

# PROPOSITIONAL LOGIC

DISCRETE MATHEMATICS

DANIEL GONZALEZ CEDRE

## Definition 1 (Proposition).

A *proposition* is a sentence (in our language) that has one (and only one) definite, consistent truth value.

## Definition 2 (Negation).

Given a proposition  $p$ , the *negation* of  $p$ , denoted  $\neg p$ , is defined by

$p$	$\neg p$
$\top$	$\perp$
$\perp$	$\top$

Some possible readings of  $\neg p$ :

- Not  $p$ .
- $p$  does not hold.
- It is not the case that  $p$ .
- We do not have that  $p$ .

## Definition 3 (Conjunction).

Given two propositions  $p$  and  $q$ , the *conjunction* of  $p$  with  $q$ , denoted  $p \wedge q$ , is defined by

$p$	$q$	$p \wedge q$
$\top$	$\top$	$\top$
$\top$	$\perp$	$\perp$
$\perp$	$\top$	$\perp$
$\perp$	$\perp$	$\perp$

Some possible readings of  $p \wedge q$ :

- $p$ , and  $q$ .
- $p$ , but  $q$ .
- $p$ ; also,  $q$ .
- $p$ ; further,  $q$ .
- In addition to  $p$ , we also have  $q$ .

## Definition 4 (Disjunction).

Given two propositions  $p$  and  $q$ , the *disjunction* of  $p$  with  $q$ , denoted  $p \vee q$ , is defined by

$p$	$q$	$p \vee q$
$\top$	$\top$	$\top$
$\top$	$\perp$	$\top$
$\perp$	$\top$	$\top$
$\perp$	$\perp$	$\perp$

Some possible readings of  $p \vee q$ :

- $p$ , or  $q$ .
- Either  $p$ , or  $q$ .

## Definition 5 (Material Implication).

Given two propositions  $p$  and  $q$ , the *conditional* formed by assuming  $p$  and concluding  $q$ , denoted  $p \rightarrow q$ , is defined by

$p$	$q$	$p \rightarrow q$
$\top$	$\top$	$\top$
$\top$	$\perp$	$\perp$
$\perp$	$\top$	$\top$
$\perp$	$\perp$	$\top$

Some possible readings of  $p \rightarrow q$ :

- If  $p$ , then  $q$ .
- $p$  implies  $q$ .
- $q$  is conditioned on  $p$ .
- $q$  only if  $p$ .
- $p$  is sufficient for  $q$ .
- $q$  is necessary for  $p$ .
- $q$  unless not  $p$ .
- $q$  or not  $p$ .

**Definition 6** (Biconditional).

Given two propositions  $p$  and  $q$ , the *biconditional* formed by  $p$  and  $q$ , denoted  $p \leftrightarrow q$ , is defined by

$p$	$q$	$p \leftrightarrow q$
$\top$	$\top$	$\top$
$\top$	$\perp$	$\perp$
$\perp$	$\top$	$\perp$
$\perp$	$\perp$	$\top$

Some possible readings of  $pq$ :

- $p$  if and only if  $q$ .
- $p$  is necessary and sufficient for  $q$ .
- $q$  is necessary and sufficient for  $p$ .

**Definition 7** (Equivalence).

If we have two expressions  $\varphi$  and  $\psi$  in our formal language, consisting of some number of (possibly shared) propositional variables, connected together by logical connectives, then with the notation  $\varphi \Leftrightarrow \psi$  we say that  $\varphi$  is equivalent to  $\psi$  : $\Leftrightarrow$  every assignment of truth values to the propositional variables of  $\varphi$  and  $\psi$  results in the same truth value for the two expressions.

**Example 1.**

Consider the two expressions  $\varphi := p \rightarrow q$  and  $\psi := \neg p \vee q$ . We can see that  $\varphi$  has two propositional variables:  $p$  and  $q$ .  $\psi$  also has two propositional variables: the same  $p$  and the same  $q$ .

If we construct the truth table for these two expressions, we will see that every assignment of truth values to  $p$  and  $q$  will result in  $p \rightarrow q$  and  $\neg p \vee q$  having the same truth value.

