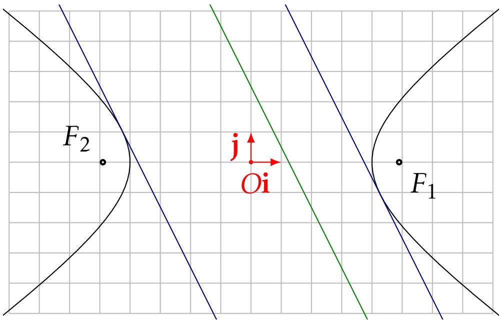
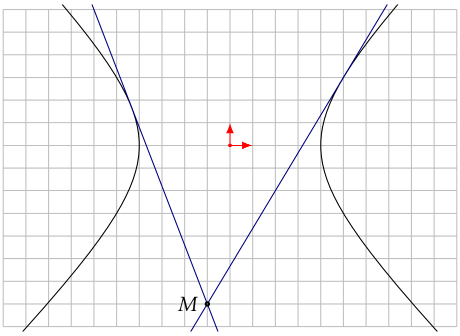
1. Determine the intersection points between the line $\ell : 2x - y - 10 = 0$ and the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$.

2. Determine the tangents to the hyperbola $\mathcal{H} : \frac{x^2}{16} - \frac{y^2}{8} - 1 = 0$ which are parallel to the line $\ell : 4x + 2y - 5 = 0$.



3. Determine the tangents to the hyperbola $\mathcal{H} : x^2 - y^2 = 16$ which contain the point $M(-1,7)$.



4. Find the area of the triangle determined by the asymptotes of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the line $\ell : 9x + 2y - 24 = 0$.

5. Find an equation for the tangent lines to:

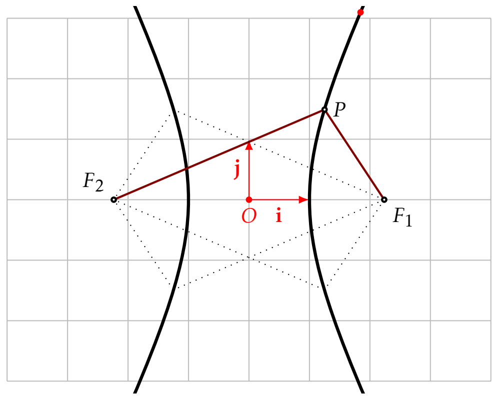
1. the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$, orthogonal to the line $\ell : 4x + 3y - 7 = 0$;
2. the parabola $\mathcal{P} : y^2 - 8x = 0$, parallel to $\ell : 2x + 2y - 3 = 0$.

6. Find an equation for the tangent lines to:

1. the hyperbola $\mathcal{H} : \frac{x^2}{3} - \frac{y^2}{5} - 1 = 0$, passing through $P(1, -5)$;
2. the paraola $\mathcal{P} : y^2 - 36x = 0$, passing through $P(2, 9)$.

7. Consider the hyperbola $\mathcal{H}: x^2 - \frac{y^2}{4} - 1 = 0$ with focal points F_1 and F_2. Find the points M situated on the hyperbola such that

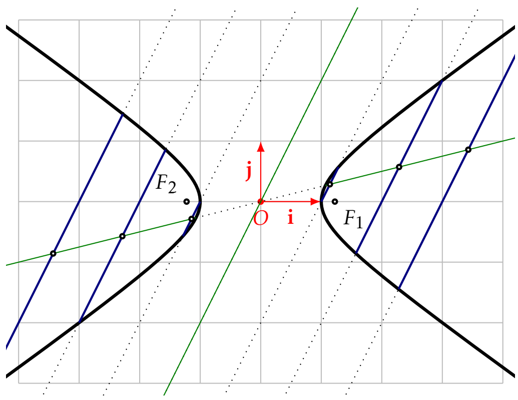
1. The angle $\measuredangle F_1 M F_2$ is right;
2. The angle $\measuredangle F_1 M F_2$ is $60°$;
3. The angle $\measuredangle F_1 M F_2$ is θ.



