Optimization of a Logical CNOT Circuit for a 3 Physical Qubit Decoherence-Free Subspace

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1 Introduction

Quantum computing's potential to revolutionize fields from cryptography to materials science hinges significantly on the development of robust quantum gates within error-resistant frameworks. Among these, the decoherence-free subspace (DFS) offers a promising avenue for protecting quantum information from environmental noise without the need for active error correction. This research focuses on the Decoherence-Free Subspace involving two sets of three-qubit systems, aiming to encode and optimize a logical Controlled-NOT (CNOT) gate—a cornerstone in quantum computation for its role in entanglement and logical operations between qubits.

Previous studies, such as those by Fong and Wandzura (2011), have laid the groundwork by demonstrating the feasibility of such encodings using minimal interaction models. However, challenges remain in achieving optimal gate operation. This study seeks to address these challenges by simulating the logical CNOT gate logic within a 3-qubit DFS with total spin quantum numbers of 0 and 1, based on the spin configurations and orientations that determine the "unleaked" state of these qubit systems.

Despite theoretical advancements, practical implementations of CNOT gates within this configuration have faced significant hurdles, primarily due to the intricate balance required between gate fidelity, quantum state preservation, and system scalability. This research contributes to this ongoing effort by:

1. Exploring the operational dynamics of CNOT gates within a DFS characterized by total spin quantum numbers,
2. Utilizing machine learning techniques to map out logical CNOT encodings, and
3. Proposing modifications to existing models based on machine learning outcomes (simulated outcomes) to enhance performance and reliability.

The methodology revolves around the construction of state representations within the DFS, employing a combination of Clebsch-Gordon coefficients for state construction and Bratelli diagrams for deeper analytical insights into system behavior under quantum operations. Through this approach, this paper aims to refine the understanding of quantum interactions within a DFS and enhance the practical deployment of quantum gates in real-world quantum computing applications.

2 Problem Setting

2.1 Defining Spin Operators and Basis States

We veered off from utilizing Bratelli diagrams given that you suggested to use the bloch diagonal matrix and used the composition of unleaked ($S_{tot} = \frac{1}{2}$) states, and ignored leaked ($S_{tot} = \frac{3}{2}$) states for N=3 DFS. We interchange S as our total spin of the qubits (can be interchanged with total angular momentum - J) and N is the number of qubits.

We define our spin operator to be:

$$\vec{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 \tag{1}$$

$$\hat{S}_j = \frac{1}{2}(\sigma_j^{(1)} + \sigma_j^{(2)} + \sigma_j^{(3)}) \tag{2}$$

where $\sigma_j^{(k)}$ is the respective pauli matrix $j \in \{x, y, z\}$ acting on the qubit $k \in \{1, 2, 3\}$.

We define $|s\rangle$ as the singlet state and $|t^+\rangle, |t^0\rangle, |t^-\rangle$ as the triplet state. Which are shown below:

$$\begin{aligned}
|s\rangle &= \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |J = 0, M = 0\rangle \\
|t^+\rangle &= |11\rangle = |J = 1, M = +1\rangle \\
|t^0\rangle &= \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |J = 1, M = 0\rangle \\
|t^-\rangle &= |00\rangle = |J = 1, M = -1\rangle
\end{aligned}$$

The N=3 DFS states are:

These are the 8 basis states written out for N=3:

$$\begin{aligned}
|s1\rangle &= \left|N = 3, J = \tfrac{1}{2}, M = +\tfrac{1}{2}, \right\rangle => |s1, \rangle \\
|s2\rangle &= \left|N = 3, J = \tfrac{1}{2}, M = -\tfrac{1}{2}, \right\rangle => |s2, \rangle \\
|s3\rangle &= \left|N = 3, J = \tfrac{1}{2}, M = +\tfrac{1}{2}, \right\rangle => |s3, \rangle \\
|s4\rangle &= \left|N = 3, J = \tfrac{1}{2}, M = -\tfrac{1}{2}, \right\rangle => |s4, \rangle \\
|s5\rangle &= \left|N = 3, J = \tfrac{3}{2}, M = +\tfrac{3}{2}, \right\rangle => |s5, \rangle \\
|s6\rangle &= \left|N = 3, J = \tfrac{3}{2}, M = +\tfrac{1}{2}, \right\rangle => |s6, \rangle \\
|s7\rangle &= \left|N = 3, J = \tfrac{3}{2}, M = -\tfrac{1}{2}, \right\rangle => |s7, \rangle \\
|s8\rangle &= \left|N = 3, J = \tfrac{3}{2}, M = -\tfrac{3}{2}, \right\rangle => |s8, \rangle
\end{aligned}$$

