Discrete Mathematics

Daniel Gonzalez Cedre

University of Notre Dame
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Chapter 7

Number Theory

7.1 Ancient Greece

Definition 7.1 ***(Divisibility).***
Given two integers $a, b \in \mathbb{Z}$, we say $a \mid b :\Leftrightarrow (\exists k \in \mathbb{Z})(ak = b)$. We read $a \mid b$ as a *divides* b, meaning $b/a \in \mathbb{Z}$. ⌟

Lemma 7.1 ***(Initial object).***
If $x \in \mathbb{Z}$, then $1 \mid x$. ⌟

Proof. Let $x \in \mathbb{Z}$ and observe that $1 \cdot x = x$. Therefore, $1 \mid x$ by definition. Q.E.D.

Lemma 7.2 ***(Terminal object).***
If $x \in \mathbb{Z}$, then $x \mid 0$. ⌟

Proof. Let $x \in \mathbb{Z}$ and observe that $0 \cdot x = 0$. Therefore, $x \mid 0$ by definition. Q.E.D.

Lemma 7.3 ***(Reflexivity).***
$(\forall x \in \mathbb{Z})\big(x \mid x\big)$. ⌟

Lemma 7.4 ***(Anti-Symmetry).***
$(\forall x \in \mathbb{Z})\big(x \mid x\big)$. ⌟

Proof. Let $x \in \mathbb{Z}$ and observe that $1 \cdot x = x$. Therefore, $x \mid x$ by definition. Q.E.D.

Lemma 7.5 ***(Divisibility is a Partial Order).***
The following statements hold for all $a, b, c \in \mathbb{Z}$:

I. $a \mid a$

   Proof. Let $a \in \mathbb{Z}$ and observe that $1 \cdot a = a$. Therefore, $a \mid a$ by definition. Q.E.D.

II. $\big(a \mid b\big) \ \wedge \ \big(b \mid a\big) \ \Rightarrow \ |a| = |b|$

   Proof. Let $a, b \in Z$ and suppose $a \mid b$ and $b \mid a$. Then, there exist $k_1, k_2 \in \mathbb{Z}$ such that $ak_1 = b$ and $bk_2 = a$ by definition. But then $bk_2 = (ak_1)k_2 = a$, so $ak_1k_2 = a$, yielding $k_1k_2 = 1$. Since the only integers with multiplicative inverses are 1 and -1, we have $\{k_1, k_2\} \subseteq \{1, -1\}$, so $a = b$ or $a = -b$. Thus, $|a| = |b|$. Q.E.D.

III. $\big(a \mid b \ \wedge \ b \mid c\big) \ \Rightarrow \ a \mid c$

   Proof. Let $a, b, c \in Z$ and suppose $a \mid b$ and $b \mid c$. Then, there exist $k_1, k_2 \in \mathbb{Z}$ such that $ak_1 = b$ and $bk_2 = c$. This yields $ak_1k_2 = c$. Since $k_1, k_2 \in \mathbb{Z}$, we observe $k_1k_2 \in \mathbb{Z}$ and conclude $a \mid c$ by definition. Q.E.D.

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Lemma 7.6 ***(Useful facts).***
The following statements hold for all $a, b, c \in \mathbb{Z}$:

I. $\big(a \mid b \ \wedge \ a \mid c\big) \ \Rightarrow \ a \mid b + c$

II. $a \mid b \ \Rightarrow \ (\forall \ell \in \mathbb{Z})\big(a \mid b\ell\big)$

III. $a \mid b \ \Rightarrow \ |a| \leqslant |b|$

The proofs of the above lemmata are left as exercises to the reader. ⌟

Corollary 7.1.
Given $a, b, c \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$, then $(\forall \ell_1, \ell_2 \in \mathbb{Z})\big(a \mid \ell_1 b + \ell_2 c\big)$. ⌟

Proof. Let $a, b, c \in \mathbb{Z}$ and suppose $a \mid b$ and $a \mid c$. Let $\ell_1, \ell_2 \in \mathbb{Z}$. From Lemma 7.6, we know $a \mid b\ell_1$ and $a \mid c\ell_2$, implying $a \mid b\ell_1 + c\ell_2$ by Lemma 7.6. Q.E.D.

Definition 7.2 ***(Prime Numbers).***
We say that a natural number $p \in \mathbb{N}$ is *prime* $:\Leftrightarrow$ $(p > 1)$ and $(\forall n \in \mathbb{N})\big(n \mid p \ \Rightarrow \ n \in \{1, p\}\big)$.
We say $n \in \mathbb{N}$ is *composite* $:\Leftrightarrow$ n is not prime. ⌟

Lemma 7.7 ***(Fundamental Lemma of Arithmetic).***
If $n \in \mathbb{N}$ and $n > 1$, then $(\exists p \in \mathbb{N})\big(p \text{ is prime} \wedge p \mid n\big)$. ⌟

Theorem 7.1 ***(Fundamental Theorem of Arithmetic).***
Every natural number greater than 1 has a *unique* prime factorization. Formally, for every natural number $n \in \mathbb{N}_{\geqslant 2}$ greater than 1, there exist *unique, distinct* primes $p_1, \ldots p_\ell \in \mathbb{N}_+$ with *unique* exponents $k_1, \ldots k_\ell \in \mathbb{N}_+$ such that

I. $\big(\forall i, j \in \{1, \ldots \ell\}\big)\big(i \neq j \ \Rightarrow \ p_i \neq p_j\big)$

II. $\big(\forall i \in \{1, \ldots \ell\}\big)(p_i \text{ is prime})$

III. $n = p_1^{k_1} p_2^{k_2} \ldots p_\ell^{k_\ell}$.

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Theorem 7.2 ***(Euclid's Theorem).***
There are infinitely-many prime numbers. ⌟