8. Consider the tangents to the parabola $\mathcal{P}: y^2 - 10x = 0$ passing through the point $P(-3, 12)$. Calculate the distance from the point P to the chord of the parabola which is formed by the two contact points.

9. Using the gradient, prove the reflective properties of the hyperbola and of the parabola.

10. Consider the hyperbola $\mathcal{H}: x^2 - 2y^2 = 1$. Determine the geometric locus described by the midpoints of the chords of $\mathcal{H}$ which are parallel to the line $2x - y = 0$.

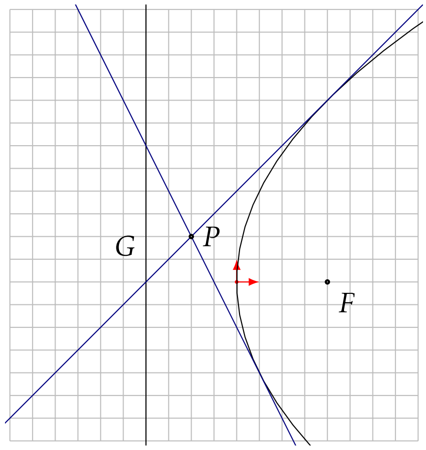


11. For which value k is the line $y = kx + 2$ tangent to the parabola $\mathcal{P}: y^2 = 4x$?

12. Consider the parabola $\mathcal{P} : y^2 = 16x$. Determine the tangents to $\mathcal{P}$ which are

1. parallel to the line $\ell : 3x - 2y + 30 = 0$;
2. perpendicular to the line $\ell : 4x + 2y + 7 = 0$.

13. Determine the tangents to the parabola $\mathcal{P} : y^2 = 16x$ which contain the point $P(-2, 2)$.



1. Determine the intersection points between the line $\ell : 2x - y - 10 = 0$ and the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$.

$$\ell \cap \mathcal{H} : \begin{cases} \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0 \\ 2x - y - 10 = 0 \end{cases} \iff \begin{cases} \frac{x^2}{20} - \frac{(2x-10)^2}{5} - 1 = 0 \\ y = 2x - 10 \end{cases}$$

$$\frac{x^2}{20} - \frac{4(x-5)^2}{5} - 1 = 0 \quad \Big| \cdot 20$$

$$x^2 - 16(x^2 - 10x + 25) - 20 = 0$$

$$-15x^2 + 160x - 420 = 0 \quad \Big| : 5$$

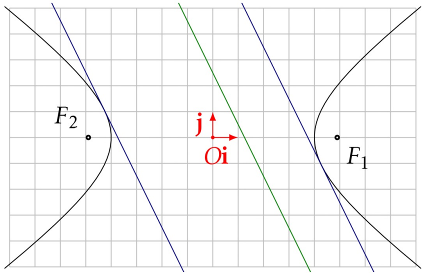
$$-3x^2 + 32x - 84 = 0$$

$$\Delta = \underset{2^{10}}{32^2} - \underset{2^2 \cdot 3 \cdot 2^2 \cdot 3 \cdot 7}{4 \cdot 3 \cdot 84} = 2^4(\underset{64}{2^6} - \underset{63}{3^2 \cdot 7}) = 2^4 \Rightarrow x_{1,2} = \frac{-32 \pm 4}{-6} = \begin{cases} \frac{-36}{-6} = 6 \\ \frac{-28}{-6} = \frac{14}{3} \end{cases}$$

$2x - 10$

So we have two intersection points $P_1(\underset{x_1}{6}, 2)$ and $P_2\left(\frac{14}{3}, -\frac{2}{3}\right)$.

2. Determine the tangents to the hyperbola $\mathcal{H} : \frac{x^2}{16} - \frac{y^2}{8} - 1 = 0$ which are parallel to the line ℓ : $4x + 2y - 5 = 0$.



