Option Pricing With Stochastic Simulation From The Heston Model

OMIS6000 - Fall 2019

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Agenda

I. Business Background

II. Objectives

III. Simulation Setup

IV. Simulation Results

V. Managerial Implications

VI. Limitations & Improvements

I. Business Background

Airline needs to buy 2 Million Gallons of fuel in a year

We wants to protect against possible increase in fuel price

We would still like to realize a savings if the price drops

I. Business Background

II. Objectives – Buying a "*Long Call*"

Unprotected = (K - St) * 2,000,000

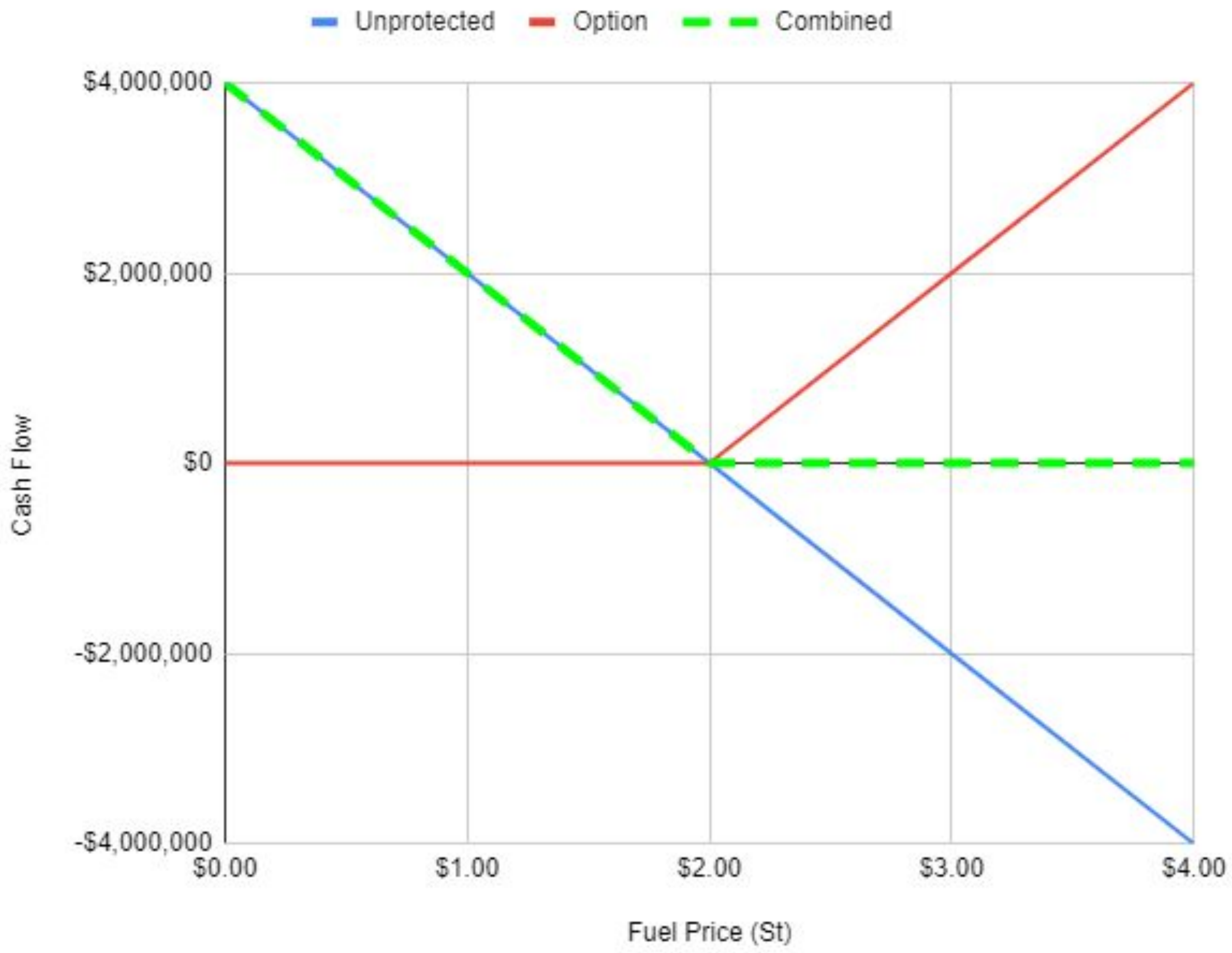
Option = max(St - K, 0) * 2,000,000

Combined = max(K - St, 0) * 2,000,000

This Option is called "Long Call"

