Monotone Catenary Degree in Numerical Monoids

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Outline

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- Definitions

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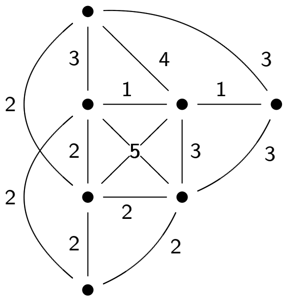

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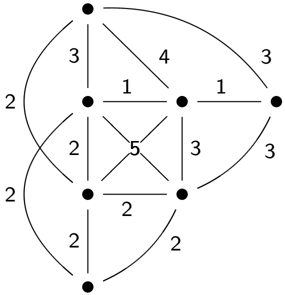

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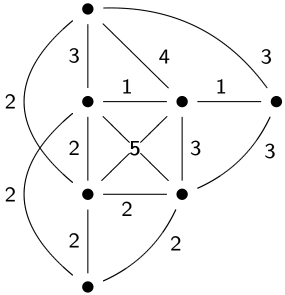

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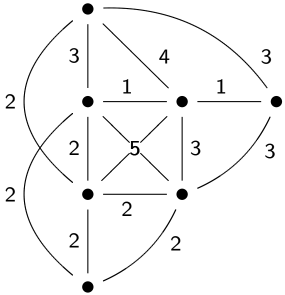

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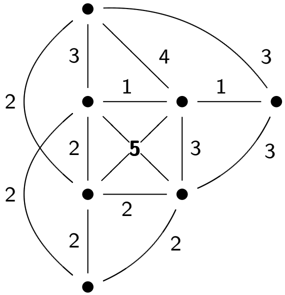

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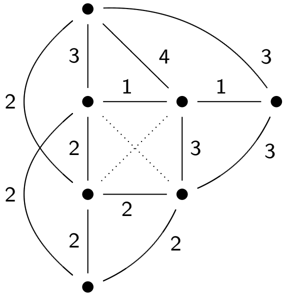

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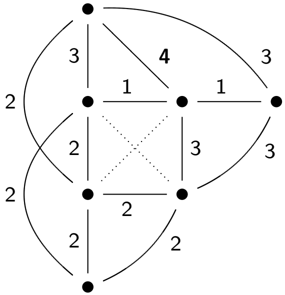

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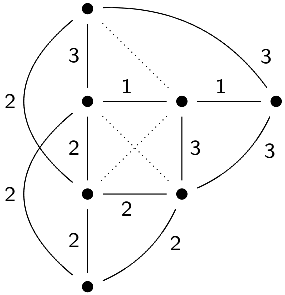

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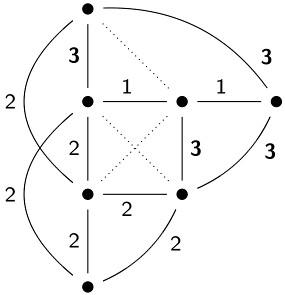

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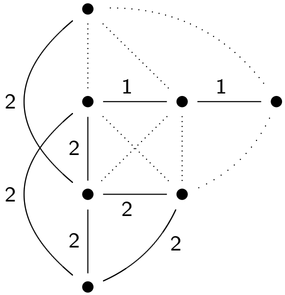

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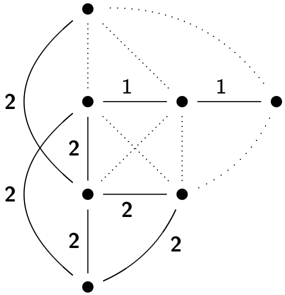

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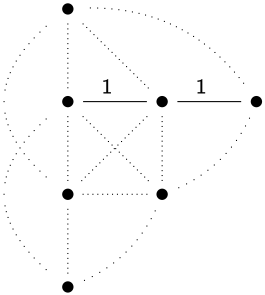

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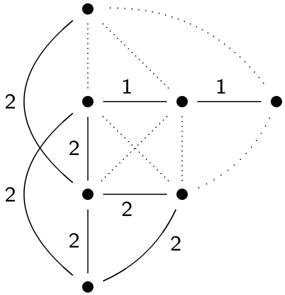

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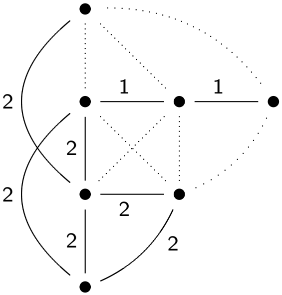

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$$c(m) = 2$$

Equivalent Catenary Degree

How does it differ?