$p$	$q$	$p \rightarrow q$	$\neg p \vee q$
$\top$	$\top$	$\top$	$\top$
$\top$	$\perp$	$\perp$	$\perp$
$\perp$	$\top$	$\top$	$\top$
$\perp$	$\perp$	$\top$	$\top$

**Definition 8** (Boolean Algebra).

A *Boolean algebra* is a collection of *terms*  $B$  with two distinguished (and distinct) terms called  $\top$  and  $\perp$ , along with a unary operation called  $\neg$  and two binary operations called  $\wedge$  and  $\vee$ , such that the following statements are true for any terms  $p, q, r$  in  $B$ :

Axioms of a Boolean Algebra	
Identity	$p \wedge \top \Leftrightarrow p$ $p \vee \perp \Leftrightarrow p$
Complement ( <i>a.k.a.</i> Negation)	$p \wedge \neg p \Leftrightarrow \perp$ $p \vee \neg p \Leftrightarrow \top$
Commutativity	$p \wedge q \Leftrightarrow q \wedge p$ $p \vee q \Leftrightarrow q \vee p$
Associativity	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
Distributive Laws	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

This kind of structure is also referred to as a *complemented, distributive lattice*. Since we are establishing the algebra of *propositions*, our terms consist only of  $\top$  and  $\perp$ .

However, since none of these axioms tell us how to use the (very useful) symbols  $\rightarrow$  and  $\leftrightarrow$ , we need two additional axioms that will turn our Boolean algebra into an example of a *Heyting algebra*:

Heyting Axioms	
Conditional Disintegration	$p \rightarrow q \Leftrightarrow \neg p \vee q$
Biconditional Disintegration	$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

By referring to the truth tables, it should be easy to see that these axioms are *truth preserving* transformations, meaning that taking an expression like  $p \wedge (q \rightarrow r)$  and applying an axiom like Identity to it does not change the truth value of the resulting expression  $(p \wedge \top) \wedge (q \rightarrow r)$ . For this reason, these are sometimes referred to as *equivalence laws* and many treatments of this subject *prove* these laws by referring to the truth tables.

For our purposes, we don't need to refer to the truth tables at all. The truth tables were a nice, intuitive, and compact way of defining the logical connectives, but we could just as easily have defined them by assuming that all of the axioms are true, without ever writing down a truth table.

**Theorem 1** (Uniqueness of Complements).

Let  $p$  be a term in a Boolean algebra  $(B, \neg, \wedge, \vee)$ . Suppose there were two terms  $x$  and  $y$  in the Boolean algebra such that

$$\begin{aligned} p \wedge x &\Leftrightarrow \perp, & p \wedge y &\Leftrightarrow \perp, \\ p \vee x &\Leftrightarrow \top, & p \vee y &\Leftrightarrow \top, \end{aligned}$$

meaning that  $x$  and  $y$  act like negations for  $p$ . Then, we have  $x \Leftrightarrow y$ .

*Proof.*

Let  $(B, \neg, \wedge, \vee)$  be a Boolean algebra and consider an arbitrary term  $p$  in the algebra. Suppose we have  $x$  and  $y$  satisfying the conditions given in the statement of the theorem above. Then, we can observe

$$\begin{aligned} x &\Leftrightarrow \top \wedge x && \text{by Identity} \\ &\Leftrightarrow (p \vee y) \wedge x && \text{by the assumption } p \vee y \Leftrightarrow \top \\ &\Leftrightarrow (p \wedge x) \vee (y \wedge x) && \text{by Distributivity} \\ &\Leftrightarrow \perp \vee (y \wedge x) && \text{by the assumption } p \wedge x \Leftrightarrow \perp \\ &\Leftrightarrow y \wedge x && \text{by Identity.} \end{aligned}$$

Similarly, we can see that

$$\begin{aligned} y &\Leftrightarrow \top \wedge y && \text{by Identity} \\ &\Leftrightarrow (p \vee x) \wedge y && \text{by the assumption } p \vee x \Leftrightarrow \top \\ &\Leftrightarrow (p \wedge y) \vee (x \wedge y) && \text{by Distributivity} \\ &\Leftrightarrow \perp \vee (x \wedge y) && \text{by the assumption } p \wedge y \Leftrightarrow \perp \\ &\Leftrightarrow x \wedge y && \text{by Identity.} \end{aligned}$$

So, from Commutativity, we can conclude that

$$x \Leftrightarrow (y \wedge x) \Leftrightarrow (x \wedge y) \Leftrightarrow y.$$

Q.E.D.