From here we can see that the unleaked states (J or $S = \frac{1}{2}$) are the states $|s1\rangle - |s4\rangle$, meanwhile the leaked states (J or $S = \frac{3}{2}$) are the states $|s5\rangle - |s8\rangle$.

Therefore representing the first four states in the qubit computational basis we get:

$$\begin{aligned}
|s1\rangle &= |s\rangle \otimes |1\rangle \\
|s2\rangle &= |s\rangle \otimes |0\rangle \\
|s3\rangle &= \tfrac{1}{\sqrt{3}}(\sqrt{2}\,|t^+\rangle \otimes |0\rangle - |t^0\rangle \otimes |1\rangle) \\
|s4\rangle &= \tfrac{1}{\sqrt{3}}(\sqrt{2}\,|t^0\rangle \otimes |0\rangle - |t^-\rangle \otimes |1\rangle)
\end{aligned}$$

These computational basis states are acquired using the Bratelli diagram along with Clebsch Gordon coefficients.

Now in order to define the states for two logical qubits (two N=3 DFS sets of qubits yielding a total of N=6), we look at the diagrams from the paper. They use the following tables taken from [1]:

	1	2	3	4	5
S_{tot}	0	0	0	0	0
$S_{z,\text{tot}}$	0	0	0	0	0
S_A	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
S_B	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
$S_{A,1,2}$	0	0	1	1	1
$S_{B,1,2}$	0	1	0	1	1

Table 2: Quantum numbers for two DFS qubits in the total spin-0 subspace. S_{tot} is the total spin of the six physical qubits, $S_{z,\text{tot}}$ the total spin-z, S_A (S_B) the total spin of DFS qubit A (B), $S_{A,1,2}$ ($S_{B,1,2}$) the spin of the first two qubits of DFS qubit A (B). The top row is the index label of the basis vectors in the total angular momentum basis. Basis vectors 1–4 are valid encoded states; basis vector 5 is a leaked state.

	6	7	8	9	10	11	12	13	14
S_{tot}	1	1	1	1	1	1	1	1	1
$S_{z,\text{tot}}$	-1	-1	-1	-1	-1	-1	-1	-1	-1
S_A	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
S_B	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
$S_{A,1,2}$	0	0	1	1	0	1	1	1	1
$S_{B,1,2}$	0	1	0	1	1	1	0	1	1

Table 3: Quantum numbers for two DFS qubits in the total spin-1, $S_{z,\text{tot}} = -1$ subspace. Basis vectors 6–9 are valid encoded states; basis vectors are leaked states, with one or both of the constituent DFS qubits leaked. Basis vectors 10 and 11 are unleaked in DFS qubit A. See Table 2 for operator definitions.

	15	16	17	18	19
S_{tot}	2	2	2	2	2
$S_{z,\text{tot}}$	-2	-2	-2	-2	-2
S_A	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
S_B	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
$S_{A,1,2}$	0	1	1	1	1
$S_{B,1,2}$	1	1	0	1	1

Table 4: Quantum numbers for two DFS qubits in the total spin-2, $S_{z,\text{tot}} = -2$ subspace. Basis vectors 15 and 16 are unleaked in DFS qubit A. See Table 2 for operator definitions.

