Schulich School of Business

FINE 6800X W20

Options, Futures and Other Derivative Securities

Valuation Project

National Bank of Canada

Auto Callable Contingent Memory Income Note

March 17, 2020

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Abstract

This report summarizes the pricing and analysis of the National Bank of Canada Auto Callable Contingent Memory Income Note Security, which is dependent on the returns of the iShares® Core S&P 500 Index ETF (CAD-Hedged). The supporting analysis is captured in the attached "fine6800-pricing.xlsm" Excel file and the Python Jupyter Notebook, which can be viewed in the attached "fine6800-pricing.html" file.

The pricing was achieved through a Monte Carlo simulation containing 1 million evolutions of the ETF prices using the antithetic variable technique, which effectively increased the number of simulations to 2 million. The change in prices (S_{i-1} *to* S_i) between evaluation dates is governed by the equation below.

$$S_i = S_{i-1} \cdot \exp\left[\left(f_i - q - \frac{1}{2}\sigma_i^2\right)dt_i + \sigma_i\sqrt{dt_i} \cdot \epsilon\right]$$

With these possible price evolutions, the associated ETF returns, security payouts and a present value of the payouts for each evolution was calculated. The mean of these present values approximate the expected value of the security and hence its price.

The Monte Carlo process is fairly standard but the uncertainty of the pricing comes from the estimation of the risk-free rate, dividend yield and volatility. The risk-free rate was derived from a spot rate curve based on the Canadian T-bill and bond prices. For both the ETF dividend yield and volatility we calculated the historical values and the implied values from options on the ETF. The implied dividend yields were very erratic so we selected the 1.7% mean historical value. For the volatility we created a piecewise linear schedule that interpolated the short term implied values but regressed to the historical mean after 2 years. Two values of all of these parameters were estimated for the separate Monte Carlo simulations for the Feb 25 and March 3, 2020 evaluation dates.

Special care also had to be taken to ensure the rules associated with the security payout were properly encoded. As a verification step we tested the payout function against the example return profiles given by NBC and 3 other sample cases (best, normal, worst).

We checked our security price for Feb and March, 2020 against the prices provided by NBC. Our values of $99.81±$0.03 and $97.16±$0.03 were $0.19 and $0.30 below the quoted values of $100 and $97.46, however both models price the security ~$2.5 higher in February. This variation is expected considering the sensitivity to the volatility used in the model.

When compared against the underlying ETF, this security is more conservative but has a much more non-linear payout function. It is ideal for Canadian investors willing to take on risk of not recovering the entire principal for the chance of limited participation in US market gains while being shielded from USD-CAD exchange rate fluctuations.

Table of Contents

Introduction

This report summarizes the pricing and analysis of the National Bank of Canada (NBC) Auto Callable Contingent Memory Income Note Security, referred to as the "*security*" in this report. The supporting analysis is captured in the attached "*fine6800-pricing.xlsm*" Excel file and a Python Jupyter Notebook, which can be viewed at the following link or in the attached "*fine6800-pricing.html*" file.

https://github.com/daniel-fudge/Maturity-Monitored-Barrier-Pricing

Payoff discussion

The security is linked to the iShares® Core S&P 500 Index ETF (CAD-Hedged), which is referred to as the "*ETF*" in this report. In particular, the security is linked to the ETF returns as defined below.

$$r_i = \frac{P_i}{P_o} - 1$$

where:

$P_i =$ Closing ETF Price the on the ith valuation date

$P_o =$ Closing ETF Price on issuance date, Nov 25, 2020

Coupon and Call Valuation Dates - There are 14 valuation dates listed on the NBC site where the ETF return is calculated as defined above to determine if the autocallable feature described below is activated and also to determine the size of the payout for the given valuation date.

Coupon and Call Payment Dates - It should also be noted that the actual coupon payment and call payout dates are approximately a week after the associated valuation dates. Additionally the call threshold is not applied to the first and last dates.

Memory - The memory feature is the number of valuation dates between the current valuation date and the date that the last coupon payment was made.

Total Coupon Payments (c) - If the ETF return is greater than the -25% barrier on the valuation date a payment is made. This payment consists of 3 different payments; Coupon, Deferred Variable and Current Variable. If the memory is greater than zero, the deferred variable payment is made otherwise the current payment is made.

$$c = \begin{cases} c_{coupon} + c_{deferred}, \text{if } memory > 0 \\ c_{coupon} + c_{current}, \text{ if } memory <= 0 \end{cases}$$

Regular Compound Payment - The coupon payment includes the $2.875 payment times 1 plus the number of missed coupons since the last coupon payment (memory).

$$c_{coupon} = face \cdot C/2 \cdot (1 + memory) = \$2.875 \cdot (1 + memory)$$

Deferred Variable Payment - The deferred payment is the product of the $100 face value, 5% deferred payment participation factor and the amount the return is above the -25% barrier.

$$c_{deferred} = face \cdot deferred\_pf \cdot (r_i - barrier) = \$5 \cdot (r_i + 25\%)$$

Current Variable Payment - The current payment is the product of the \$100 face value, 1% current payment participation factor and amount the ETF return is above 0%.

$$c_{current} = face \cdot current\_pf \cdot max(0, r_i) = \$1 \cdot max(0, r_i)$$

Autocall Feature - If the ETF return on a valuation date is greater than the 10% call threshold, the security is automatically called, the maturity redemption payment is delivered and no more payments are made.