Gallons of Fuel =	2,000,000
Strike Price (K) =	$2

Fuel Price (St)	Unprotected	Option	Combined
$0.00	$4,000,000	$0	$4,000,000
$1.00	$2,000,000	$0	$2,000,000
$2.00	$0	$0	$0
$3.00	-$2,000,000	$2,000,000	$0
$4.00	-$4,000,000	$4,000,000	$0



II. Objectives – How much to pay for option?

1. Estimate the expected value of the fuel price in 1 year, $E[S_T]$
2. Calculate the expected option payout, $P_T = E[max(S_T - K, 0)]$
3. Calculate present value, $P_0 = e^{-rT} P_T$
4. This is the fare option price, $c = P_0$

So all we need to do is predict the future fuel price!!

III. Simulation Setup - Heston Model

Heston Model [1]

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_S$$
$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t} dW_V$$

where:

$\sqrt{V_t} = \text{Volatility of } S_t$

$\sigma = \text{Volatility of } \sqrt{V_t}$

$\theta = \text{long-term price variance}$

$k = \text{rate of reversion to } \theta$

$dt = \text{Time step}$

$W_S = \text{Brownian Motion of } S_t$

$W_V = \text{Brownian Motion of } \sqrt{V_t}$

Correlation of Brownian Motion

$$dW_S = \rho dW_V + \sqrt{1-\rho^2} dW_i$$

where:

$W_i = \text{Independent Brownian Motion}$

$\rho = \text{Correlation Coefficient}$

Constants[2]

$r = 0.0319$
$K = S_0 = 2$
$T = 1.0$
$V_0 = 0.010201$
$\sigma = 0.61$
$\theta = 0.019$
$\kappa = 6.21$
$\rho \in [-0.5, -0.7, -0.9]$ with probabilities $[0.25, 0.5, 0.25]$

[1] Akhilesh Ganti. “Heston Model.” *Investopedia*, Nov. 12, 2019. https://www.investopedia.com/terms/h/heston-model.asp
[2] Mark Broadie and Ozgur Kaya. “Exact Simulation of Stochastic Volatility and other Affine Jump Diffusion Processes.” Operations Research, Vol. 54, No. 2, March-April, 2006, DOI: 10.1287/opre.1050.0247. http://www.columbia.edu/~mnb2/broadie/Assets/broadie_kaya_exact_sim_or_2006.pdf

III. Simulation Setup - Random Variables

dW_V & dW_i - Independent Brownian Motions

Normally distributed with a mean of zero and standard deviation of $\sqrt{dt}$

Each simulation from t=0 (now) to t=T (1 year from now) is divided into n time steps so $dt = T/n$

Therefore each simulation samples both dW_v and dW_i n times from a normal distribution

ϱ - Correlation Coefficient between dW_V & dW_S

Constant during each simulation but randomly assigned a value for each om the m simulations

Sampled from the values [-0.5, -0.7, -0.9] with the probabilities [0.25, 0.5, 0.25]

-0.7 is a standard correlation used in the Broadie paper [2]

[2] Mark Broadie and Ozgur Kaya. "Exact Simulation of Stochastic Volatility and other Affine Jump Diffusion Processes." *Operations Research*, Vol. 54, No. 2, March-April, 2006, DOI: 10.1287/opre.1050.0247. http://www.columbia.edu/~mnb2/broadie/Assets/broadie_kaya_exact_sim_or_2006.pdf

III. Simulation Setup - Process

1) Generate a single random value of ϱ and n independent random values of dW_v and dW_i
2) Calculate the n values of dW_S based on ϱ, dW_v and dW_i with $dW_S = \rho dW_V + \sqrt{1-\rho^2}dW_i$
3) For each of the n time steps, first calculate the changes in V and S at time t with the following equations

$$dV_t = k(\theta - V_{t-1})dt + \sigma\sqrt{V_{t-1}}dW_{Vt}$$
$$dS_t = rS_{t-1}dt + \sqrt{V_{t-1}}S_{t-1}dW_{St}$$

