

PROPOSITIONAL LOGIC

DISCRETE MATHEMATICS

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Definition 1 (Proposition).

A *proposition* is a sentence (in our language) that has one (and only one) definite, consistent truth value.

Definition 2 (Negation).

Given a proposition p , the *negation* of p , denoted $\neg p$, is defined by

p	$\neg p$
\top	\perp
\perp	\top

Some possible readings of $\neg p$:

- Not p .
- p does not hold.
- It is not the case that p .
- We do not have that p .

Definition 3 (Conjunction).

Given two propositions p and q , the *conjunction* of p with q , denoted $p \wedge q$, is defined by

p	q	$p \wedge q$
\top	\top	\top
\top	\perp	\perp
\perp	\top	\perp
\perp	\perp	\perp

Some possible readings of $p \wedge q$:

- p , and q .
- p , but q .
- p ; also, q .
- p ; further, q .
- In addition to p , we also have q .

Definition 4 (Disjunction).

Given two propositions p and q , the *disjunction* of p with q , denoted $p \vee q$, is defined by

p	q	$p \vee q$
\top	\top	\top
\top	\perp	\top
\perp	\top	\top
\perp	\perp	\perp

Some possible readings of $p \vee q$:

- p , or q .
- Either p , or q .

Definition 5 (Material Implication).

Given two propositions p and q , the *conditional* formed by assuming p and concluding q , denoted $p \rightarrow q$, is defined by

p	q	$p \rightarrow q$
\top	\top	\top
\top	\perp	\perp
\perp	\top	\top
\perp	\perp	\top

Some possible readings of $p \rightarrow q$:

- If p , then q .
- p implies q .
- q is conditioned on p .
- q only if p .
- p is sufficient for q .
- q is necessary for p .
- q unless not p .
- q or not p .

Definition 6 (Biconditional).

Given two propositions p and q , the *biconditional* formed by p and q , denoted $p \leftrightarrow q$, is defined by

p	q	$p \leftrightarrow q$
\top	\top	\top
\top	\perp	\perp
\perp	\top	\perp
\perp	\perp	\top

Some possible readings of pq :

- p if and only if q .
- p is necessary and sufficient for q .
- q is necessary and sufficient for p .

Definition 7.

If we have two expressions φ and ψ in our formal language, consisting of some number of (possibly shared) propositional variables, connected together by logical connectives, then with the notation $\varphi \Leftrightarrow \psi$ we say that φ is equivalent to ψ : \Leftrightarrow every assignment of truth values to the propositional variables of φ and ψ results in the same truth value for the two expressions.

Example 1.

Consider the two expressions $\varphi := p \rightarrow q$ and $\psi := \neg p \vee q$. We can see that φ has two propositional variables: p and q . ψ also has two propositional variables: the same p and the same q .

If we construct the truth table for these two expressions, we will see that every assignment of truth values to p and q will result in $p \rightarrow q$ and $\neg p \vee q$ having the same truth value.

p	q	$p \rightarrow q$	$\neg p \vee q$
\top	\top	\top	\top
\top	\perp	\perp	\perp
\perp	\top	\top	\top
\perp	\perp	\top	\top

Definition 8.

A *Boolean algebra* is a collection of *terms* B with two distinguished (and distinct) terms called \top and \perp , along with a unary operation called \neg and two binary operations called \wedge and \vee , such that the following statements are true for any terms p, q, r in B :

Axioms of a Boolean Algebra	
Identity	$p \wedge \top \Leftrightarrow p$ $p \vee \perp \Leftrightarrow p$
Complement (a.k.a. Negation)	$p \wedge \neg p \Leftrightarrow \perp$ $p \vee \neg p \Leftrightarrow \top$
Commutativity	$p \wedge q \Leftrightarrow q \wedge p$ $p \vee q \Leftrightarrow q \vee p$
Associativity	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
Distributive Laws	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

For our purposes in establishing the algebra of *propositions*, our set of terms consists only of \top and \perp since our propositions can only be true or false.

However, since none of these axioms tell us how to use the (very useful) symbols \rightarrow and \leftrightarrow , we need two additional axioms that will turn our Boolean algebra into an example of a *Heyting algebra*:

Heyting Axioms	
Conditional Disintegration	$p \rightarrow q \Leftrightarrow \neg p \vee q$
Biconditional Disintegration	$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

By referring to the truth tables, it should be easy to see that these axioms are *truth preserving* transformations, meaning that taking an expression like $p \wedge (q \rightarrow r)$ and applying an axiom like Identity to it does not change the truth value of the resulting expression $(p \wedge \top) \wedge (q \rightarrow r)$. For this reason, these are sometimes referred to as *equivalence laws* and many treatments of this subject *prove* these laws by referring to the truth tables.

For our purposes, we don't need to refer to the truth tables at all. The truth tables were a nice, intuitive, and compact way of defining the logical connectives, but we could just as easily have defined them by assuming that all of the axioms are true, without ever writing down a truth table.