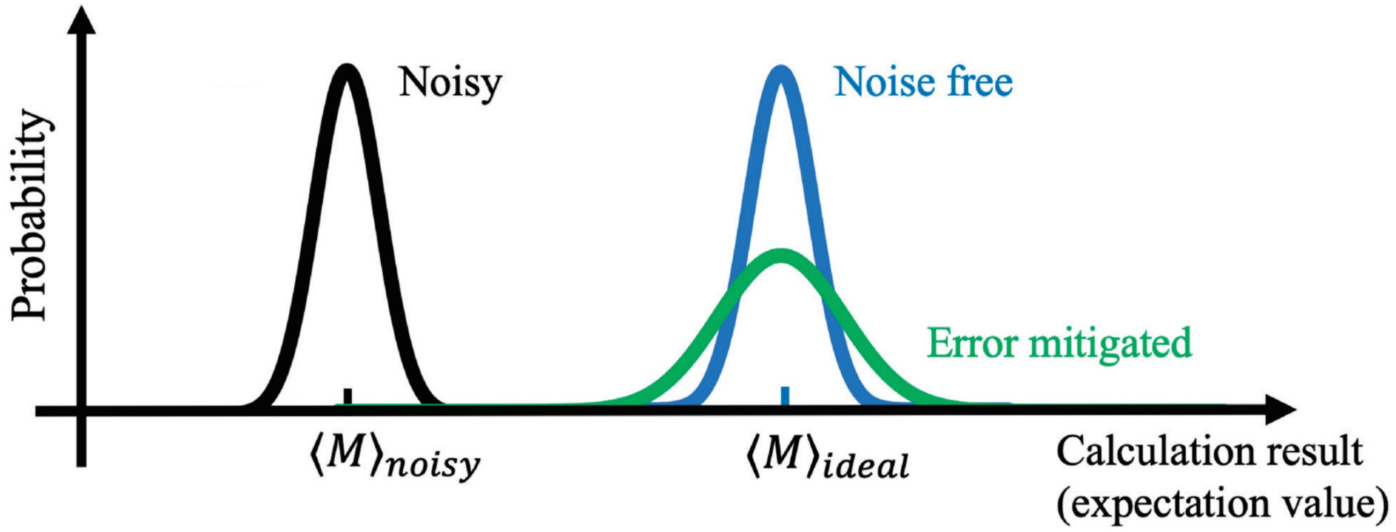
Reduce noise effects via classical post-processing.

Quantum Error Correction



Takeda et al., “Quantum error correction with silicon spin qubits”, Nature 608, 682–686 (2022)

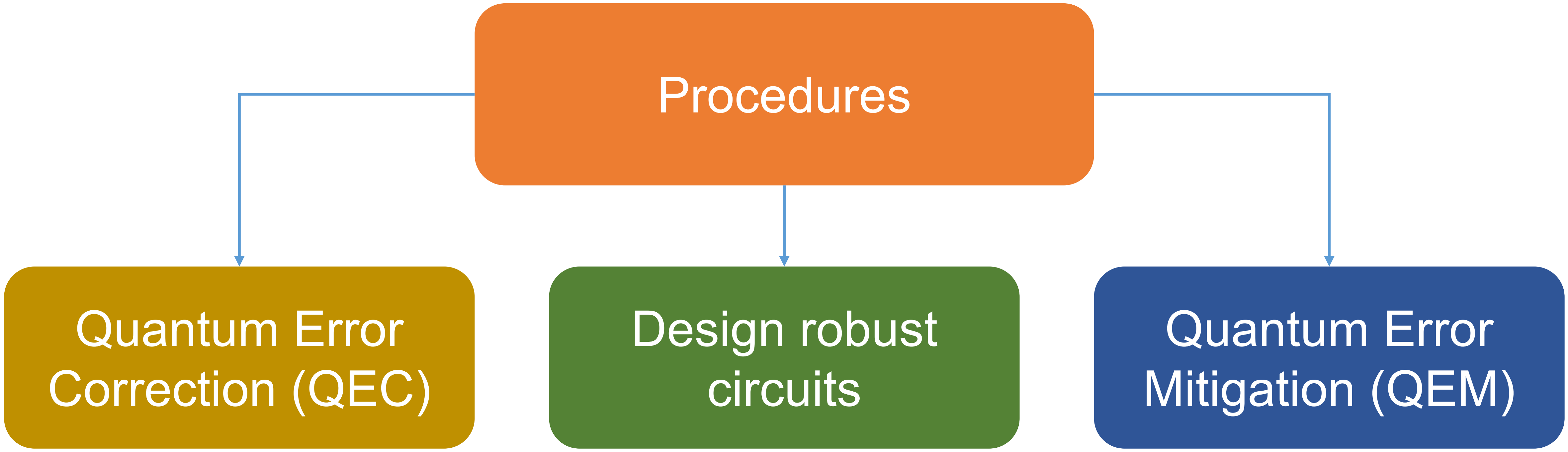
Quantum Error Mitigation



Endo et al., “Hybrid Quantum-Classical Algorithms and Quantum Error Mitigation”,
Journal of the Physical Society of Japan, 90, 032001 (2021)

Our Approach

Our Approach



Design robust circuits in the first place.
Can be used together with QEC/QEM.

Framework

- Noise-free circuit $\left|\hat{\psi}\right\rangle = \hat{\mathrm{U}}_N \hat{\mathrm{U}}_{N-1} \dots \hat{\mathrm{U}}_1 |\psi_0\rangle$

- Noisy circuit $|\psi(x)\rangle = \hat{\mathrm{U}}_N(x_N)\hat{\mathrm{U}}_{N-1}(x_{N-1}) \dots \hat{\mathrm{U}}_1(x_1)|\psi_0\rangle$

  with noise $x \in \mathbb{R}^N$

  $|\psi(0)\rangle = \left|\hat{\psi}\right\rangle$

- Noise level $\epsilon \in \mathbb{R}^+$ such that $\|x\|_2 < \epsilon$

Framework

Definition

A scalar $L > 0$ is a Lipschitz bound of $x \longmapsto |\psi(x)\rangle$ if

$$\| |\psi(x)\rangle - |\psi(x')\rangle \|_2 \leq L \|x - x'\|_2$$

for all $x, x' \in \mathbb{R}^N$.

Framework

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for all $x, x' \in \mathbb{R}^N$.

