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DISCRETE MATHEMATICS

UNIVERSITY OF NOTRE DAME

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These notes are intended for students of CSE 20110 Discrete Mathematics at the University of Notre Dame.

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Contents

Notation

SYNTAX		SEMANTICS
$\top$	*"True."*	A *true* sentence; a tautology.
$\bot$	*"False."*	A *false* sentence; a contradiction.
$x := y$	*"x is, by definition, y."*	The name x has been assigned to the object referenced by y.
$x = y$	*"p equals q."*	p and q refer to the same object.
$p \equiv q$ $p \Leftrightarrow q$	*"p is equivalent to q."* *"p if and only if q."*	The sentence p is logically equivalent to the sentence q.
$p \vdash q$ $p \Rightarrow q$	*"p proves q."* *"p implies q."*	By assuming the sentence p, we can prove the sentence q.
$\varnothing$	*"the empty set"*	The set containing no elements.
$\{a, b, c\}$	*"the set containing a, b, and c"*	The collection containing only a, b, c.
$\{x \mid \varphi(x)\}$	*"the set of all x such that* $\varphi(x)$*"*	The collection whose elements are all possible objects x for which the sentence $\varphi(x)$ is *true.*
$\{x \in \mathcal{A} \mid \varphi(x)\}$	*"the set of all x in* $\mathcal{A}$ *such that* $\varphi(x)$*"*	The collection of all x from $\mathcal{A}$ for which the sentence $\varphi(x)$ is *true.*
$f : \mathcal{A} \to \mathcal{B}$	*"f is a function from* $\mathcal{A}$ *to* $\mathcal{B}$*."*	A function named f with domain $\mathcal{A}$ and codomain $\mathcal{B}$.
$f(x)$	*"f of x"*	The output of f on the input x, where $x \in \mathcal{A}$ and $f(x) \in \mathcal{B}$.
$\mathfrak{s}(n)$	*"The successor of n."*	The next natural number after n.
$\mathbb{N}$	*"enn"*	The set of natural numbers.
$\mathbb{Z}$	*"zee"*	The set of integers.
$\mathbb{Q}$	*"queue"*	The set of rational numbers.
$\mathbb{R}$	*"arr"*	The set of real numbers.
$\mathbb{P}(x)$	*"the power set of x"*	The set of all subsets of x.

Table 1: An overview of some important notation. Note that some expressions, like $p \equiv q$ and $p \vdash q$, have more than one equivalent notation. The middle column gives some common ways of *reading* each notation in English. The last column provides the *meaning* of each expression.

COLOR	INTERPRETATION
	Emphasis
	Pronunciation
	Definition
	External link
	Internal link

Table 2: Color legend.

MARK	MEANING
直覺	***idea***
公理	***axiom***
引理	***lemma***
定理	***theorem***
推論	***corollary***
定義	***definition***
演算法	***algorithm***

Table 3: Notation for organizing topics. These glyphs will be used to demarcate definitions, theorems, lemmas, *etc.*

GLYPH	NAME	IPA	GLYPH	NAME	IPA
$A\ \alpha$	*alpha*	[a]	$N\ \nu$	*nu*	[n]
$B\ \beta$	*beta*	[v]	$\Xi\ \xi$	*xi*	[ks]
$\Gamma\ \gamma$	*gamma*	[ɣ]	$O\ o$	*omicron*	[o]
$\Delta\ \delta$	*delta*	[ð]	$\Pi\ \pi$	*pi*	[p]
$E\ \varepsilon$	*epsilon*	[e]	$P\ \rho$	*rho*	[r]
$Z\ \zeta$	*zeta*	[z]	$\Sigma\ \sigma$	*sigma*	[s]
$H\ \eta$	*eta*	[ɛː]	$T\ \tau$	*tau*	[t]
$\Theta\ \theta$	*theta*	[θ]	$\Upsilon\ \upsilon$	*upsilon*	[yː]
$I\ \iota$	*iota*	[iː]	$\Phi\ \varphi$	*phi*	[f]
$K\ \kappa$	*kappa*	[k]	$X\ \chi$	*chi*	$[\mathrm{k}^{\mathrm{h}}]$
$\Lambda\ \lambda$	*lambda*	[l]	$\Psi\ \psi$	*psi*	[ps]
$M\ \mu$	*mu*	[m]	$\Omega\ \omega$	*omega*	[ɔː]

Table 4: The Greek alphabet. Each glyph in the alphabet is given first in upper-case and then in lower-case along with its English name and the IPA pronunciation.

GLYPH	NAME	IPA	GLYPH	NAME	IPA
א	*aleph*	[ø]	ל	*lamed*	[l]
ב	*bet*	[v]	ם	*mem*	[m]
ג	*gimel*	[ɣ]	ן	*nun*	[n]
ד	*dalet*	[ð]	ס	*samech*	[s]
ה	*he*	[h]	ע	*ayin*	[ʔ]
ו	*waw*	[v]	ף	*pe*	[f]
ז	*zayin*	[z]	ץ	*tsadi*	[ts]
ח	*chet*	[χ]	ק	*qof*	[k]
ט	*tet*	[t]	ר	*resh*	[ʁ]
י	*yod*	[j]	ש	*shin*	[ʃ]
ך	*kaf*	[x]	ת	*tav*	[θ]

Table 5: The Hebrew abjad. Only non-final variations of each glyph are shown.

Logic

0

Language

> *"No language is justly studied merely as an aid to other purposes. It will in fact better serve other purposes, philological or historical, when it is studied for love, for itself."*
>
> – J. R. R. Tolkien

images/elements.png

Figure 1: A fragment of book 2 from Euclid's *Elements* taken from the Oxyrhynchus papyri, dated ca. 100 AD.

We communicate our thoughts to others with the use of language. This is worth reflecting on. You are probably reading this because you have some interest in computation, mathematics, logic, or are incurably bored; the goal of these notes is—in part—to provide the mathematical background necessary to study these fields at a higher level. This is particularly true for aspiring *computer scientists*, who may have some misconceptions about their field because of its misleading name,[1] and who may not be aware that the field properly and historically falls under the grand umbrella of *mathematics*.

[1] *It's not about computers, nor is it science.*

This ambitious undertaking must therefore involve engaging with the tumultuous and violent history of mathematics. Although modern computer science is now richly interdisciplinary, the field was born during a particularly turbulent period in the late 19th and early 20th centuries AD[2] agitated by an existential crisis in mathematics: a crisis caused by our flagrant use of language. Here's a short summary.

[2] We will see later that its roots span at least to the time of Euclid in 300 BC.

0.1 *A Brief History of...*

The serious study of rhetoric—the art of argumentation and persuasion— as a subject in its own right dates back to at least the 5th century BC.[3] Around the 3rd century BC, Euclid's 13 books of the *Elements* heralded the birth of geometry, algorithmic computation, and the first theory of numbers,[4] where he *proved* certain statements followed from a list of *axiomatic* assumptions. This was a great achievement, establishing mathematical *proof* as a form of *argumentation* that logically deduces conclusions from a list of common assumptions. The contemporaneous Greek philosopher Theophrastus further pushed the envelope by describing the *form* of these arguments and establishing their validity.

[3] The time of the ancient Greek sophists, who were notably opposed by Socrates, Plato, and Aristotle.

[4] The only evidence of algorithms before this time—for multiplying, factoring, and finding square roots—dates back to Egypt and Babylon before 1600 BC.

The ancient Greeks laid the foundation for the two instrumental aspects of mathematical thought: *abstraction* and *argumentation*. Euclid abstracted what were thought to be the fundamental truths of geometry into a list of 12 *axioms*[1] so that, instead of thinking about *that* particular wall or *that* particular stick or *that* particular roof, he could make statements and observations about *quadrilaterals,* and *lines,* and *triangles* in general. These axioms were meant to encode the *universal truths* of geometry: the nature of what it fundamentally means to construct and measure distances, angles, and (simple) shapes. The last of these axioms would quickly become infamous.

axiom

[1] An *axiom* is a statement that we assume is true without justification nor proof.

Axiom (Parallel Postulate).
If two straight lines meet a third straight line making two interior angles that are each less than right angles, then the two lines—if they were to be extended—must intersect on that side of the interior angles. 公理

Figure 2: The parallel postulate says that any two lines and that make acute interior angles and with a third line must intersect at a point.

If you stop to think for a moment, this postulate says something very obvious. Assuming all of Euclid's other axioms, there are a few *equivalent* ways to restate the parallel postulate:

1. For any line L and point P not on L, there is exactly one line parallel to L passing through P.
2. The sum of interior angles in any triangle is 180 degrees.
3. A right triangle with side lengths A, B, C satisfies $A^2 + B^2 = C^2$.

You'll recognize this third statement as the Pythagorean *theorem*,[2] which is not merely an assumption![3] For the next 2000 years, the mathematical community was haunted by the thought that it was possible to *prove* the parallel postulate using the other axioms. It seemed like the rest of the axioms did such a perfectly good job of characterizing geometry that the parallel postulate *must necessarily* follow from the other axioms.

[2] A *theorem* is a statement that has a proof.

[3] The first two are called *Playfair's axiom* and the *triangle postulate* respectively.

Figure 3: Four views of the same triangle whose angles sum to 270 degrees. Notice how the notions of *straight* and *parallel* differ on the surface of a sphere.

However, between 1810–1832 AD, no less than *three* papers on *hyperbolic* geometry were published, and by 1854 Bernhardt Riemann had developed a theory of *Riemannian* geometry on manifolds. These were all different examples of consistent models of geometry that *denied* the parallel postulate! These ideas were intensely contested: many mathematicians and natural philosophers of the time refused to accept the notion that geometry could be non-Euclidean because *it went against their intuitive notion of how geometry should behave.*

This whole ordeal was only foreshadowing what would come at the turn of the century. In 1874, Georg Cantor would make a series of discoveries[4] surrounding the nature of infinity so fundamentally opposed to common mathematical thought that he would be antagonized and ostracized for decades, causing him to suffer serious depressive crises.

[4] We will study these later.

Once again, mathematicians' *intuitive notions of how infinity should behave* were being contradicted. Cantor's discoveries sparked not only a civil war within the mathematical community but also a concerted effort by many mathematicians and logicians in the early 20th century to *fix* mathematics by establishing it on a firm *logical foundation.*[1]

[1] This ambitious project would eventually fail with the discovery of Kurt Gödel's infamous *incompleteness theorems.*

The cause of all this turmoil was, fundamentally, a *lack of precision and rigor* in the way people would communicate mathematical ideas and arguments. What does it *mean* for a line to be straight, or for two straight lines to be parallel? What does it *mean* to have two lines, or to have infinitely many lines? What *is* infinity? Is infinity a number? What *are* numbers? How do we *know* we are saying anything *true* at all?

If we hope to answer any of these questions, we must first develop a language for *precise* mathematical communication. This necessarily begins with a systematic deconstruction and analysis of *language* itself.

0.2 *Syntax and Semantics*

Languages encode ideas into sequences of symbols.[2] These symbols represent objects, ideas, actions, and concepts. The *meaning* behind a particular cluster of symbols is called its *semantics*. The *form* the language takes, dictated by its *grammatical rules* for composing symbols into valid sentences, is called its *syntax*. We refer to objects by giving them names. A *variable* is a symbol[3] that stands in place for an object that has not been determined yet.[4] We can assign a name to a *particular* object with the $:=$ symbol. We call these the *terms* of an expression.

semantics

syntax

variable

term

[2] For our purposes, we will focus only on written—as opposed to spoken or signed—languages.

[3] We typically denote variables using single Latin or Greek letters, though there are no strict universal rules. Some common examples are listed below.

- $a, b, c, i, j, k, \ell, m, n, p, q, u, v, w, x, y, z$
- $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{G}, \mathcal{H}, \mathcal{M}, \mathcal{N}, \mathcal{R}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$
- $\alpha, \beta, \gamma, \delta, \epsilon, \eta, \theta, \lambda, \mu, \pi, \sigma, \tau, \varphi, \psi, \omega$

[4] A variable does not *necessarily* refer one particular object, or even any object at all.

Definition* 0.1 *(Sentences).

A *sentence* is the expression of a complete thought or idea in accordance with the syntactic and grammatical rules of a given language. A statement is called *atomic* if it can't be broken down into smaller semantic components in any way that obeys the language's syntax and grammar.

sentence

atomic

1. A *declarative* sentence is one that describes something. They typically consist of a *subject* being described and a *predicate* property it has.
2. An *interrogative* sentence asks a non-rhetorical question.
3. An *imperative* sentence heralds a command or request.

"Oft hope is born when all is forlorn."

"What has it got in its pocketses?"

"Keep your forked tongue behind your teeth."

定義

Mathematical practice principally involves *making and justifying observations about mathematical objects.*[5] As such, we are only really interested in crafting *declarative* sentences—sentences that describe *terms*. We will systematically deconstruct and analyse these kinds of sentences, extract their *logical essence*, and build up a new language.

[5] We leave the problem of *what* a mathematical object actually *is* for later.

0.3 *A Recurring Theme*

Before going any further, we should make a brief detour to discuss a topic that lies at the *heart* of computing, logic, and the 20th century foundational crisis in mathematics: *recursion*. In a very strong sense, what we *mean* when we say that some *thing* is *computable* is that there is a *recursive procedure* that produces that *thing*.

Idea (Church-Turing Thesis). We say something is *computable* if it is expressible as a *general recursive process*, is a *term in the λ-calculus*, or could described by a *Turing machine*. 直覺

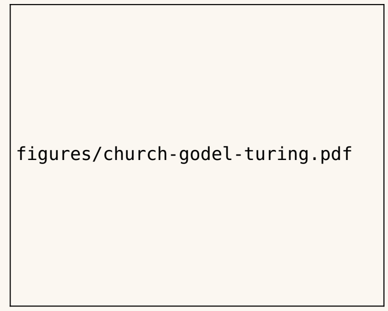

Figure 4: The Church-Turing thesis states that these three concepts—which are all *formally* equivalent—correspond with our *informal* notion of computability. In modern times, many people now take this as a definition for computability.

Actually, the three concepts described above are all *equivalent* to each other. It should then be no surprise that *recursion* (and its twin *induction*) will play a central role in our studies, so we will take this brief moment to quickly describe the fundamental idea at underlying recursion.[1]

[1] We leave Turing machines and the λ-calculus for a future time.

First, an example: how do we *compute* the sum of a list of n numbers?

$$3 + 5 + 9 + 2$$

With some hard work and determination and access to the internet, we can see that $3 + 5 + 9 + 2 = 19$, but *how* did we get that answer? At the most basic level, we started by taking two of the numbers, 3 and 5 say, computing their sum $3 + 5 = 8$, and adding this intermediate result to another number from the list, 9 say, to get $8 + 9 = 17$, and adding that again to yet another element of the list—in this case, only 2 remains—to finally arrive at $17 + 2 = 19$.

$$\begin{aligned}
3 + 5 + 9 + 2 &= 3 + 5 + 9 + 2 && (1)\\
&= 8 + 9 + 2 && (2)\\
&= 8 + 9 + 2 && (3)\\
&= 17 + 2 && (4)\\
&= 17 + 2 && (5)\\
&= 19 && (6)
\end{aligned}$$

This might seem so obvious it physically hurts, but let's analyse what we just did more closely. Suppose we have a list of n arbitrary numbers.[2]

[2] Notice that, being the sophisticates we are, we start counting at 0, so that a list of n numbers will be indexed starting at 0 and ending at $n - 1$.

$$x_0 + x_1 + x_2 + \cdots + x_{n-2} + x_{n-1}$$

Once again, we begin by taking the first two numbers and computing $x_0 + x_1$, then adding *this* result to x_2, then adding *that* result to x_3, then adding *that* result to x_4, and so on until we reach the end of the list. So, in order to compute $x_0 + x_1 + x_2 + \ldots x_{n-2} + x_{n-1}$, we *first* need to compute $x_0 + x_1 + x_2 + \ldots x_{n-2}$ and then add that result to x_{n-1}.

But wait, isn't $x_0 + x_1 + x_2 + \ldots x_{n-2}$ also the sum of a list? It is, it's just that the list has one less element! So *how do we compute the sum of elements in a list*? We first *compute the sum of elements in a list*, and then add one more element to that result. So, it seems like in order to do what we want, we need to already know how to do what we want; the key here is that we only need to know how to sum the elements of a *smaller* list in order to get the result we want for the *larger* list. As long as we can *eventually* get a result for one of these *"smaller"* sums, we will be able to build up a solution to our original problem by passing this result *"back up"* the chain of computation. Back to our first example.

$$\begin{aligned} 3+5+9+2 &= 3+5+9+2 && (1)\\ &= 3+5+9+2 && (2)\\ &= 3+5+9+2 && (3)\\ &= 8+9+2 && (4)\\ &= 17+2 && (5)\\ &= 19 && (6) \end{aligned}$$

Steps (1) through (3) continually decompose the given list into sublists on the left until we have no more lists we can break up. Each one of these lists is a smaller version of the original problem, and we compute the sums of these smaller lists by breaking them down and computing *their* sublists' sums, recombining these results at the end.

This paragraph describes the *recurrence.*

This now brings us to an important point: *we can't decompose* 3 *any further,* because this list only has one element in it. Do we know what the sum of all numbers in a list with one element is? Of course we do: it's just *that* number. Now we can return this result *back up* to the 5 that was waiting to be added to it, and when we add them together, we can return *that* result back to the 9 that was waiting, and then return *that* result to the 2 that was waiting, finally letting us conclude that the sum over the whole list is 19. The *recurrence relation* below summarizes this.[1]

This paragraph encounters the *basis.*

recurrence relation

$$\texttt{sum}(x_0, x_1, \ldots x_{n-1}) = \begin{cases} 0 & \text{if } n = 0 \\ \texttt{sum}(x_0, x_1, \ldots x_{n-2}) + x_{n-1} & \text{if } n \geqslant 1 \end{cases}$$

[1] Notice that this is actually written slightly differently than the procedure we've just described; think about *how* this is different and whether or not it actually computes the same result as the procedure we were just analysing.

We've exposed here a *recurrence* and a *basis*—the two key components underlying recursion (and, later, induction). The *recurrent* part of this procedure explains how to express a problem in terms of *"smaller"* instances of the *same problem,* describing how to combine the solutions to those subproblems into a solution for the original problem. Obviously, though, if you just keep decomposing problem into subproblems forever, you'll never be able to actually generate an answer to anything. Eventually, you need to *stop* and actually say what the answer to something is. The *basis*—*a.k.a.* *base case*—does exactly this by providing explicit answers to the *smallest* versions of the problem.

recurrence

basis

1

Zeroth-Order Logic

"The limits of my language means the limits of my world."
– Ludwig Wittgenstein

As we saw in the previous chapter, sentences can be broadly classified based on the kind of information they convey—their *functional role* in language. How do we begin deconstructing the descriptive fragment of our language? Naturally, we can think to classify the descriptive sentences by asking the fundamental question: *is this description true?*

images/moby-dick_v2_pg54.jpg

Figure 1.1: Illustration by Rockwell Kent from "Moby Dick: or, The Whale."

1.1 *Truth Values*

Let's consider the following declarative sentence.

$$\text{"Ahab is a captain."} \tag{1.1}$$

Here we have a descriptive sentence about the term *Ahab*—a man and thus an object of our discourse—asserting he *is a captain*. In the context of Herman Melville's *Moby Dick*, this is an accurate description. Referring to the above sentence as $\sigma_{1.1}$, we would then say $\sigma_{1.1}$ is *true*. We introduce the symbol $\top$ to denote these kinds of sentences.

true
$\top$

$$\text{"Ishmael is a whale."} \tag{1.2}$$

The above sentence, however, which we will name $\sigma_{1.2}$, immediately furrows the brow and strikes at the heart of our conscience. We know from the story that Ishmael is a sailor, and thus human, and therefore *not* a whale! We should then want to say that $\sigma_{1.2}$ is *false*, reserving the symbol $\perp$ for sentences of this kind.

false
$\perp$

The symbols $\top$ and $\perp$ are also sometimes called *"top"* and *"bot"* respectively.

The attributes *true* and *false* that we are attaching to these sentences are what we call *truth values*, and they are *the* essential component of the kinds of sentences we want to express. Sentences that are *true* all exhibit a quality that makes them similar to each other but dissimilar to *false* sentences, regardless what the actual sentences themselves *mean*

truth value

semantically. What we've just done is *abstract* the fundamental concept of truth value from descriptive sentences. This abstraction allows us to notice that *all true sentences are essentially the same as each other,* at least from the perspective of their truth values, with the same applying to *false* sentences. On the other hand, *true* and *false* sentences are complete opposites. This relationship inspires our first definition below.

Definition 1.1 (Propositional Equivalence).

equivalence
$\equiv$

We say that two sentences φ and ψ are *equivalent* when they have the same truth value. We denote this by writing $\varphi \equiv \psi$.[1] 定義

[1] *"φ is (logically) equivalent to ψ."*

Axiom (Propositional Equivalence is an Equivalence Relation).

We will take the following three properties to be *true* for any sentences φ, ψ, and ξ that are carriers of truth values.

1. $\varphi \equiv \varphi$. *reflexivity*
2. If $\varphi \equiv \psi$, then $\psi \equiv \varphi$. *symmetry*
3. If $\varphi \equiv \psi$ and $\psi \equiv \xi$, then $\varphi \equiv \xi$. *transitivity*

This establishes $\equiv$ is an example of an *equivalence relation.* 公理

With this new definition, we can formalize our observations from the preceding paragraph as $\sigma_{1.1} \equiv \top$ and $\sigma_{1.2} \equiv \bot$ as well as $\sigma_{1.1} \not\equiv \sigma_{1.2}$. Notice that each of these three expressions is a complete sentence describing properties[2] held by some objects.[3] In fact, these statements were themselves *true* declarative sentences. Now, let's ponder the following sentence, which we will call $\sigma_{1.3}$.

[2] being (or not) logically equivalent

[3] the sentences $\sigma_{1.1}$ and $\sigma_{1.2}$

$$\text{"Colorless green ideas sleep furiously."} \tag{1.3}$$

Like the previous examples, this is a grammatically correct, declarative sentence, but what does this sentence *mean?* Is it *true?* Is it *false?* Taking the normal English definitions for each of the words in this sentence, it doesn't seem to make any sense. We then clearly can't call it an accurate description of anything, so it can't possibly be *true.* Does that mean it must be *false?* Well, if we assume it is *false,* then what about the following sentence?

$$\text{"Colorless green ideas \textit{do not} sleep furiously."} \tag{1.4}$$

This one, which we will call $\sigma_{1.4}$, seems to be saying the opposite of whatever $\sigma_{1.3}$ was saying, so if the other one is *false,* then this one must be *true.* The question then becomes: what is $\sigma_{1.4}$ accurately describing? *This* sentence seems to make just as little sense as the original! This should lead us to conclude that $\sigma_{1.3}$ could not have been *false* either, so

nonsense

that sentence *has no truth value!* We call expressions like this *nonsensical* because they *carry no semantic meaning.*

Let's now analyse the following statement, which we will call $\sigma_{1.5}$.

$$\text{“This sentence is } \textit{false.}\text{”} \tag{1.5}$$

Expressed a little more *formally*, this is the sentence—named $\sigma_{1.5}$—that says $\sigma_{1.5} \equiv \bot$. This certainly doesn't seem like nonsense; it says something clear about a well-understood object. So, what is the truth value of this sentence? We can try reasoning about this like we did before by examining the two possible truth values the $\sigma_{1.5}$ can take.