Looking directly at these diagrams (tables 2,3,4) for states that they defined to be states 1-19 we can see that we only need the states 1-4 and 6-9 since these are the only states comprised of combinations of N=3 unleaked states. Let us rename the unleaked states $|c_1\rangle - |c_4\rangle$ (for the states 1-4 from table 2) and states $|c_5\rangle - |c_8\rangle$ (for the states 6-9 from table 3). We define those states in the N=3 basis (using the Clebsch-Gordon coefficients) to be:

$$
\begin{aligned}
|c_1\rangle &= \frac{1}{\sqrt{2}}(|s1\rangle \otimes |s2\rangle - |s2\rangle \otimes |s1\rangle)\\
|c_2\rangle &= \frac{1}{\sqrt{2}}(|s1\rangle \otimes |s4\rangle - |s2\rangle \otimes |s3\rangle)\\
|c_3\rangle &= \frac{1}{\sqrt{2}}(|s3\rangle \otimes |s2\rangle - |s4\rangle \otimes |s1\rangle)\\
|c_4\rangle &= \frac{1}{\sqrt{2}}(|s3\rangle \otimes |s4\rangle - |s4\rangle \otimes |s3\rangle)\\
|c_5\rangle &= |s2\rangle \otimes |s2\rangle\\
|c_6\rangle &= |s2\rangle \otimes |s4\rangle\\
|c_7\rangle &= |s4\rangle \otimes |s2\rangle\\
|c_8\rangle &= |s4\rangle \otimes |s4\rangle
\end{aligned}
$$

We know these are the states because of the all the spins defined from tables 2 and 4, which helps us to know which states we will be working with, then combine them according to total Spin using the Clebsch-Gordon coefficients. We also know they can represent a basis for N=6 since they are computed to demonstrate that they are orthogonal to each other, and thus have the potential to optimize and find a logical CNOT.

We will define the ideal logical CNOT operator, U_{CNOT}, acting on 6 physical qubits (2 logical qubits). We expect U_{CNOT} to behave in the following way:

$$
\begin{aligned}
U_{CNOT}|c_1\rangle &= |c_1\rangle\\
U_{CNOT}|c_2\rangle &= |c_2\rangle\\
U_{CNOT}|c_3\rangle &= |c_4\rangle\\
U_{CNOT}|c_4\rangle &= |c_3\rangle\\
U_{CNOT}|c_5\rangle &= |c_6\rangle\\
U_{CNOT}|c_6\rangle &= |c_7\rangle\\
U_{CNOT}|c_7\rangle &= |c_9\rangle\\
U_{CNOT}|c_8\rangle &= |c_8\rangle
\end{aligned}
$$

When taking the U_{CNOT} unitary matrix to act correctly on both state spaces ($S_Z = 0$ & $S_Z = -1$), we observe that the first 3 qubits (of the first logical state) $|s1\rangle$ and $|s2\rangle$ act as the logical $|0\rangle$ state, and the $|s3\rangle$ and $|s4\rangle$ act as the logical $|1\rangle$ state when it comes to being the control qubit. When the logical X gate (X_L) is applied to the four states, we should get the following:

$$
\begin{aligned}
X_L|s1\rangle &= |s3\rangle\\
X_L|s3\rangle &= |s1\rangle\\
X_L|s2\rangle &= |s4\rangle\\
X_L|s4\rangle &= |s2\rangle
\end{aligned}
$$

2.1.1 Additional Extended States (to be included or excluded)

Additionally, we will define other states to optimize over that make or may not be included and we'll define them to be:

$$
\begin{aligned}
|c_{e9}\rangle &= |s4\rangle \otimes |s1\rangle\\
|c_{e10}\rangle &= |s4\rangle \otimes |s3\rangle\\
|c_{e11}\rangle &= |s3\rangle \otimes |s2\rangle\\
|c_{e12}\rangle &= |s3\rangle \otimes |s4\rangle
\end{aligned}
$$

where

$$
\begin{aligned}
U_{CNOT}|c_{e9}\rangle &= |c_{e10}\rangle\\
U_{CNOT}|c_{e10}\rangle &= |c_{e9}\rangle\\
U_{CNOT}|c_{e11}\rangle &= |c_{e12}\rangle\\
U_{CNOT}|c_{e12}\rangle &= |c_{e11}\rangle
\end{aligned}
$$

We use these states to test while including and excluding through the optimization process to see if there is any added benefit. We obtained these states from the states $|c_3\rangle$ & $|c_4\rangle$ to see if there is any added benefit.