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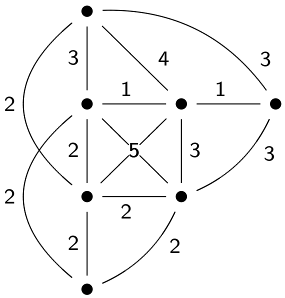

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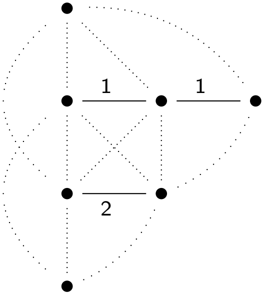

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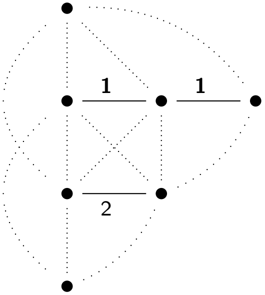

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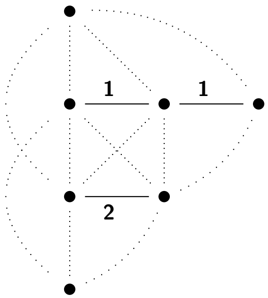

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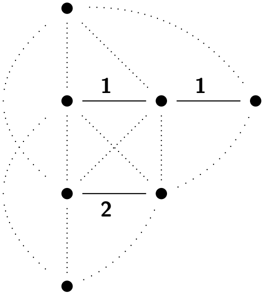

$$c_{eq}(m) = \max\{1, 2\} = 2$$

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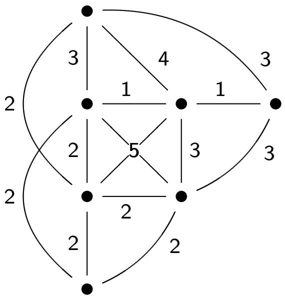

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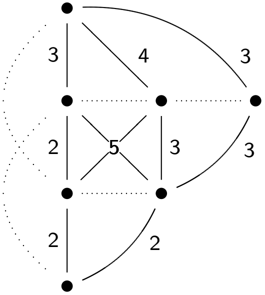

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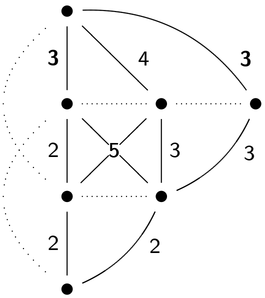

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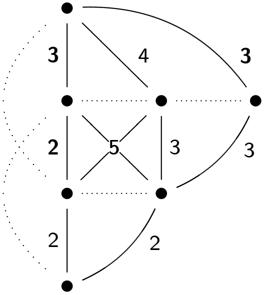

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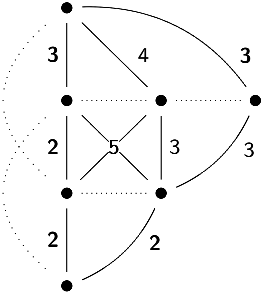

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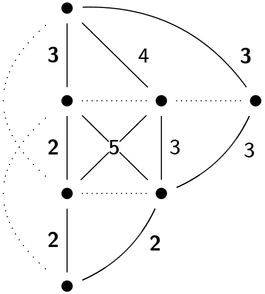

$$c_{adj}(m) = \max\{3, 2, 2\} = 3$$

Monotone Catenary Degree

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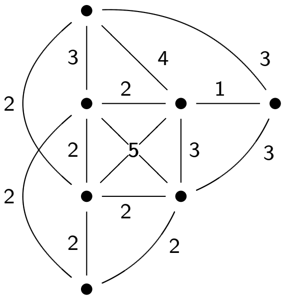

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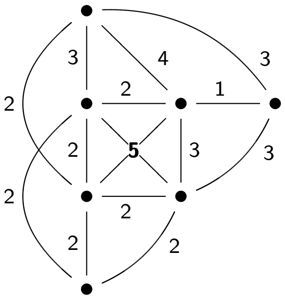

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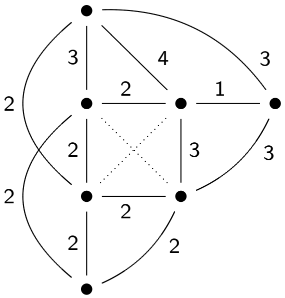

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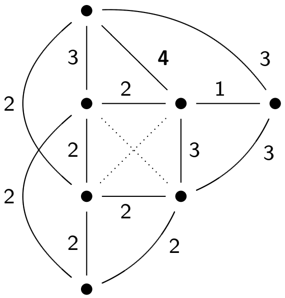

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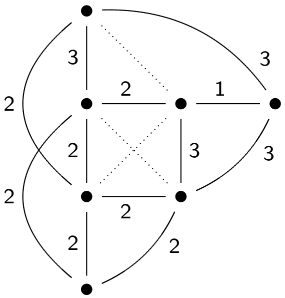

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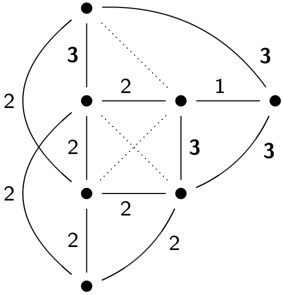