First, let's assume $\sigma_{1.5}$ is *true*, which we write formally as $\sigma_{1.5} \equiv \top$. By definition, this would imply $\sigma_{1.5}$ is an accurate description of some object, so we should believe what the sentence says about that object. In this case, the object is $\sigma_{1.5}$ and the description is that $\sigma_{1.5} \equiv \bot$. This *contradicts* our initial assumption! ↯ Therefore, $\sigma_{1.5}$ is *not true!*[1]

[1] We conclude this because this is the opposite of our initial assumption, which lead us to a contradiction.

That rules out one truth value. What happens then if we assume $\sigma_{1.5}$ is *false?* Again, we can write this formally as $\sigma_{1.5} \equiv \bot$. By definition, this implies we should *reject* what $\sigma_{1.5}$ is asserting, leaving us with $\sigma_{1.5} \not\equiv \bot$. As before, a *contradiction* emerges! ↯ Therefore, $\sigma_{1.5}$ is *not false* either!

paradox

From this simple analysis, we can see that $\sigma_{1.5}$ *does not have a truth value!* Sentences that *contradict themselves* like this are called *paradoxes*.[2] In the preceding analysis, we relied on the idea that $\top$ and $\bot$ are opposed to each other, so that the same sentence can't meaningfully be both $\top$ and $\bot$ at the same time. This should be intuitive based on our natural understanding and usage of the words *true* and *false*, but we will make it a point to *formally* introduce this idea now.

[2] The word *paradox* is unfortunately overload and context-dependent. When referring to specific sentences, we will use it to specifically mean a self-contradictory sentence such as $\sigma_{1.5}$, but it is also commonly used in some contexts to refer to situations that are simply *unintuitive* rather than outright contradictory.

Axiom (Principle of Bivalence).
Sentences expressing truth values are either *true* or *false* but not both.
公理

What this analysis has hopefully shown us is that *not every* well-formed, declarative sentence expresses a truth value. In order for a sentence to express a truth value, it must satisfy the following three properties.

1. The sentence must be grammatically well-formed.
2. The sentence must be declarative.
3. The sentence must be semantically meaningful.

These are the kinds of statements are *eligible to carry a truth value*—the ones for which *it would make sense* to say they are either *true* or *false*—so they will form the foundation of our new language. We will eventually call these *propositions*, but beware that this is not (yet) a *formal* definition of what a proposition is. First, we need to get a better sense of *what* propositions are linguistically and *how* they are formed.

1.2 *Logical Connectives*

The examples of sentences we've seen so far have all been *atomic*—meaning they can't be broken down into simpler sentences that themselves are complete thoughts—but we can obviously express thoughts that are more than merely atomic. These *compounded* propositions are formed by taking smaller propositional sentences and *connecting* them together based on what our intended meaning is.

p	q	$\neg p$	$p \land q$	$p \lor q$	$p \to q$	$p \leftrightarrow q$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$

Table 1.1: A truth table summarizing the basic connectives of classical logic. The two left-most columns represent the *input* values of the propositions p and q. The remaining columns describe the *output* of each expression given the corresponding inputs on each row.

Each of these different ways of connecting sentences together suggests a different way of *transforming* between truth values by combining the truth values of the component propositions into a truth value for the compound expression.

logical connective

In this section, we will uncover these different transformations—which we will call *logical connectives*—and encode them using *truth tables*, which specify the output truth values for every combination of inputs.

Negations

Table 1.2: Truth table for negations.

p	$\neg p$
$\top$	$\bot$
$\bot$	$\top$

Suppose we encountered the following sentence, which we call $\sigma_{1.6}$.

$$\text{“Espresso is not delicious.”} \tag{1.6}$$

Immediately, the moral observer will realize the offensive absurdity of this sentence, compelled by the force of conscience to declare $\sigma_{1.6} \equiv \bot$! With this, we could simply carry on with our day; however, pausing to think for a moment, we can see that $\sigma_{1.6}$ is intimately related to the following (much more pleasant) sentence, which we call $\sigma_{1.7}$.

$$\text{“Espresso is delicious.”} \tag{1.7}$$

negation

This sentence is *clearly true*, letting us sigh $\sigma_{1.7} \equiv \top$ in relief. Not only that, it is the saying exactly the opposite of what $\sigma_{1.6}$ asserted! We call propositions like these *negations* of each other. This is our first example of a *transformation* of truth value: the negation of a proposition is another proposition with the opposite truth value. To denote this formally, we introduce the $\neg$ symbol, allowing us to write $\sigma_{1.6} \equiv \neg\sigma_{1.7}$.

$\neg$

We can now think of $\neg$ formally as a *unary function* that operates on truth values.[1] This function works by mapping $\neg\top$ to $\bot$ and by

[1] A function is *unary* if it takes only one input argument. We will study functions in more detail later.

mapping $\neg\bot$ to $\top$. This gives us a way of abstracting negations at the level of truth values, so that we can formally define what it means to *negate* a proposition. We provide this definition now in *table* 1.2, where the left-most column represents the inputs[1] to $\neg$ and the right-most column shows the truth values of the resulting output expression.[2]

[1] ... shown with white backgrounds ...

[2] ... shown with colored backgrounds ...

Conjunctions and Disjunctions

Table 1.3: Truth table for logical conjunctions and disjunctions.

p	q	$p \wedge q$	$p \vee q$
$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$

But we can obviously connect two (and sometimes more) sentences together to create larger sentences in English. For example,

"Espresso is delicious, and it nourishes the soul." (1.8)

This sentence is composed of two smaller atomic sentences, namely *"espresso is delicious"* and *"espresso nourishes the soul,"* which we know are both independently *true.* Connecting them together with the word *"and"* should then, based on the way this word works in English, produce another *true* sentence. Conversely, if either of the subexpressions had been *false,* the compound result should also be *false.* This *binary* connective is called the logical *conjunction*, and we denote it using the $\wedge$ symbol. It is defined in *table* 1.3.

conjunction
$\wedge$

There are several distinct ways this connective can appear in English that are nonetheless equivalent. Some examples are listed below.

"Espresso is delicious, *and* it nourishes the soul."
"Espresso is delicious *and* soul-nourishing."
"Espresso is delicious, *but* it nourishes the soul."
"Espresso is delicious, *yet* nourishing to the soul."
"Espresso is delicious; *further*, it nourishes the soul."
"*Although* espresso is delicious, it *also* nourishes the soul."

Table 1.4: These sentences are all logically equivalent to $\sigma_{1.8}$, though this list is obviously not exhaustive.

disjunction
$\vee$

The conjunction has a *logical dual* called the *disjunction*, defined in *table* 1.3 using the $\vee$ symbol and exemplified by the following sentence.

"Espresso is delicious, or it nourishes the soul." (1.9)

We call these connectives *dual* to each other because negating all of the inputs to one of them is equivalent to negating the output of the other.

Definition* 1.2 *(Logical Duality).

logical duality

We say two logical connectives f and g are *logically dual* if negating the inputs of f is always logically equivalent to negating the output of g. Equivalently, we can say f is *logically dual* to g if applying f after $\neg$ gives the same result as applying $\neg$ after g on *all possible* inputs. 定義

Conjunctions and disjunctions are just one example of a dual connective pair. In fact, every logical connective is dual to some other connective![1] For now, we present this result about $\wedge$ and $\vee$ *without proof*; we will *prove* this statement when we discuss *theorem* 1.5 in a short while.

[1] Why might this be? Think about this.

Table 1.5: Truth table for conditionals.

p	q	$p \to q$	$p \leftrightarrow q$
$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\top$	$\top$

Conditional Statements

We turn our attention now to sentence $\sigma_{1.10}$ below.

$$\text{“If espresso nourishes the soul, then I will drink it.”} \tag{1.10}$$

This is a *conditional* sentence, composed of two subclauses called the *antecedent* and the *consequent*.[2] When we use this sort of linguistic construction, we mean to say that *if* the premise happens, *then* the conclusion must also happen. Said another way: the conclusion must occur *whenever* the premise is satisfied. Notice *we are not asserting anything* about the antecedent or consequent individually! We are only establishing a *relationship* where the consequent occurs *every time* that the premise is satisfied. We call this the *material implication*, denoted by the $\to$ symbol and defined in *table* 1.5.

implication
$\to$

[2] Synonyms for *antecedent* & *consequent*.

protasis	apodosis
sufficient	necessary
premise	inference
assumption	conclusion
supposition	deduction
implicant	implicand
hypothesis	thesis

$$p_{1.10} := \text{“Espresso nourishes the soul.”}$$
$$q_{1.10} := \text{“I will drink espresso.”}$$

The antecedent and consequent for $\sigma_{1.10}$ are defined above. With these definitions, we can now write $\sigma_{1.10} \equiv p_{1.10} \to q_{1.10}$ and observe that $\sigma_{1.10}$ simply says: *if* $p_{1.10} \equiv \top$, *then* $q_{1.10} \equiv \top$. Importantly, this is *the only thing* that $\sigma_{1.10}$ is asserting! This sentence *is not saying* that if $p_{1.10} \equiv \bot$, then $q_{1.10} \equiv \bot$. In fact, if the premise is *false*, then $\sigma_{1.10}$ says *nothing* about whether or not $q_{1.10}$ is *true* or *false*.

To make this concrete, suppose I told you the following.

$$\text{“If you make an } \mathcal{A} \text{ in this class, then I will eat my shoe.”} \tag{1.11}$$

If you do happen to make an $\mathcal{A}$ in this class, then I'll be forced to physically eat my shoe in order to keep up my end of the bargain; in that case, the sentence was *true*.[3] On the other hand, if you make an $\mathcal{B}$ instead, then I can go home with both shoes and conscience intact; in this case, the sentence was also *true*.[4] However, what if you make the $\mathcal{B}$ but I decide to eat my shoe anyways? Did I lie? No; just because you failed to make an $\mathcal{A}$ doesn't mean I *can't* eat my shoe! All I said was that I definitely would if you made an $\mathcal{A}$.[5] That sentence is only a lie when you *do* make an $\mathcal{A}$ in the class, but I refuse to eat my shoe, since I really am breaking my promise then.[6]

[3] $\top \to \top \equiv \top$

[4] $\bot \to \bot \equiv \top$

[5] $\bot \to \top \equiv \top$

[6] $\top \to \bot \equiv \bot$

In *table* 1.6, we list several ways of verbalising $p \to q$ in English. Since this connective can be worded in so many unintuitive ways; careful attention must be paid to phrases involving conditionals.

"I will drink espresso *if* it nourishes the soul."
"Espresso nourishes the soul *only if* I drink it."
"It is *sufficient* that espresso nourish the soul for me to drink it."
"It is *necessary* that I drink espresso for it to nourish the soul."
"I will drink espresso *unless* it doesn't nourish the soul."

Table 1.6: These sentences are *all* logically equivalent to $\sigma_{1.10}$. Pay close attention to *grammar* of each sentence, and make special note of *where* the connectives appear.

biconditional
$\leftrightarrow$

Finally, the *material equivalence*,[1] also called the *biconditional* and written $p \leftrightarrow q$, is *true* exactly when p and q have the same truth value and is *false* otherwise. With these connectives all defined, we are now ready to formally introduce the *recursive definition* of a proposition.

[1] This is often written *"if and only if"* in English, abbreviated *iff* .

A Formal Proposition

Definition* 1.3 *(Proposition).

proposition

We say that λ is a *proposition* *iff* λ satisfies the following recurrence.

1. $\lambda = \top$ or $\lambda = \bot$.
2. $\lambda = \neg(\varphi)$, where φ is a proposition.
3. $\lambda = (\varphi) \wedge (\psi)$ where φ and ψ are propositions.
4. $\lambda = (\varphi) \vee (\psi)$, where φ and ψ are propositions.
5. $\lambda = (\varphi) \to (\psi)$ where φ and ψ are propositions.
6. $\lambda = (\varphi) \leftrightarrow (\psi)$, where φ and ψ are propositions.

定義

Notice the use of *equality* $=$ rather than *equivalence* $\equiv$ throughout this definition. In each statement here, we are saying that the statement λ is *equal* to the expression on the right-hand side of the $=$ symbol, meaning *they are the same sentence written in the same way.* This gives a *syntactic* definition of what a proposition is.
The use of parentheses in this definition is to avoid issues with order of operations; in situations where the meaning is clear, we can *carefully* drop parentheses.

This definition works by first establishing as our *basis* that $\top$ and $\bot$ are propositions in (1). We then, in (2) through (6), specify larger propositions *recursively* by composing together smaller, already-existing propositions using logical connectives. This lets us verify statements like $((\neg\top) \wedge (\bot \wedge \top)) \to \top$ are indeed propositions by recursively decomposing it until we reach the bases.

figures/proposition-recursive.pdf

Figure 1.2: In this example, we have dropped some unambiguous parentheses for clarity. Notice, however, that some parentheses *cannot* be dropped: for example, those around the premise of the $\to$ conditional, and those separating the arguments of the two $\wedge$ conjunctions. If those parentheses had been placed like $((\neg\top) \wedge \bot) \wedge \top$ instead, we would have parsed instead of as in the figure.

Alternatively, think of this as *inductive bootstrapping*.[1] Beginning with $\top$ and $\bot$ from (1) as our initial instances of propositions, we then build larger propositions like $\neg\bot$ and $\top \wedge \bot$, which fall into (2) and (3) respectively. We can then take those expressions, conjunct them again using (2), and place an implication between that result and $\top$ using (5) to arrive at our final expression $((\neg\top) \wedge (\bot)) \to \top$. By taking basis expressions and connecting them together according to the rules laid out in the definition, we *computed* a way of building the final expression in a way that satisfies the definition, verifying that it is a proposition.

[1] *"Pulling itself up by the bootstraps."*

`figures/proposition-inductive.pdf`

Figure 1.3: The inductive way of building up the expression, as contrasted with the recursive way of tearing down the expression in the previous figure.

Definition 1.4 (Propositional Formula).

propositional formula

A *propositional formula* is an expression that evaluates as a proposition when all of its *variables* are themselves replaced by propositions. 定義

Logical Equivalence

The astute reader may have noticed that some expressions are logically equivalent to each other even if they look different when written out.

p	q	$\neg(p \wedge q)$	$\neg p \vee \neg q$	$p \to q$	$\neg q \to \neg p$
$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\top$	$\top$	$\top$

Table 1.7: A truth table verifying two equivalences. First, that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are equivalent as predicted by DeMorgan. Second, that $p \to q$ is equivalent to its *contrapositive* $\neg q \to \neg p$.

For example, it's clear that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$, as the name *"if and only if"* would suggest. We saw another example of an equivalence when we examined the duality of $\wedge$ and $\vee$, illustrated in *table* 1.7. We can see that statements like these are logically equivalent because the output truth values are always the same whenever we assign the same input truth values to the variables in these expressions. In their joint truth table, the output columns for the two expressions are identical.

Equivalent propositions are essentially the same when we view them through the lens of truth values.[1]

Following this idea means having to construct a joint truth table whenever we want to check whether or not two formulæ are equivalent. Although it would be a straightforward to automate, doing all of our work by hand would be *extremely* tedious. If we are given two propositions $\varphi(p_1, p_2, \ldots p_n)$ and $\psi(p_1, p_2, \ldots p_n)$ consisting of the same variables, then answering $\varphi(p_1, p_2, \ldots p_n) \stackrel{?}{\equiv} \psi(p_1, p_2, \ldots p_n)$ requires computing truth values for φ and ψ with *all possible combinations* of truth assignments to $p_1, p_2, \ldots p_n$ and checking that they match.

[1] The idea of blurring the lines between objects that are *essentially the same* according to some salient characteristics is a fundamental idea in mathematics that shows up basically everywhere. This is, fundamentally, *why* abstractions are useful and interesting: we abstract in order to draw equivalences between things we previously thought of as distinct.

Now, p_1 can either be $\top$ or $\bot$. For each of these truth values, we then have check both truth values p_2 can take. Then, for each of those, we need to check the two truth values for p_3, and so on until we reach p_n. Each particular assignment of truth values to all of the propositional variables corresponds to one row in our truth table.

If $n = 1$, so our propositions each involve one variable, this means we only need two rows in our truth table to exhaust the entire search space: one row if the variable is $\top$, and one row if it's $\bot$. However, with each new variable we introduce, we *double* the size of our search space because this new variable comes with *two new possible truth values* that we need to check *for each* of the rows we've already computed. We summarize this phenomenon with the following *recurrence relation*.[2]

[2] The degenerate case of $n = 0$, when neither expression has any propositional variables, would just require one row in our truth table since each proposition only has one, unchanging truth value.

$$\texttt{rows}(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 2 \cdot \texttt{rows}(n-1) & \text{if } n \geqslant 2 \end{cases} \tag{1.12}$$

This shows us that answering the equivalence question for propositional formulæ of n variables involves computing a truth table with 2^n rows. Obviously, *this doesn't scale;* it quickly becomes infeasible to even *allocate enough space* for our output columns, much less actually compute and check these outputs. The thinking man's alternative is to instead *prove* that the two expressions are equivalent, constructing a formal, logical argument that derives $\varphi(p_1, p_2, \ldots p_n) \equiv \psi(p_1, p_2, \ldots p_n)$ from assumptions—called *axioms*—using *rules of inference.*

proof

axiom

Logical Nonequivalence

Showing that two propositional expressions are *not* equivalent is computationally easier than showing that they *are*. Checking that two propositional formulæ are equivalent involves either writing proof or computing *every row* of an exponentially sized truth table. However, checking that two formulæ are *not* equivalent requires *just one example* of a truth assignment on which the propositions disagree. Instead of an entire truth table, all we need is a *single row*.

p	q	$\neg(p \wedge q)$	$\neg p \wedge \neg q$
$\top$	$\top$	$\bot$	$\bot$
$\top$	$\bot$	$\top$	$\bot$
$\bot$	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\top$	$\top$

Table 1.8: A truth table showing negations do not distribute over conjunctions.

For example, to show that $p \to q \not\equiv q \to p$, all we have to do is let $p \coloneqq \top$ and $q \coloneqq \bot$. We can then observe that $p \to q \equiv \top \to \bot \equiv \bot$. Meanwhile, $q \to p \equiv \bot \to \top \equiv \top$. Thus, we conclude $p \to q \not\equiv q \to p$.

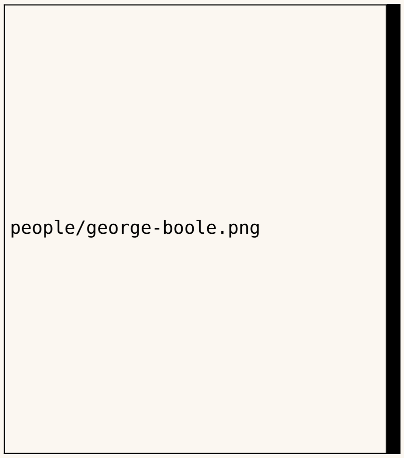

Figure 1.4: George Boole, a largely self-taught mathematician, logician, and philosopher, first described the eponymous *Boolean algebra* in his 1854 monograph *The Laws of Thought*.

Definition 1.5 (Logical Equivalence & Nonequivalence).

logical equivalence

$\equiv$

Let φ and ψ be propositional formulæ both consisting of the *same* variables $p_1, \ldots p_n$. We say that φ is *equivalent* to ψ if *every* assignment of truth values to the variables of φ and ψ produces the same truth value. In this case, we write $\varphi \equiv \psi$.

logical non equivalence

$\not\equiv$

We say that φ is *not equivalent* to ψ if *there is* an assignment of truth values to the formulæ's variables that makes the truth values of φ and ψ different. In this case, we write $\varphi \not\equiv \psi$. 定義

1.3 *The Propositional Logic*

Axioms and Proofs

The axioms of propositional logic encode the *foundational assumptions* we are making about the nature of truth-value-based reasoning. We take these truths to be self-evident *without justification*.

IDENTITY	$\top \wedge p \equiv p$	$\bot \vee p \equiv p$
COMPLEMENT	$\neg p \wedge p \equiv \bot$	$\neg p \vee p \equiv \top$
COMMUTATIVITY	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
ASSOCIATIVITY	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
DISTRIBUTIVITY	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
CONDITIONAL DISINTEGRATION		$p \to q \equiv \neg p \vee q$
BICONDITIONAL DISINTEGRATION	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$	

Table 1.9: The axioms of classical logic. The first five specify a *Boolean algebra*; notice that each of these first five axioms has a conjunctive fragment (left) and a *dual* disjunctive fragment (right).

Each of the statements in this table is a logical equivalence establishing that the two expressions are *interchangeable in all contexts.* We could verify each of these by constructing the appropriate truth table; however, the attitude we will take is that each statement in the table simply *is* *true a priori*, without any need for verification. Instead, they will form the *basis* upon which we build proofs of *other* statements.

The *complement* axiom in the second row of *table* 1.9 shows us two important facts about the negation of any proposition. If we take a proposition p and conjunct it with its negation $\neg p$, that axiom tells us that we get $\bot$; dually, disjuncting p with its negation gives us $\top$. Is this behavior *characteristic* of $\neg p$? The following theorem tells us *yes*, that any proposition that *behaves like* the negation of p must be *indistinguishable* from $\neg p$ through the lens of truth values! With that said, let's try to prove our first *theorem*.[1]

theorem

[1] A *theorem* is a provable proposition.

Theorem* 1.1 *(Uniqueness of Complements).
For any p and q, if $p \wedge q \equiv \bot$ and $p \vee q \equiv \top$, then $\neg p \equiv q$. 定理

Proof. Let p and q be arbitrary propositions.[2] Assume $p \wedge q \equiv \bot$ and $p \vee q \equiv \top$.[3] We will prove $\neg p \equiv q$ by showing that $\neg p$ and q are both equivalent to the same expression. First, observe the following.

[2] Since we need to prove this statement *for any two propositions* p and q, we introduce two *arbitrary* propositions at the beginning of our proof.

[3] These assumptions are warranted because they are the *premise* of the *conditional* statement we are proving.

$$\begin{aligned}
\neg p &\equiv \top \wedge \neg p && \text{by } \textit{identity} \\
&\equiv \neg p \wedge \top && \text{by } \textit{commutativity} \\
&\equiv \neg p \wedge (p \vee q) && \text{because we assumed } p \vee q \equiv \top \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge q) && \text{by } \textit{distributivity} \\
&\equiv \bot \vee (\neg p \wedge q) && \text{by } \textit{complement} \\
&\equiv \neg p \wedge q && \text{by } \textit{identity}
\end{aligned}$$

As a result, $\neg p \equiv \neg p \wedge q$. Similarly, we can now observe the following.

$$\begin{aligned}
q &\equiv \top \wedge q && \text{by } \textit{identity} \\
&\equiv q \wedge \top && \text{by } \textit{commutativity} \\
&\equiv q \wedge (p \vee \neg p) && \text{by } \textit{complement} \\
&\equiv (q \wedge p) \vee (q \wedge \neg p) && \text{by } \textit{distributivity} \\
&\equiv (p \wedge q) \vee (\neg p \wedge q) && \text{by } \textit{commutativity} \\
&\equiv \bot \vee (\neg p \wedge q) && \text{because we assumed } p \wedge q \equiv \bot \\
&\equiv \neg p \wedge q && \text{by } \textit{identity}
\end{aligned}$$

This gives us $q \equiv \neg p \wedge q$. Thus, we conclude $\neg p \equiv \neg p \wedge q \equiv q$. Q.E.D.