2.2 Defining Our Operator and Loss Function

In order to construct such interactions between nearby physical qubits, we have to define some terminology. We utilize the Heisenberg exchange operator denoted by $H_{ex}^{m,n} \equiv \frac{1}{4}(\sigma_x^m \otimes \sigma_x^n + \sigma_y^m \otimes \sigma_y^n + \sigma_z^m \otimes \sigma_z^n)$ where m and and n are the qubits the operator shall act on.

We construct a unitary operator following:

$$U_{ex}^{m,n}(p) = \exp\{(-i\pi p H_{ex}^{m,n})\}, \text{ where } p \in [-1, 1] \tag{3}$$

We have a secondary a unitary operator:

$$U_{ex(mod)}^{m,n}(p) = \exp\{(-ipH_{ex}^{m,n})\}, \text{ where } p \in [-\pi, \pi] \tag{4}$$

We define two operators ($U_{ex}^{m,n}(p)$ and $U_{ex(mod)}^{m,n}(p)$) to see if the larger space or smaller space is more effective at optimizing (minimizing the loss) for the large number of parameters.

We use these Unitary exchange operators only for nearest neighbor interactions to resemble the realistic architecture of superconducting qubits. When $p = \pm 1$ for (3), it corresponds to a full SWAP operation, up to a global phase, of physical qubits. When $p = 0$, it trivially corresponds to an identity gate.

2.3 Loss Function

The loss function from [1] for the logical CNOT is defined as:

$$\text{primary\_loss} = \tfrac{1}{4}(|\langle U_{1,1}\rangle|^2 + |\langle U_{2,2}\rangle|^2 + |\langle U_{3,4}\rangle|^2 + |\langle U_{4,3}\rangle|^2) + \tfrac{1}{4}(|\langle U_{6,6}\rangle|^2 + |\langle U_{7,7}\rangle|^2 + |\langle U_{8,9}\rangle|^2 + |\langle U_{9,8}\rangle|^2)$$

$$f_{CNOT}(U) = \sqrt{2 - \text{primary\_loss}} \tag{5}$$

$$\text{where } |\langle U_{i,j}\rangle|^2 = |\langle c_i| U |c_j\rangle|^2$$

We also define a new loss function for the extended basis states to be:

$$f_{CNOT}^{(add)}(U) = \sqrt{3 - \text{primary\_loss} - \frac{1}{4}(|\langle U_{e9,e10}\rangle|^2 + |\langle U_{e10,e9}\rangle|^2 + |\langle U_{e11,e12}\rangle|^2 + |\langle U_{e12,e11}\rangle|^2)} \tag{6}$$

Since U is unitary by construction, its entries have a modulus at most 1. The objective function is zero only when all entries $U_{i,j}$ have modulus 1 for either (5) or (6). Our motive in this project is to attempt both loss functions f_{CNOT} and $f_{CNOT}^{(add)}$ for the normal states ($|c_1\rangle - |c_8\rangle$) and also including the extended states respectively to test for any improvements. We discuss this later on.

2.4 The Results from [1]

The original paper [1] constructed an efficient circuit with each box denoting the Unitary operator given by their respective p value within the box. The author suggested following circuit for performing the CNOT operation in figure 1.

The logical CNOT operation in figure 1 is able to function in 22 pulses and 13 time steps. DFS qubits A (control) and B (target) are arranged as shown in figure 1 with the first qubit of subsystem A nearest to the first qubit of subsytem B. Subscripts on A and B label the constituent physical qubits. Each gate responds to an exchange unitary from (3).

When running the circuit exactly as Figure 1 on the states $|c_1\rangle - |c_8\rangle$ and applying it to the f_{CNOT} function from (5), we get $f_{CNOT} = 1.2497750693297125$ which we expected to get close to 0. We did this solely through numpy matrix manipulation.

This signifies to me that either the states $|c_1\rangle - |c_8\rangle$ are incorrect, but don't seem to be when following the tables (2,3,4), or I'm not sure what could've gone wrong. We defined the physical states to be in the order according to the qubits A_3, A_2, A_1, B_1, B_2, B_3 respectively.

