Discrete Mathematics

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Spring of 2023

Chapter 4

Mathematical Induction

4.1 Weak Induction

Definition 4.1 (Well-Order)**.**

If X is a set with a relation $\leqslant$ defined on it, we say that $\leqslant$ is a *well-order* on $X :\Leftrightarrow$ every non-empty subset of X has a minimal element. Formally, $\leqslant$ is a well-order on $X :\Leftrightarrow \big(\forall A \in \mathcal{P}(X)\big)(\exists a \in A)(\forall b \in A) a \leqslant b$.

Theorem 4.1 ($\mathbb{N}$ is Well-Ordered)**.**

$$(\forall A \subseteq \mathbb{N})\big(A \neq \varnothing \;\Rightarrow\; (\exists a \in A)(\forall b \in A)(a \leqslant b)\big)$$

This says that every nonempty subset of $\mathbb{N}$ has a least element (according to the $\leqslant$ order defined on $\mathbb{N}$).

Proof.

This proof is left as an exercise.

Q.E.D.

Theorem 4.2 (Weak Induction)**.**

If $\varphi(\cdot)$ is a wff, then

$$(\forall n \in \mathbb{N})\big(\varphi(n)\big) \quad\Leftrightarrow\quad \varphi(0) \wedge (\forall k \in \mathbb{N})\big(\varphi(k) \;\Rightarrow\; \varphi(k+1)\big).$$

Proof.

There are two fragments to this proof: the forward ($\Rightarrow$) direction and the backward ($\Leftarrow$) direction.

Fragment 1 ($\Rightarrow$):

Suppose that $(\forall n \in \mathbb{N})\big(\varphi(n)\big)$. Since $0 \in \mathbb{N}$, we then obviously have $\varphi(0)$. Now, let $k \in \mathbb{N}$ and assume $\varphi(k)$. Since $k+1 \in \mathbb{N}$, we know from our initial assumption that $\varphi(k+1)$. Thus, we have $(\forall k \in \mathbb{N})\big(\varphi(k) \;\Rightarrow\; \varphi(k+1)\big)$, and we have reached both of our desired conclusions.

Fragment 2 ($\Leftarrow$):

Assume $\varphi(0)$ and $(\forall k \in \mathbb{N})\big(\varphi(k) \;\Rightarrow\; \varphi(k+1)\big)$. Towards a contradiction, suppose that there is some $n \in \mathbb{N}$ such that $\neg\varphi(a)$. Consider $A \coloneqq \big\{x \in \mathbb{N} \;\big|\; \neg\varphi(x)\big\}$, which we clearly know exists by Axiom 4. We know that $n \in A$ because we assumed that $\neg\varphi(n)$, which implies that $A \neq \varnothing$. Then, we can use Theorem 4.1 to conclude that there is a minimal element a in A.

Since we know that $\varphi(0)$, it follows that $a \neq 0$, so a must be a successor number. This means there is a $b \in \mathbb{N}$ such that $b+1 = a$.

If $b \in A$, then that would mean that $a \leqslant b$ since a is minimal in A. However, we know that $b < b+1 = a$, so we would then have $a \leqslant b < a$. $\lightning$ Therefore, $b \notin A$.

Q.E.D.

Notice that the above theorem actually generalizes beyond just $\mathbb{N}$. In fact, we can generalize the proof of Theorem 4.2 to *any* well-ordered set X by replacing $k+1$ with the least element of the non-empty subset $X \setminus \{\ell \in X \mid \ell \leqslant k\}$.

Let's practice induction by proving the following few theorems.

Theorem 4.3 (Gaussian Summation Formula)**.**

$$(\forall n \in \mathbb{N})\left(\sum_{i=0}^{n} i = \frac{n(n+1)}{2}\right).$$

Proof.

We will prove the claim by induction on $n \in \mathbb{N}$.

Base Case:

Observe that $\sum_{i=0}^{0} i = 0 = \frac{0*(0+1)}{2}$. Therefore, the statement is satisfied at 0.

Inductive Step:

Let $k \in \mathbb{N}$ and assume $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$ (this is our *inductive hypothesis*). Now, observe

$$\begin{aligned}
\sum_{i=0}^{k+1} i &= \left(\sum_{i=0}^{k} i\right) + (k+1) && \text{by definition} \\
&= \left(\frac{k(k+1)}{2}\right) + (k+1) && \text{by the } \textit{inductive hypothesis} \\
&= (k+1)\left(\frac{k}{2} + 1\right) && \text{by factoring out } (k+1) \\
&= \frac{(k+1)(k+2)}{2} && \text{because } \frac{k}{2} + 1 = \frac{k}{2} + \frac{2}{2} = \frac{k+2}{2}.
\end{aligned}$$

Thus, we have that $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$, as desired.

Therefore, we can conclude that $(\forall n \in \mathbb{N})\left(\sum_{i=0}^{n} i = \frac{n(n+1)}{2}\right)$.

Q.E.D.

Theorem 4.4 (Gaussian Summation Formula)**.**

$$\left(\forall n \in \mathbb{N}_+\right)\left(3n \leqslant 3^n\right).$$

Proof.

We will prove the claim by induction on $n \in \mathbb{N}_+$.

Base Case:

Observe that $3 \cdot 0 = 0 \leqslant 1 = 3^0$.

Inductive Step:

Let $k \in \mathbb{N}_+$ and assume $3 \cdot k \leqslant 3^k$ (this is our *inductive hypothesis*). Now, observe

$$\begin{aligned}
3 \cdot (k+1) &= 3 \cdot k + 3 && \text{by the distributive property of multiplication} \\
&\leqslant 3^k + 3 && \text{by the } \textit{inductive hypothesis} \\
&\leqslant 3^k + 3^k && \text{because } 1 \leqslant k \;\Rightarrow\; a^1 \leqslant a^k \text{ if } a < 0^{\dagger\ddagger} \\
&\leqslant 3^k + 3^k + 3^k && \text{since } a^k \text{ is always positive if } 0 < a^{\dagger\ddagger} \\
&= 3 \cdot 3^k && \text{because } \textstyle\sum_{i=1}^{m} a = m \cdot a \text{ for any } a^{\dagger\ddagger} \\
&= 3^{k+1} && \text{because } a^b a^c = a^{b+c} \text{ for any } a^{\dagger\ddagger}
\end{aligned}$$

So, $3 \cdot (k+1) \leqslant 3^{k+1}$, as desired.

Therefore, $(\forall n \in \mathbb{N})(3n \leqslant 3^n)$.

Q.E.D.

† This is true in any ordered semiring where exponentiation is defined recursively in terms of multiplication.
‡ You can assume "basic grade-school" properties of algebra like these without special mention.