

# Discrete Mathematics

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## Chapter 4

# Mathematical Induction

### 4.1 Weak Induction

**Definition 4.1** (Well-Order).

Let  $X$  be a set with a relation  $\leq$  defined on it. We say that  $X$  is *well-ordered* by this relation *iff* every non-empty subset of  $X$  has a minimal element with respect to  $\leq$ . In other words, we say that  $\leq$  is a *well-order* on  $X$  : $\Leftrightarrow$

$$(\forall A \in \mathcal{P}(X))(\exists a \in A)(\forall b \in A)a \leq b.$$

**Theorem 4.1** ( $\mathbb{N}$  is Well-Ordered).

$$(\forall A \subseteq \mathbb{N})(A \neq \emptyset \Rightarrow (\exists a \in A)(\forall b \in A)(a \leq b))$$

*This says that every nonempty subset of  $\mathbb{N}$  has a least element (according to the  $\leq$  order defined on  $\mathbb{N}$ ).*

*Proof.* This proof is left as an exercise.

Q.E.D.

**Theorem 4.2** (Weak Induction).

*If  $\varphi(\cdot)$  is a wff, then*

$$(\forall n \in \mathbb{N})(\varphi(n)) \Leftrightarrow \varphi(0) \wedge (\forall k \in \mathbb{N})(\varphi(k) \Rightarrow \varphi(k+1)).$$

*Proof.* There are two fragments to this proof: the forward ( $\Rightarrow$ ) direction and the backward ( $\Leftarrow$ ) direction.

**Fragment 1** ( $\Rightarrow$ ):

Suppose that  $(\forall n \in \mathbb{N})(\varphi(n))$ . Since  $0 \in \mathbb{N}$ , we then obviously have  $\varphi(0)$ . Now, let  $k \in \mathbb{N}$  and assume  $\varphi(k)$ . Since  $k+1 \in \mathbb{N}$ , we know from our initial assumption that  $\varphi(k+1)$ . Thus, we have  $(\forall k \in \mathbb{N})(\varphi(k) \Rightarrow \varphi(k+1))$ , and we have reached both of our desired conclusions.

**Fragment 2** ( $\Leftarrow$ ):

Assume  $\varphi(0)$  and  $(\forall k \in \mathbb{N})(\varphi(k) \Rightarrow \varphi(k+1))$ . Towards a contradiction, suppose that there is some  $n \in \mathbb{N}$  such that  $\neg\varphi(n)$ . Consider  $A := \{x \in \mathbb{N} \mid \neg\varphi(x)\}$ , which we clearly know exists by [Axiom 4](#). We know that  $n \in A$  because we assumed that  $\neg\varphi(n)$ , which implies that  $A \neq \emptyset$ . Then, we can use [Theorem 4.1](#) to conclude that there is a minimal element  $a$  in  $A$ .

Since we know that  $\varphi(0)$ , it follows that  $a \neq 0$ , so  $a$  must be a successor number. This means there is a  $b \in \mathbb{N}$  such that  $b+1 = a$ .

If  $b \in A$ , then that would mean that  $a \leq b$  since  $a$  is minimal in  $A$ . However, we know that  $b < b+1 = a$ , so we would then have  $a \leq b < a$ .  $\nexists$  Therefore,  $b \notin A$ .

With these two directions proven, we finally have  $(\forall n \in \mathbb{N})(\varphi(n)) \Leftrightarrow \varphi(0) \wedge (\forall k \in \mathbb{N})(\varphi(k) \Rightarrow \varphi(k+1))$ .

Q.E.D.

**Note.** The above theorem actually generalizes beyond just  $\mathbb{N}$ . In fact, we can generalize the proof of [Theorem 4.2](#) to *any* well-ordered set  $X$  by replacing  $k+1$  with the least element of the non-empty subset  $X \setminus \{\ell \in X \mid \ell \leq k\}$ .

Let's practice induction by proving the following few theorems.

**Lemma 4.1** (Gaussian Summation Formula).

$$(\forall n \in \mathbb{N}) \left( \sum_{i=0}^n i = \frac{n(n+1)}{2} \right).$$

*Proof.* We will prove the claim by induction on  $n \in \mathbb{N}$ .

**Base Case:**

Observe that  $\sum_{i=0}^0 i = 0 = \frac{0 \cdot (0+1)}{2}$ . Therefore, the statement is satisfied at 0.

