HW8 Report

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1. In order to implement a PCA analysis, we first define functions 'mean_vector', 'covariance_matrix', 'sample_corr_matrix', and 'standardize'. The 'standardize' function is used for standardization as

$$\underset{\sim}{Z} = V^{-\frac{1}{2}}(\underset{\sim}{X} - \underset{\sim}{\mu}) \text{ where } V^{\frac{1}{2}} = \begin{bmatrix} \sqrt{\sigma_{11}} & & & \underset{\sim}{0} \\ & \sqrt{\sigma_{22}} & & \\ & & \ddots & \\ \underset{\sim}{0} & & & \sqrt{\sigma_{pp}} \end{bmatrix}$$

$$Z_1 = \frac{X_1 - \mu_1}{\sqrt{\sigma_{11}}}, \quad Z_2 = \frac{X_2 - \mu_2}{\sqrt{\sigma_{22}}}, \quad \cdots, \quad Z_p = \frac{X_p - \mu_p}{\sqrt{\sigma_{pp}}}$$

The corresponding function is implemented as below:

```
def standardize(data):
    X = data
    p = len(data)
    mu=pd.concat([mean_vector(data)]*len(data), ignore_index = True)
    S=covariance_matrix(data)

    #V^(-1/2)
    V = pd.DataFrame(0, index=np.arange(len(S)),columns = range(len(S)))
    for i in range(len(S)):
        V.iloc[i,i] = 1/np.sqrt(S.iloc[i,i])
    V.index, V.columns = data.columns, data.columns

    Z=(V.dot((X-mu).T)).T
    return Z
```

We create a class 'PCA_corr' which incorporates functions that entail the features (a) ~ (d):

(a) PCA on correlation matrix

We obtain and sort the eigenvalues and eigenvectors of the correlation matrix as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$

The sorted eigenvectors make up the matrix of principal components.

(b) Proportion of variance explained by principal components:

$$= \frac{\lambda_k}{\sum_{i=1}^{p} \lambda_i} = \frac{\lambda_k}{p} \quad (k = 1, 2, \cdots, p)$$

(c) Scree plot

The scree plot is obtained by simply plotting the eigenvalues in decreasing order.

(d) PC scores

The PC scores are obtained by using the formula

$Y_i = \underset{\sim}{e}_i' \underset{\sim}{Z}$

For convenience, our function can also specify the number of principal components that the user desires to print out.

The code that implements PCA analysis is as follows:

```python
In [3]: class PCA_corr:
    def __init__(self, data):
        self.data = data
        self.R = sample_corr_matrix(self.data)
        self.eigval, self.eigvec = np.linalg.eig(self.R)
        self.p = len(self.data.T) #number of original variables

    #(a) PCA on correlation matrix
    def analysis(self):
        #sort
        self.order = np.argsort(self.eigval)[::-1]
        self.eigval = self.eigval[self.order]
        self.eigvec = self.eigvec[:,self.order]

        #principal components matrix
        self.PC = pd.DataFrame(index = self.data.columns, columns = ['Y1','Y2','Y3','Y4','Y5','Y6','Y7','Y8'])
        for i in range(self.p):
            self.PC.iloc[:,i]=self.eigvec[i]

        return self.PC

    #(b) Proportion of variance explained by principal components
    def prop_var(self):
        prop = pd.DataFrame(index=range(self.p), columns=['eigenvalues', 'proportion', 'cumulative'])
        prop['eigenvalues'] = self.eigval
        prop['proportion'] = self.eigval/self.p

        prop.loc[0,'cumulative']=prop.loc[0,'proportion']
        for i in range(1,self.p):
            prop.loc[i,'cumulative']=prop.loc[i,'proportion']+prop.loc[i-1,'cumulative']

        return prop

    #(c) Scree plot
    def scree_plt(self):
        plt.title('Scree Plot of Eigenvalues')
        plt.xlabel('Number')
        plt.plot(self.eigval, 'o-')
        plt.show()

    #(d) PC Scores
    def PC_score(self, PC_num):
        self.score = pd.DataFrame(index=range(len(self.data)), columns = ['Y1','Y2','Y3','Y4','Y5','Y6','Y7','Y8'])
        self.Z = standardize(self.data)
        for i in range(len(self.score.columns)):
            self.score.iloc[:,i] = self.eigvec[:,i].dot(self.Z.T)
        return self.score.iloc[:,:PC_num]
```

