NAME: ______________________ NETID: ______________________

Exam 2.p.2024.spring

Discrete Mathematics

26th of April, 2024

1 *Answer the following questions by marking either True or False.*

(a) Every function is either injective or surjective.

- ○ *True*
- ○ *False*

(b) If $|X| = n \in \mathbb{N}$, then there are $n!$ surjections from X to X.

- ○ *True*
- ○ *False*

(c) There is a set the same size as its power set.

- ○ *True*
- ○ *False*

(d) Every random number generator must eventually repeat a number.

- ○ *True*
- ○ *False*

(e) $\forall A \forall B \Big(|A| = |B| \Leftrightarrow (\forall f : A \to B)\big((\forall a_1, a_2 \in A)(a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)) \Leftrightarrow (\forall b \in B)(\exists a \in A)(f(a) = b)\big)\Big)$.

- ○ *True*
- ○ *False*

(f) There are countably many *eventually periodic* decimal strings.

- ○ *True*
- ○ *False*

A string $f : \mathbb{N} \to X$ is called *periodic* if $(\forall n \in \mathbb{N})(f(n) = f(n+p))$ for some $p \in \mathbb{N}$ called the *period* of f. A string f is called *eventually periodic* if $f = s +\!\!+\, t$ where s is finite and t is periodic. A string that is not eventually periodic is *aperiodic*.

(g) If $P : \mathbb{N} \to 10$ is *aperiodic*, every finite decimal string appears in P.

- ○ *True*
- ○ *False*

(h) Let X and Y be sets. If $f : X \to Y$, then $|f| = |X|$.

- ○ *True*
- ○ *False*

(i) If φ_e is Euler's totient function, then $\varphi_e(99) = 80$.

- ○ *True*
- ○ *False*

(j) There are countably many propositions.

- ○ *True*
- ○ *False*

2 *Answer the following questions without proof.*

(a) What is $\left|\left\{ s : k \to \{0,1\} \;\middle|\; \sum_{i=0}^{k-1} s(i) = n \right\}\right|$ when $k, n \in \mathbb{N}$? ______________

(b) How many ways are there to scramble the string `"caesar"`? ______________

(c) How many even natural numbers can be written using $k \in \mathbb{N}$ decimal digits such that $k \geqslant 2$ and no consecutive digits are repeated? ______________

(d) What is $\left|\left\{ S \subseteq \{1,2,\ldots,50\} \;\middle|\; (\forall x, y \in S)\left(x \neq y \Rightarrow |x-y| > 25\right) \right\}\right|$? ______________

(e) Given $k \in \mathbb{N}$ such that $k \geqslant 3$, how many ways are there to write k as a sum of positive integers *without* using the numbers 1 and 2? ______________

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3 *You may rely on any theorems we have proven or studied.*

An archaeologist on a recent expedition has discovered, for each $n \in \mathbb{N}_+$, a manuscript written in a language called *n*glish, seemingly used by the native inhabitants of *n*gland in ancient times. Each manuscript is exactly n pages long and contains precisely $2n$ distinct words.

Prove each manuscript contains a page with two distinct words on it.

4 *You may rely on any theorems we have proven or studied.*

Prove there are uncountably many infinite strings of prime numbers.