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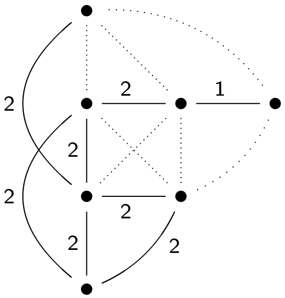

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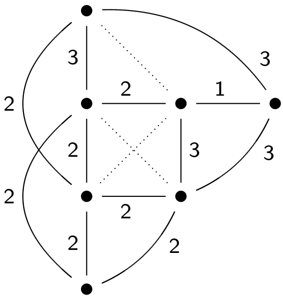

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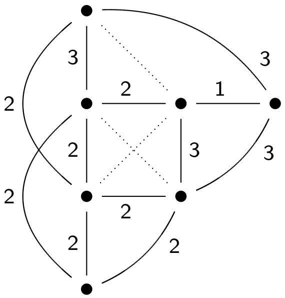

$c_{mon}(m) = 3$

Motivation

What is known about the monotone catenary degree?

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- For elements of a numerical monoid M, $c(m) \leq c_{mon}(m)$.

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We seek to investigate the inequality between monotone and regular catenary degree.

Arithmetic Monoids

Consider a monoid M generated by a finite arithmetic sequence $a, a+d, \ldots, a+kd$, so $M = \langle a, a+d, \ldots, a+kd \rangle$. We will refer to a monoid of this form as an *arithmetic monoid*.

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Examples:

- $\langle 5, 7, 9 \rangle$

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Examples:

- $\langle 5, 7, 9 \rangle$
- $\langle 11, 15, 19, 23 \rangle$

Monotone vs. Regular Catenary Graph

Factorizations of 96 in $\langle 11, 15, 19, 23, 27 \rangle$

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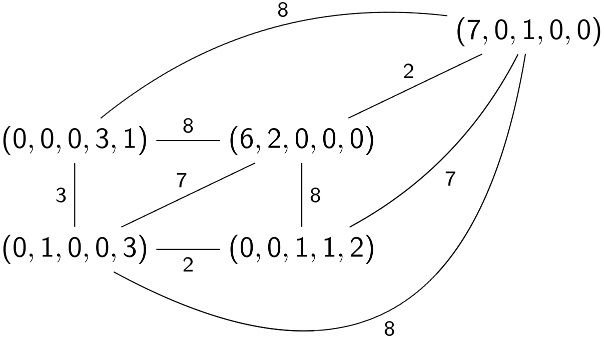

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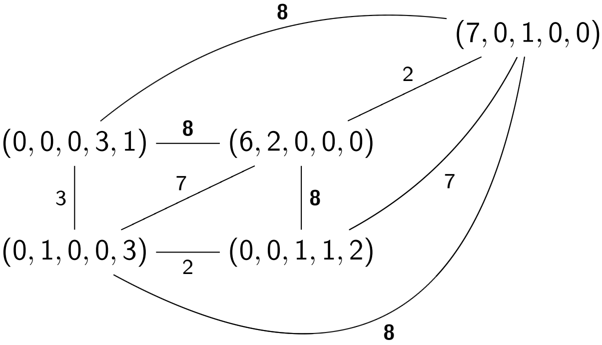

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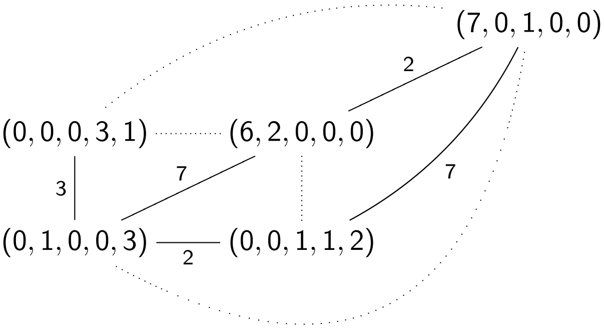

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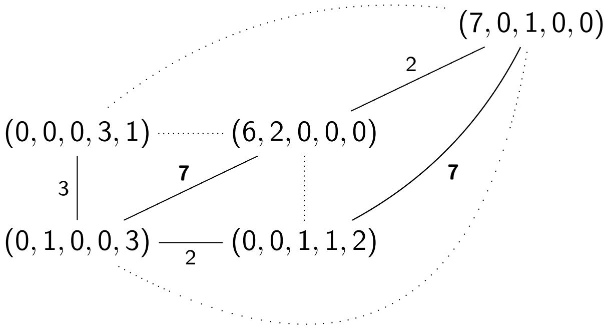

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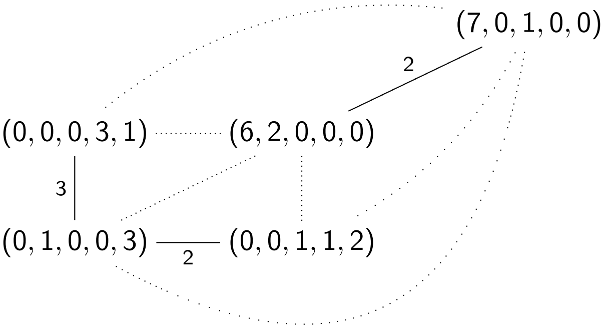