2. We first save U.S. Navy data set as 'navy'. We process the data by adding 1 to 'CUA' variable, because it contains zero. The processed data is saved as 'data'.

(a) After creating an instance 'PCA_navy' of class 'PCA_corr', we utilize the 'analysis' function defined in #1 to perform PCA on 'data.' We obtain the following principal components.

```python
In [6]: #(a) PCA on correlation matrix
PCA = PCA_navy.analysis()
PCA
```

Out[6]:

	Y1	Y2	Y3	Y4	Y5	Y6	Y7	Y8
ADO	-0.373020	-0.360289	-0.275727	-0.318853	-0.357864	-0.376906	-0.380274	-0.372321
MAC	-0.107594	-0.023598	-0.863570	0.439212	0.124525	0.134437	0.124512	0.011066
WHR	-0.220503	-0.128714	0.384548	0.805176	-0.231831	-0.061904	-0.064751	-0.277228
CUA	-0.157743	-0.717924	0.141830	-0.041530	0.641783	0.088827	0.124557	-0.050707
WNGS	0.250931	-0.466126	-0.025390	-0.093121	-0.596117	0.456799	0.382212	0.018382
OBC	-0.036979	-0.279348	-0.000481	0.152192	-0.163712	-0.333877	-0.176688	0.853194
RMS	0.839078	-0.199899	-0.097884	0.149238	0.083474	-0.363990	-0.188948	-0.221309
MMH	-0.085963	0.048830	0.001324	-0.035868	-0.040877	-0.612796	0.779215	-0.067618

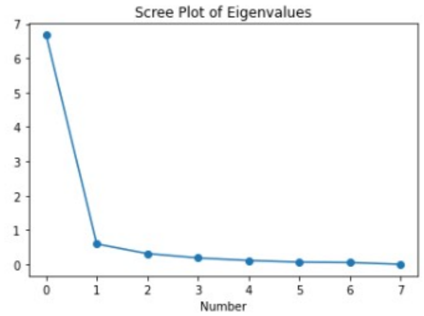
(b) We obtain the proportion of variance explained by principal components and its scree plot using #1 as shown below.

In [7]:
```
#(b) How many PCs required to adequately describe the space in which these data actually fall?
PCA_navy.prop_var()
```

Out[7]:

	eigenvalues	proportion	cumulative
0	6.674089	0.834261	0.834261
1	0.592580	0.074073	0.908334
2	0.313358	0.039170	0.947503
3	0.187037	0.023380	0.970883
4	0.112785	0.014098	0.984981
5	0.065302	0.008163	0.993144
6	0.053104	0.006638	0.999782
7	0.001744	0.000218	1.0

In [8]:
```
PCA_navy.scree_plt()
#elbow is at index=1
#Only one principal component (index=0) is required
```



The proportion drops rapidly from λ_1 (83.42%) to λ_2 (7.40%). We can also see in the scree plot that the elbow is at index =1. Thus, only one principal component (index =0) is required.