The tangents to $\mathcal{H}$ of slope k are

$$\ell_k : y = kx \pm \sqrt{a^2k^2 - b^2} \qquad \text{if} \quad k \in \left(-\infty, -\frac{b}{a}\right) \cup \left(\frac{b}{a}, \infty\right)$$

In our case $a = 4$ and $b = 2\sqrt{2}$ and $k = -2 < -\frac{2\sqrt{2}}{4}$

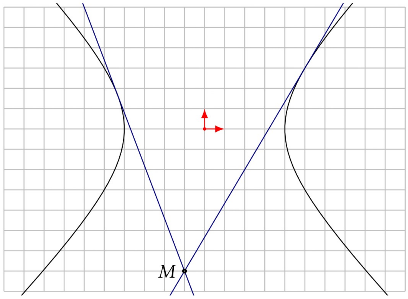
So $\sqrt{a^2k^2 - b^2} = \sqrt{16 \cdot 4 - 8} = 2\sqrt{14}$

and the two tangent lines are

$$\ell_1 : y = -2x + 2\sqrt{14} \qquad \text{and} \qquad \ell_2 : y = -2x - 2\sqrt{14}$$

Remark when we deduce the equation of a tangent line for a given slope k, it is clear that $\sqrt{a^2k^2 - b^2}$ has to be a real number in order for the equation to describe a line in the plane $\mathbb{E}^2$

3. Determine the tangents to the hyperbola $\mathcal{H} : x^2 - y^2 = 16$ which contain the point $M(-1,7)$.



We search for the possible tangents passing through M in the form

$$\ell : \ y = kx \pm \sqrt{a^2k^2 - b^2} \qquad \text{with } k \in \left(-\infty, -\tfrac{b}{a}\right) \cup \left(\tfrac{b}{a}, \infty\right)$$

since any tangent line to $\mathcal{H}$ has such an equation, in particular those who pass through M

We know that $M(-1,7) \in \ell$ and that $a = b = 4$

$$7 = -k \pm \sqrt{16k^2 - 16}$$

$$\Rightarrow \quad 7 + k = \pm \sqrt{16k^2 - 16} \quad \Big| \, (\,)^2$$

$$\Rightarrow \quad 49 + 14k + k^2 = 16k^2 - 16 \quad \left(\text{since } k^2 \geq \frac{b^2}{a^2}\right)$$

$$\Rightarrow \quad 15k^2 - 14k - 65 = 0$$

$$\Delta = 14^2 + 4 \cdot \underbrace{15 \cdot 65}_{3 \cdot 5^2 \cdot 13} = 4\left(49 + 975\right) = 4 \cdot \underbrace{(1024)}_{2^{10}} = 2^{12}$$

$$\Rightarrow \quad k_{1,2} = \frac{14 \pm 64}{30} \ \begin{cases} -\frac{50}{30} = -\frac{5}{3} \\ \frac{78}{30} = \frac{13}{5} \end{cases}$$

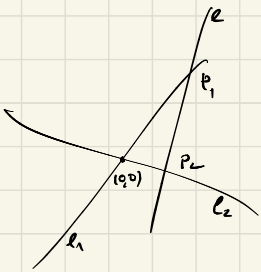
4. Find the area of the triangle determined by the asymptotes of the hyperbola $\mathcal{H}: \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the line $\ell: 9x + 2y - 24 = 0$.

$a = 2 \quad b = 3$

$\Rightarrow$ the asymptotes of $\mathcal{H}$ are $\quad \ell_1: y = \frac{3}{2}x \quad$ and $\quad \ell_2: y = \frac{-3}{2}x$

$\ell \cap \ell_1: \quad 9x + 2\frac{3}{2}x - 24 = 0 \qquad x = \frac{24}{11} \Rightarrow y = \frac{36}{11}$

$\ell \cap \ell_2: \quad 9x + 2\frac{-3}{2}x - 24 = 0 \qquad x = 4 \Rightarrow y = -6$



area of triangle $= \frac{1}{2}\left| \begin{vmatrix} 0 & 0 & 1 \\ \frac{24}{11} & \frac{36}{11} & 1 \\ 4 & -6 & 1 \end{vmatrix} \right| = \frac{144}{11}$

5. Find an equation for the tangent lines to:

1. the hyperbola $\mathcal{H}: \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$, orthogonal to the line $\ell: 4x + 3y - 7 = 0$;
2. the parabola $\mathcal{P}: y^2 - 8x = 0$, parallel to $\ell: 2x + 2y - 3 = 0$.