Note: S_0 = 2 and V_0 = 0.010201

4) Then add the deltas to the previous value to get the new value with the equations below

$$V_t = V_{t-1} + dV_t$$
$$S_t = S_{t-1} + dS_t$$

5) Repeat steps 1 to 4 m times to get m values of S_T, then take the mean to get the expected value $E[S_T]$

Note: Error $O(1/\sqrt{m})$and $O(1/n)$, therefore we set m = n^2 to minimize error for m*n samples

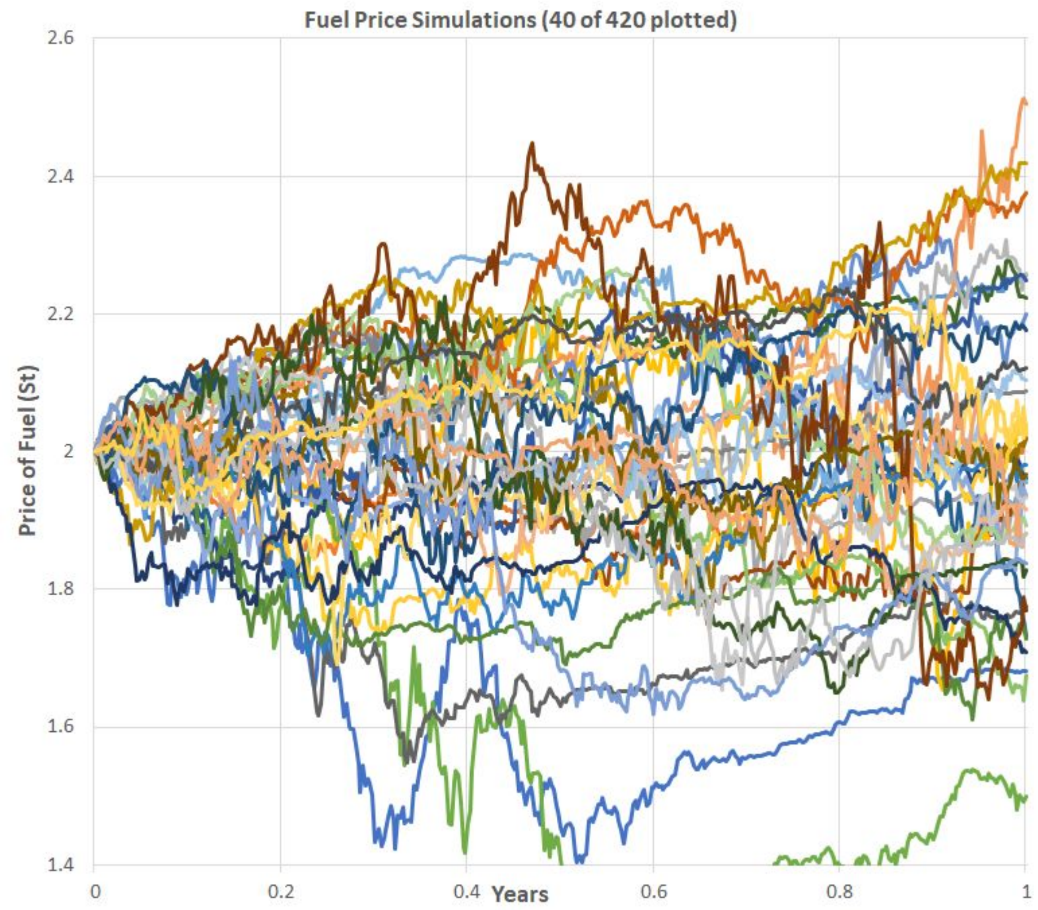
6) Calculate the fare option price from the following equation

$$c = e^{-rT}E[max(S_T - K, 0)]$$

IV. Simulation Results – Excel

Due to Excel limitations, we only ran 100 simulation, resulting in a standard error of \$0.0076

Several of these time evolutions are illustrated below, yielding an option price of \$0.1430 (\$0.1354 - \$0.1506)



R	0.0319			
K	2			
T	1			
V0	0.0102			
Sigma	0.61			
Theta	0.019			
Kappa	6.21			
Rho Dist.	-0.5	-0.7	-0.7	-0.9
n	400			
m	420			
dt	0.0025			
S0	2			

Results	
Mean Pt	**0.1477**
Std Pt	**0.1566**
Std Err	**0.0076**
Opt. Price	**0.1430**