**Inductive Step:**

Let  $k \in \mathbb{N}$  and assume  $\sum_{i=0}^k i = \frac{k(k+1)}{2}$  (this is our *inductive hypothesis*). Now, observe

$$\begin{aligned} \sum_{i=0}^{k+1} i &= \left( \sum_{i=0}^k i \right) + (k+1) && \text{by definition} \\ &= \left( \frac{k(k+1)}{2} \right) + (k+1) && \text{by the inductive hypothesis} \\ &= (k+1) \left( \frac{k}{2} + 1 \right) && \text{by the distributive property of multiplication}^{*\dagger} \\ &= \frac{(k+1)(k+2)}{2} && \text{because } \frac{k}{2} + 1 = \frac{k}{2} + \frac{2}{2} = \frac{k+2}{2}.^{\dagger\ddagger} \end{aligned}$$

Thus, we have that  $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$ , as desired.

Therefore, we can conclude that  $(\forall n \in \mathbb{N}) \left( \sum_{i=0}^n i = \frac{n(n+1)}{2} \right)$ .

Q.E.D.

**Lemma 4.2.**

$3n \leq 3^n$  for all  $n \in \mathbb{N}_+$ .

*Proof.* We will prove the claim by induction on  $n \in \mathbb{N}_+$ .

**Base Case:**

Observe that  $3 \cdot 0 = 0 \leq 1 = 3^0$ .

**Inductive Step:**

Let  $k \in \mathbb{N}_+$  and assume  $3 \cdot k \leq 3^k$  (this is our *inductive hypothesis*). Now, observe

$$\begin{aligned} 3 \cdot (k+1) &= 3 \cdot k + 3 && \text{by the distributive property of multiplication} \\ &\leq 3^k + 3 && \text{by the inductive hypothesis} \\ &\leq 3^k + 3^k && \text{because } 1 \leq k \Rightarrow a^1 \leq a^k \text{ if } a < 0^{*\dagger} \\ &\leq 3^k + 3^k + 3^k && \text{since } a^k \text{ is always positive if } 0 < a^{*\dagger} \\ &= 3 \cdot 3^k && \text{because } \sum_{i=1}^m a = m \cdot a \text{ for any } a^{*\dagger} \\ &= 3^{k+1} && \text{because } a^b a^c = a^{b+c} \text{ for any } a.^{*\dagger} \end{aligned}$$

So,  $3 \cdot (k+1) \leq 3^{k+1}$ , as desired.

Therefore,  $(\forall n \in \mathbb{N}) (3n \leq 3^n)$ .

Q.E.D.

\*These “basic grade-school” algebraic properties will now be assumed without special mention.

<sup>†</sup>This is true in any *ordered semiring* where exponentiation is defined in terms of multiplication.

<sup>‡</sup>This is true in any *field*.

**Lemma 4.3.**

$$\sum_{i=0}^n 2^i = 2^{n+1} - 1 \text{ for all } n \in \mathbb{N}.$$

*Proof.* We will prove the claim by induction on  $n \in \mathbb{N}_+$ .

**Base Case:**

$$\text{Observe that } \sum_{i=0}^0 2^i = 2^0 = 1 = 2 - 1 = 2^{1+1} - 1$$

**Inductive Step:**

Let  $k \in \mathbb{N}_+$  and assume that  $\sum_{i=0}^k 2^i = 2^{k+1} - 1$  (this is our *inductive hypothesis*). Observe that

$$\begin{aligned} \sum_{i=0}^{k+1} 2^i &= \left( \sum_{i=0}^k 2^i \right) + 2^{k+1} && \text{by definition} \\ &= (2^{k+1} - 1) + 2^{k+1} && \text{by the inductive hypothesis} \\ &= 2 \cdot 2^{k+1} - 1 && \text{using some basic algebra} \\ &= 2^{k+2} - 1 && \text{using some basic algebra.} \end{aligned}$$

So, we get  $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$ , as desired.

Therefore, we can conclude  $(\forall n \in \mathbb{N}) \left( \sum_{i=0}^n 2^i = 2^{n+1} - 1 \right)$ .

Q.E.D.

## 4.2 Strong Induction

**Theorem 4.3** (Strong Induction).

If  $\varphi(\cdot)$  is a wff and  $r \in \mathbb{N}$ , then

$$(\forall n \in \mathbb{N}) (\varphi(n)) \Leftrightarrow \left( \bigwedge_{i=0}^r \varphi(i) \right) \wedge (\forall k \in \mathbb{N}) \left( (\forall \ell \leq k) (\varphi(\ell)) \Rightarrow \varphi(k+1) \right).$$

**Lemma 4.1.**

$(\forall n \in \mathbb{N}) (\exists r \in \mathbb{N}) (\exists c_0, \dots, c_r \in \{0, 1\}) \left( n = \sum_{i=0}^r c_i 2^i \right)$ ; i.e., every natural number admits a binary representation.

*Proof.* We will prove the claim by using [Theorem 4.3](#).

Q.E.D.