Hybrid-dynamic Order-Sorted Algebra

1 Order-Sorted Algebra

The order-sorted formalism that we consider in this paper, hereafter abbreviated **OSA**, is a variation of the order-sorted algebra discussed in [5] that supports both relational and equational atoms – as opposed to only equational atoms.

Signatures The signatures are of the form $\Sigma = (S, \leqslant, F, P)$, where

a) $(S, \leqslant)$ is a preorder, i.e. a set equipped with a reflexive and transitive relation,

b) (S, F, P) is a many-sorted first-order signature.

Given a sort $s \in S$, we denote by $[s]$ the set of its connected components $[s] = \{s' \in S \mid s' \equiv_{\leqslant} s\}$, where $\equiv_{\leqslant}$ is the equivalence on S generated by the preorder $\leqslant$. The signature Σ is called *sensible* if
- for any operators $\sigma : \mathtt{ar} \to s, \sigma : \mathtt{ar}' \to s' \in F$ we have $\mathtt{ar} \equiv_{\leqslant} \mathtt{ar}'$ implies $s \equiv_{\leqslant} s'$.

The notion of sensible signature is a minimal syntactic requirement to avoid excessive ambiguity [5]. It is a much weaker requirement than preregularity [4].

Assumption 1. *For the sake of simplicity we assume that all connected components have a top sort. By a slightly abuse of notation we let $[s]$ to denote both the connected component of s and its top sort.*

Throughout this paper, we let Σ, Σ' and Σ_i to range over **OSA** signatures of the form $(S, \leqslant, F, P)$, $(S', \leqslant', F', P')$ and $(S_i, \leqslant_i, F_i, P_i)$, respectively.

Signature morphisms. Signature morphisms $\varphi : (S, \leqslant, F, P) \to (S', \leqslant', F', P')$ are first-order signature morphisms $\varphi : (S, F, P) \to (S', F', P')$ such that
a) $\varphi : (S, \leqslant) \to (S', \leqslant')$ is monotonic, and
b) it preserves the sub-sort overloading:
- for any operations $\sigma : \mathtt{ar}_1 \to s_1, \sigma : \mathtt{ar}_2 \to s_2 \in F$ such that $\mathtt{ar}_1 \equiv_{\leqslant} \mathtt{ar}_2$ we have $\varphi_{\mathtt{ar}_1,s_1}(\sigma) = \varphi_{\mathtt{ar}_2,s_2}(\sigma)$, this means that operations with the same name (and in the same connected component) are mapped to operations with the same name,
- for any relations $\pi : \mathtt{ar}_1$ and $\pi : \mathtt{ar}_2$ such that $\mathtt{ar}_1 \equiv_{\leqslant} \mathtt{ar}_2$ we have $\varphi_{\mathtt{ar}_1}(\pi) = \varphi_{\mathtt{ar}_2}(\pi)$, this means relations with the same name (and in the same connected component) are mapped to relations with the same name.

We let $\mathtt{Sig}^{\mathbf{OSA}}$ to denote the category of **OSA** signature morphisms.

Fact 1. *For any non-sensible signature there exists an isomorphic sensible signature obtained by adding to the name of each operation symbol the connected components of their arities and sorts: for example, an operation symbol $\sigma : \mathtt{ar} \to s$ will become $\sigma_{[\mathtt{ar}][s]} : \mathtt{ar} \to s$.*

Models Given a signature $\Sigma = (S, \leqslant, F, P)$, the Σ-models $\mathfrak{A}$ interpret

1. each sort $s \in S$ as a set $\mathfrak{A}_s$
2. each operation $(\sigma : \mathtt{ar} \to s) \in F$ as a function $\mathfrak{A}_{\sigma:\mathtt{ar}\to s} : \mathfrak{A}_{\mathtt{ar}} \to \mathfrak{A}_s$,
3. each relation symbol $(\pi : \mathtt{ar}) \in P$ as a relation $\mathfrak{A}_{\pi:\mathtt{ar}} \subseteq \mathfrak{A}_{\mathtt{ar}}$,

such that:

