Problem Set 9

Discrete Mathematics

Due on the 15^{th} of April, 2024

As always, all answers must be justified with a proof.

(10 pts) 1. A *palindrome* of length $k \in \mathbb{N}$ over an alphabet X is a string $s : k \to X$ such that $(\forall i \in k)(s(i) = s(k-1-i))$. How many possible palindromic words of length $k \in \mathbb{N}$ are there in the English language?[1]

[1] For simplicity, you may assume words are the same regardless of capitalization.

(20 pts) 2. Given a natural number $k \in \mathbb{N}$, how many ordered pairs (a, b) are there such that $a, b \in \mathbb{N}$ and $a + b = k$?

(20 pts) 3. Let $n \in \mathbb{N}_+$ and suppose you have an $n \times n$ chess board. We say two pieces on the board *threaten each other* if it is possible for one to capture the other by moving to occupy its square on the next move.

In how many ways can $n \in \mathbb{N}_+$ `rooks` possibly be arranged on an $n \times n$ chess board so that no two `rooks` threaten each other?

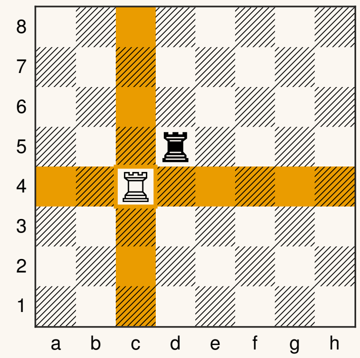

Fig. 1: Two `rooks` placed on an 8×8 chess board so that they do not threaten each other. The *movement pattern* for the white `rook` is highlighted above.

(20 pts) 4. We can write 4 as a sum of positive integers in the following ways.

$$\begin{array}{cccc} & 1+1+2 & 1+3 & \\ 1+1+1+1 & 1+2+1 & 2+2 & 4 \\ & 2+1+1 & 3+1 & \end{array}$$

Given a natural number $n \in \mathbb{N}$, how many distinct ways are there to write n as a sum of positive integers?

(30 pts) 5. A hermetic monk in meditation is repeatedly ascending and descending a ladder with $n \in \mathbb{N}$ rungs. The burdens of wisdom and devotion have left the monk's body frail, so he can only move by one or two rungs at-a-time. One day, as the monk ascends the ladder, a sudden tempest blows through his soul, and out of the whirlwind a fathomless voice calls out to him.

> *"In how many ways can you do this?"*

The monk ponders this question. After many years, he finds the answer, is enlightened, and immediately dies. What was the answer?

Hint: It may be helpful to use *strong induction* to prove your result; as a reminder, the scheme of strong induction is given below for an arbitrary predicate φ of one free variable.

$$\Big(\varphi(0) \wedge (\forall k \in \mathbb{N})\big((\forall \ell \in \mathbb{N})(\ell \leqslant k \Rightarrow \varphi(\ell)) \Rightarrow \varphi(k+1)\big)\Big) \Rightarrow (\forall n \in \mathbb{N})(\varphi(n))$$