Variable Return - If the ETF return is greater than the variable return threshold of 0%, a variable return is added to the maturity redemption payment. However, this variable return value is a product of the 0% participation factor. Therefore the variable return is always zero and is eliminated from the maturity redemption payment discussion below for clarity.

Maturity Redemption Payment (p) - Since the variable return is always zero as discussed above, and the -25% barrier is below zero and the 10% call threshold, the maturity redemption payment is \$100 if the ETF return is above the -25% barrier. Otherwise it is \$100 times 1 plus the ETF return. There is also a clause that states that the maturity redemption payment must be at least 1% of the \$100 principal amount resulting in the max function below. Note that the ETF return is from the current valuation date if the autocall feature is activated or the final valuation date if it is not.

$$p = \begin{cases} \$100, & \text{if } r_f >= -25\% \\ \$100 \cdot max(1\%, 1 + r_f), & \text{if } r_f < -25\% \end{cases}$$

Attractive features - Clearly the memory feature is very attractive to the investor. It allows the missed coupons to be recovered when the ETF return rises above the barrier. The deferred payment is also very attractive as it partially compensates the investor for the interest lost on the missed coupons as well as the missed current variable return payments. Having the deferred variable return reference the -25% barrier is also very nice since even a 0% ETF return would result in a \$1.25 deferred payment. The current variable return allows the investor to participate in the positive returns of the ETF, however the 1% current participation factor reduces the payments considerably. For instance a 10% ETF return would only result in a \$0.1 or 0.1% of face value current payment.

Unattractive Features - The -25% barrier is a negative feature to investors and they would want it to be as low as possible. When the return is below this value, no coupon payment is made and if the final valuation return is below the barrier, the redemption payment is greatly reduced. For instance, if the return drops from -25% to -26%, the maturity payment drops from \$100 to \$74. The autocallable feature is also unattractive since it eliminates the possibility of collecting future coupon payments.

Payoff Examples - To illustrate the limits of the payout function we created the following 3 return profiles. The first is a terrible profile with the returns all under -25% so no coupons are paid and the final return is a 6-sigma loss of -61.81% under the risk-neutral measure ($\mu = r_{rf}$). The -61.81% assumes a dividend yield (q) of 1.7%, risk-free rate (r_{rf}) of 0.94%, volatility (σ) of 13.1% and the March 3, 2020 price evaluation. Therefore the only payment is \$38.19 at maturity with a present value of **\$35.78**.

The second profile is very good with all returns at 9.9%, just under the call threshold and the final return at the positive 6-sigma risk-neutral return of 108.8%. This would give 13 coupon payments of $2.97 and a final payment of $103.96, resulting in a present value of **$134.83**.

The third profile is "normal" with the returns growing linearly from 0 at the issue date to the deterministic risk-neutral value of -10.7% at the final valuation date. This results in 13 coupon payments of $2.88 and a final payment of $102.88 resulting in a present value of **$132.57**. These values will be referenced again when we review the results of the Monte-Carlo simulation.

Parameter Estimation: February 25 & March 3, 2020

To perform the Monte Carlo simulation we will need to estimate the volatility (σ) and dividend yield (q) of the ETF to calculate the possible price evolutions and the associated returns on the valuation dates. Since we will be performing this security pricing under the risk-free measure, we will also set the ETF expected return (μ) to the risk free rate ($\mu = r_{rf}$) during the price simulation and then use this r_{rf} distribution to calculate the present value of all of the resulting security payouts. These parameters are estimated twice; once with all the information available on February 25, 2020 and the second on March 3, 2020. For the remainder of the report these are simply referred to as "Feb" and "Mar" evaluations.

Volatility (σ)

Historical Volatility - To calculate the historical volatility we downloaded the daily adjusted closing ETF prices for the last seven years prior to both the evaluation dates. From this we calculate the daily log returns and then multiply the standard deviation of these returns over the two seven year periods by the square root of the number of trading days per year (252). This produces a historical volatility of **12.94%** and **13.14%** for the seven year period before February 25 and March 3, 2020. For fun the volatility from February 7 to March 6 was calculated to be 33.23%!