Q.E.D. stands for *Quod Erat Demonstrandum*, which is Latin for "what was to be shown has been demonstrated," after the Greek Ὅπερ ἔδει δεῖξαι. This is called a *tombstone*, and it is a traditional way of denoting the end of a proof. Modern authors might use □ or ■ instead.

Notice how *every* statement in the proof above is written with *purpose*, and much of the proof is inspired by *the form of the theorem* we are trying to prove. Let's analyze what just happened. Before we begin writing the proof, we first read the theorem focussing on two things: the *form* of the statement, and *what* the statement says.

First and foremost, this theorem says something about *any propositions.* We have two options for proving something is true about every single proposition: we can check all of them individually, or we can show that the thing we are trying to prove is an *inherent quality of being a proposition.* The former approach is clearly unworkable whenever we have infinitely many—or even just a large amount of—things to check, as we do here. Instead, we will take the later approach: by taking an *arbitrary* proposition and *making no assumptions, imposing no constraints,* then any argument we make about this particular proposition will also apply to any other proposition we encounter.[1] The first sentence of the proof introduces these two arbitrary propositions.

[1] As an example, suppose we wanted to prove that the square of any positive number is also positive. We obviously can't check all of the positive numbers one-by-one. Instead, we can take an *arbitrary* number x such that $x > 0$, and then argue that $x^2 > 0$. If we do this successfully, then we can take *any* particular number, such as 5, substitute it for x in our argument, and obtain a proof that $5^2 > 0$. However, if we couldn't have written our *original argument* in terms of 5; this would have meant imposing the *additional constraint* that $x = 5$, preventing our argument from generalizing to *all* positive numbers.

Now that we know we are proving something *universal* about propositions, we keep reading the theorem and see that it's a statement of the form *"if ___, then ___."* This is a *conditional* statement, and the most straight-forward way to show a conditional statement is *true* is to *demonstrate the conclusion is fulfilled whenever the premise is true.* Thus, we can *assume* the premise of the conditional is *true,* and our task then is to derive the conclusion. The second sentence of our proof assumes the premise, which happens to be a conjunction of two statements.

Up to this point, everything we've done has been determined solely by the *form* of the theorem we are trying to prove. Now, our task is to take what we have and show the conclusion.[2] What follows next is a sequence of logical statements, each of which is *justified,*[3] which ends at the conclusion we wanted. *How* you decide to craft this sequence of statements—what statements to make in what order, what proof techniques to use, what intuition inspired your approach—is entirely dependent on *your style* as long as all of the logic is clear, all of the logical rules are followed, and all of the justification is correct.

[2] If our conclusion were a longer, compound statement, we would continue breaking the problem down *recursively* until we were left with something *atomic.*

[3] ... either by a *definition*, an *axiom*, an *assumption* we've made, or a prior *theorem* we've proven ...

Proof-writing is an art form in much the same way building a musical instrument is. When a luthier makes a guitar, the process is guided by the particular luthier's traditions, experiences, style, and tastes; so long as the final product is truly a guitar that sounds and plays like a guitar should, the luthier has complete liberty. While two master luthiers might take radically different approaches that lead to guitars with unique aesthetic qualities, they will nonetheless produce two functioning guitars and preference of one over the other will be a matter of judgement and taste. This is much the same when it comes to writing proofs; the analogue to programming should be clear.

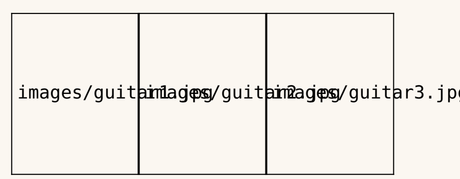

Figure 1.5: Examples of three distinct bracing styles for the classical guitar.

Since we proved *theorem* 1.1, we can now use this result in the future when proving more complicated statements. For example, it should be easy to see intuitively that $\top \equiv \neg\bot$ and $\bot \equiv \neg\top$, based on the way we use the words *true* and *false* in natural language and how $\top$ and $\bot$ are meant to correspond to those truth values. We can now prove this as a *corollary*—a simple consequence—of *theorem* 1.1.

***Corollary* 1.1.**
$\top \equiv \neg\bot$ and $\bot \equiv \neg\top$. 推論

Proof. Observe that $\bot \wedge \top \equiv \bot$ by the *identity* axiom. Similarly, we have that $\bot \vee \top \equiv \top \vee \bot \equiv \top$ by *commutativity* and the *identity* axiom again. So, we can apply *theorem* 1.1[1] and conclude $\top \equiv \neg\bot$. Similarly, we can observe that $\top \wedge \bot \equiv \bot \wedge \top \equiv \bot$ by *commutativity* and *identity*, and $\top \vee \bot \equiv \bot$ by the *identity* axiom. Thus, $\bot \equiv \neg\top$ by *theorem* 1.1. Q.E.D.

[1] We can invoke the theorem here because we have just *proven* the premises of the theorem are *true* for the particular propositions we are looking at (in this case, $p := \bot$ and $q := \top$). That means, having satisfied the premises, we get to assert the conclusion, justified by that theorem.

A proof gives us more than just a formal verification of a statement. It tells us that the statement is a *necessary consequence* of the axioms we assumed in setting up our logical system, and every instance of a proof gives us insight into *why* that's the case. These past two proofs show us that we didn't have to explicitly *define* or *assume* $\top$ to be the opposite of $\bot$ because this is a face satisfied by *any* instance of a Boolean algebra.

Let's prove another simple, but useful, theorem.

***Corollary* 1.2.**
For any propositions p and q, if $p \equiv q$, then $\neg p \equiv \neg q$. 推論

Proof. Let p and q be propositions such that $p \equiv q$ and observe.

$$\begin{aligned} q \wedge \neg p &\equiv p \wedge \neg p && \text{because we assumed } p \equiv q \\ &\equiv \bot && \text{by } \textit{commutativity} \text{ and } \textit{complement} \end{aligned}$$

We can do a very similar thing in the disjunctive case.

$$\begin{aligned} q \vee \neg p &\equiv p \vee \neg p && \text{because we assumed } p \equiv q \\ &\equiv \top && \text{by } \textit{commutativity} \text{ and } \textit{complement} \end{aligned}$$

Therefore, applying *theorem* 1.1, we conclude that $\neg p \equiv \neg q$. Q.E.D.

***Corollary* 1.3.**
For any propositions p, q, r, s such that $p \equiv q$ and $r \equiv s$, the following.

$$p \wedge r \equiv q \wedge s$$
$$p \vee r \equiv q \vee s$$
$$p \rightarrow r \equiv q \rightarrow s$$
$$p \leftrightarrow r \equiv q \leftrightarrow s$$

推論

We include *corollary* 1.3 above just for completeness, so that some of the basic properties of $\equiv$ are codified somewhere; their proofs are not particularly interesting. We are now ready to tackle the proof of a claim you probably find so obvious as to not even be worth mentioning.

Theorem* 1.2 *(Double Negation).
For any proposition p, we have that $p \equiv \neg\neg p$. 定理

Proof. Let p be a proposition. We will show p acts like the negation of $\neg p$. Observe $\neg p \wedge p \equiv p \wedge \neg p \equiv \bot$ by *commutativity* and the *complement* axiom. Similarly, $\neg p \vee p \equiv p \vee \neg p \equiv \top$ by *commutativity* and *complement*. Therefore, $p \equiv \neg(\neg p)$ by *theorem* 1.1. Q.E.D.

Theorem* 1.3 *(Idempotence).
For any proposition p, we have $p \wedge p \equiv p$ and $p \vee p \equiv p$. 定理

Proof. Let p be a proposition. For the conjunctive statement, observe.

$$\begin{aligned}
p \wedge p &\equiv \bot \vee (p \wedge p) && \text{by } \textit{identity} \\
&\equiv (p \wedge p) \vee \bot && \text{by } \textit{commutativity} \\
&\equiv (p \wedge p) \vee (p \wedge \neg p) && \text{by } \textit{complement} \\
&\equiv p \wedge (p \vee \neg p) && \text{by } \textit{distributivity} \\
&\equiv p \wedge \top && \text{by } \textit{complement} \\
&\equiv \top \wedge p && \text{by } \textit{commutativity} \\
&\equiv p && \text{by } \textit{identity}
\end{aligned}$$

An analogous chain of reasoning takes us through the disjunctive case.[1]

$$\begin{aligned}
p \vee p &\equiv (p \vee p) \wedge \top && \text{by } \textit{identity} \text{ and } \textit{commutativity} \\
&\equiv (p \vee p) \wedge (p \vee \neg p) && \text{by } \textit{complement} \\
&\equiv p \vee (p \wedge \neg p) && \text{by } \textit{distributivity} \\
&\equiv p \vee \bot && \text{by } \textit{complement} \\
&\equiv \bot \vee p && \text{by } \textit{commutativity} \\
&\equiv p && \text{by } \textit{identity}
\end{aligned}$$

Therefore, we have $p \wedge p \equiv p$ and $p \vee p \equiv p$ as desired. Q.E.D.

[1] Notice that we have combined some steps here involving *commutativity;* when it is clear, we can save some space by combining *commutativity* with the step directly proceeding it. We *do not* yet have the maturity to combine any other steps.

Theorem* 1.4 *(Domination).
For any proposition p, we have $\top \vee p \equiv \top$ and $\bot \wedge p \equiv \bot$. 定理

Proof. Let p be a proposition. We first prove the conjunctive fragment.

$$\begin{aligned}
\top \vee p &\equiv p \vee \top && \text{by } \textit{commutativity} \\
&\equiv p \vee (p \vee \neg p) && \text{by } \textit{complement} \\
&\equiv (p \vee p) \vee \neg p && \text{by } \textit{associativity}
\end{aligned}$$

$$
\begin{aligned}
&\equiv p \lor \neg p && \text{by } \textit{idempotence} \\
&\equiv \top && \text{by } \textit{complement}
\end{aligned}
$$

The disjunctive fragment works out similarly.

$$
\begin{aligned}
\bot \land p &\equiv p \land \bot && \text{by } \textit{commutativity} \\
&\equiv p \land (p \land \neg p) && \text{by } \textit{complement} \\
&\equiv (p \land p) \land \neg p && \text{by } \textit{associativity} \\
&\equiv p \land \neg p && \text{by } \textit{idempotence} \\
&\equiv \bot && \text{by } \textit{complement}
\end{aligned}
$$

We therefore conclude $p \lor \top \equiv \top$ and $p \land \bot \equiv \bot$. Q.E.D.

Theorem 1.5 (De Morgan's Laws).
$\neg(p \land q) \equiv \neg p \lor \neg q$ and $\neg(p \lor q) \equiv \neg p \land \neg q$ for any p and q. 定理

Proof. Let p and q be propositions. We will leave the proof of $\neg(p \lor q) \equiv \neg p \land \neg q$ as an exercise to the reader.

$$
\begin{aligned}
(p \land q) \land (\neg p \lor \neg q) &\equiv p \land (q \land (\neg p \lor \neg q)) && \text{by } \textit{associativity} \\
&\equiv p \land ((q \land \neg p) \lor (q \land \neg q)) && \text{by } \textit{distributivity} \\
&\equiv p \land ((q \land \neg p) \lor \bot) && \text{by } \textit{complement} \\
&\equiv p \land (\bot \lor (\neg p \land q)) && \text{by } \textit{commutativity} \\
&\equiv p \land (\neg p \land q) && \text{by } \textit{identity} \\
&\equiv (p \land \neg p) \land q && \text{by } \textit{associativity} \\
&\equiv \bot \land q && \text{by } \textit{complement} \\
&\equiv \bot && \text{by } \textit{domination} \quad \blacksquare
\end{aligned}
$$

In the conjunctive branch above, we derived $(p \land q) \land (\neg p \lor \neg q) \equiv \bot$. We show $(p \land q) \lor (\neg p \lor \neg q) \equiv \top$ in the disjunctive branch below.

$$
\begin{aligned}
(p \land q) \lor (\neg p \lor \neg q) &\equiv ((p \land q) \lor \neg p) \lor \neg q && \text{by } \textit{associativity} \\
&\equiv (\neg p \lor (p \land q)) \lor \neg q && \text{by } \textit{commutativity} \\
&\equiv ((\neg p \lor p) \land (\neg p \lor q)) \lor \neg q && \text{by } \textit{distributivity} \\
&\equiv ((p \lor \neg p) \land (\neg p \lor q)) \lor \neg q && \text{by } \textit{commutativity} \\
&\equiv (\top \land (\neg p \lor q)) \lor \neg q && \text{by } \textit{complement} \\
&\equiv (\neg p \lor q) \lor \neg q && \text{by } \textit{identity} \\
&\equiv \neg p \lor (q \lor \neg q) && \text{by } \textit{associativity} \\
&\equiv \neg p \lor \top && \text{by } \textit{complement} \\
&\equiv \top \lor \neg p && \text{by } \textit{commutativity} \\
&\equiv \top && \text{by } \textit{domination} \quad \blacksquare
\end{aligned}
$$

Therefore, by *theorem* 1.1, we conclude $\neg(p \land q) \equiv \neg p \lor \neg q$ as desired. Q.E.D.

Figure 1.6: Augustus De Morgan, after whom these laws are named, is also notable for his work on logical quantification and mathematical induction.

Rules of Inference

THE DEDUCTION RULE	$(p \vdash q) \vdash (p \to q)$	If, by assuming p, we can prove q, then we can write $p \to q$.
MODUS PONENS	$p, (p \to q) \vdash q$	If we have $p \to q$ and we know p, then we can deduce q.
MODUS TOLLENS	$\neg q, (p \to q) \vdash \neg p$	If we have $p \to q$ but also $\neg q$, then we can infer $\neg p$.
REDUCTIO AD ABSURDUM	$(\neg p \vdash q), (\neg p \vdash \neg q) \vdash p$	If $\neg p$ leads to a contradiction, then $\neg p$ is absurd; we conclude p.

Table 1.10: The rules of inference.

So far, we've developed a modestly-powerful formal language—capable of expressing some basic logical ideas—founded on *axioms*. This gives us a formal syntactic framework for expressing logical ideas, along with a basic semantics that relates these formal symbols to our natural language. The axioms in *table* 1.9 are all *equivalences*—substitution rules between propositions that preserve truth values.

Yet, you may have noticed that *some* of our reasoning in those proofs *was not based on equivalences.* This is most apparent in the proof of *theorem* 1.1, our very first theorem. We began that proof by introducing two arbitrary propositions and then *immediately assuming* that their conjunction was $\perp$ and their disjunction as $\top$. Making those assumptions was not justified on any of the equivalence axioms we'd introduced, so why were we allowed to say that in our proof? By a similar token, in the proof of *corollary* 1.1, we apply *theorem* 1.1 by saying that, since we'd satisfied the premises of that theorem, we were allowed to write down the conclusion of that theorem. Why were we allowed to say that? In short: *because it makes sense!* The problem, of course, is that nothing yet in our system *formally* gives us the right or power to do these things, even though they make logical sense. This then calls for the introduction of more axioms—ones that permit *one-way*, *inferential* arguments. We call these the *rules of inference*.

The rules in *table* 1.10 each take the form $\Gamma \vdash \varphi$,[1] where Γ represents a set of assumptions and φ is the conclusion that follows from them. The $\vdash$ symbol, sometimes called a turnstile, signifies that we can *prove* φ by assuming the statements in Γ and using the equivalence axioms, the rules of inference, and any theorems we've already proven. If there is nothing written to the left of the $\vdash$ symbol, this simply means that the conclusion φ can be derived *without* any additional assumptions.

prove

[1] *"Γ proves φ"* or *"φ follows from Γ."*

The most important of the rules of inference is *modus ponens*, enabling us to *follow through* on chains of conditional reasoning.[2] *Modus ponens* is, in a sense, the essence of classical rhetoric. Without it, the conclusion of a conditional statement's conclusion would not be meaningfully

[2] *Modus ponens* is short for the Latin phrase *modus ponendo ponens*, literally *"the method of putting by placing."*

conditioned on its premise. There would be no point in establishing hypothetical arguments because the conditional chains of reasoning would never actually have any point to work towards. This rule has a sister—*modus tollens*—which conversely allows *breaking down* arguments counterfactually, denying antecedents with false consequents.[1]

The next rule, named *reductio ad absurdum*,[2] gives us the ability to construct proofs by contradiction. Suppose we are interested in proving some proposition p. One way to reason about the validity of p is to think about what would happen if p were not the case. Hypothetically, assuming $\neg p$, if we were able to derive both q and also $\neg q$, then we would have derived a falsity ($q \wedge \neg q \equiv \bot$). If we were starting from *true* premises, this would be impossible since all of our axioms and rules of inference are *truth-preserving*. Clearly, this must mean that our assumption $\neg p$ was *not true*, leaving p as the only logical conclusion. This form of argumentation is like *"is like arguing with a hammer,"* according to a dear professor of mine from undergrad. It is incredibly powerful and has been in use since at least the year 400 BC.[3]

Finally, the *deduction rule* is a technical rule of inference that ties together the meta-symbol $\vdash$ with the logical $\rightarrow$ symbol. It enshrines the parallel between a deductive *"q follows from p"* statement and a formal *"if p then q"* statement. If this distinction is confusing, just keep in mind that we are constructing a formal language to express mathematical ideas with; the *propositions* we express are written in our language, but we write our *proofs* of these propositions in our natural language, and our natural language is what we use to write down the rules and axioms that our language must obey. The *deduction rule* tells us that the result of our *proofs* can be converted into statements *within the formal language.*

Although this is a rather small collection of rules, it is capable of representing any kind of expressible propositional rhetoric. Despite that, it's not a *minimal* set of rules for the zeroth-order logic. In fact, it's possible to have an even smaller set of rules without sacrificing the rhetorical strength of our language. *Modus tollens,* for instance, could actually be shown to follow from the other rules of inference as a *theorem*, reducing our total number of assumptions. Let's prove it now.

Theorem 1.6 (Modus Tollens).
We have $\neg q, (p \rightarrow q) \vdash \neg p$ for any propositions p and q. 定理

Proof. Let p and q be arbitrary propositions, and suppose $\neg q$ and also $p \rightarrow q$. We know that $p \rightarrow q \equiv \neg q \rightarrow \neg p$,[4] so we have $\neg q \rightarrow \neg p$. Then, by *modus ponens,* we can conclude $\neg p$. Q.E.D.

The interested reader might be excited to learn that *all* of propositional logic can be encoded using *just two connectives* ($\neg$ and $\rightarrow$) and *just three*

[1] *Modus tollens* is short for the Latin phrase *modus tollendo tollens,* literally *"the method of removing by taking away."*

[2] *Reductio ad absurdum* is a Latin phrase meaning to *"reduce to absurdity."* This has also been called *argumentum ad absurdum*.

[3] In Plato's dialogues, Socrates frequently engages in this sort of reasoning by showing his opponents' seemingly-sensible statements can be systematically dismantled to absurdity.

[4] This result—that a conditional statement is equivalent to its *contrapositive*—is left as an exercise to the reader.

axioms along with *modus ponens*. There are several classical *syllogisms* that have been studied since the time of the ancient Greeks. Before discussing these, we will first prove three important theorems.

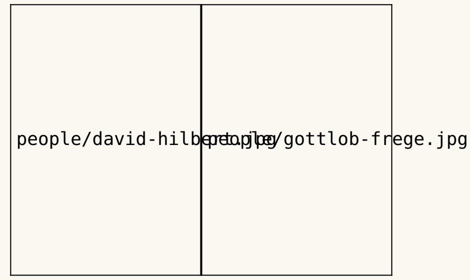

Figure 1.7: David Hilbert and Gottlob Frege were two of the most influential figures in the *logicist program* that was attempting to reduce mathematics to pure logic. Outside of logic, Hilbert was an extremely accomplished algebraist (maybe you've heard of Hilbert spaces in the context of linear algebra). Frege, while underappreciated during his life, is now recognized as one of the greatest and most profound mathematicians and philosophers of language of human history.

Hilbert's System

The logical system we've set up so far—the axioms that establish the propositional calculus as a Boolean algebra, and our comprehensive rules of inference—is very user-friendly, but for this reason it is not *minimal.* We could have made our logical system more *"elegant"*—in some eyes—by choosing a shorter list of axioms and relying on only one rule of inference, at the consequence of having *much uglier* theorems and substantially more tedious proofs. Nonetheless, there is still benefit to be had by studying one of these *more minimal* axiomatizations, as it will provide us invaluable insight into proving a very important theorem: *conjunction elimination.* This alternative axiomatization for the propositional calculus is attributed to Hilbert and Frege. A modern, more condensed version of their system can be written using only two axioms, which we now prove as theorems below.

Theorem 1.7 (Hilbert's First Axiom).
$\vdash \varphi \to (\psi \to \varphi)$ for any propositions φ and ψ. 定理

Proof. Let φ and ψ be arbitrary propositions and assume φ. Suppose ψ. We have φ by assumption. Thus, we have $\psi \vdash \varphi$ since we derived φ from ψ. By the *deduction rule*, we then obtain $\psi \to \varphi$. We now have $\varphi \vdash (\psi \to \varphi)$, since we derived $\psi \to \varphi$ from φ. Therefore, we conclude $\varphi \to (\psi \to \varphi)$ by the *deduction rule.* Q.E.D.

Theorem 1.8 (Hilbert's Second Axiom).
$\vdash (\varphi \to (\psi \to \xi)) \to ((\varphi \to \psi) \to (\varphi \to \xi))$ for any φ, ψ, ξ. 定理

Proof. Let φ, ψ, and ξ be propositions and assume $\varphi \to (\psi \to \xi)$. We want to show $(\varphi \to \psi) \to (\varphi \to \xi)$. Towards that goal, assume $\varphi \to \psi$. We now want to show $\varphi \to \xi$; so, towards this goal, assume φ. Now,

$$\varphi, (\varphi \to (\psi \to \xi)) \vdash \psi \to \xi$$

by *modus ponens* using our earlier assumption, so we obtain $\psi \to \xi$. Again, by applying *modus ponens* to our prior assumption, we see that

$$\varphi, (\varphi \to \psi) \vdash \psi$$

leaves us with ψ. We now take these two intermediate results to deduce

$$\psi, (\psi \to \xi) \vdash \xi$$

using *modus ponens.* Since we derived ξ from φ, we can assert $\varphi \vdash \xi$.

