

PROPOSITIONAL LOGIC

DISCRETE MATHEMATICS

DANIEL GONZALEZ CEDRE

Definition 1 (Proposition).

A *proposition* is a sentence (in our language) that has one (and only one) definite, consistent truth value.

Definition 2 (Negation).

Given a proposition p , the *negation* of p , denoted $\neg p$, is defined by

p	$\neg p$
\top	\perp
\perp	\top

Some possible readings of $\neg p$:

- Not p .
- p does not hold.
- It is not the case that p .
- We do not have that p .

Definition 3 (Conjunction).

Given two propositions p and q , the *conjunction* of p with q , denoted $p \wedge q$, is defined by

p	q	$p \wedge q$
\top	\top	\top
\top	\perp	\perp
\perp	\top	\perp
\perp	\perp	\perp

Some possible readings of $p \wedge q$:

- p , and q .
- p , but q .
- p ; also, q .
- p ; further, q .
- In addition to p , we also have q .

Definition 4 (Disjunction).

Given two propositions p and q , the *disjunction* of p with q , denoted $p \vee q$, is defined by

p	q	$p \vee q$
\top	\top	\top
\top	\perp	\top
\perp	\top	\top
\perp	\perp	\perp

Some possible readings of $p \vee q$:

- p , or q .
- Either p , or q .

Definition 5 (Material Implication).

Given two propositions p and q , the *conditional* formed by assuming p and concluding q , denoted $p \rightarrow q$, is defined by

p	q	$p \rightarrow q$
\top	\top	\top
\top	\perp	\perp
\perp	\top	\top
\perp	\perp	\top

Some possible readings of $p \rightarrow q$:

- If p , then q .
- p implies q .
- q is conditioned on p .
- q only if p .
- p is sufficient for q .
- q is necessary for p .
- q unless not p .
- q or not p .

Definition 6 (Biconditional).

Given two propositions p and q , the *biconditional* formed by p and q , denoted $p \leftrightarrow q$, is defined by

p	q	$p \leftrightarrow q$
\top	\top	\top
\top	\perp	\perp
\perp	\top	\perp
\perp	\perp	\top

Some possible readings of pq :

- p if and only if q .
- p is necessary and sufficient for q .
- q is necessary and sufficient for p .

Definition 7 (Equivalence).

If we have two expressions φ and ψ in our formal language, consisting of some number of (possibly shared) propositional variables, connected together by logical connectives, then with the notation $\varphi \Leftrightarrow \psi$ we say that φ is equivalent to ψ \Leftrightarrow every assignment of truth values to the propositional variables of φ and ψ results in the same truth value for the two expressions.

Example 1.

Consider the two expressions $\varphi := p \rightarrow q$ and $\psi := \neg p \vee q$. We can see that φ has two propositional variables: p and q . ψ also has two propositional variables: the same p and the same q .

If we construct the truth table for these two expressions, we will see that every assignment of truth values to p and q will result in $p \rightarrow q$ and $\neg p \vee q$ having the same truth value.

p	q	$p \rightarrow q$	$\neg p \vee q$
\top	\top	\top	\top
\top	\perp	\perp	\perp
\perp	\top	\top	\top
\perp	\perp	\top	\top

Definition 8 (Boolean Algebra).

A *Boolean algebra* is a collection of *terms* B with two distinguished (and distinct) terms called \top and \perp , along with a unary operation called \neg and two binary operations called \wedge and \vee , such that the following statements are true for any terms p, q, r in B :

Axioms of a Boolean Algebra	
Identity	$p \wedge \top \Leftrightarrow p$ $p \vee \perp \Leftrightarrow p$
Complement (a.k.a. Negation)	$p \wedge \neg p \Leftrightarrow \perp$ $p \vee \neg p \Leftrightarrow \top$
Commutativity	$p \wedge q \Leftrightarrow q \wedge p$ $p \vee q \Leftrightarrow q \vee p$
Associativity	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
Distributive Laws	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

This kind of structure is also referred to as a *complemented, distributive lattice*. Since we are establishing the algebra of *propositions*, our terms consist only of \top and \perp .

However, since none of these axioms tell us how to use the (very useful) symbols \rightarrow and \leftrightarrow , we need two additional axioms that will turn our Boolean algebra into an example of a *Heyting algebra*:

Heyting Axioms	
Conditional Disintegration	$p \rightarrow q \Leftrightarrow \neg p \vee q$
Biconditional Disintegration	$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

By referring to the truth tables, it should be easy to see that these axioms are *truth preserving* transformations, meaning that taking an expression like $p \wedge (q \rightarrow r)$ and applying an axiom like Identity to it does not change the truth value of the resulting expression $(p \wedge \top) \wedge (q \rightarrow r)$. For this reason, these are sometimes referred to as *equivalence laws* and many treatments of this subject *prove* these laws by referring to the truth tables.

For our purposes, we don't need to refer to the truth tables at all. The truth tables were a nice, intuitive, and compact way of defining the logical connectives, but we could just as easily have defined them by assuming that all of the axioms are true, without ever writing down a truth table.