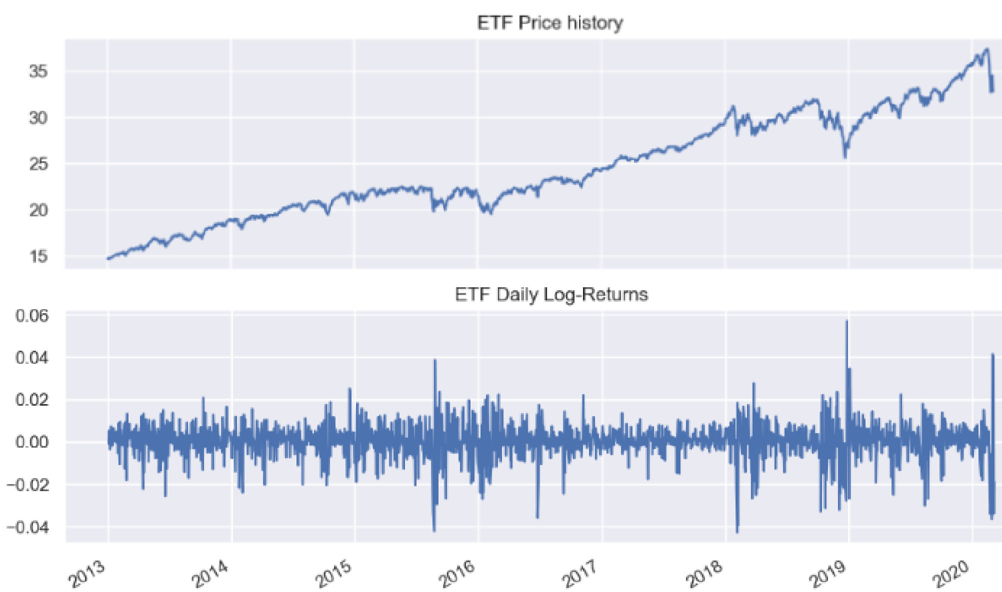

Figure 1. ETF price and return history used to calculate historical volatility.
Source: https://ca.finance.yahoo.com/quote/XSP.TO

Implied Volatility - We use the Black-Scholes-Merton model described below to find the implied volatility required to match the given call (*c*) or put (*p*) price, strike price (*K*), ETF price (*S*), dividend yield (*q*), option maturity (*T*), and risk-free rate (r_{rf}). There are three sources of uncertainty in this calculation; what values of *q* and r_{rf} to use and which option to reference.

$$c = S \cdot e^{-qT} \cdot N(d_1) - K \cdot e^{-r_{rf}T} \cdot N(d_2))$$

$$p = K \cdot e^{-r_{rf}T} \cdot N(-d_2)) - S \cdot e^{-qT} \cdot N(-d_1)$$

$$d_1 = \frac{ln(\frac{S}{K}) + (r_{rf} - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

When selecting which options to reference, we first eliminated all options with zero open interests. Next we eliminated options with strike prices more than $1 away from the ETF price. The selected continuously compounded risk-free rate was different for each option and evaluation date and was linearly interpolated from the yield curve discussed in the “Risk-Free Rate” section below. As will be discussed in the “Dividend Yield” section below, there is uncertainty concerning what dividend yield to use so we tried 1.70% and 2.05% to see the sensitivity as shown in the figure below.

The figure below illustrates the 58 separate implied volatilities that were calculated. The first thing to note was the impact of the dividend yield used in the calculation was very small relative to the total variation in the implied volatilities. There are also some extremely large volatility values in the short maturity options. This may be a function of the low open interest values but it may also reflect the period of truly extreme volatility. Even the historical volatility calculated in the period from Feb 7 through Mar 6 was 33.2%. Many of these options also have small open interests. The March 19, 2021 put had 243 & 259 open interest, which was more than twice any other option, had an implied volatility of **15.5%** and **19.4%** using the 1.7% dividend yield. It is interesting that none of the shorter maturity options have larger open interests.

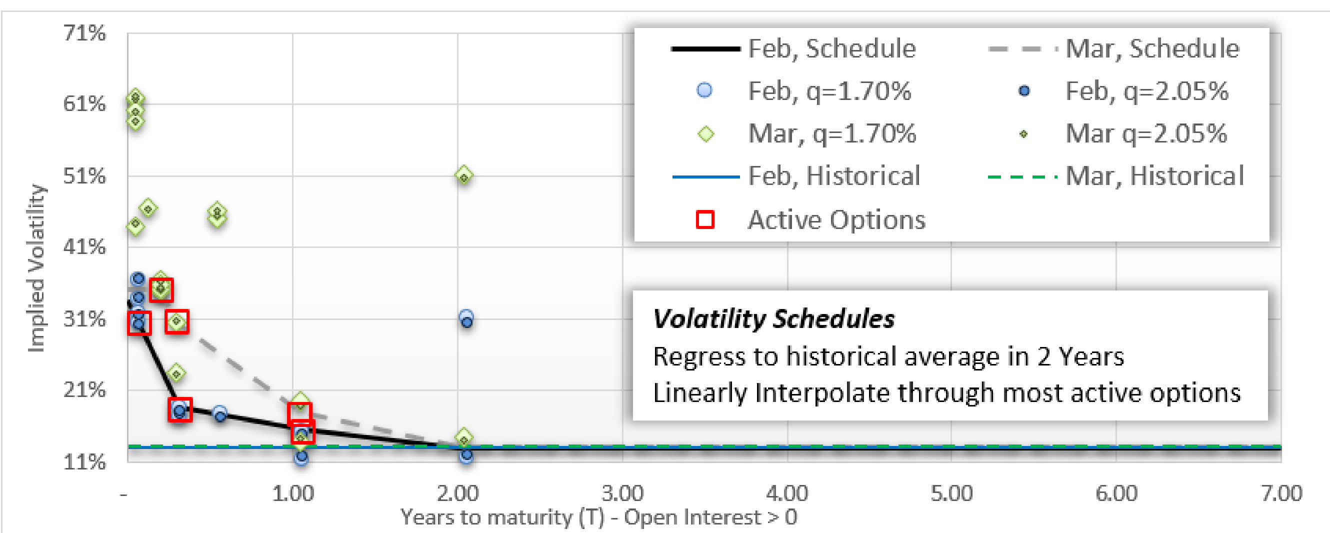

Figure 2. Volatility Schedule.

