

Discrete Mathematics

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Chapter 4

Mathematical Induction

4.1 Weak Induction

Theorem 4.1 (\mathbb{N} is Well-Ordered).

$$(\forall A \subseteq \mathbb{N})(A \neq \emptyset \Rightarrow (\exists a \in A)(\forall b \in A)(a \leq b))$$

This says that every nonempty subset of \mathbb{N} has a least element (according to the \leq order defined on \mathbb{N}).

Proof.

This proof is left as an exercise.

Q.E.D.

Theorem 4.2 (Weak Induction).

If $\varphi(\cdot)$ is a wff, then

$$(\forall n \in \mathbb{N})(\varphi(n)) \Leftrightarrow \varphi(0) \wedge (\forall k \in \mathbb{N})(\varphi(k) \Rightarrow \varphi(k+1)).$$

Proof.

There are two fragments to this proof: the forward (\Rightarrow) direction and the backward (\Leftarrow) direction.

Fragment 1 (\Rightarrow):

Suppose that $(\forall n \in \mathbb{N})(\varphi(n))$. Since $0 \in \mathbb{N}$, we then obviously have $\varphi(0)$. Now, let $k \in \mathbb{N}$ and assume $\varphi(k)$. Since $k+1 \in \mathbb{N}$, we know from our initial assumption that $\varphi(k+1)$. Thus, we have $(\forall k \in \mathbb{N})(\varphi(k) \Rightarrow \varphi(k+1))$, and we have reached both of our desired conclusions.

Fragment 2 (\Leftarrow):

Assume $\varphi(0)$ and $(\forall k \in \mathbb{N})(\varphi(k) \Rightarrow \varphi(k+1))$. Towards a contradiction, suppose that there is some $n \in \mathbb{N}$ such that $\neg\varphi(n)$. Consider $A := \{x \in \mathbb{N} \mid \neg\varphi(x)\}$, which we clearly know exists by [Axiom 4](#). We know that $n \in A$ because we assumed that $\neg\varphi(n)$, which implies that $A \neq \emptyset$. Then, we can use [Theorem 4.1](#) to conclude that there is a minimal element a in A .

Since we know that $\varphi(0)$, it follows that $a \neq 0$, so a must be a successor number. This means there is a $b \in \mathbb{N}$ such that $b+1 = a$.

If $b \in A$, then that would mean that $a \leq b$ since a is minimal in A . However, we know that $b < b+1 = a$, so we would then have $a \leq b < a$. \nexists

Therefore, $b \notin A$.

Q.E.D.

Let's practice induction by proving the following theorem.

Theorem 4.3 (Gaussian Summation Formula).

$$(\forall n \in \mathbb{N}) \left(\sum_{i=0}^n i = \frac{n(n+1)}{2} \right).$$

Proof.

We will prove the claim by induction on $n \in \mathbb{N}$.

Base Case:

Observe that $\sum_{i=0}^0 i = 0 = \frac{0*(0+1)}{2}$. Therefore, the statement is satisfied at 0.

Inductive Step:

Let $k \in \mathbb{N}$ and assume $\sum_{i=0}^k i = \frac{k(k+1)}{2}$ (this is our *inductive hypothesis*). Now, observe

$$\begin{aligned} \sum_{i=0}^{k+1} i &= \left(\sum_{i=0}^k i \right) + (k+1) && \text{by definition} \\ &= \left(\frac{k(k+1)}{2} \right) + (k+1) && \text{by the inductive hypothesis} \\ &= (k+1) \left(\frac{k}{2} + 1 \right) && \text{by factoring out } (k+1) \\ &= \frac{(k+1)(k+2)}{2} && \text{because } \frac{k}{2} + 1 = \frac{k}{2} + \frac{2}{2} = \frac{k+2}{2}. \end{aligned}$$

Thus, we have that $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$, as desired.

Therefore, we can conclude that $(\forall n \in \mathbb{N}) \left(\sum_{i=0}^n i = \frac{n(n+1)}{2} \right)$.

Q.E.D.