Problem Set 4

Discrete Mathematics

Due on the 19^{th} of February, 2024

(20 pts) 1.(a) Show $\forall x(\varnothing \subseteq x)$.

(b) Show $\forall x(x \subseteq x)$.

(c) Show $\forall x\big(\varnothing \in \mathcal{P}(x)\big)$.

(d) Show $\forall x\big(x \in \mathcal{P}(x)\big)$.

(e) Show $\forall x \forall y \forall z\Big(\big((x \subseteq y) \land (y \subseteq z)\big) \;\Rightarrow\; x \subseteq z\Big)$.

(10 pts) 2. We define the *intersection* and *difference* of any two sets x and y below.

$$x \cap y := \{z \mid z \in x \land z \in y\}$$
$$x \setminus y := \{z \mid z \in x \land z \notin y\}$$

(a) Show $\forall x \forall y \exists z\big(z = x \cap y\big)$.

(b) Show $\forall x \forall y \exists z\big(z = x \setminus y\big)$.

(20 pts) 3. We define the *union* of any two sets x and y below.

$$x \cup y := \{z \mid z \in x \lor z \in y\}$$

(a) Show $\forall x \forall y(x \cap y \subseteq x)$.

(b) Show $\forall x \forall y(x \subseteq x \cup y)$.

(c) Show $\forall x \forall y\Big(\mathcal{P}(x) \cup \mathcal{P}(y) \subseteq \mathcal{P}(x \cup y)\Big)$.

(d) Show $\forall x \forall y\Big(x \cap y = x \;\Leftrightarrow\; x \in \mathcal{P}(y)\Big)$.

(50 pts) 4. We define the *union over x* and *intersection over x* for any set x below.

$$\cup x := \Big\{z \;\Big|\; \exists y\big(y \in x \land z \in y\big)\Big\}$$
$$\cap x := \Big\{z \;\Big|\; \forall y\big(y \in x \;\Rightarrow\; z \in y\big)\Big\}$$

(a) Show that $\forall x\big(\cup \mathcal{P}(x) = x\big)$.

(b) What is $\cup\varnothing$? Justify your answer with a proof.

(c) What is $\cap\varnothing$? Justify your answer with a proof.

(d) Is $\varnothing = \{z \mid z \in \varnothing\}$? Justify your answer with a proof.

(e) Is $\varnothing = \{z \mid z \notin \varnothing\}$? Justify your answer with a proof.