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Charge-Density-Wave Formation in the Doped Two-Leg Extended Hubbard Ladder

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We investigate the electronic properties of a doped two-leg Hubbard ladder with both the onsite and nearest-neighbor Coulomb repulsions by using the weak-coupling renormalization-group method. It is shown that, for strong nearest-neighbor repulsions, the charge-density-wave state coexisting with the p-density-wave state becomes a dominant fluctuation where spins form intrachain singlets. By increasing doping rate, we have also shown that the effects of the nearest-neighbor repulsions are reduced and the system exhibits a quantum phase transition into the d-wave-like (or rung-singlet) superconducting state. We derive the effective fermion theory which describes the critical properties of the transition point with the gapless excitation of magnon. The phase diagram of the two-leg ladder compound, $Sr_{14-x}Ca_x$-$Cu_{24}O_{41}$, is discussed.

KEYWORDS: doped Hubbard ladder, intersite Coulomb repulsion, spin gap, charge-density wave, superconductivity
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Electronic properties in ladder systems have been studied intensively both theoretically and experimentally, since the superconducting (SC) state was discovered in the self-doped two-leg ladder material $Sr_{14-x}Ca_xCu_{24}O_{41}$ with $x \gtrsim 12$ under a pressure of more than 3 GPa.[1,2] The substitution of Ca for Sr changes effectively the hole-doping rate in ladder Cu sites, where the rate varies monotonically from 0.07 to 0.25 with increasing x from 0 to 12.[3] A characteristic feature is the presence of a gap in magnetic excitations at a temperature much higher than the SC transition temperature.[4,5] In addition to the SC state, recent experimental studies have focused on the charge dynamics in slightly doped materials and verified collective modes from the sliding of the charge-density wave (CDW) that developed on ladder sites.[6–9] A global phase diagram on the plane of x and temperature shows that hole doping suppresses the CDW state followed by the insulating state without the CDW order, and that high doping leads to the SC state under pressure.[9] Quite recently, the CDW collective modes are also suggested in the highly doped material $Sr_2Ca_{12}Cu_{24}$-O_{41}.[10] Therefore, it is of particular interest to investigate the competition between the SC state and the CDW state in doped ladder systems.

From a theoretical point of view, the origin of the spin gap in ladder compounds seems to be explained successfully for both the undoped[11,12] and doped[12] cases. In doped ladder systems, the d-wave-like SC (SCd) state appears[12] and, in addition, the charge-ordered state emerges when intersite interactions are included.[13] Furthermore, the competition between the SCd state and the charge-ordered or CDW state has been examined.[12,13,15] However, this critical behavior is not yet fully understood. In the present paper, the possible scenario of the instability of the CDW state and the competition between the CDW state and the SC state are proposed in a doped two-leg ladder of an extended Hubbard model (EHM) with nearest-neighbor repulsive interactions. The critical behavior is analyzed in more detail by extending the previous analytical calculations.[14,15]

We consider the two-leg EHM given by $H = H_0 + H_{\rm int}$. The first term describes the hopping energies along and between legs:

$$H_0 = -t_\parallel \sum_{j,\sigma,l} \left(c^\dagger_{j,l,\sigma} c_{j+1,l,\sigma} + \text{H.c.} \right) - t_\perp \sum_{j,\sigma} \left(c^\dagger_{j,1,\sigma} c_{j,2,\sigma} + \text{H.c.} \right), \tag{1}$$

where $c_{j,l,\sigma}$ annihilates an electron of spin σ $(=\uparrow,\downarrow)$ on rung j and leg l $(=1,2)$. The Hamiltonian $H_{\rm int}$ denotes interactions between electrons:

$$H_{\rm int} = U \sum_{j,l} n_{j,l,\uparrow} n_{j,l,\downarrow} + V_\parallel \sum_{j,l} n_{j,l} n_{j+1,l} + V_\perp \sum_j n_{j,1} n_{j,2}, \tag{2}$$

where $n_{j,l,\sigma} = c^\dagger_{j,l,\sigma} c_{j,l,\sigma}$ and $n_{j,l} = n_{j,l,\uparrow} + n_{j,l,\downarrow}$. U represents an on-site repulsion and $V_\parallel$ ($V_\perp$) represents an intrachain (interchain) nearest-neighbor repulsion. The H_0 term is diagonalized by using the Fourier transform of $c_\sigma(\boldsymbol{k})$, where $\boldsymbol{k} = (k_\parallel, k_\perp)$ with $k_\perp = 0$ or π. The energy dispersion is given by $\varepsilon(\boldsymbol{k}) = -2t_\parallel \cos k_\parallel - t_\perp \cos k_\perp$. Here, we consider the case with the finite hole doping δ satisfying $t_\perp < 2t_\parallel \cos^2 \frac{\pi}{2}\delta$, in which both the bonding ($k_\perp = 0$) and antibonding ($k_\perp = \pi$) energy bands are partially filled and Fermi points are located at $k_{\rm F,0} = \frac{\pi}{2}(1-\delta) + \lambda$ and $k_{\rm F,\pi} = \frac{\pi}{2}(1-\delta) - \lambda$ with $\lambda \equiv \sin^{-1}[t_\perp/(2t_\parallel \cos\frac{\pi}{2}\delta)]$. We examine the case of the small δ by neglecting the differences in the Fermi velocities of the bonding and antibonding bands, i.e., $v_{\rm F,0} = v_{\rm F,\pi}$ $(\equiv v_{\rm F})$.

