Discrete Mathematics

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Chapter 4

Mathematical Induction

4.1 Weak Induction

Definition 4.1 (Well-Order)**.** Let X be a set with a relation $\leqslant$ defined on it. We say that X is *well-ordered* by this relation *iff* every non-empty subset of X has a minimal element with respect to $\leqslant$. In other words, we say that $\leqslant$ is a *well-order* on X :⇔

$$\big(\forall A \in \mathcal{P}(X)\big)(\exists a \in A)(\forall b \in A)a \leqslant b.$$

Theorem 4.1 ($\mathbb{N}$ is Well-Ordered)**.**

$$(\forall A \subseteq \mathbb{N})\big(A \neq \varnothing \;\Rightarrow\; (\exists a \in A)(\forall b \in A)(a \leqslant b)\big)$$

This says that every nonempty subset of $\mathbb{N}$ *has a least element (according to the* $\leqslant$ *order defined on* $\mathbb{N}$*).*

Proof. This proof is left as an exercise.

Q.E.D.

Theorem 4.2 (Weak Induction)**.** *If* $\varphi(\cdot)$ *is a wff, then*

$$(\forall n \in \mathbb{N})\big(\varphi(n)\big) \;\Leftrightarrow\; \varphi(0) \wedge (\forall k \in \mathbb{N})\big(\varphi(k) \;\Rightarrow\; \varphi(k+1)\big).$$

Proof. There are two fragments to this proof: the forward ($\Rightarrow$) direction and the backward ($\Leftarrow$) direction.

Fragment 1 ($\Rightarrow$):
Suppose that $(\forall n \in \mathbb{N})\big(\varphi(n)\big)$. Since $0 \in \mathbb{N}$, we then obviously have $\varphi(0)$. Now, let $k \in \mathbb{N}$ and assume $\varphi(k)$. Since $k+1 \in \mathbb{N}$, we know from our initial assumption that $\varphi(k+1)$. Thus, we have $(\forall k \in \mathbb{N})\big(\varphi(k) \;\Rightarrow\; \varphi(k+1)\big)$, and we have reached both of our desired conclusions.

Fragment 2 ($\Leftarrow$):
Assume $\varphi(0)$ and $(\forall k \in \mathbb{N})\big(\varphi(k) \;\Rightarrow\; \varphi(k+1)\big)$. Towards a contradiction, suppose that there is some $n \in \mathbb{N}$ such that $\neg\varphi(a)$. Consider $A \coloneqq \big\{x \in \mathbb{N} \;\big|\; \neg\varphi(x)\big\}$, which we clearly know exists by Axiom 4. We know that $n \in A$ because we assumed that $\neg\varphi(n)$, which implies that $A \neq \varnothing$. Then, we can use Theorem 4.1 to conclude that there is a minimal element a in A.

Since we know that $\varphi(0)$, it follows that $a \neq 0$, so a must be a successor number. This means there is a $b \in \mathbb{N}$ such that $b+1 = a$.

If $b \in A$, then that would mean that $a \leqslant b$ since a is minimal in A. However, we know that $b < b+1 = a$, so we would then have $a \leqslant b < a$. ↯ Therefore, $b \notin A$.

With these two directions proven, we finally have $(\forall n \in \mathbb{N})\big(\varphi(n)\big) \;\Leftrightarrow\; \varphi(0) \wedge (\forall k \in \mathbb{N})\big(\varphi(k) \;\Rightarrow\; \varphi(k+1)\big)$.

Q.E.D.

Notice that the above theorem actually generalizes beyond just $\mathbb{N}$. In fact, we can generalize the proof of Theorem 4.2 to *any* well-ordered set X by replacing $k+1$ with the least element of the non-empty subset $X \setminus \{\ell \in X \mid \ell \leqslant k\}$.

Let's practice induction by proving the following few theorems.

Theorem 4.3 (Gaussian Summation Formula)**.** $(\forall n \in \mathbb{N})\left(\sum_{i=0}^{n} i = \frac{n(n+1)}{2}\right)$.

Proof. We will prove the claim by induction on $n \in \mathbb{N}$.