(c) The principal component scores can be computed as $Y_i = \underline{e}_i' \underline{Z}$

In [9]:
```
#(c) Calculate principal component scores
PCA_navy.PC_score(1)
```

Out[9]:

	Y1
0	5.380969
1	4.824661
2	3.470424
3	3.763104
4	2.991037
5	2.594652
6	2.369429
7	0.255922
8	0.115610
9	0.607836
10	-1.202995
11	0.206310
12	-1.251468
13	-0.193598
14	-1.764154
15	-1.954653
16	-1.644965
17	-2.787862
18	-1.455821
19	-1.790250
20	-1.561804
21	-3.386554
22	-2.712675
23	-3.154529
24	-1.718625

(d) The correlations between the variables and PC scores can be computed as

$Corr(Y_i, Z_k) = e_{ik}\sqrt{\lambda_i}$ $(i, k = 1, 2, \cdots, p)$

In [10]:
```
#(d) Calculate correlations between variables(Z) and PC scores(Y)
corr = pd.DataFrame(PCA_navy.eigvec.copy(), index = ['Z1','Z2','Z3','Z4','Z5','Z6','Z7','Z8'], columns = ['Y1','Y2','Y3','Y4','Y5
for i in range(8):
    corr.iloc[:,i]=corr.iloc[:,i]*np.sqrt(PCA_navy.eigval[i])
corr
```

Out[10]:

	Y1	Y2	Y3	Y4	Y5	Y6	Y7	Y8
Z1	-0.963669	-0.082825	-0.123434	-0.068221	0.084271	-0.009450	0.193360	-0.003590
Z2	-0.930779	-0.018165	-0.072052	-0.310486	-0.156542	-0.071385	-0.046065	0.002039
Z3	-0.712319	-0.664770	0.215264	0.061338	-0.008527	-0.000123	-0.022557	0.000055
Z4	-0.823733	0.338102	0.450725	-0.017961	-0.031273	0.038892	0.034391	-0.001498
Z5	-0.924516	0.095858	-0.129775	0.277557	-0.200197	-0.041835	0.019236	-0.001707
Z6	-0.973709	0.103489	-0.034653	0.038416	0.153409	-0.085320	-0.083879	-0.025590
Z7	-0.982409	0.095848	-0.036247	0.053868	0.128360	-0.045151	-0.043542	0.032540
Z8	-0.961863	0.008518	-0.155188	-0.021930	0.006173	0.218028	-0.050999	-0.002824

(e) We take the first principal component and apply Shapiro-Wilk's, Kolmogoroc-Smirnov, Cramer-von Mises, and Anderson-Darling test. The null hypothesis is that the principal component score follows a normal distribution. If the data follows a multivariate normal distribution, its principal components should follow a univariate normal distribution. Although we may reject the null hypothesis for the first test, based on the p-values and test statistic of the other three tests, we cannot reject the null hypothesis. Thus, we conclude that the data follows a multivariate normal distribution.

In [13]:
```
print("Shapiro-Wilk:",shapiro(x),"\n")
print("Kolmogoroc-Smirnov:",kstest(x, 'norm', args = (m,s)),"\n")
print("Cramer-von Mises:",cramervonmises(x, 'norm', args = (m,s)),"\n")
print("Anderson-Darling:",anderson(x, 'norm'))

#The tests are conducted under the null hypothesis that the principal component score follows a normal distribution.
#If the data follows a multivariate normal distribution, its principal components should follow a univariate normal distribution.
#From Kolmogoroc-Smirnov, Cramer-von Mises, and Anderson-Darling tests, by observing the p-values and test statistic, we cannot r
#We conclude that the data follows a multivariate normal distribution.
```

```
Shapiro-Wilk: ShapiroResult(statistic=0.9071464538574219, pvalue=0.026336275041103363)

Kolmogoroc-Smirnov: KstestResult(statistic=0.19927007628143245, pvalue=0.239925809351193)

Cramer-von Mises: CramerVonMisesResult(statistic=0.16719470124004313, pvalue=0.3427804517896553)

Anderson-Darling: AndersonResult(statistic=0.927302602507396, critical_values=array([0.514, 0.586, 0.703, 0.82 , 0.975]), signi
ficance_level=array([15. , 10. ,  5. ,  2.5,  1. ]))
```