Propositional Logic

Discrete Mathematics

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Definition 1 (Proposition)**.**

A *proposition* is a sentence (in our language) that has one (and only one) definite, consistent truth value.

Definition 2 (Negation)**.**

Given a proposition p, the *negation* of p, denoted $\neg p$, is defined by

p	$\neg p$
$\top$	$\bot$
$\bot$	$\top$

Some possible readings of $\neg p$:

- Not p.
- p does not hold.
- It is not the case that p.
- We do not have that p.

Definition 3 (Conjunction)**.**

Given two propositions p and q, the *conjunction* of p with q, denoted $p \wedge q$, is defined by

p	q	$p \wedge q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$

Some possible readings of $p \wedge q$:

- p, and q.
- p, but q.
- p; also, q.
- p; further, q.
- In addition to p, we also have q.

Definition 4 (Disjunction)**.**

Given two propositions p and q, the *disjunction* of p with q, denoted $p \vee q$, is defined by

p	q	$p \vee q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$

Some possible readings of $p \vee q$:

- p, or q.
- Either p, or q.

Definition 5 (Material Implication)**.**

Given two propositions p and q, the *conditional* formed by assuming p and concluding q, denoted $p \to q$, is defined by

p	q	$p \to q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\top$

Some possible readings of $p \to q$:

- If p, then q.
- p implies q.
- q is conditioned on p.
- q only if p.
- p is sufficient for q.
- q is necessary for p.
- q unless not p.
- q or not p.

Definition 6 (Biconditional)**.**

Given two propositions p and q, the *biconditional* formed by p and q, denoted $p \leftrightarrow q$, is defined by

p	q	$p \leftrightarrow q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\top$

Some possible readings of pq:

- p if and only if q.
- p is necessary and sufficient for q.
- q is necessary and sufficient for p.

Definition 7.

If we have two expressions φ and ψ in our formal language, consisting of some number of (possibly shared) propositional variables, connected together by logical connectives, then with the notation $\varphi \Leftrightarrow \psi$ we say that φ is equivalent to ψ $:\Leftrightarrow$ every assignment of truth values to the propositional variables of φ and ψ results in the same truth value for the two expressions.

Example 1.

Consider the two expressions $\varphi \coloneqq p \to q$ and $\psi \coloneqq \neg p \vee q$. We can see that φ has two propositional variables: p and q. ψ also has two propositional variables: the same p and the same q.

If we construct the truth table for these two expressions, we will see that every assignment of truth values to p and q will result in $p \to q$ and $\neg p \vee q$ having the same truth value.

p	q	$p \to q$	$\neg p \vee q$
$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\top$

Definition 8.

A *Boolean algebra* is a collection of *terms* B with two distinguished (and distinct) terms called $\top$ and $\bot$, along with a unary operation called $\neg$ and two binary operations called $\wedge$ and $\vee$, such that the following statements are true for any terms p, q, r in B:

Axioms of a Boolean Algebra	
Identity	$p \wedge \top \Leftrightarrow p$ $p \vee \bot \Leftrightarrow p$
Complement (*a.k.a.* Negation)	$p \wedge \neg p \Leftrightarrow \bot$ $p \vee \neg p \Leftrightarrow \top$
Commutativity	$p \wedge q \Leftrightarrow q \wedge p$ $p \vee q \Leftrightarrow q \vee p$
Associativity	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
Distributive Laws	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

This kind of structure is also referred to as a *complemented, distributive lattice*. Since we are establishing the algebra of *propositions*, our terms consist only of $\top$ and $\bot$.

However, since none of these axioms tell us how to use the (very useful) symbols $\to$ and $\leftrightarrow$, we need two additional axioms that will turn our Boolean algebra into an example of a *Heyting algebra*:

Heyting Axioms	
Conditional Disintegration	$p \to q \Leftrightarrow \neg p \vee q$
Biconditional Disintegration	$p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$

By referring to the truth tables, it should be easy to see that these axioms are *truth preserving* transformations, meaning that taking an expression like $p \wedge (q \to r)$ and applying an axiom like Identity to it does not change the truth value of the resulting expression $(p \wedge \top) \wedge (q \to r)$. For this reason, these are sometimes referred to as *equivalence laws* and many treatments of this subject *prove* these laws by referring to the truth tables.

For our purposes, we don't need to refer to the truth tables at all. The truth tables were a nice, intuitive, and compact way of defining the logical connectives, but we could just as easily have defined them by assuming that all of the axioms are true, without ever writing down a truth table.