- The minimal L is called the Lipschitz constant.
- L bounds the worst-case amplification of a perturbation x:

$$\left\| |\psi(x)\rangle - |\hat{\psi}\rangle \right\|_2 \leq L \|x\|_2$$

Framework

Theorem

For any $x \in \mathbb{R}^N$ with $\|x\|_2 < \epsilon$, and any initial state $|\psi_0\rangle$, we have

$$\left|\langle\psi(x)|\hat{\psi}\rangle\right| \geq 1 - \frac{L^2\epsilon^2}{2},$$

with $L > 0$ being a Lipschitz bound of $x \mapsto |\psi(x)\rangle$.

Framework

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Theorem

For any $x \in \mathbb{R}^N$ with $\|x\|_2 < \epsilon$, and any initial state $|\psi_0\rangle$, we have

$$\left|\langle\psi(x)|\hat{\psi}\rangle\right| \geq 1 - \frac{L^2\epsilon^2}{2},$$

with $L > 0$ being a Lipschitz bound of $x \longmapsto |\psi(x)\rangle$.

- The Lipschitz constant can be hard to compute.
- We show a way how to calculate Lipschitz bounds.

Framework

Lipschitz bounds

Norm-based bounds

Interaction-based bounds

Lipschitz Bounds

Theorem: Norm-based bounds

The following is a Lipschitz bound of $x \longmapsto |\psi(x)\rangle$:

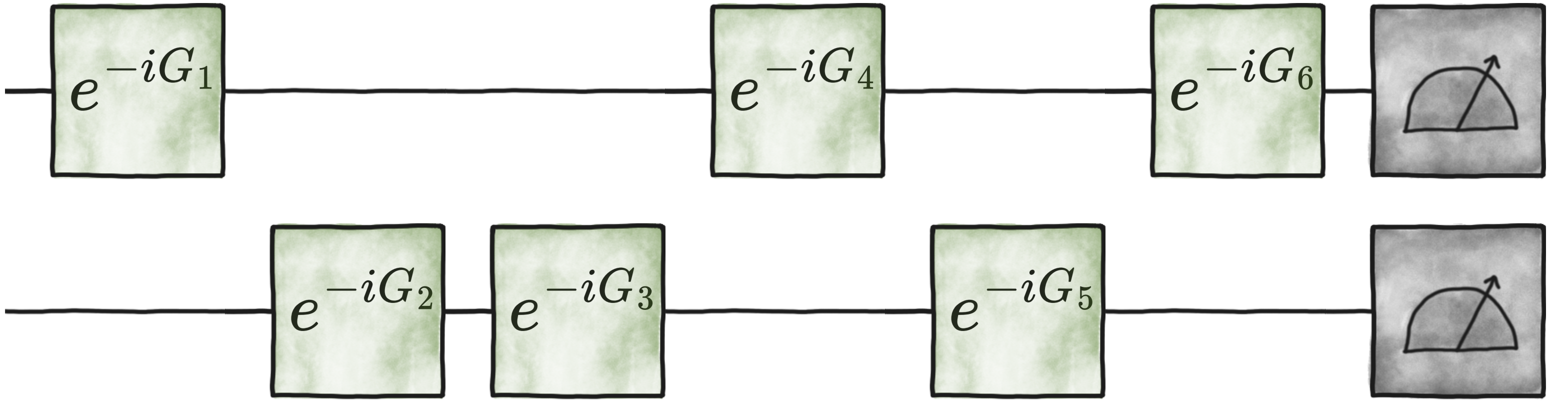
$$L = \sum_{i=1}^{N} \|\mathrm{G}_i\|_2$$

Lipschitz Bounds

Theorem: Norm-based bounds

The following is a Lipschitz bound of $x \mapsto |\psi(x)\rangle$:

$$L = \sum_{i=1}^{N} \|G_i\|_2$$



$$L = \|G_1\|_2 + \|G_2\|_2 + \|G_3\|_2 + \|G_4\|_2 + \|G_5\|_2 + \|G_6\|_2$$

Lipschitz Bounds

Theorem: Interaction-based bounds

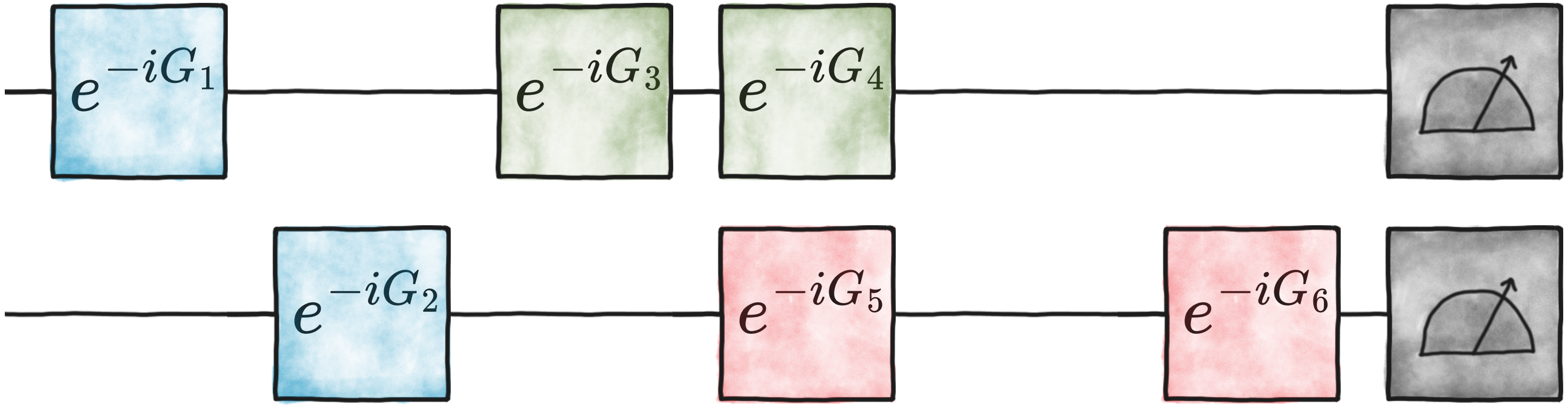
The following are Lipschitz bounds of $x \mapsto |\psi(x)\rangle$:

If N is even: $$\sum_{i=1}^{\frac{N}{2}} \left\| \begin{bmatrix} G_{2i-1} & G_{2i} \end{bmatrix} \right\|_2$$

