Problem Set 10

Discrete Mathematics

Due on the 22nd of April, 2024

(20 pts) 1. Construct—*with proof*—the *explicit* functions requested below.

(a) A bijection from $\{x \in \mathbb{R} \mid -1 < x < 1\}$ to $\{x \in \mathbb{R} \mid -\pi < x < \pi\}$.

(b) A surjection from $\mathbb{N}$ to $\{p \mid p \text{ is prime}\}$.

(c) An injection from X to $\mathbb{P}(X)$ for *every* set X.

(d) A surjection from $\mathbb{P}(X)$ to X for *every* set X.

(20 pts) 2. Justify each answer below *with proof.*

(a) How many infinite binary strings have digits whose sum is 2?

(b) How many infinite decimal strings have *finitely many* 7s?

(c) How many infinite decimal strings have *infinitely many* 7s?

(d) Is there a set x such that $\forall y\Big(|x| \geqslant |y|\Big)$?

(30 pts) 3. Let A be an arbitrary finite set of cardinality $|A| = n$, where $n \in \mathbb{N}$. How many finite strings over A are there?

(30 pts) 4. Imagine that, one day, you encounter a library. At the entrance of this library is an enormous tome $\mathscr{B}$ listing *all* of the *possible* sentences in the English language, indexed by natural numbers.

Walking past the entrance, you see that the library has rows of bookshelves numbered $0, 1, 2, \ldots$, so that there is exactly one row of books for each $n \in \mathbb{N}$. A great owl—perched on the pedestal that supports $\mathscr{B}$—informs you that, for each $n \in \mathbb{N}$, the n^{th} row of bookshelves contains *all* of the books that could possibly ever be that begin with the n^{th} sentence in $\mathscr{B}$. With the sound of granite scraping against marble, the doors to the library close behind you. The owl makes you the following proposal: you will be free to go *if and only if* you can read every book in the library *in a countable amount of time.*

Will you be set free? The owl demands a proof to justify your answer.