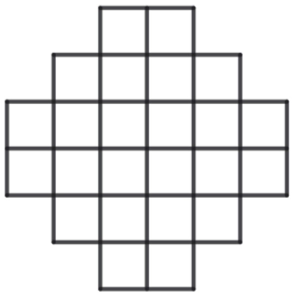
INTEGIRLS Shanghai Spring 2025

Individual Round   60 minutes

Question 1 (3 points)

The region shown below consists of 24 squares, each with side length 1 centimeter. What is the radius, in centimeter, of the smallest circle that the region can fit inside?

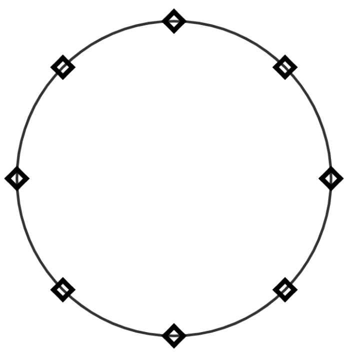
下图中含有 24 个正方形，每个边长为 1 厘米。该图形的最小外接圆的半径是多少（以厘米为单位)?



Question 2 (3 points)

8 friends are seated evenly around a round table. What is the angle formed by Anna, Carmen and Beth in degrees if Anna and Beth has 1 friend in between while Beth and Carmen has 2 friends in between?

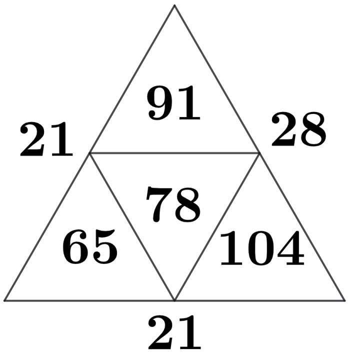
8 位朋友均匀地坐在一张圆桌。如果 Anna 和 Beth 中间有 1 位朋友，而 Beth 和 Carmen 中间有 2 位朋友，那么 Anna、Carmen 和 Beth 之间形成的夹角是多少度?



Question 3 (3 points)

A net of a regular tetrahedron is shown with numbers on each face, including the values 65, 78, 91, and 104. When folded into a tetrahedron, three faces meet at each vertex and each vertex represents a number, including 7 (which is not shown), 14, 21, and 28. What is the largest possible product between a vertex number and its opposite face number?

下图可以沿所示线折叠成一个正四面体。每一面上都有一个数字，分别为 65、78、91 和 104。三个数字面在四面体的每个角上相交，而每个角也代表一个数字，分别为 7、14、21 和 28。那么角上的数字与彼此相对的面的最大乘积是多少?



Question 4 (3 points)

A three-digit number and a two-digit number are selected uniformly at random. What is the probability that they are congruent modulo 100? (i.e., the last two digits of both numbers match)

随机选取 1 个两位数和 1 个三位数，它们模 100 时同余的概率是多少?

Question 5 (3 points)

How many subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ contain at least one odd number?

给定集合 $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$，请问它有多少个包含至少一个奇数的子集?

Question 6 (3 points)

There are 9 infinite lines in a plane, all pairwise intersecting. Among their intersections, there are 2 points where exactly 3 lines meet, and 3 points where exactly 4 lines meet. Assuming no other intersection points with more than 2 lines, how many total points of intersection exist in the plane?

平面上有 9 条直线，它们都与彼此相交，其中 3 条直线经过的交点有 2 个，4 条直线经过的交点有 3 个。请问平面上共有多少个交点?

Question 7 (3 points)

Joe the farmer is raising sheep on his farm. He estimates that with 40 sheep, the grass will be eaten completely after 30 days. If he buys 10 fewer sheep (30 sheep), the grass will last 50 days. Assuming the grass grows at a constant rate, how many days will the grass last if he purchases 45 sheep?

农民 Joe 计划在农场养羊。他估计，如果他购买 40 只羊，农场里的草 30 天后就会被吃光。如果他少买 10 只羊，草还能再维持 20 天（共维持 50 天）。假设草以恒定的速度生长，如果他最终决定购买 45 只羊，草能维持多少天?

Question 8 (3 points)

What is the largest 3-digit integer k such that the sum of all its factors less than 10 equals 9?

3 位数的整数 k 满足以下条件：其所有小于 10（不包括 10）的因子之和是 9. 那么 k 的最大值是多少?

Question 9 (5 points)

Given that $f(x)$ is an odd function and $g(x)$ is an even function, satisfying $-5 \leq f(x) - g(x) \leq 2$ for all $x \in \mathbb{R}$. Determine the minimum value of $f(2025x) + g(2025x)$.

