Department of Mathematical Sciences

TMA4500 - Industrial Mathematics, Specialization Project

Long term behavior of dense vehicular traffic with two lanes

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Abstract

In the following, we aim to study the long term behavior of traffic density on a road with two lanes, based on a Lighthill-Whitham-Richards model. For this we introduce an operator splitting scheme, which we propose to be a useful tool in the analysis of asymptotic behavior.

1 Introduction

The field of analysis of vehicular traffic the Lighthill-Whitham-Richards(LWR) model for unidirectional flow in a single lane, reading

$$\partial_t u + \partial_x (uv(u)) = 0,$$

where $u = u(x,t)$ denotes the density of vehicles at position x and time t, with velocity function $v = v(u)$. This is a classic model, derived from conservation of the total amount of vehicles. This model can also be expanded to handle multiple lanes, where vehicles have the option to switch lanes. As done by [3], with a model reading

$$\begin{aligned} \partial_t u_1 + \partial_x (u_1 v_1(u_1)) &= -S(u_1, u_2) \\ \partial_t u_2 + \partial_x (u_2 v_2(u_2)) &= S(u_1, u_2) \end{aligned} \tag{1}$$

where $u_i = u_i(x,t)$ is the density of vehicles in lane i, at position x and time t. $v_i = v_i(u)$ is then the velocity function for lane i, with $i = 1, 2$. The source term $S(u_1, u_2)$ models vehicles switching lanes, and is defined as

$$S(u_1, u_2) = K(v_2(u_2) - v_1(u_1)) \begin{cases} u_1 & v_2(u_2) \geq v_1(u_1) \\ u_2 & v_2(u_2) < v_1(u_1) \end{cases} \tag{2}$$

where K is some constant of proportionality, which is assumed to be equal to one for the following. The source function is derived from the assumption that drivers in the slower lane will wish to change lane, and that if the difference in velocity is great they will do this more often than if the difference in velocity is small. The rate of lane switching is also assumed to be proportional to the density of vehicles in the slow lane. Therefore, we multiply with the cases at the end.

The velocity functions v_1, v_2 are dependent only on the density in the lane, and are assumed to be strictly decreasing, twice differentiable and concave functions. We also make a small remark that $v_i(0)$ denotes the "speed limit" in the lane, not the actual velocity of the zero cars in that lane. The model is of course a hyperbolic conservation law, with flux function $f(u) = \{f_1(u_1), f_2(u_2)\} = \{u_1 v_1(u_1), u_2 v_2(u_2)\}$.

The main goal of the following will be to study the asymptotic behavior when $t \to \infty$. Which can be useful in order to see how the density of vehicles in traffic will change over a long time. Due to [3], We know that there exists a unique entropy solution. For the sake of ease we restate the definition of an entropy solution here.

Definition 1.1. *Let $v_i = v_i(u_i)$ be as described above. Assume that $u_{i,0} \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$ for $i = 1, 2$. We say that $u = \{u_1, u_2\}$, where $u_i \in C([0,\infty); L^1(\mathbb{R}))$ with $u_i(t, \cdot) \in BV(\mathbb{R})$ for $t \in [0, \infty)$, is a weak solution of* (1) *if*

$$\int_0^\infty \int_{\mathbb{R}} (u_1 \varphi_t + u_1 v_1(u_1)\varphi_x - S(u_1, u_2)\varphi) dx dt + \int_{\mathbb{R}} u_{1,0} \varphi|_{t=0} dx = 0 \tag{3a}$$

$$\int_0^\infty \int_{\mathbb{R}} (u_2 \varphi_t + u_2 v_2(u_2)\varphi_x + S(u_1, u_2)\varphi) dx dt + \int_{\mathbb{R}} u_{2,0} \varphi|_{t=0} dx = 0 \tag{3b}$$

for all $\varphi \in C_0^\infty([0,\infty) \times \mathbb{R})$.

Furthermore, the solution is called an entropy solution if

$$\int_0^\infty \int_{\mathbb{R}} (\lambda(u_1)\varphi_t + q_1(u_1)\varphi_x)dxdt + \int_{\mathbb{R}} \lambda(u_{1,0})\varphi|_{t=0}dx \geq \int_0^\infty \int_{\mathbb{R}} \lambda'(u_1)\varphi S(u_1,u_2)dxdt \tag{4a}$$

$$\int_0^\infty \int_{\mathbb{R}} (\lambda(u_2)\varphi_t + q_2(u_2)\varphi_x)dxdt + \int_{\mathbb{R}} \lambda(u_{2,0})\varphi|_{t=0}dx \geq -\int_0^\infty \int_{\mathbb{R}} \lambda'(u_2)\varphi S(u_1,u_2)dxdt \tag{4b}$$

for all twice differentiable convex functions λ where $q_i'(u) = \lambda'(u)f_i'(u)$ with $f_i(u) = uv_i(u)$, and for all $\varphi \in C_0^\infty([0,\infty)\times\mathbb{R}), \varphi \geq 0$.

