

Discrete Mathematics

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Chapter 4

Mathematical Induction

4.1 Weak Induction

Definition 4.1 (Well-Order). Let X be a set with a relation \leq defined on it. We say that X is *well-ordered* by this relation *iff* every non-empty subset of X has a minimal element with respect to \leq . In other words, we say that \leq is a *well-order* on X $:\Leftrightarrow$

$$(\forall A \in \mathcal{P}(X))(\exists a \in A)(\forall b \in A)a \leq b.$$

Theorem 4.1 (\mathbb{N} is Well-Ordered).

$$(\forall A \subseteq \mathbb{N})(A \neq \emptyset \Rightarrow (\exists a \in A)(\forall b \in A)(a \leq b))$$

This says that every nonempty subset of \mathbb{N} has a least element (according to the \leq order defined on \mathbb{N}).

Proof. This proof is left as an exercise.

Q.E.D.

Theorem 4.2 (Weak Induction). *If $\varphi(\cdot)$ is a wff, then*

$$(\forall n \in \mathbb{N})(\varphi(n)) \Leftrightarrow \varphi(0) \wedge (\forall k \in \mathbb{N})(\varphi(k) \Rightarrow \varphi(k+1)).$$

Proof. There are two fragments to this proof: the forward (\Rightarrow) direction and the backward (\Leftarrow) direction.

Fragment 1 (\Rightarrow):

Suppose that $(\forall n \in \mathbb{N})(\varphi(n))$. Since $0 \in \mathbb{N}$, we then obviously have $\varphi(0)$. Now, let $k \in \mathbb{N}$ and assume $\varphi(k)$. Since $k+1 \in \mathbb{N}$, we know from our initial assumption that $\varphi(k+1)$. Thus, we have $(\forall k \in \mathbb{N})(\varphi(k) \Rightarrow \varphi(k+1))$, and we have reached both of our desired conclusions.

Fragment 2 (\Leftarrow):

Assume $\varphi(0)$ and $(\forall k \in \mathbb{N})(\varphi(k) \Rightarrow \varphi(k+1))$. Towards a contradiction, suppose that there is some $n \in \mathbb{N}$ such that $\neg\varphi(n)$. Consider $A := \{x \in \mathbb{N} \mid \neg\varphi(x)\}$, which we clearly know exists by [Axiom 4](#). We know that $n \in A$ because we assumed that $\neg\varphi(n)$, which implies that $A \neq \emptyset$. Then, we can use [Theorem 4.1](#) to conclude that there is a minimal element a in A .

Since we know that $\varphi(0)$, it follows that $a \neq 0$, so a must be a successor number. This means there is a $b \in \mathbb{N}$ such that $b+1 = a$.

If $b \in A$, then that would mean that $a \leq b$ since a is minimal in A . However, we know that $b < b+1 = a$, so we would then have $a \leq b < a$. \nexists Therefore, $b \notin A$.

With these two directions proven, we finally have $(\forall n \in \mathbb{N})(\varphi(n)) \Leftrightarrow \varphi(0) \wedge (\forall k \in \mathbb{N})(\varphi(k) \Rightarrow \varphi(k+1))$.

Q.E.D.

Notice that the above theorem actually generalizes beyond just \mathbb{N} . In fact, we can generalize the proof of [Theorem 4.2](#) to *any* well-ordered set X by replacing $k+1$ with the least element of the non-empty subset $X \setminus \{\ell \in X \mid \ell \leq k\}$.

Let's practice induction by proving the following few theorems.

Theorem 4.3 (Gaussian Summation Formula). $(\forall n \in \mathbb{N}) \left(\sum_{i=0}^n i = \frac{n(n+1)}{2} \right)$.

Proof. We will prove the claim by induction on $n \in \mathbb{N}$.

Base Case:

Observe that $\sum_{i=0}^0 i = 0 = \frac{0 \cdot (0+1)}{2}$. Therefore, the statement is satisfied at 0.