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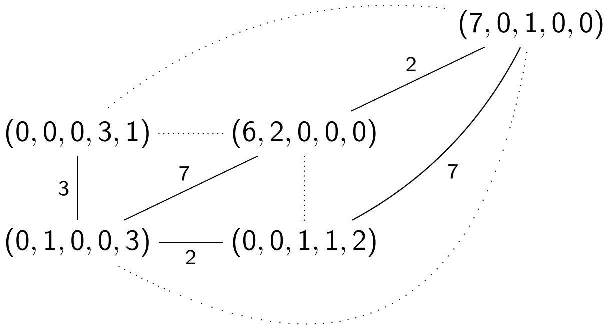

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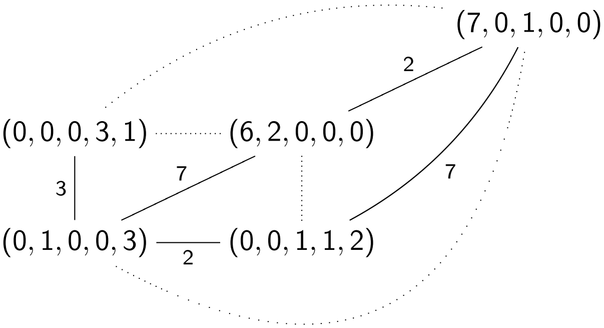

$c(96) = 7$

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$$
\begin{tikzcd}
l = 8 : & & (6,2,0,0,0) \arrow[r, dash, "2"] \arrow[d, dash, "7"] & (7,0,1,0,0) \arrow[d, dash, "7"] \\
l = 4 : & (0,0,0,3,1) \arrow[r, dash, "3"] & (0,1,0,0,3) \arrow[r, dash, "2"] & (0,0,0,1,2)
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\end{array}
$$

$$c_{mon}(96) = c(96) = 7.$$

Main Result for Arithmetic Sequences

Theorem

Given an arithmetic monoid $M = \langle a, a+d, \ldots, a+kd \rangle$*, for all elements* $m \in M$*,*

$$c_{mon}(m) = c(m).$$

As a result of this, $c_{mon}(M) = c(M)$*.*

Generalized Arithmetic Sequences

Consider $M = \langle a, ah + d, \ldots, ah + kd \rangle$. We will refer to a monoid of this form as a *generalized arithmetic monoid*. We will discuss generalized arithmetic monoids in embedding dimension three: $M = \langle a, ah + d, ah + 2d \rangle$.

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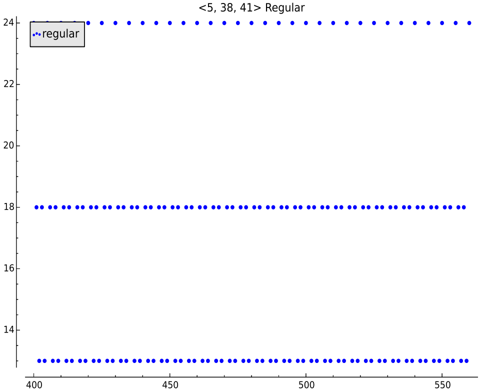
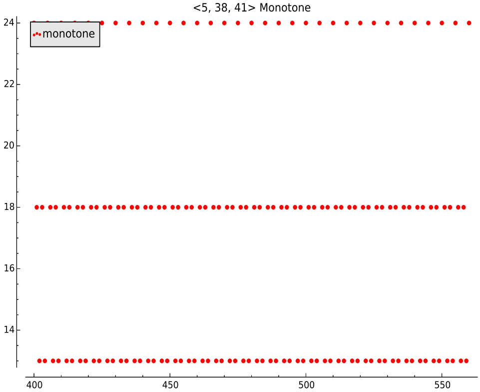
  $a = 5,\ h = 7,\ d = 3$

Generalized Arithmetic Sequences





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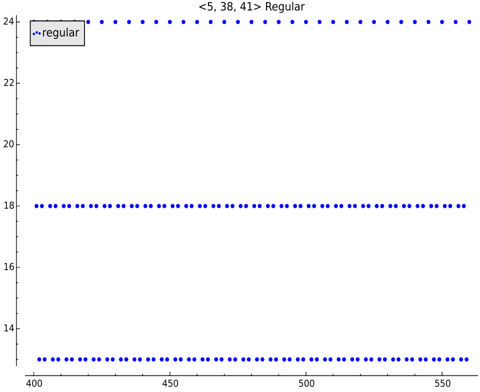