Following the standard weak-coupling approach (g-ology), the linearized kinetic energy is given by $H_0 = \sum_{\boldsymbol{k},p,\sigma} v_{\rm F}(pk_\parallel - k_{F,k_\perp}) c^\dagger_{p,\sigma}(\boldsymbol{k}) c_{p,\sigma}(\boldsymbol{k})$, where the index $p = +/-$ denotes the right-/left-moving electron. By introducing field operators $\psi_{p,\sigma,\zeta}(x) = L^{-1/2} \sum_{k_\parallel} {\rm e}^{{\rm i}k_\parallel x} c_{p,\sigma}(k_\parallel, k_\perp)$ where $\zeta = +(-)$ for $k_\perp = 0(\pi)$ and L is the system size, the interactions near the Fermi points are rewritten as $H_{\rm int} = (1/4)\int {\rm d}x \sum_{p,\sigma} \sum'_{\zeta_i=\pm} \mathscr{H}_{\rm int}$, where $\mathscr{H}_{\rm int}$ is given by

$$g^{\epsilon\bar\epsilon}_{1(2)\parallel} \psi^\dagger_{p,\sigma,\zeta_1} \psi^\dagger_{-p,\sigma,\zeta_2} \psi_{+(-)p,\sigma,\zeta_4} \psi_{-(+)p,\sigma,\zeta_3} + g^{\epsilon\bar\epsilon}_{1(2)\perp} \psi^\dagger_{p,\sigma,\zeta_1} \psi^\dagger_{-p,\bar\sigma,\zeta_2} \psi_{+(-)p,\bar\sigma,\zeta_4} \psi_{-(+)p,\sigma,\zeta_3}, \tag{3}$$

and $\bar\sigma = \uparrow(\downarrow)$ for $\sigma = \downarrow(\uparrow)$, $\epsilon = \zeta_1\zeta_3$ and $\bar\epsilon = \zeta_1\zeta_2$. The summation of the band index ζ_i $(i = 1,\ldots,4)$ is taken under the condition $\zeta_1\zeta_2\zeta_3\zeta_4 = +1$. The coupling constants $g_{i\parallel}^{\epsilon\bar{\epsilon}}$ and $g_{i\perp}^{\epsilon\bar{\epsilon}}$ with $i = 1(2)$ corresponding to backward (forward) scattering are given by $g_{i\parallel}^{\epsilon\bar{\epsilon}} = (l_\epsilon V_\perp + m_{i,\epsilon} V_\parallel)$ and $g_{i\perp}^{\epsilon\bar{\epsilon}} = (U + l_\epsilon V_\perp + m_{i,\epsilon} V_\parallel)$, where $l_\pm = \pm 1$, $m_{1,+} = -2\cos\pi\delta \cos 2\lambda$, $m_{1,-} = -2\cos\pi\delta$, $m_{2,+} = +2$, and $m_{2,-} = +2\cos 2\lambda$. We neglect umklapp scattering processes at finite doping and also neglect forward scattering processes within the same branch p.[16)]