Standard Error $= \sigma_s / \sqrt{m}$

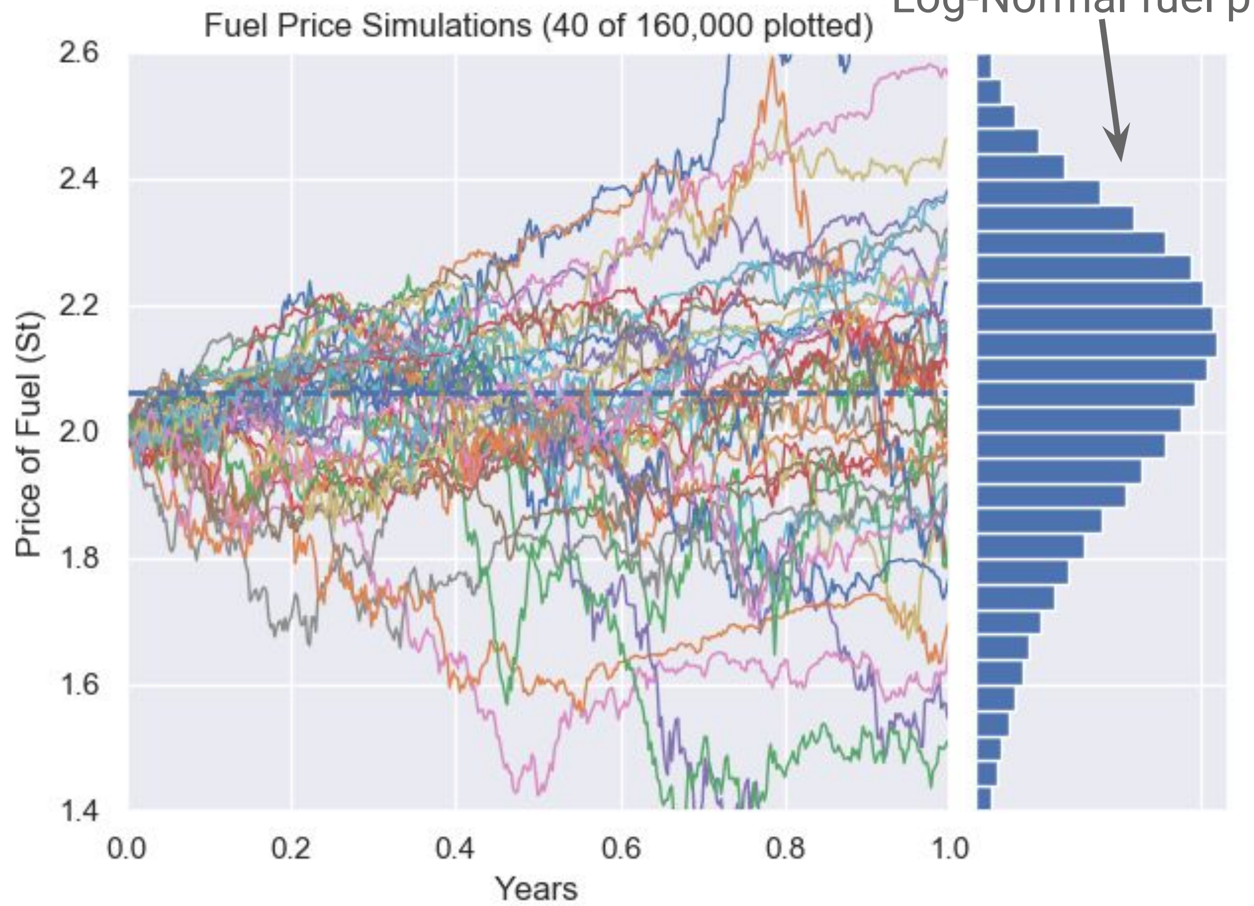
Cells with functions from Risk Solver are in yellow.

