NAME: ______________________ NETID: ______________________

Exam f.p.2024.spring

Discrete Mathematics

9th of May, 2024

UNIVERSITY OF NOTRE DAME

1 *Answer the following questions either True or False.*

(a) If $\mathcal{A}$ and $\mathcal{B}$ are sets, then $|\mathcal{A} \cup \mathcal{B}| = |\mathcal{A}| + |\mathcal{B}|$.

- ◯ *True*
- ◯ *False*

(b) If a and b are odd, then $(\exists c \in \mathbb{Z})\Big(c^2 = a^2 + b^2\Big)$.

- ◯ *True*
- ◯ *False*

(c) If $\mathcal{A}$ and $\mathcal{B}$ are both finite, then $|\mathcal{A} \setminus \mathcal{B}| = |\mathcal{A}| - |\mathcal{B}|$.

- ◯ *True*
- ◯ *False*

(d) If this sentence is a proposition, then every countable set is finite.

- ◯ *True*
- ◯ *False*

(e) $2^{21} \equiv 1 \pmod 7$.

- ◯ *True*
- ◯ *False*

(f) This sentence implies $\forall x(\varnothing \subseteq x)$.

- ◯ *True*
- ◯ *False*

(g) $\big(\forall \mathcal{A} \in \mathbb{P}(\mathbb{N})\big)\big(\mathcal{A} \neq \varnothing \implies (\exists a \in \mathcal{A})(\forall b \in \mathcal{A})(a \leqslant b)\big)$.

- ◯ *True*
- ◯ *False*

(h) $\mathbb{Z}/n\mathbb{Z}$ is a *group* under *multiplication* for every $n \in \mathbb{N}_+$.

- ◯ *True*
- ◯ *False*

(i) $\big\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y = \gcd(x, 15)\big\}$ is a function.

- ◯ *True*
- ◯ *False*

(j) $\forall x \forall y \forall z\Big(x \setminus \big(y \cap z\big) = \big(x \setminus y\big) \cup \big(x \setminus z\big)\Big)$.

- ◯ *True*
- ◯ *False*

2 *Answer the following questions without proof.*

(a) Compute $\gcd(386, 352)$. ______________

(b) Find all $x \in \mathbb{Z}$ such that $4x + 6 \equiv 3 + 2x \pmod 9$. ______________

(c) Find all integer solutions to $3x^{49} + 3x + 2 \equiv 4 \pmod 7$. ______________

(d) List the elements of the following set using set-builder notation: ______________
$\left\{ x \in \mathbb{N} \;\middle|\; (\exists k \in \mathbb{Z})(xk = 28) \wedge (\forall y \in \mathbb{N})\big(y \mid x \;\Rightarrow\; y \in \{1, x\}\big) \right\}$.

(e) Provide an example of a transitive set x such that $x \notin \mathbb{N}$. ______________

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3 *Answer the following questions without proof.*

(a) Provide a surjection from $\left\{ f \;\middle|\; (\exists n \in \mathbb{N})\big(f : n \to \{0,1\}\big) \right\}$ to $\mathbb{Z}$. ______________________

(b) In how many different ways can the strings `"fighting"` and `"irish"` be scrambled and then concatenated together? ______________________

(c) Let $k \in \mathbb{N}$ such that $k \geqslant 3$. How many strings $s : k \to \{0, 1, \ldots, 9\}$ satisfy $(\exists i \in k)\big(s(i) = s(i+1) - 1 = s(i+2) - 2\big)$? ______________________

(d) Let $n \in \mathbb{N}_+$. How many binary strings b satisfy $|b| + \sum_{i=0}^{|b|-1} b(i) = n$? ______________________

(e) How many ways are there to move from the bottom-left square to the top-right square on an $n \times n$ chess board if $n \in \mathbb{N}_+$? ______________________

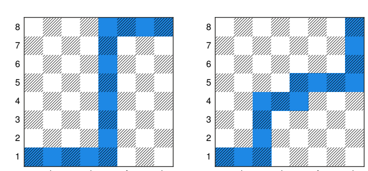

Fig. 1: Two examples of valid paths from the bottom-left corner to the top-right corner on an 8×8 chessboard.

4 *Any use of logical axioms, rules of inference, or theorems must be stated. You may not appeal to truth tables.*

Prove $\neg(p \to q) \to p$ is a tautology for any propositions p and q.

5 *You may rely on any theorems we have proven or studied.*

1. Give a recursive definition of the Fibonacci sequence $\mathcal{F} : \mathbb{N} \to \mathbb{N}$.

2. Show that $\mathcal{F}(n) < 2^n$ for all $n \in \mathbb{N}_+$.

6 *You may rely on any theorems we have proven or studied.*

Let $\mathfrak{F} := \{\mathcal{F}(i) \mid i \in \mathbb{N}\}$ be the set of all Fibonacci numbers and observe $|\mathfrak{F}| = \aleph_0$. Show $\exists i, j \in \mathbb{N}$ such that $i \neq j$ and $\mathcal{F}(i) \equiv \mathcal{F}(j) \pmod{2024}$.

7 *You may rely on any theorems we have proven or studied.*

Let $n \in \mathbb{N}_+$ and let $b : n \to \{0, 1, \ldots, 9\}$ be a decimal string representing a natural number k. Suppose the sum of the digits of b is divisible by 3. Prove that $3 \mid k$.

8 *You may rely on any theorems we have proven or studied.*

Show that there are uncountably many infinite hexadecimal strings.