

## Problem Set 10

### Discrete Mathematics

Due on the 22<sup>nd</sup> of April, 2024

- (40 pts) 1. Construct—with proof—the explicit functions requested below.
- (a) A bijection from  $\{x \in \mathbb{R} \mid -1 < x < 1\}$  to  $\{x \in \mathbb{R} \mid -\pi < x < \pi\}$ .
  - (b) A surjection from  $\mathbb{N}$  to  $\{p \mid p \text{ is prime}\}$ .
  - (c) An injection from  $X$  to  $\mathbb{P}(X)$  for every set  $X$ .
  - (d) A surjection from  $\mathbb{P}(X)$  to  $X$  for every set  $X \neq \emptyset$ .
- (30 pts) 2. Let  $A$  be an arbitrary finite set of cardinality  $|A| = n$ , where  $n \in \mathbb{N}$ . How many finite strings over  $A$  are there?
- (30 pts) 3. Imagine that, one day, you encounter a library.<sup>1</sup> At the entrance of this library is an enormous tome  $\mathcal{B}$  listing *all* of the *possible* sentences in the English language, indexed by natural numbers.

Walking past the entrance, you see that the library has rows of bookshelves numbered  $0, 1, 2, \dots$ , so that there is exactly one row of books for each  $n \in \mathbb{N}$ . A great owl—perched on the pedestal that supports  $\mathcal{B}$ —informs you that, for each  $n \in \mathbb{N}$ , the  $n^{\text{th}}$  row of bookshelves contains *all* of the books that could possibly ever be that begin with the  $n^{\text{th}}$  sentence in  $\mathcal{B}$ . With the sound of granite scraping against marble, the doors to the library close behind you. The owl makes you the following proposal: you will be free to go *if and only if* you can read every book in the library *in a countable amount of time*.

Will you be set free? The owl demands a proof to justify your answer.

<sup>1</sup> This library is so massive and so complete a collection of literary works that it even contains books of *infinite length*.