1. Determine the intersection of the hyperboloid

$$\mathcal{H}^1_{4,3,1}: \frac{x^2}{16}+\frac{y^2}{9}-\frac{z^2}{1}=1 \quad \text{with the line} \quad \ell=\begin{bmatrix}4\\-2\\1\end{bmatrix}+\langle\begin{bmatrix}4\\0\\1\end{bmatrix}\rangle.$$

Write down the equations of the tangent planes in the intersection points.

2. Determine the tangent plane of the hyperboloid

$$\mathcal{H}^1_{2,3,1}: \frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{1}=1$$

in the point $M(2,3,1)$. Show that the tangent plane intersects the surface in two lines.

3. Determine the generators of the hyperboloid

$$\frac{x^2}{36}+\frac{y^2}{9}-\frac{x^2}{4}=1$$

which are parallel to the plane $x+y+z=0$.

4. Determine the intersection of the hyperboloid

$$\mathcal{H}^2_{2,1,3}: \frac{x^2}{4}+\frac{y^2}{1}-\frac{z^2}{9}=-1 \quad \text{with the line} \quad \ell=\begin{bmatrix}3\\1\\6\end{bmatrix}+\langle\begin{bmatrix}1\\1\\3\end{bmatrix}\rangle.$$

Write down the equations of the tangent planes in the intersection points.

5. Determine the intersection of the paraboloid

$$\mathcal{P}^h_{2,\frac{1}{2}}: x^2-4y^2=4z \quad \text{with the line} \quad \ell=\begin{bmatrix}2\\0\\3\end{bmatrix}+\langle\begin{bmatrix}2\\1\\-2\end{bmatrix}\rangle.$$

Write down the equations of the tangent planes in the intersection points.

6. Determine the tangent plane of

1. the elliptic paraboloid $\frac{x^2}{5}+\frac{y^2}{3}=z$ and of
2. the hyperbolic paraboloid $x^2-\frac{y^2}{4}=z$

which are parallel to the plane $x-3y+2z-1=0$.

7. Determine the plane which contains the line

$$\begin{bmatrix}-1\\0\\0\end{bmatrix}+\langle\begin{bmatrix}2\\-1\\0\end{bmatrix}\rangle \quad \text{and is tangent to the quadric} \quad x^2+2y^2-z^2+1=0.$$

8. Show that the parabolid $\mathcal{P}^e_{p,p}$ is the locus of points for which the distance from a point equals the distance to a plane. Such a surface is called *elliptic paraboloid of revolution*.

9. Use a parametrization of a parabola and a rotation matrix to deduce a parametrization of an elliptic paraboloid of revolution.

10. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S}: \begin{cases} x = \sqrt{1+t^2}\cos(s) \\ y = \sqrt{1+t^2}\sin(s) \\ z = 2t \end{cases}$$

- Give the equation of $\mathcal{S}$.
- Find the parameters of the point $P(1,1,2)$.
- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give the equation of $T_P\mathcal{S}$.

11. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S}: \begin{cases} x = s \\ y = t \\ z = s^2 - t^2 \end{cases}$$

- Give the equation of $\mathcal{S}$.
- Find the parameters of the point $P(1,1,0)$.
- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give the equation of $T_P\mathcal{S}$.

12. Determine the generators of the paraboloid

$$4x^2 - 9y^2 = 36z$$

containing the point $P(3\sqrt{2},2,1)$.

13. Determine the generators of the paraboloid

$$\frac{x^2}{16} - \frac{y^2}{4} = z$$

which are parallel to the plane $3x + 2y - 4z = 0$.

14. Which of the following is a hyperboliod?

1. $\mathcal{S}: 2xz + 2xy + 2yz = 1$
2. $\mathcal{S}: 5x^2 + 3y^2 + xz = 1$
3. $\mathcal{S}: 2xy + 2yz + y + z = 2$

1. Determine the intersection of the hyperboloid

$$\mathcal{H}^1_{4,3,1} : \frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{1} = 1 \quad \text{with the line} \quad \ell = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + \left\langle \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \right\rangle.$$

Write down the equations of the tangent planes in the intersection points.

$$\mathcal{H} \cap \ell : \quad \frac{(4+4t)^2}{16} + \frac{(-2)^2}{9} - (1+t)^2 = 0$$

$$\Leftrightarrow \quad \frac{4}{9} = 1$$

so there is no solution, i.e no intersection point

$\Rightarrow$ no tangent planes to write down

2. Determine the tangent plane of the hyperboloid

$$\mathcal{H}^1_{2,3,1} : \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$$

in the point $M(2,3,1)$. Show that the tangent plane intersects the surface in two lines.