已知 $f(x)$ 是一个奇函数，$g(x)$ 是一个偶函数。不等式 $-5 \leq f(x) - g(x) \leq 2$ 对于所有实数 x 成立。请求出 $f(2025x) + g(2025x)$ 的最小值。

Question 10 (5 points)

a, b, c, d are positive integers that satisfy $a^2 = b^3$, $c^3 = d^4$, and $a = d + 19$. Find a.

a, b, c, d 都是正整数。如果有 $a^2 = b^3$, $c^3 = d^4$, $a = d + 19$, 求 a 的值。

Question 11 (5 points)

The sides of $\triangle ABC$ satisfy $AB : BC : CA = 9 : 11 : 10$. The angle bisector of $\angle BAC$ meets BC at D, and M is the midpoint of AC. The lines AD and BM intersect at point O. Given that $DO = 18$, find the length of AO.

如图所示，$AB : BC : CA = 9 : 11 : 10$，$AD$ 平分 $\angle BAC$，M 是 AC 的中点，AD 与 BM 交于点 O。如果 $DO = 18$，求 AO 的长度。

Question 12 (5 points)

Consider the sequence defined by $a_n = 4a_{n-1} - 3a_{n-2}$. Find $a_{2025} \bmod 100$ for $a_0 = 2$ and $a_1 = 5$.

有一个数列满足 $a_n = 4a_{n-1} - 3a_{n-2}$。已知 $a_0 = 2$，$a_1 = 5$，求 $a_{2025} \bmod 100$ 的值。

Question 13 (5 points)

Compute

$$\frac{\sum_{p=1}^{2024} p^3}{(\sum_{q=1}^{9} q)^2 \cdot \sum_{r=1}^{2024} r}$$

计算下列式子：

$$\frac{\sum_{p=1}^{2024} p^3}{(\sum_{q=1}^{9} q)^2 \cdot \sum_{r=1}^{2024} r}$$

Question 14 (5 points)

An urn contains 2 red, 5 blue, and 6 green marbles. If 5 marbles are drawn at random, the probability of getting marbles of exactly two colors is $\frac{m}{n}$, where m and n are coprime positive integers. Find $m+n$.

在一个罐子里，有若干个弹珠。有 2 个红色的，5 个蓝色的和 6 个绿色的弹珠。从其中取出五个，如果这五个弹珠中只有两种颜色的概率为 $\frac{m}{n}$，且 m 与 n 为互素的正整数。请求出 $m + n$ 的值。

Question 15 (5 points)

There is a $\frac{91}{100}$ probability of seeing a shooting star in the next hour. What's the probability that Eli sees a shooting star in the next half hour?

已知再接写来的一个小时里看到流星的概率是 $\frac{91}{100}$，那么 Eli 在接下来的半个小时里看到流星的概率是多少?

Question 16 (5 points)

Define $f(x) = \tan(x)\tan(2x)\tan(3x)$. What is the value of

$$\sum_{i=1}^{15} f(\frac{2i\pi}{15} + \frac{\pi}{15})?$$

有函数满足 $f(x) = \tan(x)\tan(2x)\tan(3x)$，求

$$\sum_{i=1}^{15} f(\frac{2i\pi}{15} + \frac{\pi}{15})$$

的值。

Question 17 (7 points)

Amy, Bob, Cathy, Danny and Eddie are preparing for a football match. Starting from Amy, each student must pass the ball to one of the other four students with equal probability. How many valid ball-passes are there if the ball returns to Amy right after 5 passes?

Amy，Bob，Cathy，Danny 和 Eddie 正在准备一场足球比赛。从 Amy 开始，每个学生必须以相同的概率把球传给其他四个学生中的一个。如果在 5 传球后球返回给 Amy，有多少次可能的有效传球?

Question 18 (7 points)

Find the largest possible area of an equilateral triangle circumscribing $\triangle XYZ$ with $XY = 3$, $YZ = 4$, and $ZX = 5$.

三角形 XYZ 满足 XY=3，YZ=4，ZX=5，求出三角形 XYZ 的最大外接等边三角形的面积。

Question 19 (7 points)

Fill a 3×3 grid with 9 non-negative integers (not necessarily different) such that the sum of each row and each column equals 9. How many different ways can this be done?

填充一个 3×3 的网格，使其包含 9 个非负整数（不一定不同），使得每一行与每一列的总和均为 9。则一共有多少种不同的填充方式?

Question 20 (7 points)

Alethea holds two identical eggs in front of a one-hundred story building on Mercury, hideously wondering, from which minimum floor should these eggs be thrown so as to ensure that they crack? Given that the two identical eggs crack at an identical minimum floor number i, (if you're at a floor higher than i, the egg also cracks), what is the minimum number of floors Alethea has to travel to to guarantee that she has determined i? Note that she has two eggs and should use them wisely.

