Transformation of an analytical solution of the 1D acoustic wave equation for a homogeneous, unbounded medium from $(\omega,\ k)$ to $(t,\ x)$ domain

D. Köhn

April 30, 2023

In this Jupyter Notebook, I derived the Greens function solution of the 1D acoustic wave equation for an unbounded, homogeneous medium in the $(\omega,\ k)$ domain $\hat{G}_1(\omega, k)$:

$$\hat{G}_1(\omega, k) = \frac{1}{4\pi^2 V_{p0}^2}\left[\frac{1}{k^2 - \frac{\omega^2}{V_{p0}^2}}\right] \tag{1}$$

where V_{p0} denotes the constant P-wave velocity, k the angular wavenumber and ω the angular frequency. To derive the solution in the time-space $(t,\ x)$ domain, we have to apply the inverse Fourier transform twice to get from $(\omega,\ k)$ to $(\omega,\ x)$ and finally $(t,\ x)$ domain. The application of the inverse Fourier transform to eq. (1) leads to the following integral:

$$\hat{G}_1(\omega, x) = \frac{1}{4\pi^2 V_{p0}^2}\int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2 - \frac{\omega^2}{V_{p0}^2}} dk \tag{2}$$

Notice, that the integral has two poles at $k = \pm\frac{\omega^2}{V_{p0}^2}$ on the real axis, so its evaluation is not straightforward. However, with some basic knowledge of complex analysis this integral is quite easy to compute and will deliver some interesting connections between mathematics and physics. Therefore, I will shortly review the basics of complex analysis in the following section. If you are interested in a thorough introduction to this topic, I refer to more extensive textbooks e.g.

- Zill & Shanahan: Complex Analysis: A First Course with Applications
- Schaum's Outlines: Complex Variables
- Saff & Snider: Fundamentals of Complex Analysis

1 Basics of Complex Analysis

In this section, I introduce Jordans Lemma, the Residue theorem and show you how to compute residues.

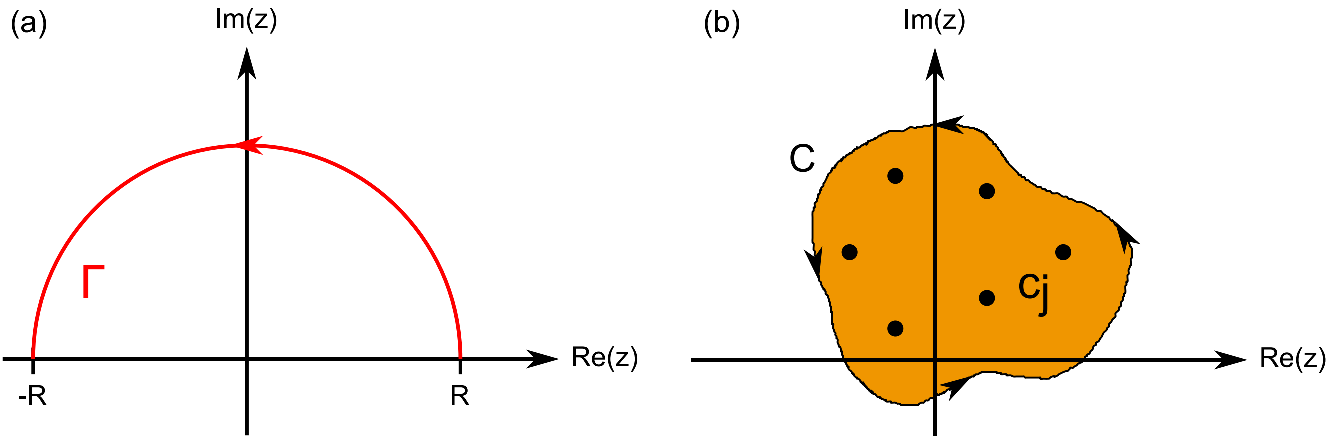

Figure 1: Basics of complex analysis: Jordans Lemma (a) and Residue theorem (b).