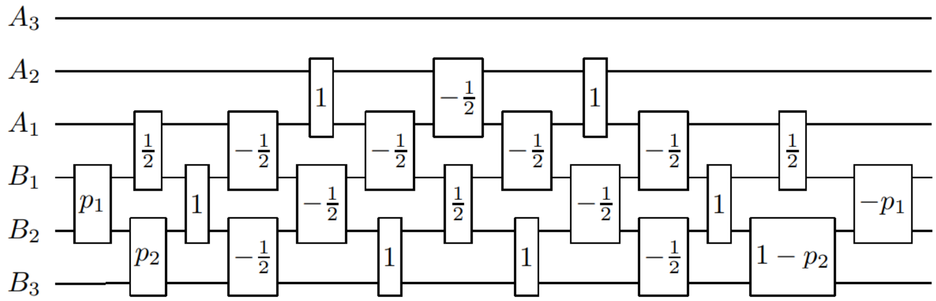

Figure 1: Logical CNOT suggested by [1]. Where $p_1 = \arccos(-1/\sqrt{3})/\pi$ and $p_2 = \arcsin(1/\sqrt{3})/\pi$

3 Building the Fixed Ansatz

In order to find a circuit with a f_{CNOT} value close to 0 and reduced pulse sequence and time steps, we used a variational quantum regression model. We utilized a layered approach, with each layer consisting of a 2 physical qubit unitary exchange operator (from (5) or (6)) with only adjacent qubits. Details for a circuit with a depth of 3 layers are described in Figure 2.

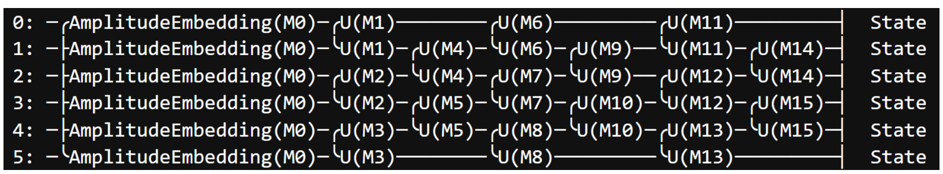

Figure 2: 3 layer circuit setup prior to parameter minimization. The parameters M1-M15 are the parameters $p_1 - p_{15}$ for the unitary exhange operators.

We initialize the parameters p_i for the respective exchange operators at random, and with each iteration, we optimize the parameters in such a way that the overall circuit reduces the $f_C NOT$ value as close to zero as possible, which in turn gets us closer to our intended logical CNOT operator.

For optimization of the parameters to fit the criteria, we used the 'Limited-Memory BFGS' optimizer.

4 Results

Now some of the results we achieved using the ansatz architecture from Figure 2 through optimizing different expected states are the following with the scipy minimize function:

Using 13 Layers

Include Extended States	Fong Qubit Orientation	Opt. Range	f_{CNOT} loss
True	False	[-1,1]	0.518468423781077
True	True	[-1,1]	0.5813965786336346
True	False	$[-\pi, \pi]$	0.5842887553791757
True	True	$[-\pi, \pi]$	0.42624192671513017
False	False	[-1,1]	0.27665417339027903
False	True	[-1,1]	0.3056212416029626
False	False	$[-\pi, \pi]$	0.47292994155429047
False	True	$[-\pi, \pi]$	0.2950255204555837

Using 7 Layers

Include Extended States	Fong Qubit Orientation	Opt. Range	f_{CNOT} loss
True	False	[-1,1]	0.8825795130608739
True	True	[-1,1]	0.6700158300423998
True	False	$[-\pi,\pi]$	0.7597794087369221
True	True	$[-\pi,\pi]$	0.9387374038187708
False	False	[-1,1]	0.5727536542728501
False	True	[-1,1]	0.6570580915016189
False	False	$[-\pi,\pi]$	0.5804529364077136
False	True	$[-\pi,\pi]$	1.0000000007696217

We define 'Fong Qubit Orientation' to be True if the orientation is A_3, A_2, A_1, B_1, B_2, B_3 in that order as in Figure 1. When 'Fong Qubit Orientation' is False we define the orientation to be A_1, A_2, A_3, B_1, B_2, B_3 in that order (basically defining nearest neighboring qubits). The 'Include Extended States' boolean is whether we include the additional states $|c_{e9}\rangle - |c_{e12}\rangle$ or not. If the 'Include Extended States' is True, we utilize f_{CNOT} from (6), else if False we utilize f_{CNOT} from (5). 'Opt. Range' (Optimization Range) is either using p=[-1,1] for $U_{ex}(p)$ from (3) or $p = [-\pi,\pi]$ for $U_{ex(mod)}(p)$ from (4).

