

# PROBLEM SET 5

DISCRETE MATHEMATICS

Due: 1<sup>st</sup> of March, 2023

1. Show that  $(\forall A \subseteq \mathbb{N})(A \neq \emptyset \Rightarrow (\exists a \in A)(\forall b \in A)(a \leq b))$ . In other words, show that  $\mathbb{N}$  is *well-ordered* by  $\leq$ .
2. Show that  $2 + 3n \leq 2^n$  for all natural numbers  $n > 3$ .
3. There are two components to this question: a proof, and a program.
  - (a) Implement the sum  $\sum_{i=0}^n i^2$  as a *recursive* function of one variable in **Python** and verify that your function sends  $n \mapsto \frac{n(n+1)(2n+1)}{6}$ . Name your function **ps05pr3a**.
  - (b) Show that  $\sum_{i=0}^n i^2 = \frac{n(n+1)(2n+1)}{6}$  for all  $n \in \mathbb{N}$ .
4. There are two components to this question: a proof, and a program.
  - (a) Implement the factorial  $n! := \prod_{i=1}^n i$  as a *recursive* function of one variable in **Python** and verify that your function satisfies  $n! \geq 3^n$  when  $n \geq 7$ . Name your function **ps05pr4a**.
  - (b) Show that  $n! \geq 3^n$  for all natural numbers  $n \geq 7$ .
5. For the rest of this problem set, let  $\varphi := \frac{1+\sqrt{5}}{2}$  and  $\hat{\varphi} := 1 - \varphi = \frac{1-\sqrt{5}}{2}$ .
  - (a) Prove that  $\varphi^2 = \varphi + 1$  and that  $\hat{\varphi}^2 = \hat{\varphi} + 1$ .
  - (b) Show that  $\varphi^n = \varphi^{n-1} + \varphi^{n-2}$  for all natural numbers  $n \geq 2$ .
  - (c) Show that  $\varphi^n = \varphi \mathcal{F}_n + \mathcal{F}_{n-1}$  for all  $n \geq 2$ .
6. There are two components to this question: a proof, and a program.
  - (a) Implement the Fibonacci sequence  $\langle \mathcal{F}_n \rangle_{n \in \mathbb{N}}$  as a *recursive* function of one variable in **Python** and verify that your function sends  $n \mapsto \frac{1}{\sqrt{5}}(\varphi^n - \hat{\varphi}^n)$ . Name your function **ps05pr6a**.
  - (b) Show that  $\mathcal{F}_n = \frac{1}{\sqrt{5}}(\varphi^n - \hat{\varphi}^n)$  for all natural numbers  $n \geq 2$ .

## Code Submission Instructions:

Several of the problems in this problem set have a programming component. The **Python** functions you define must be named as the instructions for each problem indicate, and they *must be recursive*. You are not permitted to use any internal or external libraries (*i.e.*, no `import <...>` statements). Your functions should all be implemented in one file, with the filename **ps05-<lastname>-<firstname>.py**; for example, a possible file name would be **ps05-gonzalez-cedre-daniel.py**.

If you are submitting the rest of your solutions to this problem set electronically, then attach your **Python** file *in the same email* as the rest of your solutions.

If you are submitting your proofs in-person on paper, then email your code separately.