

# Problem Set 9

## Discrete Mathematics

Due on the 17<sup>th</sup> of April, 2024

As always, all answers must be justified with a proof.

- (10 pts) 1. A *palindrome* of length  $k \in \mathbb{N}$  over an alphabet  $X$  is a string  $s : k \rightarrow X$  such that  $(\forall i \in k)(s(i) = s(k - 1 - i))$ . How many possible palindromic words of length  $k \in \mathbb{N}$  are there in the English language?<sup>1</sup>
- (20 pts) 2. Given a natural number  $k \in \mathbb{N}$ , how many ordered pairs  $(a, b)$  are there such that  $a, b \in \mathbb{N}$  and  $a + b = k$ ?
- (20 pts) 3. Let  $n \in \mathbb{N}_+$  and suppose you have an  $n \times n$  chess board. We say two pieces on the board *threaten each other* if it is possible for one to capture the other by moving to occupy its square on the next move.
- In how many ways can  $n \in \mathbb{N}_+$  rooks possibly be arranged on an  $n \times n$  chess board so that no two rooks threaten each other?
- (20 pts) 4. We can write 4 as a sum of positive integers in 8 distinct ways.

$$\begin{array}{rclcl}
 & & 1 + 1 + 2 & 1 + 3 & \\
 1 + 1 + 1 + 1 & 1 + 2 + 1 & 2 + 2 & 4 & \\
 & 2 + 1 + 1 & 3 + 1 & & 
 \end{array}$$

Given a natural number  $n \in \mathbb{N}$ , how many distinct ways are there to write  $n$  as a sum of positive integers?

- (30 pts) 5. A hermetic monk in meditation is repeatedly ascending and descending a ladder with  $n \in \mathbb{N}$  rungs. The burdens of wisdom and devotion have left the monk's body frail, so he can only move by one or two rungs at-a-time. One day, as the monk ascends the ladder, a sudden tempest blows through his soul, and out of the whirlwind a fathomless voice calls out to him.

"In how many ways can you do this?"

The monk ponders this question. After many years, he finds the answer, is enlightened, and immediately dies. What was the answer?

*Hint:* It may be helpful to use *strong induction* to prove your result; as a reminder, the scheme of strong induction is given below for an arbitrary predicate  $\varphi$  of one free variable.

$$\left( \varphi(0) \wedge (\forall k \in \mathbb{N}) \left( (\forall \ell \in \mathbb{N}) (\ell \leq k \Rightarrow \varphi(\ell)) \Rightarrow \varphi(k + 1) \right) \right) \Rightarrow (\forall n \in \mathbb{N}) (\varphi(n))$$

<sup>1</sup> For simplicity, you may assume words are the same regardless of capitalization.

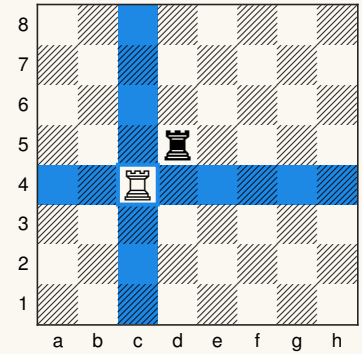


Fig. 1: Two rooks placed on an  $8 \times 8$  chess board so that they do not threaten each other. The *movement pattern* for the white rook is highlighted above.