Problem Set 5

Discrete Mathematics

Due on the 27^{th} of February, 2024

(20 pts) 1. Show that $\forall x(x \neq x \cup \{x\})$.

(15 pts) 2. We will work up to a proof of the commutativity of addition on $\mathbb{N}$.

(a) Show $(\forall x \in \mathbb{N})(x + 0 = 0 + x)$.

(b) Show $(\forall x, y \in \mathbb{N})(x + \mathfrak{s}(y) = \mathfrak{s}(y) + x)$.

(c) Show $(\forall x, y \in \mathbb{N})(x + y = y + x)$.

(15 pts) 3. Show $(\forall x, y, z \in \mathbb{N})\big(x \cdot (y + z) = (x \cdot y) + (x \cdot z)\big)$.

(15 pts) 4. Show $(\forall x, y, z \in \mathbb{N})\big(x \cdot (y \cdot z) = (x \cdot y) \cdot z\big)$.

(15 pts) 5. For this problem, you may assume the commutativity and associativity of addition and multiplication over $\mathbb{N}$. You may also assume multiplication distributes over addition on $\mathbb{N}$. Prove the following statement for all $n \in \mathbb{N}$.

$$1 + \sum_{i=0}^{n} 2^n = 2^{n+1}$$

(20 pts) 6. We say x is *$\in$-transitive* by definition when $(\forall y \in x)(\forall z \in y)(z \in x)$. Show that every natural number is $\in$-transitive.

Recall that the natural numbers are defined recursively as follows.

$$0 := \varnothing$$
$$\mathfrak{s}(n) := n \cup \{n\}$$

Addition on $\mathbb{N}$ is defined below.

$$n + 0 := n$$
$$n + \mathfrak{s}(m) := \mathfrak{s}(n + m)$$

Multiplication on $\mathbb{N}$ is defined below.

$$n \cdot 0 := 0$$
$$n \cdot \mathfrak{s}(m) := (n \cdot m) + n$$

Exponentiation on $\mathbb{N}$ is defined below.

$$n^0 := 1$$
$$n^{\mathfrak{s}(m)} := n \cdot n^m$$

We define the iterated sum of a sequence of terms $f(0), f(1), f(2), \ldots$ as follows.

$$\sum_{i=0}^{0} f(i) := f(0)$$
$$\sum_{i=0}^{\mathfrak{s}(n)} f(i) := \left(\sum_{i=0}^{n} f(i)\right) + f(\mathfrak{s}(n))$$

You may rely on the following theorems:
$(\forall x \in \mathbb{N})(\mathfrak{s}(x) = x + 1)$.
$(\forall x \in \mathbb{N})(\mathfrak{s}(x) = 1 + x)$.
$(\forall x, y, z \in \mathbb{N})(x + (y + z) = (x + y) + z)$.