

Discrete Mathematics

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Chapter 1

Propositional Logic

1.1 Propositions & Connectives

Definition 1.1 (Proposition).

A *proposition* is a sentence (in our language) that has one (and only one) definite, consistent truth value.

Definition 1.2 (Negation).

Given a proposition p , the *negation* of p is denoted $\neg p$ and can be defined by the following truth table:

p	$\neg p$
\top	\perp
\perp	\top

Some possible readings of $\neg p$:

- Not p .
- p does not hold.
- It is not the case that p .
- We do not have that p .

Definition 1.3 (Conjunction).

Given two propositions p and q , the *conjunction* of p with q is denoted $p \wedge q$ and can be defined by the following truth table:

p	q	$p \wedge q$
\top	\top	\top
\top	\perp	\perp
\perp	\top	\perp
\perp	\perp	\perp

Some possible readings of $p \wedge q$:

- p , and q .
- p , but q .
- p ; also, q .
- p ; further, q .
- In addition to p , we also have q .

Definition 1.4 (Disjunction).

Given two propositions p and q , the *disjunction* of p with q is denoted $p \vee q$ and can be defined by the following truth table:

p	q	$p \vee q$
\top	\top	\top
\top	\perp	\top
\perp	\top	\top
\perp	\perp	\perp

Some possible readings of $p \vee q$:

- p , or q .
- Either p , or q .

Definition 1.5 (Material Implication).

Given two propositions p and q , the *conditional* formed by assuming p and concluding q is denoted $p \rightarrow q$ and can be defined by the following truth table:

p	q	$p \rightarrow q$
\top	\top	\top
\top	\perp	\perp
\perp	\top	\top
\perp	\perp	\top

Some possible readings of $p \rightarrow q$:

- If p , then q .
- p implies q .
- q is conditioned on p .
- q only if p .
- p is sufficient for q .
- q is necessary for p .
- q unless not p .
- q or not p .

Definition 1.6 (Material Equivalence).

Given two propositions p and q , the *biconditional* formed by p and q is denoted $p \leftrightarrow q$ and can be defined by the following truth table:

p	q	$p \leftrightarrow q$
\top	\top	\top
\top	\perp	\perp
\perp	\top	\perp
\perp	\perp	\top

Some possible readings of $p \leftrightarrow q$:

- p if and only if q .
- p is necessary and sufficient for q .
- q is necessary and sufficient for p .

Definition 1.7 (Equivalence).

If we have two expressions φ and ψ in our formal language, consisting of some number of (possibly shared) propositional variables, connected together by logical connectives, then with the notation $\varphi \Leftrightarrow \psi$ we say that φ is equivalent to ψ \Leftrightarrow every assignment of truth values to the propositional variables of φ and ψ results in the same truth value for the two expressions.

Example 1.1.

Consider the two expressions $\varphi := p \rightarrow q$ and $\psi := \neg p \vee q$. We can see that φ has two propositional variables: p and q . ψ also has two propositional variables: the same p and the same q . If we construct the truth table for these two expressions, we will see that every assignment of truth values to p and q will result in $p \rightarrow q$ and $\neg p \vee q$ having the same truth value.

p	q	$p \rightarrow q$	$\neg p \vee q$
\top	\top	\top	\top
\top	\perp	\perp	\perp
\perp	\top	\top	\top
\perp	\perp	\top	\top

Definition 1.8 (Satisfiability).

A propositional expression φ consisting the propositional variables p_1, \dots, p_n is *satisfiable* \Leftrightarrow there exists an assignment of truth values to p_1, \dots, p_n that results in φ being equivalent to \top .

1.2 Boolean Algebras

Definition 1.9 (Boolean Algebra).

A *Boolean algebra* is a collection of *terms* B with two distinguished (and distinct) terms called \top and \perp , along with a unary operation called \neg and two binary operations called \wedge and \vee , such that the following statements are true for any terms p, q, r in B :

Axioms of a Boolean Algebra	
Identity	$p \wedge \top \Leftrightarrow p$ $p \vee \perp \Leftrightarrow p$
Complement (a.k.a. Negation)	$p \wedge \neg p \Leftrightarrow \perp$ $p \vee \neg p \Leftrightarrow \top$
Commutativity	$p \wedge q \Leftrightarrow q \wedge p$ $p \vee q \Leftrightarrow q \vee p$
Associativity	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
Distributive Laws	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

This kind of structure is also referred to as a *complemented, distributive lattice*. Since we are establishing the algebra of *propositions*, our terms consist only of \top and \perp .