When finding such a long term behavior numerical simulations were first performed giving some initial suspicions that the asymptotic behavior will be for the traffic to even out, and the velocity of the two lanes to become closer and closer. Further some initial observations and special cases help indicate that this indeed might be a potential long term behavior of the system.

Finally in order to study the asymptotic behavior in a manageable way we introduce an operator splitting type scheme. We provide proof that this scheme indeed converges to an entropy solution of our system. Finally we apply this scheme to the Riemann problem to study the asymptotic behavior of more general settings.

2 Preliminaries

2.1 Preliminaries on ordinary differential equations

In our analysis we will need some results related to ordinary differential equations(ODEs). This section will be mainly based on [4]. We suppose for this section we are given an ODE

$$\mathbf{x}' = F(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{b} \tag{5}$$

We start by stating the classic existence and uniqueness result for ODEs(Theorem 3.2.1, and Theorem 3.3.3 in [4]).

Theorem 2.1. *Let $\mathcal{U} \subset \mathbb{R}^d$ be open and contain the initial data $\mathbf{b}$, and let $F : \mathcal{U} \to \mathbb{R}^d$ be locally Lipschitz. Then there exist an interval (η, η) and a C^1 function $x : (-\eta,\eta) : \mathcal{U}$ such that (5) is satisfied. Furthermore, let x_1, x_2 be two solutions of (5), assumed to be defined for $\alpha_j < t < \beta_j$, $j =$ 1, 2. Then for all t in the range $\max\{\alpha_1,\alpha_2\} < t < \min\{\beta_1,\beta_2\}$ where both solutions are defined, $x_1(t) = x_2(t)$.*

This simple theorem gives a lot of value when satisfied, though the existence part is only local in nature. We also state the always useful Gronwall's inequality, which can be found in lemma 3.3.1 in [4]

Theorem 2.2. *Let $g : [0,T] \to \mathbb{R}$ be continuous, and suppose there are non-negative constants C, K such that*

$$g(t) \leq C + K\int_0^t g(s)ds \tag{6}$$

Then we have the inequality

$$g(t) \leq Ce^{Kt}, \quad 0 < y \leq T$$

2.2 Hyperbolic conservation laws

One important tool for our analysis will be to apply front tracking methods. Therefore we introduce some main theorems from this theory here. This short presentation will be based mostly on results from chapter two of [2]. We start with a general scalar conservation law,

$$\partial_t u + \partial_x(f(u)) = 0, \quad u(x,0) = u_0(x) \tag{7}$$

where $u_0(x)$ is the initial condition, and $f(x)$ is the flux function. We then say that the function $u(x,t)$ is a Kruzkov entropy solution to (7) on $\mathbb{R} \times [0,T]$ if the inequality

$$\int_0^T \int_{\mathbb{R}} (\eta(u)\phi_t + q(u)\phi_x) dx dt - \int_{\mathbb{R}} \eta(u)\phi|_{t=T} dx + \int_{\mathbb{R}} \eta(u)\phi|_{t=0} dx \geq 0$$

holds for any test function $\phi \in C_0^\infty(\mathbb{R} \times [0,T])$, any convex two times differentiable function η, and with $q'(u) = f'(u)\eta'(u)$.

Front tracking is a method for finding Kruzkov entropy solutions to (7), when u_0 is a step function, and f a continuous piecewise linear function. Therefore, for the remainder of this section we let f be a continuous piecewise linear function $f : [-K,K] \to \mathbb{R}$ for some constant K. We can then denote the breakpoints, i.e. the points where the slope of f changes by $-K = u_0 < u_1 < ... < u_n = K$. We then state what is the main tool for solving front tracking problems, which is Corollary 2.4 in [2].

Theorem 2.3. *Then the Riemann problem*

$$\partial_t u + \partial_x(f(u)) = 0, \quad u(x,0) = \begin{cases} u_l, & \text{for } x < 0 \\ u_r, & \text{for } x \geq 0 \end{cases}$$

has a piecewise constant solution, satisfying the Kruzkov entropy inequality. If $u_l < u_r$, let $u_l = v_1 < v_2 < ... < v_m = u_r$ denote the breakpoints of $f_\smile$, and if $u_l \geq u_r$, let $u_l = v_1 > v_2 > ... > v_m = u_r$ denote the breakpoints of $f_\frown$. The weak solution of the Riemann problem is then given by

$$u(x,t) = \begin{cases} v_1 & \text{for } x \leq s_1 t \\ v_2 & \text{for } s_1 t < x \leq s_2 t \\ ... \\ v_i & \text{for } s_{i-1} t < x \leq s_i t \\ ... \\ v_m & \text{for } s_{m-1} t < x \end{cases}$$