Alethea 手拿两个完全相同的鸡蛋，在水星上一栋一百层高的建筑前思索：为了确保鸡蛋摔碎，这些鸡蛋至少需要从第几层扔下？已知这两枚鸡蛋会在同一个最低楼层数 i 时摔碎（若从高于 i 的楼层扔下，鸡蛋同样会碎）。问：Alethea 至少需要尝试多少次扔鸡蛋，才能保证一定能确定 i 的值？注意，她只有两枚鸡蛋，必须合理利用。

Question 21 (7 points)

Lola is relearning Giselle hops *en pointe* after her ACL surgery. She starts at the edge of the barre, making a decision at the end of each time interval. She will either make a successful hop forward, stand still and agonize in pain, or fall backwards towards the barre. These cases happen with probabilities 0.2, 0.5, and 0.3, respectively. She will never go towards the center of the barre, so if she's at the barre, she'll have a probability 0.8 of staying still and a probability 0.2 of moving forward. Given that she aspires to be a Lingling and practices this step for an infinite amount of time, what proportion of the time would she be staying still?

Lola 在接受 ACL 手术后重新学习足尖跳跃。她从扶杆的边缘开始练习，并在每个时间间隔结束时做出一个决定。她有 0.2 的概率可以成功向前跳跃，0.5 的概率站在原地忍受疼痛，0.3 的概率向扶杆方向摔倒。如果她已经在扶杆处，她不会退到里面，而是有 0.8 的概率保持原地不动，和 0.2 的概率向前跳跃。Lola 练习这一动作无数次，希望成为 Lingling。请问她停留在原地的时间占所有时间的多少?

Question 22 (7 points)

Consider the polynomial $f_n(z) \in \mathbb{C}[x]$ (the ring of complex polynomials) where $f_n(z) = z^n - (1+z)^n$. What is the greatest imaginary part among the roots of $f_{2025}(z)$?

考虑多项式 $f_n(z) \in \mathbb{C}[x]$（复数多项式环），其中 $f_n(z) = z^n - (1+z)^n$。求 $f_{2025}(z)$ 的根中虚部的最大值。

Question 23 (7 points)

Consider the sum of the reciprocal of the terms in the infinite diagonals of the Pascal Triangle, as indicated below. We know the series $S_0 = \sum 1$ and $S_1 = \sum \frac{1}{n}$ diverge, but fact is, S_i converges for every $i \geq 2$. What is S_{2025}?

考虑帕斯卡三角形每一条对角线上的项的倒数之和。已知 $S_0 = \sum 1$ 和 $S_1 = \sum \frac{1}{n}$ 发散，但是对于所有 $i \geq 2$ 的 i，S_{2025} 收敛。求 S_{2025}。

Question 24 (10 points)

Consider the Fibonacci sequence defined as $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$. What is the smallest possible positive value of n such that

$$\sin\frac{F_n\pi}{14}\cos\frac{F_n\pi}{11} = 1?$$

斐波那契数列的定义是 $F_n = F_{n-1} + F_{n-2}$，$F_0 = 0$，$F_1 = 1$。求使

$$\sin\frac{F_n\pi}{14}\cos\frac{F_n\pi}{11} = 1$$

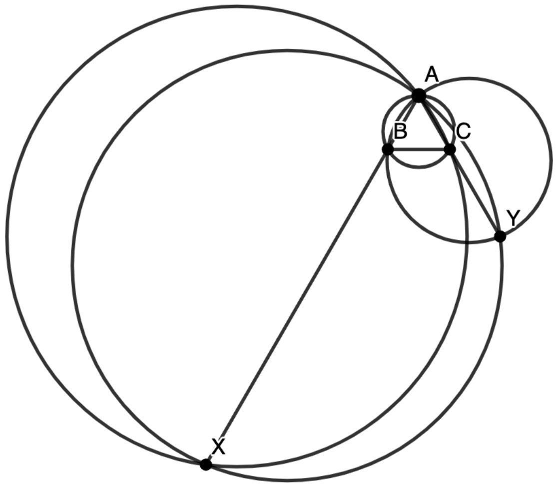
成立的最小正整数 n。

CONTINUED ON THE NEXT PAGE.

Question 25 (10 points)

Let equilateral $\triangle ABC$ be inscribed in circle ω_0. Circle ω_1 centered at O_1 passes through A and B, while circle ω_2 with center O_2 passes through A and C. We have $O_1A \perp O_2A$. Suppose rays AB, AC meet circles ω_2, ω_1 at X, Y, respectively, and let ω_3 be the circle through A, X, and Y centered at O_3. If the angle bisector of $\angle BAC$ also bisects $\alpha = \angle O_3AO_1$, what is $\tan^2\left(\frac{\angle\alpha}{2}\right)$?

正三角形 $\triangle ABC$ 内接于圆 ω_0。圆 ω_1 以 O_1 为圆心，过点 A 和 B；圆 ω_2 以 O_2 为圆心，过点 A 和 C。已知 $O_1A \perp O_2A$。设射线 AB 于圆 ω_2 交于 X，射线 AC 与圆 ω_1 交于 Y，以及 ω_3 是 O_3 的圆心。若 $\angle BAC$ 的角平分线也平分 $\alpha = \angle O_3AO_1$，求 $\tan^2\left(\frac{\angle\alpha}{2}\right)$。



END OF TEST.