1.1 Jordans Lemma

Fig. 1a shows a half-circle contour Γ in the complex plane. The radius of the contour is denoted by R. If we take a complex valued function $f(z)$ with $z \in \mathbb{C}$, which continuously converges to zero in the case $R \to \infty$, the integral over the half-circle contour Γ of $f(z)e^{iaz}$ becomes zero:

$$\int_{\Gamma} f(z)e^{iaz}dz = 0, \tag{3}$$

if $a > 0$.

1.2 Residue Theorem

Fig. 1b shows a closed contour C enclosing an area in the complex plane containing some isolated singularities c_j. Then the integral over the contour C of a complex valued function $f(z)$ is equal to

$$\oint_C f(z)dz = 2\pi i \sum_j Res(f(z), c_j) \tag{4}$$

where $Res(f(z), c_j)$ denotes the residues of the singularities contained in the area enclosed by the contour C. So, in order to evaluate the integral $\oint_C f(z)dz$, we only have to compute the sum of the residues of the singularities.

1.3 Computation of residues for simple poles

The poles in eq. (2) are poles of order one, also denoted as "simple". We can compute their residues via

$$Res(f(z), c_j) = \lim_{z \to c_j} (z - c_j)f(z) \tag{5}$$

Effectively, we are removing the pole from f(z) and compute the limit $z \to c_j$.

2 Inverse Fourier transform from $(\omega,\ k)$ to $(\omega,\ x)$ domain

With this basic complex analysis knowledge, we are able to tackle integral (2). First, I factorize the denominator of the integral to simplify the computation of the residues

$$\hat{G}_1(\omega, x) = \frac{1}{4\pi^2 V_{p0}^2} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2 - \frac{\omega^2}{V_{p0}^2}} dk = \frac{1}{4\pi^2 V_{p0}^2} \int_{-\infty}^{\infty} \frac{e^{ikx}}{(k - \frac{\omega}{V_{p0}})(k + \frac{\omega}{V_{p0}})} dk \tag{6}$$

where I used the simple relation $(a^2 - b^2) = (a - b)(a + b)$. Next, we define a closed contour C in the complex plane (fig. 2), starting on the real axis at -R, add a half-circle contour below the pole at $k = -\omega/V_{p0}$ denoted by c_1, integrate further along the real axis, add a half-circle contour above the pole at $k = \omega/V_{p0}$ denoted by c_2, extend the contour further to R and close the contour over the upper half-space of the complex plane using the half-circle contour Γ in counter-clockwise direction. Notice, that we only included the pole at $k = -\omega/V_{p0}$ in the area enclosed by the contour, I will explain this decision later.