where the speeds s_i are computed by $s_i = \frac{f(v_{i+1}) - f(v_i)}{v_{i+1} - v_i}$. Furthermore,

$$\|u(\cdot,x) - u_0(x)\|_{L^1} \leq t\|f\|_{Lip}|u_r - u_l|. \tag{8}$$

Her we made use of the lower convex envelope, $f_\smile$ and the upper concave envelope, $f_\frown$, which we define as

$$f_\smile = \sup\{g \leq f | g \text{ a convex function}\}$$

and $f_\frown = (-1)(-f)_\smile$. This result can be used to find a solution to the more general problem where u_0 is a piecewise constant function, with a finite number of discontinuities. For such a functions we will sometimes refer to the discontinuities as fronts. The general idea is to treat each front individually as a translated Riemann problem until a front collides with another. Collisions then give new fronts, meaning different Riemann problems to be solved. Before introducing our final tool from the theory of scalar hyperbolic equations, we define the total variation of a function,

$$T.V.(u) = \sup \sum_i |u(x_{i+1}) - u(x_i)|$$

where the supremum is taken over all possible intervals. We now introduce what will be one of our main tools later, which can be found as corollary 2.8 in [2].

Theorem 2.4. *Assume u_0 to be a piecewise continuous functions with a finite number of discontinuities, $u_0 : \mathbb{R} \to [-K, K]$. Then the initial value problem*

$$\partial_t u + \partial_x(f(u)) = 0, \quad u(x, 0) = u_0(x)$$

admits a weak Kruzkov entropy solution $u = u(x, t)$. This function will be a step function for any given time t, and $\{u(x, t)\} \subset \{u_0(x)\} \cup$ breakpoints of f. Furthermore the total variation will be non-increasing,

$$T.V.(u(\cdot, t)) \leq T.V.(u_0). \tag{9}$$

Finally, we have the inequality

$$\|u(\cdot, t) - u(\cdot, s)\|_{L^1} \leq \|f\|_{Lip} T.V.(u_0)|t - s|. \tag{10}$$

2.3 L^p spaces and Bochner spaces

We will also need some important compactness results. We start by stating the compactness result for L^p spaces, the Kolmogorov-Riesz compactness theorem. This statement of the theorem is based on [1].

Theorem 2.5. *A subset $\mathcal{F}$ of $L^p(\mathbb{R}^d)$, with $1 \leq p < \infty$, is totally bounded if and only if,*

(a) (Localization) for every $\varepsilon > 0$ there is some R so that

$$\int_{|x|>R} |f(x)|^p dx < \varepsilon^p$$

for any $f \in \mathcal{F}$

(b) (Tightness) for every $\varepsilon > 0$ there is some $\rho > 0$ so that, for every $f \in \mathcal{F}$, and $y \in \mathbb{R}^d$ with $|y| < \rho$,

$$\int_{\mathbb{R}^d} |f(x + y) - f(x)|^p dx < \varepsilon^p$$

We next move on to make some definition for Bochner spaces, and state a similar compactness theorem for these. Let x be some Banach space, and let I be some interval. Then, given a function $f : I \to X$ we define the norm

$$\|f\|_{L^p(I;X)} = \begin{cases} (\int_I \|f(t)\|^p dt)^{\frac{1}{p}} & \text{if } p \in [1, \infty), \\ \text{ess sup}_{t \in I} \|f\| & \text{if } p = \infty \end{cases}$$

We can then define the corresponding Bochner space by

$$L^p(I; X) = \{f : I \to X | \|f\|_{L^p(I;x)} < \infty\}, \quad p \in [1, \infty]$$

We also define $C^k(I; X)$ as the space of functions $f : I \to X$, that may be differentiated up to k times with respect to $t \in I$. Here we will however only really care about the case $k = 0$. Finally we state a compactness theorem for Bochner spaces, which is similar to Kolmogorov-Riesz.

Theorem 2.6. *A subset $\mathcal{F} \subset L^p(I; X), p \in [1, \infty)$ is relatively compact in $L^p(I; X)$ if and only if*

(i) $\{\int_{t_1}^{t_2} dt | f \in \mathcal{F}\}$ is relatively compact in X for all $t_1, t_2 \in I$ with $t_1 < t_2$.

(ii) $\int_0^{T-s} \|f(t - s) - f(t)\|^p dt \to 0$ as $s \to 0$, uniformly in $\mathcal{F}$.

If $\mathcal{F} \subset C(I; X)$ then $\mathcal{F}$ is relatively compact if and only if (i) holds, and (ii) replaced by $\text{ess sup}_{t \in I} \|f(s + t) - f(t)\| \to 0$ as $x \to 0$, uniformly in $\mathcal{F}$

3 Numerical simulation

For some initial exploration of what asymptotic behavior we might expect a numerical simulation has been implemented. The simulations have been performed using finite volume schemes. We discretize in time and space, with stepsize Δt and Δx respectively. We will then use the standard notation $u_i^{j,k} = u_i(j\Delta x, k\Delta t)$. We will also use the notation $\lambda = \frac{\Delta t}{\Delta x}$. Simulations were ran testing with different schemes, both a Lax-Friedrichs and an Engquist-Osher type scheme have been implemented. For simplicity of implementation, periodic boundary conditions were used for both schemes.