As we can see, adding additional states $|c_{e9}\rangle - |c_{e12}\rangle$ made it worse for 13 layers, but slightly better for 7 layers. Overall, these values are no where near 0, where we would like our ideal loss to be. Also, there doesn't seem to be a notable improvement when increasing the space of the parameter optimization bounds from [-1,1] to $[-\pi,\pi]$, nor when modifying the Fong Qubit Orientation. Even when attempting to use 18 layers to see if the accuracy would increase it still did not improve significantly in comparison to 13 layers.

Now, defining a new and simplified $f^{(4)}_{CNOT}$ that we use on 4 qubits, it is:

$$f^{(4)}_{CNOT}(U) = \sqrt{1 - \frac{1}{4}(|\langle U_{i,i}\rangle|^2 + |\langle U_{i+1,i+1}\rangle|^2 + |\langle U_{i+2,i+3}\rangle|^2 + |\langle U_{i+3,i+2}\rangle|^2)} \tag{7}$$

The set of states for (7) are either i=1 (states $|c_1\rangle - |c_4\rangle$) or i=5 (states $|c_5\rangle - |c_8\rangle$)

Now, we attempt to run an ansatz using with also 13 layers, but this time only along the space of 4 states using the f_{CNOT} from (7).

Using 13 Layers

Set of 4 States	Fong Qubit Orientation	Opt. Range	$f^{(4)}_{CNOT}$ loss
First 4	False	[-1,1]	0.11289527726167026
First 4	False	$[-\pi,\pi]$	0.13727806092228126
Second 4	False	[-1,1]	0.14017060530001393
Second 4	False	$[-\pi,\pi]$	0.13232137854327394
First 4	True	[-1,1]	0.22975482460624957
First 4	True	$[-\pi,\pi]$	0.19287227519506245
Second 4	True	[-1,1]	0.28040396252586397
Second 4	True	$[-\pi,\pi]$	0.18874626539730371

Using 7 Layers

Set of 4 States	Fong Qubit Orientation	Opt. Range	$f^{(4)}_{CNOT}$ loss
First 4	False	[-1,1]	0.23543665341332728
First 4	False	$[-\pi,\pi]$	0.19769972079465944
Second 4	False	[-1,1]	0.37588902010261915
Second 4	False	$[-\pi,\pi]$	0.323694714172568
First 4	True	[-1,1]	0.37956673174485744
First 4	True	$[-\pi,\pi]$	0.3963069579208583
Second 4	True	[-1,1]	0.45013494357193
Second 4	True	$[-\pi,\pi]$	0.4895734198666129

We define 'Set of 4 States' to be either 'First 4' which occur over states $|c_1\rangle - |c_4\rangle$ or 'Second 4' which occur over the states $|c_5\rangle - |c_8\rangle$. The 'Fong Qubit Orientation' is to be True if the orientation is A_3, A_2, A_1, B_1, B_2, B_3 in that order as in Figure 1. When 'Fong Qubit Orientation' is False we define the orientation to be A_1, A_2, A_3, B_1, B_2, B_3 in that order (basically defining nearest neighboring qubits). The 'Include Extended States' boolean is whether we include the additional states $|c_{e9}\rangle - |c_{e12}\rangle$ or not. 'Opt. Range' (Optimization Range) is either using p=[-1,1] for $U_{ex}(p)$ from (3) or $p = [-\pi, \pi]$ for $U_{ex(mod)}(p)$ from (4).

