Problem Set 2

Discrete Mathematics

Due on the 4^{th} of February, 2024

We say that a propositional formula is a *tautology* if it is logically equivalent to $\top$ under any assignment of truth values to its variables.

You may rely on the following theorems throughout this problem set in addition to the axioms of classical logic.

- *Uniqueness of Negations*
- $\neg\top \equiv \bot$ and $\neg\bot \equiv \top$
- *Double Negation*
- *Idempotency*
- *Domination*
- *De Morgan's Laws*

(5 pts) 1. Consider the following proof of $p \to (q \to r) \equiv (p \to q) \to r$.

> ***Proof.*** Observe the following chain of reasoning.
>
> $$\begin{aligned} p \to (q \to r) &\equiv p \vee \neg(q \to r) && \text{by } \textit{conditional disintegration} \\ &\equiv p \vee \neg(q \vee \neg r) && \text{by } \textit{conditional disintegration} \\ &\equiv p \vee \neg q \vee \neg r && \text{by } \textit{associativity} \\ &\equiv (p \vee \neg q) \vee \neg r && \text{by } \textit{associativity} \\ &\equiv (p \to q) \vee \neg r && \text{by } \textit{conditional disintegration} \\ &\equiv (p \to q) \to r && \text{by } \textit{conditional disintegration} \end{aligned}$$
>
> Therefore, $p \to (q \to r) \equiv (p \to q) \to r$.
>
> Q.E.D.

Find all of the mistakes, if any, in this proof, and *explain why.*

(40 pts) 2. Prove the claims below *without truth tables* for all propositions p, q, r.

(a) $p \to q \equiv \neg q \to \neg p$.

(b) $(p \wedge (p \to q)) \to q$ is a tautology.

(c) $(\neg q \wedge (p \to q)) \to \neg p$ is a tautology.

(d) $(p \to q) \to ((p \to \neg q) \to \neg p)$ is a tautology.

(40 pts) 3. In this problem, we will progressively establish that the alternative axioms Hilbert proposed are all tautologies *without truth tables.*. Here, the variables p, q, and r all represent arbitrary propositions.

(a) Show $p \to p$ is a tautology.

(b) Show $(p \to q) \to (\neg q \to \neg p)$ is a tautology.

(c) Show $p \to (q \to p)$ is a tautology.

(d) Show $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ is a tautology.

(10 pts) 4. Show that $\neg$ and $\wedge$ are sufficient to express *any* proposition.

(5 pts) 5. Is there a *single connective* capable of expressing *any* proposition?[1] Justify your answer with a proof.

[1] This does not necessarily have to be one of $\neg$, $\wedge$, $\vee$, $\to$, nor $\leftrightarrow$. You can define new logical connectives using truth tables.