Econometría Tarea I

1. <u>Def proceso estacionario.</u>

$\{X_t\}$ es un proceso estacionario $:\Leftrightarrow \Big[n\in\mathbb{N} \Rightarrow \big[T, t_1, \dots t_n \in \mathbb{N} \Rightarrow \mathbb{P}(t_1, \dots t_n) = \mathbb{P}(t_{1+T}, \dots, t_{n+T}) \big] \Big]$

<u>Def caminata aleatoria simple:</u>

$\{S_t\}$ es una caminata aleatoria simple

$:\Leftrightarrow$

El ruido blanco $\{X_t\}$ con media 0 existe, de modo que

$S_t = X_1 + \dots + X_t$

S_t no es necesario estacionario. si $VAR(X_t) > 0$ S_t no es estacionario.

<u>Prueba:</u>

$\cancel{\mathbb{P}(S_2 = a) = \mathbb{P}(X_1 + X_2 = a) \overset{indep}{=} \mathbb{P}(X_1 = a-b)\,\mathbb{P}(X_2 = b)}$

$\cancel{\text{assumir } \mathbb{P}(X_2 = b) > 0 \quad = \mathbb{P}(X_1 + X_2 = a \mid X_2 = b)\,\mathbb{P}(X_2 = b)}$

$\cancel{\overset{indep.}{=} \mathbb{P}(X_1 = a-b)\,\mathbb{P}(X_2 = b)}$

$\cancel{= \mathbb{P}(S_1 = a-b)\,\mathbb{P}(S_1 = b)}$

$VAR(S_1) = \sigma_x^2 \qquad VAR(S_2) \overset{indep}{=} \sigma_x^2 + \sigma_x^2$

Que eso $\sigma_x^2 \neq 0$, $VAR(S_2) \neq VAR(S_1)$.

Entonces $\mathbb{P}(S_1) \neq \mathbb{P}(S_{1+1})$

Sea $\{X_t\}$ un processo $MA(1)$ con el ruido blanco $\{Z_t\}$.
Sea $\sigma_z > 0$.

2. a) verificar $\gamma(0) > 0$.

$$\gamma(0) = VAR(X_t) = \underbrace{\sigma_z^2}_{>0} + \underbrace{\theta_1^2 \sigma_z^2}_{>0} > 0$$

b) sea $|k| \geq 2$

$$\gamma(k) = COV(X_t, X_{t+k})$$

$$= \mathbb{E}[Z_t Z_{t+k} - \theta Z_{t-1} Z_{t+k} - \theta Z_t Z_{t+k-1} + \theta^2 Z_{t-1} Z_{t+k-1}]$$

$t \neq t+k$, $t-1 \neq t+k$, $t \neq t+k-1$, $t-1 \neq t+k-1$ $= 0$

3. Sea $\{Z_t\}$ el ruido blanco de $\{\tilde{X}_t\}$

$$\mathbb{E}(\tilde{X}_t) = \mathbb{E}[Z_t] - \underbrace{\mathbb{E}[\theta_1 Z_{t-1}]}_{0} + \underbrace{\mathbb{E}[\theta_2 Z_{t-2}]}_{0} = 0$$

$$Var(\tilde{X}_t) = Var(Z_t - \theta_1 Z_{t-1} - \theta_2 Z_{t-2})$$

$$= \sigma_z^2 + \theta_1^2 \sigma_z^2 + \theta_2^2 \sigma_z^2 = \sigma_z^2(1 + \theta_1^2 + \theta_2^2)$$

$i \neq j \Rightarrow cov(Z_i, Z_j) = 0$

4. $\gamma(0) = \sigma_z^2(1 + \theta_1^2 + \theta_2^2)$

- $\gamma(1) = \mathbb{E}((Z_t - \theta_1 Z_{t-1} - \theta_2 Z_{t-2})(Z_{t-1} - \theta_1 Z_{t-2} - \theta_2 Z_{t-3}))$

$$= \mathbb{E}(-\theta_1 Z_{t-1}^2 + \theta_2\theta_1 Z_{t-2}^2) = \sigma_z^2(\theta_1\theta_2 - \theta_1)$$

$$\gamma(2) = \mathbb{E}[-Z_{t-2}\,\theta_2\,Z_{t-2}] = -\theta_2\,\sigma_Z^2$$

$|k| \geq 2$

$$\gamma(k) = \mathbb{E}[0] = 0$$

b.