Selected Volatility - So the question is what volatility to use in the Monte Carlo simulation. Since this option has a 10% autocall feature we can't ignore the extreme volatility in the short term since it could end the security very quickly . The ETF return is also relative to the issue price so a large price drop at the start will impact all future returns. However it is also not realistic to assume this high volatility will persist. As a compromise, we have implemented the volatility schedule shown on the previous page for the Feb and March evaluations. We assume that in 2 years the ETF will regress to the historical mean after all of the market turmoil subsides. Before 2 years we linearly interpolate back through the implied volatilities of selected options. This data is very noisy at short maturities so some discretion was made in selecting options. For instance the ~60% volatilities were ignored since they are most likely caused by market panic and not a function of the underlying ETF, which we are modeling. When multiple active options have the same maturity, a weighted sum of the implied volatilities and open interest was calculated.

Dividend Yield (*q*)

Historical Dividend Yield - The historical annualized continuously compounded dividend yields (*q*) are easily calculated from the semi-annual dividend payments (*D*) and ETF price (*S*) in the equation below. The **2.05%** yield quoted in the security documentation is one of the highest yields over the last seven years and it isn't realistic to assume it would continue, especially considering the economic turmoil before both evaluation dates. Therefore using the **1.70%** historical mean would be more realistic.

$$q_i = 2 \cdot ln(1 + \frac{D_i}{S_i})$$

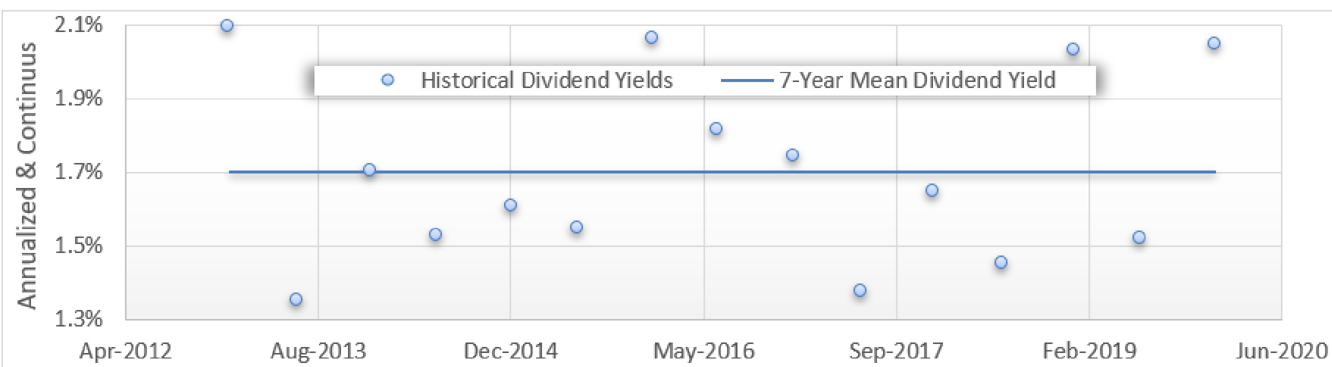

Figure 3. ETF historical dividend yields.

Implied Dividend Yield - Using the equation below we can calculate the implied dividend yield from the call (c) and put (p) prices on the ETF. Unfortunately these values do not appear realistic since they are outside the range of historical values and even include negative values. The most reasonable call-put pair with 0.2 years to maturity and 20+10 open interests gives an implied volatility of 2.94%. However, this is well above the max historical value and appears to be a function of market instability.

$$q_i = -\frac{1}{T_i} ln \frac{c_i - p_i + K_i e^{-r_i T_i}}{S_o}$$

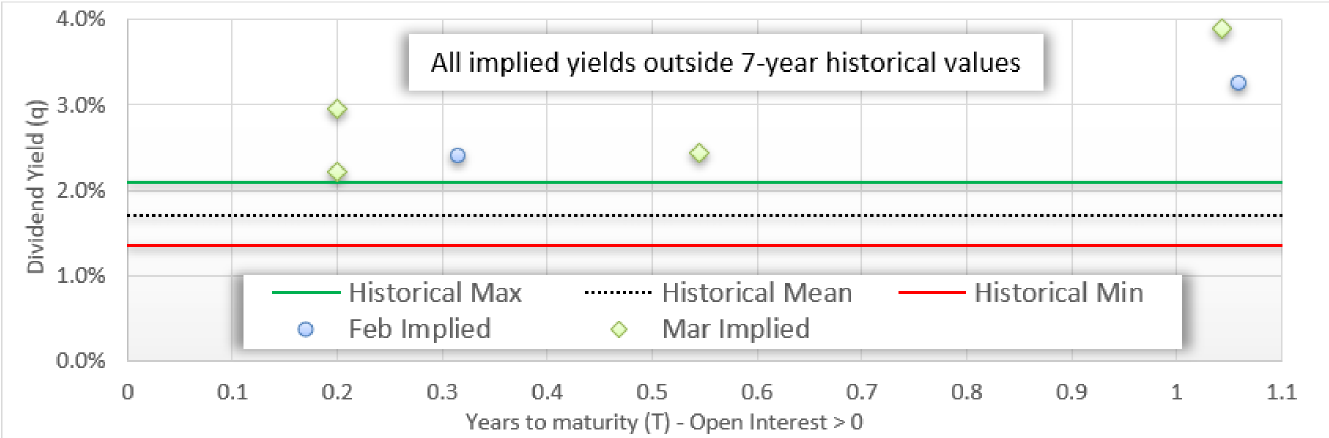

Figure 4. ETF implied dividend yields.