$$T_M \mathcal{H}^1 : \quad \frac{x}{2} + \frac{y}{3} - z = 1$$

$$T_M \mathcal{H}^1 \cap \mathcal{H}^1 : \quad \begin{cases} \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1 \\ \frac{x}{2} + \frac{y}{3} - z = 1 \end{cases}$$

so $z = \frac{x}{2} + \frac{y}{3} - 1$

$$\Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{9} - \left(\frac{x}{2} + \frac{y}{3} - 1\right)^2 = 1$$

$$\frac{x^2}{4} + \frac{y^2}{9} + 1 + \frac{xy}{3} - x - \frac{2y}{3}$$

$$\Rightarrow \quad xy - 3x - 2y - 6 = 0$$

$\Rightarrow \quad (x-2)(y-3)=0$

So $T_m \mathcal{H}' \cap \mathcal{H}' : \begin{cases} \frac{x}{2}+\frac{y}{3}-z=1 \\ (x-2)(y-3)=0 \end{cases}$

the solutions to this system is the union of solutions to the two systems

$$l_1 : \begin{cases} \frac{x}{2}+\frac{y}{3}-z=1 \\ x-2=0 \end{cases} \quad \text{and} \quad l_2 : \begin{cases} \frac{x}{2}+\frac{y}{3}-z=1 \\ y-3=0 \end{cases}$$

i.e. $T_m \mathcal{H}' \cap \mathcal{H}'$ is the union of the two lines l_1 and l_2

3. Determine the generators of the hyperboloid

$$\frac{x^2}{36}+\frac{y^2}{9}-\frac{x^2}{4}=1$$

which are parallel to the plane $x+y+z=0$.

$$\mathcal{H} : \left(\frac{x}{6}-\frac{z}{2}\right)\left(\frac{x}{6}+\frac{z}{2}\right)=\left(1-\frac{y}{3}\right)\left(1+\frac{y}{3}\right)$$

One family of generators is $\quad l_\alpha : \begin{cases} \frac{x}{6}-\frac{z}{2}=\alpha\left(1-\frac{y}{3}\right) \\ \alpha\left(\frac{x}{6}+\frac{z}{2}\right)=1+\frac{y}{3} \end{cases} \quad \alpha \in \mathbb{R}$

$$\Leftrightarrow \begin{cases} x-3z=6\alpha-2\alpha y \\ \alpha x+3\alpha z=6+2y \end{cases}$$

$$\begin{pmatrix} 1 & 2\alpha & -3 & 6\alpha \\ \alpha & -2 & 3\alpha & 6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2\alpha & -3 & 6\alpha \\ 0 & -2-2\alpha^2 & 6\alpha & 6-6\alpha^2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{-3(1-\alpha^2)}{1+\alpha^2} & \frac{8\alpha}{1+\alpha^2} \\ 0 & 1 & \frac{-3\alpha}{1+\alpha^2} & -\frac{3(1-\alpha^2)}{1+\alpha^2} \end{pmatrix}$$

you can also calculate a direction vector with the vector product.

$$\Rightarrow \ell_\alpha : \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \frac{1}{1+\alpha^2} \begin{bmatrix} 8\alpha \\ -3(1-\alpha^2) \\ 0 \end{bmatrix} + z \underbrace{\begin{bmatrix} \frac{3(1-\alpha^2)}{1+\alpha^2} \\ \frac{3\alpha}{1+\alpha^2} \\ 1 \end{bmatrix}}_{=v}$$

a direction vector

$$\ell_\alpha \parallel \pi \Leftrightarrow v \parallel \pi$$

$$\Leftrightarrow \frac{3(1-\alpha^2)}{1+\alpha^2} + \frac{3\alpha}{1+\alpha^2} + 1 = 0$$

$$\Leftrightarrow -2\alpha^2 + 3\alpha + 4 = 0$$

$$\Delta = 41 \Rightarrow \alpha_{1,2} = \frac{3 \pm \sqrt{41}}{4}$$

In this family of generators there are two lines parallel to π

$$\ell_{\alpha_1} : \{ \quad \dots \qquad \ell_{\alpha_2} : \{ \quad \dots$$

The second family of generators is $\tilde{\ell}_\alpha$: $\begin{cases} \frac{x}{6} - \frac{z}{2} = \alpha\left(1 + \frac{y}{3}\right) \\ \alpha\left(\frac{x}{6} + \frac{z}{2}\right) = 1 - \frac{y}{3} \end{cases}$