However, since none of these axioms tell us how to use the (very useful) symbols \rightarrow and \leftrightarrow , we need two additional axioms that will turn our Boolean algebra into an example of a *Heyting algebra*:

Heyting Axioms	
Conditional Disintegration	$p \rightarrow q \Leftrightarrow \neg p \vee q$
Biconditional Disintegration	$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

By referring to the truth tables, it should be easy to see that these axioms are *truth preserving* transformations, meaning that taking an expression like $p \wedge (q \rightarrow r)$ and applying an axiom like Identity to it does not change the truth value of the resulting expression $(p \wedge \top) \wedge (q \rightarrow r)$. For this reason, these are sometimes referred to as *equivalence laws* and many treatments of this subject *prove* these laws by referring to the truth tables.

For our purposes, we don't need to refer to the truth tables at all. The truth tables were a nice, intuitive, and compact way of defining the logical connectives, but we could just as easily have defined them by assuming that all of the axioms are true, without ever writing down a truth table. This provides a more *algebraic* approach to the study of logic, which is more in-line with the way logic is used to actually prove theorems in mathematics.

As such, while the while the truth tables provided a nice way of defining the logical connectives, we will be taking the algebraic approach by *assuming the axioms* of a Boolean algebra are true about our propositional logic and using them to prove theorems about our logical system.

Definition 1.10 (Tautology).

We say a proposition φ is a *tautology* $:\Leftrightarrow$ we have that $\varphi \Leftrightarrow \top$.

Definition 1.11 (Contradiction).

We say a proposition φ is a *contradiction* $:\Leftrightarrow$ we have that $\varphi \Leftrightarrow \perp$.

Example 1.2.

$p \vee \neg p$ is a tautology, and $p \wedge \neg p$ is a contradiction.

Theorem 1.1 (Uniqueness of Complements).

Let p be a term in a Boolean algebra (B, \neg, \wedge, \vee) . Suppose there were two terms x and y in the Boolean algebra such that

$$\begin{aligned} p \wedge x &\Leftrightarrow \perp, & p \wedge y &\Leftrightarrow \perp, \\ p \vee x &\Leftrightarrow \top, & p \vee y &\Leftrightarrow \top, \end{aligned}$$

meaning that x and y act like negations for p . Then, we have $x \Leftrightarrow y$.

Proof.

Let (B, \neg, \wedge, \vee) be a Boolean algebra and consider an arbitrary term p in the algebra. Suppose we have x and y satisfying the conditions given in the statement of the theorem above. Then, we can observe

$$\begin{aligned} x &\Leftrightarrow \top \wedge x && \text{by Identity} \\ &\Leftrightarrow (p \vee y) \wedge x && \text{by the assumption } p \vee y \Leftrightarrow \top \\ &\Leftrightarrow (p \wedge x) \vee (y \wedge x) && \text{by Distributivity} \\ &\Leftrightarrow \perp \vee (y \wedge x) && \text{by the assumption } p \wedge x \Leftrightarrow \perp \\ &\Leftrightarrow y \wedge x && \text{by Identity.} \end{aligned}$$

Similarly, we can see that

$$\begin{aligned} y &\Leftrightarrow \top \wedge y && \text{by Identity} \\ &\Leftrightarrow (p \vee x) \wedge y && \text{by the assumption } p \vee x \Leftrightarrow \top \\ &\Leftrightarrow (p \wedge y) \vee (x \wedge y) && \text{by Distributivity} \\ &\Leftrightarrow \perp \vee (x \wedge y) && \text{by the assumption } p \wedge y \Leftrightarrow \perp \\ &\Leftrightarrow x \wedge y && \text{by Identity.} \end{aligned}$$

So, from Commutativity, we can conclude that $x \Leftrightarrow (y \wedge x) \Leftrightarrow (x \wedge y) \Leftrightarrow y$.

Q.E.D.

Lemma 1.1.

For every Boolean algebra (B, \neg, \wedge, \vee) , we have

$$\begin{aligned} \neg \top &\Leftrightarrow \perp \\ \neg \perp &\Leftrightarrow \top. \end{aligned}$$

Proof.