1. $\mathfrak{A}_s \subseteq \mathfrak{A}_{s'}$ whenever $s \leqslant s'$,
2. $\mathfrak{A}_{\sigma:\mathtt{ar}\to s}$ and $\mathfrak{A}_{\sigma:\mathtt{ar}'\to s'}$ agree on $\mathfrak{A}_{\mathtt{ar}} \cap \mathfrak{A}_{\mathtt{ar}'}$ for all operations $\sigma : \mathtt{ar} \to s$ and $\sigma : \mathtt{ar}' \to s'$ in F such that $[\mathtt{ar}] = [\mathtt{ar}']$, and
3. $\mathfrak{A}_{\pi:\mathtt{ar}}$ and $\mathfrak{A}_{\pi:\mathtt{ar}'}$ coincide on $\mathfrak{A}_{\mathtt{ar}} \cap \mathfrak{A}_{\mathtt{ar}'}$ for all relations $\pi : \mathtt{ar}$ and $\pi : \mathtt{ar}'$ in P such that $[\mathtt{ar}] = [\mathtt{ar}']$.

When there is no danger of confusion we may denote $\mathfrak{A}_{\sigma:\mathtt{ar}\to s}$ and $\mathfrak{A}_{\pi:\mathtt{ar}}$ simply by $\mathfrak{A}_\sigma$ and $\mathfrak{A}_\pi$, respectively. A Σ-homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is a monotonic S-sorted function $h = \{h_s : \mathfrak{A}_s \to \mathfrak{B}_s\}_{s\in S}$ such that

1. if $s \leqslant s'$ then h_s and $h_{s'}$ agree on $\mathfrak{A}_s$,
2. $\mathfrak{A}_\sigma ; h_s = h_{\mathtt{ar}} ; \mathfrak{B}_\sigma$ for all $\sigma \colon \mathtt{ar} \to s \in F$, and
3. $h_{\mathtt{ar}}(\mathfrak{A}_\pi) \subseteq \mathfrak{B}_\pi$ for all $\pi \colon \mathtt{ar} \in P$.

We let $\mathtt{Mod}^{\mathbf{OSA}}(\Sigma)$ to denote the category of Σ-models.

Terms The terms over a signature $\Sigma = (S, \leqslant, F, P)$ are defined inductively, as usual in the order-sorted-algebra literature. For every sort $s \in S$, the set $T_{\Sigma,s}$ of Σ-terms is the least set such that:

- $\sigma(t) \in T_{\Sigma,s}$ for all $\sigma \colon \mathtt{ar} \to s$ in F and $t \in T_{\Sigma,\mathtt{ar}}$;
- $T_{\Sigma,s_0} \subseteq T_{\Sigma,s}$ whenever $s_0 \leqslant s$.

We denote by $T_{\Sigma,[s]}$ the set of all terms whose sort is in the same connected component as s; that is, $T_{\Sigma,[s]} = \bigcup\{T_{\Sigma,s_0} \mid s_0 \in S \text{ and } [s_0] = [s]\}$.

Sentences Let $\Sigma = (S, \leqslant, F, P)$ be a signature. There are three types of atomic sentences:

1. *equations* $t = t'$, where $t, t' \in T_{\Sigma,[s]}$ and $s \in S$,
2. *relations* $\pi(t)$, where $\pi : \mathtt{ar} \in P$ and $t \in T_{\Sigma,\mathtt{ar}}$.

The set of Σ-sentences $\mathtt{Sen}^{\mathbf{OSA}}(\Sigma)$ are constructed from the above atomic sentences by applying Boolean connectives and quantification over finite sets of variables. In order to avoid clashes of variables with constants from the target signatures when translating quantified sentences along signature morphisms we define variables for a signature Σ as triples (v, s, Σ), where (a) v is the name of the variable, (b) s is the sort of the variable, and (c) Σ is the signature for which it was defined. Then we define quantified Σ-sentences as triples $\mathcal{Q}X \cdot \gamma$, where $\mathcal{Q}$ is a quantifier ($\forall$ or $\exists$), X is a set of variables over Σ, and γ is a sentence over $\Sigma[X]$ — the signature obtained from Σ by adding the variables in X as constants to Σ.

