PROBLEM SET 5
DISCRETE MATHEMATICS
Due: 1st of March, 2023

1. Show that $(\forall A \subseteq \mathbb{N})\big(A \neq \varnothing \Rightarrow (\exists a \in A)(\forall b \in A)(a \leqslant b)\big)$. In other words, show that $\mathbb{N}$ is *well-ordered* by $\leqslant$.

2. Show that $2 + 3n \leqslant 2^n$ for all natural numbers $n > 3$.

3. There are two components to this question: a proof, and a program.

   (a) Implement the sum $\sum_{i=0}^{n} i^2$ as a *recursive* function of one variable in `Python` and verify that your function sends $n \mapsto \frac{n(n+1)(2n+1)}{6}$. Name your function `ps05pr3a`.

   (b) Show that $\displaystyle\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

4. There are two components to this question: a proof, and a program.

   (a) Implement the factorial $n! := \prod_{i=1}^{n} i$ as a *recursive* function of one variable in `Python` and verify that your function satisfies $n! \geqslant 3^n$ when $n \geqslant 7$. Name your function `ps05pr4a`.

   (b) Show that $n! \geqslant 3^n$ for all natural numbers $n \geqslant 7$.

5. For the rest of this problem set, let $\varphi := \frac{1+\sqrt{5}}{2}$ and $\hat{\varphi} := 1 - \varphi = \frac{1-\sqrt{5}}{2}$.

   (a) Prove that $\varphi^2 = \varphi + 1$ and that $\hat{\varphi}^2 = \hat{\varphi} + 1$.

   (b) Show that $\varphi^n = \varphi^{n-1} + \varphi^{n-2}$ for all natural numbers $n \geqslant 2$.

   (c) Show that $\varphi^n = \varphi \mathcal{F}_n + \mathcal{F}_{n-1}$ for all $n \geqslant 2$.

6. There are two components to this question: a proof, and a program.

   (a) Implement the Fibonacci sequence $\langle \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ as a *recursive* function of one variable in `Python` and verify that your function sends $n \mapsto \frac{1}{\sqrt{5}}(\varphi^n - \hat{\varphi}^n)$. Name your function `ps05pr6a`.

   (b) Show that $\mathcal{F}_n = \frac{1}{\sqrt{5}}(\varphi^n - \hat{\varphi}^n)$ for all natural numbers $n \geqslant 2$.

Code Submission Instructions:

Several of the problems in this problem set have a programming component. The `Python` functions you define must be named as the instructions for each problem indicate, and they *must be recursive*. You are not permitted to use any internal or external libraries (*i.e.*, no `import <...>` statements). Your functions should all be implemented in one file, with the filename `ps05-<lastname>-<firstname>.py`; for example, a possible file name would be `ps05-gonzalez-cedre-daniel.py`.

If you are submitting the rest of your solutions to this problem set electronically, then attach your `Python` file *in the same email* as the rest of your solutions.

If you are submitting your proofs in-person on paper, then email your code separately.