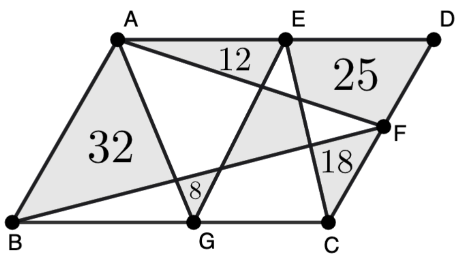
INTEGIRLS Shanghai Spring 2025

Team Round 60 minutes

Question 1 (5 points)

In the diagram below, $ABCD$ is a parallelogram. Points E, F, and G lie on sides AD, CD, and BC respectively. Given the areas of certain marked regions, find the area of the remaining shaded region.

在下图中，四边形 $ABCD$ 为平行四边形，点 E、F 和 G 分别位于边 AD、CD 和 BC 上。已知部分区域的面积，求阴影部分的面积。



Question 2 (5 points)

Alice and Bob each have a pair of identical dice. They take turns rolling theirs simultaneously, with Alice rolling first. Let P_n be the probability that Alice is the first to roll a sum exceeding n. Compute $\frac{1}{P_7} - \frac{1}{P_6}$.

Alice 和 Bob 各自拥有一对相同的骰子。他们轮流同时掷骰子，Alice 先掷。设 P_n 为 Alice 首次掷出和大于 n 的概率。计算 $\frac{1}{P_7} - \frac{1}{P_6}$。

Question 3 (5 points)

If

$$\begin{cases} 6x(x+y)+3 = 13x+13y \\ 12(x^2+xy+y^2)+\frac{9}{(x+y)^2} = 85 \end{cases}$$

find the value of $x-y$.

假如方程组

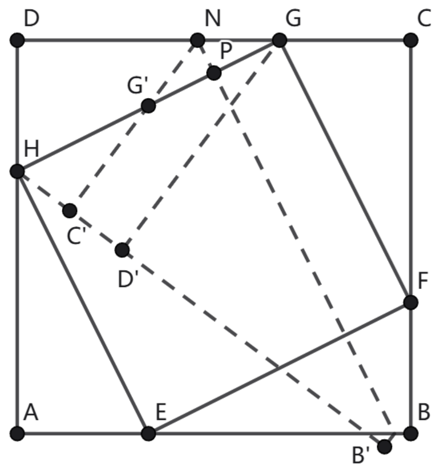
$$\begin{cases} 6x(x+y)+3 = 13x+13y \\ 12(x^2+xy+y^2)+\frac{9}{(x+y)^2} = 85 \end{cases}$$

成立，求 $x-y$ 的值。

Question 4 (5 points)

Eliza is skilled at origami. She begins with a square piece of paper $ABCD$. She selects points E and F on sides AB and BC respectively, then folds the paper along a line perpendicular to EF, creating crease MN. This fold maps points B and C to new positions B' and C' in the plane. After unfolding, she makes another fold perpendicular to MN, creating crease GH with G on CD and H on AD, mapping D to D'. Given that D' lies on segment $B'C'$, quadrilateral $EFGH$ is a square, and $AE = 4$, $EB = 8$; let P be the intersection of MN and GH, and let G' be the intersection of NC' and GH'. Determine the length of segment PH.

Eliza 擅长折纸。一天，她计划按如下方式折叠一张正方形纸片 $ABCD$：在边 AB 和 BC 上分别取点 E 和 F。沿垂直于线段 EF 的直线折叠，形成折痕 MN。点 B 和 C 分别落在平面中的点 B' 和 C' 上，然后将其恢复。接着，沿垂直于 MN 的直线折叠，形成折痕 GH。点 G 和 H 分别位于边 CD 和 AD 上。点 D 落在平面中的点 D' 上，然后将其恢复。已知点 D' 位于线段 $B'C'$ 上，且四边形 $EFGH$ 为正方形，给定 $AE = 4$，$EB = 8$。此外，MN 与 GH 的交点为 P，NC' 与 GH' 的交点为 G'（如图所示）。求线段 PH 的长度。



Question 5 (5 points)

In a carnival game, players win a prize if they can completely cover a target circle using three given unit circles. The game operator wants to maximize the area of the target circle to minimize winning chances. If the maximum possible area of such a target circle that can be fully covered by three unit circles is $\frac{m}{n}\pi$, where m and n are coprime positive integers, find the value of $m + n$.

在一个嘉年华游戏中，玩家需要使用三个给定的单位圆完全覆盖目标圆，即可获得奖励。游戏运营者希望最大化目标圆的面积，以降低获胜的频率。若可以被三个单位圆完全覆盖的最大目标圆的面积为 $\frac{m}{n}\pi$，其中 m 和 n 为互质的正整数，则求 $m + n$。

Question 6 (7 points)

Two non-overlapping circles have an area difference of 165π. Their internal and external common tangents are 7 and 15 units long, respectively. What is the distance between the centers of the two circles?

两个不重叠的圆的面积差为 165π。它们的内公切线和外公切线的长度分别为 7 和 15 个单位。求两个圆的圆心距离。

Question 7 (7 points)

On a 3×3 square pegboard, the 9 pegs are randomly colored red or blue. What is the probability that a rubber band cannot be stretched over 4 pegs of the same color to form a rectangle (or a square)?

在一个 3×3 的方形钉板上，9 个钉子被随机涂成红色或蓝色。求无法用橡皮筋围住 4 个同色钉子形成矩形（包括正方形）的概率。

Question 8 (10 points)

Let $f(n)$ denote the number of ways to arrange n blocks in a row, where each block is either blue or orange, with the restriction that no two orange blocks are adjacent. (An arrangement with no orange blocks is considered valid.) Compute the value of $f(10) + f(11)$.

$f(n)$ 表示将 n 个蓝色或橙色方块排成一排的不同排列方式，其中任意两个橙色方块不能相邻（不含橙色方块的排列满足要求）。求 $f(10) + f(11)$。

Question 9 (10 points)

There are 30 number 2's with a space between them like this: $2\ 2\ 2\ \cdots 2$. To decide what to write in the 29 gaps, a coin is flipped 29 times. If it lands on heads, a $+$ is written; if it lands on tails, a $\times$ is written. What is the expected value of the resultant expression?

有 30 个数字 2，之间有空格，如下所示：$2\ 2\ 2\ \cdots 2$。为了决定在 29 个空格中填写什么，抛掷一枚硬币 29 次。如果硬币正面朝上，则填写一个 $+$；如果硬币反面朝上，则填写一个 $\times$。求最终表达式的期望值是多少?

Question 10 (10 points)

In right $\triangle ABC$, $\angle B = 90°$, $AB > BC$. Let I be the incenter of $\triangle ABC$, and let its incircle meet sides AB, BC, and AC at F, D, and E, respectively. Suppose lines DE and AB meet at X, and lines AI and DE meet at H. Let K be the intersection of AI and CF, and let G be the intersection of BH and XI. If KG bisects $\angle HGI$, what is $\tan\angle CAB$?

在直角三角形 $\triangle ABC$ 中，$\angle B = 90°$，且 $AB > BC$。设 I 为 $\triangle ABC$ 的内心，其内切圆分别与边 AB、BC 和 AC 相交于点 F、D 和 E。假设直线 DE 与 AB 相交于点 X；直线 AI 与 DE 相交于点 H。设 K 为直线 AI 与 CF 的交点，G 为直线 BH 与 XI 的交点。如果 KG 平分 $\angle HGI$，求 $\tan\angle CAB$ 的值。

END OF TEST.