Inductive Step:

Let $k \in \mathbb{N}$ and assume $\sum_{i=0}^k i = \frac{k(k+1)}{2}$ (this is our *inductive hypothesis*). Now, observe

$$\begin{aligned} \sum_{i=0}^{k+1} i &= \left(\sum_{i=0}^k i \right) + (k+1) && \text{by definition} \\ &= \left(\frac{k(k+1)}{2} \right) + (k+1) && \text{by the inductive hypothesis} \\ &= (k+1) \left(\frac{k}{2} + 1 \right) && \text{by the distributive property of multiplication}^{*\dagger} \\ &= \frac{(k+1)(k+2)}{2} && \text{because } \frac{k}{2} + 1 = \frac{k}{2} + \frac{2}{2} = \frac{k+2}{2}.^{\ddagger} \end{aligned}$$

Thus, we have that $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$, as desired.

Therefore, we can conclude that $(\forall n \in \mathbb{N}) \left(\sum_{i=0}^n i = \frac{n(n+1)}{2} \right)$.

Q.E.D.

Theorem 4.4. $3n \leq 3^n$ for all $n \in \mathbb{N}_+$.

Proof. We will prove the claim by induction on $n \in \mathbb{N}_+$.

Base Case:

Observe that $3 \cdot 0 = 0 \leq 1 = 3^0$.

Inductive Step:

Let $k \in \mathbb{N}_+$ and assume $3 \cdot k \leq 3^k$ (this is our *inductive hypothesis*). Now, observe

$$\begin{aligned} 3 \cdot (k+1) &= 3 \cdot k + 3 && \text{by the distributive property of multiplication} \\ &\leq 3^k + 3 && \text{by the inductive hypothesis} \\ &\leq 3^k + 3^k && \text{because } 1 \leq k \Rightarrow a^1 \leq a^k \text{ if } a < 0^{*\dagger} \\ &\leq 3^k + 3^k + 3^k && \text{since } a^k \text{ is always positive if } 0 < a^{*\dagger} \\ &= 3 \cdot 3^k && \text{because } \sum_{i=1}^m a = m \cdot a \text{ for any } a^{*\dagger} \\ &= 3^{k+1} && \text{because } a^b a^c = a^{b+c} \text{ for any } a.^{*\dagger} \end{aligned}$$

So, $3 \cdot (k+1) \leq 3^{k+1}$, as desired.

Therefore, $(\forall n \in \mathbb{N}) (3n \leq 3^n)$.

Q.E.D.

*These “basic grade-school” algebraic properties will now be assumed without special mention.

[†]This is true in any *ordered semiring* where exponentiation is defined in terms of multiplication.

[‡]This is true in any *field*.

Theorem 4.5. $\sum_{i=0}^n 2^i = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

Proof. We will prove the claim by induction on $n \in \mathbb{N}_+$.

Base Case:

Observe that $\sum_{i=0}^0 2^i = 2^0 = 1 = 2 - 1 = 2^{1+1} - 1$

Inductive Step:

Let $k \in \mathbb{N}_+$ and assume that $\sum_{i=0}^k 2^i = 2^{k+1} - 1$ (this is our *inductive hypothesis*). Observe that

$$\begin{aligned} \sum_{i=0}^{k+1} 2^i &= \left(\sum_{i=0}^k 2^i \right) + 2^{k+1} && \text{by definition} \\ &= (2^{k+1} - 1) + 2^{k+1} && \text{by the inductive hypothesis} \\ &= 2 \cdot 2^{k+1} - 1 && \text{using some basic algebra} \\ &= 2^{k+2} - 1 && \text{using some basic algebra.} \end{aligned}$$

So, we get $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$, as desired.

Therefore, we can conclude $(\forall n \in \mathbb{N}) \left(\sum_{i=0}^n 2^i = 2^{n+1} - 1 \right)$.

Q.E.D.