If N is odd: $$\|G_N\|_2 + \sum_{i=1}^{\frac{N-1}{2}} \left\| \begin{bmatrix} G_{2i-1} & G_{2i} \end{bmatrix} \right\|_2$$

- $\begin{bmatrix} A & B \end{bmatrix}$ is a block matrix: 

Lipschitz Bounds



$$L = \|G_1\|_2 + \|G_2\|_2 + \|[G_3 \quad G_4]\|_2 + \|[G_5 \quad G_6]\|$$

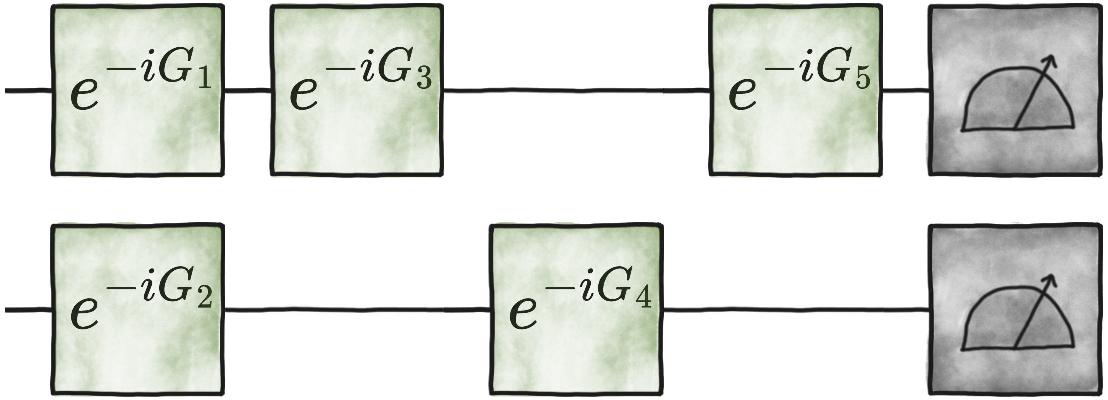
Framework

- Which bounds are better?
- Generally: $\|[G_1 \quad G_2]\|_2 \leq \|G_1\|_2 + \|G_2\|_2$
- Almost always: $\|[G_1 \quad G_2]\|_2 < \|G_1\|_2 + \|G_2\|_2$
- How big is the gap?

  → Current investigation

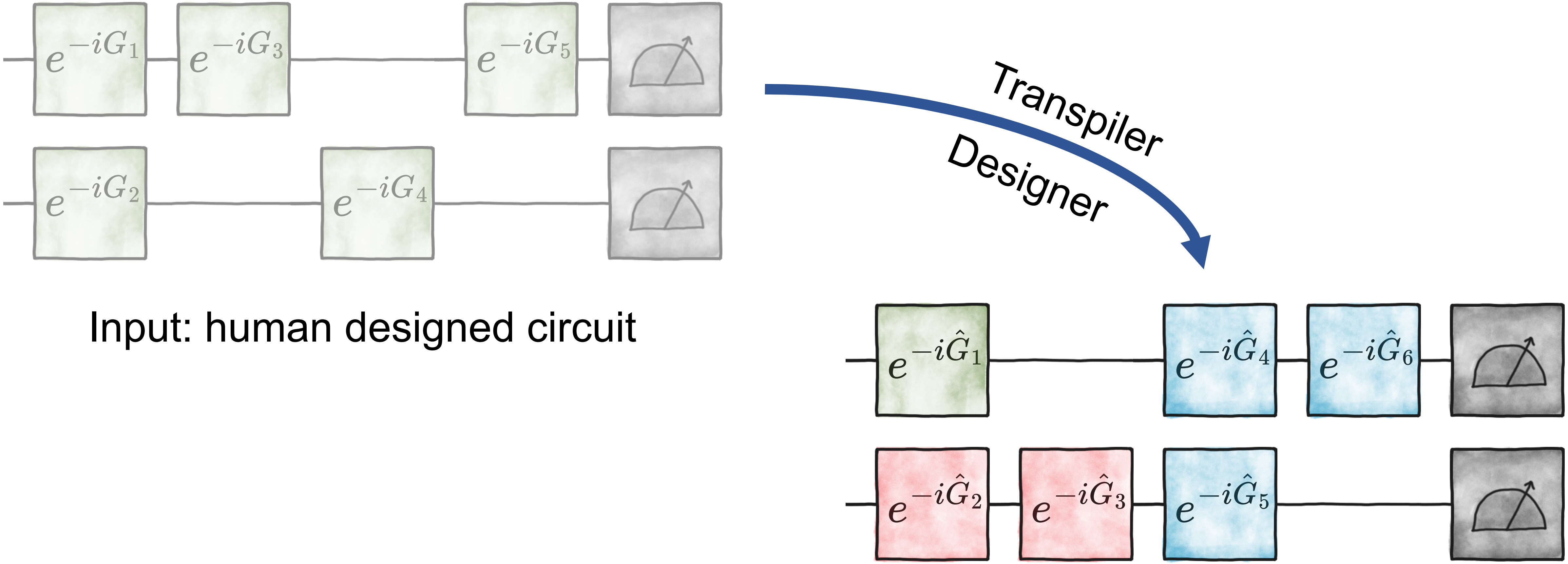
Design Guidelines

Design Guidelines



Input: human designed circuit

Design Guidelines



Output: transpiled resilient circuit

Experiments

Experiments

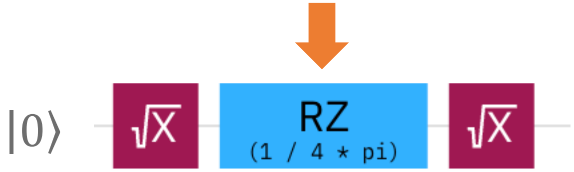
- Target: compare circuits with different Lipschitz bounds
- Goal: show lower Lipschitz bounds imply robustness

Experiments

- Target: compare circuits with different Lipschitz bounds
- Goal: show lower Lipschitz bounds imply robustness
- Procedure:
  1. Choose a set of noise levels $\{\epsilon\} \subseteq \mathbb{R}^{+}$
  2. For each ϵ, draw several $x \in B_{\epsilon}(0) \subseteq \mathbb{R}^{N}$ uniformly
  3. Insert CCE gates: $e^{-i\frac{\theta}{2}Z} \rightarrow e^{-i(1+x_i)\frac{\theta}{2}Z}$
  4. Calculate $\mathcal{F} = \left|\langle \psi_{\epsilon}(x) | \hat{\psi} \rangle\right|$