Risk-free Rate (r_{rf})

When performing the Monte Carlo simulation under the risk-neutral measure we use the risk-free rate to estimate the non-stochastic component of the price evolution as well as to discount the payout to the present value to determine the fair security price. Instead of selecting a single value for the next 7 years, we will use the Canadian T-bill and bond prices to construct the yield curve. By interpolating these values we can get the spot rates to use in the present value calculation. We can also calculate the forward rates between each of the valuation periods (spot rate for first valuation) when performing Monte Carlo time steps between valuation periods. The figure below illustrates the spot rates extracted from the Canadian T-bills and Bonds.

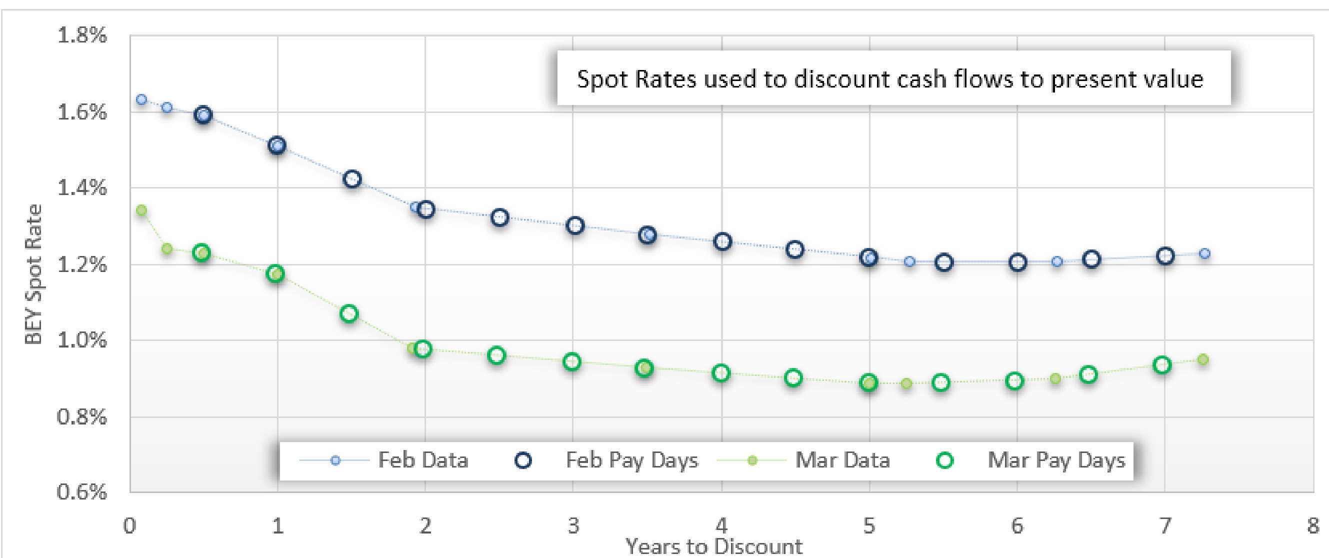

Figure 5. Spot rates used for discounting security payouts to either February 25 or March 3, 2020.
Source: https://www.marketwatch.com/investing/bond

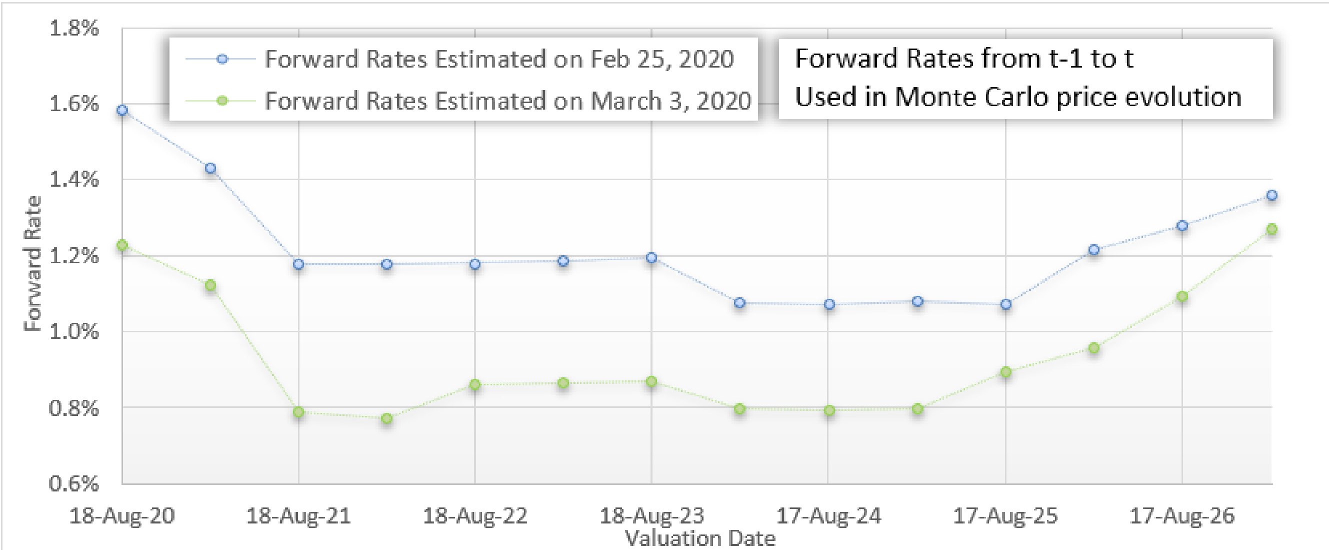

Figure 6. Continuously compounded forward rates between the previous and current valuation dates.

