ITESO, Universidad Jesuita de Guadalajara

Maestría de Ciencia de Datos

Optimización Convexa

Tarea 9: Funciones Convexas

Estudiante: Daniel Nuño
Profesor: Dr. Juan Diego Sanchez Torres
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Introduction

When an optimization problem is convex, there appear several desirable characteristics that should be known and exploited. Therefore, although up to this point, the notion of convexity has been used implicitly in the solution of several problems, it is now necessary to go deeper into this concept, its properties, and methods.

Activities

Consider the following exercises on convex functions:

Problem 1: Basic Exercises on Convex Functions

Prove (using math) the following claims:

1. The function $f_1(x) = |x|$ is convex.
2. The function $f_2(x) = x^2$ with $x > 0$ is strongly convex.
3. The function $f_3(x) = \exp(x)$ is convex.
4. The sum of convex functions is a convex function.
5. The convex combination of convex functions is a convex function.
6. The point-wise maximum of convex functions is a convex function.
7. The norms are convex functions.
8. The composition of a convex function with an affine function is a convex function.
9. If $f(x)$ is an affine function, then $g(x) = \|f(x)\|_2^2$ is a convex function.
10. If $f(x)$ is an affine function and $h(x)$ is a convex function, then $g(x) = \|f(x)\|_2^2 + \alpha h(x)$ is a convex function for $\alpha > 0$.

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Problema 1

1. $f(x) = |x|$ es convexa

$$f(\theta x + (1-\theta)y) = |\theta x + (1-\theta)y|$$
$$\leq |\theta x| + |(\theta+1)y| \rightarrow \text{propiedad de la distancia}$$
$$= \theta|x| + (1-\theta)|y| \rightarrow \text{propiedad homogeneidad}$$
$$= \theta f(x) + (1-\theta)f(y)$$

2. $f(x) = x^2$ con $x > 0$

$$f(\theta x^2 + (1-\theta)y^2) = (\theta x + (1-\theta)y)^2$$
$$\theta x^2 + (1-\theta)y^2 \geq \theta x^2 + 2\theta(1-\theta)xy + (1-\theta)^2 y^2$$
$$\theta x^2 + y^2 - \theta y^2 \geq \theta^2 x^2 + 2\theta(1-\theta)xy + y^2 - 2\theta \cancel{x}^2 + \theta^2 x^2$$
$$0 \geq (\theta^2 - \theta)x^2 + 2\theta(1-\theta)xy + (\theta^2-\theta)y^2$$
$$0 \geq (\theta^2-\theta)x^2 - 2(\theta^2-\theta)xy + (\theta^2-\theta)y^2$$
$$0 \geq (\theta^2-\theta)(x-y)^2$$

3. $f(x) = \exp(x)$

$$f(\theta x + (1-\theta)y) \leq \exp(\theta x) + \exp((1-\theta)y)$$
$$= \theta\exp(x) + (1-\theta)\exp(y)$$

4. $f(x)$ y $g(x)$ son convexas entonces $f(x)+g(x)$ es convexa

$$h(\theta x + (1-\theta)y) = f(\theta x + (1-\theta)y) + g(\theta x + (1-\theta)y)$$
$$\leq \theta(f(x) + g(x)) + (1-\theta)(f(y)+g(y))$$

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5. $\sum_{i=1}^{N} \alpha_i f_i(x) = F(x)$

$\alpha_i \geq 0$ Todos los coeficientes deben ser no negativos

$$F(\theta x + (1-\theta)y) \leq \sum_{i}^{N} \alpha_i f_i(\theta x + (1-\theta)y)$$
$$\leq \sum_{1} \alpha_i (\theta f_i(x) + (1-\theta) f_i(y))$$
$$= \theta F(x) + (1-\theta) F(y)$$

6. Sea $f_i(x)$ una familia de funciones convexa $f(x) = \max\{f_i(x)\}$

$$f_i(\theta x + (1-\theta)y) \leq \theta f_i(x) + (1-\theta) f_i(y)$$

buscando los maximos de ambos lados

$$\max\{f_i(\theta x + (1-\theta)y)\} \leq \max\{\theta f_i(x) + (1-\theta) f_i(y)\}$$
$$\max\{f_i(\theta x + (1-\theta)y)\} \leq \max\{\theta f_i(x)\} + \max\{(1-\theta) f_i(y)\}$$

se reescribe como

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

7. sea $h(x)$ la funcion que describe la norma y siguiendo reglas de la norma

$$h(\theta x + (1-\theta)y) \leq h(\theta x) + h((1-\theta)y)$$
$$h(\theta x + (1-\theta)y) \leq \theta h(x) + (1-\theta) h(y)$$

8. Sea $f(x) = Ax + b$ una funcion afin

$$f(\theta x + (1-\theta)y) = A(\theta x + (1-\theta)y) + b$$
$$\leq A\theta x + A(1-\theta)y + b + b\theta - b\theta$$
$$= \theta(Ax + b) + (1-\theta)A(y+b)$$
$$= \theta(Ax + b) + (1-\theta)(Ay + b)$$

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9. $g(x) = \|f(x)\|_2^2 = \|Ax + b\|_2^2$

$g(x\theta + (1-\theta)y) = \|A(x\theta + (1-\theta)y + b\|_2^2$

$\leq \|A\theta x + A(1-\theta)y + b\|_2^2$

$= \|\theta(A\theta x + b)\|_2^2 + \|(1-\theta)(Ay + b)\|_2^2$

10. para cualquier funcón es una norma si la funcion satisface

a. $f(\alpha x) = |\alpha| f(x), \forall \alpha \in \mathbb{R}$

b. $f(x+y) \leq f(x) + f(y)$

c. $f(x) \geq 0, \forall x, \; f(x) = 0 \Rightarrow x = 0$

Sabemos que la suma de funciones también es convexa

$g(\theta x + (1-\theta)y) = \|f(Ax\theta + (1-\theta)y)\|_2^2 + \alpha h(x\theta + (1-\theta)y)$

$\leq \|f(A\theta x + A(1-\theta)y + b)\|_2^2 + \alpha(h(x\theta) + h((1-\theta)y))$

$= \|\theta f(Ax + b) + (1-\theta) f(Ay + b)\|_2^2 + \alpha(\theta h(x) + (1-\theta)h(y))$