Discrete Mathematics

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Spring of 2023

Chapter 6

Cardinality

6.1 Functions

Definition 6.1 (Injectivity)**.**
We say that a function $f : X \to Y$ is an *injection* $:\Leftrightarrow$ either of the following two statements holds:

I. $(\forall x_1 \in X)(\forall x_2 \in X)\big(x_1 \neq x_2 \;\Rightarrow\; f(x_1) \neq f(x_2)\big)$

II. $(\forall x_1 \in X)(\forall x_2 \in X)\big(f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2\big)$

Notice that these two statements are equivalent since the leading quantifiers are identical and the unquantified implications are contrapositives of each other, and we know from the propositional logic that $(p \to q) \;\Leftrightarrow\; (\neg q \to \neg p)$.

Definition 6.2 (Surjectivity)**.**
We say that a function $f : X \to Y$ is a *surjection* $:\Leftrightarrow (\forall y \in Y)(\exists x \in X)\big(f(x) = y\big)$.

Definition 6.3 (Bijectivity)**.**
We say that a function $f : X \to Y$ is a *bijection* $:\Leftrightarrow f$ is both injective and surjective.

Example 6.1.
Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(z) = z - 1$. This function is a bijection.

Proof (injectivity). Let $x_1, x_2 \in \mathbb{Z}$ and suppose $f(x_1) = f(x_2)$. Then, we can observe

$$\begin{aligned} f(x_1) = f(x_2) &\;\Rightarrow\; x_1 - 1 = x_2 - 1 && \text{by definition} \\ &\;\Rightarrow\; x_1 = x_2 && \text{by basic algebra} \end{aligned}$$

Therefore, f is an injection.

Q.E.D.

Proof (surjectivity). Let $y \in \mathbb{Z}$ and note $y + 1 \in \mathbb{Z}$. Since $f(y+1) = (y+1) - 1 = y$ by definition, f is surjective.

Q.E.D.