We now apply the *deduction rule* several times to arrive at the conclusion. From $(\varphi \vdash \xi)$, we deduce $(\varphi \rightarrow \xi)$. Next, from $((\varphi \rightarrow \psi) \vdash (\varphi \rightarrow \xi))$, we deduce $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \xi))$. Lastly, we take our expression $((\varphi \rightarrow (\psi \rightarrow \xi)) \vdash ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \xi)))$ and finally derive $((\varphi \rightarrow (\psi \rightarrow \xi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \xi)))$. Q.E.D.

Classical Syllogisms

We now follow in the footsteps of classical students of rhetoric, who in antiquity would ponder over these (and other) *syllogisms*—a traditional term referring to an argument where a conclusion is drawn from some collection of premises—as a way to hone our skills in the sister arts of proof-writing and deductive reasoning.

The following theorem allows us to construct and follow extended chains of conditional reasoning. Combined with *modus ponens,* this fundamentally forms the basis for any nontrivial argument.

Theorem 1.9 (Hypothetical Syllogism).
We have $(p \rightarrow q), (q \rightarrow r) \vdash p \rightarrow r$ for any propositions p, q, r. 定理

Proof. Let p, q, and r be arbitrary propositions, and suppose $p \rightarrow q$ and $q \rightarrow r$. We will first show that $p \vdash r$. Assume p. Since $p \rightarrow q$, we have q by *modus ponens.* Further, since we have $q \rightarrow r$, we get r by *modus ponens.* Thus, $p \vdash r$. Therefore, by applying the *deduction rule,* we can conclude $p \rightarrow r$. Q.E.D.

The next theorem is the converse of the *deduction rule.* When these two are taken together, they establish the formal, syntactic equivalence between the $\rightarrow$ and $\vdash$ symbols, which are semantically distinct.

Theorem 1.10 (Implication Elimination).
We have $(p \rightarrow q) \vdash (p \vdash q)$ for any propositions p and q. 定理

Proof. Let p and q be arbitrary propositions, and suppose $p \rightarrow q$. We will now show that $p \vdash q$. Assume p. Then, since we have $p \rightarrow q$, we can derive q by *modus ponens.* Thus, $p \vdash q$. Q.E.D.

Theorem 1.11 (Conjunction Introduction).
We have $p, q \vdash p \wedge q$ for any propositions p and q. 定理

Proof. Let p and q be arbitrary propositions. Assume p, and also separately assume q. Towards a contradiction, suppose $\neg(p \wedge q)$. [1] We can plainly see the following chain of equivalences.

$$\begin{aligned} \neg(p \wedge q) &\equiv \neg p \vee \neg q && \text{by \textit{De Morgan's laws}} \\ &\equiv p \rightarrow \neg q && \text{by \textit{conditional disintegration}} \end{aligned}$$

[1] When beginning a *proof by contradiction,* it is good form to explicitly alert the reader to this fact with a phrase like *"towards a contradiction."*

So, we have $p \to \neg q$, from which we can derive $\neg q$ by *modus ponens.* However, since already we had q, we now have a contradiction. $\lightning$ [1]

Therefore, we can conclude $p \wedge q$ by *reductio ad absurdum*. Q.E.D.

[1] The symbol $\lightning$ is useful in *proofs by contradiction* to highlight to the reader where the *contradiction* is and when it is reached.

In *table* 1.11, we summarize these results and some other theorems. We leave the proofs of these as an important list of exercises to the reader.

MODUS TOLLENS	$\neg q, (p \to q) \vdash \neg p$	
HYPOTHETICAL SYLLOGISM	$(p \to q), (q \to r) \vdash p \to r$	
IMPLICATION ELIM.	$(p \to q) \vdash (p \vdash q)$	*a.k.a. the consolidation rule*
CONJUNCTION INTRO.	$p, q \vdash p \wedge q$	*a.k.a. adjunction*
CONJUNCTION ELIM.	$p \wedge q \vdash p$	*a.k.a. simplification*
DISJUNCTION INTRO.	$p \vdash p \vee q$	*a.k.a. addition*
DISJUNCTION ELIM.	$(p \to r), (q \to r), (p \vee q) \vdash r$	*a.k.a. proof by cases*
EX FALSO QUODLIBET	$p, \neg p \vdash q$	*a.k.a. explosion*
CONSTRUCTIVE DILEMMA	$(\alpha \to \gamma), (\beta \to \delta), (\alpha \vee \beta) \vdash \gamma \vee \delta$	

Table 1.11: Some useful theorems.

2

First-Order Logic

"I am in a charming state of confusion."

– Ada Lovelace

The language we have described so far is often called the *classical logic*—since this is a modern development on Aristotelian logic—or the *propositional logic* because its basic syntactic unit is the proposition. Having the proposition as the most granular accessible referent helps keep this language manageable, but it will hold us back from being as expressive as we'd like to be. For example, suppose we are hungry, and in the course of our ruminations we discover that shepherd's pie is irresistibly delicious. We also happen to know the same thing about paella. Having recognized these facts, no simple substitute will do: we *must* have one of these two meals if we are to be satisfied at all. How might we express this logically? Let's introduce some definitions.

$$s := \text{“We eat \textit{shepherd's pie}.”}$$
$$p := \text{“We eat \textit{paella}.”}$$
$$n := \text{“We do not eat \textit{anything}.”}$$

The claim we are trying to express would formally look as follows.

$$(\neg s \wedge \neg p) \rightarrow n \tag{2.1}$$

From the syntax above, it doesn't seem like there is any relationship between the premise of that conditional statement and its conclusion. In fact, there doesn't even appear to be a relationship between s and p, even though they are both saying something really similar, because *syntactically* they just look like two distinct propositions! Suppose our friend felt the same way as we do about food, but he additionally knew about a *secret third food:* the tostada. Our friend might then resolve to have *that* meal as a fall-back if he can't get his hands on shepherd's pie or paella. He would let $t :=$ "We eat *a tostada*." and say the following.

$$(\neg s \wedge \neg p) \rightarrow t \tag{2.2}$$

images/shepherds-pie.jpg

English *shepherd's pie*, as God intended.

images/paella.jpg

The humble *paella*, national dish of Spain.

images/tostada.jpg

Tostada & café, a classic Cuban breakfast.

Now, despite our two claims having the *exact same syntactic form,* they express remarkably different ideas. To realize this, think about what it would take to prove (2.1): after verifying $\neg s$ and $\neg p$, we would then need to show we did not eat *any other food!* This is a *universal claim* we are making about *all* possible meals. However, our friend is not making this kind of claim: his conclusion is simply that *there exists* a particular meal he eats if $\neg s$ and $\neg p$ are satisfied. To prove himself right, he simply has to show that he ate that particular meal.

2.1 *A More Expressive Language*

It will quickly become frustrating for our language to limit our expressivity like this. The missing component in our language is the ability to distinguish the *object* of our speech from the *predicate* description we make about it when we declare a proposition.

$$\begin{array}{ll} & \text{Every man is mortal.} \\ & \text{Socrates is a man.} \\ \hline \therefore & \text{Socrates is mortal.} \end{array}$$

The argument above seems like a clear, sensible argument; it in fact looks like a simple application of *modus ponens.* Yet, we realize that a proof of this argument in the propositional logic could not actually invoke *modus ponens.* There is no way to symbolize the first sentence in such a way that we obtain a conditional $x \to y$ where the premise is *"Socrates is a man,"* and if we can't do that then we can't apply *modus ponens.* We fix this issue by augmenting our language with the ability to *syntactically* distinguish between *predicates* and the *terms* they describe.

Definition 2.1 (Term).

term

A *term* is a symbol denoting an object. Specific terms—*e.g.*, the natural number 5, Socrates, shepherd's pie—are called *constants*. Placeholder terms denoting objects that have not been specifically determined are called *variables*. Notice that *terms, on their own, do not form complete sentences! A term does not have a truth value!* 定義

Definition 2.2 (Predicate).

predicate

universe of discourse

Let $x_1, \dots x_n$ be variable symbols. We say $\varphi(x_1, \dots x_n)$ is an *n-ary predicate* if replacing each of the n variables $x_1, \dots x_n$ by terms $t_1, \dots t_n$ from our results in a *proposition* $\varphi(t_1, t_2, \dots t_n)$, carrying a truth value. The collection of all terms that our language has referential access to is our *universe of discourse.* 定義

We've now introduced a new problem into our language though. Suppose we have define the predicates $\mu(x) :=$ "x is a man" and

$\theta(x) :=$ "x is mortal" in an attempt to translate the previous argument. We can now translate the second premise and conclusion as μ(Socrates) and θ(Socrates) respectively. But we still can't translate the first line. For this, we need the ability to express *quantities*.

universal
$\forall$

Let $\varphi(x_1, \ldots x_i, \ldots x_n)$ be an n-ary predicate containing a variable x_i. The *universal quantification* of the variable x_i appearing in φ is denoted $\forall x\big(\varphi(x_1, \ldots x, \ldots x_n)\big)$ and says *any constant* replacing x will satisfy φ.

"For all x, $\varphi(x_1, \ldots x, \ldots x_n)$."

```
def forall(universe, predicate):
    for x in universe:
        if not predicate(x):
            return False
    return True
```

Figure 2.1: A hypothetical implementation of $\forall x(\varphi(x))$. If it returns `False`, then there is at least one `x` in `universe` such that `predicate(x) == False`, which is equivalent to $\forall x(\varphi(x)) \equiv \bot$. Otherwise, every `x` satisfies `predicate(x) == True`, meaning $\forall x(\varphi(x)) \equiv \top$.

existential
$\exists$

free variable

The *existential quantification* of x_i is denoted $\exists x\big(\varphi(x_1, \ldots x, \ldots x_n)\big)$ and claims that *there is at least one* constant that, in place of x, satisfies φ. The *scope* of a quantifier is denoted by parentheses specifying its variable's lifetime; that variable is *bound* to that quantifier within that scope. A variable that is not bound to any quantifier is called *free*. Statements with free variables *cannot have truth values*, they do not carry *meaning.* If a statement has free variables, those variables need to either be replaced by *terms*, or be bound to a *quantifier*. Because this will be useful in the

"There exists x such that $\varphi(x_1, \ldots x, \ldots x_n)$."

```
def exists(universe, predicate):
    for x in universe:
        if predicate(x):
            return True
    return False
```

Figure 2.2: A hypothetical implementation of $\exists x(\varphi(x))$. If `True` is returned, then there must be an `x` in `universe` such that `predicate(x) == True`, which is equivalent to $\exists x(\varphi(x)) \equiv \top$. Otherwise, every `x` satisfies `predicate(x) == False`, so that $\exists x(\varphi(x)) \equiv \bot$.

unique existential

$\exists!$

future, we also introduce the *unique existential quantification* of x_i as a way of saying that *there is exactly one* constant satisfying φ in place for x. We use the notation $\exists! x\big(\varphi(x_1, \ldots x, \ldots x_n)\big)$ to denote this, and read this in English as *"there exists a unique x such that $\varphi(x)$."*

$$\exists! x\big(\varphi(x)\big) \;:\Leftrightarrow\; \exists x\Big(\varphi(x) \wedge \forall y\Big(\varphi(y) \rightarrow (y = x)\Big)\Big).$$

This is a special case of existential quantification; using the unique existential quantifier means making an existential claim *and additionally* asserting that only one such example exists. So, we define the $\exists!$ quantifier *in terms of* the $\exists$ quantifier. Be careful to note that the ! symbol in $\exists!$ *does not correspond with negating anything!* Do not make the mistake of confusing ! with $\neg$ if you have experience with a programming language where the ! syntax corresponds to logical negation.

Forming Formulæ Well

Definition 2.3 (Formulæ).

atomic formula

We say a formula φ is *atomic* if it satisfies the following recurrence.

1. $\varphi = \top$ or $\varphi = \bot$.
2. $\varphi = \psi(t_1, \dots t_n)$, where ψ is an n-ary predicate, $t_1, \dots t_n$ are terms.

well-formed formula

We say λ is a *well-formed formula*—often abbreviated *wff*—if it satisfies the recurrence relation below.

1. λ is an atomic formula.
2. $\lambda = \neg(\varphi)$, where φ is a *wff*.
3. $\lambda = (\varphi) \wedge (\psi)$, where φ and ψ are *wff*.
4. $\lambda = (\varphi) \vee (\psi)$, where φ and ψ are *wff*.
5. $\lambda = (\varphi) \rightarrow (\psi)$, where φ and ψ are *wff*.
6. $\lambda = (\varphi) \leftrightarrow (\psi)$, where φ and ψ are *wff*.
7. $\lambda = \forall x\big(\varphi\big)$, where φ is a *wff*.
8. $\lambda = \exists x\big(\varphi\big)$, where φ is a *wff*.

sentence

A well-formed formula with no free variables is called a *sentence in the first-order logic.* Looking at the above definitions, a *wff* that has no free variables will boil down to a *proposition,* meaning it will have a definite, unambiguous truth value. Sentences will be our primary mode for expressing conjectures, theorems, and proofs. 定義

2.2 *Rules of Inference*

UNIVERSAL INTRODUCTION	$\varphi(t)$ for an *arbitrary* t $\vdash$ $\forall x\big(\varphi(x)\big)$	If we know $\varphi(t)$ and t is *arbitrary*, then we can say $\forall x\big(\varphi(x)\big)$.
UNIVERSAL ELIMINATION	$\forall x\big(\varphi(x)\big)$ $\vdash$ $\varphi(t)$ for *any term* t	If we have $\forall x\big(\varphi(x)\big)$, then we can pick *any* t and say $\varphi(t)$.
EXISTENTIAL INTRODUCTION	$\varphi(t)$ for a *particular* t $\vdash$ $\exists x\big(\varphi(x)\big)$	If we know $\varphi(t)$ for a *specific* term t, then we can say $\exists x\big(\varphi(x)\big)$.
EXISTENTIAL ELIMINATION	$\exists x\big(\varphi(x)\big)$ $\vdash$ $\varphi(t)$ for a *new term* t	If we have $\exists x\big(\varphi(x)\big)$, then we have $\varphi(t)$ for some t *that has not yet appeared*.

Table 2.1: The rules of inference for quantified expressions involving predicates. Note that the *"new term"* referred to by *existential elimination* must be a symbol that has not yet appeared in your proof.

When we were building the propositional logic, we first defined a *syntax* for our logic by introducing the logical connectives and some other special symbols; we then gave it an *algebraic semantics* when we introduced the equivalence axioms and the rules of inference. Now that we are augmenting our language with *terms, predicates, and quantifiers,* we have a similar need to establish semantics for interpreting our new symbols. We introduce these rules in *table* 2.1. In addition, we have three important theorems involving quantified expressions, each containing a *universal* fragment and an *existential* fragment. This first theorem establishes a form of *De Morgan duality* between the $\forall$ and $\exists$ quantifiers: *negating* a quantified sentence is equivalent to quantifying the *negated* sentence using the *other* quantifier.

Theorem 2.1 (Negation of Quantifiers).
If φ is a predicate of at most one free variable, these equivalences hold.

$$\neg\forall x\big(\varphi(x)\big) \equiv \exists x\big(\neg\varphi(x)\big) \qquad \neg\exists x\big(\varphi(x)\big) \equiv \forall x\big(\neg\varphi(x)\big)$$

定理

The next theorem illustrates a sort of *distributive law* for quantifiers. Be sure to *pay careful attention to the parentheses* in the following theorem.

Theorem 2.2 (Distribution of Quantifiers).
Let φ be a predicate of at most one free variable and p be a proposition. The four equivalences below are then satisfied; mind the parentheses.

$$\begin{array}{ll} \forall x\big(\varphi(x)\big) \wedge p \equiv \forall x\big(\varphi(x) \wedge p\big) & \exists x\big(\varphi(x)\big) \wedge p \equiv \exists x\big(\varphi(x) \wedge p\big) \\ \forall x\big(\varphi(x)\big) \vee p \equiv \forall x\big(\varphi(x) \vee p\big) & \exists x\big(\varphi(x)\big) \vee p \equiv \exists x\big(\varphi(x) \vee p\big) \end{array}$$

Further, if ψ is also a predicate with at most one free variable and t is a term, then the following four one-way inferences hold.

$$\begin{array}{ll} \forall x\big(\varphi(x) \wedge \psi(x)\big) \vdash \forall x\big(\varphi(x)\big) \wedge \psi(t) & \exists x\big(\varphi(x)\big) \wedge \psi(t) \vdash \exists x\big(\varphi(x) \wedge \psi(x)\big) \\ \forall x\big(\varphi(x) \vee \psi(x)\big) \vdash \forall x\big(\varphi(x)\big) \vee \psi(t) & \exists x\big(\varphi(x)\big) \vee \psi(t) \vdash \exists x\big(\varphi(x) \vee \psi(x)\big) \end{array}$$

However, those inferences above are *not* equivalences, as shown below.

$$\begin{array}{ll} \forall x\big(\varphi(x)\big) \wedge \psi(t) \nvdash \forall x\big(\varphi(x) \wedge \psi(x)\big) & \exists x\big(\varphi(x) \wedge \psi(x)\big) \nvdash \exists x\big(\varphi(x)\big) \wedge \psi(t) \\ \forall x\big(\varphi(x)\big) \vee \psi(t) \nvdash \forall x\big(\varphi(x) \vee \psi(x)\big) & \exists x\big(\varphi(x) \vee \psi(x)\big) \nvdash \exists x\big(\varphi(x)\big) \vee \psi(t) \end{array}$$

Finally, the following four equivalences hold for conditional statements.

$$\begin{array}{ll} \forall x\big(\varphi(x) \to p\big) \equiv \exists x\big(\varphi(x)\big) \to p & \forall x\big(p \to \varphi(x)\big) \equiv p \to \forall x\big(\varphi(x)\big) \\ \exists x\big(\varphi(x) \to p\big) \equiv \forall x\big(\varphi(x)\big) \to p & \exists x\big(p \to \varphi(x)\big) \equiv p \to \exists x\big(\varphi(x)\big) \end{array}$$

定理

The third and final theorem concerns the *order of quantifiers*, importantly pointing out that *quantifiers don't necessarily commute with each other*.

Theorem 2.3 (Quantifier Shift).
If φ is a predicate of at most two free variables, then the following hold.

$$\begin{array}{ll} \forall x \forall y\big(\varphi(x,y)\big) \equiv \forall y \forall x\big(\varphi(x,y)\big) & \exists x \exists y\big(\varphi(x,y)\big) \equiv \exists y \exists x\big(\varphi(x,y)\big) \\ \forall x \exists y\big(\varphi(x,y)\big) \nvdash \exists y \forall x\big(\varphi(x,y)\big) & \exists x \forall y\big(\varphi(x,y)\big) \vdash \forall y \exists x\big(\varphi(x,y)\big) \end{array}$$

定理

2.3 *The Art of Writing Proofs*

The way approach a proof of a statement principally depends on the *form* of the what we're trying to prove. Depending on what the statement *looks* like, a valid proof may be allowed to take certain liberties or be required to satisfy certain constraints. We will end this chapter with some words of advice for writing proofs based on the rules of inference we have established and the semantic interpretation we have attached to our various logical symbols. Since propositions and sentences in the first-order logic are *recursive* constructions, the first thing we should do when presented a statement to prove is to *recursively* analyze its *form.*

people/paul-erdos-thought.jpg

Figure 2.3: *"The purpose of life is to prove and to conjecture."* – Paul Erdős

Quantified Formulæ

If we are trying to prove a statement like $\forall x\big(\varphi(x)\big)$, we can *check* $\varphi(t)$ *for all possible values of* t. This is usually not possible, as our universe of discourse often contains infinitely many objects. The natural alternative is to *introduce an arbitrary term* t and, without making any assumptions about t, to show that t satisfies φ. If we manage to do this without relying on any details pertaining to t specifically, then our argument *will generalize universally.* On the other hand, to prove a statement of the form $\exists x\big(\varphi(x)\big)$, the task is to *find a specific object* t that we can prove satisfies φ. Existential claims are often the *most difficult* kind to prove because there is, generally, no clear strategy for *how* t should be found.

people/paul-erdos-cafe.jpg

Figure 2.4: *"Another roof, another proof."* – Paul Erdős

Conditional Statements

Suppose we have a statement we want to prove that takes the form of a conditional $p \to q$. These are *by far the most common* kinds of statements we will be interested in proving. This involves showing we can derive q from p, so we first *assume* p in order to get to q. After assuming p is the case, we can think of how to derive q based on its *form* by again going through this analysis. Alternatively, instead of showing $p \to q$ directly, we can always think to prove $\neg q \to \neg p$ and apply our knowledge that a conditional statement is always equivalent to its contrapositive.

Junctions

Statements that look like $p \wedge q$ are relatively straight-forward: we have to show that *both* p *and* q are true. Similarly, showing $p \vee q$ requires deriving *one of either* p *or* q, but we are free to choose which one to pursue. Naturally, this will depend on what forms p and q take.

Nonconstructive Proofs

When, in the course of human events, it becomes necessary for one people to encounter a *contradiction*, a decent respect to the opinions of mankind requires that they should *reject the assumptions* that impelled them there. What we mean by this is: if you are ever feeling like a proposition p is *obviously true*, but its proof feels insurmountable, try *assuming* $\neg p$ and seeing what happens. If this leads you to a *contradiction,* then you can invoke *reductio ad absurdum* and conclude p, washing your hands of the situation.

Ex falso quodlibet can be treated as a cousin to *reductio ad absurdum.* It is nowhere near as commonly used as a mode of reasoning, and to many it is far less intuitive than a simple proof by contradiction would be, but there are situations when it can be used to shortcut a proof in only a couple of lines. Keep an eye out for situations in which you are asked to prove a conditional statement $p \equiv \perp \to q$ with a *false premise* because this rule will let you immediately reach your conclusion.

Mathematics

3

Foundations

> *"Finally I am becoming stupider no more."*
>
> – Paul Erdős

With the development of the first-order logic, we finally have a formal language for rigorous communication. This language has several incredibly nice properties: it's sufficiently expressive to prove any *universal* truth, while not being so unwieldy as to admit falsehoods or contradictions. The development of the first-order logic—along with Gödel's completeness and incompleteness theorems—marks one of humanity's greatest intellectual achievements, which would have ramifications throughout nearly every field of philosophy and natural science. With this language in hand, we are now ready to embark on our studies of *mathematics* proper. The natural first question we have to answer is: *what is our universe of discourse?* What *are* mathematical objects?

Figure 3.1: Kurt Gödel was an absolutely monumental figure in mathematical logic. He famously showed all universal truths in the first-order logic are provable (a property known as *completeness*). Despite this, he then infamously demonstrated there are *mathematical* truths that *cannot* be proven (the *incompleteness* theorems).

3.1 *Informal Notions*

Thanks to insights made throughout the 20th and 21st centuries, there are actually several competing ways to answer this question (though the most *modern* and *"computer science"* of these formalisms would have required us to take a different logical foundation than the one we did). We will be taking a mainstream perspective that is fundamentally based on the concept of a *set*, but we will introduce two other useful kinds of objects in this section for convenience. *Technically* speaking, *every* object in our universe of discourse will be, or could be, *implemented* as a set, but it's often distracting to think of things like numbers as sets. As an analogy, think about the files on your computer. The PDF file you're reading these notes from is, fundamentally, a long binary number stored somewhere in your computer's memory. That number *represents* this PDF in the same way a set can represent a function, or the number 15, but if all you want to do is read these notes then it wouldn't be useful to interact with the binary *implementation* of the PDF.