If $\varphi \colon \Sigma \to \Sigma'$ and $\forall X \cdot \gamma \in \mathtt{Sen}^{\mathbf{OSA}}(\Sigma)$ then $\varphi(\forall X \cdot \gamma) = \forall X \cdot \varphi'(\gamma)$, where $X' = \{(v, \varphi(s), \Sigma') \mid (v, s, \Sigma) \in X\}$ and $\varphi' : \Sigma[X] \to \Sigma[X']$ maps every symbol in Σ as φ and any variable $(v, s, \Sigma) \in X$ to $(v, \varphi(s), \Sigma') \in X'$. When there is no danger of confusion, we identify each variable by its name. For more details about this topic, one may look into the quantification spaces defined in [2].

Satisfaction relation. Given a signature $\Sigma = (S, \leqslant, F, P)$ and a Σ-model $\mathfrak{A}$, the satisfaction of atomic sentences is based on the interpretation of terms:

1. $\mathfrak{A} \models_\Sigma t = t'$ iff $\mathfrak{A}_t = \mathfrak{A}_{t'}$,
2. $\mathfrak{A} \models \pi(t)$ iff $\mathfrak{A}_t \in \mathfrak{A}_\pi$.

The satisfaction of sentences obtained by applying Boolean connectives and quantification is defined in the standard way.

1.1 Congruences and quotients

Definition 2 (Order-sorted relation)**.** *Let $\Sigma = (S, \leqslant, F, P)$ be an* **OSA***-signature, and $\mathfrak{A}$ a Σ-model. An $(S, \leqslant)$-relation on $\mathfrak{A}$ is an S-sorted relation $\sim$ on $\mathfrak{A}$ such that $\sim_s$ and $\sim_{s_0}$ coincide on $\mathfrak{A}_s \cap \mathfrak{A}_{s_0}$ for all sorts $s, s_0 \in S$ such that $[s] = [s_0]$.*

Definition 3 (Order-sorted congruence)**.** *An order-sorted Σ-congruence $\equiv$ on a Σ-model $\mathfrak{A}$ is an $(S, \leqslant)$-relation on $\mathfrak{A}$ such that*

- *for all function symbols $\sigma \colon \mathtt{ar}_1 \to s_1$ and $\sigma \colon \mathtt{ar}_2 \to s_2$ with $[\mathtt{ar}_1] = [\mathtt{ar}_2]$, and elements $a_1, a_2 \in \mathfrak{A}_{\mathtt{ar}_1} \cap \mathfrak{A}_{\mathtt{ar}_2}$, if $a_1 \equiv_{\mathtt{ar}_i} a_2$ then $\mathfrak{A}_\sigma(a_1) \equiv_{s_i} \mathfrak{A}_\sigma(a_2)$;*
- *for all relation symbols $\pi \colon \mathtt{ar}_1$ and $\pi \colon \mathtt{ar}_2$ with $[\mathtt{ar}_1] = [\mathtt{ar}_2]$, and elements $a_1, a_2 \in \mathfrak{A}_{\mathtt{ar}_1} \cap \mathfrak{A}_{\mathtt{ar}_2}$, if $a_1 \equiv_{\mathtt{ar}_i} a_2$ and $a_1 \in \mathfrak{A}_\pi$ then $a_2 \in \mathfrak{A}_\pi$;*

Proposition 4 (Quotient)**.** *Every* **OSA***-congruence $\equiv$ on a model $\mathfrak{A}$ determines a* quotient model *$\widehat{\mathfrak{A}}$, also denoted $\mathfrak{A}/_\equiv$, as follows:*

