Discrete Mathematics

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Chapter 4

Mathematical Induction

4.1 Weak Induction

Definition 4.1 (Well-Order)**.**
Let X be a set with a relation $\leqslant$ defined on it. We say that X is *well-ordered* by this relation *iff* every non-empty subset of X has a minimal element with respect to $\leqslant$. In other words, we say that $\leqslant$ is a *well-order* on X :⇔

$$\big(\forall A \in \mathcal{P}(X)\big)(\exists a \in A)(\forall b \in A)a \leqslant b.$$

Theorem 4.1 ($\mathbb{N}$ is Well-Ordered)**.**

$$(\forall A \subseteq \mathbb{N})\big(A \neq \varnothing \;\Rightarrow\; (\exists a \in A)(\forall b \in A)(a \leqslant b)\big)$$

This says that every nonempty subset of $\mathbb{N}$ *has a least element (according to the* $\leqslant$ *order defined on* $\mathbb{N}$*).*

Proof. This proof is left as an exercise.

Q.E.D.

Theorem 4.2 (Weak Induction)**.**
If $\varphi(\cdot)$ *is a wff, then*

$$(\forall n \in \mathbb{N})\big(\varphi(n)\big) \;\Leftrightarrow\; \varphi(0) \wedge (\forall k \in \mathbb{N})\big(\varphi(k) \;\Rightarrow\; \varphi(k+1)\big).$$

Proof. There are two fragments to this proof: the forward ($\Rightarrow$) direction and the backward ($\Leftarrow$) direction.

Fragment 1 ($\Rightarrow$):
Suppose that $(\forall n \in \mathbb{N})\big(\varphi(n)\big)$. Since $0 \in \mathbb{N}$, we then obviously have $\varphi(0)$. Now, let $k \in \mathbb{N}$ and assume $\varphi(k)$. Since $k+1 \in \mathbb{N}$, we know from our initial assumption that $\varphi(k+1)$. Thus, we have $(\forall k \in \mathbb{N})\big(\varphi(k) \;\Rightarrow\; \varphi(k+1)\big)$, and we have reached both of our desired conclusions.

Fragment 2 ($\Leftarrow$):
Assume $\varphi(0)$ and $(\forall k \in \mathbb{N})\big(\varphi(k) \;\Rightarrow\; \varphi(k+1)\big)$. Towards a contradiction, suppose that there is some $a \in \mathbb{N}$ such that $\neg\varphi(a)$. Consider $A \coloneqq \big\{x \in \mathbb{N} \;\big|\; \neg\varphi(x)\big\}$, which we clearly know exists by Axiom 4. We know that $n \in A$ because we assumed that $\neg\varphi(n)$, which implies that $A \neq \varnothing$. Then, we can use Theorem 4.1 to conclude that there is a minimal element a in A.

Since we know that $\varphi(0)$, it follows that $a \neq 0$, so a must be a successor number. This means there is a $b \in \mathbb{N}$ such that $b + 1 = a$.

If $b \in A$, then that would mean that $a \leqslant b$ since a is minimal in A. However, we know that $b < b+1 = a$, so we would then have $a \leqslant b < a$. ↯ Therefore, $b \notin A$.

With these two directions proven, we finally have $(\forall n \in \mathbb{N})\big(\varphi(n)\big) \;\Leftrightarrow\; \varphi(0) \wedge (\forall k \in \mathbb{N})\big(\varphi(k) \;\Rightarrow\; \varphi(k+1)\big)$.

Q.E.D.

Note. The above theorem actually generalizes beyond just $\mathbb{N}$. In fact, we can generalize the proof of Theorem 4.2 to *any* well-ordered set X by replacing $k+1$ with the least element of the non-empty subset $X \setminus \{\ell \in X \mid \ell \leqslant k\}$.

Let's practice induction by proving the following few theorems.

Example 4.1 (Gaussian Summation Formula)**.**
$(\forall n \in \mathbb{N})\left(\sum_{i=0}^{n} i = \frac{n(n+1)}{2}\right)$.

Proof. We will prove the claim by induction on $n \in \mathbb{N}$.

Base Case:
Observe that $\sum_{i=0}^{0} i = 0 = \frac{0 * (0+1)}{2}$. Therefore, the statement is satisfied at 0.

Inductive Step:
Let $k \in \mathbb{N}$ and assume $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$ (this is our *inductive hypothesis*). Now, observe

$$\begin{aligned}
\sum_{i=0}^{k+1} i &= \left(\sum_{i=0}^{k} i\right) + (k+1) && \text{by definition} \\
&= \left(\frac{k(k+1)}{2}\right) + (k+1) && \text{by the } \textit{inductive hypothesis} \\
&= (k+1)\left(\frac{k}{2} + 1\right) && \text{by the distributive property of multiplication}^{*\dagger} \\
&= \frac{(k+1)(k+2)}{2} && \text{because } \frac{k}{2} + 1 = \frac{k}{2} + \frac{2}{2} = \frac{k+2}{2}.^{\dagger\ddagger}
\end{aligned}$$

Thus, we have that $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$, as desired.

