

# Discrete Mathematics

Daniel Gonzalez Cedre

University of Notre Dame  
Spring of 2023

# Chapter 1

## Propositional Logic

### 1.1 Propositions & Connectives

#### Definition 1.1 (Proposition).

A *proposition* is a sentence (in our language) that has one (and only one) definite, consistent truth value.

#### Definition 1.2 (Negation).

Given a proposition  $p$ , the *negation* of  $p$  is denoted  $\neg p$  and can be defined by the following truth table:

$p$	$\neg p$
$\top$	$\perp$
$\perp$	$\top$

Some possible readings of  $\neg p$ :

- Not  $p$ .
- $p$  does not hold.
- It is not the case that  $p$ .
- We do not have that  $p$ .

#### Definition 1.3 (Conjunction).

Given two propositions  $p$  and  $q$ , the *conjunction* of  $p$  with  $q$  is denoted  $p \wedge q$  and can be defined by the following truth table:

$p$	$q$	$p \wedge q$
$\top$	$\top$	$\top$
$\top$	$\perp$	$\perp$
$\perp$	$\top$	$\perp$
$\perp$	$\perp$	$\perp$

Some possible readings of  $p \wedge q$ :

- $p$ , and  $q$ .
- $p$ , but  $q$ .
- $p$ ; also,  $q$ .
- $p$ ; further,  $q$ .
- In addition to  $p$ , we also have  $q$ .

#### Definition 1.4 (Disjunction).

Given two propositions  $p$  and  $q$ , the *disjunction* of  $p$  with  $q$  is denoted  $p \vee q$  and can be defined by the following truth table:

$p$	$q$	$p \vee q$
$\top$	$\top$	$\top$
$\top$	$\perp$	$\top$
$\perp$	$\top$	$\top$
$\perp$	$\perp$	$\perp$

Some possible readings of  $p \vee q$ :

- $p$ , or  $q$ .
- Either  $p$ , or  $q$ .

**Definition 1.5** (Material Implication).

Given two propositions  $p$  and  $q$ , the *conditional* formed by assuming  $p$  and concluding  $q$  is denoted  $p \rightarrow q$  and can be defined by the following truth table:

$p$	$q$	$p \rightarrow q$
$\top$	$\top$	$\top$
$\top$	$\perp$	$\perp$
$\perp$	$\top$	$\top$
$\perp$	$\perp$	$\top$

Some possible readings of  $p \rightarrow q$ :

- If  $p$ , then  $q$ .
- $p$  implies  $q$ .
- $q$  is conditioned on  $p$ .
- $q$  only if  $p$ .
- $p$  is sufficient for  $q$ .
- $q$  is necessary for  $p$ .
- $q$  unless not  $p$ .
- $q$  or not  $p$ .

**Definition 1.6** (Material Equivalence).

Given two propositions  $p$  and  $q$ , the *biconditional* formed by  $p$  and  $q$  is denoted  $p \leftrightarrow q$  and can be defined by the following truth table:

$p$	$q$	$p \leftrightarrow q$
$\top$	$\top$	$\top$
$\top$	$\perp$	$\perp$
$\perp$	$\top$	$\perp$
$\perp$	$\perp$	$\top$

Some possible readings of  $p \leftrightarrow q$ :

- $p$  if and only if  $q$ .
- $p$  is necessary and sufficient for  $q$ .
- $q$  is necessary and sufficient for  $p$ .

**Definition 1.7** (Equivalence).

If we have two expressions  $\varphi$  and  $\psi$  in our formal language, consisting of some number of (possibly shared) propositional variables, connected together by logical connectives, then with the notation  $\varphi \Leftrightarrow \psi$  we say that  $\varphi$  is equivalent to  $\psi$   $\Leftrightarrow$  every assignment of truth values to the propositional variables of  $\varphi$  and  $\psi$  results in the same truth value for the two expressions.

