Problem Set 2

Discrete Mathematics

Due: 6^{th} of February, 2023

For the following problems, make sure any application of the equivalence rules (the axioms of a Boolean algebra) is clearly stated and that all steps are justified. Do not use truth tables as a source of justification.

1. Show that $p \to q$ is logically equivalent to $\neg q \to \neg p$.
2. Show that $(\neg p \to \bot) \to p$ is a tautology.
3. Show that $(p \to r) \vee (q \to r)$ is equivalent to $(p \wedge q) \to r$.
4. Show that $\big((p \vee q) \wedge (\neg p \vee r)\big) \to (q \vee r)$ is a tautology.
5. Using the predicate definitions below, let our universe of discourse be the collection of all characters in the play Macbeth and translate the following propositions of the first-order logic into English sentences.

$$\kappa(x) \coloneqq \text{“}x\text{ is a king.”} \qquad \omega(x) \coloneqq \text{“}x\text{ is man woman-borne.”}$$
$$\vartheta(x) \coloneqq \text{“}x\text{ is a thane.”} \qquad \mu(x,y) \coloneqq \text{“}x\text{ murders }y\text{.”}$$
$$\upsilon(x) \coloneqq \text{“}x\text{ is a witch.”}$$

   (a) Macbeth is a king, and Banquo is a thane.
   (b) Every king is murdered by someone.
   (c) Someone murders every king.
   (d) No witches murder anyone.
   (e) No kings murder any thanes.
   (f) Macbeth is not murdered by any man woman-borne.
   (g) Every king is a thane.
   (h) Banquo is not a king, yet someone murders him.
   (i) Any witch could not possibly be a king.
   (j) The only kings are those woman-borne.

6. Using the predicate definitions below, let our universe of discourse be the collection of all living beings on the Obra-Dinn and translate the following propositions of the first-order logic into English sentences.

$$\kappa(x) \coloneqq \text{“}x\text{ is the Captain.”} \qquad \delta(x,y) \coloneqq \text{“}x\text{ is drowned by }y\text{.”}$$
$$\mu(x) \coloneqq \text{“}x\text{ is a mate.”} \qquad \rho(x,y) \coloneqq \text{“}x\text{ was terribly ravaged by }y\text{.”}$$
$$\sigma(x) \coloneqq \text{“}x\text{ is a sea monster.”} \qquad \gamma(x,y) \coloneqq \text{“}x\text{ was shot by }y\text{.”}$$

   (a) $\kappa(\text{“Robert Witterel”}) \wedge \forall x\big(\mu(x) \to \gamma(x, \text{“Robert Witterel”})\big)$.
   (b) $\neg\forall x \exists y\big(\sigma(y) \wedge \rho(x,y)\big)$.
   (c) $\exists x \forall y\big(\sigma(x) \wedge (\mu(y) \to \delta(y,x))\big)$.
   (d) $\Bigg(\neg\exists x\big(\sigma(x)\big) \wedge \Big(\neg\exists x\big(\sigma(x)\big) \to \exists x\big(\gamma(\text{“Robert Witterel”}, x)\big)\Big)\Bigg) \to \exists x\big(\gamma(\text{“Robert Witterel”}, x)\big)$.

7. Let φ and ψ be predicates. Rewrite the following first-order sentences so that no quantifier has a $\neg$ to its left.
   (a) $\neg\forall x \exists y(x \vee y)$
   (b) $\neg\forall x \forall y(\neg x \vee (x \leftrightarrow y))$
   (c) $\neg\exists x \forall y \forall z\big((x \to y) \to z\big)$
   (d) $\neg\exists x\Big(\varphi(x) \vee \forall y \exists z\big(\psi(y,z) \to \neg x\big)\Big)$