

Problem Set 3

Discrete Mathematics

Due on the 12th of February, 2024

In addition to the axioms and rules of inference, you may rely on: all proven theorems, *Implication Elimination*, *Hilbert's First & Second Axioms*.

(10 pts) 1. Prove each of the following statements for any propositions φ, ψ, ξ .

(a) $(\varphi \rightarrow \psi), (\psi \rightarrow \xi) \vdash (\varphi \rightarrow \xi)$

Hypothetical Syllogism

(b) $\varphi, \psi \vdash \varphi \wedge \psi$

Conjunction Introduction

(40 pts) 2. Prove each of the following statements for any propositions φ, ψ, ξ .

(a) $\vdash \varphi \rightarrow \varphi$

(b) $\vdash (\neg\varphi \rightarrow \varphi) \rightarrow \varphi$

Consequentia Mirabilis, a.k.a. *Lex Clavia*

(c) $\vdash \neg\varphi \rightarrow (\varphi \rightarrow \neg\psi)$

Ex Contradictione Quodlibet

(d) $\varphi \wedge \psi \vdash \varphi$

Conjunction Elimination

(e) $\vdash \top$

The Truth Theorem

(30 pts) 3. Prove each of the following statements for any propositions φ, ψ, ξ, χ .

(a) $\varphi \vdash (\varphi \vee \psi)$

Disjunction Introduction, a.k.a. *Addition*

(b) $(\varphi \rightarrow \xi), (\psi \rightarrow \xi), (\varphi \vee \psi) \vdash \xi$

Disjunction Elimination, a.k.a. *Proof by Cases*

(c) $\varphi, \neg\varphi \vdash \psi$

Ex Falso Quodlibet, a.k.a. *Explosion*

(d) $(\varphi \vee \psi), \neg\varphi \vdash \psi$

Disjunctive Syllogism

(e) $(\varphi \rightarrow \xi), (\psi \rightarrow \chi), (\varphi \vee \psi) \vdash \xi \vee \chi$

Constructive Dilemma

(10 pts) 4. Let \mathcal{L} be a binary predicate. Prove the following statement.¹

$$\vdash \neg\exists x\forall y(\mathcal{L}(x, y) \leftrightarrow \neg\mathcal{L}(y, y))$$

¹ Hint: try a proof by contradiction.

(10 pts) 5. Consider a universe of discourse consisting of every natural number. Recall that a positive integer is *prime* when it has *exactly two* positive divisors: one and itself.

As a fun side note: 2 is a prime number.

Let $\omega(x) := "x \text{ is an odd number}."$

Let $\pi(x) := "x \text{ is a prime number}."$

Further, suppose the following statements only contain propositions.

(a) Prove φ , where φ is the statement $\varphi \vdash \forall x(\omega(x) \rightarrow \pi(x))$.

(b) Prove $\forall x(\omega(x) \rightarrow \pi(x))$.