As possible states, we consider the SCd state, the CDW state, and the p-density-wave (PDW) state.[14)] The PDW state corresponds to the spin–Peierls state in the limit of $\delta \to 0$. The order parameter of the SCd state is $O_{\mathrm{SC}d} = N^{-1}\sum_j (c_{j,1,\uparrow}c_{j,2,\downarrow} - c_{j,1,\downarrow}c_{j,2,\uparrow})$, while those of the density waves are $O_A = N^{-1}\sum_{\boldsymbol{k},\sigma} f_A(\boldsymbol{k}) c_\sigma^\dagger(\boldsymbol{k}) c_\sigma(\boldsymbol{k}+\boldsymbol{Q})$, with $\boldsymbol{Q} = (\pi(1-\delta),\pi)$, $f_{\mathrm{CDW}} = 1$ and $f_{\mathrm{PDW}} = \sin k_\parallel$. These operators are rewritten in terms of bosonic phase fields by applying the Abelian bosonization method.[16,17)] The field operators of the right- and left-moving electrons are written as $\psi_{p,\sigma,\zeta}(x) = \eta_{\sigma,\zeta}(2\pi a)^{-1/2}\exp[\mathrm{i}pk_{\mathrm{F},k_\perp}x + \mathrm{i}p\varphi_{p,s,\zeta}(x)]$, where $s = +$ for $\sigma = \uparrow$ and $s = -$ for $\sigma = \downarrow$. These fields satisfy the commutation relations: $[\varphi_{p,s,\zeta}(x), \varphi_{p,s',\zeta'}(x')] = \mathrm{i}p\pi\,\mathrm{sgn}(x-x')\delta_{s,s'}\delta_{\zeta,\zeta'}$ and $[\varphi_{+,s,\zeta}, \varphi_{-,s',\zeta'}] = \mathrm{i}\pi\delta_{s,s'}\delta_{\zeta,\zeta'}$. The Klein factors $\eta_{\sigma,\zeta}$ are introduced in order to retain correct anticommutation relations.[14)] For calculating physical quantities, the field $\varphi_{p,s,\zeta}$ is replaced by new bosonic fields: $\phi_{\nu r} = (\phi_{\nu r}^+ + \phi_{\nu r}^-)$ and $\theta_{\nu r} = (\phi_{\nu r}^+ - \phi_{\nu r}^-)$, where $\varphi_{p,s,\zeta} = (\phi_{\rho+}^p + \zeta\phi_{\rho-}^p + s\phi_{\sigma+}^p + s\zeta\phi_{\sigma-}^p)$ with $p = \pm$, $s = \pm$, and $\zeta = \pm$. The phase fields $\phi_{\rho\pm}$ and $\phi_{\sigma\pm}$ represent charge and spin fluctuations, respectively, and the suffices $\pm$ refer to the even and odd sectors. They satisfy $[\phi_{\nu r}(x), \theta_{\nu' r'}(x')] = -\mathrm{i}\pi\Theta(-x+x')\delta_{r,r'}$ with $\Theta(x)$ being the Heaviside step function. In terms of $\phi_{\nu r}$ and $\theta_{\nu r}$, the order parameters $O = \int \mathrm{d}x\mathcal{O}$ are given by

$$\begin{aligned}\mathcal{O}_{\mathrm{SC}d} &\propto \mathrm{e}^{\mathrm{i}\theta_{\rho+}}\cos\theta_{\rho-}\cos\phi_{\sigma+}\cos\phi_{\sigma-} \\ &\quad - \mathrm{i}\mathrm{e}^{\mathrm{i}\theta_{\rho+}}\sin\theta_{\rho-}\sin\phi_{\sigma+}\sin\phi_{\sigma-}, \end{aligned}\tag{4a}$$

$$\begin{aligned}\mathcal{O}_{\mathrm{CDW}} &\propto \cos\phi_{\rho+}\sin\theta_{\rho-}\cos\phi_{\sigma+}\cos\theta_{\sigma-}, \\ &\quad - \sin\phi_{\rho+}\cos\theta_{\rho-}\sin\phi_{\sigma+}\sin\theta_{\sigma-}, \end{aligned}\tag{4b}$$

$$\begin{aligned}\mathcal{O}_{\mathrm{PDW}} &\propto \cos\phi_{\rho+}\cos\theta_{\rho-}\sin\phi_{\sigma+}\sin\theta_{\sigma-}, \\ &\quad + \sin\phi_{\rho+}\sin\theta_{\rho-}\cos\phi_{\sigma+}\cos\theta_{\sigma-}. \end{aligned}\tag{4c}$$

We can also rewrite the Hamiltonian in terms of bosonic phase variables. In eq. (3), the phase field $\phi_{\rho-}$ appears in the form $\cos(2\phi_{\rho-} + 4\lambda x)$. Since we can safely assume that $t_\perp$ is a relevant perturbation for $t_\perp$ being not so small,[18,19)] the term with $\cos(2\phi_{\rho-} + 4\lambda x)$ would become irrelevant, and thus we discard it in the following. We also neglect the $\cos 2\phi_{\sigma-}\cos 2\theta_{\sigma-}$ term since its scaling dimension is larger than 2. Then our Hamiltonian reduces to $H = \int \mathrm{d}x\,\mathcal{H}$ with