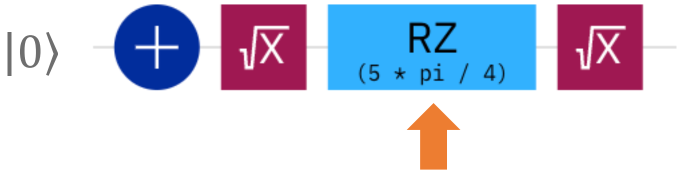
Experiment I

U_A:

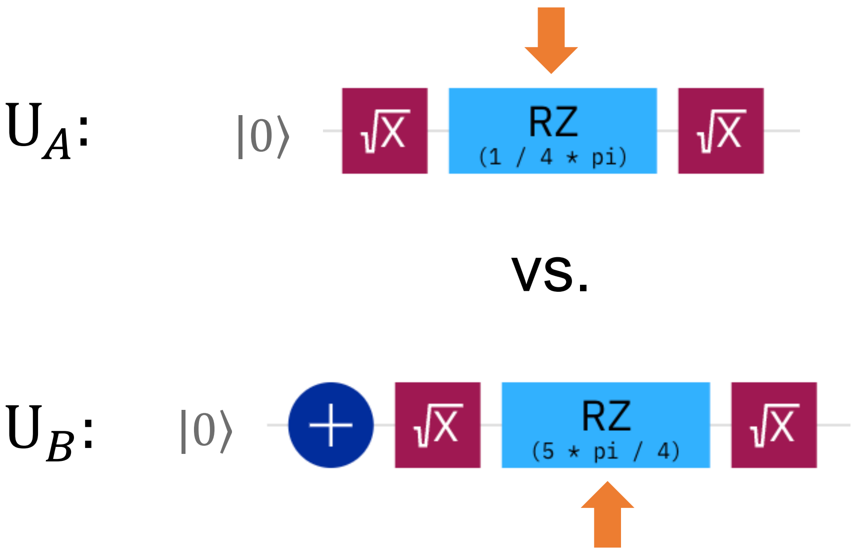


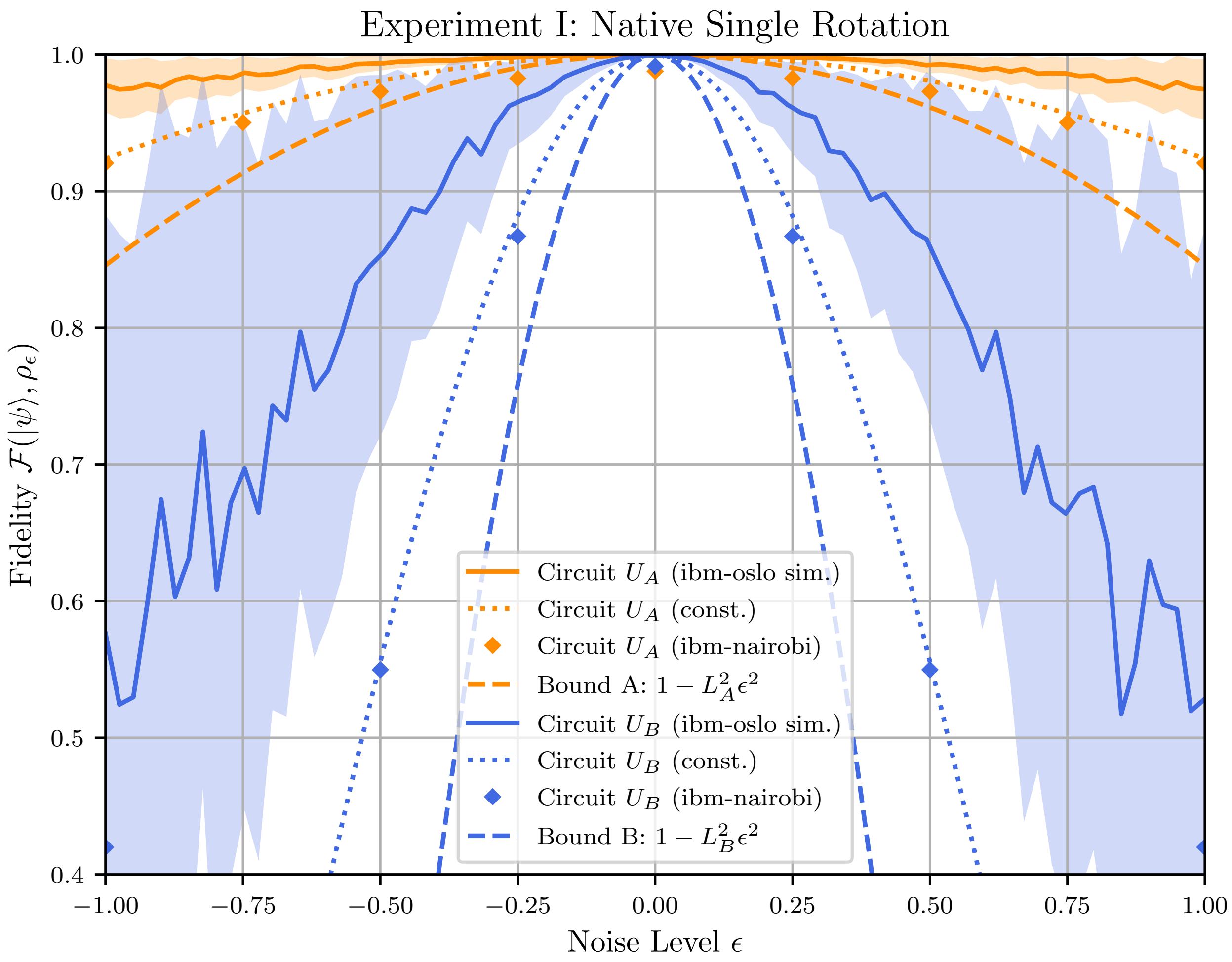
vs.