Numbers

number

The most natural kinds of objects we should feel impelled to discuss are the *numbers*, and the most fundamental kind of number is, naturally, the *natural number.* Informally, these correspond precisely with the *non-negative whole numbers.* We can elegantly characterize these kinds of numbers with the following recurrence.

1. Zero is a natural number.
2. If n is a natural number, then $\mathfrak{s}(n)$ is also a natural number.

$\mathfrak{s}(n)$

In the above recurrence, the notation $\mathfrak{s}(n)$—read *"the successor of n"*—is referring to the *"next (whole) number after n."* This is the defining characteristic of the natural numbers, from which every other arithmetical property springs forth: begin somewhere (*i.e.*, at zero), and proceed by taking steps (*i.e.*, if n is a natural number, then so its the *next* one).

figures/numberline.pdf

Figure 3.2: An initial segment of the natural number line, which begins at zero.

These will be a very important class of object for us to talk about, so we introduce them into our universe of discourse here. For now, we will be philosophical Platonists in the sense that we will simply believe the natural numbers exist *"out there, somewhere, in the ideal platonic realm of forms."* After we develop a bit more theory, we will be able to be more concrete about *what* precisely a number *is* formally-speaking.

Functions

function

co/domain

A crucial part of the description of the natural numbers we just made is this notion of the *successor* of a natural number n. This idea is usually expressed in terms of the *successor function,* which begs us to define what a *function* is. For the moment, we will say a *function* is an object that *maps inputs from a domain* to *outputs in a codomain* in a *deterministic way.* Specifically, a function *must produce exactly one output* for each of its valid inputs—the output will not change unless the input changes.[1] If we have a function named f and a valid input x, then the notation we will use to denote the output value f realizes on the input x is $f(x)$.[2]

$$\forall x \forall y \big(x = y \Rightarrow f(x) = f(y)\big)$$

With this notation, we express this idea more formally above, taking note that the quantifiers range over the collection of valid inputs for f. We throw function into our universe for now and revisit this later.

[1] Think about this informal definition and see if it agrees with the kinds of things you have been calling *"functions"* throughout your life so far.

[2] *"f of x,"* or *"f at x."*

Sets

Since functions are maps that transform inputs into outputs, we are finally driven to ask *"inputs from where?"* All roads eventually lead to the idea of *a collection of things.* Functions map *collections of inputs to outputs.* Polygons are *collection of points.* Numbers measure the sizes of *collection of things.* In fact, any form of speech will find it hard to avoid invoking the concept of a *collection of things* eventually.

A notion of such *fundamental* importance to mathematics should therefore have a central place in our universe of discourse. In the same way binary numbers form a foundation for the files in your computer, we will be building our mathematical universe using *collections* as our fundamental unit of reference. We will call these collections *sets*, and refer to the objects they contain as their *elements.* For example, we might say that the number 0 is an element of the set of all natural numbers.

set

element

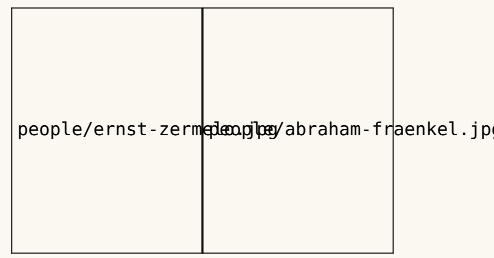

Figure 3.3: Ernst Zermelo (left) produced one of the first axiomatizations of set theory in 1908, which was augmented in 1922 by Abraham Fraenkel (right) and also, independently, by Thoralf Skolem.

As the most fundamental and basic object in our universe, we will study these first and *encode* their behavior in the form of *axioms.* Each axiom will incorporate some *intuitive property* that we would expect to be *true* about sets based on their inspiration as *"abstract collections of things."* This system of axioms—which we will study in the next section—is called *Zermelo-Fraenkel set theory*.

A Note on Notation

	ENTAILMENT	EQUIVALENCE
Language	$\rightarrow$	$\leftrightarrow$
Metalanguage	$\vdash$	$\equiv$
Mathematics	$\Rightarrow$	$\Leftrightarrow$

Table 3.1: With the rules of inference and the theorems in *table* 1.11, we recognize the equivalence between $\rightarrow, \leftrightarrow$ syntactically and $\vdash, \equiv$ semantically. To simplify our notation, we replace these symbols with $\Rightarrow, \Leftrightarrow$ respectively.

As a final note, *we will be simplifying our notation from this section forward*. We had previously been introduced to the symbols $\rightarrow$ and $\leftrightarrow$ for expressing conditional statements *within* the language of the first-order logic. In the *metalanguage*—the language we are using right now to *talk about* the formal system we built—we used the $\vdash$ symbol to denote that some conclusions are derivable from some premises, and we used $\equiv$ to denote that two statements were logically indistinguishable. Given the theorems we proved in the last few chapters, the line between these two classes of symbols has been made blurrier, and it's typical in mainstream mathematical practice to ignore this distinction entirely. So, we now introduce the symbol $\Rightarrow$ to denote entailment as a replacement for the $\rightarrow$ and $\vdash$ symbols. Similarly, we introduce $\Leftrightarrow$ as a replacement for $\leftrightarrow$ and $\equiv$, denoting logical equivalence in all contexts.

$\Rightarrow$

$\Leftrightarrow$

3.2 *Set Theory*

A *set* is an abstraction of the idea of a *collection of objects.* This idea, carried forward, naturally implies the need to communicate two kinds of relationships between objects: *equality* and *elementhood.* These will be the two basic predicate symbols of our *theory of sets.*

In order to identify objects that are the same, we introduce the binary *equality* predicate: given two objects x and y, we say $x = y$ precisely when x is *identical* to y. If you've seen the $=$ symbol before in your life, this is exactly the same symbol you're used to, and it has the natural properties you would expect of a predicate called *"equality."*

$=$

1. $\forall x(x = x)$ — *reflexivity*
2. $\forall x \forall y\big((x = y) \Rightarrow (y = x)\big)$ — *symmetry*
3. $\forall x \forall y \forall z\Big(\big((x = y \wedge y = z)\big) \Rightarrow (x = z)\Big)$ — *transitivity*

You'll notice that these are precisely the same three properties that *logical equivalence* had; these are both examples of equivalence relations. We will assume these three statements about equality axiomatically.

The second, and more interesting, predicate relates sets to the elements they contain. We call this predicate *elementhood* and denote it with the $\in$ symbol. These two predicates are enough to express anything we could possibly want about sets. As an example, suppose that $\mathcal{A}$ is a set. By saying $(0 \in A) \wedge (1 \in A)$, we are saying that $\mathcal{A}$ contains both 0 and 1 as elements, and by saying $2 \notin \mathcal{A}$ we claim that 2 is *not* an element of $\mathcal{A}$. However, saying $(0 \in \mathcal{A}) \wedge (1 \in \mathcal{A})$ doesn't prevent $\mathcal{A}$ from possibly containing *more* elements. If we wanted to say that $\mathcal{A}$ contains *only* the elements 0 and 1, we would have to assert that $0 \in \mathcal{B}$ and that $1 \in \mathcal{B}$, but we would also need to say $\forall x(x \in \mathcal{B} \Rightarrow (x = 0 \vee x = 1))$. This asserts that not only are 0 and 1 among the elements of $\mathcal{A}$, but that any element of $\mathcal{A}$ *must* be one of those two. Now, notice that we can rewrite $0 \in \mathcal{A} \wedge 1 \in \mathcal{A}$ as $\forall x((x = 0 \vee x = 1) \Rightarrow x \in \mathcal{A})$. So, saying that $\mathcal{A}$ contains *exactly* the elements 0 and 1 tells us that being 0 or being 1 is an *equivalent condition* for being an element of $\mathcal{A}$. We can write this as $\forall x(x \in \mathcal{A} \Leftrightarrow (x = 0 \vee x = 1))$. This would be a lot to write every time, so let's introduce some *notation.*

$\in$

Definition 3.1 (Set Notation).

$\{x_0, \ldots x_{n-1}\}$ Given finitely many terms $x_0, x_1, \ldots x_{n-1}$, we denote by $\big\{x_0, x_1, \ldots x_{n-1}\big\}$ the set whose elements are *exactly* the objects $x_0, x_1, \ldots x_{n-1}$. We write out each element of the set explicitly, separating the elements with commas, with the understanding that the following is *true* for any z.

"The set containing $x_0, x_1, \ldots x_{n-1}$."

$$z \in \big\{x_0, x_1, \ldots x_{n-1}\big\} \;:\Leftrightarrow\; (z = x_0) \vee (z = x_1) \vee \ldots (z = x_{n-1})$$

This is often called set builder notation. From *figure* 3.6, we can use this notation to say $\mathcal{A} = \{0, 0, 1, 2\}$ and $\mathcal{B} = \{0, 2, 1\}$ whereas $\mathcal{C} = \{0, 1, 3, 2\}$. Notice that set builder notation is extremely restrictive; it only lets us describe sets with *finitely many elements,* and it forces us to *write them all out.* What if we want to talk about a set with so many elements that it would be annoying—or impossible—to write them all down? How would we write down the set of *even* natural numbers, or the set of *prime* numbers, or even the set of natural numbers *itself*? To solve this problem, we introduce *set comprehension notation.*

$$z \in \{x \mid \varphi(x)\} \;:\Leftrightarrow\; \varphi(z)$$

With this notation, we can pick a predicate φ and refer to the collection of all those things that satisfy that predicate by writing $\{x \mid \varphi(x)\}$. In this way, we can refer to the set of even natural numbers by writing $\{x \mid x \in \mathbb{N} \land \text{"}x \textit{ is even}\text{"}\}$. We don't yet have a formal way of expressing "*x is even;*" once we do, $x \in \mathbb{N} \land$ "*x is even*" will be a predicate.

$\{x \mid \varphi(x)\}$

"The set of all x such that $\varphi(x)$."

定義

figures/set-subset.pdf

Figure 3.4: The *orange*, *red*, and *purple* sets are all subsets of the *yellow* set. We can see *purple* $\subseteq$ *orange*, but *orange* $\not\subseteq$ *purple*. Further, *orange* $\not\subseteq$ *red*, and *red* $\not\subseteq$ *orange*, implying *purple* $\not\subseteq$ *red*, and *red* $\not\subseteq$ *purple*.

The elementhood predicate is our fundamental relational symbol (apart from equality) between sets, but this predicate naturally implies another interesting relationship that two sets can share. For example, the blue set in *figure* 3.5 represents the set of all odd natural numbers, which we just learned can be written as $\{x \mid x \in \mathbb{N} \land \text{"}x \textit{ is odd}\text{"}\}$ using set comprehension notation. It should be pretty clear that every element of this set is a natural number. The same is true about $\{0, 1, 2\}$ and $\{0, 1, 2, 3\}$. Every element of each one of these sets is *also* an element of $\mathbb{N}$. Taking this further, the elements of $\{0, 1, 2\}$ are each $\{0, 1, 2, 3\}$. This emergent relationship is captured by the definition below.

Definition 3.2 (Subset).
Given two sets x and y, we say that *x is a subset of y*, denoted with the notation $x \subseteq y$, when every element of x is also an element of y.

$x \subseteq y$

$$x \subseteq y \;:\Leftrightarrow\; \forall z\big(z \in x \Rightarrow z \in y\big)$$

We can now see $\{x \mid x \in \mathbb{N} \land \text{"}x \textit{ is odd}\text{"}\} \subseteq \mathbb{N}$ and $\{0, 1, 2\} \subseteq \{0, 1, 2, 3\}$. However, $\{0, 1, 2, 3\} \not\subseteq \{0, 1, 2\}$ because $3 \in \{0, 1, 2, 3\}$ but $3 \notin \{0, 1, 2\}$.
定義

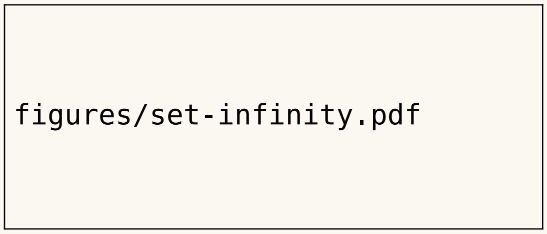

Figure 3.5: The set of *all natural numbers*, the smallest set $\mathbb{N}$ for which $0 \in \mathbb{N}$ and $\forall x(x \in \mathbb{N} \Rightarrow \mathfrak{s}(x) \in \mathbb{N})$, shown with the subset of *odd natural numbers*.

Infinity

We should keep one thing clear: these *definitions do not assert anything!* Just because we now have the ability to write something down with this new notation *doesn't* mean the notation *refers* to an existing object. To *formally* have sets to talk about, we need to introduce them with either an *axiom* or a *proof.* There is, of course, a set that has been looming over us this whole time—the set of *natural numbers*—that we certainly want to exist. Towards that goal, we introduce one more definition.

Definition 3.3 (Inductive Set).

inductive

We say a set $\mathcal{I}$ is *inductive* if $0 \in \mathcal{I}$ and $\forall x\big(x \in \mathcal{I} \Rightarrow \mathfrak{s}(x) \in \mathcal{I}\big)$. 定義

Axiom 0 (Infinity).

$\exists x\Big(x \textit{ is inductive} \wedge \forall y\big(y \textit{ is inductive} \Rightarrow x \subseteq y\big)\Big)$. 公理

The set described by *axiom* 0 is—in a sense—the *"smallest" inductive set,* which is precisely the set of natural numbers.[1] Therefore, *axiom* 0 establishes the *existence of the set of natural numbers*. Once this understanding is clear, it is common to make the following recursive declaration.

[1] The fact that the smallest inductive set is actually the set of natural numbers is a *theorem,* but we will not take the time nor effort to prove it here.

$$\mathbb{N} := \Big\{x \;\Big|\; x = 0 \vee \exists y\big(y \in \mathbb{N} \wedge x = \mathfrak{s}(y)\big)\Big\}$$

The rest of this chapter introduces *six more axioms* for set theory, each encoding a particular piece of *intuition* about *how sets should behave.*

Extensionality

Sets are entirely determined by their elements. Because sets abstract the idea of *a collection of objects*, everything we need to know about a set should determined by the elements it contains. We should expect that *looking inside* the and *comparing the elements* of two sets to answer the question *"are these two sets equal?"*

In *figure* 3.6, we have the sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. We can see that $0 \in \mathcal{A}$, $1 \in \mathcal{A}$, and $2 \in \mathcal{A}$, and we also have that $0 \in \mathcal{B}$, $1 \in \mathcal{B}$, and $2 \in \mathcal{B}$. Even though the elements appear with different frequencies and in different positions between the two sets, it must be that $\mathcal{A} = \mathcal{B}$ because they have all the same elements. However, we can see that $3 \in \mathcal{C}$ while

$3 \notin \mathcal{A}$, implying that $\mathcal{A} \neq \mathcal{C}$. In general, we should then expect sets to be equal precisely when they have the same elements, and that sets with the same elements should always be equal.

Axiom 1 (Extensionality).
$\forall x \forall y \Big((x = y) \Leftrightarrow \forall z (z \in x \Leftrightarrow z \in y) \Big)$. 公理

This relationship between $=$ and $\in$ is exactly what the *axiom of extensionality* encodes. In our example above, we can now use this axiom to *prove* that $\mathcal{A} = \mathcal{B}$ by showing that $\forall z(z \in \mathcal{A} \Leftrightarrow z \in \mathcal{B})$. In fact, this is essentially what we've done in the preceding paragraph; because $\mathcal{A}$ and $\mathcal{B}$ are both small, finite sets, by listing all the elements of each set and showing that they're all the same, we have a proof of $\forall z(z \in \mathcal{A} \Leftrightarrow z \in \mathcal{B})$. *Extensionality* tells us this means $\mathcal{A} = \mathcal{B}$.

figures/set-extensionality.pdf

Figure 3.6: A visual representation of two sets. The *purple* set has the same elements as the *red* set, the figures refer to the same set, letting us infer $\mathcal{A} = \mathcal{B}$. The *orange* set contains an element not present in the other two sets, so $\mathcal{C} \neq \mathcal{A}$ and $\mathcal{C} \neq \mathcal{B}$. The *yellow* set has no elements, so it is empty.

By the same token, if we wanted to show that $\mathcal{A} \neq \mathcal{C}$, we would need to show $\neg \forall z(z \in \mathcal{A} \Leftrightarrow z \in \mathcal{C})$. We decompose this statement below.

$$\begin{aligned}
\neg \forall z(z \in \mathcal{A} \Leftrightarrow z \in \mathcal{C}) &\equiv \exists z \neg (z \in \mathcal{A} \Leftrightarrow z \in \mathcal{C}) \\
&\equiv \exists z \neg \big((z \in \mathcal{A} \Rightarrow z \in \mathcal{C}) \wedge (z \in \mathcal{C} \Rightarrow z \in \mathcal{A}) \big) \\
&\equiv \exists z \big(\neg (z \in \mathcal{A} \Rightarrow z \in \mathcal{C}) \vee \neg (z \in \mathcal{C} \Rightarrow z \in \mathcal{A}) \big) \\
&\equiv \exists z \big(\neg (z \notin \mathcal{A} \vee z \in \mathcal{C}) \vee \neg (z \notin \mathcal{C} \vee z \in \mathcal{A}) \big) \\
&\equiv \exists z \big((z \in \mathcal{A} \wedge z \notin \mathcal{C}) \vee (z \in \mathcal{C} \wedge z \notin \mathcal{A}) \big)
\end{aligned}$$

So, what we would need to do is *find* an element z that's *in* one of the two sets but *not in* the other. Since we saw that $3 \in \mathcal{C}$ but $3 \notin \mathcal{A}$, that's exactly what it means for $\mathcal{A} \neq \mathcal{C}$ according to the *axiom of extensionality*.

Lemma 3.1.
$\forall x \forall y \big(x = y \Leftrightarrow (x \subseteq y) \wedge (y \subseteq x) \big)$. 引理

Proof. Let x and y be sets. Observe the following chain of equivalences.

$$\begin{aligned}
x = y &\Leftrightarrow \forall z (z \in x \Leftrightarrow z \in y) && \text{by } \textit{extensionality} \\
&\Leftrightarrow \forall z \Big((z \in x \Rightarrow z \in y) \wedge (z \in y \Rightarrow z \in x) \Big) \\
&\Leftrightarrow \forall z (z \in x \Rightarrow z \in y) \wedge \forall z (z \in y \Rightarrow z \in x) \\
&\Leftrightarrow (x \subseteq y) \wedge (y \subseteq x) && \text{by definition}
\end{aligned}$$

Therefore, $x = y \Leftrightarrow (x \subseteq y) \wedge (y \subseteq x)$. Q.E.D.

What about the set $\mathcal{D}$? In the figure, it would seem like $\mathcal{D}$ has no elements at all. *Extensionality* reveals to us that this means $\mathcal{D}$ cannot equal any of $\mathcal{A}$, $\mathcal{B}$, nor $\mathcal{C}$. In fact, $\mathcal{D}$ can't be equal to *any* set containing *any* elements because that set would contain something $\mathcal{D}$ doesn't.

Definition 3.4 (Empty Set).

empty

We say that a set x is *empty* *iff* $\forall y(y \notin x)$. We also define the following.

$$\varnothing := \{z \mid z \neq z\}$$

$\varnothing$

The referent of the $\varnothing$ symbol above is called *the empty set*. 定義

If we think of sets as *abstract containers*, it should be easy to conceptualize an empty container, which is exactly what $\varnothing$ would correspond to. With such a suggestive name, we should be able to say that $\varnothing$ is empty, right? Let's prove this as our first real theorem of set theory.

Theorem 3.1 (The Empty Set is Empty).
$\forall x(x \notin \varnothing)$. 定理

Proof. Let x be a set. Suppose, towards a contradiction, that $x \in \varnothing$. Then, we know $x \in \{z \mid z \neq z\}$ by definition of the empty set. This further tells us, by the definition of set comprehension notation, that $x \neq x$. However, we know $x = x$. $\lightning$ Therefore, $x \notin \varnothing$. Q.E.D.

Based on how $\mathcal{D}$ is drawn in *figure* 3.6, $\mathcal{D}$ empty since $\forall x(x \notin \mathcal{D})$. Does that mean that $\mathcal{D} = \varnothing$, or is it possible to have multiple distinct empty sets? As you might have guessed by what the *axiom of extensionality* says, *there is only one empty set* because all empty sets are equal to each other, justifying the name *the empty set* for $\varnothing$.

Theorem 3.2 (The Empty Set is Unique).
$\forall x\Big(\forall y\big(y \notin x\big) \Rightarrow x = \varnothing\Big)$. 定理

Proof. Let x be a set such that $\forall y\big(y \notin x\big)$. We will show x has all the same elements as $\varnothing$. Let z be a set. We will show $z \in x \Leftrightarrow z \in \varnothing$.

If $z \in x$, notice $z \notin x$ follows from $\forall y\big(y \notin x\big)$. Thus, $z \in \varnothing$ by *explosion*. If $z \in \varnothing$, then $z \neq z$ by definition; but, $z = z$. So, $z \in x$ by *explosion*.

Recall *ex falso quodlibet;* anything follows from a contradiction.

Thus, $\forall z(z \in x \Leftrightarrow z \in \varnothing)$. So, $x = \varnothing$ by the *axiom of extensionality*. Q.E.D.

It's important to note that none of the prior analyses nor theorems *prove* that $\varnothing$ exists, only that there can be at most one empty set. We will need to wait until the *axiom of separation* to discuss this.

As you may have guessed, the empty set is the *smallest* set in a precise sense. Given any two sets $\mathcal{X}$ and $\mathcal{Y}$, we can define an *ordering* by saying that $\mathcal{X}$ is *"less than"* $\mathcal{Y}$ in when $\mathcal{X} \subseteq \mathcal{Y}$. With this notion of ordering induced by the $\subseteq$ relation, we can see that the $\varnothing$ is *ordered below* every other set, making it *minimal* in the $\subseteq$ ordering among all sets. Since there is only one empty set, $\varnothing$ is the *minimum* of this ordering.

***Theorem* 3.3.**
$\forall x(\varnothing \subseteq x)$. 定理

Proof. Let x be a set. Towards a contradiction, suppose $\varnothing \nsubseteq x$. Then, there exists some z such that $z \in \varnothing \land z \notin x$ by definition. This implies $z \in \varnothing$; however, we know $\forall w(w \notin \varnothing)$. ↯ Therefore, $\varnothing \subseteq x$. Q.E.D.

We might be lead to ask: is there a *maximum* set with respect to this $\subseteq$ ordering? We will answer this question in a short while. In the meantime, this is not the only nice property of the *set inclusion* ordering induced by the $\subseteq$ relation. In fact, this relation has all the defining properties of a *partial order*: *reflexivity*, *antisymmetry*, and *transitivity.*

Theorem* 3.4 *(Set Inclusion is a Partial Order).
The following three statements hold about the $\subseteq$ relation.