1.) A tangent line to $\mathcal{H}$ has an equation of the form

$$\ell_k: y = kx \pm \sqrt{a^2k^2 - b^2}$$

Since $\ell_k \perp \ell: y = -\frac{4}{3}x + \frac{7}{3}$ $\Rightarrow$ the slope k of the tangent is $\frac{3}{4}$ $(*)$

$$\Rightarrow \quad \ell_k: \quad y = \frac{3}{4}x \pm \sqrt{20 \cdot \frac{9}{16} - 5} = \frac{3}{4}x \pm \frac{5}{2}$$

so the two tangents are $\ell_1: y = \frac{3}{4}x + \frac{5}{2}$ and $\ell_2: y = \frac{3}{4}x - \frac{5}{2}$

2.) We consider the tangent lines to $\mathcal{P}$ in the form

$$\ell_k: y = kx + \frac{p}{2k}$$

Since $\ell_k \parallel \ell: y = -x + \frac{3}{2}$ $\Rightarrow$ the slope k of the tangent is 1

$p = 4$

So, the tangent is $y = x + 2$

6. Find an equation for the tangent lines to:

1. the hyperbola $\mathcal{H} : \frac{x^2}{3} - \frac{y^2}{5} - 1 = 0$, passing through $P(1, -5)$;
2. the paraola $\mathcal{P} : y^2 - 36x = 0$, passing through $P(2, 9)$.

Method I

$\ell_k : \quad y = kx \pm \sqrt{3k^2 - 5}$

$P(1,-5) \in \ell_k \quad \Rightarrow \quad -5 = k \pm \sqrt{3k^2-5}$

$\Rightarrow \quad -5 - k = \sqrt{3k^2-5} \quad |(\,)^2$

$\Rightarrow \quad 25 + 10k + k^2 = 3k^2 - 5$

$\Rightarrow \quad 2k^2 - 10k - 30 = 0$

$\Rightarrow \quad k^2 - 5k - 15 = 0 \qquad \Delta = 25 + 60 \qquad k_{1,2} = \frac{10 \pm \sqrt{85}}{4}$

So the possible tangent lines are $y = \frac{10 \pm \sqrt{85}}{4} x \pm \sqrt{3\left(\frac{10 \pm \sqrt{85}}{4}\right)^2 - 5}$

we have four lines here
now we can check which of these four equations are satisfied by the coordinates of P

Method II we look for the tangent lines in the form

$\ell_{(x_0,y_0)} : \frac{x x_0}{3} - \frac{y y_0}{5} = 1$

since $P \in \ell_{(x_0,y_0)}$ we have $\frac{x_0}{3} + y_0 = 1 \Rightarrow y_0 = 1 - \frac{x_0}{3}$

since $(x_0,y_0) \in \mathcal{H}$: $\quad \frac{x_0^2}{3} - \frac{\left(1 - \frac{x_0}{3}\right)^2}{5} - 1 = 0$

$$\ldots \quad x_{1,2} = -\frac{3}{14} \pm \frac{3\sqrt{85}}{14}$$

so we obtain the points

$$(x_0,y_0) = \left(-\frac{3}{14} + \frac{3\sqrt{85}}{14}, \frac{15}{14} - \frac{\sqrt{85}}{14}\right) \text{ and } (x_0,y_0) = \left(-\frac{3}{14} - \frac{3\sqrt{85}}{14}, \frac{15}{14} + \frac{\sqrt{85}}{14}\right)$$

and the corresponding lines in these two points.

