Discrete Mathematics

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Chapter 4

Mathematical Induction

4.1 Weak Induction

Theorem 4.1 ($\mathbb{N}$ is Well-Ordered)**.**

$$(\forall A \subseteq \mathbb{N})\big(A \neq \varnothing \Rightarrow (\exists a \in A)(\forall b \in A)(a \leqslant b)\big)$$

This says that every nonempty subset of $\mathbb{N}$ has a least element (according to the $\leqslant$ order defined on $\mathbb{N}$).

Proof.

This proof is left as an exercise.

Q.E.D.

Theorem 4.2 (Weak Induction)**.**

If $\varphi(\cdot)$ is a wff, then

$$(\forall n \in \mathbb{N})\big(\varphi(n)\big) \;\Leftrightarrow\; \varphi(0) \wedge (\forall k \in \mathbb{N})\big(\varphi(k) \Rightarrow \varphi(k+1)\big).$$

Proof.

There are two fragments to this proof: the forward ($\Rightarrow$) direction and the backward ($\Leftarrow$) direction.

Fragment 1 ($\Rightarrow$):

Suppose that $(\forall n \in \mathbb{N})\big(\varphi(n)\big)$. Since $0 \in \mathbb{N}$, we then obviously have $\varphi(0)$. Now, let $k \in \mathbb{N}$ and assume $\varphi(k)$. Since $k+1 \in \mathbb{N}$, we know from our initial assumption that $\varphi(k+1)$. Thus, we have $(\forall k \in \mathbb{N})\big(\varphi(k) \Rightarrow \varphi(k+1)\big)$, and we have reached both of our desired conclusions.

Fragment 2 ($\Leftarrow$):

Assume $\varphi(0)$ and $(\forall k \in \mathbb{N})\big(\varphi(k) \Rightarrow \varphi(k+1)\big)$. Towards a contradiction, suppose that there is some $n \in \mathbb{N}$ such that $\neg\varphi(a)$. Consider $A \coloneqq \{x \in \mathbb{N} \mid \neg\varphi(x)\}$, which we clearly know exists by Axiom 4. We know that $n \in A$ because we assumed that $\neg\varphi(n)$, which implies that $A \neq \varnothing$. Then, we can use Theorem 4.1 to conclude that there is a minimal element a in A.

Since we know that $\varphi(0)$, it follows that $a \neq 0$, so a must be a successor number. This means there is a $b \in \mathbb{N}$ such that $b + 1 = a$.

If $b \in A$, then that would mean that $a \leqslant b$ since a is minimal in A. However, we know that $b < b + 1 = a$, so we would then have $a \leqslant b < a$. $\lightning$

Therefore, $b \notin A$.

Q.E.D.

Let's practice induction by proving the following theorem.

Theorem 4.3 (Gaussian Summation Formula)**.**

$$(\forall n \in \mathbb{N})\left(\sum_{i=0}^{n} i = \frac{n(n+1)}{2}\right).$$

Proof.

We will prove the claim by induction on $n \in \mathbb{N}$.

Base Case:

Observe that $\sum_{i=0}^{0} i = 0 = \frac{0*(0+1)}{2}$. Therefore, the statement is satisfied at 0.

Inductive Step:

Let $k \in \mathbb{N}$ and assume $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$ (this is our *inductive hypothesis*). Now, observe

$$\begin{aligned}
\sum_{i=0}^{k+1} i &= \left(\sum_{i=0}^{k} i\right) + (k+1) && \text{by definition} \\
&= \left(\frac{k(k+1)}{2}\right) + (k+1) && \text{by the \textit{inductive hypothesis}} \\
&= (k+1)\left(\frac{k}{2} + 1\right) && \text{by factoring out } (k+1) \\
&= \frac{(k+1)(k+2)}{2} && \text{because } \frac{k}{2} + 1 = \frac{k}{2} + \frac{2}{2} = \frac{k+2}{2}.
\end{aligned}$$

Thus, we have that $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$, as desired.

Therefore, we can conclude that $(\forall n \in \mathbb{N})\left(\sum_{i=0}^{n} i = \frac{n(n+1)}{2}\right)$.

Q.E.D.