**Theorem 2** (Double Negation).

For any Boolean algebra  $(B, \neg, \wedge, \vee)$  and any term  $p$  in the algebra, we have  $\neg\neg p \Leftrightarrow p$ .

*Proof.*

Let  $(B, \neg, \wedge, \vee)$  be a Boolean algebra and consider an arbitrary term  $p$  in the algebra. We want to show that  $\neg\neg p \Leftrightarrow p$ .

By the Complement axiom, we know  $p \wedge \neg p \Leftrightarrow \perp$  and  $p \vee \neg p \Leftrightarrow \top$ , meaning that  $p$  is the complement of  $\neg p$ . Similarly, we can see that  $\neg p \wedge \neg(\neg p) \Leftrightarrow \perp$  and  $\neg p \vee \neg(\neg p) \Leftrightarrow \top$ , showing us that  $\neg\neg p$  is the complement of  $\neg p$ .

Since complements are unique, as seen in [Theorem 1](#), and both  $p$  and  $\neg\neg p$  are complements of  $\neg p$ , we must have that  $p \Leftrightarrow \neg\neg p$ .

Q.E.D.

**Theorem 3** (Idempotency).

For every Boolean algebra  $(B, \neg, \wedge, \vee)$  and every term  $p$  in the algebra, we have

$$\begin{aligned} p \wedge p &\Leftrightarrow p \\ p \vee p &\Leftrightarrow p. \end{aligned}$$

*Proof.*

Let  $p$  be a term in an arbitrary Boolean algebra  $(B, \neg, \wedge, \vee)$ . Observe

$p \wedge p \Leftrightarrow (p \wedge p) \vee \perp$	by Identity	$p \vee p \Leftrightarrow (p \vee p) \wedge \top$	by Identity
$\Leftrightarrow (p \wedge p) \vee (p \wedge \neg p)$	by Complement	$\Leftrightarrow (p \vee p) \wedge (p \vee \neg p)$	by Complement
$\Leftrightarrow p \wedge (p \vee \neg p)$	by Distributivity	$\Leftrightarrow p \vee (p \wedge \neg p)$	by Distributivity
$\Leftrightarrow p \wedge \top$	by Complement	$\Leftrightarrow p \vee \perp$	by Complement
$\Leftrightarrow p$	by Identity,	$\Leftrightarrow p$	by Identity.

Therefore,  $p \wedge p \Leftrightarrow p$  and  $p \vee p \Leftrightarrow p$ , as desired.

Q.E.D.

**Theorem 4** (Domination).

For every Boolean algebra  $(B, \neg, \wedge, \vee)$  and every term  $p$  in the algebra, we have

$$\begin{aligned} p \wedge \perp &\Leftrightarrow \perp \\ p \vee \top &\Leftrightarrow \top. \end{aligned}$$

*Proof.*

Let  $p$  be a term in an arbitrary Boolean algebra  $(B, \neg, \wedge, \vee)$ . Observe

$p \wedge \perp \Leftrightarrow p \wedge (p \wedge \neg p)$	by Complement	$p \vee \top \Leftrightarrow p \vee (p \vee \neg p)$	by Complement
$\Leftrightarrow (p \wedge p) \wedge \neg p$	by Associativity	$\Leftrightarrow (p \vee p) \vee \neg p$	by Associativity
$\Leftrightarrow p \wedge \neg p$	by Idempotency	$\Leftrightarrow p \vee \neg p$	by Idempotency
$\Leftrightarrow \perp$	by Complement,	$\Leftrightarrow \top$	by Complement,

Therefore,  $p \wedge \perp \Leftrightarrow \perp$  and  $p \vee \top \Leftrightarrow \top$ .

Q.E.D.