9. $\mathcal{P}: y^2 - 36x = 0 \qquad P(2,9)$

$$\ell_k: \quad y = kx + \frac{p}{2k} \ni P(2,9)$$

$$9 = 2k + \frac{18}{2k} \quad \Rightarrow \quad 9k = 2k^2 + 9$$

$$2k^2 - 9k + 9 = 0$$

$$\Delta = 81 - 8\cdot 9 = 9 \qquad k_{1,2} = \frac{9 \pm 3}{4} \begin{cases} 3 \\ \frac{3}{2} \end{cases}$$

$k = 3: \qquad y = 3x + 3 \ni P(2,9) \qquad \leftarrow (\cancel{*}*)$

$k = \frac{3}{2} \qquad y = \frac{3}{2}x + \frac{9}{\frac{3}{2}} \iff y = \frac{3}{2}x + 6 \ni P(2,9) \qquad \leftarrow (*)$

Method II $\qquad \ell_{(x_0,y_0)}: \quad yy_0 = 18(x + x_0) \ni P(2,9)$

$$y_0 9 = 36 + 18x_0 \quad \Big|:9$$

$$y_0 = 4 + 2x_0$$

$$(x_0,y_0) \in \mathcal{P} \qquad (4 + 2x_0)^2 = 36x_0$$

$$(2 + x_0)^2 = 9x_0$$

$$4 + 4x_0 + x_0^2 = 9x_0$$

$$x_0^2 - 5x_0 + 4 = 0 \qquad \Delta = 25 - 16 = 9 \quad x_{1,2} = \frac{5 \pm 3}{2} \begin{cases} 4 \\ 1 \end{cases}$$

So $(x_0, y_0) = (4, 12)$ or $(x_0, y_0) = (1, 6)$

$\downarrow$ $\downarrow$

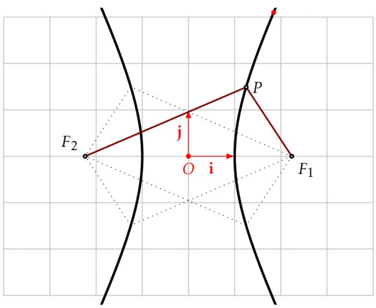
$12y = 18(x+4)$ $6y = 18(x+1)$

$2y = 3x + 12$ $y = 3x + x$

this is (*) this is (**)

7. Consider the hyperbola $\mathcal{H} : x^2 - \frac{y^2}{4} - 1 = 0$ with focal points F_1 and F_2. Find the points M situated on the hyperbola such that

1. The angle $\measuredangle F_1 M F_2$ is right;
2. The angle $\measuredangle F_1 M F_2$ is $60°$;
3. The angle $\measuredangle F_1 M F_2$ is θ.



1. the points are the intersection of $\mathcal{H}$ with a circle centered in the origin and passing through the focal points F_1 and F_2 (see similar problem in previous problem set)

2. One could use a geometric argument as in 1:
   - consider the equilateral triangle F_1F_2A with A above the x-axis
   - intersect $\mathcal{H}$ with the circumcenter of this triangle
   - the intersection points above the x-axis are two of the points we need
   - the other two are obtain with A below the x-axis.

3. With an algebraic calculation we can give a solution for all θ

We know that for a point $P \in \mathcal{H}$ we have

$$\cos\theta = \cos \measuredangle F_1 P F_2 = \frac{\overrightarrow{PF_1} \cdot \overrightarrow{PF_2}}{\|\overrightarrow{PF_1}\| \cdot \|\overrightarrow{PF_2}\|}$$

so, ... we can calculate

$P(x_0, y_0)$ $F_1(c, 0)$ $F_2(-c, 0)$

$$\Rightarrow \overrightarrow{PF_1}\left(c - x_0, y_0\right), \quad \overrightarrow{PF_2}\left(-c - x_0, y_0\right)$$

$$\frac{y^2}{b^2} = \frac{x^2}{a} - 1$$

$$\|\overrightarrow{PF_1}\| = \sqrt{(c - x_0)^2 + y_0^2} = \dots = |c x_0 - a| = c x_0 - a$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\|\overrightarrow{PF_2}\| = \sqrt{(-c - x_0)^2 + y_0^2} = \dots = |c x_0 + a| = c x_0 + a$$

$$\Rightarrow \frac{\overrightarrow{PF_1} \cdot \overrightarrow{PF_2}}{\|\overrightarrow{PF_1}\| \cdot \|\overrightarrow{PF_2}\|} = \frac{(c - x_0)(-c - x_0) + y_0^2}{(c x_0 - a)(c x_0 + a)} = \frac{-c^2 + x_0^2 + y_0^2}{c^2 x_0^2 - a^2} = \frac{x_0^2 - c^2 + b^2\left(\frac{x_0^2}{a^2} - 1\right)}{c^2 x_0^2 - a^2}$$