U_B:

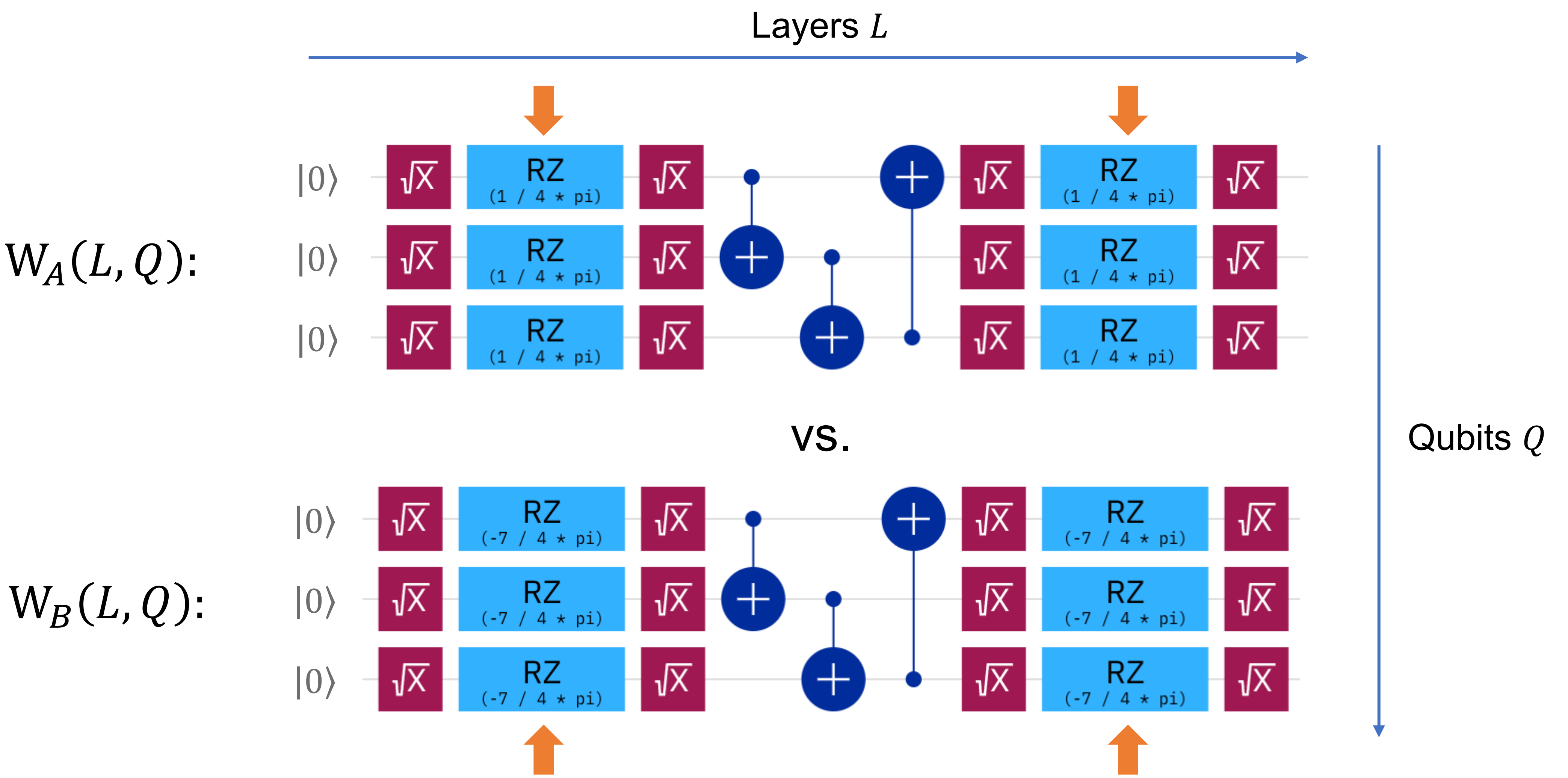


Experiment I

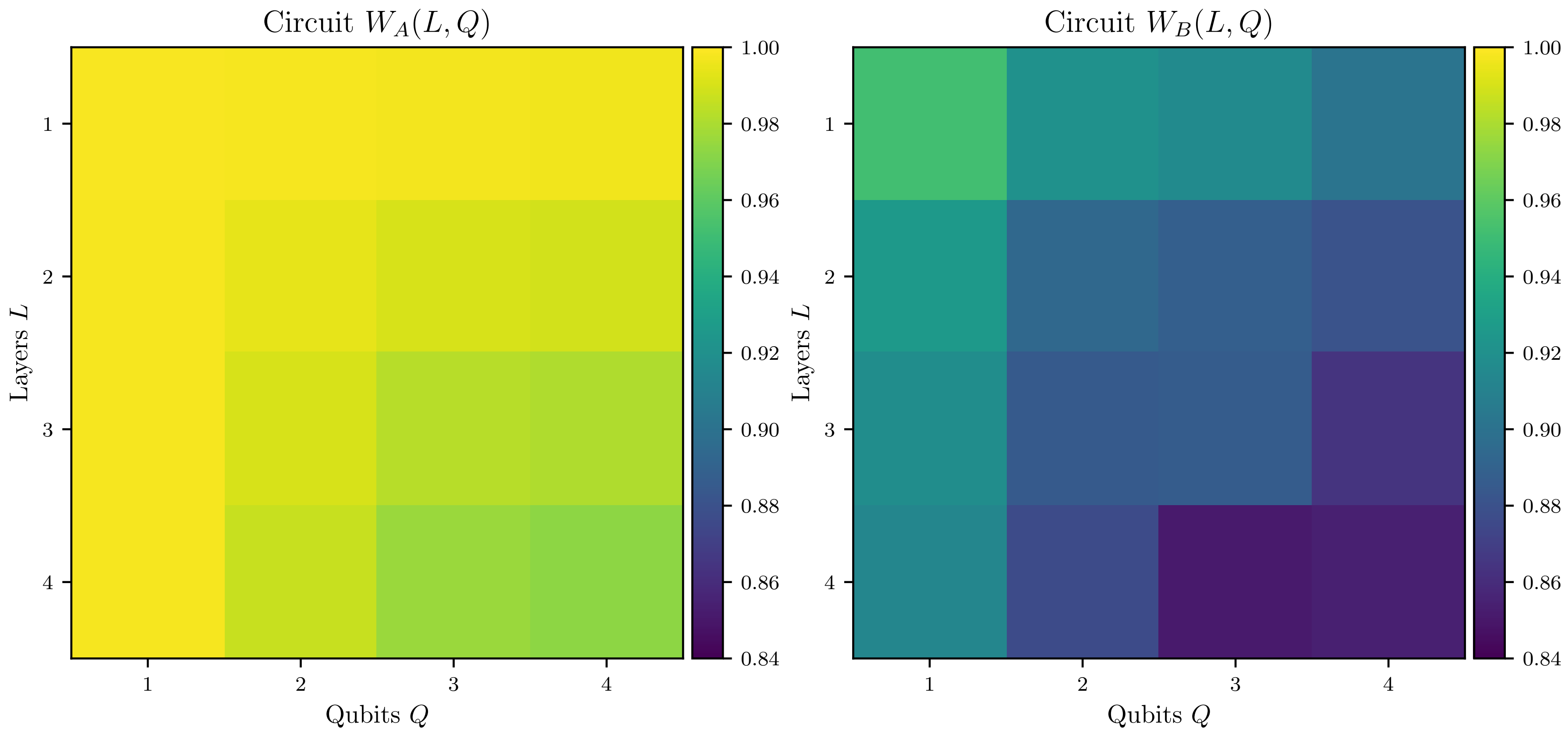




Experiment II



Experiment II



Conclusion
&
Outlook

Conclusion

- Framework for robustness analysis for C(C)E
- Derived worst-case error bounds
- Defined guidelines for quantum algorithm design
- Performed numerical validation

Outlook

- Extend the framework to account for decoherent errors $|\psi\rangle \to \rho$
- Connect worst-case Lipschitz bounds with QEC/QEM
- Integrate the framework into QML

→ Lipschitz bounds give rise to the use of regularization in QML

Thanks!

Let’s discuss!

Appendix

Framework

$$
\begin{aligned}
\left\| \begin{bmatrix} H_1 & H_2 \end{bmatrix} \right\|_2 &= \sqrt{\lambda_{\max}(H_1^\dagger H_1 + H_2^\dagger H_2)} \\
&\leq \sqrt{\lambda_{\max}(H_1^\dagger H_1) + \lambda_{\max}(H_2^\dagger H_2)} \\
&\leq \sqrt{\lambda_{\max}(H_1^\dagger H_1)} + \sqrt{\lambda_{\max}(H_2^\dagger H_2)} \\
&= \|H_1\|_2 + \|H_2\|_2 .
\end{aligned}
$$

Framework

$$\lambda_{\max}(H_1^\dagger H_1 + H_2^\dagger H_2)$$
$$\leq \lambda_{\max}(H_1^\dagger H_1) + \lambda_{\max}(H_2^\dagger H_2)$$

This inequality is strict if and only if the eigenvectors corresponding to the maximum eigenvalues of $H_1^\dagger H_1$ and $H_2^\dagger H_2$ do not align, i.e.,

$$\operatorname{argmax}_{\|v\|_2=1} v^\dagger H_1^\dagger H_1 v \cap \operatorname{argmax}_{\|v\|_2=1} v^\dagger H_2^\dagger H_2 v$$
$$= \emptyset.$$

Framework

Theorem Reversed