**Theorem 5** (Absorption).

For every Boolean algebra  $(B, \neg, \wedge, \vee)$  and any two terms  $p$  and  $q$  in the algebra, we have

$$\begin{aligned} p \wedge (p \vee q) &\Leftrightarrow p \\ p \vee (p \wedge q) &\Leftrightarrow q. \end{aligned}$$

*Proof.*

Let  $p$  and  $q$  be terms in an arbitrary Boolean algebra  $(B, \neg, \wedge, \vee)$ . Observe

$$\begin{array}{ll} p \wedge (p \vee q) \Leftrightarrow (p \vee \perp) \wedge (p \vee q) & \text{by Identity} \\ \Leftrightarrow p \vee (\perp \wedge q) & \text{by Distributivity} \\ \Leftrightarrow p \vee \perp & \text{by Domination} \\ \Leftrightarrow p & \text{by Identity,} \end{array} \quad \begin{array}{ll} p \vee (p \wedge q) \Leftrightarrow (p \wedge \top) \vee (p \wedge q) & \text{by Identity} \\ \Leftrightarrow p \wedge (\top \vee q) & \text{by Distributivity} \\ \Leftrightarrow p \wedge \top & \text{by Domination} \\ \Leftrightarrow p & \text{by Identity,} \end{array}$$

Therefore,  $p \wedge (p \vee q) \Leftrightarrow p$  and  $p \vee (p \wedge q) \Leftrightarrow q$ .

Q.E.D.

**Theorem 6** (De Morgan's Laws).

For every Boolean algebra  $(B, \neg, \wedge, \vee)$  and any two terms  $p$  and  $q$  in the algebra, we have

$$\begin{aligned} \neg(p \wedge q) &\Leftrightarrow \neg p \vee \neg q \\ \neg(p \vee q) &\Leftrightarrow \neg p \wedge \neg q. \end{aligned}$$

*Proof.*

Let  $p$  and  $q$  be terms in an arbitrary Boolean algebra  $(B, \neg, \wedge, \vee)$ . For the first statement, we need to show that  $p \wedge q$  is a complement for  $\neg p \vee \neg q$ , meaning  $(p \wedge q) \wedge (\neg p \vee \neg q) \Leftrightarrow \perp$  and  $(p \wedge q) \vee (\neg p \vee \neg q) \Leftrightarrow \top$ .

Observe that

$$\begin{aligned} (p \wedge q) \wedge (\neg p \vee \neg q) &\Leftrightarrow (q \wedge p) \wedge (\neg p \vee \neg q) && \text{by Commutativity} \\ &\Leftrightarrow q \wedge (p \wedge (\neg p \vee \neg q)) && \text{by Associativity} \\ &\Leftrightarrow q \wedge ((p \wedge \neg p) \vee (p \wedge \neg q)) && \text{by Distributivity} \\ &\Leftrightarrow q \wedge (\perp \vee (p \wedge \neg q)) && \text{by Complement} \\ &\Leftrightarrow q \wedge (p \wedge \neg q) && \text{by Identity} \\ &\Leftrightarrow q \wedge (\neg q \wedge p) && \text{by Commutativity} \\ &\Leftrightarrow (q \wedge \neg q) \wedge p && \text{by Associativity} \\ &\Leftrightarrow \perp \wedge p && \text{by Complement} \\ &\Leftrightarrow \perp && \text{by Domination.} \end{aligned}$$

Further, we have

$$\begin{aligned} (p \wedge q) \vee (\neg p \vee \neg q) &\Leftrightarrow (p \wedge q) \vee (\neg q \vee \neg p) && \text{by Commutativity} \\ &\Leftrightarrow ((p \wedge q) \vee \neg q) \vee \neg p && \text{by Associativity} \\ &\Leftrightarrow ((p \vee \neg q) \wedge (q \vee \neg q)) \vee \neg p && \text{by Distributivity} \\ &\Leftrightarrow ((p \vee \neg q) \wedge \top) \vee \neg p && \text{by Complement} \\ &\Leftrightarrow (p \vee \neg q) \vee \neg p && \text{by Identity} \\ &\Leftrightarrow (\neg q \vee p) \vee \neg p && \text{by Commutativity} \\ &\Leftrightarrow \neg q \vee (p \vee \neg p) && \text{by Associativity} \\ &\Leftrightarrow \neg q \vee \top && \text{by Complement} \\ &\Leftrightarrow \top && \text{by Domination.} \end{aligned}$$

So, we can see that  $p \wedge q$  is a complement for  $\neg p \vee \neg q$ . Since complements are unique by [Theorem 1](#), we can conclude that  $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ .

Showing  $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$  is left as an exercise to the reader.

Q.E.D.