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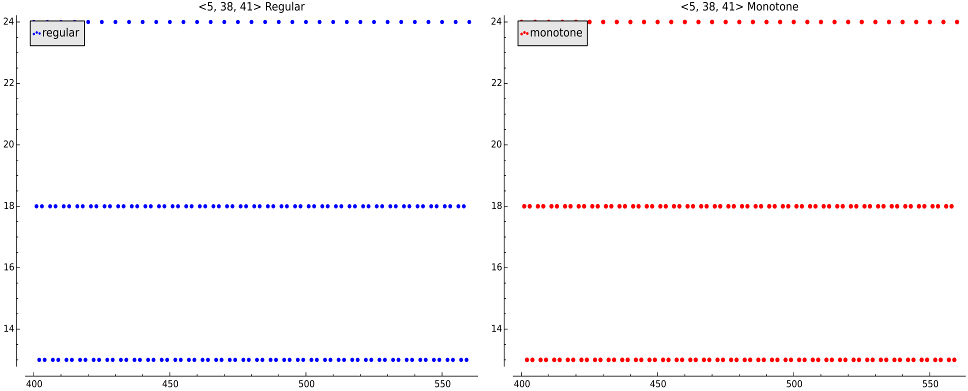

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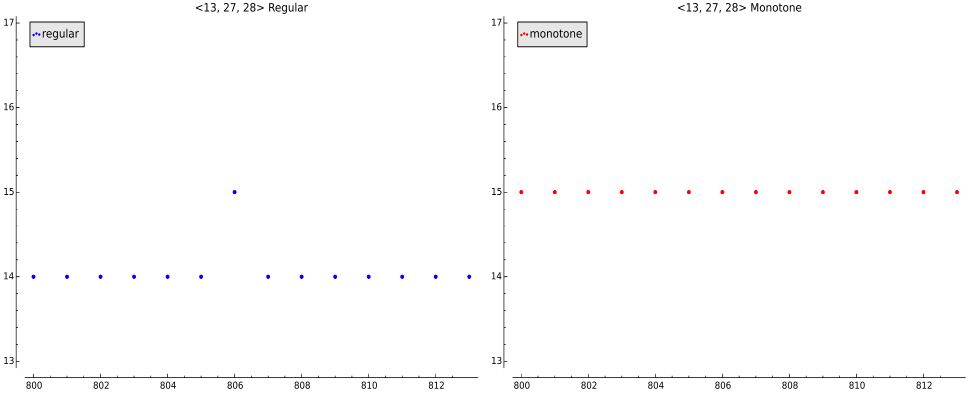

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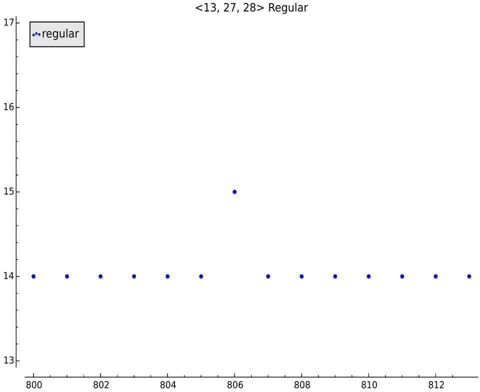

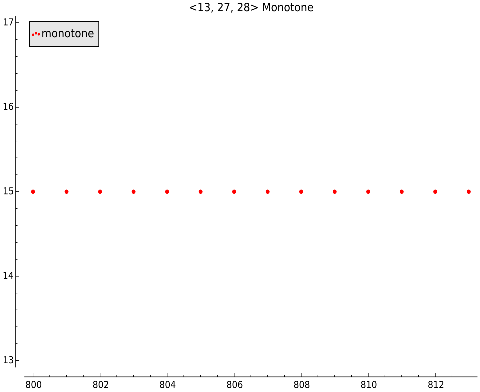

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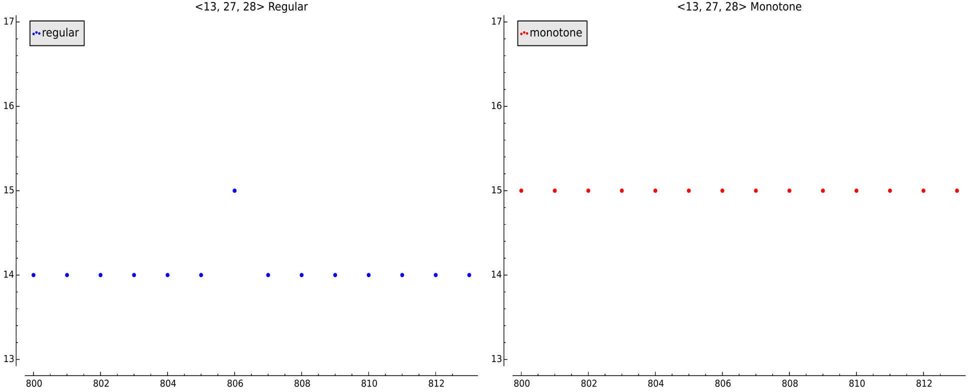