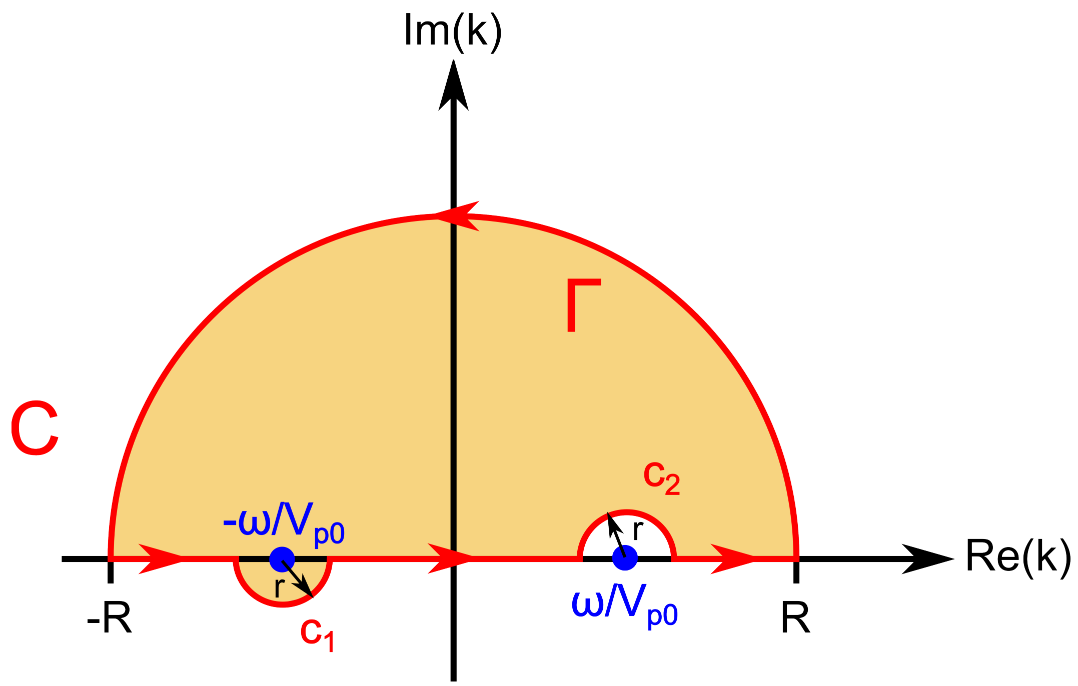

Figure 2: Closed contour C in the complex k-plane used for the solution of integral (6) with the Residue theorem.

For $R \rightarrow \infty$

$$\frac{1}{(k - \frac{\omega}{V_{p0}})(k + \frac{\omega}{V_{p0}})} \rightarrow 0$$

converges to zero. Therefore, the integral along the contour Γ of

$$\frac{1}{4\pi^2 V_{p0}^2} \int_{\Gamma} \frac{e^{ikx}}{(k - \frac{\omega}{V_{p0}})(k + \frac{\omega}{V_{p0}})} dk = 0$$

will become zero according to Jordans Lemma eq. (3). However, this is only true for $a > 0$ in e^{iaz}. In our case $a = x$, so in order to use Jordans Lemma, we have to put a constraint on x by replacing x with $|x|$. The physical consequence of this assumption will be, that we excite two waves in the time-space-domain. One traveling to the left, the other to the right from the source position. This is a really interesting connection between Jordans Lemma and the physics of wave propagation. The integrals along the half-circle contours c_1 and c_2 also become zero, if their radii $r \to 0$. So all half-circle contributions of the closed contour integral vanish, except the integration along the real axis which we need to evaluate our original integral eq. (6). In the next step, we will use the residue theorem to evaluate the integral. Remember, only the pole at $k = -\omega/V_{p0}$ is contained in the area enclosed by the closed contour C, so only the residue of this pole has to be evaluated. Defining the function $f(k)$ as

$$f(k) = \frac{e^{ik|x|}}{(k - \frac{\omega}{V_{p0}})(k + \frac{\omega}{V_{p0}})},$$

we can compute the residue of the pole at $k = -\omega/V_{p0}$ according to section 1.3

$$\begin{aligned}
Res(f(k), -\frac{\omega}{V_{p0}}) &= \lim_{k \to -\frac{\omega}{V_{p0}}} (k + \frac{\omega}{V_{p0}}) \frac{e^{ik|x|}}{(k - \frac{\omega}{V_{p0}})(k + \frac{\omega}{V_{p0}})} \\
&= \lim_{k \to -\frac{\omega}{V_{p0}}} \frac{e^{ik|x|}}{(k - \frac{\omega}{V_{p0}})} \\
&= \frac{e^{-i|x|\frac{\omega}{V_{p0}}}}{(-\frac{\omega}{V_{p0}} - \frac{\omega}{V_{p0}})} \\
&= \frac{e^{-i|x|\frac{\omega}{V_{p0}}}}{(-2\frac{\omega}{V_{p0}})}
\end{aligned}$$