**Example 1.1.**

Consider the two expressions  $\varphi := p \rightarrow q$  and  $\psi := \neg p \vee q$ . We can see that  $\varphi$  has two propositional variables:  $p$  and  $q$ .  $\psi$  also has two propositional variables: the same  $p$  and the same  $q$ . If we construct the truth table for these two expressions, we will see that every assignment of truth values to  $p$  and  $q$  will result in  $p \rightarrow q$  and  $\neg p \vee q$  having the same truth value.

$p$	$q$	$p \rightarrow q$	$\neg p \vee q$
$\top$	$\top$	$\top$	$\top$
$\top$	$\perp$	$\perp$	$\perp$
$\perp$	$\top$	$\top$	$\top$
$\perp$	$\perp$	$\top$	$\top$

**Definition 1.8** (Satisfiability).

A propositional expression  $\varphi$  consisting the propositional variables  $p_1, \dots, p_n$  is *satisfiable*  $\Leftrightarrow$  there exists an assignment of truth values to  $p_1, \dots, p_n$  that results in  $\varphi$  being equivalent to  $\top$ .

## 1.2 Boolean Algebras

**Definition 1.9** (Boolean Algebra).

A *Boolean algebra* is a collection of *terms*  $B$  with two distinguished (and distinct) terms called  $\top$  and  $\perp$ , along with a unary operation called  $\neg$  and two binary operations called  $\wedge$  and  $\vee$ , such that the following statements are true for any terms  $p, q, r$  in  $B$ :

Axioms of a Boolean Algebra	
Identity	$p \wedge \top \Leftrightarrow p$ $p \vee \perp \Leftrightarrow p$
Complement (a.k.a. Negation)	$p \wedge \neg p \Leftrightarrow \perp$ $p \vee \neg p \Leftrightarrow \top$
Commutativity	$p \wedge q \Leftrightarrow q \wedge p$ $p \vee q \Leftrightarrow q \vee p$
Associativity	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
Distributive Laws	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

This kind of structure is also referred to as a *complemented, distributive lattice*. Since we are establishing the algebra of *propositions*, our terms consist only of  $\top$  and  $\perp$ .

However, since none of these axioms tell us how to use the (very useful) symbols  $\rightarrow$  and  $\leftrightarrow$ , we need two additional axioms that will turn our Boolean algebra into an example of a *Heyting algebra*:

Heyting Axioms	
Conditional Disintegration	$p \rightarrow q \Leftrightarrow \neg p \vee q$
Biconditional Disintegration	$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

By referring to the truth tables, it should be easy to see that these axioms are *truth preserving* transformations, meaning that taking an expression like  $p \wedge (q \rightarrow r)$  and applying an axiom like Identity to it does not change the truth value of the resulting expression  $(p \wedge \top) \wedge (q \rightarrow r)$ . For this reason, these are sometimes referred to as *equivalence laws* and many treatments of this subject *prove* these laws by referring to the truth tables.

For our purposes, we don't need to refer to the truth tables at all. The truth tables were a nice, intuitive, and compact way of defining the logical connectives, but we could just as easily have defined them by assuming that all of the axioms are true, without ever writing down a truth table. This provides a more *algebraic* approach to the study of logic, which is more in-line with the way logic is used to actually prove theorems in mathematics.

As such, while the truth tables provided a nice way of defining the logical connectives, we will be taking the algebraic approach by *assuming the axioms* of a Boolean algebra are true about our propositional logic and using them to prove theorems about our logical system.

**Definition 1.10** (Tautology).

We say a proposition  $\varphi$  is a *tautology*  $:\Leftrightarrow$  we have that  $\varphi \Leftrightarrow \top$ .

**Definition 1.11** (Contradiction).

We say a proposition  $\varphi$  is a *contradiction*  $:\Leftrightarrow$  we have that  $\varphi \Leftrightarrow \perp$ .

**Example 1.2.**

$p \vee \neg p$  is a tautology, and  $p \wedge \neg p$  is a contradiction.