3.1 Numerical schemes

The Lax-Friedrichs scheme used can be written up as follows

$$u_i^{j,k+1} = \left(\frac{u_i^{j+1,k} + u_i^{j-1,k}}{2} \right) - \frac{\lambda}{2} \left(f_i(u_i^{j+1,k}) - f_i(u_i^{j-1,k}) \right) + (-1)^i \Delta t S(u_1^{j,k}, u_2^{j,k})$$

where we of course have $f_i(u_i) = v_i(u_i)u_i$, for $i \in \{1,2\}$. This has the primary advantage of being simple to implement, though it is know to not preserve shocks. We will not comment much on the results of these simulations. Stating only that they gave results similar to those from the Engquist-Osher, though the results seemed to become smoother faster.

The Engquist-Osher scheme implemented is in general of the form

$$u_i^{j,k+1} = u_i^{j,k} + \lambda(f_i^{EO}(u_i^{j,k}, u_i^{j+1,k}) - f_i^{EO}(u_i^{j-1,k}, u_i^{j,k}) + (-1)^i \Delta t S(u_1^{j,k}, u_2^{j,k})$$

where $f_i^{EO}(u,v)$ is the Engquist-Osher flux, defined as

$$f_i^{EO}(u,v) = \frac{1}{2} \left(f_i(u) + f_i(v) - \int_u^v |f_i'(s)| ds \right).$$

The integral in the final term can be troublesome to compute, requiring either numerical integration, or a nice flux which is easy to integrate exactly. We note the very simple case when the derivative of the flux has a constant sign. Which makes this a very simple case of evaluating the flux, and gives the upwind scheme. However, we have one more special case which also yields a simpler scheme. This special case is when we have a flux which is convex or concave in the region where u lives, and the flux has a unique minima/maxima in the convex/concave region which includes $[0,1]$. In these cases we get the following simplification,

$$f_i^{EO}(u,v) = f_i(u \vee \omega) + f_i(v \wedge \omega) - f_i(\omega) \quad \text{Convex} \tag{11a}$$

$$f_i^{EO}(u,v) = f_i(v \vee \omega) + f_i(u \wedge \omega) + f_i(\omega) \quad \text{Concave} \tag{11b}$$

We show that this simplification holds for the case with convex flux, and note that the proof is similar for a concave flux.

We assume that f is a $\mathcal{C}^1$ convex function on $[a,b]$, with some unique $\omega \in [a,b]$ such that $f'(\omega) = 0$. From this we get that

$$\int_u^v |f'(s)| ds = \int_{u \vee \omega}^{v \vee \omega} f'(s) ds - \int_{u \wedge \omega}^{v \wedge \omega} f'(s) ds = f(v \vee \omega) - f(v \wedge \omega) - (f(u \vee \omega) - f(u \wedge \omega))$$

Using the notation

$$\max\{a,b\} = a \vee b, \qquad \min\{a,b\} = a \wedge b$$

Assuming here that $u, v \in [a,b]$, we need only test the different possibilities for which is greater of $\{u, v, \omega\}$, and see that the formula from (11) indeed holds.