As you can see, we first removed the pole from $f(k)$ and then computed the limit $k \to -\frac{\omega}{V_{p0}}$. Using the Residue theorem eq.(4), we are finally able to evaluate integral eq.(6)

$$\begin{aligned}
\hat{G}_1(\omega, x) &= \frac{1}{4\pi^2 V_{p0}^2} \oint_C \frac{e^{ikx}}{(k - \frac{\omega}{V_{p0}})(k + \frac{\omega}{V_{p0}})} dk \\
&= \frac{1}{4\pi^2 V_{p0}^2} \int_{-\infty}^{\infty} \frac{e^{ikx}}{(k - \frac{\omega}{V_{p0}})(k + \frac{\omega}{V_{p0}})} dk \\
&= \frac{1}{4\pi^2 V_{p0}^2} \left[2\pi i Res(f(k), -\frac{\omega}{V_{p0}}) \right] \\
&= \frac{1}{4\pi^2 V_{p0}^2} \left[2\pi i \frac{e^{-i|x|\frac{\omega}{V_{p0}}}}{(-2\frac{\omega}{V_{p0}})} \right]
\end{aligned}$$

After applying some basic simplifications

$$\begin{aligned}
\hat{G}_1(\omega, x) &= \frac{1}{4\pi^2 V_{p0}^2}\left[2\pi i \frac{e^{-i|x|\frac{\omega}{V_{p0}}}}{-2\frac{\omega}{V_{p0}}}\right] \\
&= \frac{1}{4\pi V_{p0}^2}\left[i \frac{e^{-i|x|\frac{\omega}{V_{p0}}}}{-\frac{\omega}{V_{p0}}}\right] \\
&= \frac{1}{4\pi V_{p0}^2}\left[i \frac{e^{-i|x|\frac{\omega}{V_{p0}}}}{i^2\frac{\omega}{V_{p0}}}\right] \\
&= \frac{1}{4\pi V_{p0}}\left[i \frac{e^{-i|x|\frac{\omega}{V_{p0}}}}{i^2\omega}\right] \\
&= \frac{1}{4\pi i V_{p0}}\left[\frac{e^{-i|x|\frac{\omega}{V_{p0}}}}{\omega}\right]
\end{aligned}$$

we get the solution in the $(\omega,\ x)$ domain

$$\hat{G}_1(\omega, x) = \frac{1}{4\pi i V_{p0}}\left[\frac{e^{-i|x|\frac{\omega}{V_{p0}}}}{\omega}\right] \tag{7}$$

3 Inverse Fourier transform from $(\omega,\ x)$ to $(t,\ x)$ domain

After computing the analytical solution in the $(\omega,\ x)$ domain, we can apply an inverse Fourier transform with respect to ω to get the solution in the $(t,\ x)$ domain

$$G_1(t, x) = \frac{1}{4\pi i V_{p0}} \int_{-\infty}^{\infty} \frac{e^{-i|x|\frac{\omega}{V_{p0}}}}{\omega} e^{i\omega t} d\omega$$

Combining the exponents of the e-functions leads to

$$G_1(t, x) = \frac{1}{4\pi i V_{p0}} \int_{-\infty}^{\infty} \frac{e^{i\omega(t-\frac{|x|}{V_{p0}})}}{\omega} d\omega \tag{8}$$

Lets analyze this problem more closely. Compared to the previous problem, we have only one pole at $\omega = 0$. To apply Jordans Lemma, we have to analyze the term $e^{ia\omega}$. In this case $a = (t - \frac{|x|}{V_{p0}})$. What does a actually mean? The term $\frac{|x|}{V_{p0}}$ denotes the travel time of the seismic wave from the source position to the receiver at location $|x|$, while t simply denotes the time. Because the time difference $a = (t - \frac{|x|}{V_{p0}})$ can be either positive or negative, we have to distinguish two cases in order to apply Jordans Lemma and the Residue theorem correctly.