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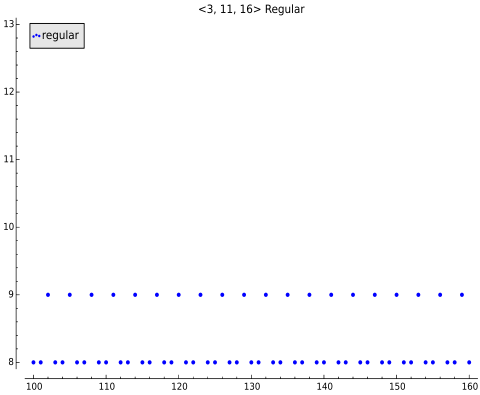

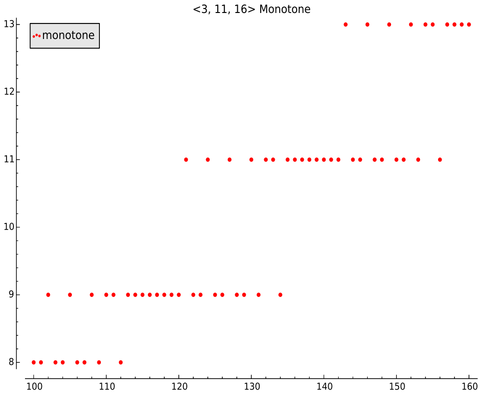

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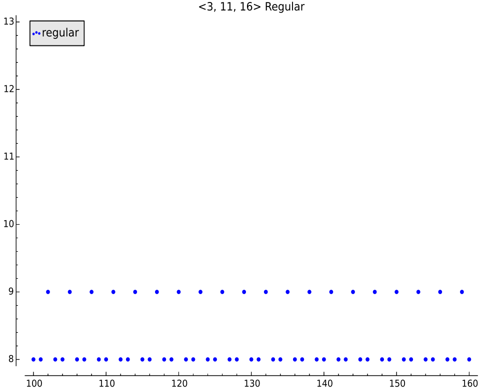

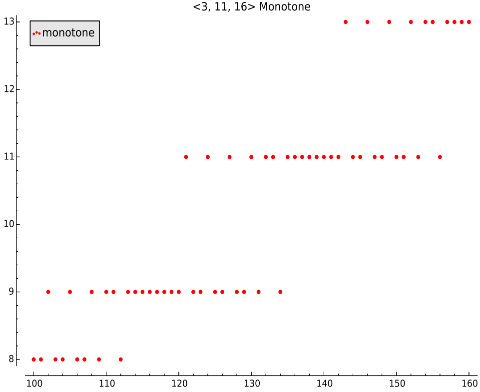

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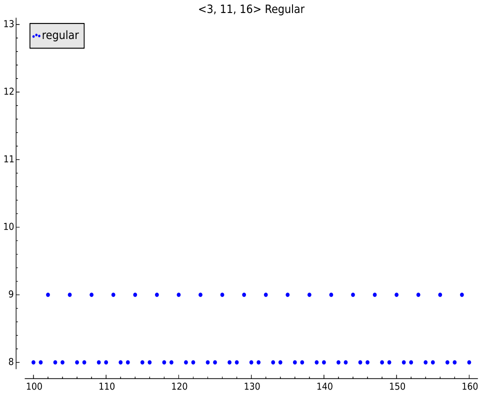

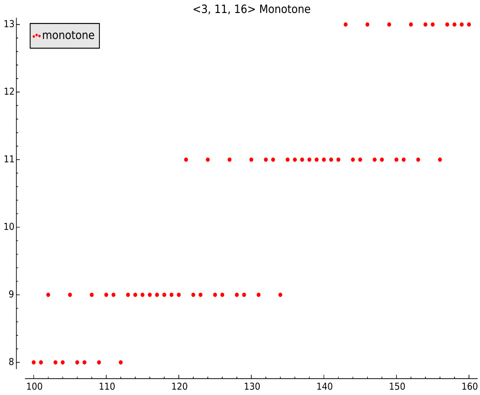

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Generalized Arithmetic Sequences

Conjecture

If $\gcd(h-1, d) > 1$, *then* $c_{mon}(M) = c(M)$.

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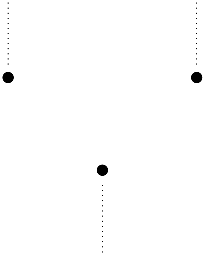

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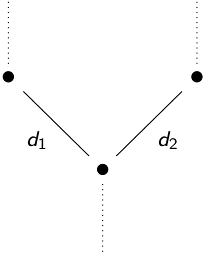

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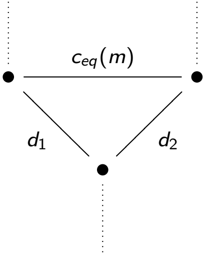

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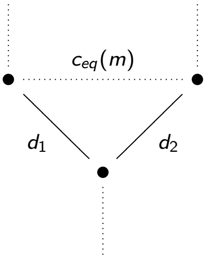

