Discrete Mathematics

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Chapter 6

Cardinality

6.1 Functions

Definition 6.1 (Function).
As a reminder, we say that $f : X \to Y$ is a function from X to Y *iff* both of the following hold:

I. $f \subseteq X \times Y$

II. $(\forall x \in X)(\exists! y \in Y)\big((x, y) \in f\big)$

Definition 6.2 (Injectivity).
We say that a function $f : X \to Y$ is an *injection* :⇔ either of the following two statements holds:

i. $(\forall x_1 \in X)(\forall x_2 \in X)\big(x_1 \neq x_2 \;\Rightarrow\; f(x_1) \neq f(x_2)\big)$

ii. $(\forall x_1 \in X)(\forall x_2 \in X)\big(f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2\big)$

Notice that these two statements are equivalent since the leading quantifiers are identical and the unquantified implications are contrapositives of each other, and we know from the propositional logic that $(p \to q) \;\Leftrightarrow\; (\neg q \to \neg p)$. It is common to denote injective functions using the notation $f : X \hookrightarrow Y$.

Definition 6.3 (Surjectivity).
We say that a function $f : X \to Y$ is a *surjection* :⇔ $(\forall y \in Y)(\exists x \in X)\big(f(x) = y\big)$.
It is common to denote injective functions using the notation $f : X \twoheadrightarrow Y$.

Definition 6.4 (Bijectivity).
We say that a function $f : X \to Y$ is a *bijection* :⇔ f is both injective and surjective.
For bijections, it is common to combine the injective and surjective notations and denote them $f : X \hookrightarrow\!\!\!\!\rightarrow Y$.

Example 6.1.
Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(z) = z - 1$. This function is a bijection.

Proof. Let $x_1, x_2 \in \mathbb{Z}$ and suppose $f(x_1) = f(x_2)$. Then, we can observe

$$\begin{aligned} f(x_1) = f(x_2) &\;\Rightarrow\; x_1 - 1 = x_2 - 1 && \text{by definition} \\ &\;\Rightarrow\; x_1 = x_2 && \text{by basic algebra} \end{aligned}$$

Therefore, f is an injection.

Now, let $y \in \mathbb{Z}$ and note $y + 1 \in \mathbb{Z}$. Since $f(y + 1) = (y + 1) - 1 = y$ by definition, we have that f is surjective.

Since f is both injective and surjective, f is a bijection by definition.

Q.E.D.

Definition 6.5 (Cardinality)**.**
Let A and B be sets. The *cardinality* of a set, which we denote by $|A|$*, corresponds to our intuitive notion of its *size* relative to other sets. If we want to compare two sets, we assess their relative cardinalities by determining whether or not one set *fits inside* the other by seeing what kinds of functions it is possible to define between them.

We say that the *cardinality* of A is *no greater than* the *cardinality* of B $:\Leftrightarrow \exists f : A \to B$ such that f is an injection. In this case, we say that $|A| \leqslant |B|$.

We say that the *cardinality* of A is *no lesser than* the *cardinality* of B $:\Leftrightarrow \exists f : A \to B$ such that f is an surjection. In this case, we say that $|A| \geqslant |B|$.

Naturally, we then say that A and B have the *same cardinality* $:\Leftrightarrow \exists f : A \to B$ such that f is a bijection, which we denote by $|A| = |B|$.

Definition 6.6 (Finite Set)**.**
We say that a set A is *finite* $:\Leftrightarrow (\exists n \in \mathbb{N})(\exists f : A \to n)(f$ is a bijection$)$. In this case, we will say that $|A| = n$.

Definition 6.7 (Countable Set)**.**
We say that a set A is *countable* $:\Leftrightarrow (\exists f : A \to \mathbb{N})(f$ is an injection$)$. In this case, we say that $|A| \leqslant \aleph_0$.

Example 6.1.
Let's prove that $|\mathbb{N}| = |\mathbb{Z}|$.

Proof. Consider the function $f : \mathbb{Z} \to \mathbb{N}$ given by

$$f(z) = \begin{cases} 2z & \text{if } z \geqslant 0 \\ 2(-z) - 1 & \text{if } z < 0 \end{cases}$$

First, let's see that this is an injection. Let $x_1, x_2 \in \mathbb{Z}$ and suppose $f(x_1) = f(x_2)$. We now have two cases.

Case 1:
If $f(x_1)$ is even, then we know $f(x_1) = 2k$ for some $k \in \mathbb{N}$ by definition. Then, we have $f(x_2) = 2k$ as well, since $f(x_1) = f(x_2)$.

Now, we claim that $x_1 \geqslant 0$: if we assume $x < 0$ towards the contrary, then we would have $f(x_1) = 2(-x_1) - 1$, which is odd. We would then have

$$\begin{aligned} 2(-x_1) - 1 = 2k \;&\Rightarrow\; 2(-x_1) - 2k = 1 \\ &\Rightarrow\; 2(k - x_1) = 1 \\ &\Rightarrow\; k - x_1 = 1/2 \end{aligned}$$

However, since k and x_1 are both integers (and $\mathbb{Z}$ is an ordered ring), $k - x_1$ must be an integer. ↯

By the same argument, we also have that $x_2 \geqslant 0$. Therefore, $2x_1 = f(x_1) = f(x_2) = 2_x2$, so $x_1 = x_2$.

Case 2:
This case is left as an exercise to the reader.

Thus, f is injective since $x_1 = x_2$ in both cases.

Now, let's show that f is a surjection. Suppose $y \in \mathbb{N}$ and again we have two cases.

Case 1:
If y is even, then $y = 2k$ for some $k \in \mathbb{N}$. But then, we can simply see $k \in \mathbb{Z}$ and $f(k) = 2k$ since $k \geqslant 0$.

Case 2:
If y is odd, then $y = 2k+1$ for some $k \in \mathbb{N}$. Then, $k \in \mathbb{Z}$ and $f(-k-1) = 2(k+1)-1 = 2k+2-1 = 2k+1$ because $k \geqslant 0 \;\Rightarrow\; -k \leqslant 0 \;\Rightarrow\; -k-1 < -k \leqslant 0$ and $-k-1 \in \mathbb{Z}$.

Therefore, since we found a preimage for y in both cases, f is surjective.

This means f is a bijection, so we can conclude that $|\mathbb{N}| = |\mathbb{Z}|$.

Q.E.D.

*The *cardinality* of a set is not always guaranteed to exist without the Axiom of Choice.

Theorem 6.1 (Cantor-Shröder-Bernstein).
Given two sets A and B, if there exist injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$, then there exists a bijection $h : A \hookrightarrow\!\!\!\!\rightarrow B$.