$$\gamma(0) = VAR(Z_t - \theta_1 Z_{t-1}) = \sigma_Z^2(1+\theta_1^2)$$

$$\gamma(1) = \mathbb{E}[(Z_t - \theta_1 Z_{t-1})(Z_{t-1} - \theta_1 Z_{t-2})]$$

$$= -\sigma_Z^2\,\theta_1$$

$$\rho(1) = \frac{-\sigma_Z^2\,\theta_1}{\sigma_Z^2(1+\theta_1^2)} = -\frac{\theta_1}{1+\theta_1^2}$$

$$\left[\forall_{\theta_1\in\mathbb{R}} \; -\frac{\theta_1}{1+\theta_1^2} \geq -0{,}5\right] \Rightarrow -0{,}5 \leq \rho(1) \leq 0{,}5$$

$\uparrow$

~~porque~~ $\cancel{\rho(1)(\theta_1) = \rho(1)(-\theta}$

porque $\rho_{\theta_1}(1) = -\rho_{-\theta_1}(1)$

minimizar $\frac{x}{1+x^2}$

$$0 = \frac{d}{dx}\frac{x}{1+x^2} = \frac{1}{1+x^2} + \frac{-2x^2}{(1+x^2)^2} =$$

$$= \frac{(1+x^2-2x^2)}{1+x^2} = \frac{1-x^2}{1+x^2}$$

$(\Leftrightarrow)$

$x \in \{-1, 1\}$

$$\frac{-1}{1+(-1)^2} = -0{,}5 \geq -0{,}5$$

$$\frac{1}{1+1^2} = 0{,}5 \geq -0{,}5$$

6.

$$\rho(1) = \frac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2}$$

$$\rho(2) = \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}$$

se puede encontrar los maximaens de estas expreciones, como functiones dependente de θ_1 y θ_2, pero no tengo mucho tiempo hasta las 12.

$\rho(1)$ ~~esta~~ y $\rho(2)$ son encarados

$$\frac{\partial}{\partial \theta_1} \rho(1) = -(1-\theta_2)\left[\frac{1}{1+\theta_1^2+\theta_2^2} + \frac{\theta_1}{2\theta_1}\right]$$

$$= -(1-\theta_2)\left(\frac{1}{1+\theta_1^2+\theta_2^2} + \frac{1}{2}\right)$$

7.

Sea $\{X_t\} \sim MA(q)$ con el ruido blanco Z_t.

Sea $n \in \mathbb{N}$ y $\tau, t_1, \dots t_n \in \mathbb{N}$.

Sea $a = \min\{t_1, \dots t_n\} - q$, $b = \max\{t_1, \dots, t_n\}$.

Sea $m = 1 + b - a$ (queremos que $m = |\{a, a+1, \dots, b\}|$).

Sea $\hat{X}_i := X_{t_i}$, y $A_i := \{t_i, t_i - 1, \dots t_i - q\}$

sea $\hat{X}_{i+\tau} := X_{t_i + \tau}$

Sea $\mathbb{1}(True) = 1$ y $\mathbb{1}(False) = 0$ (por exemplo $\mathbb{1}(1+1=2) = 1$ y $\mathbb{1}(a \in \emptyset \Rightarrow a \supseteq \emptyset) = 1$)