1. $\forall x(x \subseteq x)$ *reflexivity*
2. $\forall x \forall y\big(((x \subseteq y) \land (y \subseteq x)) \Rightarrow x = y\big)$. *antisymmetry*
3. $\forall x \forall y \forall z\big(((x \subseteq y) \land (y \subseteq z)) \Rightarrow x \subseteq z\big)$. *transitivity*

This makes $\subseteq$ an example of a *partial order* on the class of sets. We prove the *reflexive* property below, leaving the rest as exercises. 定理

Proof. Let x be a set. Let z be a set and recall that $(z \in x) \Rightarrow (z \in x)$. Therefore, since z was arbitrary, we have $x \subseteq x$ by definition. Q.E.D.

Pairing

If you remember from our earlier discussion of set notation, we have a way of expressing *"the set containing a, b, and c"* by writing down $\{a, b, c\}$. However, just having the ability to *say* something doesn't make what we're saying *meaningful.* If we want to be sure that $\{a, b, c\}$ actually refers to an object that *exists,* then we will either need *proof* that it exists, or we'll need to rely on an *axiom* to grant us its existence. This next axiom partially addresses the problem with our set notation by guaranteeing that *set builder notation* always refers to an existing set so long as all of its elements also exist.

Axiom* 2 *(Pairing).
$\forall x \forall y \exists z(z = \{x, y\})$. 公理

figures/set-pairing.pdf

Figure 3.7: Given the sets $\{0,1,7\}$ and $\{2,3\}$ exist, the *pairing axiom* asserts the existence of $\{\{0,1,7\},\{2,3\}\}$.

By definition, we call a set a *singleton* if it contains exactly *one* element and a *doubleton* if it contains exactly *two* elements. The *pairing axiom* makes the straight-forward assertion that $\{x,y\}$ exists so long as x and y also exist—this set $\{x,y\}$ may be a *singleton* if $x=y$ or a *doubleton* if $x \neq y$. In other words, this axiom lets us construct *unordered pairs.*

Separation

We can similarly express *"the set of all things that make $\varphi(\cdot)$ true"* for any predicate φ by writing $\{x \mid \varphi(x)\}$, but again we have the same problem regarding the existence of the referent. If you think this obsession might be needlessly neurotic, let's take a moment to see what happens if we pretend that $\{x \mid \varphi(x)\}$ exists for any predicate we feel like; after all, it should be natural to say that *"the set of all x with some property"* exists if a set is simply an abstract collection of objects.

people/bertrand-russell.jpg

Figure 3.8: Russell's paradox is named after eminent mathematician and philosopher Bertrand Russell. He first mentioned this paradox in a letter to logician and philosopher Gottlob Frege as a critique of his *"Basic Law V,"* which was essentially an *unrestricted* form of comprehension for logical functions.

Define the predicate $\rho(s) :=$ "$s \notin s$." Just as a sanity check, remind yourself about $\mathcal{A}$ from *figure* 3.6 and notice that $\mathcal{A} \notin \mathcal{A}$ because $\mathcal{A}$ is not 0, 1, nor 2. This means $\mathcal{A}$ satisfies ρ, making $\rho(\mathcal{A})$ is *true*—the point here being that ρ is sometimes *true* for some sets. Let's consider $\mathfrak{R} := \{x \mid (x)\} = \{x \mid x \notin x\}$ and analyse the truth value of $\rho(\mathfrak{R})$.

If $\rho(\mathfrak{R})$ is the case, then $\mathfrak{R} \notin \mathfrak{R}$ by the definition of ρ. That means that $\mathfrak{R} \in \{x \mid \rho(x)\}$, implying $\mathfrak{R} \in \mathfrak{R}$. $\lightning$ What happens if $\neg\rho(\mathfrak{R})$ instead? Then, $\neg(\mathfrak{R} \notin \mathfrak{R})$, which simply says $\mathfrak{R} \in \mathfrak{R}$, implying $\mathfrak{R} \in \{x \mid \rho(x)\}$ by definition. However, this would mean $\rho(\mathfrak{R})$, so that $\mathfrak{R} \notin \mathfrak{R}$. $\lightning$

It seems like no matter what we do, we run into a problem. The mere *existence* of something like $\mathfrak{R}$ is *inherently contradictory*. We cannot allow things like $\mathfrak{R}$ to exist or they would introduce a contradiction into our system. This observation—that the *"set"* of all sets that don't contain themselves doesn't exist—is known as *Russell's paradox.*

Russell's paradox

This paradox stemmed from our reckless use of *unrestricted comprehension*. If we *restrict* comprehension only to *existing* sets, we avoid this issue. Instead of demanding $\{x \mid \varphi(x)\}$ always exists, we should *separate* off those elements satisfying φ from an *already existing set.*

Technically, *separation* is called an *axiom schema* because it is actually one axiom for each predicate φ. We can't write this as just one sentence because we can only quantify over *objects*, not *predicates.*

Axiom 3 (Schema of Separation).
For any predicate φ with at most one free variable, the following is *true.*
$\forall x \exists y \big(y = \{z \mid z \in x \wedge \varphi(z)\}\big)$. 公理

Power

Since the *axiom of separation* gives us the ability to take arbitrary subsets of existing sets, you would hope to be able to talk about the collection of *all* those subsets as its own set.

Definition* 3.5 *(Power Set).

power set

$\mathbb{P}(x)$

Given a set x, we define the *power set of x* to be the set of *all possible subsets* of x. We denote this by writing $\mathbb{P}(x) := \{z \mid z \subseteq x\}$. 定義

Remarkably, despite the litany of axioms we have so far, we don't actually have any *guarantee* that the power set of an arbitrary set exists! We need to introduce a whole new axiom to assert this fact.

Axiom* 4 *(Power).
$\forall x \exists y (y = \{z \mid z \subseteq x\})$. 公理

As a small example, consider the sets $\mathcal{G} := \{0, 1\}$ and $\mathcal{H} := \{2, 3, 5\}$. Their respective power sets are given below.

$$\begin{aligned}
\mathbb{P}(\mathcal{G}) &= \{\{\}, \{0\}, \{1\}, \{0, 1\}\} \\
&= \{\varnothing, \{0\}, \{1\}, \mathcal{G}\} \\
\mathbb{P}(\mathcal{H}) &= \{\{\}, \{2\}, \{3\}, \{5\}, \{2, 3\}, \{3, 5\}, \{2, 5\}, \{2, 3, 5\}\} \\
&= \{\varnothing, \{2\}, \{3\}, \{5\}, \{2, 3\}, \{3, 5\}, \{2, 5\}, \mathcal{H}\}
\end{aligned}$$

You'll notice that $\mathbb{P}(\mathcal{G})$ has 4 elements while $\mathcal{G}$ has 2, and $\mathbb{P}(\mathcal{H})$ has 8 elements while $\mathcal{H}$ has 3; this is no coincidence: power sets grow *exponentially* in the size of their input—hence the name *power* set.[1] You might also notice that $\varnothing$ and the set itself are each elements of the power sets in our example above; this generalizes to *all* sets.

[1] We will prove this interesting fact later.

***Lemma* 3.2.**
$\forall x (\varnothing \in \mathbb{P}(x) \land x \in \mathbb{P}(x))$. 引理

Union

So far, the we can only make new sets by pairing up existing sets using *axiom* 2, by taking subsets using *axiom* 3, and collecting all those subsets together using *axiom* 4. We would also like to *merge* two sets together, combining all of their elements all in one set.

Definition* 3.6 *(Union of Two Sets).

union

$x \cup y$

Given two sets x and y, we define the *union* of those two sets as $x \cup y := \{z \mid z \in x \lor z \in y\}$. This is the set consisting of all of the elements of x in addition to all of the elements of y together. 定義

figures/set-union-intersection-difference.pdf

Figure 3.9: In this figure, the *orange set* is $\{0,1,2,3,4,9\}$ and the *red set* is $\{3,5,6,7,8,9\}$. The *yellow set* is the *union* of the two sets, consisting of $\{0,1,2,3,4,5,6,7,8,9\}$. The *purple set* is their *intersection*, consisting of $\{3,9\}$. The *green set* consisting of $\{0,1,2,4\}$ is the difference $\{0,1,2,3,4,9\} \setminus \{3,9\}$.

Now, if we were to stop here and introduce an axiom along the lines of *"the union of two existing sets always exists,"* then we would only ever be able to take the union of *finitely many* sets.[1] Why should we limit ourselves like this? If we've reasonably gathered some amount of sets together, why shouldn't we be allowed to take the union of *all* of them together? Along the same lines, why not give ourselves the freedom to *iterate* the *"union operation"* over the elements of an arbitrary set?

[1] Convince yourself of this. How would you take the union of infinitely many sets if you're only allowed pairwise unions?

Definition 3.7 (Union Over a Set).

union over

$\cup x$

Given a set x, we define the *union over x*, meaning the *iterated union* over the elements of x, as $\cup x := \{z \mid \exists y(y \in x \wedge z \in y)\}$. 定義

You'll notice that the definition above takes a set and gathers the *elements of all of its elements* into a set by themselves. As an example, consider the set $\mathcal{J} := \{\{0,1,2\},\{3,\{5,7\}\},\{\{8\},9\}\}$. The union over $\mathcal{J}$ is then given by $\cup\mathcal{J} = \{0,1,2,3,\{5,7\},\{8\},9\}$. We will dedicate our next axiom to these kinds of iterated unions, asserting that *"the iterated union over the elements of an existing set exists."*

Axiom 5 (Union).
$\forall x \exists y(y = \cup x)$. 公理

Notice that the *union axiom* only asserts the existence of unions *over* sets that exist; it does *not* say that the union of *two* existing sets exists. It's up to us now to prove it for ourselves.

Theorem 3.5 (Existence of Unions).
$\forall x \forall y \exists z(z = x \cup y)$. 定理

Proof. Let x and y be sets. By the *pairing axiom*, $\tau := \{x,y\}$ exists. Then, we know that $\cup\tau$ exists by the union axiom, with the recognition that $\cup\tau = \{b \mid \exists a(a \in \tau \wedge b \in a)\}$ by definition. Recall that, by definition, $x \cup y = \{w \mid w \in x \vee w \in y\}$. We now witness the following for any z.

$$
\begin{aligned}
z \in \cup\tau &\Leftrightarrow z \in \big\{b \;\big|\; \exists a(a \in \tau \wedge b \in a)\big\} && \text{by definition of } \cup\tau \\
&\Leftrightarrow \exists a(a \in \tau \wedge z \in a) && \text{by definition} \\
&\Leftrightarrow \exists a(a \in \{x,y\} \wedge z \in a) && \text{by definition of } \tau
\end{aligned}
$$

$$
\begin{aligned}
&\Leftrightarrow \exists a((a = x \vee a = y) \wedge z \in a) && \text{because } \tau = \{x, y\} \\
&\Leftrightarrow z \in x \vee z \in y && \text{by } \textit{extensionality} \\
&\Leftrightarrow z \in \{w \mid w \in x \vee w \in y\} && \text{by set comprehension notation} \\
&\Leftrightarrow z \in x \cup y && \text{by definition of } x \cup y
\end{aligned}
$$

Thus, $\cup\tau = x \cup y$, so $x \cup y$ exists. Q.E.D.

The *schema of separation* synergizes well with the *union axiom*, allowing us to prove that many useful set-theoretic constructions are possible. Two important ones that we would be remiss to leave out are the *intersection* and the *difference* of two sets. If x and y are sets, then their *intersection* is the set of all elements they *share in common*. This is defined as $x \cap y := \{z \mid z \in x \wedge z \in y\}$. The *axiom of separation* easily guarantees us that $x \cap y$ always exists.

intersection

$x \cap y$

Theorem 3.6 (Existence of Intersections).
$\forall x \forall y \exists z(z = x \cap y)$. 定理

As with $\cup x$, we define what it means to iterate the *intersection over x*, collecting those things that are shared in common by *all* elements of x. We define this by $\cap x := \{z \mid \forall y(y \in x \Rightarrow z \in y)\}$. Although we *axiom* 5 tells us that iterated unions always exist, do not mistakenly presuppose that $\cap x$ should behave the same way! As an exercise, think about $\cap\varnothing$.

intersection over

$\cap x$

The *difference* of x and y is the set obtained by *removing* every element of y from x. This is bizarrely denoted $x \setminus y := \{z \mid z \in x \wedge z \notin y\}$, notation which we are not responsible for. As with unions and intersections of two sets, the difference of two arbitrary sets always exists.

set minus

$x \setminus y$

Theorem 3.7 (Existence of Differences).
$\forall x \forall y \exists z(z = x \setminus y)$. 定理

Regularity

You may have wondered by this point, either based on the problem sets or out of your own curiosity, whether or not sets can contain themselves as elements. You may even believe that, because of results like Russell's paradox, sets obviously can't contain themselves. While your intuition would be inline with mainstream mathematics and all of the physical intuition surrounding sets, there is actually nothing so far that would *formally* prohibit $x \in x$ to be *true* about some set x. As simple people—interested in doing *reasonable* and *mostly computable* mathematics—we should adopt the mainstream view that sets like $x = \{x\}$ and $\{x, y\} = \{\{y\}, \{x\}\}$ shouldn't exist.

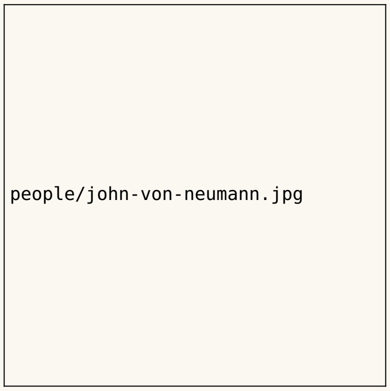

Figure 3.10: The axiom of regularity was introduced by John von Neumann to facilitate the study of the ordinal numbers. An important practical consequence of this axiom is that sets are not allowed to be elements of themselves.

Axiom* 6 *(Regularity).
$\forall x\big(x \neq \varnothing \Rightarrow \exists y(y \in x \land x \cap y = \varnothing)\big).$ 公理

This strangely written axiom has far-reaching consequences, one of which is that *there are no infinitely descending* $\in$*-chains.* For our purposes, we only need it to establish the fact that *sets do not contain themselves*.

Theorem* 3.8 *(Well-Foundedness of Elementhood).
$\forall x(x \notin x).$ 定理

Proof. Let x be a set and suppose, towards a contradiction, that $x \in x$. Consider $y \coloneqq \{z \mid z \in x \land z = x\}$, which exists by the *axiom of separation*. Observe that $y = \{x\}$ because $\forall z\big(z \in y \Leftrightarrow z = x\big)$. Since y is nonempty, the *axiom of regularity* tells us $\exists z(z \in y \land y \cap z = \varnothing)$.

$$\begin{aligned}\exists z\big(z \in y \land y \cap z = \varnothing\big) &\Leftrightarrow \exists z\big(z \in \{x\} \land \{x\} \cap z = \varnothing\big)\\ &\Leftrightarrow \{x\} \cap x = \varnothing\end{aligned}$$

This implies $y \cap x = \varnothing$. However, since $x \in y$ and $x \in x$, we know $x \in y \cap x$, so that $y \cap x \neq \varnothing$. $\lightning$ Therefore, $x \notin x$. Q.E.D.

It also prohibits the existence of *"the set of all sets,"* which we might refer to as the *"universal set"* and is typically denoted by $\mathcal{U} \coloneqq \{z \mid z = z\}$. This *"set"* would have the property $\forall x(x \in \mathcal{U})$, and it's a common feature of many *naïve* approaches to set theory, but it introduces many *formal* issues. We will see one such issue now.

Theorem* 3.9 *(The Universe Does Not Exist).
$\forall x\big(x \neq \{z \mid z = z\}\big).$ 定理

Proof. Suppose, towards a contradiction, that $\mathcal{U} \coloneqq \{z \mid z = z\}$ exists. Observe that $\mathcal{U} = \mathcal{U}$, so $\mathcal{U} \in \mathcal{U}$ by definition. However, $\forall x(x \notin x)$. $\lightning$ Therefore, $\{z \mid z = z\}$ does not exist. Q.E.D.

Another Note on Notation

We will introduce one last bit of incredibly convenient notation here. Given any set $\mathcal{X}$ and predicate φ, we have a more compact way of expressing "*$\varphi(x)$ for all x in $\mathcal{X}$*" and *"there exists x in $\mathcal{X}$ such that $\varphi(x)$."*

$$\begin{aligned}(\forall x \in \mathcal{X})\big(\varphi(x)\big) &:\Leftrightarrow \forall x\big(x \in \mathcal{X} \Rightarrow \varphi(x)\big)\\ (\exists x \in \mathcal{X})\big(\varphi(x)\big) &:\Leftrightarrow \exists x\big(x \in \mathcal{X} \land \varphi(x)\big)\end{aligned}$$

"For all x in $\mathcal{X}$, $\varphi(x)$."

"There is some x in $\mathcal{X}$ such that $\varphi(x)$."

Notice, when we say $(\forall x \in \mathcal{X})(\varphi(x))$, that this is *all one sentence.* We are *not* saying "$(\forall x \in \mathcal{X})$" nor "$(\varphi(x))$" nor any combination of those statements by themselves because these independent expressions are not sentences! *They do not mean anything by themselves!*

A statement like "$(\forall x \in \mathcal{X})$" is nonsense on its own because nothing is actually being said about the x elements of $\mathcal{X}$; there is no *clause* in this expression, so it's not a sentence. Similarly, "$(\varphi(x))$" would be nonsense *unless we know who x is;* sentences can't contain *free variables.*

$$z \in \{x \in \mathcal{X} \mid \varphi(x)\} \;:\Leftrightarrow\; z \in \mathcal{X} \wedge \varphi(z)$$

We finish by introducing, above, a compact analogue of the *restricted* set comprehension notation that *axiom* 5 facilitates. This new notation $\{x \in \mathcal{X} \mid \varphi(x)\}$ is read as follows: *"the set of all x in $\mathcal{X}$ such that $\varphi(x)$."*

3.3 *Functions*

Central to the history, tradition, and practice of mathematics is the concept of a *function*—is a special kind of *relation* between two sets in which *every* element of the first set *has* a *unique* corresponding element in the second set. We spoke about these intuitively in section 3.1, but it has come time to think about how to define these within set theory.

Suppose we have two sets $\mathcal{A}$ and $\mathcal{B}$. A function from $\mathcal{A}$ to $\mathcal{B}$ establishes an associating between the elements $a \in \mathcal{A}$ and the elements $b \in \mathcal{B}$ in a way that corresponds intuitively with our notions of *input* and *output* respectively. If we wanted to pair up these inputs with their corresponding outputs, we might first think to construct the unordered pair $\{a, b\}$; however, it should be clear that is fails to represent which element of $\{a, b\}$ was the *input* and which one was the *output*, since $\{a, b\}$ and $\{b, a\}$ are indistinguishable in set theory. We need a way of establishing sets in which the *order* of the elements also matters.

To distinguish them from unordered pairs, we will denote an *ordered pair* using $(\cdot, \cdot)$ parentheses instead of $\{\cdot, \cdot\}$ brackets. Two ordered pairs (x_1, y_1) and (x_2, y_2) should be equal *iff* all of their *corresponding coordinates* are equal in all the same positions.

$$(x_1, y_1) = (x_2, y_2) \quad \Leftrightarrow \quad (x_1 = x_2 \wedge y_1 = y_2)$$

This is the *characterization* of ordered pairs; any definition or implementation using sets that we come up with *must* enforce this relationship, or it wouldn't really capture what we *mean* by *"ordered pair."* The following definition given by Kazimierz Kuratowski accomplishes precisely this.

Definition* 3.8 *(Ordered Pair).

ordered pair
(x, y)

Given sets x and y, we define the *ordered pair* whose first coordinate is x and second coordinate is y as $(x, y) := \{\{x\}, \{x, y\}\}$. 定義

***Lemma* 3.3.**
$\forall a \forall b \forall x \forall y \Big(\big((a, b) = (x, y)\big) \Leftrightarrow \big(a = x \wedge b = y\big) \Big).$ 引理

This gives us a way of associating the elements of two sets by constructing sets of ordered pairs whose coordinates are elements of each respective set. Thus, a *relation* between $\mathcal{A}$ and $\mathcal{B}$ is nothing more than a particular set of ordered pairs (a, b) where $a \in \mathcal{A}$ and $b \in \mathcal{B}$. More precisely, we say $\mathcal{R}$ is a *relation between $\mathcal{A}$ and $\mathcal{B}$* when $\mathcal{R} \subseteq \{(a, b) \mid a \in \mathcal{A} \land b \in \mathcal{B}\}$ for any two sets $\mathcal{A}$ and $\mathcal{B}$. The *largest* relation between two sets is the set of *all* such possible ordered pairs. This important construction—named in honor of René Descartes—is defined below with its own dedicated notation.

relation

Definition 3.9 (Cartesian Product).
The *Cartesian product* of two sets x and y is the set of *all possible ordered pairs between them.* Formally, $x \times y := \{(a, b) \mid a \in x \land b \in y\}$. 定義

set product
$x \times y$

Importantly, the Cartesian product of any two sets always exists. This conveniently means that whenever we are interested in relating the elements of two sets—or of constructing a function between two sets—we won't have to worry about existence questions thanks to the *axiom of separation* because it will simply be a subset of the Cartesian product.

Theorem 3.10 (Existence of Cartesian Products).
$\forall x \forall y \exists z (z = x \times y)$. 定理

A function, as we previously motivated, is a special kind of relation: one in which *every element of the domain* has a *unique image in the codomain*. This means that a function f from $\mathcal{A}$ to $\mathcal{B}$ should, first and foremost, be a *relation* $f \subseteq \mathcal{A} \times \mathcal{B}$. Then, we should impose the special condition on the ordered pairs $(a, b) \in f$ that, *for every $a \in \mathcal{A}$*, there always *exists exactly one $b \in \mathcal{B}$* such that a is paired up with b in f.

Definition 3.10 (Function).
Given sets $\mathcal{X}$ and $\mathcal{Y}$, we introduce the notation $f : \mathcal{X} \to \mathcal{Y}$ to indicate that *f is a function from $\mathcal{X}$ to $\mathcal{Y}$*. We define what this means below.

$f : \mathcal{X} \to \mathcal{Y}$

$$f \subseteq \mathcal{X} \times \mathcal{Y} \quad \land \quad (\forall x \in \mathcal{X})(\exists! y \in \mathcal{Y})\Big((x, y) \in f\Big)$$

The sets $\mathcal{X}$ and $\mathcal{Y}$ are called the *domain* and *codomain* of f respectively. When we know that f is a function, we can replace the ordered pair notation above with the traditional *functional* notation below.