3.1 Case 1: $t < \frac{|x|}{V_{p0}}$

In the case $t < \frac{|x|}{V_{p0}}$, the current time t is smaller than the travel time of the seismic wave from the source to the receiver. Therefore, the seismic wave has not reached the detector yet and we should not record any signal. To fulfill this causality condition, we have to choose a specific closed contour line in the complex ω-plane (fig. 3a). We start on the real axis at the point $\omega = -R$, then move in positive direction along the real axis, exclude the singularity at $\omega = 0$, by using a

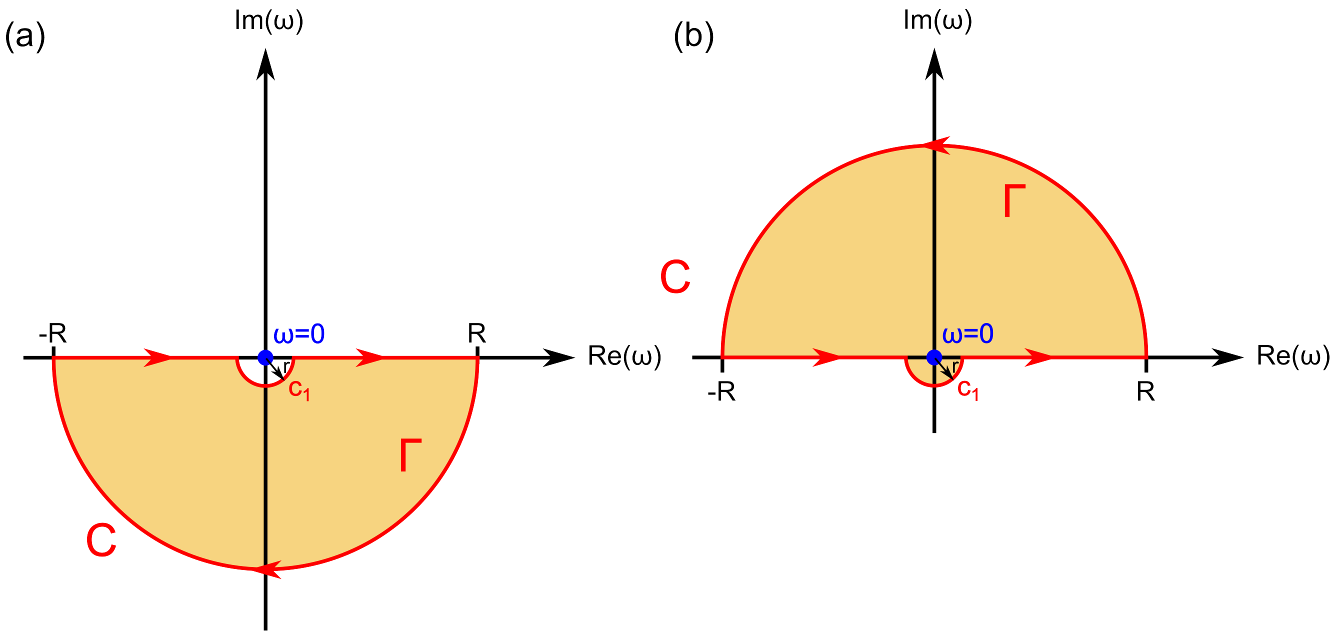

Figure 3: Closed contour C in the complex ω-plane used for the solution of integral (8) with the Residue theorem for the case $t < \frac{|x|}{V_{p0}}$ (a) and $t \geq \frac{|x|}{V_{p0}}$ (b).

half-circle contour below the pole and move further to $\omega = R$. Because we assumed $t < \frac{|x|}{V_{p0}}$ we get $a = (t - \frac{|x|}{V_{p0}}) < 0$. So in order to use Jordans Lemma, we have to close the contour along a half-circle in the negative half-space of the complex plane in clock-wise direction. As in the previous integral, the contribution of the half-circle contour Γ is zero if $R \to \infty$. The same happens for the half-circle contour c_1 if $r \to 0$. Therefore, the remaining contribution is the integral along the real axis from $\omega = -\infty$ to ∞, which is equal to integral (8), that we want to solve. Application of the residue theorem in this case is really easy, because we excluded the only singularity at $\omega = 0$ from the area enclosed by the closed contour C. So the residues are zero leading to

$$
\begin{aligned}
G_1(t,x) &= \frac{1}{4\pi i V_{p0}} \oint_C \frac{e^{i\omega(t-\frac{|x|}{V_{p0}})}}{\omega} d\omega \\
&= \frac{1}{4\pi i V_{p0}} \int_{-\infty}^{\infty} \frac{e^{i\omega(t-\frac{|x|}{V_{p0}})}}{\omega} d\omega \\
&= \frac{1}{4\pi i V_{p0}} \left[2\pi i 0 \right] = 0,
\end{aligned}
$$

which is exactly the solution we wanted to achieve: before the wave arrives at the receiver ($t < \frac{|x|}{V_{p0}}$), we should detect no signal at all.