**Theorem 1.1** (Uniqueness of Complements).

Let  $p$  be a term in a Boolean algebra  $(B, \neg, \wedge, \vee)$ . Suppose there were two terms  $x$  and  $y$  in the Boolean algebra such that

$$\begin{aligned} p \wedge x &\Leftrightarrow \perp, & p \wedge y &\Leftrightarrow \perp, \\ p \vee x &\Leftrightarrow \top, & p \vee y &\Leftrightarrow \top, \end{aligned}$$

meaning that  $x$  and  $y$  act like negations for  $p$ . Then, we have  $x \Leftrightarrow y$ .

*Proof.*

Let  $(B, \neg, \wedge, \vee)$  be a Boolean algebra and consider an arbitrary term  $p$  in the algebra. Suppose we have  $x$  and  $y$  satisfying the conditions given in the statement of the theorem above. Then, we can observe

$$\begin{aligned} x &\Leftrightarrow \top \wedge x && \text{by Identity} \\ &\Leftrightarrow (p \vee y) \wedge x && \text{by the assumption } p \vee y \Leftrightarrow \top \\ &\Leftrightarrow (p \wedge x) \vee (y \wedge x) && \text{by Distributivity} \\ &\Leftrightarrow \perp \vee (y \wedge x) && \text{by the assumption } p \wedge x \Leftrightarrow \perp \\ &\Leftrightarrow y \wedge x && \text{by Identity.} \end{aligned}$$

Similarly, we can see that

$$\begin{aligned} y &\Leftrightarrow \top \wedge y && \text{by Identity} \\ &\Leftrightarrow (p \vee x) \wedge y && \text{by the assumption } p \vee x \Leftrightarrow \top \\ &\Leftrightarrow (p \wedge y) \vee (x \wedge y) && \text{by Distributivity} \\ &\Leftrightarrow \perp \vee (x \wedge y) && \text{by the assumption } p \wedge y \Leftrightarrow \perp \\ &\Leftrightarrow x \wedge y && \text{by Identity.} \end{aligned}$$

So, from Commutativity, we can conclude that  $x \Leftrightarrow (y \wedge x) \Leftrightarrow (x \wedge y) \Leftrightarrow y$ .

Q.E.D.

**Lemma 1.1.**

For every Boolean algebra  $(B, \neg, \wedge, \vee)$ , we have

$$\begin{aligned} \neg \top &\Leftrightarrow \perp \\ \neg \perp &\Leftrightarrow \top. \end{aligned}$$

*Proof.*

Let  $(B, \neg, \wedge, \vee)$  be a Boolean algebra with the distinguished terms  $\top$  and  $\perp$ . We will show that  $\neg \top \Leftrightarrow \perp$  by showing that  $\neg \top$  and  $\perp$  both act like negations for  $\top$ .

First, observe that  $\neg \top \wedge \top \Leftrightarrow \perp$  by the Complement axiom. Similarly, we have  $\neg \top \vee \top \Leftrightarrow \top$  by the Complement axiom.

Now, notice that  $\perp \wedge \top \Leftrightarrow \perp$  by the Identity axiom. Similarly, we get  $\perp \vee \top \Leftrightarrow \top \vee \perp \Leftrightarrow \perp$  by the Identity axiom (and Commutativity).

This shows us that  $\neg \top$  and  $\perp$  are both negations for  $\top$ . Therefore, recalling that complements are unique by [Theorem 1.1](#), we can conclude  $\neg \top \Leftrightarrow \perp$ .

Showing  $\neg \perp \Leftrightarrow \top$  is left as an exercise to the reader.

Q.E.D.

**Theorem 1.2** (Double Negation).

For any Boolean algebra  $(B, \neg, \wedge, \vee)$  and any term  $p$  in the algebra, we have  $\neg\neg p \Leftrightarrow p$ .