Therefore, we can conclude that $(\forall n \in \mathbb{N})\left(\sum_{i=0}^{n} i = \frac{n(n+1)}{2}\right)$.

Q.E.D.

Example 4.2.
$3n \leqslant 3^n$ for all $n \in \mathbb{N}_+$.

Proof. We will prove the claim by induction on $n \in \mathbb{N}_+$.

Base Case:
Observe that $3 \cdot 1 = 3 \leqslant 3 = 3^1$.

Inductive Step:
Let $k \in \mathbb{N}_+$ and assume $3 \cdot k \leqslant 3^k$ (this is our *inductive hypothesis*). Now, observe

$$\begin{aligned}
3 \cdot (k+1) &= 3 \cdot k + 3 && \text{by the distributive property of multiplication} \\
&\leqslant 3^k + 3 && \text{by the } \textit{inductive hypothesis} \\
&\leqslant 3^k + 3^k && \text{because } 1 \leqslant k \ \Rightarrow\ a^1 \leqslant a^k \text{ if } a < 0^{*\dagger} \\
&\leqslant 3^k + 3^k + 3^k && \text{since } a^k \text{ is always positive if } 0 < a^{*\dagger} \\
&= 3 \cdot 3^k && \text{because } \textstyle\sum_{i=1}^{m} a = m \cdot a \text{ for any } a^{*\dagger} \\
&= 3^{k+1} && \text{because } a^b a^c = a^{b+c} \text{ for any } a.^{*\dagger}
\end{aligned}$$

So, $3 \cdot (k+1) \leqslant 3^{k+1}$, as desired.

Therefore, $(\forall n \in \mathbb{N}_+)(3n \leqslant 3^n)$.

Q.E.D.

[*] These "basic grade-school" algebraic properties will now be assumed without special mention.
[†] This is true in any *ordered semiring* where exponentiation is defined in terms of multiplication.
[‡] This is true in any *field*.

Example 4.3.
$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

Proof. We will prove the claim by induction on $n \in \mathbb{N}$.

Base Case:
Observe that $\sum_{i=0}^{0} 2^i = 2^0 = 1 = 2 - 1 = 2^{1+1} - 1$

Inductive Step:
Let $k \in \mathbb{N}_+$ and assume that $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ (this is our *inductive hypothesis*). Observe that

$$\begin{aligned}
\sum_{i=0}^{k+1} 2^i &= \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1} && \text{by definition} \\
&= \left(2^{k+1} - 1\right) + 2^{k+1} && \text{by the } \textit{inductive hypothesis} \\
&= 2 \cdot 2^{k+1} - 1 && \text{using some basic algebra} \\
&= 2^{k+2} - 1 && \text{using some basic algebra.}
\end{aligned}$$

So, we get $\sum_{i=0}^{k+1} 2^i = 2^{k+1} - 1$, as desired.

Therefore, we can conclude $(\forall n \in \mathbb{N})\left(\sum_{i=0}^{n} 2^i = 2^{n+1} - 1\right)$.

Q.E.D.

Example 4.4.
Every natural number is either even or odd.

Proof. We will prove the claim by using *strong* induction on $\mathbb{N}$.

Base Case:
Firstly, observe that $0 = 2 \cdot 0$ by definition; since $0 \in \mathbb{N}$, we now know that 0 is even.
Secondly, observe that $1 = 0 + 1 = 2 \cdot 0 + 1$ by definition; since $0 \in \mathbb{N}$, we also know that 1 is odd.

Inductive Step:
Let $k \in \mathbb{N}$ and assume, for any $\ell \leqslant k$, that ℓ is either even or odd. In particular, k is either even or odd.

Case 1:
If k is even, then there is an $\ell \in \mathbb{N}$ such that $k = 2\ell$. But then $k + 1 = 2\ell + 1$, so $k + 1$ is odd.

Case 2:
If k is odd, then there is an $\ell \in \mathbb{N}$ such that $k = 2\ell + 1$. Similarly then, we can see that $k + 1 = (2\ell + 1) + 1 = 2\ell + (1 + 1) = 2\ell + 2 = 2(\ell + 1)$, so that $k + 1$ is even.

Thus, from these two cases, $k + 1$ is either even or odd.

Therefore, $(\forall n \in \mathbb{N})\big((n \text{ is even}) \vee (n \text{ is odd})\big)$.

Q.E.D.

Lemma 4.1.
For any $n, m \in \mathbb{N}$, if $n = 2m$ or $n = 2m + 1$, then $n \geqslant m$.

Proof. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. There are two cases.

Case 1:
If $n = 2m$, then suppose towards a contradiction that $m > n$. We would then have $n = 2m > 2n = n + n \geqslant n$, so that $n > n$, implying $n \neq n$. $\lightning$ This shows us $m \leqslant n$.