	Iteration	0	1	2	3	4	5	6	7	8	9
	Time	0	0.0025	0.005	0.0075	0.01	0.0125	0.015	0.0175	0.02	0.0225
rho	Nv		-0.0054	0.0133	-0.1111	0.0349	0.1143	-0.0260	-0.0131	-0.0902	0.0264
-0.5	Ni		-0.0350	-0.0360	-0.0340	0.0889	0.0129	0.0136	0.0425	-0.0112	-0.0235
	dWs		-0.0276	-0.0378	0.0261	0.0595	-0.0460	0.0247	0.0434	0.0354	-0.0336
Sim	dWv		-0.0054	0.0133	-0.1111	0.0349	0.1143	-0.0260	-0.0131	-0.0902	0.0264
1	dVt		-0.0002	0.0009	-0.0070	0.0016	0.0054	-0.0015	-0.0006	-0.0050	0.0012
	dSt		-0.0054	-0.0074	0.0056	0.0076	-0.0067	0.0053	0.0086	0.0068	-0.0040
Pt	Vt	0.0102	0.0100	0.0110	0.0040	0.0056	0.0110	0.0094	0.0088	0.0038	0.0050
0.166	St	2	1.9946	1.9872	1.9928	2.0004	1.9937	1.9991	2.0076	2.0145	2.0104
rho	Nv		-0.0061	-0.0020	-0.0771	0.0935	0.0364	0.0144	0.0332	0.0294	-0.0450
-0.7	Ni		0.0297	0.0450	0.0715	0.0493	0.0015	-0.0603	-0.0247	0.0359	-0.0393
	dWs		0.0255	0.0335	0.1050	-0.0302	-0.0244	-0.0531	-0.0409	0.0051	0.0034
Sim	dWv		-0.0061	-0.0020	-0.0771	0.0935	0.0364	0.0144	0.0332	0.0294	-0.0450
2	dVt		-0.0002	0.0000	-0.0046	0.0044	0.0023	0.0011	0.0024	0.0023	-0.0037
	dSt		0.0053	0.0069	0.0213	-0.0044	-0.0048	-0.0117	-0.0093	0.0014	0.0011
Pt	Vt	0.0102	0.0100	0.0100	0.0054	0.0098	0.0122	0.0133	0.0157	0.0180	0.0143
0	St	2	2.0053	2.0122	2.0335	2.0291	2.0243	2.0126	2.0033	2.0048	2.0058