- *for every sort $s \in S$, $\widehat{\mathfrak{A}}_s = \{\widehat{a} \in \widehat{\mathfrak{A}}_{[s]} \mid a \in \mathfrak{A}_s\}$, where $\widehat{\mathfrak{A}}_{[s]}$ is the quotient of the set $\mathfrak{A}_{[s]}$ determined by $\equiv_{[s]}$;*

- *for every function symbol* $\sigma\colon \mathtt{ar} \to s \in F$ *and* $a \in \mathfrak{A}_{\mathtt{ar}}$*,* $\widehat{\mathfrak{A}}_\sigma(\widehat{a}) = \widehat{\mathfrak{A}_\sigma(a)}$*;*
- *for every relation* $\pi\colon \mathtt{ar} \in P$*,* $\widehat{a} \in \widehat{\mathfrak{A}}_\pi$ *iff* $a \in \mathfrak{A}_\pi$*.*

2 Hybrid-Dynamic Order-Sorted Algebra

Hybrid-Dynamic Order-Sorted Algebra with rigid symbols (**HDOSA**) is defined based on the ideas used to define rigid first-order hybrid logic [1] and first-order hybrid logic with user-defined sharing [3, 2]. The signatures contain a distinguished subset of "rigid" symbols such as sorts, operation and relation symbols which have the same interpretation in each world. This makes it possible to use a "rigid" semantics for the quantification, where the variables are interpreted uniformly across the worlds.

Signatures *The signatures* are of the form $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$ where:

1. $\Sigma^{\mathtt{n}} = (S^{\mathtt{n}}, \leqslant^{\mathtt{n}}, F^{\mathtt{n}}, P^{\mathtt{n}})$ is an **OSA** signature of nominals with a single connected component,
2. $\Sigma^{\mathtt{r}} = (S^{\mathtt{r}}, \leqslant, F^{\mathtt{r}}, P^{\mathtt{r}})$ is an **OSA** signature of rigid symbols, and
3. $\Sigma = (S, \leqslant, F, P)$ is an **OSA** signature.

We let $S^{\mathtt{f}} = S \backslash S^{\mathtt{r}}$ be set of flexible sorts, and $F^{\mathtt{f}} = F \backslash F^{\mathtt{r}}$ and $P^{\mathtt{f}} = P \backslash P^{\mathtt{r}}$ be the subsets of F and P that consist of flexible symbols (obtained by removing rigid symbols). Throughout this paper we let Δ, Δ' and Δ_i range over signatures of the form $(\Sigma^{\mathtt{n}}, \Sigma^{\mathtt{r}} \subseteq \Sigma)$, $(\Sigma'^{\mathtt{n}}, \Sigma'^{\mathtt{r}} \subseteq \Sigma')$ and $(\Sigma_i^{\mathtt{n}}, \Sigma_i^{\mathtt{r}} \subseteq \Sigma_i)$, respectively.

Signature morphisms. The signature morphisms $\varphi : \Delta \to \Delta'$ are pairs $\varphi = (\varphi^n, \varphi)$, where:
a) $\varphi^{\mathtt{n}} : \Sigma^{\mathtt{n}} \to \Sigma'^{\mathtt{n}}$ is an **OSA**-signature morphism mapping the top sorts of Σ^n to the top sorts of $\Sigma'^{\mathtt{n}}$, and
b) $\varphi : \Sigma \to \Sigma'$ is an **OSA**-signature morphism such that $\varphi(\Sigma^{\mathtt{r}}) \subseteq \Sigma'^{\mathtt{r}}$.