Monte Carlo

We will perform two separate Monte Carlo simulations starting at the evaluation dates of Feb 25 and March 3, 2020. Each simulation will include 1 million price evolutions and 14 time steps starting at the known price (S_o) at the evaluation date (t_o). We have also included the antithetic variable technique so when we draw the 14 independent random samples from the normal distribution we create the mirror sample by multiplying by -1 and then taking the average of the two resulting prices. This effectively makes 2 million price evolutions. Note that we do not need to increase the number of time steps since we do not need the price history between the 14 valuation dates and we are assuming the volatility and risk-free rate are constant between valuation dates. At each time step the current price (S_i) is determined from from the previous value (S_{i-1}) and the equation below.

$$S_i = S_{i-1} \cdot \exp\left[\left(f_i - q - \frac{1}{2}\sigma_i^2\right) dt_i + \sigma_i \sqrt{dt_i} \cdot \epsilon\right]$$

Note that f_i is the forward rate from time step *i*-1 to *i* calculated above, q is the constant dividend yield of 1.70%, σ_i is sampled from the piecewise linear distribution discussed previously, dt_i is the number of years between t_i and t_{i-1} and ε is a random sample from the standard normal distribution *N(0, 1)*.

The figures on the following pages illustrate these simulations. The histogram on the right side shows the log-normal distribution with a negative drift under the risk-neutral measure as expected since $f_i - q - ½\sigma_i^2$ is negative. The histogram on the bottom shows the present value distribution, which highlights several of the payout features discussed earlier.