$f(x) = y$

$$f(x) = y \;:\Leftrightarrow\; (x, y) \in f$$

This convenient notation lets us rewrite the right-hand side of our definition as $(\forall x \in \mathcal{X})(\exists! y \in \mathcal{Y})(f(x) = y)$. 定義

3.4 *Lifting the Veil*

Equipped with the axioms of set theory, we are now ready to discover *who* the natural numbers *really are* with, of course, a recursive definition.

$$0 := \varnothing$$
$$\mathfrak{s}(n) := n \cup \{n\}$$

We begin by establishing that *the first natural number* is *the empty set*. We then obtain the *successors* of zero by iteratively *adding one new element* to the previous natural number. If we apply this definition, we can compute that the natural number 1 is actually the set containing 0.

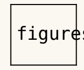

Figure 3.11: The natural 0 as a set.

$$1 := \mathfrak{s}(0) = 0 \cup \{0\} = \varnothing \cup \{\varnothing\} = \{\varnothing\} = \{0\}$$

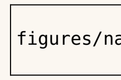

Figure 3.12: The natural 1 as a set.

A similar computation reveals that 2 is the set containing both 0 and 1.

$$2 := \mathfrak{s}(1) = 1 \cup \{1\} = \{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\} = \{0, 1\}$$

Figure 3.13: The natural 2 as a set.

If we continue this process, you'll start to notice a pattern emerging.

$$\begin{aligned}
0 &= \varnothing && = \{\} \\
1 &= \{\varnothing\} && = \{0\} \\
2 &= \{\varnothing, \{\varnothing\}\} && = \{0,1\} \\
3 &= \Big\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\Big\} && = \{0,1,2\} \\
4 &= \Bigg\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \Big\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\Big\}\Bigg\} && = \{0,1,2,3\} \\
&\vdots && \vdots
\end{aligned}$$

Figure 3.14: The natural 3 as a set.

This characterization results from $0 = \varnothing$ and the following theorems.

Theorem 3.11 (Successor Function has no Fixed Points).
$\forall x\big(x \neq x \cup \{x\}\big)$. 定理

Theorem 3.12 (Every Natural Number is Transitive).
$(\forall x \in \mathbb{N})(\forall y \in x)(\forall z \in y)(z \in x)$. 定理

$m < n$

These two facts show us *every natural number is the set of all the natural numbers that came before it*. This lets us define $(m < n) :\Leftrightarrow (m \in n)$ for any natural numbers $m, n \in \mathbb{N}$,[1] inspiring the following notation.[2]

$$[\![n]\!] := \{x \in \mathbb{N} \mid x \in n\} = \{x \in \mathbb{N} \mid x < n\}$$

We can clearly see that $n = [\![n]\!]$ for any $n \in \mathbb{N}$. While this might seem like useless notation at first, it will be useful in the future when we need to make a natural number *as a set* and *as a number*. It should be less confusing if we use notation like $m + n$ when treating them like numbers and $[\![m]\!] \cup [\![n]\!]$ when treating them like sets.

[1] We say $m \leqslant n$ if $(m < n) \vee (m = n)$.

[2] Note that this notational definition *only applies to natural numbers.*

4

Arithmetic

"Don't for heaven's sake, be afraid of talking nonsense! But you must pay attention to your nonsense."

– Ludwig Wittgenstein

4.1 The Categorical Structure of Arithmetic

Now that we know *who* the natural numbers are, we'd like to be able to *use* them for something, so we need to understand their basic structure and behavior. First, let's remind ourselves of an obvious fact.

$$0 \in \mathbb{N}.$$

Secondly, the successor of any natural number is also a natural number.

$$(\forall n \in \mathbb{N})(\mathfrak{s}(n) \in \mathbb{N}).$$

However, zero being the *first* natural number means *it has no predecessors.*

$$(\forall n \in \mathbb{N})(0 \neq \mathfrak{s}(n)).$$

Further, numbers are *equal* precisely when they have *the same successor.*

$$(\forall n, m \in \mathbb{N})\Big((\mathfrak{s}(n) = \mathfrak{s}(m)) \Rightarrow (n = m)\Big).$$

Finally, and most importantly, *every natural number can eventually be reached by starting at zero and iteratively finding successors*. This gives us a remarkably powerful way to prove statements about the naturals.

As a note: it is not difficult to show that the reverse direction of this statement is also *true*, but it is much less interesting than the forward direction given here.

$$\Big(\varphi(0) \wedge (\forall k \in \mathbb{N})\Big(\varphi(k) \Rightarrow \varphi\big(\mathfrak{s}(k)\big)\Big)\Big) \Rightarrow (\forall n \in \mathbb{N})(\varphi(n))$$

Given a predicate φ, the above statement proclaims that $\varphi(x)$ is *true* about every natural number x if we first know $\varphi(0)$ is *true* and then, whenever $\varphi(k)$ is *true* for an arbitrary $k \in \mathbb{N}$, the statement $\varphi(\mathfrak{s}(k))$ about the next natural number is *induced* into being *true* as well. This is known as *mathematical induction*, a concept spiritually dual to *recursion.*

induction

Each of the aforementioned statements about $\mathbb{N}$ is a theorem of set theory, and we have taken great care in setting up our axioms and definitions so that this would be the case. Although some parts of this journey may have felt delicate, arbitrary, or contrived, the remarkable fact of the matter is that these five rules establish a *canonical representation* for the natural numbers *as an idea.* Not only do the natural numbers have these properties, but *any structure or representation or system* that has these five properties *encodes* a copy of the numbers $0, 1, 2, \ldots$ as we humans have known them our whole lives. *Any structure that looks like the natural numbers must act like the natural numbers.*

At the end of the day, the specific choices we made to *implement* the natural numbers set-theoretically were fundamentally unimportant. What matters is that we *have* a representation of $\mathbb{N}$ so that we can reason about them formally. The following section will define many of the operations on $\mathbb{N}$ you may be familiar with, but you should keep in mind that any definitions we make—any theorems we prove—about $\mathbb{N}$ will also be true about *anything that looks like* $\mathbb{N}$.

Definition* 4.1 *(Addition & Multiplication).
The two basic algebraic operations on $\mathbb{N}$ are *addition* and *multiplication*.

$$n + 0 := n \qquad\qquad n \cdot 0 := 0$$
$$n + \mathfrak{s}(m) := \mathfrak{s}(n+m) \qquad\qquad n \cdot \mathfrak{s}(m) := (n \cdot m) + n$$

We define these binary operations above through recursion on the *second argument* while keeping the *first argument* fixed. 定義

Definition* 4.2 *(Exponentiation & Tetration).
We also define how to *exponentiate* and *tetrate* natural numbers below.

$$n^0 := 1 \qquad\qquad n \uparrow\uparrow 0 := 1$$
$$n^{\mathfrak{s}(m)} := n \cdot n^m \qquad\qquad n \uparrow\uparrow \mathfrak{s}(m) := n^{n \uparrow\uparrow m}$$

Again, these are recursive definitions in the *second argument* that take an arbitrary natural number as their *first argument*. 定義

Definition* 4.3 *(Sums & Products).
Given a function $f : \mathbb{N} \to \mathbb{N}$, we define the *sum* and *product* of the first n values of this function recursively below.

$$\sum_{i=0}^{0} f(i) := f(0) \qquad\qquad \prod_{i=0}^{0} f(i) := f(0)$$
$$\sum_{i=0}^{\mathfrak{s}(n)} f(i) := \left(\sum_{i=0}^{n} f(i)\right) + f(\mathfrak{s}(n)) \qquad \prod_{i=0}^{\mathfrak{s}(n)} f(i) := \left(\prod_{i=0}^{n} f(i)\right) \cdot f(\mathfrak{s}(n))$$

We can generalize these definitions to cases where the lower index is nonzero as long as the upper index dominates the lower index. 定義

***Theorem* 4.1.**
$(\forall n \in \mathbb{N})(\mathfrak{s}(n) = n + 1)$. 定理

Proof. Let $n \in \mathbb{N}$ and observe the following.

$$\begin{aligned} n + 1 &= n + \mathfrak{s}(0) && \text{since } 1 \coloneqq \mathfrak{s}(0) \\ &= \mathfrak{s}(n + 0) && \text{by definition of addition} \\ &= \mathfrak{s}(n) && \text{by definition of addition} \end{aligned}$$

Therefore, we have $\mathfrak{s}(n) = n + 1$. Q.E.D.

***Theorem* 4.2.**
$(\forall n \in \mathbb{N})(n + 0 = n)$. 定理

Proof. Let $n \in \mathbb{N}$ and notice that $n + 0 = n$ by the definition of addition. Q.E.D.

***Theorem* 4.3.**
$(\forall n \in \mathbb{N})(0 + n = n)$. 定理

Proof. We will prove this by induction.

Basis Step: Observe that $0 + 0 = 0$ by the definition of addition.

Inductive Step: Let $k \in \mathbb{N}$ and assume $0 + k = k$. We will now show that $0 + \mathfrak{s}(k) = \mathfrak{s}(k)$. Bear witness to the following deduction.

$$\begin{aligned} 0 + \mathfrak{s}(k) &= \mathfrak{s}(0 + k) && \text{by definition of addition} \\ &= \mathfrak{s}(k) && \text{by the \textit{inductive hypothesis}} \end{aligned}$$

Therefore, we conclude $(\forall n \in \mathbb{N})(0 + n = n)$. Q.E.D.

Theorem* 4.4 *(Associativity of Addition).
$(\forall x, y, z \in \mathbb{N})(x + (y + z) = (x + y) + z)$. 定理

Proof. Let $x, y \in \mathbb{N}$. We will prove this by induction.

Basis Step: Observe the following chain of reasoning.

$$\begin{aligned} x + (y + 0) &= x + y && \text{by definition of addition} \\ &= (x + y) + 0 && \text{by definition of addition} \end{aligned}$$

Inductive Step: Let $k \in \mathbb{N}$ and assume $x + (y + k) = (x + y) + k$. Observe.

$$\begin{aligned} x + (y + \mathfrak{s}(k)) &= x + \mathfrak{s}(y + k) && \text{by definition of addition} \\ &= \mathfrak{s}(x + (y + k)) && \text{by definition of addition} \\ &= \mathfrak{s}((x + y) + k) && \text{by the \textit{inductive hypothesis}} \\ &= (x + y) + \mathfrak{s}(k) && \text{by definition of addition} \end{aligned}$$

Thus, $x + (y + \mathfrak{s}(k)) = (x + y) + \mathfrak{s}(k)$ as desired.

Therefore, we conclude $(\forall x, y, z \in \mathbb{N})(x + (y + z) = (x + y) + z)$. Q.E.D.

4.2 Abstraction and Extension

$x \leqslant y$

Given $n, m \in \mathbb{N}$, we say $n \leqslant m \Leftrightarrow (\exists x \in \mathbb{N})(n + x = m)$, meaning n is *less than or equal to* m. We also define a *strict* version of this order by saying $n < m \Leftrightarrow (n \leqslant m) \wedge (n \neq m)$. Knowing this, we realize the natural numbers have all the defining properties of an *ordered semiring*.

$x < y$

Theorem 4.5 (The Naturals are an Ordered Semiring).
$\mathbb{N}$ is a commutative monoid under addition with identity element 0.

1. $(\exists e \in \mathbb{N})(\forall x \in \mathbb{N})(e + x = x)$. *existence of additive identity*
2. $(\forall x, y, z \in \mathbb{N})(x + (y + z) = (x + y) + z)$. *associativity of addition*
3. $(\forall x, y \in \mathbb{N})(x + y = y + x)$. *commutativity of addition*

$\mathbb{N}$ is a commutative monoid under multiplication with identity element 1.

4. $(\exists e \in \mathbb{N})(\forall x \in \mathbb{N})(e \cdot x = x)$. *existence of multiplicative identity*
5. $(\forall x, y, z \in \mathbb{N})(x \cdot (y \cdot z) = (x \cdot y) \cdot z)$. *associativity of multiplication*
6. $(\forall x, y \in \mathbb{N})(x \cdot y = y \cdot x)$. *commutativity of multiplication*

Multiplication *distributes* over addition, and the additive identity is also the *multiplicative annihilator.* This makes $\mathbb{N}$ a *commutative semiring*.

7. $(\forall x, y, z \in \mathbb{N})(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$. *distributivity*
8. $(\forall x \in \mathbb{N})(0 \cdot x = 0)$. *annihilation*

Addition and multiplication are *monotonic*, making $\mathbb{N}$ an *ordered semiring*.

9. $(\forall x, y, z \in \mathbb{N})((x \leqslant y) \Rightarrow (x + z \leqslant y + z))$. *addition is monotonic*
10. $(\forall x, y, z \in \mathbb{N})((x \leqslant y \wedge 0 \leqslant z) \Rightarrow (x \cdot z \leqslant y \cdot z))$. *multiplication is monotonic*

We usually say that $\mathbb{N}$ is the *canonical* ordered semiring because any other algebraic structure that has all of these same properties *must contain a copy of $\mathbb{N}$ within it as a substructure.* 定理

The Integer Ring

$\mathbb{Z}$

The *integers* $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ extend $\mathbb{N}$ by introducing *additive inverses*[1] for every element and inheriting all of the previous properties. An algebraic structure with all the properties of a monoid, but which also has inverses for every element, is called a *group*. Since addition is also *commutative* on $\mathbb{Z}$, also say $\mathbb{Z}$ is a *commutative group*.

group

[1] If $\mathfrak{A}$ with operation $\star$ is an algebraic structure with identity element $e_\star$, then we say $b \in \mathfrak{A}$ is an *inverse* for $a \in \mathfrak{A}$ with respect to $\star$ if $a \star b = e_\star$. Depending on the context, we may denote the inverse of a by $-a$ or a^{-1} when it exists.

Theorem 4.6 (The Integers are a Group).
$(\forall z \in \mathbb{Z})(\exists w \in \mathbb{Z})(z + w = 0)$. 定理

An algebraic structure with two operations that is a *commutative group* under one and a *monoid* under the other and where the latter operation distributes over the former is called a *ring*. If the operations are both monotonic with respect to a linear order $\leqslant$, then we call it an *ordered ring*. The integers $\mathbb{Z}$ with standard $+$ and $\cdot$ operations, ordered by $\leqslant$ as usual, are the *canonical* example of an ordered ring.

There is an intimate relationship between $\mathbb{N}$ and $\mathbb{Z}$ that is revealed by the *absolute value function*, denoted $|\cdot| : \mathbb{Z} \to \mathbb{N}$ and defined below.

$$|z| := \begin{cases} z & \textit{if} \quad z \geqslant 0 \\ -z & \textit{if} \quad z < 0 \end{cases}$$

$|z|$ The *absolute value* of an integer $z \in \mathbb{Z}$ is then denoted $|z|$.

The Rational Field

$\mathbb{Q}$ The set of *rational* numbers $\mathbb{Q} = \left\{ {}^{p}/_{q} \;\middle|\; p \in \mathbb{Z} \wedge q \in \mathbb{N}_{+} \right\}$ extends $\mathbb{Z}$ by introducing *multiplicative inverses* for every *nonzero* element. Every ring with this additional property is called a *field*. With the inherited properties from the integers, $\mathbb{Q}$ is the canonical *ordered field*.

Theorem* 4.7 *(The Rationals are a Field).
$(\forall q \in \mathbb{Q})\big(q \neq 0 \Rightarrow (\exists r \in \mathbb{Q})(q \cdot r = 1)\big).$ 定理

The Continuum

$\mathbb{R}$ The set of real numbers $\mathbb{R}$ augments the set of rationals by ensuring every Cauchy-convergent sequence has a limit that exists in the set.

Zero-Product Property

All of these algebraic structures happen to be *cancellative* with respect to both of their operations. This implies there are *no nonzero zero divisors*.

***Theorem* 4.8.**
Let $\mathfrak{A}$ be any of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ with its standard addition and multiplication operations. Then, the following three statements are *true*.

1. $(\forall x, y, z \in \mathfrak{A})\Big((x + z = y + z) \Rightarrow (x = y)\Big).$ *additive cancellation*
2. $(\forall x, y, z \in \mathfrak{A})\Big((x \cdot z = y \cdot z \;\wedge\; z \neq 0) \Rightarrow (x = y)\Big).$ *multiplicative cancellation*
3. $(\forall x, y \in \mathfrak{A})\Big((x \cdot y = 0) \Leftrightarrow (x = 0 \vee y = 0)\Big).$ *domain property*

定理

5

Number Theory

> "Αὐτός μέν Πῡθᾰγόρας, ἐν τῷ ἱέρῳ λογῳ διαρρήδην μορφών καί ἰδέων κράντορα τόν ἀριθμόν ἐλεγεν ἐιναι, καί θέων καί δαιμόνων αἶτιον· καί τῷ πρέσβυτατῳ καί κρατιστεύοντι τέχνιτῃ θέῳ κανονα, καί λογον τεχνικόν, νουν τε καί σταθμάν ἀκλῐνέσταταν τόν ἀριθμόν ὗπεικε συστάσιος καί γενέσεως τῶν πάντων."
>
> – Ἰάμβλιχος

> *"Number is the ruler of forms and ideas and the cause of Gods and dæmons."*
>
> – Pythagoras

Figure 5.1: Pythagoras (Πυθαγόρης).

5.1 *Ancient Greece*

Definition* 5.1 *(Divisibility).

divides For any $a, b \in \mathbb{Z}$, we say that *a divides b* when b is a multiple of a.

$a \mid b$

$$a \mid b \;:\Leftrightarrow\; (\exists k \in \mathbb{Z})(a \cdot k = b)$$

Note that this is a *sentence* establishing a relation on $\mathbb{Z}$. 定義

Theorem* 5.1 *(Absolute Monotonicity of Divisibility).
Let $a, b \in \mathbb{Z}$ such that $b \neq 0$. Then, $a \mid b$ implies $|a| \leqslant |b|$. 定理

***Lemma* 5.1.**
Let $z \in \mathbb{Z}$. Then, $1 \mid z$ and $z \mid 0$. Further, we have $(0 \mid z) \Leftrightarrow (z = 0)$. Finally, $(z \mid 1) \Leftrightarrow (z \in \{-1, 1\})$. 引理

Definition* 5.2 *(Parity).

even *odd* Let $z \in \mathbb{Z}$. We say that z is *even* by definition if $2 \mid z$. Analogously, we z is *odd* if $2 \mid z - 1$. This characteristic of z is called its *parity*. 定義

Theorem* 5.2 *(Even-Odd Dichotomy).
For every $z \in \mathbb{Z}$, we know z is either even or odd but not both. 定理

***Theorem* 5.3.**
Let $n, a, b, x, y \in \mathbb{Z}$ such that $n \mid x$ and $n \mid y$. Then, $n \mid ax + by$. 定理

Theorem* 5.4 *(Divisibility is a Partial Order).
The divisibility relation on $\mathbb{N}$ has the three following properties.

1. $(\forall a \in \mathbb{N})(a \mid a)$. *reflexivity*
2. $(\forall a, b \in \mathbb{N})((a \mid b \wedge b \mid a) \Rightarrow a = b)$. *antisymmetry*
3. $(\forall a, b, c \in \mathbb{N})((a \mid b \wedge b \mid c) \Rightarrow a \mid c)$. *transitivity*

This makes divisibility on $\mathbb{N}$ an example of a *partial order.* 定理

Definition* 5.3 *(Primality).
prime We say that a natural number $p \in \mathbb{N}$ is *prime* when $p > 1$ and p is *minimally divisible*, meaning $(\forall n \in \mathbb{N})(n \mid p \Rightarrow n \in \{1, p\})$. Any natural *composite* number that is *not prime* is called *composite* by definition. 定義

Lemma* 5.2 *(Fundamental Lemma of Arithmetic).
Let $n \in \mathbb{N}$ such that $n \geqslant 2$. Then, $(\exists p \in \mathbb{N})(p \text{ is prime} \wedge p \mid n)$. 引理

Theorem* 5.5 *(Fundamental Theorem of Arithmetic).
Let $n \in \mathbb{N} \setminus [\![2]\!]$. Then, $\exists! k \in \mathbb{N}_+$ and $\exists!(p_0, \alpha_0), \ldots (p_k, \alpha_k) \in \mathbb{N} \times \mathbb{N}$ such that $p_0, \ldots p_k$ are *distinct prime numbers* and the following holds.

$$n = \prod_{i=0}^{k} p_i^{\alpha_i} = p_0^{\alpha_0} p_1^{\alpha_1} \ldots p_k^{\alpha_k}$$

定理

Theorem* 5.6 *(Euclid's Theorem).
There are infinitely many prime numbers. 定理

Proof. We know there is at least one prime number since 2 is prime. Towards a contradiction, suppose $p_0, \ldots p_k \in \mathbb{N}$ is a complete list of *all* the prime numbers, where $k \in \mathbb{N}$. Consider the product $\mathcal{P} := \prod_{i=0}^{k} p_i$ of all of these prime numbers. Since $p_i \geqslant 2$ for each $i \in [\![k+1]\!]$, we know $\mathcal{P} \geqslant 2$, meaning $\mathcal{P}$ has a prime divisor by *lemma* 5.2. Let p_j be that prime divisor, so that $p_j \mid \mathcal{P} + 1$, and observe the following.

$$p_j \left(\prod_{i=0, i \neq j}^{k} p_i \right) = \prod_{i=0}^{k} p_i = \mathcal{P}$$

This observation implies $p_j \mid \mathcal{P}$. Since p_j divides both $\mathcal{P}$ and $\mathcal{P} + 1$, *theorem* 5.3 leads us to the following astonishing revelation.

$$p_j \mid (\mathcal{P} + 1) - \mathcal{P}$$

This implies $p_j \mid 1$, so $p_j \leqslant 1$. However, $p_j > 1$ since p_j is prime. ↯

Therefore, $p_0, \ldots p_k$ must *not* have been a complete list of the primes. Applying this argument to any *finite* set of primes leads us to our conclusion: *there are not finitely many prime numbers*. Q.E.D.

Definition* 5.4 *(Greatest Divisors and Least Multiples).

$\gcd(a, b)$ The *greatest common divisor* of two integers $a, b \in \mathbb{Z}$—denoted $\gcd(a, b)$—is a natural number $d \in \mathbb{N}$ that lives up to its name: d is a *common divisor* of a and b, and d is *greatest* among all possible common divisors.

1. $\gcd(a, b) \mid a$
2. $\gcd(a, b) \mid b$
3. $(\forall z \in \mathbb{Z})\Big((z \mid a \wedge z \mid b) \Rightarrow z \mid \gcd(a, b)\Big)$

Note that we define the *greatness* of $\gcd(a, b)$ with respect to *divisibility* as opposed to the traditional $\leqslant$ linear ordering. This allows us to observe $\gcd(0, 0) = 0$ where it would otherwise not be well-defined.

$\operatorname{lcm}(a, b)$ We define the *least common multiple* of $a, b \in \mathbb{Z}$ *dually* as the common multiple that is *least* among all possible common multiples.