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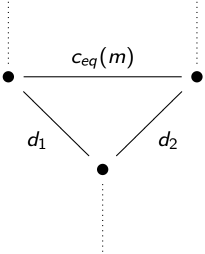

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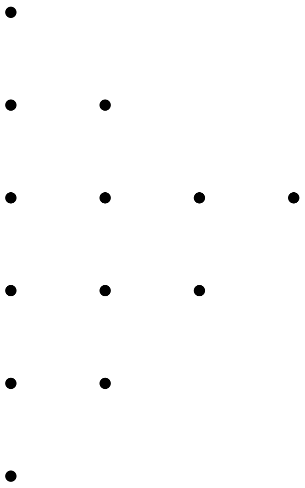

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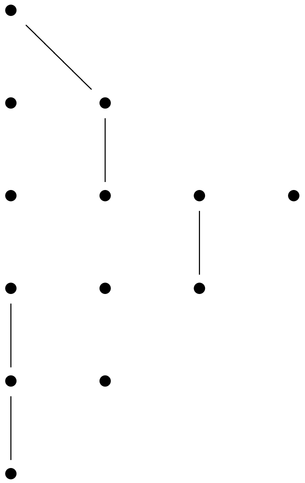

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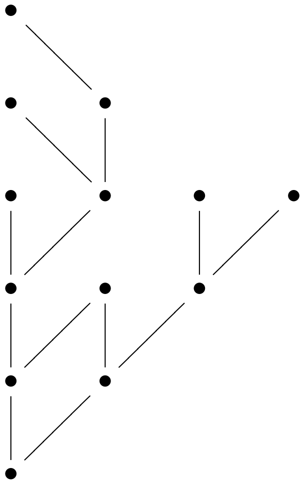

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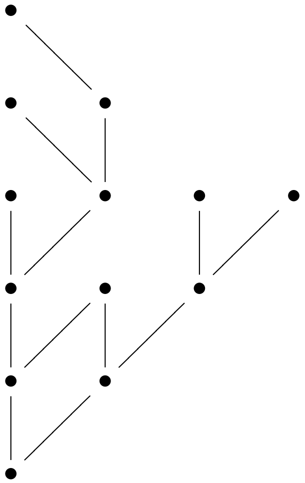

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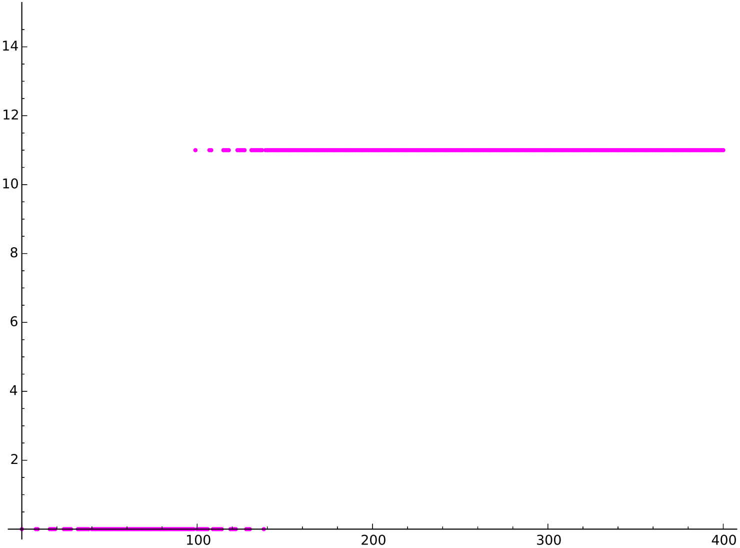
Equivalent Catenary Degree in $\langle 8, 9, 19 \rangle$



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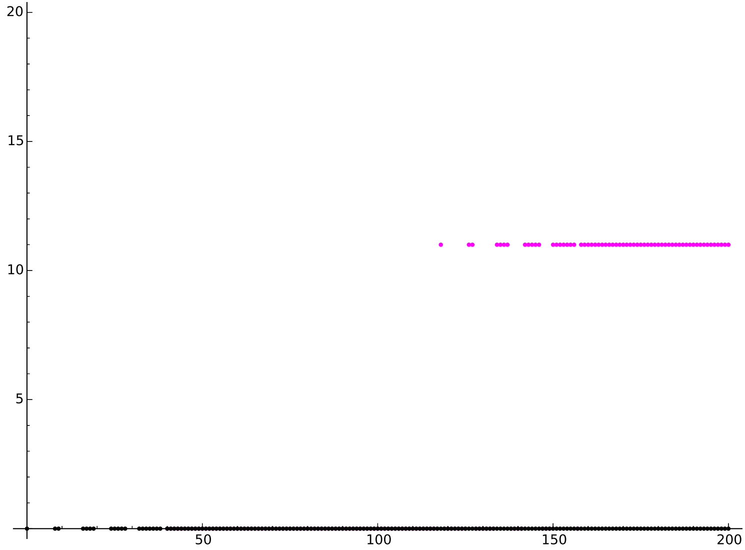

