

# Problem Set 5

## Discrete Mathematics

Due on the 27<sup>th</sup> of February, 2024

- (20 pts) 1. Show that  $\forall x(x \neq x \cup \{x\})$ .
- (20 pts) 2. We will work up to a proof of the commutativity of addition on  $\mathbb{N}$ .
- (a) Show  $(\forall x \in \mathbb{N})(x + 0 = 0 + x)$ .
- (b) Show  $(\forall x, y \in \mathbb{N})(x + s(y) = s(y) + x)$ .
- (c) Show  $(\forall x, y \in \mathbb{N})(x + y = y + x)$ .
- (20 pts) 3. Show  $(\forall x, y, z \in \mathbb{N})(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$ .
- (20 pts) 4. For this problem, you may assume the commutativity and associativity of addition and multiplication over  $\mathbb{N}$ . You may also assume multiplication distributes over addition on  $\mathbb{N}$ . Prove the following statement for all  $n \in \mathbb{N}$ .

$$1 + \sum_{i=0}^n 2^i = 2^{n+1}$$

- (20 pts) 5. We say  $x$  is *∈-transitive* by definition when  $(\forall y \in x)(\forall z \in y)(z \in x)$ . Show that every natural number is ∈-transitive.

Recall that the natural numbers are defined recursively as follows.

$$0 := \emptyset$$

$$s(n) := n \cup \{n\}$$

Addition on  $\mathbb{N}$  is defined below.

$$n + 0 := n$$

$$n + s(m) := s(n + m)$$

Multiplication on  $\mathbb{N}$  is defined below.

$$n \cdot 0 := 0$$

$$n \cdot s(m) := (n \cdot m) + n$$

Exponentiation on  $\mathbb{N}$  is defined below.

$$n^0 := 1$$

$$n^{s(m)} := n \cdot n^m$$

We define the iterated sum of a sequence of terms  $f(0), f(1), f(2), \dots$  as follows.

$$\sum_{i=0}^0 f(i) := f(0)$$

$$\sum_{i=0}^{s(n)} f(i) := \left( \sum_{i=0}^n f(i) \right) + f(s(n))$$

You may rely on the following theorems:

$$(\forall x \in \mathbb{N})(s(x) = x + 1).$$

$$(\forall x \in \mathbb{N})(s(x) = 1 + x).$$

$$(\forall x, y, z \in \mathbb{N})(x + (y + z) = (x + y) + z).$$