$$\begin{aligned}\mathcal{H} &= \frac{v_\mathrm{F}}{\pi}\sum_{r=\pm}\left[\sum_{p=\pm}\left(\partial_x\phi_{\rho r}^p\right)^2 + \frac{g_{\rho r}}{\pi v_\mathrm{F}}\left(\partial_x\phi_{\rho r}^+\right)\left(\partial_x\phi_{\rho r}^-\right)\right] \\ &\quad + \frac{v_\mathrm{F}}{\pi}\sum_{r=\pm}\left[\sum_{p=\pm}\left(\partial_x\phi_{\sigma r}^p\right)^2 - \frac{g_{\sigma r}}{\pi v_\mathrm{F}}\left(\partial_x\phi_{\sigma r}^+\right)\left(\partial_x\phi_{\sigma r}^-\right)\right] \\ &\quad + \frac{1}{2\pi^2 a^2}\left(g_{\overline{\mathrm{c}-},\mathrm{s}+}\cos 2\phi_{\sigma+} + g_{\overline{\mathrm{c}-},\mathrm{s}-}\cos 2\phi_{\sigma-}\right. \\ &\quad \left. + g_{\overline{\mathrm{c}-},\overline{\mathrm{s}-}}\cos 2\theta_{\sigma-}\right)\cos 2\theta_{\rho-} \\ &\quad + \frac{1}{2\pi^2 a^2}\left(g_{\mathrm{s}+,\mathrm{s}-}\cos 2\phi_{\sigma+}\cos 2\phi_{\sigma-}\right. \\ &\quad \left. + g_{\mathrm{s}+,\overline{\mathrm{s}-}}\cos 2\phi_{\sigma+}\cos 2\theta_{\sigma-}\right), \end{aligned}\tag{5}$$

where the coupling constants of the harmonic terms are given by $g_{\rho(\sigma)r} = \sum_{\epsilon=\pm} f_{\rho(\sigma)r}^\epsilon (g_{2\parallel}^{+\epsilon} + (-)g_{2\perp}^{+\epsilon} - g_{1\parallel}^{\epsilon\epsilon})/2$ with $r = \pm$, $f_{\rho+}^\epsilon = 1$, $f_{\rho-}^\epsilon = \epsilon$, $f_{\sigma+}^\epsilon = -1$ and $f_{\sigma-}^\epsilon = -\epsilon$. The coupling constants of the nonlinear terms are $g_{\overline{\mathrm{c}-},\mathrm{s}+} \equiv -g_{1\perp}^{-+}$, $g_{\overline{\mathrm{c}-},\mathrm{s}-} \equiv -g_{2\perp}^{-+}$, $g_{\overline{\mathrm{c}-},\overline{\mathrm{s}-}} \equiv (g_{2\parallel}^{-+} - g_{1\parallel}^{-+})$, $g_{\mathrm{s}+,\mathrm{s}-} \equiv g_{1\perp}^{++}$, $g_{\mathrm{s}+,\overline{\mathrm{s}-}} \equiv g_{1\perp}^{--}$. From these nine coupling constants, a set of independent parameters can be selected since the global spin-rotation SU(2) symmetry leads to[14)] $(g_{\sigma+} + g_{\sigma-} - g_{\mathrm{s}+,\mathrm{s}-}) = (g_{\sigma+} - g_{\sigma-} - g_{\mathrm{s}+,\overline{\mathrm{s}-}}) = (g_{\overline{\mathrm{c}-},\mathrm{s}+} - g_{\overline{\mathrm{c}-},\mathrm{s}-} - g_{\overline{\mathrm{c}-},\overline{\mathrm{s}-}}) = 0$. We choose the following six coupling constants:

$$g_{\overline{\mathrm{c}-},\mathrm{st}} = +U - V_\perp - 2V_\parallel\cos\pi\delta, \tag{6a}$$

$$g_{\overline{\mathrm{c}-},\mathrm{ss}} = +U - V_\perp + 2V_\parallel(\cos\pi\delta + 2\cos 2\lambda), \tag{6b}$$

$$g_{\rho+} = +U + 2V_\perp + V_\parallel[4 + \cos\pi\delta(1 + \cos 2\lambda)], \tag{6c}$$

$$g_{\rho-} = -V_\perp - V_\parallel\cos\pi\delta(1 - \cos 2\lambda), \tag{6d}$$

$$g_{\sigma+} = +U - V_\parallel\cos\pi\delta(1 + \cos 2\lambda), \tag{6e}$$

$$g_{\sigma-} = +V_\perp + V_\parallel\cos\pi\delta(1 - \cos 2\lambda), \tag{6f}$$

where $g_{\overline{\mathrm{c}-},\mathrm{st}} \equiv -g_{\overline{\mathrm{c}-},\mathrm{s}+}$ and $g_{\overline{\mathrm{c}-},\mathrm{ss}} \equiv (-g_{\overline{\mathrm{c}-},\mathrm{s}-} + g_{\overline{\mathrm{c}-},\overline{\mathrm{s}-}})$. The present model and the above treatment are quite similar to those in ref. 15. However, the application of the renormalization-group (RG) method to eq. (5) is complicated when the excitation gaps of spin modes should be estimated properly. Therefore, we fermionize the spin part of eq. (5)[14)] by introducing the spinless fermion fields $\psi_{\pm,r}(x) = \eta_r(2\pi a)^{-1/2}\exp\left[\pm\mathrm{i}\,2\phi_{\sigma r}^\pm(x)\right]$, where $r = \pm$ and $\{\eta_r, \eta_{r'}\} = 2\delta_{r,r'}$. By using the SU(2) constraints and the Majorana fermions ξ^n ($n = 1$–4), eq. (5) is rewritten as

