1. ARMA (1,2)

$$(1-\phi B) X_t = (1-\theta_1 B - \theta_2 B^2)\, \cancel{X}\, Z_t$$

$$\gamma_0 = \mathbb{E}[X_t^2] = \mathbb{E}\left[X_t \left[(1-\theta_1 B - \theta_2 B^2) Z_t + \phi X_{t-1}\right]\right]$$

$$= \phi \gamma_1 + \underbrace{\mathbb{E}[X_t Z_t]}_{\sigma_z^2} - \theta_1 \mathbb{E}[X_t Z_{t-1}] - \theta_2 \mathbb{E}[X_t Z_{t-2}]$$

$$= \phi \gamma_1 + \sigma_z^2 \left(1 - \theta_1(\phi - \theta_1) - \theta_2(-\theta_2 + \phi^2 - \phi\theta_1)\right)$$

$$\gamma_1 = \mathbb{E}[X_t X_{t-1}] = \mathbb{E}\left[X_{t-1} \left[(1-\theta_1 B - \theta_2 B^2) Z_t + \phi X_{t-1}\right]\right]$$

$$= \phi \gamma_0 + \underbrace{\mathbb{E}[Z_t X_{t-1}]}_{0} + \theta_1 \underbrace{\mathbb{E}[Z_{t-1} X_{t-1}]}_{\sigma_z^2} + \theta_2 \mathbb{E}[Z_{t-2} X_{t-1}]$$

$$= \phi \gamma_0 + \theta_1 \sigma_z^2 \left(\theta_1 + \theta_2(\phi - \theta_1)\right)$$

$k > 1$

$$\gamma_k = \mathbb{E}[X_t X_{t-k}] = \mathbb{E}\left[X_{t-k} \left[(1-\theta_1 B - \theta_2 B^2 Z_t + \phi X_{t-1}\right]\right]$$

$$= \phi \gamma_{k-1} + 0$$

$$\mathbb{E}[X_t Z_{t-1}] (= \mathbb{E}[X_{t-1} Z_{t-2}])$$

$$= \mathbb{E}\left[Z_{t-1} \left[(-\theta_1 B - \theta_2 B^2) Z_t + \phi X_{t-1}\right]\right]$$

$$= \sigma_z^2 \cdot (\phi - \theta_1)$$

$$\mathbb{E}[X_t Z_{t-2}] = \mathbb{E}\left[Z_{t-2} \left[-\theta_2 B^2 Z_t + \phi X_{t-1}\right]\right]$$

$$= \sigma_z^2 (-\theta_2) + \phi\, \mathbb{E}[X_t Z_{t-1}]$$

$$= \sigma_z^2 \left(-\theta_2 + \phi^2 - \phi\theta_1\right)$$

2,

a)

I $\quad 0.9 < 1, \quad 0.8 < 1 \Rightarrow$ invertible, estacionario

II $\quad 0.1 < 1, \quad 0.8 < 1 \Rightarrow$ invertible, estacionario

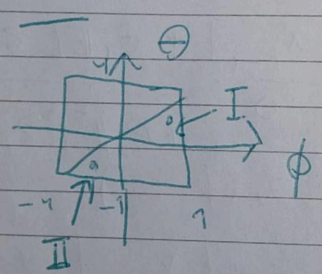
III $\quad 0.1 < 1 \Rightarrow$ estacionario

$x_{1,2}$: roots of $-0.7x^2 + 0.2x + 1$

$x_1 = 1.3 > 1$

$|x_2| = 1.06 > 1$

$\Rightarrow$ invertible



b) I. $\rho_0 = 1$

$$\rho_1 = 0.9 \cdot \frac{(1 + 0.9 \cdot 0.8)(0.9 + 0.8)}{1 - 2 \cdot 0.9 \cdot (-0.8) + 0.8^2} = \frac{731}{770} \approx 0.949\ldots$$

$\rho_2 = 0.854 \quad \rho_3 = 0.769 \quad \rho_4 = 0.692 \quad \rho_5 = 0.622$

II. $\rho_0 = 1$

$$\rho_1 = (-0.1) \cdot \frac{(1 - 0.1 \cdot 0.8)(-0.1 + 0.8)}{1 - 2 \cdot 0.08 + 0.8^2} = -0.435$$

$\rho_2 = -0.043 \quad \rho_3 = 4.35 \cdot 10^{-3} \quad \rho_4 = -4.35 \cdot 10^{-4} \quad \rho_5 = 4.3$[illegible]

III

$\cancel{X_t + 0.1 X_{t-1}}$ $\quad \phi_1 = -0.1$

$\theta_1 = -0.2 \quad \theta_2 = 0.7$

$a = \cancel{1 - \theta_1 \phi_1 + \theta_2 \cdot \phi_1^2}$

$= 1 - 0.02 + 0.2^2 - 0.1^2 \cdot 0.7 + 0.02 \cdot 0.7 + 0.7^2$

$= 1.517$

$b = -0.2 \cdot 0.7 + 0.2 + 0.1 \cdot 0.7 = +0.13$

$\rho_0 = 1 \qquad \rho_1 = \dfrac{\phi_1^0 b + \phi_1^1 a - \phi_1^{-1}\theta_2 + \theta_2 \phi_1^1}{\phi_1 b + a} = -46.3$

↑ not an autocorrelation

b) I

$\phi_{11} = \rho_1 = 0.949$

$\phi_{22} = \dfrac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = -0.4650$

$\phi_{33} = \dfrac{\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}} = \dfrac{\rho_3 + \rho_1 \rho_2^2 + 2\rho_1 \rho_2 - \rho_1^2 \rho_3}{1 + 2\rho_1^2 \rho_2 - \rho_2^2 - 2\rho_1^2} = 0.290$

II $\phi_{11} = \rho_1 = 0.357 \qquad \phi_{22} = \dfrac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = -0.187$

$\phi_{33} = \dfrac{\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}} = 0.102$

3.