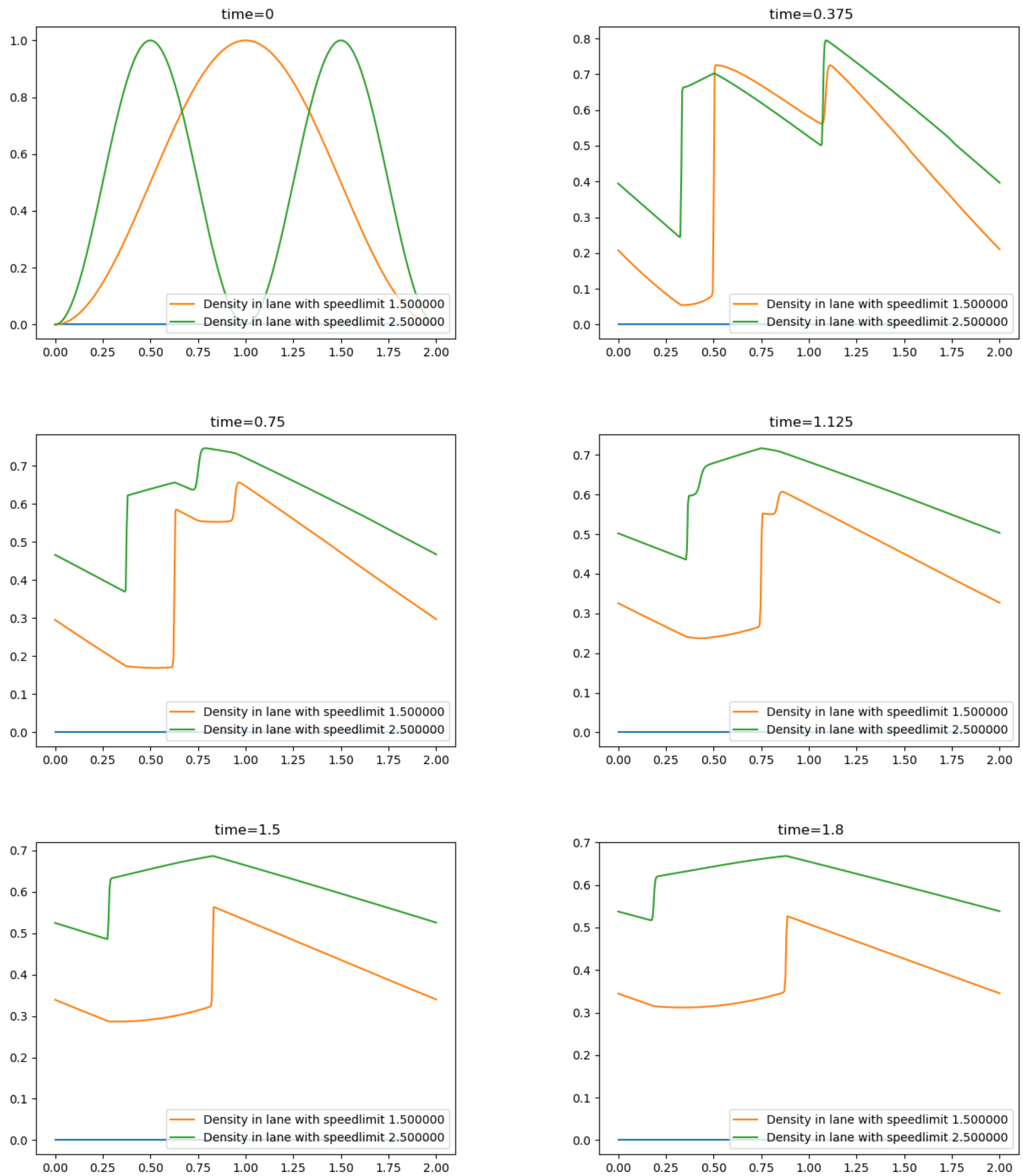

Figure 1: Graphs from one single simulation, as can be seen, the graphs seem to flatten out. At the same time the velocity in the two lanes gets closer and closer to being equal.

3.2 Numerical results

We have done some experimentation with various models for velocity, and various initial conditions. The results from these initial experiments gave results that would indicate that the total variation will decrease, indicating an approach towards constant densities in each lane. We also see that over time the velocities in the two lanes get closer and closer. This trend has been seen for both the Lax-Friedrichs scheme, as well as the Engquist-Osher scheme.

We have in particular included in (1) the figure from doing a simulation with length of a period equal to two, and $K = 1$, with velocity functions

$$v_1(u) = 1.5(1 - u), \quad \text{and} \quad v_2(u) = 2.5(1 - u)$$

and initial data

$$u_{1,0}(x) = \sin^2(\pi \frac{x}{2}), \quad \text{and} \quad u_{2,0} = u_{1,0}(x) = \sin^2(\pi x)$$

4 Properties and special cases

Before we start our study of the asymptotic behavior of the system we are studying we will go over some known properties that might be relevant, and look at some simpler special cases. Due to [3], from lemma 2.4, we get that given initial conditions contained between zero and one, then the range of the densities stay between zero and one for all time. From the same paper theorem 3.2 gives existence and uniqueness for an entropy solution.

4.1 Constant densities

We start our investigations by looking at the simplest possible traffic distribution in the lanes, this being a uniform distribution of cars in space. Let us first assume the first lane to have uniform density in both time and space, $u_1(t, x) = u_1$. If we assume this to be the case we obviously get that

$$\partial_x(u_1 v_1(u_1)) = \partial_t u_1 = 0$$

which will in turn imply that $S(u_1, u_2) = 0$. There are a couple of ways for this to hold, the first is for u_1 to be zero, and the velocity still being the greatest in lane two, implying not to dense traffic in lane two. Second, we can have $u_2 = 0$, while $v_1(u_1) > v_2(0)$, which of course implies the traffic to be uniformly distributed in both lanes. Lastly we can have that $v_1(u_1) = v_2(u_2(x, t))$. Since we have assumed u_1 to be constant we have that $v_1(u_1)$ must also be constant. This combines into the requirement that $v_1(u_1) = v_2(u_2) = v$, where v is some constant. The strict monotonicity of the velocity functions then gives that u_2 must also be constant in space and time.