$$\begin{aligned}\mathcal{H} &= \frac{v_\mathrm{F}}{\pi}\sum_r\left[\sum_p\left(\partial\phi_{\rho r}^p\right)^2 + \frac{g_{\rho r}}{\pi v_\mathrm{F}}\left(\partial_x\phi_{\rho r}^+\right)\left(\partial_x\phi_{\rho r}^-\right)\right] \\ &\quad - \mathrm{i}\frac{v_\mathrm{F}}{2}\left(\boldsymbol{\xi}_+\cdot\partial_x\boldsymbol{\xi}_+ - \boldsymbol{\xi}_-\cdot\partial_x\boldsymbol{\xi}_-\right) - \frac{g_{\sigma+}}{2}\left(\boldsymbol{\xi}_+\cdot\boldsymbol{\xi}_-\right)^2 \\ &\quad - \mathrm{i}\frac{v_\mathrm{F}}{2}\left(\xi_+^4\partial_x\xi_+^4 - \xi_-^4\partial_x\xi_-^4\right) - g_{\sigma-}\left(\boldsymbol{\xi}_+\cdot\boldsymbol{\xi}_-\right)\xi_+^4\xi_-^4 \\ &\quad - \frac{\mathrm{i}}{2\pi a}\left(g_{\overline{\mathrm{c}-},\mathrm{st}}\boldsymbol{\xi}_+\cdot\boldsymbol{\xi}_- + g_{\overline{\mathrm{c}-},\mathrm{ss}}\xi_+^4\cdot\xi_-^4\right)\cos 2\theta_{\rho-}, \end{aligned}\tag{7}$$

where $\psi_{p,+} = (\xi_p^1 + \mathrm{i}\xi_p^2)/\sqrt{2}$, $\psi_{p,-} = (\xi_p^4 + \mathrm{i}\xi_p^3)/\sqrt{2}$, and $\boldsymbol{\xi}_p = (\xi_p^1, \xi_p^2, \xi_p^3)$. Thus, the effective theory for the spin sector becomes $\mathrm{O}(3)\times\mathrm{Z}_2$ symmetric, as seen in the isotropic Heisenberg[17)] and half-filled Hubbard ladders.[14)]

We investigate the low-energy behavior by using the perturbative RG method with the lattice constant $a \to a\mathrm{e}^{\mathrm{d}l}$. The following six scaling equations are obtained:

$$\frac{\mathrm{d}}{\mathrm{d}l}G_{\rho-} = -\frac{3}{4}G_{\overline{\mathrm{c}-},\mathrm{st}}^2 - \frac{1}{4}G_{\overline{\mathrm{c}-},\mathrm{ss}}^2, \tag{8a}$$

$$\frac{\mathrm{d}}{\mathrm{d}l}G_{\sigma+} = -G_{\sigma+}^2 - G_{\sigma-}^2 - \frac{1}{2}G_{\overline{\mathrm{c}-},\mathrm{st}}^2, \tag{8b}$$

$$\frac{\mathrm{d}}{\mathrm{d}l}G_{\sigma-} = -2G_{\sigma+}G_{\sigma-} - \frac{1}{2}G_{\overline{\mathrm{c}-},\mathrm{st}}G_{\overline{\mathrm{c}-},\mathrm{ss}}, \tag{8c}$$

$$\frac{\mathrm{d}}{\mathrm{d}l}G_{\overline{\mathrm{c}-},\mathrm{st}} = -G_{\rho-}G_{\overline{\mathrm{c}-},\mathrm{st}} - 2\,G_{\sigma+}G_{\overline{\mathrm{c}-},\mathrm{st}} - G_{\sigma-}G_{\overline{\mathrm{c}-},\mathrm{ss}}, \tag{8d}$$

$$\frac{\mathrm{d}}{\mathrm{d}l}G_{\overline{\mathrm{c}-},\mathrm{ss}} = -G_{\rho-}G_{\overline{\mathrm{c}-},\mathrm{ss}} - 3G_{\sigma-}G_{\overline{\mathrm{c}-},\mathrm{st}}, \tag{8e}$$

and $\mathrm{d}G_{\rho+}/\mathrm{d}l = 0$, where $G(0) = g/(2\pi v_\mathrm{F})$. We note that these RG equations can also be derived directly from eq. (5). We analyze RG equations numerically for $U > 0$, $V_\parallel > 0$ and $V_\perp > 0$. For the small $V_\perp/U$ and $V_\parallel/U$, the limiting behavior of the RG equations is given by $(G_{\rho-}^*, G_{\sigma+}^*, G_{\sigma-}^*,$

Table I. Possible states and the corresponding pattern of phase locking, where I is an integer and the symbol $*$ indicates an unlocked bosonic phase field. The signs $+$ and $-$ denote those for the renormalized masses $m_{\rm t}$ and $m_{\rm s}$, where we have assumed I being an even number, i.e., $c_{\overline{\rho-}} > 0$.