II. • $\phi(B)X_t = \Theta(B)Z_t$

• $\frac{\phi(B)}{\Theta(B)}X_t = Z_t$

$\Leftrightarrow \pi(B) = \frac{\phi(B)}{\Theta(B)}$

$\Leftrightarrow \boxed{\cancel{\pi(B)\Theta(B) = \phi(B)}}$

$X_t = \frac{\Theta(B)}{\phi(B)} Z_t$

$\Leftrightarrow \boxed{\cancel{\psi(B)\phi(B) = \Theta(B)}}$

I.

$\pi(B)(1+0.8B) = (1-0.9B)$

$\Leftrightarrow$

$\cancel{(1-\pi_1 B - \pi_2 B^2)(1+0.8B) = 1-0.9B}$

$\cancel{1 + (0.8-\pi_1)B - \pi_1 0.8B^2}$

$(\pi_0 \mp \pi_1 B \mp \pi_2 B^2 \mp \pi_3 B^3 \dots)(1+0.8B) = \dots$

$\cancel{\pi_0(1+0.8B}$

$\pi_0 \mp \pi_1 B \mp \pi_2 B^2 \mp \pi_3 B^3 \dots + 0.8\pi_0 B \mp 0.8\pi_1 B^2 \mp 0.8\pi_2 B^3 \dots = \dots$

$\pi_0 + (-\pi_1 + 0.8\pi_0)B + (-\pi_2 \mp 0.8\pi_1)B^2 + (\pi_3 + 0.8)B^3 + \dots = 1-0.9B$

$\Leftrightarrow \pi_0 = 1$ $\quad -\pi_n \mp \pi_{n-1}\cdot 0.8 = 0$ for $n>1$

$-\pi_1 + 0.8 = -0.9$

$\Rightarrow \pi_1 = 1.7$, $\pi_n = -\pi_{n-1}\cdot 0.8$ for $n>1$

4.

$$(1-0.4B)\nabla Z_t = (1+0.5B-0.44B^2)\,a_t$$

a) Z_t:

$$(1-0.4B)(1-B)Z_t = 1+0.5B-0.44B^2\,a_t$$

$$(1-B-0.4B+0.4B^2)Z_t = \ldots$$

$$(1-1.4B+0.4B^2)Z_t = \ldots$$

Invertible:

· roots ~~dx~~ x_1, x_2 of $-0.44x^2+0.5x+1$

$|x_1| = 5 > 1$, $|x_2| = 1.4\ldots > 1$

prozess invertible ✓

Stationary:

roots x_1, x_2 of $0.4x^2-1.4x+1$

$x_1 = \frac{5}{2} > 1$, $x_2 = 1 \not> 1$

prozess is not stationary

b) ∇Z_t from a, we conclude invertibility

$\cancel{(1-0.4B)\, Z_t}$

$1-0.4x=0$

$x=\frac{1}{0.4}>1 \Rightarrow$ stationarity

c) $\nabla^2 Z_t$:

$$\nabla(1-0.4B)\nabla Z_t = \nabla(1+0.5B-0.14B^2)\, a_t$$

$$(1-0.4B)\nabla^2 Z_t = (1+0.5B-0.14B^2-B-0.5B^2+0.14B^3)\, a_t$$

From b, we conclude stationarity.

Invertibility:

Let x_1, x_2, x_3 be roots of : $0.14x^3-0.64x^2-0.5x+1$

$x_1=5>1$, $\underline{x_2=1}$ $x_3=-\frac{10}{7}$

$\Rightarrow$ non invertible

d) $\frac{1}{2} \nabla (1+B) Z_t = \frac{1}{2} \nabla Z_t + \frac{1}{2} \nabla Z_{t-1}$

$\frac{1}{2}(1+B)(1-0,4B) \nabla Z_t = \frac{1}{2}(1+B)(1+0,5B-0,14B^2)\, a_t$

$\Rightarrow$

$(1-0,4B)\left(\frac{1}{2}\nabla Z_t + \frac{1}{2}\nabla Z_{t-1}\right) = \frac{1}{2}\left(1+(0,5+1)B+(-0,14+0,5)B^2 - 0,14B^3\right) a_t$

- Z_t is stationary since $|-0,4| < 1$

- let x_1, x_2, x_3 be roots of $\frac{1}{2}(-0,14x^3 + 0,36x^2 + 1,5x + 1)$

$x_1 = 5$, $\underline{x_2 = -1}$, $x_3 = -\frac{10}{7}$

$\Rightarrow$ non invertible