Consideramos la distribution de $(\hat{X}_1, \dots, \hat{X}_n)$:

$$\mathbb{P}(\hat{X}_1 = y_1, \dots \hat{X}_n = y_n) = \int_\Omega \mathbb{P}(Z_a = \tilde{z}_a)\, \mathbb{P}(\hat{X}_1 = y_1, \dots, \hat{X}_n = y_n \mid Z_a = \tilde{z}_a)\, d\tilde{z}_a$$

$$= \underbrace{\int_\Omega \int_\Omega \cdots \int_\Omega}_{m\text{-veces}} \mathbb{P}(Z_a = \tilde{z}_a)\, \mathbb{P}(Z_{a+1} = \tilde{z}_{a+1} \mid Z_a = \tilde{z}_a) \cdots \mathbb{P}(Z_b = \tilde{z}_b \mid Z_a = \tilde{z}_a, Z_{a+1} = \tilde{z}_{a+1}, \dots Z_{b-1} = \tilde{z}_{b-1})$$

$$\mathbb{1}\Big(\sum_{j \in A_1} \tilde{z}_j = y_1, \sum_{j \in A_2} \tilde{z}_j = y_2, \dots, \sum_{j \in A_n} \tilde{z}_j = y_n\Big)\, d\tilde{z}_a\, d\tilde{z}_{a+1} \cdots d\tilde{z}_b$$

$$\overset{Z_i \text{ son independientes}}{=} \int_\Omega \cdots \int_\Omega \mathbb{P}(Z_a = \tilde{z}_a) \cdots \mathbb{P}(Z_b = \tilde{z}_b)\, \mathbb{1}\Big(\sum_{j \in A_1} \tilde{z}_j = y_1, \dots, \sum_{j \in A_n} \tilde{z}_j = y_n\Big)\, d\tilde{z}_a \cdots d\tilde{z}_b$$

$$\overset{Z_i \text{ son de la misma distribution}}{=} \int_\Omega \cdots \int_\Omega \mathbb{P}(Z_{a+\tau} = \tilde{z}_a) \cdots \mathbb{P}(Z_{b+\tau} = \tilde{z}_b)\, \mathbb{1}\Big(\sum_{j \in A_1} \tilde{z}_j = y_1, \dots, \sum_{j \in A_n} \tilde{z}_j = y_n\Big)\, d\tilde{z}_a \cdots d\tilde{z}_b$$

$$= \mathbb{P}(\hat{X}_{1+\tau} = y_1, \dots, \hat{X}_{n+\tau} = y_n) \Rightarrow \{X_t\} \text{ es estacionario.}$$

~~Find Z s.t $Z \neq \emptyset \wedge z \in Z \Rightarrow z \subsetneq Z$~~

~~$Z = \{\{\emptyset\}\}$ $\{\{Z, 1\}\}$~~

7. cont.   assume $|k| \leq q$ $\theta_0 := -1$

$$\gamma(k) = \mathbb{E}(X_t X_{t-k})$$

$$= \mathbb{E}\left[\left(-\sum_{i=1}^{q} Z_{t-i}\,\theta_i\right)\left(-\sum_{i=1}^{q} Z_{t-i-k}\,\theta_i\right)\right]$$

$$= \sum_{i=k}^{q} \mathbb{E}\left[Z_{t-i}^2\,\theta_i\,\theta_{i+k}\right] = \sum_{i=k}^{q} \sigma_Z^2\,\theta_i\,\theta_{i+k}$$

$$= \sigma_Z^2 \sum_{i=k}^{q} \theta_i\,\theta_{i+k}$$

8. 1)

8. a) Todos los procesos AR son invertibles, e.g. i, ii, iii, iv, x

Todos los procesos MR son estacionarios, e.g. v, ..., ix

Todos los procesos AR son estacionario $\Leftrightarrow$ todos los raíces x_i son $|x_i| \gtrless 1$.

Todos los procesos MR son ~~...~~ invertibles $\Leftrightarrow$ todos los raíces x_i son $|x_i| \gtrless 1$

b) Hay una formula por los autocorrelaciones de AR y hay una formula por los autocorrelaciones de MR.

c) ~~$X_t = c + z_t - \phi_1 X_{t-1} \cdots - \phi_q X_{t-q}$~~

i) $\{X_t\}$ es estacionario porque $|0.9| < 1$

$\Rightarrow \mathbb{E}[X_t] = \mathbb{E}[X_{t+1}]$.

