Discrete Mathematics

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Chapter 2

First-Order Logic

2.1 Predicates & Quantification

Definition 2.1 (Universe of Discourse).

Our *universe of discourse*, usually denoted by the letter Ω, denotes the collection of objects under consideration. This specifies the objects that are admissible as inputs to predicates, meaning these are the objects we can actually make concrete, descriptive claims about using our formal language.

Definition 2.2 (Predicate).

Given a universe of discourse Ω and a non-negative integer n, we say that $\varphi(x_1, \dots x_n)$ is an n-ary *predicate* $:\Leftrightarrow$ the substitution of an object ω_i from Ω for each x_i in φ results in a proposition $\varphi(\omega_1, \dots \omega_n)$. These are sometimes referred to as *propositional functions.* Notice that if $n = 0$ then the 0-ary predicate is simply a proposition.

The placeholders $x_1, \dots x_n$ in φ are called *variables*, and the process of assigning objects to these variables is called *instantiation* or *variable assignment.*

Example 2.1.

If we choose a universe of discourse Ω consisting of some collection of people, then

$$\mu(x) \coloneqq \text{“}x \text{ is a mathematician.”} \qquad \gamma(x, y) \coloneqq \text{“}x \text{ drinks a Guinness with } y\text{.”}$$
$$\varepsilon(x) \coloneqq \text{“}x \text{ loves espresso.”} \qquad \alpha(x, y, z) \coloneqq \text{“}x, y, \text{ and } z \text{ are colleagues.”}$$

are unary, binary, and ternary predicates respectively.

Definition 2.3 (Universal Quantifier).

Given a unary predicate $\varphi(x)$, the *universal quantification* of the variable x in φ is denoted by $\forall x\big(\varphi(x)\big)$ and expresses that $\varphi(\omega)$ is true for any arbitrary ω from our universe of discourse.

Example 2.2.

Using the definitions from Example 2.1, we can rewrite “Every mathematician loves espresso” as $\forall x\big(\varepsilon(x)\big)$ or as $\forall x\big(\mu(x) \to \varepsilon(x)\big)$ depending on whether our universe Ω is the collection of all mathematicians or the collection of all people.

Definition 2.4 (Existential Quantifier).

Given a unary predicate $\varphi(x)$, the *existential quantification* of the variable x in φ is denoted by $\exists x\big(\varphi(x)\big)$ and expresses that $\varphi(\omega)$ is true for at least one ω from our universe of discourse.

Example 2.3.

Using the definitions from Example 2.1, we can rewrite “Some mathematician loves espresso” as $\exists x\big(\varepsilon(x)\big)$ or as $\exists x\big(\mu(x) \wedge \varepsilon(x)\big)$ depending on whether our universe Ω is the collection of all mathematicians or the collection of all people.

2.2 Rules of Inference

Definition 2.5 (Rule of Inference).

A *rule of inference* is a construction in the meta-language that tells us how we're allowed to take previous statements in our formal language and derive new statements from them. They essentially describe the allowable algebraic manipulations we can make to the symbols in our language, and in this way they function analogously to the instruction set of a computer (if you're familiar with that).

Another way to think of this is that they are truth-preserving axioms for our formal system; they are fundamental assumptions about the semantics of our formal language that allow us to reinterpret the statements we formulate in it. They usually take one of two forms: equivalences and inferences. For each of these rules, there is an *underlying tautology*, which is a tautological sentence in our formal language expressing the same idea as the rule.

An *equivalence rule* would take the form $\varphi \Leftrightarrow \psi$, where φ and ψ are sentences, indicating that any instance of φ can be replaced by an instance of ψ. The underlying tautology of such a rule is $p \leftrightarrow q$. A popular alternative notation is $\varphi \equiv \psi$.

An *inference rule* would take the form $\Gamma \Rightarrow \psi$ where Γ is a collection of sentences $\varphi_1, \ldots \varphi_n$ in our formal language and ψ is some other sentence, indicating that whenever we write down all of the sentences in Γ, we can then write down ψ afterward. The underlying tautology of such a rule is $\varphi_1 \wedge \ldots \varphi_n \Rightarrow \psi$. A popular alternative notation is $\Gamma \vdash \psi$, or $\varphi_1, \ldots \varphi_n \vdash \psi$, indicating that we can deduce ψ from the sentences in Γ.

The axioms for a Boolean algebra given in Definition 1.8 in Chapter 1 are examples of *equivalence rules of inference*, and we will take these as part of our collection of rules of inference for the formal language we are building.

Equivalence Rules		
Identity	$p \wedge \top \Leftrightarrow p$ $p \vee \bot \Leftrightarrow p$	$p \wedge \top \equiv p$ $p \vee \bot \equiv p$
Complement (*a.k.a.* Negation)	$p \wedge \neg p \Leftrightarrow \bot$ $p \vee \neg p \Leftrightarrow \top$	$p \wedge \neg p \equiv \bot$ $p \vee \neg p \equiv \top$
Commutativity	$p \wedge q \Leftrightarrow q \wedge p$ $p \vee q \Leftrightarrow q \vee p$	$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$
Associativity	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ $p \vee (q \vee r) \equiv (p \vee q) \vee r$
Distributive Laws	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Inference Rules		
Deduction	If by assuming p we are able to derive q, then we can derive $p \to q$	$\dfrac{p \vdash q}{p \to q}$
Modus Ponens	If we know that $p \to q$ and we have p, then we can derive q	$\dfrac{\begin{array}{l}p\\ p \to q\end{array}}{q}$
Modus Tollens	If we know that $p \to q$ and we have $\neg q$, then we can derive $\neg p$	$\dfrac{\begin{array}{l}\neg q\\ p \to q\end{array}}{\neg p}$
Reductio ad Absurdum	If by assuming p we can derive both q and $\neg q$, then we can derive $\neg p$	$\dfrac{\begin{array}{l}p \vdash q\\ p \vdash \neg q\end{array}}{\neg p}$