Problem Set 5

Discrete Mathematics

Due on the 27^{th} of February, 2024

Recall that the natural numbers are defined recursively as follows.

$$0 := \varnothing$$
$$\mathfrak{s}(n) := n \cup \{n\}$$

Addition on $\mathbb{N}$ is defined below.

$$n + 0 := n$$
$$n + \mathfrak{s}(m) := \mathfrak{s}(n + m)$$

Multiplication on $\mathbb{N}$ is defined below.

$$n \cdot 0 := 0$$
$$n \cdot \mathfrak{s}(m) := (n \cdot m) + n$$

Exponentiation on $\mathbb{N}$ is defined below.

$$n^0 := 1$$
$$n^{\mathfrak{s}(m)} := n \cdot n^m$$

We define the iterated sum of a sequence of terms $f(0), f(1), f(2), \ldots$ as follows.

$$\sum_{i=0}^{0} f(i) := f(0)$$
$$\sum_{i=0}^{\mathfrak{s}(n)} f(i) := \left(\sum_{i=0}^{n} f(i)\right) + f(\mathfrak{s}(n))$$

You may rely on the following theorems:
$(\forall x \in \mathbb{N})(\mathfrak{s}(x) = x + 1)$.
$(\forall x \in \mathbb{N})(\mathfrak{s}(x) = 1 + x)$.
$(\forall x, y, z \in \mathbb{N})(x + (y + z) = (x + y) + z)$.

(10 pts) 1. Find and explain the flaw(s) in this argument.

> We prove every nonempty set of people all have the same age.
> ***Proof.*** We denote the age of a person p by $\alpha(p)$.
>
> *Basis Step:*
> Suppose $P = \{p\}$ is a set with one person in it. Clearly, all the people in P have the same age as each other.
>
> *Inductive Step:*
> Let $k \in \mathbb{N}_+$ and suppose any set of k-many people all have the same age. Let $P = \{p_1, p_2, \ldots p_k, p_{k+1}\}$ be a set with $k + 1$ people in it. Consider $L := \{p_1, \ldots p_k\}$ and $R := \{p_2, \ldots p_{k+1}\}$. Since L and R both have k people, we know everyone in these sets has the same age by the *inductive hypothesis.*
>
> Let $\ell, r \in P$. If $\ell \in L \wedge r \in L$, then $\alpha(\ell) = \alpha(r)$. Similarly, if $\ell \in R \wedge r \in R$, then $\alpha(\ell) = \alpha(r)$. Now, suppose $\ell \in L \wedge r \in R$.
>
> $$\alpha(\ell) = \alpha(p_1) = \alpha(p_2) = \alpha(p_{k+1}) = \alpha(r)$$
>
> So, all people in P have the same age.
>
> Therefore, everyone on Earth has the same age. Q.E.D.

(20 pts) 2. Show that $\forall x(x \neq x \cup \{x\})$.

(15 pts) 3. We will work up to a proof of the commutativity of addition on $\mathbb{N}$.

(a) Show $(\forall x \in \mathbb{N})(x + 0 = 0 + x)$.

(b) Show $(\forall x, y \in \mathbb{N})(x + \mathfrak{s}(y) = \mathfrak{s}(y) + x)$.

(c) Show $(\forall x, y \in \mathbb{N})(x + y = y + x)$.

(15 pts) 4. Show $(\forall x, y, z \in \mathbb{N})\big(x \cdot (y + z) = (x \cdot y) + (x \cdot z)\big)$.

(20 pts) 5. For this problem, you may assume the commutativity and associativity of addition and multiplication over $\mathbb{N}$. You may also assume multiplication distributes over addition on $\mathbb{N}$. Prove the following statement for all $n \in \mathbb{N}$.

$$1 + \sum_{i=0}^{n} 2^i = 2^{n+1}$$

(20 pts) 6. We say x is *$\in$-transitive* by definition when $(\forall y \in x)(\forall z \in y)(z \in x)$. Show that every natural number is $\in$-transitive.