*Proof.*

Let  $(B, \neg, \wedge, \vee)$  be a Boolean algebra and consider an arbitrary term  $p$  in the algebra. We want to show that  $\neg\neg p \Leftrightarrow p$ , which we can do by establishing that  $p$  and  $\neg\neg p$  are both complements of  $\neg p$ .

By the Complement axiom, we know  $p \wedge \neg p \Leftrightarrow \perp$  and  $p \vee \neg p \Leftrightarrow \top$ , meaning that  $p$  is the complement of  $\neg p$ . Similarly, we can see that  $\neg p \wedge \neg(\neg p) \Leftrightarrow \perp$  and  $\neg p \vee \neg(\neg p) \Leftrightarrow \top$ , showing us that  $\neg\neg p$  is the complement of  $\neg p$ .

Since complements are unique, as seen in [Theorem 1.1](#), and both  $p$  and  $\neg\neg p$  are complements of  $\neg p$ , we can therefore conclude that  $p \Leftrightarrow \neg\neg p$ .

Q.E.D.

**Theorem 1.3** (Idempotency).

For every Boolean algebra  $(B, \neg, \wedge, \vee)$  and every term  $p$  in the algebra, we have

$$\begin{aligned} p \wedge p &\Leftrightarrow p \\ p \vee p &\Leftrightarrow p. \end{aligned}$$

*Proof.*

Let  $p$  be a term in an arbitrary Boolean algebra  $(B, \neg, \wedge, \vee)$ . Observe

$\begin{aligned} p \wedge p &\Leftrightarrow (p \wedge p) \vee \perp && \text{by Identity} \\ &\Leftrightarrow (p \wedge p) \vee (p \wedge \neg p) && \text{by Complement} \\ &\Leftrightarrow p \wedge (p \vee \neg p) && \text{by Distributivity} \\ &\Leftrightarrow p \wedge \top && \text{by Complement} \\ &\Leftrightarrow p && \text{by Identity,} \end{aligned}$		$\begin{aligned} p \vee p &\Leftrightarrow (p \vee p) \wedge \top && \text{by Identity} \\ &\Leftrightarrow (p \vee p) \wedge (p \vee \neg p) && \text{by Complement} \\ &\Leftrightarrow p \vee (p \wedge \neg p) && \text{by Distributivity} \\ &\Leftrightarrow p \vee \perp && \text{by Complement} \\ &\Leftrightarrow p && \text{by Identity.} \end{aligned}$
---	--	---

Therefore,  $p \wedge p \Leftrightarrow p$  and  $p \vee p \Leftrightarrow p$ , as desired.

Q.E.D.

**Theorem 1.4** (Domination).

For every Boolean algebra  $(B, \neg, \wedge, \vee)$  and every term  $p$  in the algebra, we have

$$\begin{aligned} p \wedge \perp &\Leftrightarrow \perp \\ p \vee \top &\Leftrightarrow \top. \end{aligned}$$

*Proof.*

Let  $p$  be a term in an arbitrary Boolean algebra  $(B, \neg, \wedge, \vee)$ . Observe

$\begin{aligned} p \wedge \perp &\Leftrightarrow p \wedge (p \wedge \neg p) && \text{by Complement} \\ &\Leftrightarrow (p \wedge p) \wedge \neg p && \text{by Associativity} \\ &\Leftrightarrow p \wedge \neg p && \text{by Idempotency} \\ &\Leftrightarrow \perp && \text{by Complement,} \end{aligned}$		$\begin{aligned} p \vee \top &\Leftrightarrow p \vee (p \vee \neg p) && \text{by Complement} \\ &\Leftrightarrow (p \vee p) \vee \neg p && \text{by Associativity} \\ &\Leftrightarrow p \vee \neg p && \text{by Idempotency} \\ &\Leftrightarrow \top && \text{by Complement.} \end{aligned}$
--	--	--

Therefore,  $p \wedge \perp \Leftrightarrow \perp$  and  $p \vee \top \Leftrightarrow \top$ .

Q.E.D.

