Punto 8 - Integración

La regla de Simpson $\frac{3}{8}$ consiste en aproximar el integrando por un polinomio interpolador de orden 3.

b) Sea $\{0, h, 2h, 3h\}$ el conjunto de discretización. Como $h = \frac{b-a}{3}$, con a el punto inicial (0) y b el punto final $(3h)$. Se tienen que los puntos intermedios están dados por:

$$x_1 = a+h = a+\frac{b-a}{3} = \frac{3a}{3}+\frac{b-a}{3} = \frac{2a+b}{3}$$

$$x_2 = a+2h = a+\frac{2b-2a}{3} = \frac{3a}{3}+\frac{2b-2a}{3} = \frac{2b+a}{3}$$

a) Sea el conjunto soporte $\{(0, f(a)), (h, f(\frac{2a+b}{3})), (2h, f(\frac{a+2b}{3})), (3h, f(b))\}$.

Se tiene $f(x) \approx P_3(x) = f(a)L_1(x) + f(\frac{2a+b}{3})L_2(x) + f(\frac{a+2b}{3})L_3(x) + f(b)L_4(x)$

Con $L_1(x) = \dfrac{(x-h)(x-2h)(x-3h)}{(-h)(-2h)(-3h)} = \dfrac{(x-h)(x-2h)(x-3h)}{-6h^3} = \dfrac{x^3-6hx^2+11h^2x-6h^3}{-6h^3}$;

$$L_2(x) = \frac{x(x-2h)(x-3h)}{h(-h)(-2h)} = \frac{x(x-2h)(x-3h)}{2h^3} = \frac{x^3-5hx^2+6h^2x}{2h^3} ;$$

$$L_3(x) = \frac{x(x-h)(x-3h)}{2h\cdot h\cdot(-h)} = \frac{x(x-h)(x-3h)}{-2h^3} = \frac{x^3-4hx^2+3h^2x}{-2h^3} ;$$

$$L_4(x) = \frac{x(x-h)(x-2h)}{3h\cdot 2h\cdot h} = \frac{x(x-h)(x-2h)}{6h^3} = \frac{x^3-3hx^2+2h^2x}{6h^3} .$$

Así, $\displaystyle\int_a^b f(x)\,dx \cong \int_0^{3h} P_3(x)\,dx = \int_0^{3h} f(a)L_1(x)\,dx + \int_0^{3h} f\left(\frac{2a+b}{3}\right)L_2(x)\,dx + \int_0^{3h} f\left(\frac{a+2b}{3}\right)L_3(x)\,dx + \int_0^{3h} f(b)L_4(x)\,dx$

Pero $\displaystyle\int_0^{3h} f(a)L_1(x) = \frac{f(a)}{-6h^3}\left[\frac{x^4}{4} - 2hx^3 + \frac{11}{2}h^2x - 6h^3x\right]_{x=0}^{x=3h} = \frac{f(a)}{-6h^3}\cdot\left(\frac{-9h^4}{4}\right) = \frac{3}{8}hf(a)$;

$$\int_0^{3h} f\left(\frac{2a+b}{3}\right)L_2(x) = \frac{f\left(\frac{2a+b}{3}\right)}{2h^3}\left[\frac{x^4}{4} - \frac{5}{3}hx^3 + \frac{3}{2}h^2x^2\right]_{x=0}^{x=3h} = \frac{f\left(\frac{2a+b}{3}\right)}{2h^3}\cdot\left(\frac{9h^4}{4}\right) = \frac{9}{8}hf\left(\frac{2a+b}{3}\right);$$

$$\int_0^{3h} f\left(\frac{a+2b}{3}\right)L_3(x) = \frac{f\left(\frac{a+2b}{3}\right)}{-2h^3}\left[\frac{x^4}{4} - \frac{4}{3}hx^3 + \frac{3}{2}h^2x^2\right]_{x=0}^{x=3h} = \frac{f\left(\frac{a+2b}{3}\right)}{-2h^3}\cdot\left(\frac{-9h^4}{4}\right) = \frac{9}{8}hf\left(\frac{a+2b}{3}\right);$$

$$\int_0^{3h} f(b)L_4(x)\,dx = \frac{f(b)}{6h^3}\left[\frac{x^4}{4} - hx^3 + h^2x^2\right]_{x=0}^{x=3h} = \frac{f(b)}{6h^3}\cdot\left(\frac{9}{4}h^4\right) = \frac{3}{8}hf(b).$$

$$\therefore \int_a^b f(x)\,dx \approx \frac{3}{8}hf(a) + \frac{9}{8}hf\left(\frac{2a+b}{3}\right) + \frac{9}{8}hf\left(\frac{a+2b}{3}\right) + \frac{3}{8}hf(b)$$

$$= \frac{3h}{8}\left(f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right)$$