Let (B, \neg, \wedge, \vee) be a Boolean algebra with the distinguished terms \top and \perp . We will show that $\neg \top \Leftrightarrow \perp$ by showing that $\neg \top$ and \perp both act like negations for \top .

First, observe that $\neg \top \wedge \top \Leftrightarrow \perp$ by the Complement axiom. Similarly, we have $\neg \top \vee \top \Leftrightarrow \top$ by the Complement axiom.

Now, notice that $\perp \wedge \top \Leftrightarrow \perp$ by the Identity axiom. Similarly, we get $\perp \vee \top \Leftrightarrow \top \vee \perp \Leftrightarrow \perp$ by the Identity axiom (and Commutativity).

This shows us that $\neg \top$ and \perp are both negations for \top . Therefore, recalling that complements are unique by [Theorem 1.1](#), we can conclude $\neg \top \Leftrightarrow \perp$.

Showing $\neg \perp \Leftrightarrow \top$ is left as an exercise to the reader.

Q.E.D.

Theorem 1.2 (Double Negation).

For any Boolean algebra (B, \neg, \wedge, \vee) and any term p in the algebra, we have $\neg\neg p \Leftrightarrow p$.

Proof.

Let (B, \neg, \wedge, \vee) be a Boolean algebra and consider an arbitrary term p in the algebra. We want to show that $\neg\neg p \Leftrightarrow p$, which we can do by establishing that p and $\neg\neg p$ are both complements of $\neg p$.

By the Complement axiom, we know $p \wedge \neg p \Leftrightarrow \perp$ and $p \vee \neg p \Leftrightarrow \top$, meaning that p is the complement of $\neg p$. Similarly, we can see that $\neg p \wedge \neg(\neg p) \Leftrightarrow \perp$ and $\neg p \vee \neg(\neg p) \Leftrightarrow \top$, showing us that $\neg\neg p$ is the complement of $\neg p$.

Since complements are unique, as seen in [Theorem 1.1](#), and both p and $\neg\neg p$ are complements of $\neg p$, we can therefore conclude that $p \Leftrightarrow \neg\neg p$.

Q.E.D.

Theorem 1.3 (Idempotency).

For every Boolean algebra (B, \neg, \wedge, \vee) and every term p in the algebra, we have

$$\begin{aligned} p \wedge p &\Leftrightarrow p \\ p \vee p &\Leftrightarrow p. \end{aligned}$$

Proof.

Let p be a term in an arbitrary Boolean algebra (B, \neg, \wedge, \vee) . Observe

$\begin{aligned} p \wedge p &\Leftrightarrow (p \wedge p) \vee \perp && \text{by Identity} \\ &\Leftrightarrow (p \wedge p) \vee (p \wedge \neg p) && \text{by Complement} \\ &\Leftrightarrow p \wedge (p \vee \neg p) && \text{by Distributivity} \\ &\Leftrightarrow p \wedge \top && \text{by Complement} \\ &\Leftrightarrow p && \text{by Identity,} \end{aligned}$		$\begin{aligned} p \vee p &\Leftrightarrow (p \vee p) \wedge \top && \text{by Identity} \\ &\Leftrightarrow (p \vee p) \wedge (p \vee \neg p) && \text{by Complement} \\ &\Leftrightarrow p \vee (p \wedge \neg p) && \text{by Distributivity} \\ &\Leftrightarrow p \vee \perp && \text{by Complement} \\ &\Leftrightarrow p && \text{by Identity.} \end{aligned}$
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Therefore, $p \wedge p \Leftrightarrow p$ and $p \vee p \Leftrightarrow p$, as desired.

Q.E.D.

Theorem 1.4 (Domination).

For every Boolean algebra (B, \neg, \wedge, \vee) and every term p in the algebra, we have

$$\begin{aligned} p \wedge \perp &\Leftrightarrow \perp \\ p \vee \top &\Leftrightarrow \top. \end{aligned}$$

Proof.

