Discrete Mathematics

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Chapter 3

Zermelo-Fraenkel Set Theory

"No one shall expel us from the paradise that Cantor has created."

—David Hilbert

3.1 The Language of Set Theory

In order to use our first-order logic as a language with which to talk about sets, we need to specify:

1. What is our universe of discourse Ω?
2. What are our fundamental predicate symbols?

With these two questions answered, we will be able to take objects from Ω and make atomic formulæ out of them, which we can then use to construct our *wff*.

Our universe of discourse will consist of *those objects that we can prove exist* using the rules of inference and the *axioms of set theory* (which we will develop in this chapter). The axioms of set theory will be sentences in the *language of Zermelo-Fraenkel set theory* that describe *what exactly sets are* and *how they work*.

We will have two predicate symbols, defined below.

Definition 3.1 (Equality)**.**
We define the binary predicate $=$ to mean that its left argument is *identically the same* as its right argument. So, if x and y are sets, then we say $x = y$ when we mean that the names x and y both refer to the same underlying object.

Definition 3.2 (Elementhood)**.**
We define the binary predicate $\in$ to mean that its left argument is contained in its right argument as an element. So, if x and y are sets, then the phrase $x \in y$ conveys that x is an element of y.

Definition 3.3 (Language of Set Theory)**.**
The *language of Zermelo-Fraenkel set theory* consists of the first-order logic along with

I. a universe of discourse consisting of those things that we can prove exist from the axioms (presented in Section 3.2) by using the rules of inference,

II. the binary predicate for equality (Definition 3.1), for which we use the $=$ symbol.

III. the binary predicate for elementhood (Definition 3.2), for which we use the $\in$ symbol.

3.2 Axioms of Set Theory

Axiom 0 (Existence)**.**
$\exists x(x = x)$

This axiom asserts that our universe of discourse is non-empty. Assuming this axiom lets us know for sure that when we make claims about sets, those claims are actually in reference to objects that provably exist (because we can use this axiom as an assumption in any proof).