Equivalent Catenary Degree in $\langle n_1, n_2, n_3 \rangle$

Theorem

Let M be a minimally generated numerical monoid $\langle n_1, n_2, n_3 \rangle$. Then $c_{eq}(M) = \frac{n_3 - n_1}{gcd(n_2 - n_1, n_3 - n_1)}$

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- If $m = n_2 \left(\frac{n_3 - n_1}{gcd(n_2 - n_1, n_3 - n_1)} \right) + x$ for some $x \in M$, then $c_{eq}(m) = c_{eq}(M)$.
- If $m = n_2 \left(\frac{n_3 - n_1}{gcd(n_2 - n_1, n_3 - n_1)} \right) + x$ for some $x \notin M$, then $c_{eq}(m) = 0$.

Equivalent Catenary Degree in $\langle n_1, n_2, n_3 \rangle$

Theorem

For all $m \in M$ such that $m > n_2 \left(\frac{n_3 - n_1}{gcd(n_2 - n_1, n_3 - n_1)} \right) + \mathcal{F}(M)$, $c_{eq}(m) = c_{eq}(M)$. Furthermore, $c_{eq}(m)$ can take on only two values: 0 and $c_{eq}(M)$.

Moving Between Factorizations

$$M = \langle n_1, n_2, n_3 \rangle$$

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Theorem

Consider a factorization $z = (b, c, d)$ of an element $s \in M$ with $|z| = x$. We can write any factorization z_i of s where $|z_i| = x - l$ as $z_i = \left(b + \frac{-ln_1 + k(n_3 - n_1)}{n_3 - n_1} - k - l, c + \frac{ln_1 - k(n_3 - n_1)}{n_3 - n_1}, d + k \right)$.

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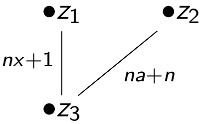

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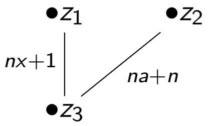

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Let $M = \langle na, na+n, 2na+nx+1 \rangle$ for $n \in \mathbb{N}$ and $x \geq 2$. Then $c_{mon}(M) > c(M)$.

$\langle a, a+1, \mathcal{F}\langle a, a+1\rangle\rangle$

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- $c_{mon}(M) = a^2 - 2a - 1$
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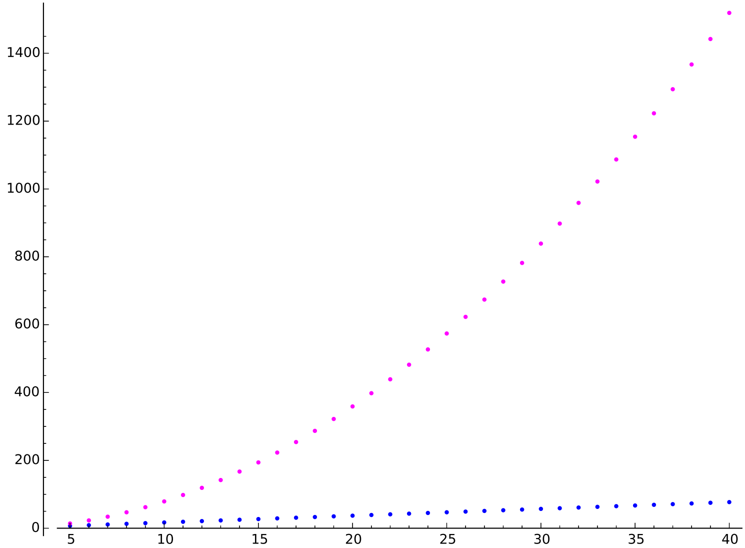
- $c_{mon}(M) = a^2 - 2a - 1$
- $c(M) = 2a - 3$
- $c_{mon}(M) - c(M) = a^2 - 4a - 4$

$\langle a, a+1, \mathcal{F}\langle a, a+1\rangle\rangle$



$c_{mon}(M) - c(M)$

Theorem

The difference between the monotone and regular catenary degrees of a monoid can be arbitrarily large.

Takeaways

- In some monoids, namely those generated by arithmetic sequences, $c_{mon}(M) = c(M)$.

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- In some monoids, namely those generated by arithmetic sequences, $c_{mon}(M) = c(M)$.
- In generalized arithmetic monoids, we can have either $c_{mon}(M) > c(M)$ or $c_{mon}(M) = c(M)$.
- In general, we expect that $c_{mon}(M) > c(M)$. In fact, the difference between the two can grow arbitrarily large.

Thank You

We would like to thank:

- The NSF and NSA
- Sam Houston State University, University of Hawai'i at Hilo, and PURE Math
- Dr. Brian Loft, Dr. Rebecca García, and Dr. Luis García
- Dr. Brian Wissman, Dr. Bob Pelayo
- Dr. Scott Chapman and Dr. Chris O'Neill
- Felix Gotti and Marly Cormar

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