

*Exam f.p.2024.spring*

*Discrete Mathematics*

*9<sup>th</sup> of May, 2024*



**1 Answer the following questions either True or False.**

- (a) ??
- ☐ True
- ☐ False
- (b) If  $a$  and  $b$  are odd, then  $(\exists c \in \mathbb{Z})(c^2 = a^2 + b^2)$ .
- ☐ True
- ☐ False
- (c) If  $\mathcal{A}$  and  $\mathcal{B}$  are both finite, then  $|\mathcal{A} \setminus \mathcal{B}| = |\mathcal{A}| - |\mathcal{B}|$ .
- ☐ True
- ☐ False
- (d) If this sentence is a proposition, then every countable set is finite.
- ☐ True
- ☐ False
- (e)  $2^{21} \equiv 1 \pmod{7}$ .
- ☐ True
- ☐ False
- (f) This sentence implies  $\forall x(\emptyset \subseteq x)$ .
- ☐ True
- ☐ False
- (g)  $(\forall \mathcal{A} \in \mathbb{P}(\mathbb{N}))(\mathcal{A} \neq \emptyset \Rightarrow (\exists a \in \mathcal{A})(\forall b \in \mathcal{A})(a \leq b))$ .
- ☐ True
- ☐ False
- (h)  $\mathbb{Z}/n\mathbb{Z}$  is a group under multiplication for every  $n \in \mathbb{N}_+$ .
- ☐ True
- ☐ False
- (i)  $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y = \gcd(x, 15)\}$  is a function.
- ☐ True
- ☐ False
- (j)  $\forall x \forall y \forall z (x \setminus (y \cap z) = (x \setminus y) \cup (x \setminus z))$ .
- ☐ True
- ☐ False

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2 *Answer the following questions without proof.*

(a) Compute  $\gcd(386, 352)$ .

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(b) Find all  $x \in \mathbb{Z}$  such that  $4x + 6 \equiv 3 + 2x \pmod{9}$ .

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(c) Find all integer solutions to  $3x^{49} + 3x + 2 \equiv 4 \pmod{7}$ .

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(d) List the elements of the following set using set-builder notation:

$$\left\{ x \in \mathbb{N} \mid (\exists k \in \mathbb{Z})(xk = 28) \wedge (\forall y \in \mathbb{N})(y \mid x \Rightarrow y \in \{1, x\}) \right\}.$$

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(e) Provide an example of a transitive set  $x$  such that  $x \notin \mathbb{N}$ .

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3 Answer the following questions without proof.

(a) Provide a surjection from  $\left\{ f \mid (\exists n \in \mathbb{N})(f : n \rightarrow \{0, 1\}) \right\}$  to  $\mathbb{Z}$ .

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(b) In how many different ways can the strings "fighting" and "irish" be scrambled and then concatenated together?

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(c) Let  $k \in \mathbb{N}$  such that  $k \geq 3$ . How many strings  $s : k \rightarrow \{0, 1, \dots, 9\}$  satisfy  $(\exists i \in k)(s(i) = s(i+1) - 1 = s(i+2) - 2)$ ?

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(d) Let  $n \in \mathbb{N}_+$ . How many binary strings  $b$  satisfy  $|b| + \sum_{i=0}^{|b|-1} b(i) = n$ ?

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(e) How many ways are there to move from the bottom-left square to the top-right square on an  $n \times n$  chess board if  $n \in \mathbb{N}_+$ ?

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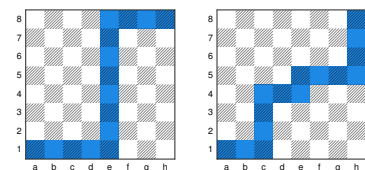


Fig. 1: Two examples of valid paths from the bottom-left corner to the top-right corner on an  $8 \times 8$  chessboard.

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**4** *Any use of logical axioms, rules of inference, or theorems must be stated. You may not appeal to truth tables.*

Prove  $\neg(p \rightarrow q) \rightarrow p$  is a tautology for any propositions  $p$  and  $q$ .

5 *You may rely on any theorems we have proven or studied.*

1. Give a recursive definition of the Fibonacci sequence  $\mathcal{F} : \mathbb{N} \rightarrow \mathbb{N}$ .

2. Show that  $\mathcal{F}(n) < 2^n$  for all  $n \in \mathbb{N}_+$ .

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**6** *You may rely on any theorems we have proven or studied.*

Let  $\mathfrak{F} := \{\mathcal{F}(i) \mid i \in \mathbb{N}\}$  be the set of all Fibonacci numbers and observe  $|\mathfrak{F}| = \aleph_0$ . Show  $\exists i, j \in \mathbb{N}$  such that  $i \neq j$  and  $\mathcal{F}(i) \equiv \mathcal{F}(j) \pmod{2024}$ .

**7** *You may rely on any theorems we have proven or studied.*

Let  $n \in \mathbb{N}_+$  and let  $b : n \rightarrow \{0, 1, \dots, 9\}$  be a decimal string representing a natural number  $k$ . Suppose the sum of the digits of  $b$  is divisible by 3. Prove that  $3 \mid k$ .

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8 *You may rely on any theorems we have proven or studied.*

Show that there are uncountably many infinite hexadecimal strings.