a direction vector for $\tilde{\ell}_\alpha$ is

$$w = \begin{vmatrix} i & j & k \\ \frac{1}{6} & -\frac{\alpha}{3} & -\frac{1}{2} \\ \frac{\alpha}{6} & \frac{1}{3} & \frac{\alpha}{2} \end{vmatrix} = i\left(\frac{-\alpha^2+1}{6}\right) - j\left(\frac{\alpha}{12} + \frac{\alpha}{12}\right) + k\left(\frac{1}{18} + \frac{\alpha^2}{18}\right)$$

$$\tilde{\ell}_\alpha \parallel \pi \Leftrightarrow w \parallel \pi$$

$$\Leftrightarrow \frac{-\alpha^2+1}{6} - \frac{\alpha}{6} + \frac{1+\alpha^2}{18} = 0$$

$$\Leftrightarrow \quad -3\alpha^2+3-3\alpha+1+\alpha^2=0$$

$$\Leftrightarrow \quad -2\alpha^2-3\alpha+4=0$$

$$\Delta = 41 \quad \Rightarrow \alpha_{1,2} = \frac{-3 \pm \sqrt{41}}{4}$$

As expected we obtain two generators also in the family $\tilde{\ell}_\alpha$

$\tilde{\ell}_{\alpha_1} : \{ \ldots$ and $\tilde{\ell}_{\alpha_2} : \{ \ldots$

4. Determine the intersection of the hyperboloid

$$\mathcal{H}^2_{2,1,3} : \frac{x^2}{4} + \frac{y^2}{1} - \frac{z^2}{9} = -1 \quad \text{with the line} \quad \ell = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} + \left\langle \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\rangle.$$

Write down the equations of the tangent planes in the intersection points.

- $\mathcal{H}^2 \cap \ell$: $\dfrac{(3+t)^2}{4} + (1+t)^2 - \dfrac{3^2(2+t)^2}{9} = -1$

$$\Leftrightarrow \quad (t-1)^2=0$$

so we obtain a double solution. This means that ℓ intersects $\mathcal{H}^2$ in a double point, i.e ℓ is tangent to $\mathcal{H}^2$

- The intersection point is $p = \begin{pmatrix} 4 \\ 2 \\ 9 \end{pmatrix}$ and the corresponding tangent plane is

$$T_p\mathcal{H}^2 : x+2y-z=-1$$

5. Determine the intersection of the paraboloid

$$\mathcal{P}^h_{2,\frac{1}{2}} : x^2 - 4y^2 = 4z \quad \text{with the line} \quad \ell = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \langle \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

• $\mathcal{P}^h \cap \ell$: $\quad (2+2t)^2 - 4t^2 = 4(3-2t)$

$\Leftrightarrow t = \frac{1}{2}$

So we have a single solution which corresponds to a simple point of intersection. The line punctures the surface in the point

$$P = \begin{bmatrix} 3 \\ 1/2 \\ 2 \end{bmatrix}$$

• $\mathcal{P}^h$: $\frac{x^2}{2} - \frac{y^2}{1/2} = 2z \Rightarrow T_P\mathcal{P}^h : \frac{3x}{2} - \frac{\frac{1}{2}y}{1/2} = z + 2$

$\Rightarrow T_P\mathcal{P}^h : \frac{3}{2}x - y - z - 2 = 0$

6. Determine the tangent plane of

1. the elliptic paraboloid $\frac{x^2}{5} + \frac{y^2}{3} = z$ and of
2. the hyperbolic paraboloid $x^2 - \frac{y^2}{4} = z$

which are parallel to the plane $x - 3y + 2z - 1 = 0$.