State	$\langle\theta_{\rho-}\rangle$	$\langle\phi_{\sigma+}\rangle$	$\langle\phi_{\sigma-}\rangle$	$\langle\theta_{\sigma-}\rangle$	$m_{\rm t}$	$m_{\rm s}$
SCd	$\frac{\pi}{2}I$	$\frac{\pi}{2}I$	$\frac{\pi}{2}I$	$*$	$+$	$+$
CDW+PDW	$\frac{\pi}{2}I$	$\frac{\pi}{2}(I+1)$	$*$	$\frac{\pi}{2}(I+1)$	$-$	$+$

$G^*_{\overline{\rm c-},{\rm st}}, G^*_{\overline{\rm c-},{\rm ss}}) = (-,-,-,+,+)$ which corresponds to $(g^*_{\overline{\rm c-},{\rm s+}}, g^*_{\overline{\rm c-},{\rm s-}}, g^*_{\overline{\rm c-},\overline{\rm s-}}, g^*_{{\rm s+},{\rm s-}}, g^*_{{\rm s+},\overline{\rm s-}}) = (-,-,0,-,0)$ in eq. (5). The relevant behavior of the coupling constants implies that the phases are locked in order to minimize the cosine potential in eq. (5). The positions of phase locking and the corresponding ground states are summarized in Table I. Since the $\theta_{\sigma-}$ field is conjugate to $\phi_{\sigma-}$, these two fields cannot be locked at the same time. From eq. (4), the nonvanishing order parameter is $\mathcal{O}_{{\rm SC}d}$. Since the correlation function of the operator ${\rm e}^{{\rm i}\theta_{\rho+}}$ exhibits a power-law behavior, we find that the SCd fluctuation becomes quasi-long-range ordered (quasi-LRO) in this case. We note that the SCd state moves to the D-Mott or D'-Mott state in the limit of $\delta \to 0$.[14] For the large $V_\perp/U$ and $V_\parallel/U$, the limiting behavior of the RG equations is now given by $(G^*_{\rho-}, G^*_{\sigma+}, G^*_{\sigma-}, G^*_{\overline{\rm c-},{\rm st}}, G^*_{\overline{\rm c-},{\rm ss}}) = (-,-,+,-,+)$, corresponding to $(g^*_{\overline{\rm c-},{\rm s+}}, g^*_{\overline{\rm c-},{\rm s-}}, g^*_{\overline{\rm c-},\overline{\rm s-}}, g^*_{{\rm s+},{\rm s-}}, g^*_{{\rm s+},\overline{\rm s-}}) = (+,0,+,0,-)$. In this case, the dominant order parameters are $\mathcal{O}_{\rm CDW}$ and $\mathcal{O}_{\rm PDW}$ both of which lead to the quasi-LRO with correlation functions of the same exponent. We call this coexisting state the CDW+PDW state.

In order to analyze the properties near the critical point of the transition between the SCd state and the CDW+PDW state, we restrict ourselves to the case where the mass of the charge mode ($\rho-$) is larger than those of the spin modes ($\sigma\pm$). The $\theta_{\rho-}$ field is locked by the cosine potential below the mass scale of the charge mode $m_{\rho-}$. By replacing $\cos 2\theta_{\rho-}$ with its average value $c_{\overline{\rho-}} \equiv \langle \cos 2\theta_{\rho-}\rangle$ in eq. (5), the effective low-energy Hamiltonian for the spin degrees of freedom is obtained as[14,17]

$$\begin{aligned}\mathcal{H}_\sigma = &-{\rm i}\frac{v_{\rm F}}{2}\left(\boldsymbol{\xi}_+\cdot\partial_x\boldsymbol{\xi}_+ - \boldsymbol{\xi}_-\cdot\partial_x\boldsymbol{\xi}_-\right) - {\rm i}m_{\rm t}^0\boldsymbol{\xi}_+\cdot\boldsymbol{\xi}_- \\ &-{\rm i}\frac{v_{\rm F}}{2}\left(\xi_+^4\partial_x\xi_+^4 - \xi_-^4\partial_x\xi_-^4\right) - {\rm i}m_{\rm s}^0\xi_+^4\xi_-^4 \\ &-\frac{g_{\sigma+}}{2}\left(\boldsymbol{\xi}_+\cdot\boldsymbol{\xi}_-\right)^2 - g_{\sigma-}\left(\boldsymbol{\xi}_+\cdot\boldsymbol{\xi}_-\right)\xi_+^4\xi_-^4, \end{aligned} \tag{9}$$