3.2 Case 2: $t \geq \frac{|x|}{V_{p0}}$

The case $t \geq \frac{|x|}{V_{p0}}$ is a little bit more interesting. The current time t is larger than the runtime of the seismic wave from the source to the receiver. Therefore, the seismic wave has arrived at the detector and we should record the signal excited at the source position. Therefore, we have to choose a closed contour line in the complex ω-plane, that incorporates the pole at $\omega = 0$ (fig.

3b). We again start on the real axis at the point $\omega = -R$, then move in positive direction along the real axis, include the singularity at $\omega = 0$, by using a half-circle contour below the pole and move further to $\omega = R$. Because we assumed $t \geq \frac{|x|}{V_{p0}}$ we get $a = (t - \frac{|x|}{V_{p0}}) \geq 0$. So in order to use Jordans Lemma, we have to close the contour along a half-circle in the positive half-space of the complex plane. As in the previous integral, the contribution of the half-circle contour Γ is zero if $R \to \infty$. The same happens for the half-circle contour c_1 if $r \to 0$. Therefore, the remaining contribution is the integral along the real axis from $\omega = -\infty$ to ∞, which is equal to integral (8), that we want to solve. To apply the Residue theorem, we have to compute the residue at the pole $\omega = 0$. Defining

$$f(\omega) = \frac{e^{i\omega(t - \frac{|x|}{V_{p0}})}}{\omega},$$

we get

$$Res(f(\omega), 0) = \lim_{\omega \to 0} \omega \frac{e^{i\omega(t - \frac{|x|}{V_{p0}})}}{\omega} = \lim_{\omega \to 0} e^{i\omega(t - \frac{|x|}{V_{p0}})} = 1.$$

With the residue of the pole at $\omega = 0$, we can finally evaluate our integral (8) using the Residue theorem:

$$\begin{aligned} G_1(t, x) &= \frac{1}{4\pi i V_{p0}} \oint_C \frac{e^{i\omega(t - \frac{|x|}{V_{p0}})}}{\omega} d\omega \\ &= \frac{1}{4\pi i V_{p0}} \int_{-\infty}^{\infty} \frac{e^{i\omega(t - \frac{|x|}{V_{p0}})}}{\omega} d\omega \\ &= \frac{1}{4\pi i V_{p0}} \Big[2\pi i Res(f(\omega), 0) \Big] \\ &= \frac{1}{4\pi i V_{p0}} \Big[2\pi i 1 \Big] \\ &= \frac{1}{2V_{p0}} \end{aligned}$$

So, after the wave has arrived at the receiver ($t \geq \frac{|x|}{V_{p0}}$), we should detect a constant signal with an amplitude of $\frac{1}{2V_{p0}}$.

3.3 Final analytical solution in the $(t,\ x)$ domain

When combining the two solutions for the cases $t < \frac{|x|}{V_{p0}}$ and $t \geq \frac{|x|}{V_{p0}}$, we finally get

$$G_1(t, x) = \frac{1}{2V_{p0}} H\left(t - \frac{|x|}{V_{p0}}\right), \tag{9}$$

where H denotes the Heaviside function:

$$H\left(t - \frac{|x|}{V_{p0}}\right) = \begin{cases} 0 & t < \frac{|x|}{V_{p0}} \\ 1 & t \geq \frac{|x|}{V_{p0}} \end{cases}$$

The recorded amplitude is zero before the acoustic wave arrives at the receiver and jumps to an amplitude of $1/(2V_{p0})$ as soon as the wave arrives at the receiver (fig. 4). This behavior of a constant amplitude jump is not what we observe in field data applications in our universe with 3 spatial dimensions. We will discuss this shortcoming of the 1D approximation and the resulting impact on the physics later.