If we want to guarantee a worst-case fidelity of no less than $\mathcal{F}$ for any $x \in \mathbb{R}^n$ with $\|x\|_2 < \epsilon$ and any initial state $|\psi_0\rangle$, then the noise level must be bounded by

$$\epsilon \leq \frac{\sqrt{2}}{L}\sqrt{1-\mathcal{F}}.$$

Framework

Theorem Reversed

If we want to guarantee a worst-case fidelity of no less than $\mathcal{F}$ for any $x \in \mathbb{R}^n$ with $\|x\|_2 < \epsilon$ and any initial state $|\psi_0\rangle$, then the noise level must be bounded by

$$\epsilon \leq \frac{\sqrt{2}}{L}\sqrt{1-\mathcal{F}}.$$

- ϵ may be determined via calibration results.
- Can be used to determine if QEC is possible.

Thanks!

Let's discuss!

Framework

Lipschitz bounds

Norm-based bounds

Interaction-based bounds

Considering $||G||_2$ of each gate e^{-iG}

Considering subsequent gates $e^{-iG}e^{-iH}$

Framework

Problem

Find L such that, for any $x \in \mathbb{R}^N$ with $\|x\|_2 < \epsilon$, and any initial state $|\psi_0\rangle$, it holds that

$$\left|\langle\psi(x)|\hat{\psi}\rangle\right| \geq 1 - g(\epsilon, L),$$

where $L > 0$ depends only on the circuit components.

Framework

Problem

Find L such that, for any $x \in \mathbb{R}^N$ with $\|x\|_2 < \epsilon$, and any initial state $|\psi_0\rangle$, it holds that

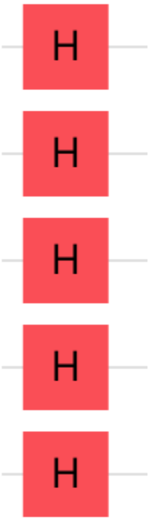
$$\left|\langle\psi(x)|\hat{\psi}\rangle\right| \geq 1 - g(\epsilon, L),$$

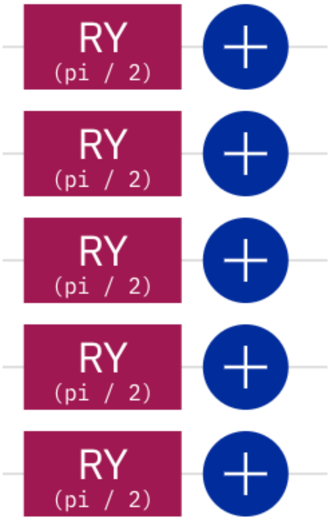
where $L > 0$ depends only on the circuit components.