4.1.1 Initial conditions constant in both lanes

Another simple case is initial conditions of the form,

$$u_i(x, 0) = c_i, \quad c_i \in [0, 1], i \in \{1, 2\}$$

We see that this gives constant velocity in both lanes as well. When this is inserted into (1), we get a reduction to an ODE depending only on the time variable.

$$\dot{u_1} = -S(u_1, u_2) \qquad \dot{u_2} = S(u_1, u_2) \tag{12}$$

Thus we can write $u_i(x, t) = k_i(t)$. We note also that the total density of cars will be conserved, meaning $k_1(t) + k_2(t) = c_1 + c_2 = K$. Which is trivially seen from the ODE. We start by looking at the existence and uniqueness of this system. The existence of a solution is found easily in all regions where $v_1(u_1) \neq v_2(u_2)$, by noting that in these regions $S(u_1, u_2)$ is a Lipschitz continuous function under the assumption that the velocity functions are as well. Thus we can conclude the existence of a solution at any point where $v_1(u_1) \neq v_2(u_2)$, by applying (2.1). Similarly this (2.1) gives uniqueness of the solution.

We now wish to do some stability analysis to (12). We obviously get a equilibrium point at $v_1(u_1) = v_2(u_2)$, which is the situation where the velocity is the same in both lanes so drivers will not wish to switch lanes. There will also be equilibrium points at $u_1 = 0, \quad v_1 > v_2$ and $u_2 = 0 \quad v_2 > v_1$. This describes a situation where one lane has a higher speed limit than the other and the total density of cars is low, so even if all the cars drive in the fast lane the density in the fast lane is still low enough that they will drive faster in the fast lane. Furthermore we see that for $v_1(u_1) > v_2(u_2)$ we get that u_1 will have positive derivative and u_2 will have negative derivative, with the opposite case if we switch the inequality. We can use these somewhat trivial observations to state the following lemma.

Lemma 4.1. *Let $u = \{u_1, u_2\}$ be a solution to* (12)*, with the source term as described above and initial conditions $u_i(0) = k_i \in [0,1]$. Then there are four possibilities for the orbit of u*

- $S(u_{1,0}, u_{2,0}) = 0$*, then u is constant.*
- $v_1(0) < v_2(k_1 + k_2)$*, then for any any $\varepsilon > 0 \quad \exists t > 0$ such that $0 < u_1 < \varepsilon$*
- $v_2(0) < v_1(k_1 + k_2)$*, then for any any $\varepsilon > 0 \quad \exists t > 0$ such that $0 < u_2 < \varepsilon$*
- *Otherwise, for any $\varepsilon > 0 \quad \exists t > 0$ such that $|v_1(u_1(t)) - v_2(u_2(t))| < \varepsilon$*

In short this lemma states that any initial condition, will give a flow which gets arbitrarily close to an equilibrium point. Note that this is really a standard result from the stability theory of ODEs, but this proof has been added for the sake of completeness.

Proof. We handle each case separately, the first case follows trivially from the equation. Next we look at the second case, suppose the assumptions of the second case holds, that $u_{1,0} = k_1 > 0$, and let $\varepsilon > 0$. Let

$$V := \{(u_1, u_2) | u_1 + u_2 = k_1 + k_2, u_1 \in [\varepsilon/2, a]\},$$

and $\gamma = \inf_{(u_1,u_2)\in V} |S(u_1, u_2)|$. Then by assumption, and since V is compact, it follows that $\gamma > 0$. Thus, there exists t such that $k_1 - \gamma t < \varepsilon$. It then follows that $u_1(t) \leq k_1 - \gamma t < \varepsilon$. Furthermore, since we have a continuous coupling function, we get that $u_1(t) > 0$, and we have proven the second case, noting that the third case is similar.

The third case we again split into two cases, one case where $v_1(u_{1,0}) > v_2(u_{2,0})$, and one where this inequality is flipped. We prove the result for the first of these, and note that the second can be proven using the same techniques. Now, suppose that we are in this case, and define the set

$$V := \{(u_1, u_2) | u_1 + u_2 = k_1 + k_2, (v_1(u_1(t)) - v_2(u_2(t))) \in [\varepsilon/2, v_1(u_{1,0}) - v_2(u_{2,0})]\}$$

Then let $\gamma = \inf_{(u_1,u_2)\in V} |S(u_1, u_2)|$. Similar to the previous case we get that there exists t such that $|v_1(u_{1,0} - t\gamma) - v_2(u_{2,0} + t\gamma)| < \varepsilon$. Then we clearly see again that $u(t)$ must satisfy the requirement of the lemma, and we have finished the proof. $\square$

5 Operator splitting

In order to get concrete results we now wish to apply a front tracking methodology to obtain a more general result. Ideally we would simply find a solution to the Riemann problem, and expand to the general case for step functions, then using step functions to approximate more general initial data. Solving the Riemann problem was not easily done analytically here. Thus we instead introduce an operator splitting approximation. Here we start with a section describing the method, and stating a few properties. Then we prove that it converges to an entropy solution of (1). Finally we make some remarks regarding the

5.1 Defining the scheme

The idea of our operator splitting scheme here is to discretize in time. Then for each time step we first solve the uncoupled system with an approximated flux function using front tracking. Then we let only the lane switching term work. This gives the next time step, and is easily computed.