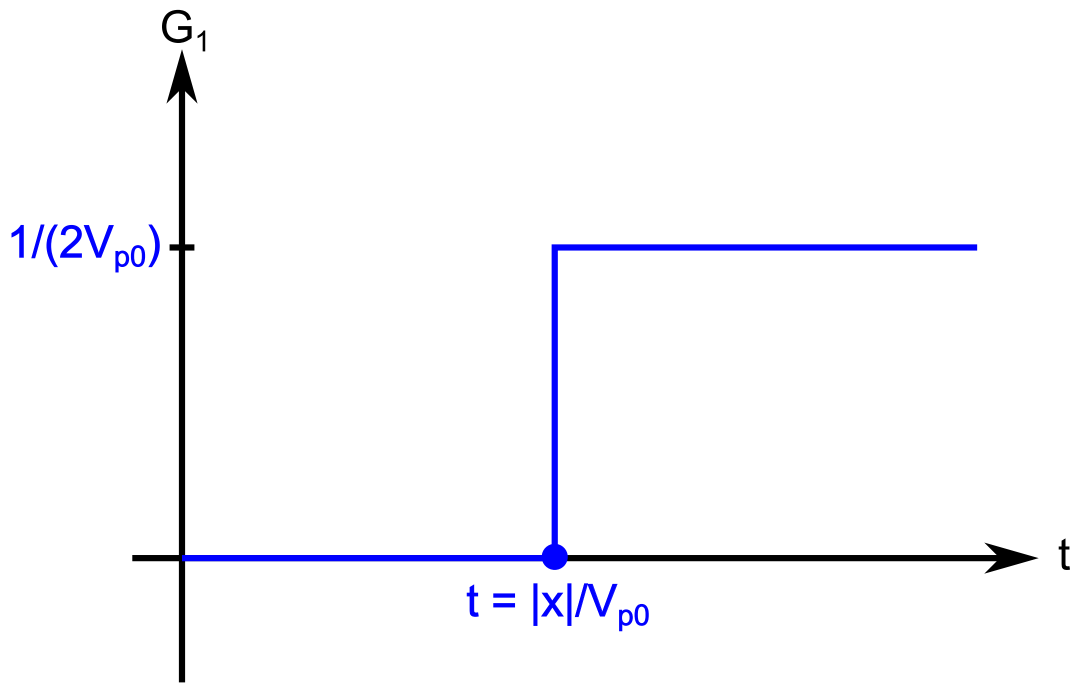

Figure 4: Final analytical solution of the 1D acoustic wave equation for an unbounded, homogeneous medium.

4 Conclusions

I hope you enjoyed this derivation of the analytical solution for the 1D acoustic wave equation. The most fascinating aspect for me is the influence of Jordans Lemma on the physics of seismic wave propagation, especially the simple condition $a > 0$. At first glance, the poles pose a significant problem to solve the integrals required for the inverse Fourier transform. However, we have seen that the incorporation or neglection of the poles in the evaluation of the integrals using the Residue theorem was essential to finally achieve a casual solution in the $(t,\ x)$ domain. You might wonder why I excluded the pole at $k = \omega/V_{p0}$ during the inverse Fourier transform from the $(\omega,\ k)$ to the $(\omega,\ x)$ domain (fig. 2). Well, if you include this pole, you get another acoustic wave - traveling backwards in time and arriving at the receiver position at $t = -|x|/V_{p0}$. A perfect example of symmetry in physics, the restriction by Jordans Lemma to $|x|$ lead to two acoustic waves traveling symmetrically from the source position in positive and negative spatial x-direction. The inclusion of both poles at $k = \pm\omega/V_{p0}$ leads to two acoustic waves, one traveling in positive, the other in negative time direction. Therefore, the analysis of poles in the complex plane using the Residue theorem and the application of Jordans Lemma have a significant impact on the physics of acoustic wave propagation.