5 Conclusion

This study has attempted to demonstrate the feasibility of optimizing a logical CNOT circuit within a 3-qubit decoherence-free subspace, offering insights into the quantum behavior of minimized error states under varied conditions. Unfortunately those results did not align with the results from [1]. I attempted to recreate the given tensor product states using the tables 2,3,4 from the paper and with the Clebsch-Gordon Coefficients to make states $|c_1\rangle - |c_8\rangle$. We tried adding additional states $|c_{e9}\rangle - |c_{e12}\rangle$ to see if that made some improvement, but it seems that it actually made it worse. For our results of optimizing only over sets of 4 states ($|c_1\rangle - |c_4\rangle$ or $|c_5\rangle - |c_8\rangle$), the f_{CNOT} loss did go down, but only because the dependence on states was a lot lower. The 'Fong Qubit Orientation' do seem to make a worse interaction when optimizing over 4 states, and slightly better for a regular qubit layout (A_1-A_3 followed by B_1-B_3). Adding 18 layers as seen in the appendix, also shows that there wasn't much of an improvement over adding those extra layers, thus yielding bad results overall.

6 Appendix

We attempted to use more layers for the 8 states (normal) and 12 states (extended)

Using 18 Layers

Include Extended States	Fong Qubit Orientation	Opt. Range	f_{CNOT} loss
True	False	[-1,1]	0.4165036183694318
True	True	[-1,1]	0.36372719304585405
True	False	$[-\pi, \pi]$	0.48653835012375013
True	True	$[-\pi, \pi]$	0.3676823674719307
False	False	[-1,1]	0.31558265583264755
False	True	[-1,1]	0.2925393751889955
False	False	$[-\pi, \pi]$	0.3334449121934268
False	True	$[-\pi, \pi]$	0.2949956528341103

We define 'Fong Qubit Orientation' to be True if the orientation is A_3, A_2, A_1, B_1, B_2, B_3 in that order as in Figure 1. When 'Fong Qubit Orientation' is False we define the orientation to be A_1, A_2, A_3, B_1, B_2, B_3 in that order (basically defining nearest neighboring qubits). The 'Include Extended States' boolean is whether we include the additional states $|c_{e9}\rangle - |c_{e12}\rangle$ or not. If the 'Include Extended States' is True, we utilize f_{CNOT} from (6), else if False we utilize f_{CNOT} from (5). 'Opt. Range' (Optimization Range) is either using p=[-1,1] for $U_{ex}(p)$ from (3) or $p = [-\pi, \pi]$ for $U_{ex(mod)}(p)$ from (4).

We attempted to use more layers for the First 4 and Second 4 states to see if any improvements arose:

Using 18 Layers

Set of 4 States	Fong Qubit Orientation	Opt. Range	$f^{(4)}_{CNOT}$ loss
First 4	False	[-1,1]	0.09375015398923997
First 4	False	$[-\pi, \pi]$	0.08684995668943028
Second 4	False	[-1,1]	0.09656780064679775
Second 4	False	$[-\pi, \pi]$	0.10732955868583335
First 4	True	[-1,1]	0.1682778481874323
First 4	True	$[-\pi, \pi]$	0.16221064962525886
Second 4	True	[-1,1]	0.12702925933238932
Second 4	True	$[-\pi, \pi]$	0.13169647075497323

We define 'Set of 4 States' to be either 'First 4' which occur over states $|c_1\rangle - |c_4\rangle$ or 'Second 4' which occur over the states $|c_5\rangle - |c_8\rangle$. The 'Fong Qubit Orientation' is to be True if the orientation is A_3, A_2, A_1, B_1, B_2, B_3 in that order as in Figure 1. When 'Fong Qubit Orientation' is False we define the orientation to be A_1, A_2, A_3, B_1, B_2, B_3 in that order (basically defining nearest neighboring qubits). The 'Include Extended States' boolean is whether we include the additional states $|c_{e9}\rangle - |c_{e12}\rangle$ or not. 'Opt. Range' (Optimization Range) is either using p=[-1,1] for $U_{ex}(p)$ from (3) or $p = [-\pi, \pi]$ for $U_{ex(mod)}(p)$ from (4).

References

[1] B. H. Fong and S. M. Wandzura, "Universal quantum computation and leakage reduction in the 3-qubit decoherence free subsystem," 2011.