Base Case:
Observe that $\sum_{i=0}^{0} i = 0 = \frac{0 * (0+1)}{2}$. Therefore, the statement is satisfied at 0.

Inductive Step:
Let $k \in \mathbb{N}$ and assume $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$ (this is our *inductive hypothesis*). Now, observe

$$\begin{aligned}
\sum_{i=0}^{k+1} i &= \left(\sum_{i=0}^{k} i\right) + (k+1) && \text{by definition} \\
&= \left(\frac{k(k+1)}{2}\right) + (k+1) && \text{by the \textit{inductive hypothesis}} \\
&= (k+1)\left(\frac{k}{2} + 1\right) && \text{by the distributive property of multiplication}^{*\dagger} \\
&= \frac{(k+1)(k+2)}{2} && \text{because } \frac{k}{2} + 1 = \frac{k}{2} + \frac{2}{2} = \frac{k+2}{2}.^{\dagger\ddagger}
\end{aligned}$$

Thus, we have that $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$, as desired.

Therefore, we can conclude that $(\forall n \in \mathbb{N})\left(\sum_{i=0}^{n} i = \frac{n(n+1)}{2}\right)$.

Q.E.D.

Theorem 4.4. $3n \leqslant 3^n$ *for all* $n \in \mathbb{N}_+$.

Proof. We will prove the claim by induction on $n \in \mathbb{N}_+$.

Base Case:
Observe that $3 \cdot 0 = 0 \leqslant 1 = 3^0$.

Inductive Step:
Let $k \in \mathbb{N}_+$ and assume $3 \cdot k \leqslant 3^k$ (this is our *inductive hypothesis*). Now, observe

$$\begin{aligned}
3 \cdot (k+1) &= 3 \cdot k + 3 && \text{by the distributive property of multiplication} \\
&\leqslant 3^k + 3 && \text{by the \textit{inductive hypothesis}} \\
&\leqslant 3^k + 3^k && \text{because } 1 \leqslant k \;\Rightarrow\; a^1 \leqslant a^k \text{ if } a < 0^{*\dagger} \\
&\leqslant 3^k + 3^k + 3^k && \text{since } a^k \text{ is always positive if } 0 < a^{*\dagger} \\
&= 3 \cdot 3^k && \text{because } \textstyle\sum_{i=1}^{m} a = m \cdot a \text{ for any } a^{*\dagger} \\
&= 3^{k+1} && \text{because } a^b a^c = a^{b+c} \text{ for any } a.^{*\dagger}
\end{aligned}$$

So, $3 \cdot (k+1) \leqslant 3^{k+1}$, as desired.

Therefore, $(\forall n \in \mathbb{N})(3n \leqslant 3^n)$.

Q.E.D.

[*] These "basic grade-school" algebraic properties will now be assumed without special mention.
[†] This is true in any *ordered semiring* where exponentiation is defined in terms of multiplication.
[‡] This is true in any *field*.

Theorem 4.5. $\displaystyle\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ *for all* $n \in \mathbb{N}$.

Proof. We will prove the claim by induction on $n \in \mathbb{N}_+$.

Base Case:

Observe that $\displaystyle\sum_{i=0}^{0} 2^i = 2^0 = 1 = 2 - 1 = 2^{1+1} - 1$

Inductive Step:

Let $k \in \mathbb{N}_+$ and assume that $\displaystyle\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$. (this is our *inductive hypothesis*). Observe that

$$\begin{aligned}
\sum_{i=0}^{k+1} 2^i &= \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1} && \text{by definition} \\
&= \left(2^{k+1} - 1\right) + 2^{k+1} && \text{by the \textit{inductive hypothesis}} \\
&= 2 \cdot 2^{k+1} - 1 && \text{using some basic algebra} \\
&= 2^{k+2} - 1 && \text{using some basic algebra.}
\end{aligned}$$

So, we get $\displaystyle\sum_{i=0}^{k+1} 2^i = 2^{k+1} - 1$, as desired.

Therefore, we can conclude $(\forall n \in \mathbb{N})\left(\displaystyle\sum_{i=0}^{n} 2^i = 2^{n+1} - 1\right)$.

Q.E.D.