1) $\mathcal{P}^e : \frac{x^2}{5} + \frac{y^2}{3} = z \qquad \pi : x - 3y + 2z - 1 = 0$

$p = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \Rightarrow T_p\mathcal{P}^e : \frac{xx_0}{5} + \frac{yy_0}{3} = \frac{z + z_0}{2}$

$T_p\mathcal{P}^e \parallel \pi \Rightarrow \frac{x_0}{5} = \frac{y_0}{-9} = \frac{-1}{4}$

$\Rightarrow x_0 = -\frac{5}{4}, \quad y_0 = \frac{9}{4}$

since $p \in \mathcal{P}^e$ we also have

$$\frac{\left(\frac{-5}{4}\right)^2}{5} + \frac{\left(\frac{9}{4}\right)^2}{3} = z_0 \Rightarrow z_0 = 2$$

$\Rightarrow p = \begin{bmatrix} -5/4 \\ 9/4 \\ 2 \end{bmatrix}$ is the point where $T_p \mathcal{P}^e \parallel \pi$

$$T_p \mathcal{P}^e: -\frac{x}{4} + \frac{3y}{4} = \frac{z+2}{2}$$

$$\Leftrightarrow \quad x - 3y + 2z + 4 = 0$$

2.) $T_p \mathcal{P}^h: \quad x x_0 - \frac{y y_0}{4} = \frac{z + z_0}{2}$

$T_p \mathcal{P}^h \parallel \pi \Rightarrow \quad x_0 = -\frac{1}{4}, \ y_0 = -3$

since $p \in \mathcal{P}^h \Rightarrow \quad z_0 = -\frac{1}{2}$

$\Rightarrow \quad p = \begin{bmatrix} -1/4 \\ -3 \\ -1/2 \end{bmatrix}$ and $T_p \mathcal{P}^h: x - 3y + 2z + 1 = 0$

7. Determine the plane which contains the line

$$\ell: \underset{Q}{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}} + \langle \underset{v}{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}} \rangle$$ and is tangent to the quadric $x^2 + 2y^2 - z^2 + 1 = 0.$

for $p = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ $\quad T_p S: \quad x_0 x + 2y_0 y - z z_0 + 1 = 0$

$n(x_0, 2y_0, -z_0)$ is a normal vector

$\ell \subseteq \pi \Rightarrow n \perp v \Rightarrow n \cdot v = 0 \quad 2x_0 - 2y_0 = 0 \Rightarrow x_0 = y_0$

$\ell \subseteq \pi \Rightarrow Q \in T_p S \Rightarrow -x_0 + 1 = 0 \Rightarrow x_0 = 1$

$\Rightarrow x_0 = y_0 = 1$

so $T_p S: \quad x + 2y - z z_0 + 1 = 0$

$p \in S \Rightarrow \quad 1 + 2 - z_0^2 + 1 = 0 \quad \Rightarrow z_0 = \pm 2$

$\Rightarrow p = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

8. Show that the parabolid $\mathcal{P}^e_{p,p}$ is the locus of points for which the distance from a point equals the distance to a plane. Such a surface is called *elliptic paraboloid of revolution*.

Similar to problem 10 from last week

9. Use a parametrization of a parabola and a rotation matrix to deduce a parametrization of an elliptic paraboloid of revolution.

Similar to problem 11 from last week

10. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S}: \begin{cases} x = \sqrt{1+t^2}\cos(s) \\ y = \sqrt{1+t^2}\sin(s) \\ z = 2t \end{cases}$$

- Give the equation of $\mathcal{S}$.
- Find the parameters of the point $P(1,1,2)$.
- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give the equation of $T_P\mathcal{S}$.

- S is an elliptic paraboloid with equation

$$x^2 + y^2 = \frac{z}{2}$$

- The parameters of the point $P = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ are $t = 1$, $s = \frac{\pi}{4}$

- $T_P S = P + \langle \frac{\partial \sigma}{\partial s}(P), \frac{\partial \sigma}{\partial t}(P) \rangle$

$$\frac{\partial \sigma}{\partial s}(P) = \begin{bmatrix} -\sqrt{1+t^2}\sin(s) \\ \sqrt{1+t^2}\cos(s) \\ 0 \end{bmatrix}(P) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{\partial \sigma}{\partial t}(P) = \begin{bmatrix} \frac{t}{\sqrt{1+t^2}}\cos(s) \\ \frac{t}{\sqrt{1+t^2}}\sin(s) \\ 2 \end{bmatrix}(P) = \begin{bmatrix} 1/2 \\ 1/2 \\ 2 \end{bmatrix}$$

$$T_P S: \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1/2 \\ 1/2 \\ 2 \end{bmatrix} \quad \alpha, \beta \in \mathbb{R}$$

- $T_P S: \frac{x x_0}{a} + \frac{y y_0}{b} = z + z_0 \Rightarrow 4x + 4y - z - 2 = 0$