It took 11.7 seconds to perform the 2 million simulation Monte Carlo on a 1.6 GHz, i5 laptop. The large number of simulations was used to reduce the standard error to $0.03 for price comparisons.

```
Expected Present Value = $99.81 ($9.51-$132.54), with a $0.026 standard error.
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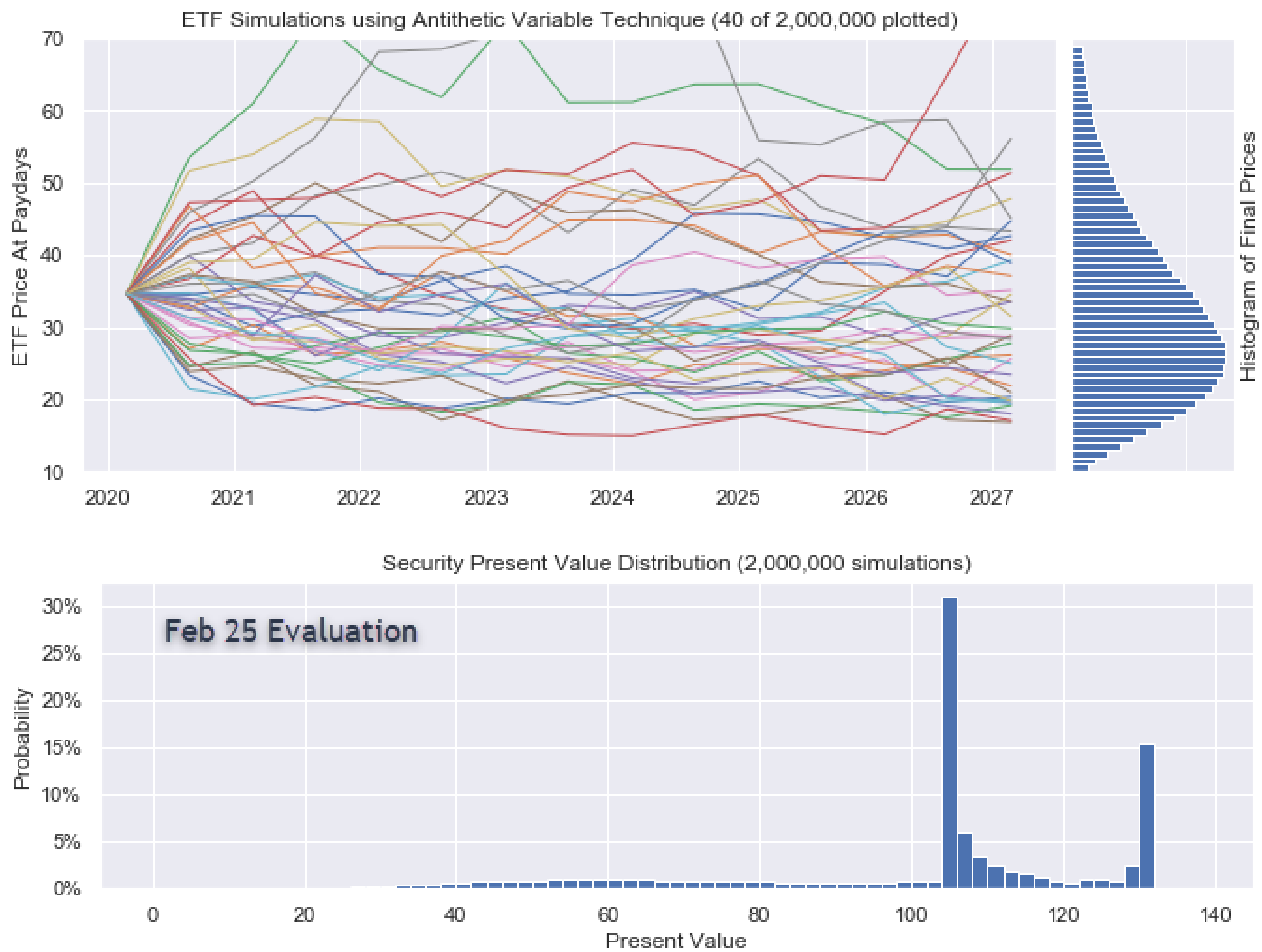

Figure 7. Monte Carlo results for February 25, 2020 evaluation.

The large \$104 to \$106 spike is a result of the Maturity Redemption Payment being \$100 as long as the final valuation return is above -25%. The autocallable feature can't be activated in the 1st valuation period so if you are going to hit the 10% limit, there is a good chance you will receive at least 2 coupon payments, resulting in a total payout greater than \$104. The 2nd spike above \$130 corresponds to the upper limit imposed by the 10% autocallable feature and was highlighted by our "normal" profile getting a present value of \$132.57 but our 6-sigma profile getting only \$2.26 more at \$134.83. Additionally our negative 6-sigma profile got a present value of \$35.78, which corresponds to the end of the left tail in the Feb histogram. This long negative tail and truncated upper tail is the reason why the mean expected value, which is our price, is less than the high-probability histogram bins.

The Monte Carlo was also repeated several times with different constant volatility values and dividend yields to assess the price sensitivity to each as shown in the figure on the following page. Clearly the dividend yield has a small effect when compared to the volatility.

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Expected Present Value = $97.16 ($6.04-$135.50), with a $0.028 standard error.
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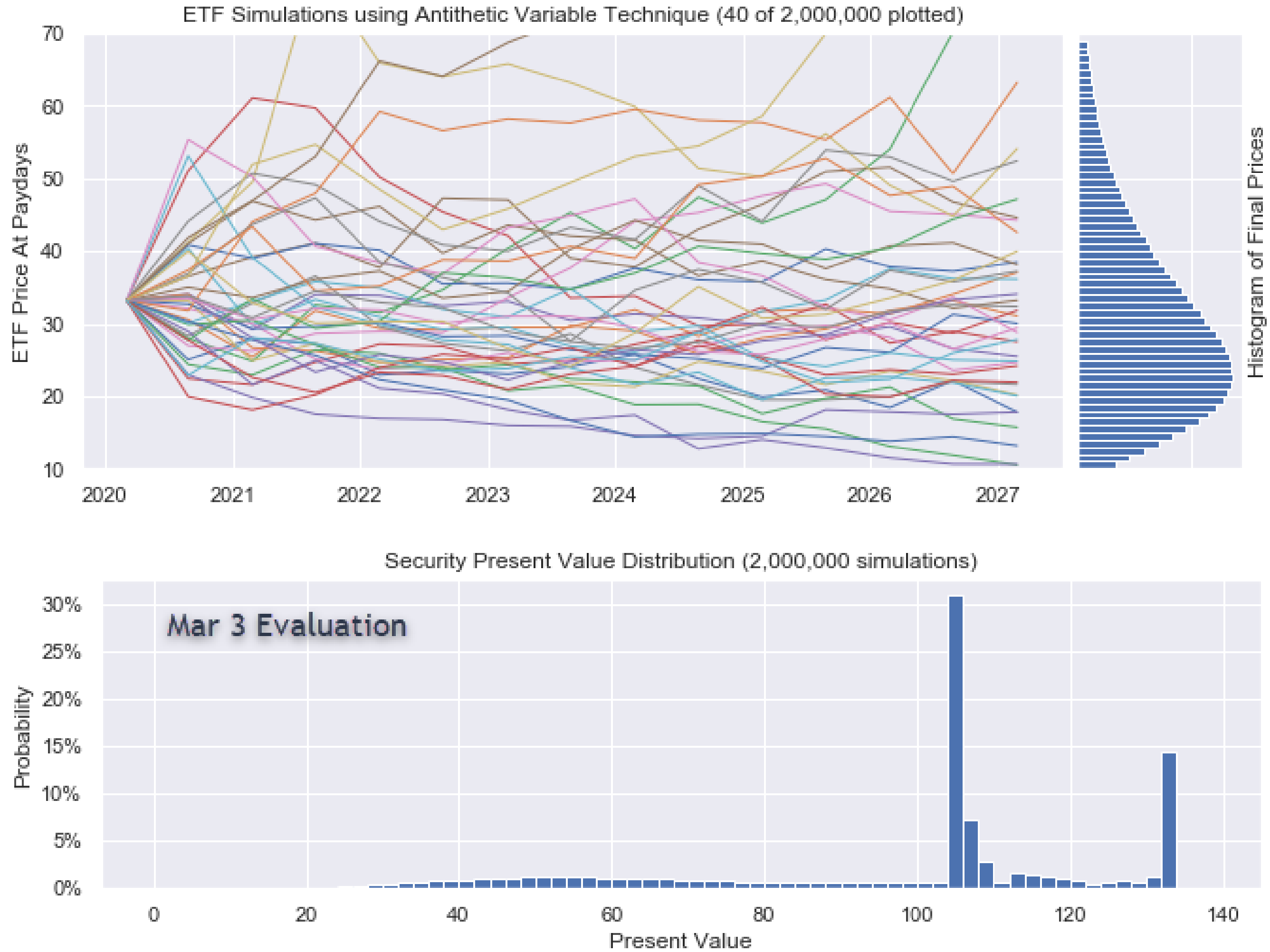

Figure 8. Monte Carlo results for March 3, 2020 evaluation.

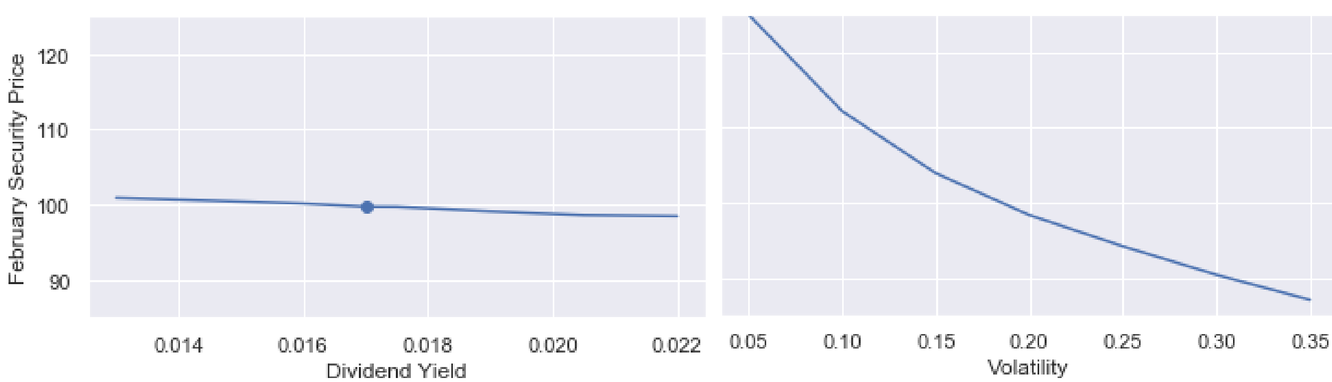

Figure 9. Security Price to Volatility and Dividend yield.

Price Comparison

NBC priced the security at \$100 and \$97.46 on Feb 25 and March 5, 2020 as compared to our values of \$99.81±\$0.03 and \$97.16±\$0.03. So we were \$0.19 and \$0.30 below the quoted values, however both models price the security ~2.6% higher in February. A quick look at figures 2 and 9 clearly indicate that any variation in how the volatility values were selected would greatly impact the final price. This was especially true in February and March when the market prices were erratic. Inspecting figure 9 indicates that they could have used a constant volatility value of ~17% in February but we don't think a constant value accurately captures the current market dynamics. They may have also used a constant value for the risk-free rate, which would impact the valuation. However, since we are modeling the security over 7 years, which is a fairly long period, incorporating the yield curve into the pricing seems to be a prudent measure. In addition, if we changed the Feb and March dividend yields from 1.7% to 1.635% and 1.587% we would get values of \$100.00.±\$0.03 and \$97.46±\$0.03, which match the NBC values perfectly. This all illustrates the extreme sensitivity to the selected parameter values.