**Theorem 1.5** (Absorption).

For every Boolean algebra  $(B, \neg, \wedge, \vee)$  and any two terms  $p$  and  $q$  in the algebra, we have

$$\begin{aligned} p \wedge (p \vee q) &\Leftrightarrow p \\ p \vee (p \wedge q) &\Leftrightarrow p. \end{aligned}$$

*Proof.*

Let  $p$  and  $q$  be terms in an arbitrary Boolean algebra  $(B, \neg, \wedge, \vee)$ . Observe

$p \wedge (p \vee q) \Leftrightarrow (p \vee \perp) \wedge (p \vee q)$	by Identity	$p \vee (p \wedge q) \Leftrightarrow (p \wedge \top) \vee (p \wedge q)$	by Identity
$\Leftrightarrow p \vee (\perp \wedge q)$	by Distributivity	$\Leftrightarrow p \wedge (\top \vee q)$	by Distributivity
$\Leftrightarrow p \vee \perp$	by Domination	$\Leftrightarrow p \wedge \top$	by Domination
$\Leftrightarrow p$	by Identity,	$\Leftrightarrow p$	by Identity.

Therefore,  $p \wedge (p \vee q) \Leftrightarrow p$  and  $p \vee (p \wedge q) \Leftrightarrow p$ .

Q.E.D.

**Theorem 1.6** (De Morgan's Laws).

For every Boolean algebra  $(B, \neg, \wedge, \vee)$  and any two terms  $p$  and  $q$  in the algebra, we have

$$\begin{aligned} \neg(p \wedge q) &\Leftrightarrow \neg p \vee \neg q \\ \neg(p \vee q) &\Leftrightarrow \neg p \wedge \neg q. \end{aligned}$$

*Proof.*

Let  $p$  and  $q$  be terms in an arbitrary Boolean algebra  $(B, \neg, \wedge, \vee)$ . Observe that

$(p \wedge q) \wedge (\neg p \vee \neg q) \Leftrightarrow (q \wedge p) \wedge (\neg p \vee \neg q)$	by Commutativity
$\Leftrightarrow q \wedge (p \wedge (\neg p \vee \neg q))$	by Associativity
$\Leftrightarrow q \wedge ((p \wedge \neg p) \vee (p \wedge \neg q))$	by Distributivity
$\Leftrightarrow q \wedge (\perp \vee (p \wedge \neg q))$	by Complement
$\Leftrightarrow q \wedge (p \wedge \neg q)$	by Identity
$\Leftrightarrow q \wedge (\neg q \wedge p)$	by Commutativity
$\Leftrightarrow (q \wedge \neg q) \wedge p$	by Associativity
$\Leftrightarrow \perp \wedge p$	by Complement
$\Leftrightarrow \perp$	by Domination.

Further, we have

$(p \wedge q) \vee (\neg p \vee \neg q) \Leftrightarrow (p \wedge q) \wedge (\neg q \vee \neg p)$	by Commutativity
$\Leftrightarrow ((p \wedge q) \vee \neg q) \vee \neg p$	by Associativity
$\Leftrightarrow ((p \vee \neg q) \wedge (q \vee \neg q)) \vee \neg p$	by Distributivity
$\Leftrightarrow ((p \vee \neg q) \wedge \top) \vee \neg p$	by Complement
$\Leftrightarrow (p \vee \neg q) \vee \neg p$	by Identity
$\Leftrightarrow (\neg q \vee p) \vee \neg p$	by Commutativity
$\Leftrightarrow \neg q \vee (p \vee \neg p)$	by Associativity
$\Leftrightarrow \neg q \vee \top$	by Complement
$\Leftrightarrow \top$	by Domination.

So, we can see that  $\neg p \vee \neg q$  is a complement for  $p \wedge q$ . Since complements are unique by [Theorem 1.1](#), we can conclude that  $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ .

Showing  $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$  is left as an exercise to the reader.

Q.E.D.