Discrete Mathematics

Daniel Gonzalez Cedre

University of Notre Dame
Spring of 2023

Chapter 2

First-Order Logic

2.1 Predicates & Quantification

Definition 2.1 (Universe of Discourse)**.**

Our *universe of discourse*, usually denoted by the letter Ω, denotes the collection of objects under consideration. This specifies the objects that are admissible as inputs to predicates, meaning these are the objects we can actually make concrete, descriptive claims about using our formal language.

Definition 2.2 (Predicate)**.**

Given a universe of discourse Ω and a non-negative integer n, we say that $\varphi(x_1, \ldots x_n)$ is an n-ary *predicate* $:\Leftrightarrow$ the substitution of an object ω_i from Ω for each x_i in φ results in a proposition $\varphi(\omega_1, \ldots \omega_n)$. These are sometimes referred to as *propositional functions*. Notice that if $n = 0$ then the 0-ary predicate is simply a proposition.

The placeholders $x_1, \ldots x_n$ in φ are called *variables*, and the process of assigning objects to these variables is called *instantiation* or *variable assignment*.

Example 2.1.

If we choose a universe of discourse Ω consisting of some collection of people, then

$$\mu(x) \coloneqq \text{“}x \text{ is a mathematician.”} \qquad \gamma(x, y) \coloneqq \text{“}x \text{ drinks a Guinness with } y\text{.”}$$
$$\varepsilon(x) \coloneqq \text{“}x \text{ loves espresso.”} \qquad \alpha(x, y, z) \coloneqq \text{“}x\text{, } y\text{, and } z \text{ are colleagues.”}$$

are unary, binary, and ternary predicates respectively.

Definition 2.3 (Universal Quantifier)**.**

Given a unary predicate $\varphi(x)$, the *universal quantification* of the variable x in φ is denoted by $\forall x\big(\varphi(x)\big)$ and expresses that $\varphi(\omega)$ is true for any arbitrary ω from our universe of discourse.

Example 2.2.

Using the definitions from Example 2.1, we can rewrite “Every mathematician loves espresso” as $\forall x\big(\varepsilon(x)\big)$ or as $\forall x\big(\mu(x) \to \varepsilon(x)\big)$ depending on whether our universe Ω is the collection of all mathematicians or the collection of all people.

Definition 2.4 (Existential Quantifier)**.**

Given a unary predicate $\varphi(x)$, the *existential quantification* of the variable x in φ is denoted by $\exists x\big(\varphi(x)\big)$ and expresses that $\varphi(\omega)$ is true for at least one ω from our universe of discourse.

Example 2.3.

Using the definitions from Example 2.1, we can rewrite “Some mathematician loves espresso” as $\exists x\big(\varepsilon(x)\big)$ or as $\exists x\big(\mu(x) \land \varepsilon(x)\big)$ depending on whether our universe Ω is the collection of all mathematicians or the collection of all people.

2.2 Rules of Inference