Overall Assessment

The National Bank of Canada Auto Callable Contingent Memory Income Note provides an investor with exposure to the US equities market through the underlying iShares® Core S&P 500 Index ETF, which is hedged to the Canadian dollar. This is ideal for Canadian investors willing to take on risk of not recovering the entire principal for the chance of limited participation in US market gains while being shielded from the impact of USD-CAD exchange rate fluctuations.

The payout function of this security discussed earlier in this paper makes an interesting comparison between simply investing in the ETF. Figures 7 & 8 show a log-normal distribution for the ETF under the risk-neutral measure but definitely not a log-normal distribution for the security's present value. The \$2.875 coupon and the \$100 maturity payments protect the investor from ETF losses greater than the -25% barrier. However the 10% call threshold and 1% current payment participation factor limits the investor's gains if the ETF performs very well. This makes the security more conservative than investing directly in the ETF. Although the security contains attractive features like recouping missed coupons and compensation for lost interest, the -25% barrier exposes the investor to a large loss if the reference return is below this value. This is shown in the very long but flat negative tail in figures 7 & 8.

Another comparison would be to a portfolio of the ETF and a regular bond. This would simply be the bond amount plus the ETF return, which we could assume to be log-normal. This would be a smooth distribution unlike the security's nearly bi-modal distribution with a long negative tail.

Investors should be aware that there is a strong possibility that the projections contained in these calculations may not materialize mainly due to the uncertainty in the volatility. The current economic climate with extremely low interest rates, the coronavirus pandemic and crude oil price turmoil, has created investor panic and irrational trading resulting in extreme volatility. This makes activating the non-linear negative payout features of this security more probable. Finally, potential investors should be aware that the security is a price return, and does not take into account dividends and/or distributions. Although from our calculation, the dividend yield has a small effect when compared to the volatility.