$a = 1$ $b = 2$ $\Rightarrow$ $c = \sqrt{a^2 + b^2} = \sqrt{5}$

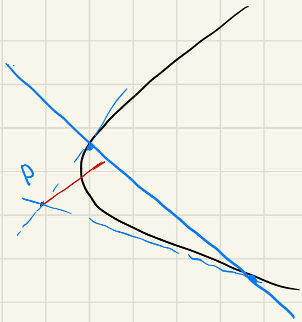
So in our case $$\cos\theta = \frac{x_0^2 - 5 + 4(x_0^2 - 1)}{5 x_0^2 - 1} = \frac{5x_0^2 - 9}{5x_0^2 - 1}$$

So in the previous case $\theta = 60°$

$$\frac{1}{2} = \cos\theta = \frac{5x_0^2 - 9}{5x_0^2 - 1} \iff x_0 = \pm\sqrt{\frac{17}{5}}$$

So the four point are $M\left(\pm\sqrt{\frac{17}{5}}, \pm 4\sqrt{\frac{3}{5}}\right)$

8. Consider the tangents to the parabola $\mathcal{P}: y^2 - 10x = 0$ passing through the point $P(-3, 12)$. Calculate the distance from the point P to the chord of the parabola which is formed by the two contact points.



We need the two contact points so it makes sense to work with a tangent line in the form $yy_0 = p(x + x_0)$

since $P(-3,12)$ lies on such a line, we have

$$12y_0 = 5(x_0 - 3) \Rightarrow x_0 = \frac{12}{5}y_0 + 3$$

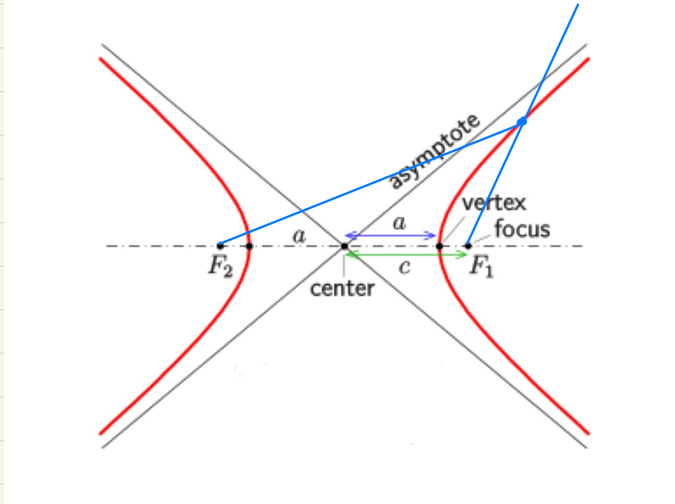
$$(x_0, y_0) \in \mathcal{P} \Rightarrow y_0^2 - 24y_0 - 30 = 0 \qquad \Delta = 24^2 + 4 \cdot 30 = 4 \cdot 174$$

$$\Rightarrow y_0 \in \{12 - \sqrt{174}, 12 + \sqrt{174}\}$$

$$\Rightarrow (x_0, y_0) \in \{P_1(\dots;\dots), P_2(\dots;\dots)\}$$

So we can write down the line P_1P_2 and calculate the distance from P to this line.