1. $a \mid \operatorname{lcm}(a, b)$
2. $b \mid \operatorname{lcm}(a, b)$
3. $(\forall z \in \mathbb{Z})\Big((a \mid z \wedge b \mid z) \Rightarrow \operatorname{lcm}(a, b) \mid z\Big)$

These definitions can naturally be extended to finite sets of more than two integers at a time. 定義

Definition* 5.5 *(Coprimality).

coprime We say $x, y \in \mathbb{Z}$ are *coprime* when $\gcd(x, y) = 1$. Given $\mathcal{Z} \subseteq \mathbb{Z}$ and $k \in \mathbb{N} \setminus [\![2]\!]$, we say the numbers in $\mathcal{Z}$ are *k-wise relatively prime* when $\gcd(z_0, z_1, \ldots z_{k-1}) = 1$ for each choice of distinct $z_0, z_1, \ldots z_{k-1} \in \mathcal{Z}$. 定義

***Lemma* 5.3.**

For any $a, b \in \mathbb{Z}$, the following two statements are *true.*

1. $\gcd(a, b) = 0 \Leftrightarrow (a = 0 \wedge b = 0)$
2. $\gcd(a, b) \geqslant 1 \Leftrightarrow (a \neq 0 \vee b \neq 0)$

Further, $\gcd(x, x) = \gcd(x, 0) = x$ and $\gcd(x, 1) = 1$ for all $x \in \mathbb{Z}$. 引理

***Theorem* 5.7.**

Given arbitrary integers $a, b \in \mathbb{Z}$, the following statement is *true.*

$$\gcd(a, b) = 1 \Leftrightarrow (\forall p \in \mathbb{N})\Big(p \text{ is prime} \Rightarrow \big(p \nmid a \vee p \nmid b\big)\Big)$$

This means precisely that *coprime numbers share no prime factors*. 定理

Lemma* 5.4 *(Euclid's Division Lemma).
If $a, b \in \mathbb{Z}$ and $b \neq 0$, there exist *unique* $q, r \in \mathbb{Z}$ satisfying the following.

$$a = q \cdot b + r \qquad \text{and} \qquad 0 \leqslant r < |b|$$

remainder
quotient

We say that r in the above equation is the *remainder* obtained from the division of a by b, and q is the *quotient*. 引理

Algorithm* 5.1 *(Euclidean Division).
Given $a, b \in \mathbb{Z}$, we compute their greatest common divisor as follows.

$$\mathtt{gcd}(a,b) := \begin{cases} a & \textit{if } b = 0 \\ \mathtt{gcd}(b,r) & \textit{if } b \neq 0, \textit{where } r \in \mathbb{Z} \textit{ satisfies} \\ & (\exists q \in \mathbb{Z})(a = qb + r) \textit{ and } 0 \leqslant r < |b| \end{cases}$$

演算法

Theorem* 5.8 *(Bézout's Identity).
For any $a, b \in \mathbb{Z}$, there exist $x, y \in \mathbb{Z}$ such that $ax + by = \gcd(a, b)$. 定理

Theorem* 5.9 *(Euclid's Lemma).
For any $a, b \in \mathbb{Z}$ and any prime $p \in \mathbb{N}$, if $p \mid ab$, then $p \mid a$ or $p \mid b$. 定理

Proof. Let $a, b \in \mathbb{Z}$ and let $p \in \mathbb{N}$ be prime such that $p \mid ab$. If $p \mid a$, then we are done; on the contrary, suppose $p \nmid a$. Since p is prime, we can derive $q \nmid p \lor q \nmid a$ for any arbitrary prime $q \in \mathbb{N}$ as follows.

$$q \mid p \quad \Rightarrow \quad q \in \{1, p\} \quad \Rightarrow \quad q = p \quad \Rightarrow \quad q \nmid a$$

This tells us p and a *share no prime factors*, so $\gcd(p, a) = 1$. Applying *Bézout's identity*, there exist $x, y \in \mathbb{Z}$ making the following equality hold.

$$1 = px + ay$$

Since $p \mid ab$, we know $pk = ab$ for some $k \in \mathbb{Z}$. Now, we can sit back.

$$\begin{aligned} 1 = xp + ya \Rightarrow 1b &= (px + ay)b \\ \Rightarrow b &= (px)b + (ay)b \\ \Rightarrow b &= p(xb) + (ab)y \\ \Rightarrow b &= p(xb) + (pk)y \\ \Rightarrow b &= p(xb) + p(ky) \\ \Rightarrow b &= p(xb + ky) \end{aligned}$$

The above reasoning then demonstrates $p \mid b$ because $xb + ky \in \mathbb{Z}$, concluding our proof. Q.E.D.

***Corollary* 5.1.**
For any $a, b, c \in \mathbb{Z}$, if $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$. 推論

6

Combinatorics

"What we can't say we can't say, and we can't whistle it either."
– Frank P. Ramsey

The study of counting.

6.1 Judging the Size of a Set

function

Recall that a *function* $f : X \to Y$ from a *domain* X to a *codomain* Y establishes a *relation* that associates *every* element $x \in X$ of the domain with *exactly one* element $f(x) \in Y$ of the codomain.

The phrase $f : X \to Y$ can be read either as the *noun* *"f from X to Y"* or as the *full sentence* *"f is a function from X to Y"* depending on context.

$$(\forall x \in X)(\exists! y \in Y)\big(f(x) = y\big)$$

image
preimage

Commonly, *the output* $f(x) \in Y$ of a given input $x \in X$ is called the *image of x under f*. Analogously, *the input* $x \in X$ that generates a given output $y := f(x) \in Y$ is referred to as the *preimage of y under f*.

When $f(x) = y$, we refer to x as the *preimage* of y, and we call y the *image* of x.

Definition* 6.1 *((·)-jectivity).
injection

Let X and Y be sets and consider a function $f : X \to Y$. We say that f is *injective* if f always maps *distinct inputs* to distinct outputs.[2] Formally, this means f satisfies the following statement.

[2] *Injections* are also known as *"one-to-one."*

$$(\forall a, b \in X)\big(f(a) = f(b) \Rightarrow a = b\big)$$

An equivalent, but often more useful, way to express this is given below.

$$(\forall a, b \in X)\big(a \neq b \Rightarrow f(a) \neq f(b)\big)$$

Once we know that f is an injection, we can denote this characteristic of f by writing $f : X \hookrightarrow Y$, reading this as *"f is an injection from X to Y"* or *"f injects X into Y"* to taste. Notice the use of the word "*into*."

surjection

We say that f is *surjective* when *every codomain element has a preimage*.[3] This means that f *"covers"* its entire codomain—that the range of f is identical to its codomain. Formally, we say this as follows.

[3] *Surjections* are sometimes called *"onto."*

$$(\forall y \in Y)(\exists x \in X)(f(x) = y)$$

Knowing that f is surjective grants access to the convenient denotational syntax $f : X \twoheadrightarrow Y$, which can be read as *"f is a surjection from X to Y"* or *"f surjects X onto Y."* Notice the use of the word *"onto."*

bijection

When f is both injective and surjective at the same time, we say that the function is *bijective* and use the combined $f : X \hookrightarrow\!\!\!\!\rightarrow Y$ syntax.[1] 定義

[1] There are *"people"* who refer to *bijections* as *"one-to-one correspondences."* They have been abandoned by God and will never feel the warm light of heaven.

It's often a good idea to have a visual in mind to ground your intuition. In the same way that we can think of functions as *"curves that pass the vertical line test,"* we can think of *injective functions* as curves that pass the "*horizontal line test.*"

We judge the relative sizes of sets by the kinds of functions that exist between them, and use the notions of injectivity and surjectivity to give formal meaning to *"the size of a set."*

Definition* 6.2 *(Equinumerosity).
We define A to be *no smaller than* B when A can be *injected* into B.

$|A| \leqslant |B|$

$$|A| \leqslant |B| \;:\Leftrightarrow\; \exists f(f : A \hookrightarrow B)$$

We define A to be *no larger than* B when A can be *surjected* onto B.

$|A| \geqslant |B|$

$$|A| \geqslant |B| \;:\Leftrightarrow\; \exists g(g : A \twoheadrightarrow B)$$

We say that two sets A and B have the *same cardinality*—meaning *same size* or *same number of elements*—there is a *bijection* between A and B.

$|A| = |B|$

$$|A| = |B| \;:\Leftrightarrow\; \exists h(h : A \hookrightarrow\!\!\!\!\rightarrow B)$$

Definitions for $|A| < |B|$ and $|A| > |B|$ spring naturally from these. 定義

Lemma* 6.1 *(Reflexivity of Cardinality).
$\forall A(|A| = |A|)$. 引理

Proof. Let A be a set and consider the function $f : A \to A$ given by $f(a) := a$ for every $a \in A$. We will show f is a bijection.

To show f is injective, suppose $a, b \in A$ and assume $f(a) = f(b)$. Then, since $f(a) = a$ and $f(b) = b$, we know $a = b$ by definition. This proves $(\forall x, y \in A)(f(x) = f(y) \Rightarrow x = y)$, meaning f is injective.

To show that f is surjective, let $a \in A$ and observe $f(a) = a$. This proves $(\forall y \in A)(\exists x \in A)(f(x) = y)$, meaning f is surjective.

Therefore, since f is both injective and surjective, we know that f is a bijection from A to A, and thus $|A| = |A|$ by definition. Q.E.D.

The above lemma involves an important construction that shows up frequently in many contexts.[1] The *identity function on a set X* is the function $\mathtt{id}_X : X \to X$ that maps every element of X back to itself; formally, $\mathtt{id}_X(x) := x$ for every $x \in X$. This function *always* exists for any X, and this function is *always* a bijection on X. We will discuss this function and its properties in more detail in the next section.

$\mathtt{id}_X$

[1] We will realize soon that this is another echo of a recurring pattern we have already seen.

These definitions expose to us a formal way of *counting* the elements of a set. Suppose we have a set $\mathcal{A} := \{a, b, c, d\}$. To count the elements of $\mathcal{A}$, we might point at a first, then b second, then c third, and finally d. This implicitly defines the function $f : \{0, 1, 2, 3\} \to \{a, b, c, d\}$ below.

$$
\begin{aligned}
0 &\xmapsto{\quad f \quad} a \\
1 &\xmapsto{\quad f \quad} b \\
2 &\xmapsto{\quad f \quad} c \\
3 &\xmapsto{\quad f \quad} d
\end{aligned}
$$

We can interpret this mapping as saying that the element a that f assigns as the output of 0 is the *first* element of $\mathcal{A}$, with the element b being the *second* because it is the output of 1 under f, and so on. If this *"counting function"* f is a bijection, then what we've done is establish a perfect association between $\mathcal{A}$ and *the natural number* $[\![4]\!] = \{0, 1, 2, 3\}$.[2] Any other set in bijection with $\mathcal{A}$ will also be in bijection with $[\![4]\!]$, so we can think of 4 as the *canonical representative of "sets with 4 elements."* We refer to these canonical representatives as *cardinal numbers*, and we use the notation $|X|$ to refer to *the cardinality of X*—the cardinal number that represents the *"size of X."*

cardinal

$|X|$

[2] Recall that $[\![n]\!] := \{0, 1, \ldots n-1\}$ when $n \in \mathbb{N}$. There is no *formal* distinction between n and $[\![n]\!]$, but we will sometimes use this notation to emphasize that we are looking at n *as a set with n elements* rather than as a *number* intended for arithmetical or algebraic manipulation.

If a set is what we call *"finite,"* then we should be able to count its elements using a natural number $[\![n]\!] = \{0, 1, \ldots n-1\}$, and in that case the *natural* choice of cardinal for X is simply $|X| = n$.

Definition* 6.3 *(Finite).
We say a set F is *finite* if there exists $n \in \mathbb{N}$ such that $|F| = |[\![n]\!]|$. In this situation, the natural number n is *unique*, so we define $|F| := n$. 定義

***Lemma* 6.2.**
For any $n \in \mathbb{N}$, we have $|\{1, 2, \ldots n\}| = n$. 引理

Proof. Let $n \in \mathbb{N}$. We will show $|\{1, 2, \ldots n\}| = |\{0, 1, \ldots n-1\}|$. Consider the function $f : \{1, 2, \ldots n\} \to \{0, 1, \ldots n-1\}$ given by $f(x) := x - 1$ for each $x \in \{1, 2, \ldots n\}$.

To see that f is an injection, consider $a, b \in \{1, 2, \ldots n\}$ and suppose $f(a) = f(b)$. We then know $a - 1 = b - 1$ by the definition of f. Cancelling on both sides then yields $a = b$ as desired.

To see that f is surjective, let $y \in \{0, 1, \ldots n-1\}$. Notice $0 \leqslant y \leqslant n-1$, so that $1 \leqslant y+1 \leqslant n$, implying $y+1 \in \{1, 2, \ldots n\}$.[1] We can now simply observe that $f(y+1) = (y+1) - 1 = y$. Q.E.D.

[1] This verifies $y+1$ is in the domain of f.

It should hopefully be intuitively straightforward to say that *"every set has a size,"* and that therefore the cardinalities of sets are always comparable: for any two sets A and B, we should know that either $|A| \leqslant |B|$ or that $|B| \leqslant |A|$. As it turns out, *this is not a theorem* that we can prove using the massive mathematical system we've established. If we want to know this fact, we need one final axiom.[2]

[2] While this is the final axiom we will be introducing for our purposes, there *is* actually one more axiom in standard ZFC: *the axiom schema of replacement,* which tersely says *"the image of a set under a definable class function is a set"* We won't be using this axiom for anything, so it won't be mentioned or discussed in the text.

Axiom 7 (Equivalent to the Axiom of Choice).
Every set has a unique cardinality. 公理

Theorem 6.1 (Dichotomy of Cardinality).
For any sets A and B, either $|A| \leqslant |B|$ or $|B| \leqslant |A|$. 定理

6.2 Compositionality and Invertibility

Definition 6.4 (Composition).
Let X, Y, and Z be sets. Given *compatible* functions $f : X \to Y$ and $g : Y \to Z$, the *composition of g with f* is a function $g \circ f : X \to Y$ defined by $(g \circ f)(x) := g\big(f(x)\big)$ for all $x \in X$. We read the name of this function as *"g composed with f"* or *"g after f."* 定義

$g \circ f$

Theorem 6.2 ((·)-jections are (·)-morphisms).
Let X and Y be sets and consider a function $f : X \to Y$. If we know f is an injection, then f must have a *surjective left inverse* and vice versa.

$|X| \leqslant |Y| \Leftrightarrow |Y| \geqslant |X|$

$$f \text{ is injective} \Leftrightarrow (\exists g : Y \twoheadrightarrow X)(g \circ f = \mathtt{id}_X)$$

Conversely, f is a surjection exactly when f has an *injective right inverse.*

$|X| \geqslant |Y| \Leftrightarrow |Y| \leqslant |X|$

$$f \text{ is surjective} \Leftrightarrow (\exists g : Y \hookrightarrow X)(f \circ g = \mathtt{id}_Y)$$

When f is a bijection, there is a *unique, bijective, two-sided inverse* for f.

$|X| = |Y| \Leftrightarrow |Y| = |X|$

$$f \text{ is bijective} \Leftrightarrow (\exists! g : Y \hookrightarrow\!\!\!\!\rightarrow X)(g \circ f = \mathtt{id}_X \wedge f \circ g = \mathtt{id}_Y)$$

In this last case, we refer to the unique two-sided inverse for f as *the inverse of f* and use f^{-1} to denote this function. 定理

f^{-1}

Theorem 6.3 (Cantor-Schöder-Bernstein).
Suppose X and Y are sets. If there exist injections $f : X \hookrightarrow Y$ and $g : Y \hookrightarrow X$ in opposite directions between the two sets, then a bijection $h : X \hookrightarrow\!\!\!\!\rightarrow Y$ exists from one set to the other. We restate this as follows.

$$\forall A \forall B \Big(\big(|A| \leqslant |B| \wedge |B| \leqslant |A|\big) \Rightarrow |A| = |B| \Big)$$

Notice that this establishes the *antisymmetry of cardinality.* 定理

6.3 Counting with Our Fingers

***Theorem* 6.4.**
If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$. 定理

This is one of the reasons why $A \times B$ is called the Cartesian *product* of A with B.

Theorem* 6.5 *(Inclusion/Exclusion Principle).
If A and B are finite, then $|A \cup B| = |A| + |B| - |A \cap B|$. In general, given n finite sets $A_1, A_2, \dots A_n$ with $n \in \mathbb{N}_+$, the following is *true*.

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \leqslant \sum_{i=1}^{n} |A_i|$$

As a consequence, the union of finitely many finite sets is finite. 定理

***Definition* 6.5.**
The *floor function* is the map $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ given below for any $x \in \mathbb{R}$.

$$\lfloor x \rfloor := \max\{z \in \mathbb{Z} \mid z \leqslant x\}$$

$\lfloor x \rfloor$ This defines $\lfloor x \rfloor$ to be the *greatest integer less than or equal to* x.[1] In other words, $\lfloor x \rfloor$ is the result of *rounding* x *down* to the nearest integer. The *ceiling function* $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$, given below, is dual to the floor function.

$$\lceil x \rceil := \min\{z \in \mathbb{Z} \mid z \geqslant x\}$$

$\lceil x \rceil$ This defines $\lceil x \rceil$ as the *least integer greater than or equal to* x, which corresponds analogously to *rounding* x *up* to the nearest integer. 定義

[1] As with any definition we make, but especially with definitions such as these where we define an *object* based on a *property we want it to have*, we should always ask the question: does such an object *actually exist?* As with the $\gcd(a,b)$ and $\operatorname{lcm}(a,b)$ for $a, b \in \mathbb{Z}$, the answer here is that *yes*, $\lfloor x \rfloor$ always exists for any $x \in \mathbb{R}$.

Theorem* 6.6 *(Pigeonhole Principle).
Consider any two sets A and B. The following two statements are *true*.

$$|A| > |B| \Leftrightarrow (\forall f : A \to B)(f \text{ is not injective})$$
$$|A| < |B| \Rightarrow (\forall f : A \to B)(f \text{ is not surjective})$$

Further, if there exist $n, k \in \mathbb{N}_+$ such that $|A| = n$ and $|B| = k$, then for any $f : A \to B$ there exists $b \in B$ for which the inequality below holds.

$$\Big|\{a \in A \mid f(a) = b\}\Big| \geqslant \left\lfloor \frac{n-1}{k} \right\rfloor + 1 = \left\lceil \frac{n}{k} \right\rceil$$

定理

6.4 Structure and Substructure

Definition* 6.6 *(Combination).
Given a finite set A of cardinality $n := |A|$, we know that the set of *all possible* subsets of A is given by $\mathbb{P}(A) = \{z \mid z \subseteq A\}$. We now know that each of those subsets $B \subseteq A$ must have cardinality $B \in \{0, \dots n\}$.
combination If $k := |B|$, we say that B in this case is a *k-combination of A*.

n choose k

For any natural numbers $n, k \in \mathbb{N}$, we define the combinatorial number *n choose k* to be the number of cardinality k subsets of $[\![n]\!]$ as below.

$$\binom{n}{k} := \left| \left\{ z \;\middle|\; z \subseteq [\![n]\!] \wedge |z| = k \right\} \right|$$

$\binom{n}{k}$

We denote n choose k with the notation $\binom{n}{k}$. Since the *identities* of the elements of a set don't influence its *size*, it should be clear to see that $\binom{n}{k}$ *measures the number of k-combinations of any set of cardinality* n. 定義

***Lemma* 6.3.**
Consider a finite set X of cardinality $n := |X|$ and let $k \in \mathbb{N}$. Then, we have that $\big|\{z \mid z \subseteq X \wedge |z| = k\}\big| = \binom{n}{k}$. 引理

Definition* 6.7 *(Simple Graph).
A *graph* G is defined by a set of *vertices*—also called *nodes*—connected together by some number of *edges* between them. 定義

***Lemma* 6.4.**
If G is a finite graph, then $0 \leqslant \mathtt{deg}(v) < |V(G)|$ for every $v \in V(G)$. 引理

Lemma* 6.5 *(Handshake Lemma).
Suppose G is a (finite, simple) graph on $n \geqslant 2$ nodes. Then, G contains two distinct vertices v and w such that $\mathtt{deg}(v) = \mathtt{deg}(w)$. 引理

6.5 *Arrangement and Derangement*

***Theorem* 6.7.**
Given two finite sets A and B, there are $|B|^{|A|}$ distinct functions with domain A and codomain B. Formally, $|\{f \mid f : A \to B\}| = |B|^{|A|}$. 定理

Inspired by this theorem, some authors denote the set $\{f \mid f : A \to B\}$ using the notation B^A so that $|B^A| = |B|^{|A|}$. For this reason, the set of functions from A to B is sometimes called an *exponential object in the category of sets.*

***Lemma* 6.6.**
If X is a finite set, then $|\mathbb{P}(X)| = 2^{|X|}$. 引理

Definition* 6.8 *(Permutation).

permutation

Given a set $\mathcal{X}$, we call bijections $f : \mathcal{X} \hookrightarrow\!\!\!\!\!\rightarrow \mathcal{X}$ *permutations on* $\mathcal{X}$. 定義

Definition* 6.9 *(String).

finite string

Given a natural number $n \in \mathbb{N}$ and a set $\mathcal{A}$, a *finite string* over $\mathcal{A}$ is simply a function $f : [\![n]\!] \to \mathcal{A}$. The *length* of the string f is given by $|f| = |[\![n]\!]| = n$; this is simultaneously the cardinality of the string itself and of its domain.[1] The codomain $\mathcal{A}$ is the set where the string f takes its characters from, and is sometimes called an *alphabet*.

If $f : \mathcal{N} \to \mathcal{A}$ is a string and $\mathcal{M} \subseteq \mathcal{N}$, we refer to the function

[1] Remember that every function is formally a set of ordered pairs, so the string $f : [\![3]\!] \to \{a, b, c\}$ given by $f =$ "*bac*" is actually the set $f = \{(0, b), (1, a), (2, c)\}$.

$g : \mathcal{M} \to \mathcal{A}$ given by $g(x) := f(x)$ for each $x \in \mathcal{M}$ as a *substring* of f. Its length is given by $|g|$, which is $|s|$ as expected. 定義

When f is a finite string, We will sometimes conveniently refer to finite strings by surrounding their outputs, written in order, with quotes. For example, suppose $s : [\![12]\!] \to \{\texttt{d}, \texttt{e}, \texttt{h}, \texttt{l}, \texttt{o}, \texttt{r}, \texttt{w}, \texttt{!}, \texttt{␣}\}$ is given as follows.

$$
\begin{aligned}
0 &\longmapsto \texttt{h} & \qquad 6 &\longmapsto \texttt{w} \\
1 &\longmapsto \texttt{e} & \qquad 7 &\longmapsto \texttt{o} \\
2 &\longmapsto \texttt{l} & \qquad 8 &\longmapsto \texttt{r} \\
3 &\longmapsto \texttt{l} & \qquad 9 &\longmapsto \texttt{l} \\
4 &\longmapsto \texttt{o} & \qquad 10 &\longmapsto \texttt{d} \\
5 &\longmapsto \texttt{␣} & \qquad 11 &\longmapsto \texttt{!}
\end{aligned}
$$

Then, we can represent $s =$ "$\texttt{hello␣world!}$", where $s(0) = \texttt{h}$, $s(1) = \texttt{e}$, and so on. We can also write "$\texttt{eoo}$" $\subseteq$ "$\texttt{hello␣world!}$" to refer to the substring of s consisting of only the vowels.

6.6 Partitions

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