

# Problem Set 5

## Discrete Mathematics

Due on the 27<sup>th</sup> of February, 2024

- (10 pts) 1. Find and explain the flaw(s) in this argument.

We prove every nonempty set of people all have the same age.

**Proof.** We denote the age of a person  $p$  by  $\alpha(p)$ .

*Basis Step:*

Suppose  $P = \{p\}$  is a set with one person in it. Clearly, all the people in  $P$  have the same age as each other.

*Inductive Step:*

Let  $k \in \mathbb{N}_+$  and suppose any set of  $k$ -many people all have the same age. Let  $P = \{p_1, p_2, \dots, p_k, p_{k+1}\}$  be a set with  $k+1$  people in it. Consider  $L := \{p_1, \dots, p_k\}$  and  $R := \{p_2, \dots, p_{k+1}\}$ . Since  $L$  and  $R$  both have  $k$  people, we know everyone in these sets has the same age by the *inductive hypothesis*.

Let  $\ell, r \in P$ . If  $\ell \in L \wedge r \in L$ , then  $\alpha(\ell) = \alpha(r)$ . Similarly, if  $\ell \in R \wedge r \in R$ , then  $\alpha(\ell) = \alpha(r)$ . Now, suppose  $\ell \in L \wedge r \in R$ .

$$\alpha(\ell) = \alpha(p_1) = \alpha(p_2) = \alpha(p_{k+1}) = \alpha(r)$$

So, all people in  $P$  have the same age.

Therefore, everyone on Earth has the same age.

Q.E.D.

- (20 pts) 2. Show that  $\forall x(x \neq x \cup \{x\})$ .

- (15 pts) 3. We will work up to a proof of the commutativity of addition on  $\mathbb{N}$ .

(a) Show  $(\forall x \in \mathbb{N})(x + 0 = 0 + x)$ .

(b) Show  $(\forall x, y \in \mathbb{N})(x + s(y) = s(y) + x)$ .

(c) Show  $(\forall x, y \in \mathbb{N})(x + y = y + x)$ .

- (15 pts) 4. Show  $(\forall x, y, z \in \mathbb{N})(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$ .

- (20 pts) 5. For this problem, you may assume the commutativity and associativity of addition and multiplication over  $\mathbb{N}$ . You may also assume multiplication distributes over addition on  $\mathbb{N}$ . Prove the following statement for all  $n \in \mathbb{N}$ .

$$1 + \sum_{i=0}^n 2^i = 2^{n+1}$$

- (20 pts) 6. We say  $x$  is  **$\in$ -transitive** by definition when  $(\forall y \in x)(\forall z \in y)(z \in x)$ . Show that every natural number is  $\in$ -transitive.

Recall that the natural numbers are defined recursively as follows.

$$0 := \emptyset$$

$$s(n) := n \cup \{n\}$$

Addition on  $\mathbb{N}$  is defined below.

$$n + 0 := n$$

$$n + s(m) := s(n + m)$$

Multiplication on  $\mathbb{N}$  is defined below.

$$n \cdot 0 := 0$$

$$n \cdot s(m) := (n \cdot m) + n$$

Exponentiation on  $\mathbb{N}$  is defined below.

$$n^0 := 1$$

$$n^{s(m)} := n \cdot n^m$$

We define the iterated sum of a sequence of terms  $f(0), f(1), f(2), \dots$  as follows.

$$\sum_{i=0}^0 f(i) := f(0)$$

$$\sum_{i=0}^{s(n)} f(i) := \left( \sum_{i=0}^n f(i) \right) + f(s(n))$$

You may rely on the following theorems:

$$(\forall x \in \mathbb{N})(s(x) = x + 1).$$

$$(\forall x \in \mathbb{N})(s(x) = 1 + x).$$

$$(\forall x, y, z \in \mathbb{N})(x + (y + z) = (x + y) + z).$$