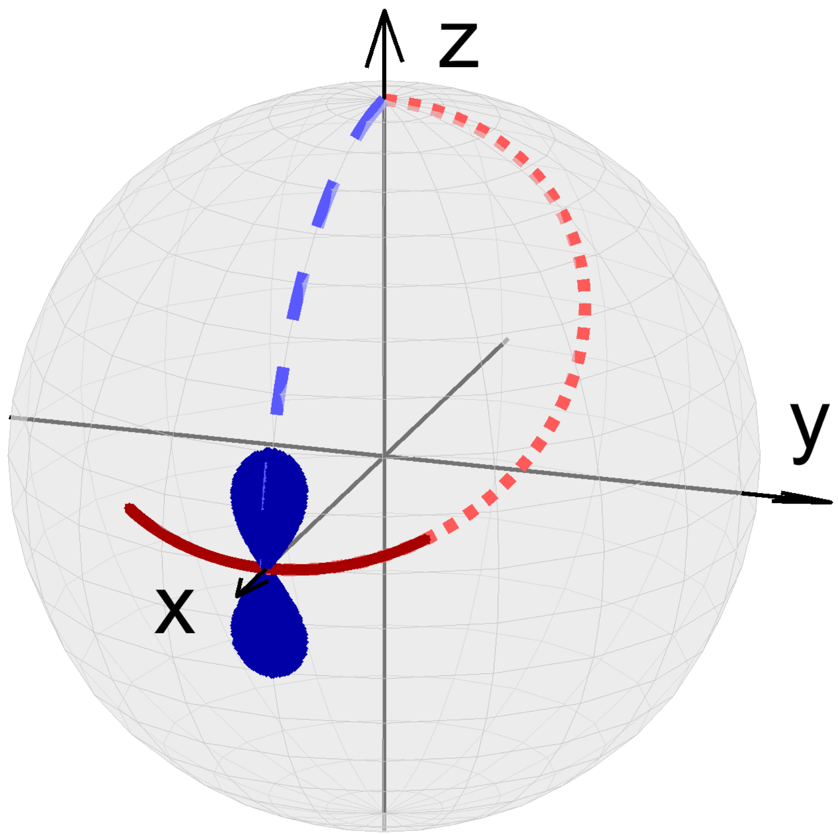
- This is a worst-case bound w.r.t. CCEs and the initial states.
- We show: L can be a Lipschitz bound of $x \mapsto |\psi(x)\rangle$.

Design Guidelines

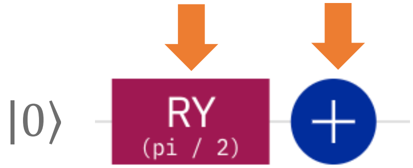
- E.g.:





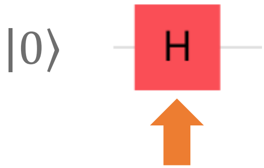


U_C:



vs.

U_D:

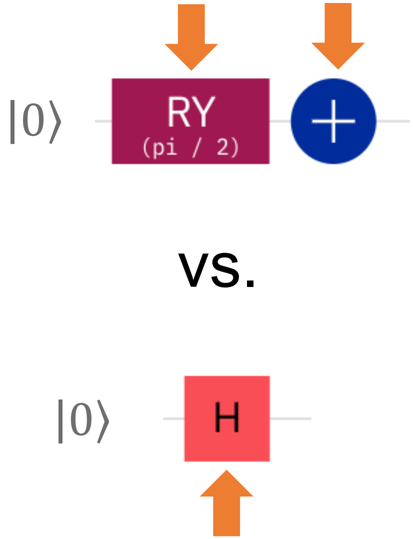


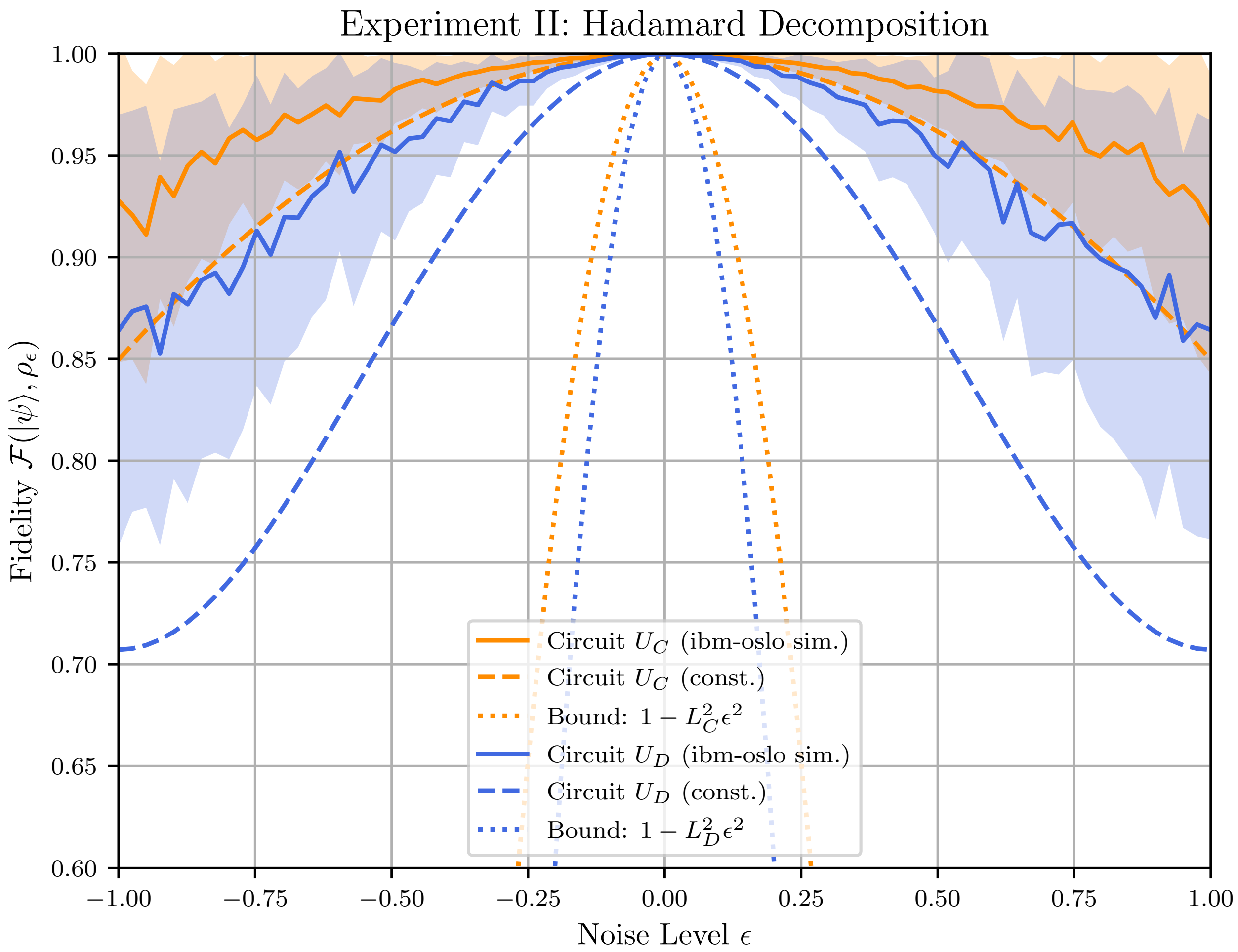
Experiment II

U_C:

vs.