9. Using the gradient, prove the reflective properties of the hyperbola and of the parabola.



Hyperbola

a ray starting in F_2 is reflected in the curve away from F_1 on a line passing through the contact point and F_1

$$\mathcal{H}_{a,b}: \ d(M, F_1) - d(M, F_2) \pm 2a = 0$$

$$\iff \underbrace{\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} \pm 2a}_{\phi(x,y)} = 0$$

$$\nabla\phi(x,y) = \left(\frac{\partial \phi}{\partial x}(x,y), \frac{\partial \phi}{\partial y}(x,y) \right)$$

$$\frac{\partial \phi}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} \pm 2a \right) = \frac{x-c}{\sqrt{(x-c)^2+y^2}} - \frac{x+c}{\sqrt{(x+c)^2+y^2}}$$

$$\frac{\partial \phi}{\partial x}(x,y) = \frac{y}{\sqrt{(x-c)^2+y^2}} - \frac{y}{\sqrt{(x+c)^2+y^2}}$$

$$\Rightarrow \nabla\phi(M) = \left(\frac{x_M - c}{\|\overrightarrow{F_1M}\|} - \frac{x+c}{\|\overrightarrow{F_2M}\|}, \frac{y}{\|\overrightarrow{F_1M}\|} - \frac{y}{\|\overrightarrow{F_2M}\|} \right)$$

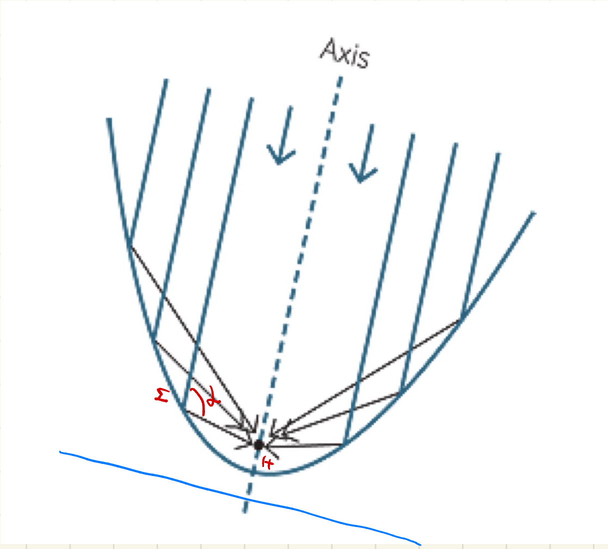
$$= \frac{1}{\|\overrightarrow{F_1M}\|}(x_M - c, y) - \frac{1}{\|\overrightarrow{F_2M}\|}(x+c, y)$$

$$= \frac{\overrightarrow{F_1M}}{\|\overrightarrow{F_1M}\|} - \frac{\overrightarrow{F_2M}}{\|\overrightarrow{F_2M}\|}$$

$\Rightarrow$ $\nabla\phi(M)$ is a direction vector for the exterior angle bisector of $\widehat{F_1MF_2}$ $(*)$

Parabola

rays parallel to the axis of the parabola which hit the curve get reflected in the focal point



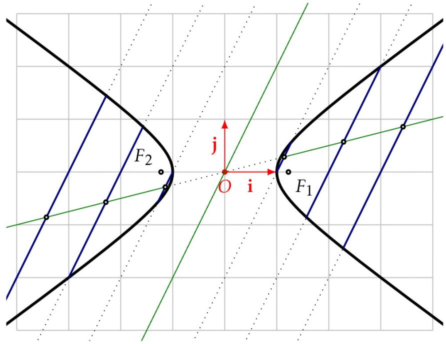
$$\mathcal{P}_p : \quad \underset{\substack{\downarrow \\ F\left(\frac{p}{2}, 0\right)}}{d(M, F)} - \underset{\substack{\downarrow \\ x = -\frac{p}{2}}}{d(M, \text{directrix})} = 0$$

$$\Leftrightarrow \quad \underbrace{\sqrt{(x_M - c)^2 + y_M^2} - \left(x + \frac{p}{2}\right)}_{\phi(x_M, y_M)} = 0$$

$$\nabla\phi(x,y) = \left(\frac{x-c}{\sqrt{(x-c)^2+y^2}} - 1, \frac{y}{\sqrt{(x-c)^2+y^2}}\right) = \frac{1}{\|\overrightarrow{FM}\|}\overrightarrow{FM} - (1,0)$$

$\Rightarrow$ $\nabla\phi(M)$ is a dir. vector for the angle α

10. Consider the hyperbola $\mathcal{H} : x^2 - 2y^2 = 1$. Determine the geometric locus described by the midpoints of the chords of $\mathcal{H}$ which are parallel to the line $2x - y = 0$.



