

Problem Set 5

Discrete Mathematics

Due on the 27th of February, 2024

- (10 pts) 1. Find and explain the flaw(s) in this argument.

We prove every nonempty set of people all have the same age.
Proof. We denote the age of a person p by $\alpha(p)$.

Basis Step: Suppose $P = \{p\}$ is a set with one person in it. Clearly, all the people in P have the same age as each other.

Inductive Step: Let $k \in \mathbb{N}_+$ and suppose any set of k -many people all have the same age. Let $P = \{p_1, p_2, \dots, p_k, p_{k+1}\}$ be a set with $k+1$ people in it. Consider $L := \{p_1, \dots, p_k\}$ and $R := \{p_2, \dots, p_{k+1}\}$. Since L and R both have k people, we know everyone in these sets has the same age by the *inductive hypothesis*.

Let $\ell, r \in P$. If $\ell \in L \wedge r \in L$, then $\alpha(\ell) = \alpha(r)$. Similarly, if $\ell \in R \wedge r \in R$, then $\alpha(\ell) = \alpha(r)$. Now, suppose $\ell \in L \wedge r \in R$.

$$\alpha(\ell) = \alpha(p_1) = \alpha(p_2) = \alpha(p_{k+1}) = \alpha(r)$$

So, all people in P have the same age.

Therefore, everyone on Earth has the same age. Q.E.D.

- (20 pts) 2. Show that $\forall x (x \neq x \cup \{x\})$.

- (15 pts) 3. We will work up to a proof of the commutativity of addition on \mathbb{N} .

- (a) Show $(\forall x \in \mathbb{N})(x + 0 = 0 + x)$.
 (b) Show $(\forall x, y \in \mathbb{N})(x + s(y) = s(y) + x)$.
 (c) Show $(\forall x, y \in \mathbb{N})(x + y = y + x)$.

- (15 pts) 4. Show $(\forall x, y, z \in \mathbb{N})(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$.

- (20 pts) 5. For this problem, you may assume the commutativity and associativity of addition and multiplication over \mathbb{N} . You may also assume multiplication distributes over addition on \mathbb{N} . Prove the following statement for all $n \in \mathbb{N}$.

$$1 + \sum_{i=0}^n 2^i = 2^{n+1}$$

- (20 pts) 6. We say x is **\in -transitive** by definition when $(\forall y \in x)(\forall z \in y)(z \in x)$. Show that every natural number is \in -transitive.

Recall that the natural numbers are defined recursively as follows.

$$0 := \emptyset$$

$$s(n) := n \cup \{n\}$$

Addition on \mathbb{N} is defined below.

$$n + 0 := n$$

$$n + s(m) := s(n + m)$$

Multiplication on \mathbb{N} is defined below.

$$n \cdot 0 := 0$$

$$n \cdot s(m) := (n \cdot m) + n$$

Exponentiation on \mathbb{N} is defined below.

$$n^0 := 1$$

$$n^{s(m)} := n \cdot n^m$$

We define the iterated sum of a sequence of terms $f(0), f(1), f(2), \dots$ as follows.

$$\sum_{i=0}^0 f(i) := f(0)$$

$$\sum_{i=0}^{s(n)} f(i) := \left(\sum_{i=0}^n f(i) \right) + f(s(n))$$

You may rely on the following theorems:

$$(\forall x \in \mathbb{N})(s(x) = x + 1).$$

$$(\forall x \in \mathbb{N})(s(x) = 1 + x).$$

$$(\forall x, y, z \in \mathbb{N})(x + (y + z) = (x + y) + z).$$