

*Exam f.p.2024.spring**Discrete Mathematics**9th of May, 2024***1 Answer the following questions either True or False.**

- (a) If \mathcal{A} and \mathcal{B} are sets, then $|\mathcal{A} \cup \mathcal{B}| = |\mathcal{A}| + |\mathcal{B}|$.
☐ True
☐ False
- (b) If a and b are odd, then $(\exists c \in \mathbb{Z})(c^2 = a^2 + b^2)$.
☐ True
☐ False
- (c) If \mathcal{A} and \mathcal{B} are both finite, then $|\mathcal{A} \setminus \mathcal{B}| = |\mathcal{A}| - |\mathcal{B}|$.
☐ True
☐ False
- (d) If this sentence is a proposition, then every countable set is finite.
☐ True
☐ False
- (e) $2^{21} \equiv 1 \pmod{7}$.
☐ True
☐ False
- (f) This sentence implies $\forall x(\emptyset \subseteq x)$.
☐ True
☐ False
- (g) $(\forall \mathcal{A} \in \mathbb{P}(\mathbb{N}))(\mathcal{A} \neq \emptyset \Rightarrow (\exists a \in \mathcal{A})(\forall b \in \mathcal{A})(a \leq b))$.
☐ True
☐ False
- (h) $\mathbb{Z}/n\mathbb{Z}$ is a group under multiplication for every $n \in \mathbb{N}_+$.
☐ True
☐ False
- (i) $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y = \gcd(x, 15)\}$ is a function.
☐ True
☐ False
- (j) $\forall x \forall y \forall z (x \setminus (y \cap z) = (x \setminus y) \cup (x \setminus z))$.
☐ True
☐ False

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2 *Answer the following questions without proof.*

(a) Compute $\gcd(386, 352)$.

(b) Find all $x \in \mathbb{Z}$ such that $4x + 6 \equiv 3 + 2x \pmod{9}$.

(c) Find all integer solutions to $3x^{49} + 3x + 2 \equiv 4 \pmod{7}$.

(d) List the elements of the following set using set-builder notation:

$$\left\{ x \in \mathbb{N} \mid (\exists k \in \mathbb{Z})(xk = 28) \wedge (\forall y \in \mathbb{N})(y \mid x \Rightarrow y \in \{1, x\}) \right\}.$$

(e) Provide an example of a transitive set x such that $x \notin \mathbb{N}$.

3 Answer the following questions without proof.

(a) Provide a surjection from $\left\{ f \mid (\exists n \in \mathbb{N})(f : n \rightarrow \{0, 1\}) \right\}$ to \mathbb{Z} .

(b) In how many different ways can the strings "fighting" and "irish" be scrambled and then concatenated together?

(c) Let $k \in \mathbb{N}$ such that $k \geq 3$. How many strings $s : k \rightarrow \{0, 1, \dots, 9\}$ satisfy $(\exists i \in k)(s(i) = s(i+1) - 1 = s(i+2) - 2)$?

(d) Let $n \in \mathbb{N}_+$. How many binary strings b satisfy $|b| + \sum_{i=0}^{|b|-1} b(i) = n$?

(e) Let $n \in \mathbb{N}_+$. How many ways are there to move from the bottom-left square to the top-right square on an $n \times n$ chess board if you can only move to the right or up one square at-a-time?

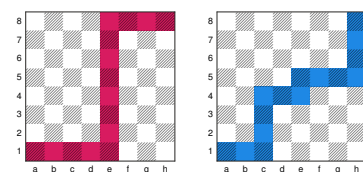


Fig. 1: Two examples of valid paths from the bottom-left corner to the top-right corner on an 8×8 chessboard.

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4 *Any use of logical axioms, rules of inference, or theorems must be stated. You may not appeal to truth tables.*

Prove $\neg(p \rightarrow q) \rightarrow p$ is a tautology for any propositions p and q .

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5 *You may rely on any theorems we have proven or studied.*

1. Give a recursive definition of the Fibonacci sequence $\mathcal{F} : \mathbb{N} \rightarrow \mathbb{N}$.

2. Show that $\mathcal{F}(n) < 2^n$ for all $n \in \mathbb{N}_+$.

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6 *You may rely on any theorems we have proven or studied.*

Let $\mathfrak{F} := \{\mathcal{F}(i) \mid i \in \mathbb{N}\}$ be the set of all Fibonacci numbers and observe $|\mathfrak{F}| = \aleph_0$. Show $\exists i, j \in \mathbb{N}$ such that $i \neq j$ and $\mathcal{F}(i) \equiv \mathcal{F}(j) \pmod{2024}$.

7 *You may rely on any theorems we have proven or studied.*

Let $n \in \mathbb{N}_+$ and let $b : n \rightarrow \{0, 1, \dots, 9\}$ be a decimal string representing a natural number k . Suppose the sum of the digits of b is divisible by 3. Prove that $3 \mid k$.

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8 *You may rely on any theorems we have proven or studied.*

Show that there are uncountably many infinite hexadecimal strings.