A line parallel to the given line has an equation of the form

$$\ell_m : y = 2x + m$$

Then $\ell_m \cap \mathcal{H}$: $\begin{cases} x^2 - 2(2x+m)^2 = 1 \\ y = 2x + m \end{cases}$ $\Leftrightarrow$ $x^2 - 8x^2 - 8xm - 2m^2 - 1 = 0$

$$-7x^2 - 8mx - 2m^2 - 1 = 0$$

$$\Delta = 64m^2 - 4 \cdot 7(2m^2 + 1)$$
$$= 4(16m^2 - 14m^2 - 7)$$
$$= 4(2m^2 - 7)$$

So ℓ_m intersects $\mathcal{H}$ if $2m^2 - 7 > 0 \Leftrightarrow m^2 > \frac{7}{2} \Leftrightarrow m \in (-\infty, -\frac{7}{2}) \cup (\frac{7}{2}, \infty)$

For such m, the two intersection points are

$$P_1\left(x_1 = \frac{8m + 2\sqrt{2m^2 - 7}}{-14}, 2x_1 + m\right) \text{ and } P_2\left(x_2 = \frac{8m - \sqrt{\Delta}}{-14}, 2x_2 + m\right)$$

The midpoint of the segment $[P_1P_2]$ is

$$M\left(\frac{x_1 + x_2}{2}, 2\frac{x_1 + x_2}{2} + m\right) = M\left(-\frac{4}{7}m, -\frac{1}{7}m\right)$$

So, the geometric locus is the set of points $\left\{\left(-\frac{4}{7}m, -\frac{1}{7}m\right) : m \in (-\infty, -\frac{7}{2}) \cup (\frac{7}{2}, \infty)\right\}$

11. For which value k is the line $\underbrace{y = kx + 2}_{\ell_k}$ tangent to the parabola $\underbrace{\mathcal{P} : y^2}_{\mathcal{P}_2} = 4x$?

$$\ell_k \cap \mathcal{P}_2 : \begin{cases} y^2 = 4x \\ y = kx + 2 \end{cases} \iff \begin{cases} (kx+2)^2 = 4x \\ y = kx + 2 \end{cases}$$

$$k^2x^2 + 4kx + 4 = 4x$$

$$k^2x^2 + 4(k-1)x + 4$$

$$\Delta = 4^2(k-1)^2 - 4\cdot 4\cdot k^2$$
$$= 4^2(k^2 - 2k - 1 - k^2)$$
$$= 4^2(-2k-1)$$

ℓ_k is tangent to $\mathcal{P}_2$ if $\Delta = 0 \iff k = -\frac{1}{2}$

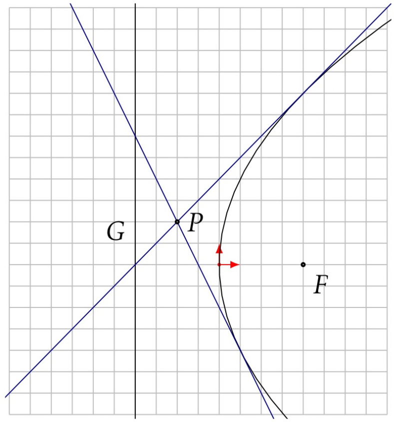
12. Consider the parabola $\mathcal{P} : y^2 = 16x$. Determine the tangents to $\mathcal{P}$ which are

1. parallel to the line $\ell : 3x - 2y + 30 = 0$;
2. perpendicular to the line $\ell : 4x + 2y + 7 = 0$.

use the form $y = kx + \frac{p}{2k}$ for a tangent

similar to exer. 5

13. Determine the tangents to the parabola $\mathcal{P} : y^2 = 16x$ which contain the point $P(-2, 2)$.



use the form $y = kx + \frac{p}{2k}$ for a tangent

similar to exer. 6