Problem Set 2
Discrete Mathematics
Due: 3$^{\text{rd}}$ of February, 2023

1. Show that $p \to q$ is logically equivalent to $\neg q \to \neg p$.

2. Show that $(\neg p \to \bot) \to p$ is a tautology.

3. Show that $(p \to r) \vee (q \to r)$ is equivalent to $(p \wedge q) \to r$.

4. Show that $(p \vee q) \wedge (\neg p \vee r) \to (q \vee r)$ is a tautology.

5. Using the predicate definitions below, let our universe of discourse be the collection of all characters in the play Macbeth and translate the following propositions of the first-order logic into English sentences.

$$\kappa(x) \coloneqq \text{``}x \text{ is a king."}$$
$$\vartheta(x) \coloneqq \text{``}x \text{ is a thane."}$$
$$\upsilon(x) \coloneqq \text{``}x \text{ is a witch."}$$

$$\omega(x) \coloneqq \text{``}x \text{ is man woman-borne."}$$
$$\mu(x, y) \coloneqq \text{``}x \text{ murders } y\text{."}$$

   (a) Macbeth is a king, and Banquo is a thane.

   (b) Every king is murdered by someone.

   (c) Someone murders every king.

   (d) No witches murder anyone.

   (e) No kings murder any thanes.

   (f) Macbeth is not murdered by any man woman-borne.

   (g) Every king is a thane.

   (h) Banquo is not a king, yet someone murders him.

   (i) Any witch could not possibly be a king.

   (j) The only kings are those woman-borne.

6. Using the predicate definitions below, let our universe of discourse be the collection of all living beings on the Obra-Dinn and translate the following propositions of the first-order logic into English sentences.

$$\kappa(x) \coloneqq \text{``}x \text{ is the Captain."}$$
$$\mu(x) \coloneqq \text{``}x \text{ is a mate."}$$
$$\sigma(x) \coloneqq \text{``}x \text{ is a sea monster."}$$

$$\delta(x, y) \coloneqq \text{``}x \text{ is drowned by } y\text{."}$$
$$\rho(x, y) \coloneqq \text{``}x \text{ was terribly ravaged by } y\text{."}$$
$$\gamma(x, y) \coloneqq \text{``}x \text{ was shot by } y\text{."}$$

   (a) $\kappa(\text{``Richard Witterel"}) \wedge \forall x \big( \mu(x) \to \gamma(x, \text{``Richard Witterel"}) \big)$.

   (b) $\neg \forall x \exists y \big(\sigma(y) \wedge \rho(x, y) \big)$.

   (c) $\exists x \forall y \big(\sigma(x) \wedge (\mu(y) \to \delta(y, x)) \big)$.

   (d) $\left(\neg \exists x \big(\sigma(x) \big) \wedge \left(\neg \exists x \big(\sigma(x) \big) \to \exists x \big(\gamma(\text{``Richard Witterel"}, x) \big) \right) \right) \to \exists x \big( \gamma(\text{``Richard Witterel"}, x) \big)$.

7. Let φ and ψ be predicates. Rewrite the following first-order sentences so that no quantifier has a $\neg$ to its left.

   (a) $\forall x \exists y (x \vee y)$

   (b) $\forall x \forall y (\neg x \vee (x \leftrightarrow y))$

   (c) $\exists x \forall y \forall z \big((x \to y) \to z \big)$

   (d) $x \Big(\varphi(x) \vee \forall y \exists z \big(\psi(y, z) \to \neg x \big) \Big)$