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Zeroth-Order Logic

We can classify sentences broadly based on their *functional role* in language. Since, in the practice of mathematics, we are interested in *describing the properties of abstract objects,* the kinds of sentences that we want to communicate are primarily *declarative.* Naturally, we can think to classify the declarative sentences based on the answer to the fundamental question: *"is this description true?"*

1.1 Truth Values

Let's consider the following declarative sentence.

$$\text{" Ahab is a captain."} \tag{1.1}$$

This is a descriptive sentence about the term *Ahab*—a man, and thus the object of our discourse—asserting that he *is a captain.* In the context of Herman Melville's *Moby Dick,* this is an accurate description. We would therefore say that sentence (1.1) is *true*. We introduce the symbol $\top$ to denote these kinds of sentences.

true

$\top$

The $\top$ symbol is sometimes called *"top."*

$$\text{" Ishmael is a whale."} \tag{1.2}$$

The above sentence, in contrast, immediately furrows the brow of the reasonable man and strikes at the heart of our collective conscience. As we all know from the story, Ishmael is a sailor, an thus human, and therefore *not* a whale! This sentence is *not* accurately describing reality as we understand it. We will refer to statements like sentence (1.2) as *false*, and sentences of this kind will be referred to with the $\perp$ symbol.

false

$\perp$

The $\perp$ symbol is sometimes called *"bot."*

The attributes *true* and *false* that we are attaching to these sentences are called *truth values*, and they are the essential component that distinguishes the kinds of sentences we want to express from those that we do not. Sentences that are *true* all seem to exhibit a quality that makes them similar to each other but dissimilar to *false* sentences, regardless of what the actual sentences themselves *mean* semanti-

truth value

cally. What we've just done is *abstract* the fundamental concept of *"truth"* from descriptive sentences by making the observation that *all true sentences are essentially the same as each other from the perspective of truth values.* This inspires our first (informal) definition.

Definition 1.1: Propositional Equivalence.
We say that two declarative sentences φ and ψ are *equivalent* when they have the same truth value. We denote this by writing $\varphi \equiv \psi$.

$\equiv$

This describes a *weaker* version of equality: sentences that are equivalent are not actually identical to each other, but they share all of the characteristics[1] that we care about. As such, we should expect this new $\equiv$ symbol to have a lot of the same properties that equality has.

[1]… their *truth values* in this case…

Axiom: Propositional Equivalence is an Equivalence Relation.
We will assume that the following three properties hold for any sentences φ, ψ, and ξ that are consistent carriers of truth values.

1. $\varphi \equiv \varphi$. *reflexivity*
2. If $\varphi \equiv \psi$, then $\psi \equiv \varphi$. *symmetry*
3. If $\varphi \equiv \psi$ and $\psi \equiv \xi$, then $\varphi \equiv \xi$. *transitivity*

These three rules establish $\equiv$ as an example of an *equivalence relation.*

With this new conception of equivalence between sentences that carry truth values, we can formalize our prior observations. If we give the names $\sigma_{(1.1)}$ and $\sigma_{(1.2)}$ to sentence (1.1) and sentence (1.2) respectively, then we can now express $\sigma_{(1.1)} \equiv \top$ and $\sigma_{(1.2)} \equiv \bot$. We can also say $\sigma_{(1.1)} \not\equiv \sigma_{(1.2)}$ since we intuitively understand $\top \not\equiv \bot$.

Now, let's ponder the following sentence.

$$\text{"Colorless green ideas sleep furiously."} \tag{1.3}$$

Like the previous examples, this is a grammatically correct, declarative, English sentence… but what does it *mean?* What *is* a "colorless green idea"? What does it *mean* to "sleep furiously"? Taking the standard definitions[2] of each of the words in sentence (1.3), it does not seem to make any sense. We do not have a clear *definition of the object* that sentence is describing, and we also do not *understand the description* being made. Even though this sentence is grammatically correct, it conveys the same meaning as a grammatically malformed sentence like "Sleep color green." If we were to treat it like a normal, sensible sentence, we could form the following sentence.

[2]*e.g.,* using those found in the Oxford English Dictionary

$$\text{"Colorless green ideas } \textit{do not} \text{ sleep furiously."} \tag{1.4}$$

Based on the way English grammar works, we would expect sentence (1.4) to say the *opposite* of what sentence (1.3) says. But since sentence (1.3) did not seem to say anything sensible, we can not say that sentence (1.4) does either! We call such expressions *nonsensical* because they *carry no semantic meaning.*

non-sense

Let's now analyze the following statement, which we will call $\sigma_{(1.5)}$.

$$\text{“This sentence is } \textit{false}\text{.”} \tag{1.5}$$

Expressed a little more formally, this is a sentence named $\sigma_{(1.5)}$ that says $\sigma_{(1.5)} \equiv \perp$. This sentence certainly does not seem nonsensical; it seems to be clearly expressing a well-understood property[3] about a well-defined object.[4] Okay, so what is its truth value? Since we can not go out and physically *grab* this sentence and inspect it with our hands and eyes, we can instead try reasoning about it *hypothetically.*

[3] namely: falsity

[4] namely: the sentence $\sigma_{(1.5)}$

First, let's suppose

Axiom: Principle of Bivalence.
Well-formed sentences that purport to express consistent truth values should be exclusively either *true* or *false.*

1.2 *Logical Connectives*

Negations

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem.

Conjunctions and Disjunctions

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem.

Conditional Statements

1.3 *A Modest Proposal*

Equivalence

Axioms

Rules of Inference

Classical Syllogisms