$\Rightarrow \mu - 0.9\mu = 4$

$\Rightarrow \mu = \frac{4}{1-0.9} = 40$

iii)

$|-0.1| < 1$, $0.9 - 0.1 < 1$
$-0.1 - (-0.9) < 1$

$\Rightarrow$ estacionario

$\Rightarrow \mu - 0.9\mu + 0.1\mu = 10$

$\Rightarrow \mu = \frac{10}{0.2} = 50$

iv)

$|0.9| < 1$, $-0.1 + 0.9 < 1$, $0.9 - (-0.1) = 1$

$\Rightarrow$ no estacionario

~~$\Rightarrow$ solution~~ ~~$\mathbb{E}[X_t$ not well defined for X_t~~

$\Rightarrow \mathbb{E}[X_t]$ is not well definded if $\mathbb{E}[X_{t-1}]$ is not well deffined, for all $t \in \mathbb{Z}$, as $\mathbb{E}[X_t]$ depends on $\mathbb{E}[X_{t-1}]$. No $\mathbb{E}[X_{t\pm1}]$ is deffined for not t.

$\Rightarrow \mu$ not well defined

9.

i $\cancel{X_t - 0.9 X_{t-1} = 4 + Z_t}$

$\cancel{(-1)}$

$\cancel{X_t = 4 + Z_t}$

ii $X_t = Z_t - 0.1 X_{t-1}$

$= Z_t - 0.1 (Z_{t-1} - 0.1 X_{t-2})$

$= Z_t - 0.1 Z_{t-1} + 0.1^2 (Z_{t-2} - 0.1 X_{t-3})$

$\overset{\forall k \in \mathbb{N}}{=} Z_t + \left[\sum_{i=1}^{k} (-0.1)^i Z_{t-i} \right] + (-0.1)^{k+1} X_{t-k-1}$

$\xrightarrow{k \to \infty} \cancel{Z_t} \sum_{i=0}^{\infty} (-0.1)^i Z_{t-i}$

$\Rightarrow$ $\psi_0 = 1$

$\psi_1 = -0.1$

$\psi_2 = \cancel{0.0}\, 10^{-2}$

$\psi_3 = -10^{-3}$

$\psi_4 = 10^{-4}$

$\psi_5 = -10^{-5}$

10.

$$Z_t = X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \phi_3 X_{t-2}$$

Yule - Walker Equations para $m>0$

$$\gamma_m = \sum_{k=1}^{3} \phi_k \gamma_{m-k}$$

$$\Rightarrow \quad \gamma_k = \gamma_{-k}$$

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

$$\overset{/\gamma_0}{\Leftrightarrow}$$

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

$\rho_1 = 0.2$, $\rho_2 = 0.5$ $\rho_3 = 0.7$

$\Leftrightarrow$

$\cancel{0.2 = \phi_1 +}$

$$\begin{pmatrix} 0.2 \\ 0.5 \\ 0.7 \end{pmatrix} = \begin{pmatrix} 1 & 0.2 & 0.5 \\ 0.2 & 1 & 0.2 \\ 0.5 & 0.2 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

Solve la equation: $\phi_1 = \cancel{\frac{[illegible]}{355}} = -\frac{18}{71}$, $\phi_2 = \cancel{\frac{[illegible]}{142}} = \frac{57}{142}$, $\phi_3 = \cancel{\frac{233}{355}} \; \frac{53}{71}$

$1 - \frac{48}{345} B - \frac{[illegible]}{[illegible]} B + \frac{[illegible]}{355} B$

Raisines

Polynom characteristico

$$1 + \frac{18}{71} x - \frac{57}{142} x^2 - \frac{53}{71} x^3$$

Raisines

$x_1 \approx 0.865 \ldots$

$x_2 \approx 0.700\ldots + 1.029\ldots i$

$x_3 \approx 0.700\ldots + 1.029\ldots i$

$\Rightarrow$ todos los raisinene x_i son $|x_i| \leq 1$

$\Rightarrow$ processo no es estacionario.

$\Rightarrow$ no se pueden aplicar los yules walker equaciones