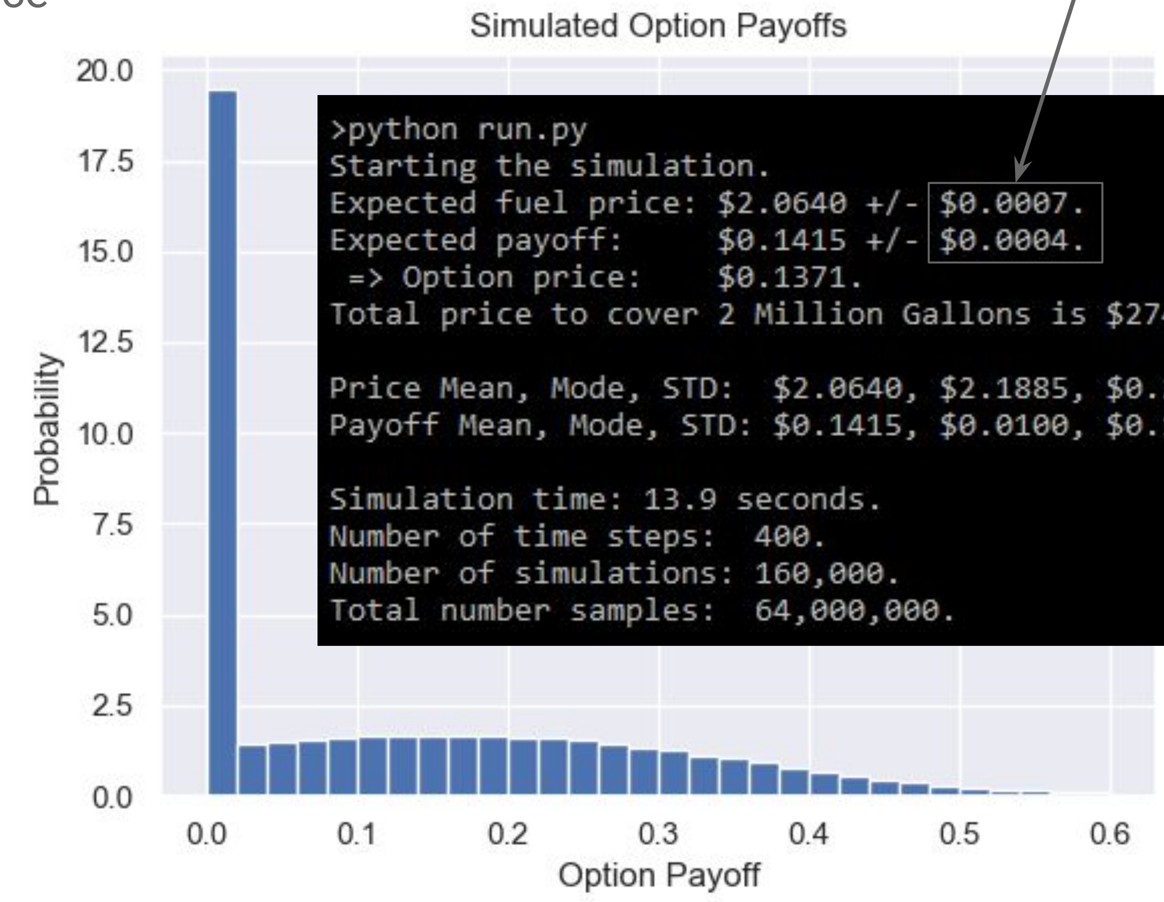
IV. Simulation Results – Python

Python Implementation: https://github.com/daniel-fudge/Heston-Option-Pricing

Option price of $0.1371 and standard error of $0.0004 (down from $0.0075 in Excel)



Standard Error $= \sigma_s / \sqrt{m}$

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>python run.py
Starting the simulation.
Expected fuel price: $2.0640 +/- $0.0007.
Expected payoff:     $0.1415 +/- $0.0004.
 => Option price:    $0.1371.
Total price to cover 2 Million Gallons is $274,201.

Price Mean, Mode, STD:  $2.0640, $2.1885, $0.2653.
Payoff Mean, Mode, STD: $0.1415, $0.0100, $0.1563.

Simulation time: 13.9 seconds.
Number of time steps:  400.
Number of simulations: 160,000.
Total number samples:  64,000,000.
```

V. Managerial Implications

Assuming the fuel price and strike price is $2 /Gallon, a fare price for the options is $0.1371 or $274,201 to protect 2 Million Gallons ($4 Million Dollars)

If these options are sold for less than $0.1371, they should buy

If not available on exchange, may be able to negotiate an over-the-counter deal

Or may be able to find another asset such as heating fuel that is strongly correlated with jet fuel

V. Managerial Implications

Unprotected = (K - St) * 2,000,000

Option-price = max(St - K, 0) * 2,000,000 - c

Combined = max(K - St, 0) * 2,000,000 - c

Profiting when St < K - c/2,000,000 = $1.8629

Protecting when St > K + c/2,000,000 = $2.1371

Gallons of Fuel =	2,000,000
Strike Price (K) =	$2
Option Price (c) =	$274,201

Fuel Price (St)	Unprotected	Option	Option-Price	Combined
$0.00	$4,000,000	$0	-$274,201	$3,725,799
$1.00	$2,000,000	$0	-$274,201	$1,725,799
$2.00	$0	$0	-$274,201	-$274,201
$3.00	-$2,000,000	$2,000,000	$1,725,799	-$274,201
$4.00	-$4,000,000	$4,000,000	$3,725,799	-$274,201

VI. Limitations and Improvements

The major limitation of this simulation was the selection of the constants

K, V_0, S_0, r and T can be determined easily

However σ, θ, κ and ϱ were taken from a paper by Broadie[2]

These should be estimated with more precision from the fuel price history and implied from the current prices of similar assets

Sensitivity studies could also be conducted by varying these parameters or replacing them with probability distributions again based on historical data

[2] Mark Broadie and Ozgur Kaya. “Exact Simulation of Stochastic Volatility and other Affine Jump Diffusion Processes.” *Operations Research*, Vol. 54, No. 2, March-April, 2006, DOI: 10.1287/opre.1050.0247. http://www.columbia.edu/~mnb2/broadie/Assets/broadie_kaya_exact_sim_or_2006.pdf

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where:

$\sqrt{V_t}$ = Volatility of S_t

σ = Volatility of $\sqrt{V_t}$

θ = long-term price variance

k = rate of reversion to θ

dt = Time step

W_S = Brownian Motion of S_t

$\boxed{W_V = \text{Brownian Motion of } \sqrt{V_t}}$

Correlation of Brownian Motion

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