where $m_{\rm t}^0$ and $m_{\rm s}^0$ represent the bare masses of the Majorana triplet and singlet sector: $m_{\rm t}^0 = (c_{\overline{\rho-}}/2\pi a)(U - V_\perp - 2V_\parallel \cos\pi\delta)$ and $m_{\rm s}^0 = (c_{\overline{\rho-}}/2\pi a)[U - V_\perp + 2V_\parallel(\cos\pi\delta + 2\cos 2\lambda)]$. The quantity $m_{\rm t}^0$ ($m_{\rm s}^0$) has physical meanings of the spin gap in the magnon (soliton) excitation.[17] Equation (9) is further analyzed in terms of the following scaling equations for coupling constants:

$$\frac{{\rm d}G_{\rm t}}{{\rm d}l} = G_{\rm t} - 2G_{\rm t}G_{\sigma+} - G_{\rm s}G_{\sigma-}, \tag{10a}$$

$$\frac{{\rm d}G_{\rm s}}{{\rm d}l} = G_{\rm s} - 3G_{\rm t}G_{\sigma-}, \tag{10b}$$

$$\frac{{\rm d}G_{\sigma+}}{{\rm d}l} = -G_{\sigma+}^2 - G_{\sigma-}^2 - G_{\rm t}^2, \tag{10c}$$

$$\frac{{\rm d}G_{\sigma-}}{{\rm d}l} = -2G_{\sigma+}G_{\sigma-} - G_{\rm t}G_{\rm s}, \tag{10d}$$

where $G_{\rm t} = m_{\rm t}^0/v_{\rm F}$, $G_{\rm s} = m_{\rm s}^0/v_{\rm F}$, and $G_{\sigma\pm} = g_{\sigma\pm}/2\pi v_{\rm F}$. The couplings $G_{\rm s}$ and $G_{\rm t}$ are relevant, while $G_{\sigma\pm}$ are marginal. In eq. (10), the $G_{\rm s}$ term as a function of l increases rapidly compared with other G's and becomes relevant at $l = l_{\rm s}$ corresponding to the energy scale of a gap in the Majorana singlet mode $m_{\rm s} \approx t_\parallel {\rm e}^{-l_{\rm s}}$, where we stop the calculation of eq. (10). The mode remaining below the energy scale of $m_{\rm s}$ is the Majorana triplet sector. The effective theory for this mode is given by $\mathcal{H}_\sigma^{\rm eff} = -{\rm i}\frac{1}{2}v_{\rm F}(\boldsymbol{\xi}_+\cdot\partial_x\boldsymbol{\xi}_+ - \boldsymbol{\xi}_-\cdot\partial_x\boldsymbol{\xi}_-) - {\rm i}m_{\rm t}^{\rm s}\boldsymbol{\xi}_+\cdot\boldsymbol{\xi}_- - \frac{1}{2}g_{\sigma+}^{\rm s}(\boldsymbol{\xi}_+\cdot\boldsymbol{\xi}_-)^2$, where $m_{\rm t}^{\rm s} = v_{\rm F}[G_{\rm t}(l_{\rm s}) - G_{\sigma-}(l_{\rm s})]$ and $g_{\sigma+}^{\rm s} = 2\pi v_{\rm F}G_{\sigma+}(l_{\rm s})$. Then we solve the RG equations, ${\rm d}G_{\rm t}/{\rm d}l = G_{\rm t} - 2G_{\rm t}G_{\sigma+}$ and ${\rm d}G_{\sigma+}/{\rm d}l = -2G_{\sigma+}^2 - G_{\rm t}^2$, with the initial conditions $G_{\rm t}(l_{\rm s}) = m_{\rm t}^{\rm s}/v_{\rm F}$ and $G_{\sigma+}(l_s) = g_{\sigma+}^{\rm s}/2\pi v_{\rm F}$. We easily find that these RG equations have two stable fixed points, $(G_{\rm t}^*, G_{\sigma+}^*) = (+\infty, -\infty)$ and $(-\infty, -\infty)$, corresponding to the SCd state and the CDW+PDW state, respectively. The magnitude of the gap in the Majorana triplet sector can be estimated from $m_{\rm t} \approx t_\parallel {\rm e}^{-l_{\rm t}}\,{\rm sgn}(G_{\rm t}^*)$ with $l_{\rm t}$ being determined by $|G_{\rm t}(l_{\rm t})| = 1$ (see Table I). There are also two unstable fixed points, $(G_{\rm t}^*, G_{\sigma+}^*) = (0,0)$ and $(0,-\infty)$, corresponding to the second-order and first-order phase transitions,[14] while only the former transition is obtained in the present numerical calculation.