We now begin with a proper definition. Let $h > 0$ be the step size, let $u_i^0(x) = u_{i,0}(x)$ for $i = 1, 2$. The discretized version at the k'th time step will be denoted $u_i^k(x)$.

In order to first perform a front tracking step we need to make a piecewise linear approximation to the flux function. Therefore, given $h > 0$, let g_i be a piecewise linear, continuous approximation to f_i, such that $\sup_{x\in[0,1]} |f_i(x) - g_i(x)| \leq h$. We require this function to be constructed by drawing secant lines under f_i. We also require that $g_i(0) = f_i(0) = g_i(1) = f_i(1) = 0$. Since f_i is a concave function we can conclude that g_i will also be concave under this construction. In addition we can apply the mean value theorem together with he concavity of f_i to obtain the bound

$$\tilde{s}(g_1, g_2) = \max_i \sup_{x,y\in[0,1]} \left| \frac{g_i(x) - g_i(y)}{x - y} \right| \leq s_{max} = \sup_{x\in[0,1]} |f_i'(x)|$$

. We define the front tracking step. Let $\mu_i^k(x,t)$ be an entropy solution of

$$\partial_t \mu_i^k + \partial_x(g_i(\mu_i^k)) = 0, \quad \mu_i^k(x,0) = u_i^k(x) \tag{13}$$

We know that in the case where u_i^k is a piecewise constant function with range between zero and one, then so is $\mu_i^k(x,t)$ for all t. This is the first step described above. The next step, which will be referred to as the coupled step, is to let the cars switch lanes. Let η_i^k solve

$$\partial_t \eta_i^k = (-1)^i S(\eta_1^k, \eta_2^k), \quad \eta_i^k(x,0) = \mu_i^k(x,h). \tag{14}$$

We again know this has a solution, as this derivative is taken only in time and not space. Furthermore we can note that if $\mu_i^k(x,h)$ is piecewise constant with range between zero and one for $i = 1, 2$, then so will η_i^k be for any time. Finally we define $u_i^{k+1}(x) = \eta_i^k(x,h)$. The following proposition gives then some nice properties of the scheme.

Proposition 5.1. *Suppose u_i^0 is a step function with compact support taking values between zero and one for $i = 1, 2$. Then, for any natural number k, u_i^k will always be a step function with compact support, and range contained in $[0, 1]$. Thus it will also be a $L^1(\mathbb{R})$ function.*

Proof. To show that the support is compact for any time step, let first U_i^k denote the support of u_i^k, and $U^k = U_1^k \cup U_2^k$. We know by assumption that U^0 is compact as the initial data in both lanes is an integrable step function. We aim to apply an inductive argument to show this, so we start by assuming that u_i^k is a step function with compact support for $i = 1, 2$. We have already argued for u_i^{k+1} being a step function with range between zero and one, so we need only show that U^{k+1} is compact. Compactness gives that $U^k \subset [a, b]$ for some real numbers a and b. Thus the front tracking step cannot expand the range beyond $[a, b + s_{max}h]$. Note that the range can not expand to the left, which can be shown by noting that $g_i(0) = 0$ and inserting into the formula for front velocity. Next, the coupled step will by definition give no change to regions where $\mu_1^k(x,h) = \mu_2^k(x,h) = 0$, and so will not expand the range at all. Thus we get that $U^{k+1} \subset [a, b + s_{max}h]$, and the inductive hypothesis holds. □

Due to this lemma, we can safely conclude that every step of the discretization is well defined. We further expand on the final part of the proof by noting that if the initial condition has support contained in $[a, b]$ then the k'th step and all steps before it will be guaranteed to have range contained in $[a, b + s_{max}kh]$.

For the next proposition we will need

$$\gamma \leq \frac{|S(\theta) - S(\eta)|}{|\theta_1 - \eta_1| + |\theta_2 - \eta_2|}$$

which is of course the Lipschitz constant of the source function with the taxicab norm on $\mathbb{R}^2$.

Proposition 5.2. *Given $u^0 = (u_1^0, u_2^0)$, with $T.V.(u^0) = T.V.(u_1^0) + T.V.(u_2^0)$. Given some time T we can find a bound*

$$T.V.(u^k) = T.V(u_1^k) + T.V.(u_2^k) \leq M(T). \tag{15}$$

where $M(T)$ does not depend on the step size h, but we require $kh \leq T$.