U_D:





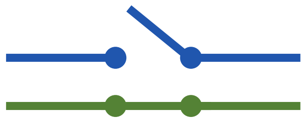
Qubits

SimTech

Bit

logical

0



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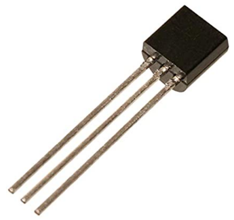


physical

notation

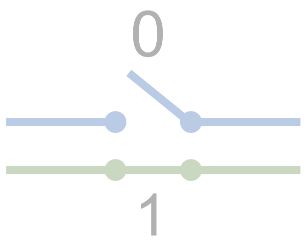
$$000 \rightarrow 0$$
$$001 \rightarrow 1$$
$$010 \rightarrow 2$$
$$011 \rightarrow 3$$

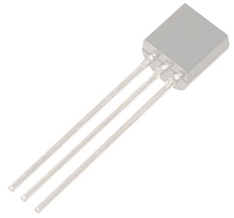
...

Qubits

Bit

logical





physical

notation

$$000 \rightarrow 0$$
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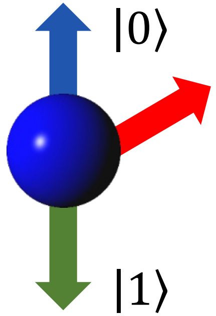
...

Qubit

logical



$^{171}Yb^{+1}$ γ e … physical

notation

$$|000\rangle \rightarrow |0\rangle$$
$$|001\rangle \rightarrow |1\rangle$$
$$|010\rangle \rightarrow |2\rangle$$
$$|011\rangle \rightarrow |3\rangle$$
…

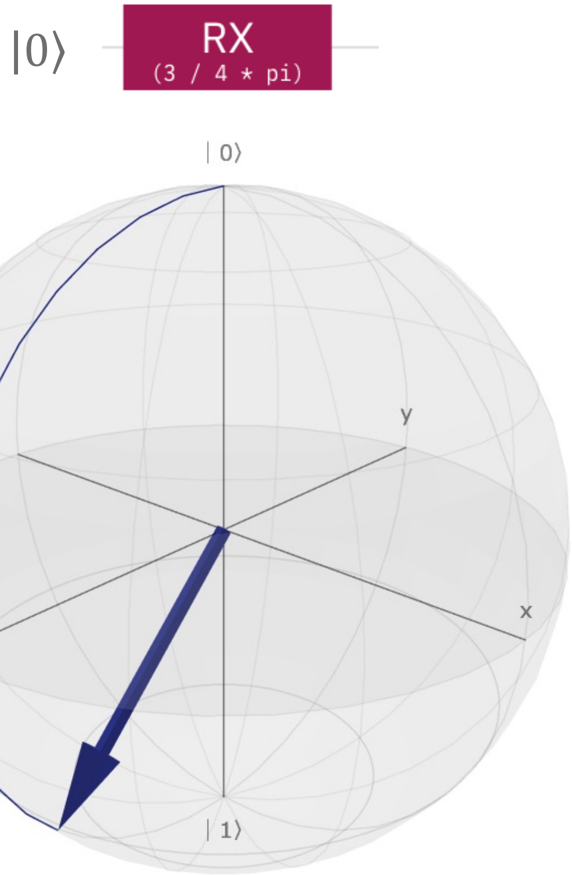
Qubits

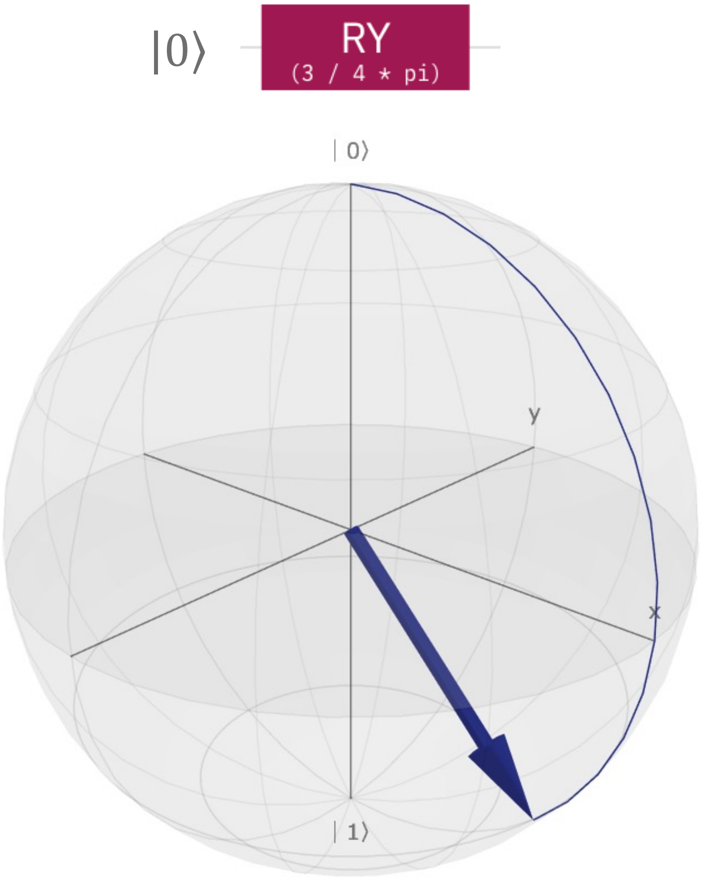
- Computational basis: $|0 \dots 0\rangle, \dots, |1 \dots 1\rangle \in H$

- General single qubit state: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$

- Measurement: $|\psi\rangle \rightarrow \begin{cases} 0 \text{ with } P(0) = |\alpha|^2 \\ 1 \text{ with } P(1) = |\beta|^2 \end{cases}$

Quantum Gates





Quantum Gates