From the numerical integration of the RG equations, we obtain the ground-state phase diagram shown in Fig. 1. The SCd state (the CDW+PDW state) is obtained for $V_\parallel/U + V_\perp/U \lesssim 0.4$ ($\gtrsim 0.4$). The SCd state is stabilized by the on-site repulsive interaction, which segregates up-spin from down-spin on the same site and leads to singlet pairing on a rung. On the other hand, the CDW+PDW state is obtained due to the nearest-neighbor repulsive interactions, which induce a density wave leading to the singlet state on the same site or chain. The effect of $V_\parallel$ is slightly larger than that of $V_\perp$, although both the intersite interactions have essentially the same effect of inducing the CDW+PDW state. Figure 2 shows the change from the CDW+PDW state to the SCd state with an increase in the doping δ (> 0.05). The novel aspect of the present paper is the competition induced by doping which reduces the effect of $V_\parallel$, as shown

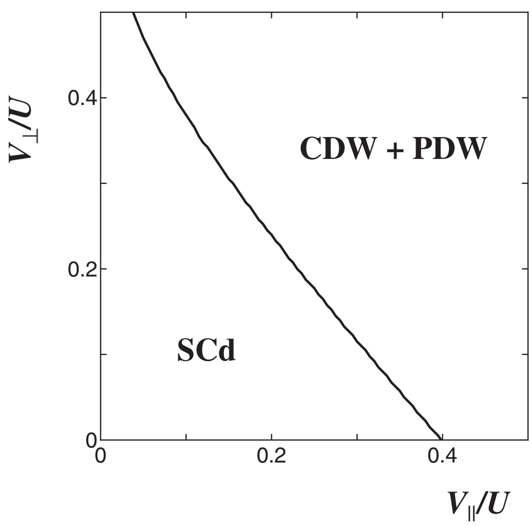

Fig. 1. Ground-state phase diagram on the plane of $V_\parallel/U$ and $V_\perp/U$, with $U/t_\parallel = 2$, $\delta = 0.1$, and $t_\perp = t_\parallel$.

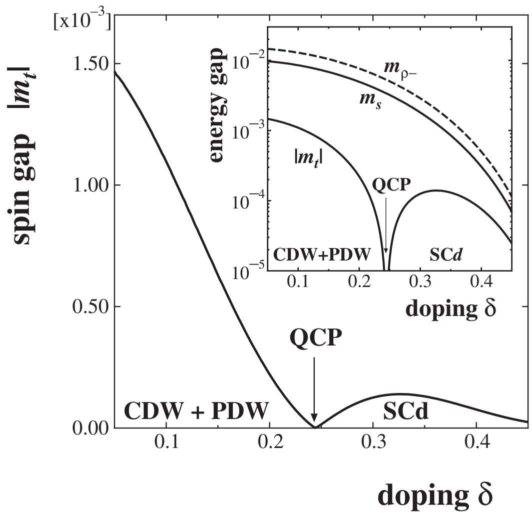

Fig. 2. Doping dependence of the magnon spin gap m_t with $U/t_\parallel = 2$, $V_\parallel/U = V_\perp/U = 0.25$, and $t_\perp = t_\parallel = 1$. In the inset, the doping dependences of $m_{\rho-}$, m_s, and m_t are shown.

in eq. (6). In the inset, we show the respective masses estimated from $|m_a| \approx t_\parallel \exp(-l_a)$ ($a = \mathrm{t}$, s, $\rho-$) by noting that the corresponding coupling constant $|G_a|$ becomes of the order of unity at $l = l_a$. Our system exhibits a *second-order* phase transition where the magnon excitation gap vanishes at the quantum critical point (QCP). The critical property for the Majorana triplet sector, which differs from that of the conventional Tomonaga–Luttinger liquid, is described by the $SU(2)_2$ Wess–Zumino–Novikov–Witten model with the central charge $c = 3/2$.[17]

In the present letter, by applying the weak-coupling RG method to the EHM on a two-leg ladder, we have shown that the doping δ suppresses the CDW+PDW quasi-LRO state and yields the system to the QCP, and that the SCd quasi-LRO state is stabilized by further doping. Here, we discuss the experimental results of the two-leg ladder compound $Sr_{14-x}Ca_xCu_{24}O_{41}$. The phase diagram of $Sr_{14-x}Ca_xCu_{24}O_{41}$ obtained in ref. 9 resembles our phase diagram in Fig. 2 if the magnitude of the gap $|m_\mathrm{t}|$ is regarded as the transition temperature. On closer look, our phase diagram exhibits features in contrast to those of $Sr_{14-x}Ca_xCu_{24}O_{41}$, that is, the resistivity above the transition temperatures shows an insulating behavior and there is no experimental evidence of the QCP between the CDW state and the SC state. In order to explain the phase diagram of $Sr_{14-x}Ca_xCu_{24}O_{41}$, the dimensionality effect and/or the disorder effect has been discussed.[9] The quantum critical behavior would be smeared out by these effects, which are not taken into account in the present paper. However, it will be still interesting to examine the competing region in the sense that the magnon gap would become extremely small and an anomalous behavior can be expected at temperatures higher than the characteristic energies of the disorder and the dimensionality. We note that the origin of the high-temperature insulating phase is still unknown and its analysis is left for a future study.

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