

# Discrete Mathematics

Daniel Gonzalez Cedre

University of Notre Dame  
Spring of 2023

## Chapter 4

# Mathematical Induction

### 4.1 Weak Induction

**Theorem 4.1** ( $\mathbb{N}$  is Well-Ordered).

$$(\forall A \subseteq \mathbb{N})(A \neq \emptyset \Rightarrow (\exists a \in A)(\forall b \in A)(a \leq b))$$

*This says that every nonempty subset of  $\mathbb{N}$  has a least element (according to the  $\leq$  order defined on  $\mathbb{N}$ ).*

*Proof.*

This proof is left as an exercise.

Q.E.D.

**Theorem 4.2** (Weak Induction).

*If  $\varphi(\cdot)$  is a wff, then*

$$(\forall n \in \mathbb{N})(\varphi(n)) \Leftrightarrow \varphi(0) \wedge (\forall k \in \mathbb{N})(\varphi(k) \Rightarrow \varphi(k+1)).$$

*Proof.*

There are two fragments to this proof: the forward ( $\Rightarrow$ ) direction and the backward ( $\Leftarrow$ ) direction.

**Fragment 1** ( $\Rightarrow$ ):

Suppose that  $(\forall n \in \mathbb{N})(\varphi(n))$ . Since  $0 \in \mathbb{N}$ , we then obviously have  $\varphi(0)$ . Now, let  $k \in \mathbb{N}$  and assume  $\varphi(k)$ . Since  $k+1 \in \mathbb{N}$ , we know from our initial assumption that  $\varphi(k+1)$ . Thus, we have  $(\forall k \in \mathbb{N})(\varphi(k) \Rightarrow \varphi(k+1))$ , and we have reached both of our desired conclusions.

**Fragment 2** ( $\Leftarrow$ ):

Assume  $\varphi(0)$  and  $(\forall k \in \mathbb{N})(\varphi(k) \Rightarrow \varphi(k+1))$ . Towards a contradiction, suppose that there is some  $n \in \mathbb{N}$  such that  $\neg\varphi(n)$ . Consider  $A := \{x \in \mathbb{N} \mid \neg\varphi(x)\}$ , which we clearly know exists by [Axiom 4](#). We know that  $n \in A$  because we assumed that  $\neg\varphi(n)$ , which implies that  $A \neq \emptyset$ . Then, we can use [Theorem 4.1](#) to conclude that there is a minimal element  $a$  in  $A$ .

Since we know that  $\varphi(0)$ , it follows that  $a \neq 0$ , so  $a$  must be a successor number. This means there is a  $b \in \mathbb{N}$  such that  $b+1 = a$ .

If  $b \in A$ , then that would mean that  $a \leq b$  since  $a$  is minimal in  $A$ . However, we know that  $b < b+1 = a$ , so we would then have  $a \leq b < a$ .  $\nexists$

Therefore,  $b \notin A$ .

Q.E.D.

Let's practice induction by proving the following theorem.

**Theorem 4.3** (Gaussian Summation Formula).

$$(\forall n \in \mathbb{N}) \left( \sum_{i=0}^n i = \frac{n(n+1)}{2} \right).$$

*Proof.*

We will prove the claim by induction on  $n \in \mathbb{N}$ .

**Base Case:**

Observe that  $\sum_{i=0}^0 i = 0 = \frac{0*(0+1)}{2}$ . Therefore, the statement is satisfied at 0.

**Inductive Step:**

Let  $k \in \mathbb{N}$  and assume  $\sum_{i=0}^k i = \frac{k(k+1)}{2}$  (this is our *inductive hypothesis*). Now, observe

$$\begin{aligned} \sum_{i=0}^{k+1} i &= \left( \sum_{i=0}^k i \right) + (k+1) && \text{by definition} \\ &= \left( \frac{k(k+1)}{2} \right) + (k+1) && \text{by the inductive hypothesis} \\ &= (k+1) \left( \frac{k}{2} + 1 \right) && \text{by factoring out } (k+1) \\ &= \frac{(k+1)(k+2)}{2} && \text{because } \frac{k}{2} + 1 = \frac{k}{2} + \frac{2}{2} = \frac{k+2}{2}. \end{aligned}$$

Thus, we have that  $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$ , as desired.

Therefore, we can conclude that  $(\forall n \in \mathbb{N}) \left( \sum_{i=0}^n i = \frac{n(n+1)}{2} \right)$ .

Q.E.D.