

Problem Set 7

Discrete Mathematics

Due on the 25th of March, 2024

- (10 pts) 1. Let X be a set. Show that $(\forall Y \in \mathbb{P}(X))(|Y| \leq |X|)$.
- (15 pts) 2. Show that $\forall X \forall Y (|X| \leq |Y| \Rightarrow \exists Z (Z \subseteq Y \wedge |X| = |Z|))$.
- (15 pts) 3. Let X, Y, Z be sets and consider $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. We define the *composition* of f with g to be the function $g \circ f : X \rightarrow Z$ given by $(g \circ f)(x) := g(f(x))$ for all $x \in X$.
- (a) Show that, if f and g are both injections, then $g \circ f$ is injective.
- (b) Show that, if f and g are both surjections, then $g \circ f$ is surjective.
- (c) Show that, if f and g are both bijections, then $g \circ f$ is bijective.
- (30 pts) 4. For this problem, let X and Y be nonempty sets and let $f : X \rightarrow Y$.
- (a) If f is injective, show there exists $g : Y \rightarrow X$ where $g \circ f = \text{id}_X$.
- (b) If f is surjective, show there exists $g : Y \rightarrow X$ where $f \circ g = \text{id}_Y$.
- (c) If f is a bijection, then show that there exists a *unique* function $g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.
- (30 pts) 5. *Euler's totient function* is the function $\varphi_e : \mathbb{N} \rightarrow \mathbb{N}$ that counts how many positive integers are *coprime* with each $n \in \mathbb{N}$, defined below.

$$\varphi_e(n) := \left| \{z \in \mathbb{N} \mid 1 \leq z \leq n \wedge \gcd(z, n) = 1\} \right|$$

- (a) If $p, k, m \in \mathbb{N}_+$ are *positive* naturals with p prime and $m \leq p^k$, then prove that $\gcd(p^k, m) \neq 1 \Leftrightarrow p \mid m$.
- (b) If p is prime, then prove that $\varphi_e(p) = p - 1$.
- (c) If p is prime and $k \in \mathbb{N}_+$, then prove that $\varphi_e(p^k) = p^k - p^{k-1}$.

Since the codomain of f and the domain of g are the same, they are *compatible*, and their composition is sensibly defined.

These are called *monomorphisms*.

These are called *epimorphisms*.

These are called *isomorphisms*.

Hint: count the multiples of p .