Case 2:
If $n = 2m + 1$, then again assume towards a contradiction that $m > n$. This would mean $n = 2m + 1 > 2n + 1 > 2n = n + n \geqslant n$, so $n > n$, implying $n \neq n$. $\lightning$ So, we get $m \leqslant n$.

Q.E.D.

4.2 Strong Induction

Theorem 4.3 (Strong Induction).
If $\varphi(\cdot)$ is a wff and $r \in \mathbb{N}$, then

$$(\forall n \in \mathbb{N})\big(\varphi(n)\big) \;\Leftrightarrow\; \left(\bigwedge_{i=0}^{r} \varphi(r)\right) \wedge (\forall k \in \mathbb{N})\Big((\forall \ell \leqslant k)\big(\varphi(\ell)\big) \;\Rightarrow\; \varphi(\ell+1)\Big).$$

Note. An equivalent way to state the *inductive hypothesis* for a proof involving strong induction is to say that we assume $(\forall \ell < k)\big(\varphi(\ell)\big)$, and then our task in the inductive step is to prove $\varphi(k)$.

Example 4.5.
$(\forall n \in \mathbb{N})(\exists r \in \mathbb{N})\big(\exists c_0, \dots c_r \in \{0,1\}\big)\Big(n = \sum_{i=0}^{r} c_i 2^i\Big)$; *i.e.*, every natural number admits a binary representation.

Proof. We will prove the claim by using *strong* induction on $\mathbb{N}$.

Base Case:
Consider $r \coloneqq 0$ and $c_0 \coloneqq 0$ and observe that $0 = 0 \cdot 1 = 0 \cdot 2^0 = \sum_{i=0}^{0} c_i 2^i = \sum_{i=0}^{r} c_i 2^i$.

Inductive Step:
Let $k \in \mathbb{N}$ and assume that $(\forall \ell \leqslant k)(\exists r \in \mathbb{N})\big(\exists c_0, \dots c_r \in \{0,1\}\big)\Big(\ell = \sum_{i=0}^{r} c_i 2^i\Big)$. Since k, being a natural number, must be either even or odd, we can consider the following two cases.

Case 1:
If k is even, then we know $k = 2\ell$ for some $\ell \in \mathbb{N}$ by definition. This implies that $\ell \leqslant k$ by Lemma 4.1, so we then we can apply the *inductive hypothesis* to get $\ell = \sum_{i=0}^{r} c_i 2^i$ for some $r \in \mathbb{N}$ and some $c_0, \dots c_r \in \{0,1\}$. We can then see that

$$\begin{aligned}
k+1 = 2\ell + 1 = 1 + 2\sum_{i=0}^{r} c_i 2^i \quad & \text{by the \textit{inductive hypothesis}} \\
= 1 + \sum_{i=0}^{r} c_i 2^{i+1} \quad & \text{by distribution and the fact that } 2 \cdot 2^i = 2^{i+1} \\
= 1 + \sum_{i=1}^{r+1} c_{i-1} 2^i \quad & \text{by shifting our indices of summation} \\
= 1 \cdot 2^0 + \sum_{i=1}^{r+1} c_{i-1} 2^i \cdot 2^0 \quad & \text{because } 1 = 2^0
\end{aligned}$$

So, if we let $\tilde{r} \coloneqq r+1$, $\tilde{c}_0 \coloneqq 1$, and $\tilde{c}_i \coloneqq c_{i-1}$ for $i \in \{1, \dots \tilde{r}\}$, we find $k+1 = \sum_{i=0}^{\tilde{r}} \tilde{c}_i 2^i$.

Case 2:
If k is odd, then we know $k = 2\ell + 1$ for some $\ell \in \mathbb{N}$ by definition. Lemma 4.1 then tells us that $\ell \leqslant k$, so again we know that $\ell = \sum_{i=0}^{r} c_i 2^i$ for some $r \in \mathbb{N}$ and some $c_0, \dots c_r \in \{0,1\}$ by the *inductive hypothesis*. We now observe that

$$\begin{aligned}
k+1 = (2\ell+1)+1 = 2\ell+2 = 2^2\ell = 2^2 \sum_{i=0}^{r} c_i 2^i \quad & \text{by the \textit{inductive hypothesis}} \\
= \sum_{i=0}^{r} c_i 2^{i+2} \quad & \text{by distribution and } 2^2 2^i = 2^{i+2} \\
= 0 + 0 + \sum_{i=2}^{r+2} c_{i-2} 2^i \quad & \text{by shifting our indices}
\end{aligned}$$

So, if we let $\tilde{r} \coloneqq r+2$, $\tilde{c}_0 \coloneqq 0$, $\tilde{c}_1 \coloneqq 0$, and $\tilde{c}_i \coloneqq c_{i-1}$ for $i \in \{2, \dots \tilde{r}\}$, we find $k+1 = \sum_{i=0}^{\tilde{r}} \tilde{c}_i 2^i$.

Therefore, every natural number has a binary representation.

Q.E.D.