Let p be a term in an arbitrary Boolean algebra (B, \neg, \wedge, \vee) . Observe

$\begin{aligned} p \wedge \perp &\Leftrightarrow p \wedge (p \wedge \neg p) && \text{by Complement} \\ &\Leftrightarrow (p \wedge p) \wedge \neg p && \text{by Associativity} \\ &\Leftrightarrow p \wedge \neg p && \text{by Idempotency} \\ &\Leftrightarrow \perp && \text{by Complement,} \end{aligned}$		$\begin{aligned} p \vee \top &\Leftrightarrow p \vee (p \vee \neg p) && \text{by Complement} \\ &\Leftrightarrow (p \vee p) \vee \neg p && \text{by Associativity} \\ &\Leftrightarrow p \vee \neg p && \text{by Idempotency} \\ &\Leftrightarrow \top && \text{by Complement.} \end{aligned}$
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Therefore, $p \wedge \perp \Leftrightarrow \perp$ and $p \vee \top \Leftrightarrow \top$.

Q.E.D.

Theorem 1.5 (Absorption).

For every Boolean algebra (B, \neg, \wedge, \vee) and any two terms p and q in the algebra, we have

$$\begin{aligned} p \wedge (p \vee q) &\Leftrightarrow p \\ p \vee (p \wedge q) &\Leftrightarrow q. \end{aligned}$$

Proof.

Let p and q be terms in an arbitrary Boolean algebra (B, \neg, \wedge, \vee) . Observe

$p \wedge (p \vee q) \Leftrightarrow (p \vee \perp) \wedge (p \vee q)$	by Identity	$p \vee (p \wedge q) \Leftrightarrow (p \wedge \top) \vee (p \wedge q)$	by Identity
$\Leftrightarrow p \vee (\perp \wedge q)$	by Distributivity	$\Leftrightarrow p \wedge (\top \vee q)$	by Distributivity
$\Leftrightarrow p \vee \perp$	by Domination	$\Leftrightarrow p \wedge \top$	by Domination
$\Leftrightarrow p$	by Identity,	$\Leftrightarrow p$	by Identity.

Therefore, $p \wedge (p \vee q) \Leftrightarrow p$ and $p \vee (p \wedge q) \Leftrightarrow q$.

Q.E.D.

Theorem 1.6 (De Morgan's Laws).

For every Boolean algebra (B, \neg, \wedge, \vee) and any two terms p and q in the algebra, we have

$$\begin{aligned} \neg(p \wedge q) &\Leftrightarrow \neg p \vee \neg q \\ \neg(p \vee q) &\Leftrightarrow \neg p \wedge \neg q. \end{aligned}$$

Proof.

Let p and q be terms in an arbitrary Boolean algebra (B, \neg, \wedge, \vee) . Observe that

$(p \wedge q) \wedge (\neg p \vee \neg q) \Leftrightarrow (q \wedge p) \wedge (\neg p \vee \neg q)$	by Commutativity
$\Leftrightarrow q \wedge (p \wedge (\neg p \vee \neg q))$	by Associativity
$\Leftrightarrow q \wedge ((p \wedge \neg p) \vee (p \wedge \neg q))$	by Distributivity
$\Leftrightarrow q \wedge (\perp \vee (p \wedge \neg q))$	by Complement
$\Leftrightarrow q \wedge (p \wedge \neg q)$	by Identity
$\Leftrightarrow q \wedge (\neg q \wedge p)$	by Commutativity
$\Leftrightarrow (q \wedge \neg q) \wedge p$	by Associativity
$\Leftrightarrow \perp \wedge p$	by Complement
$\Leftrightarrow \perp$	by Domination.

Further, we have

$(p \wedge q) \vee (\neg p \vee \neg q) \Leftrightarrow (p \wedge q) \wedge (\neg q \vee \neg p)$	by Commutativity
$\Leftrightarrow ((p \wedge q) \vee \neg q) \vee \neg p$	by Associativity
$\Leftrightarrow ((p \vee \neg q) \wedge (q \vee \neg q)) \vee \neg p$	by Distributivity
$\Leftrightarrow ((p \vee \neg q) \wedge \top) \vee \neg p$	by Complement
$\Leftrightarrow (p \vee \neg q) \vee \neg p$	by Identity
$\Leftrightarrow (\neg q \vee p) \vee \neg p$	by Commutativity
$\Leftrightarrow \neg q \vee (p \vee \neg p)$	by Associativity
$\Leftrightarrow \neg q \vee \top$	by Complement
$\Leftrightarrow \top$	by Domination.

So, we can see that $\neg p \vee \neg q$ is a complement for $p \wedge q$. Since complements are unique by [Theorem 1.1](#), we can conclude that $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$.

Showing $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ is left as an exercise to the reader.

Q.E.D.