# Propositional Logic
## Discrete Mathematics
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**Definition 1** (Proposition)**.**

A *proposition* is a sentence (in our language) that has one (and only one) definite, consistent truth value.

**Definition 2** (Negation)**.**

Given a proposition $p$, the *negation* of $p$, denoted $\neg p$, is defined by

| $p$ | $\neg p$ |
|:---:|:---:|
| $\top$ | $\bot$ |
| $\bot$ | $\top$ |

Some possible readings of $\neg p$:

· Not $p$.
· $p$ does not hold.
· It is not the case that $p$.
· We do not have that $p$.

**Definition 3** (Conjunction)**.**

Given two propositions $p$ and $q$, the *conjunction* of $p$ with $q$, denoted $p \wedge q$, is defined by

| $p$ | $q$ | $p \wedge q$ |
|:---:|:---:|:---:|
| $\top$ | $\top$ | $\top$ |
| $\top$ | $\bot$ | $\bot$ |
| $\bot$ | $\top$ | $\bot$ |
| $\bot$ | $\bot$ | $\bot$ |

Some possible readings of $p \wedge q$:

· $p$, and $q$.
· $p$, but $q$.
· $p$; also, $q$.
· $p$; further, $q$.
· In addition to $p$, we also have $q$.

**Definition 4** (Disjunction)**.**

Given two propositions $p$ and $q$, the *disjunction* of $p$ with $q$, denoted $p \vee q$, is defined by

| $p$ | $q$ | $p \vee q$ |
|:---:|:---:|:---:|
| $\top$ | $\top$ | $\top$ |
| $\top$ | $\bot$ | $\top$ |
| $\bot$ | $\top$ | $\top$ |
| $\bot$ | $\bot$ | $\bot$ |

Some possible readings of $p \vee q$:

· $p$, or $q$.
· Either $p$, or $q$.

**Definition 5** (Material Implication)**.**

Given two propositions $p$ and $q$, the *conditional* formed by assuming $p$ and concluding $q$, denoted $p \rightarrow q$, is defined by

| $p$ | $q$ | $p \rightarrow q$ |
|:---:|:---:|:---:|
| $\top$ | $\top$ | $\top$ |
| $\top$ | $\bot$ | $\bot$ |
| $\bot$ | $\top$ | $\top$ |
| $\bot$ | $\bot$ | $\top$ |

Some possible readings of $p \rightarrow q$:

· If $p$, then $q$.
· $p$ implies $q$.
· $q$ is conditioned on $p$.
· $q$ only if $p$.
· $p$ is sufficient for $q$.
· $q$ is necessary for $p$.
· $q$ unless not $p$.
· $q$ or not $p$.

**Definition 6** (Biconditional).

Given two propositions $p$ and $q$, the *biconditional* formed by $p$ and $q$, denoted $p \leftrightarrow q$, is defined by

| $p$ | $q$ | $p \leftrightarrow q$ |
|:---:|:---:|:---:|
| $\top$ | $\top$ | $\top$ |
| $\top$ | $\bot$ | $\bot$ |
| $\bot$ | $\top$ | $\bot$ |
| $\bot$ | $\bot$ | $\top$ |

Some possible readings of $pq$:

    · $p$ if and only if $q$.
    · $p$ is necessary and sufficient for $q$.
    · $q$ is necessary and sufficient for $p$.

**Definition 7.**

If we have two expressions $\varphi$ and $\psi$ in our formal language, consisting of some number of (possibly shared) propositional variables, connected together by logical connectives, then with the notation $\varphi \Leftrightarrow \psi$ we say that $\varphi$ is equivalent to $\psi :\Leftrightarrow$ every assignment of truth values to the propositional variables of $\varphi$ and $\psi$ results in the same truth value for the two expressions.

**Example 1.**

Consider the two expressions $\varphi := p \to q$ and $\psi := \neg p \lor q$. We can see that $\varphi$ has two propositional variables: $p$ and $q$. $\psi$ also has two propositional variables: the same $p$ and the same $q$.

If we construct the truth table for these two expressions, we will see that every assignment of truth values to $p$ and $q$ will result in $p \to q$ and $\neg p \lor q$ having the same truth value.

| $p$ | $q$ | $p \to q$ | $\neg p \lor q$ |
|:---:|:---:|:---:|:---:|
| $\top$ | $\top$ | $\top$ | $\top$ |
| $\top$ | $\bot$ | $\bot$ | $\bot$ |
| $\bot$ | $\top$ | $\top$ | $\top$ |
| $\bot$ | $\bot$ | $\top$ | $\top$ |

**Definition 8.**

A *Boolean algebra* is a collection of *terms $B$* with two distinguished (and distinct) terms called $\top$ and $\bot$, along with a unary operation called $\neg$ and two binary operations called $\land$ and $\lor$, such that the following statements are true for any terms $p, q, r$ in $B$:

| Axioms of a Boolean Algebra | |
|:---|:---|
| Identity | $p \land \top \Leftrightarrow p$ <br> $p \lor \bot \Leftrightarrow p$ |
| Complement (*a.k.a.* Negation) | $p \land \neg p \Leftrightarrow \bot$ <br> $p \lor \neg p \Leftrightarrow \top$ |
| Commutativity | $p \land q \Leftrightarrow q \land p$ <br> $p \lor q \Leftrightarrow q \lor p$ |
| Associativity | $p \land (q \land r) \Leftrightarrow (p \land q) \land r$ <br> $p \lor (q \lor r) \Leftrightarrow (p \lor q) \lor r$ |
| Distributive Laws | $p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r)$ <br> $p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r)$ |

This kind of structure is also referred to as a *complemented, distributive lattice*. For our purposes in establishing the algebra of *propositions*, our set of terms consists only of $\top$ and $\bot$ since our propositions

can only be true or false.

However, since none of these axioms tell us how to use the (very useful) symbols $\rightarrow$ and $\leftrightarrow$, we need two additional axioms that will turn our Boolean algebra into an example of a *Heyting algebra*:

| Heyting Axioms | |
|---|---|
| Conditional Disintegration | $p \rightarrow q \Leftrightarrow \neg p \vee q$ |
| Biconditional Disintegration | $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$ |

By referring to the truth tables, it should be easy to see that these axioms are *truth preserving* transformations, meaning that taking an expression like $p \wedge (q \rightarrow r)$ and applying an axiom like Identity to it does not change the truth value of the resulting expression $(p \wedge \top) \wedge (q \rightarrow r)$. For this reason, these are sometimes referred to as *equivalence laws* and many treatments of this subject *prove* these laws by referring to the truth tables.

For our purposes, we don't need to refer to the truth tables at all. The truth tables were a nice, intuitive, and compact way of defining the logical connectives, but we could just as easily have defined them by assuming that all of the axioms are true, without ever writing down a truth table.