Kripke structures The Δ-models are pairs (W, M), where

1. $W \in |\mathtt{Mod}^{\mathbf{OSA}}(\Sigma^{\mathtt{n}})|$, and
2. $M\colon W_{[s_1]} \times W_{[s_n]} \to |\mathtt{Mod}^{\mathbf{OSA}}(\Sigma)|$ is a mapping such that
   (a) $[s_1], \ldots, [s_n]$ are all the top sorts of $\Sigma^{\mathtt{n}}$,
   (b) the rigid symbols have the same interpretations across the worlds, i.e. $M_{w_1}\upharpoonright_{\Sigma^{\mathtt{r}}}$ and $M_{w_2}\upharpoonright_{\Sigma^{\mathtt{r}}}$ are equal as order-sorted models, for all worlds $w_1, w_2 \in |W|$; this means that M_{w_1} and M_{w_2} have the same carrier sets for the rigid sorts, and the same interpretations of the rigid operation or predicate symbols.

Rigid terms Let Δ be a **HDOSA** signature such that $[s_1], \ldots, [s_n]$ are all the top nominal sorts. The *rigidification* of Σ with respect to $\Sigma^{\mathtt{n}}$ is the signature $\Sigma_@ = (S_@, \leqslant_@, F_@, P_@)$, where

1. $S_@ = \{@_{\overline{k}} s \mid \overline{k} \in T_{\Sigma^{\mathtt{n}},[s_1]} \times \cdots \times T_{\Sigma^{\mathtt{n}},[s_n]} \text{ and } s \in S\}$,
2. $\leqslant_@ = \{(@_{\overline{k}}\, s_1, @_{\overline{k}}\, s_2) \mid \overline{k} \in T_{\Sigma^{\mathtt{n}},[s_1]} \times \cdots \times T_{\Sigma^{\mathtt{n}},[s_n]} \text{ and } s_1 \leqslant s_2\}$,
3. $F_@ = \{@_{\overline{k}}\sigma \colon @_{\overline{k}}\mathtt{ar} \to @_{\overline{k}} s \mid \overline{k} \in T_{\Sigma^{\mathtt{n}},[s_1]} \times \cdots \times T_{\Sigma^{\mathtt{n}},[s_n]} \text{ and } \sigma \colon \mathtt{ar} \to s \in F\}$, [1]
4. $P_@ = \{@_{\overline{k}}\pi \colon @_{\overline{k}}\mathtt{ar} \mid \overline{k} \in T_{\Sigma^{\mathtt{n}},[s_1]} \times \cdots \times T_{\Sigma^{\mathtt{n}},[s_n]} \text{ and } \pi \colon \mathtt{ar} \in P\}$.

Since the rigid symbols have the same interpretation across the worlds, we further define $@_{\overline{k}} x = x$ for all tuples of nominals $\overline{k} \in T_{\Sigma^{\mathtt{n}},[s_1]} \times \cdots \times T_{\Sigma^{\mathtt{n}},[s_n]}$ and all symbols x in $\Sigma^{\mathtt{r}}$. The set of *rigid Δ-terms* is $T_{\Sigma_@}$

Sentences Given a signature Δ, the proper atomic Δ-sentences consist of

1. rigid equations $t =_s t'$, where $t, t' \in T_{\Sigma_@,[s]}$, and $[s] \in [S_@]$, and
2. rigid relations $\pi(t)$, where $\pi \colon \mathtt{ar} \in P_@$ and $t \in T_{\Sigma_@,\mathtt{ar}}$.

The set of Δ-sentences is given by the following grammar:

$$e ::= t_1 = t_2 \mid \pi(t) \mid k_1 = k_2 \mid \mathfrak{a}(k_1, k_2) \mid \neg e \mid \vee E \mid @_k\, e \mid \langle \mathfrak{a} \rangle e \mid {\downarrow} z \cdot e' \mid \exists X, Y \cdot e''$$

where (a) $t_1 = t_2$ is a rigid equation, (b) $\pi(t)$ is a rigid relation, (c) $k_1 = k_2$ is a nominal equation, (d) $\mathfrak{a}(k_1, k_2)$ is a action relation, (e) E is a finite set of Δ-sentences, (f) z is a nominal variable and e' is a $\Delta[z]$-sentence, (g) X is a finite set of nominal variables, Y is a finite set of variables of rigid sorts, and e'' is a $\Delta[X, Y]$-sentence.

References

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