MAP 5615 - Monte Carlo Methods

Homework 2 (Due February 25, 2019)

1. Prove
$$\begin{aligned} & \max\left[F(X_1), \max_{k=1,\ldots,N-1}\left(F(x_{k+1})-\frac{k}{N},\frac{k}{N}-F(x_k)\right), 1-F(X_N)\right] \\ = \; & \max\left[\max_{k=1,\ldots,N}\left(\frac{k}{N}-F(x_k)\right), \max_{k=1,\ldots,N}\left(F(x_k)-\frac{k-1}{N}\right)\right] \end{aligned}$$
which was used in the derivation of D_N.

2. Consider the Fibonacci generator: $x_{n+1} \equiv x_n + x_{n-1} \mod 2^{31}$, where $x_0 = x_1 = 1$. Apply the Kolmogorov-Smirnov test to the first 1000 numbers (including the seeds) obtained from this generator. What are your conclusions?

3. Show that the serial correlation coefficient ρ is equal to -1, if $n = 2$, provided the denominator is not zero.

4. In this problem, you can use any good pseudorandom generator. Let $u(i)$ be the runs-up of length i, for $i = 1, 2, 3$, and $u(4)$ be the runs-up of length 4 or more. Compute the serial correlation coefficient ρ_4 for $u(1), u(2), u(3), u(4)$. Repeat this procedure 100 times to obtain 100 correlation coefficients. Does the serial coefficient test indicate dependency? Can you explain the result intuitively? Based on your conclusions, explain whether one can apply the χ^2- test directly to the run-up counts (assuming that we know the probability of run-up of length i) like it was applied in the gap test? (Hint: think about the basic assumptions required to apply the χ^2- test, and investigate whether these assumptions are satisfied by "run-up events".)

5. Consider the following modification of the run test. Let's start with the same sequence used in the run test description.

   $$2,\ 7,\ 8,\ 1,\ 9,\ 6,\ 4,\ 0,\ 3,\ 11,\ 10,\ 17$$

   The first run-up is | 2 7 8 |, and it is a run-up of length three. Now we discard the number that comes after the run-up, in this case, it is 1, and use the next number in the sequence, 9, to start the next run. Since after 9 comes 6, we have a run-up of length one, | 9 |, and we discard the number 6, and continue in this way, discarding the number that comes next after a run-up event occurs. Here are the new runs-up events for the above sequence:

   |2 7 8|9|4|3 11|17|

   Under this modified run test, we have run-up of lengths 3,1,1,2,1.

   (a) Argue that in the modified run test, the run-up events are independent, and a simple χ^2-test can be used to design a modified run test. (You can use the approach in the previous exercise to investigate independence numerically).

   (b) Prove that in the modified run test, the probability of having a run-up of length n is $\frac{1}{n!} - \frac{1}{(n+1)!}$, and the probability of having a run-up of length n or more is $\frac{1}{n!}$.

   (c) Design a modified run test using parts (a) and (b), and apply it to Mersenne twister.

6. Design a statistical test for random number generators, based on the following result. Then apply the test to any generator you want and explain the results.
   Fact: A coin is flipped consecutively until the number of heads obtained equals the number of tails. The output of a flip is heads with probability p. Define the random variable X as: $X =$ the first time the total number of heads is equal to the total number of tails. Observe that X takes values 2,4,6,.... For example, if the outcomes of one experiment are: H,H,T,H,T,T then the value of X for this outcome is 6. Here is the probability density function of X :
   $$P\{X = 2n\} = \frac{1}{2n-1}\binom{2n}{n}p^n(1-p)^n.$$
   (A proof of this statement can be found in "Introduction to Probability Models", Sheldon Ross, 8th edition, page 128.)