5. Theoretical: Show that the $D^4 f$ operator is given by:

$$D^4(x_j) \cong \frac{f(x_{j+2}) - 4f(x_{j+1}) + 6f(x_j) - 4f(x_{j-1}) + f(x_{j-2})}{h^4}$$

Considere los siguientes desarrollos de Taylor:

$$f(x_{j+2}) = \cancel{f(x_j)} \cancel{+2h f'(x_j)} + \cancel{\frac{4h^2}{2} f''(x_j)} + \cancel{\frac{8h^3}{6} f'''(x_j)} + \frac{16h^4}{24} f^{(4)}(x_j) + \cancel{\frac{32h^5}{120} f^{(5)}(x_j)} + O(h^6)$$

$$-4f(x_{j+1}) = \cancel{-4f(x_j)} - \cancel{4h f'(x_j)} - \cancel{\frac{4h^2}{2} f''(x_j)} - \cancel{\frac{4h^3}{6} f'''(x_j)} - \frac{4h^4}{24} f^{(4)}(x_j) - \cancel{\frac{4h^5}{120} f^{(5)}(x_j)} + O(h^6)$$

$$6f(x_j) = \cancel{6f(x_j)}$$

$$-4f(x_{j-1}) = \cancel{-4f(x_j)} + \cancel{4h f'(x_j)} - \cancel{\frac{4h^2}{2} f''(x_j)} + \cancel{\frac{4h^3}{6} f'''(x_j)} - \frac{4h^4}{24} f^{(4)}(x_j) + \cancel{\frac{4h^5}{120} f^{(5)}(x_j)} + \dots$$

$$f(x_{j-2}) = \cancel{f(x_j)} - \cancel{2h f'(x_j)} + \cancel{\frac{4h^2}{2} f''(x_j)} - \cancel{\frac{8h^3}{6} f'''(x_j)} + \frac{16h^4}{24} f^{(4)}(x_j) - \cancel{\frac{32}{120} f^{(5)}(x_j)} + \dots$$

Sumando, obtenemos que:

$$f(x_{j+2}) - 4f(x_{j+1}) + 6f(x_j) - 4f(x_{j-1}) + f(x_{j-2}) = h^4 f^{(4)}(x_j) + O(h^6)$$

$$\therefore f^{(4)}(x_j) = D^4 f(x_j) = \frac{f(x_{j+2}) - 4f(x_{j+1}) + 6f(x_j) - 4f(x_{j-1}) + f(x_{j-2})}{h^4} + O(h^2)$$

y la aproximación de este operador es del orden de $O(h^2)$.