Proof. We look first at a single step. We already know that (13) is total variation diminishing from (9), meaning $T.V.(\mu^k(\cdot, h)) \leq T.V.(u^k)$. Next we look at the coupled step, suppose first $\eta = (\eta_1, \eta_2)$ and $\theta = (\theta_1, \theta_2)$ are solutions to (14) with different initial conditions. Then define $w = \eta - \theta$, let $|w| = |\eta_1 - \theta_1| + |\eta_2 - \theta_2|$. Then we have

$$\partial_t(|w|) = \operatorname{sign}(w_1)(S(\theta) - S(\eta)) + \operatorname{sign}(w_2)(S(\eta) - S(\theta)) \leq 2|S(\theta) - S(\eta)| \leq 2\gamma|w|$$

Using here the notation, $S(\eta) = S(\eta_1, \eta_2)$. We apply Gronwall's inequality here to obtain

$$|w(t)| \leq e^{\gamma t}|w(0)|$$

Using the two initial values $\mu^k(x, h)$ and $\mu^k(x + y, h)$ gives that

$$\left|\eta^k(x, h) - \eta^k(x + y, h)\right| \leq e^{\gamma h}\left|\mu^k(x, h) - \mu^k(x + y, h)\right|$$

This implies that

$$T.V.(u^{k+1}) \leq e^{\gamma h} T.V.(u^k)$$

An inductive argument then gives that

$$T.V.(u^k) \leq e^{\gamma k h} T.V.(u^0).$$

Thus we can set $M = e^{\gamma T} T.V.(u^0)$, which will satisfy (15). □

5.2 Convergence of discretization

We now study the convergence properties of this discretization. Let $\{h_j\}$ be a sequence such that $h_j > 0$, and $h_j \to 0$ as $j \to \infty$. Let $\{g_i^j\}$ denote a sequence of approximate flux functions as described above for h_j. Let $u_i^{n,j}$ denote the discretization using h_j as temporal step size. We define the linear interpolation in the standard way

$$w_i^j(x, t) = u_i^{k-1,j}(x) + \frac{t - h_j(k-1)}{h_j}(u_i^{k,j}(x) - u_i^{k-1,j}(x)) \quad \text{for } t \in ((k-1)h_j, kh_j] \tag{16}$$

We recognize this sequence of interpolants to be a subset of $C([0, \infty), L^1(R))$. We now show first that this sequence will indeed converge to some element in $C([0, \infty), L^1(R))$. Let first $\mathcal{F}_i$ denote the family $\{w_i^j\}$.

Lemma 5.1. *Let $T > 0$ be some time. The family $\mathcal{F}_i$ is relatively compact in $C([0, T], L^1(\mathbb{R}))$.*

Proof. We show this by applying (2.6). We begin with (i), to show this we will apply the Kolmogorov-Riesz compactness theorem. Assume $t_1 = nh_j + \beta_j$ and $t_2 = mh_j + \gamma_j$ such that $0 \leq t_1 < t_2 \leq T$. Then we see immediately that

$$\int_{t_1}^{t_2} f(t)dt = \frac{h_j - \beta_j}{2}(u_i^{n-1,j} + u_i^{n,j}) + \frac{\gamma_j}{2}(u_i^{m+1,j} + u_i^{m,j}) + \sum_{k=n}^{m-1} \frac{h_j}{2}(u_i^{k+1,j} + u_i^{k,j}) \tag{17}$$

Which is a sum of step functions, all with support contained in $[a, b + s_{max}T]$. To see the localization requirement of Kolmogorov-Riesz is satisfied we simply apply (5.1) to see that there is a maximal possible support at any time t, and thus that this will be well localized. Next, we show the tightness requirement of Kolmogorov-Riesz. Lemma 11 from [1], gives that

$$\int_{\mathbb{R}} |f(x+y) - f(x)| dx \leq |y| T.V.(f).$$

We then note that

$$T.V. \left(\int_{t_1}^{t_2} w_i^j(t) dt \right) \leq (t_2 - t_1) M(T),$$

and so tightness is also satisfied. Thus Kolmogorov-Riesz gives that $\{\int_{t_1}^{t_2} f(t)dt | f \in \mathcal{F}_i\}$ is relatively compact in $L^1(\mathbb{R})$, and (i) is satisfied.

We now move on to (ii), meaning we need to show that

$$\text{ess sup}_{t \in I} \int_{\mathbb{R}} \left| w_i^j(x, t+s) - w_i^j(x,t) \right| dx \to 0 \text{ as } s \to 0$$

uniformly in $\mathcal{F}_i$. We start this argument by noting that for a given time w_i^j will, independently of j, have support contained within $[a, b + s_{max}T]$, which has length $l := b - a + 2Ts_{max}$. We look at how big the change will be over one time step,

$$\Delta_h = \int_{\mathbb{R}} \left| u_i^{k+1,j}(x) - w_i^{k,j}(x) \right| dx.$$

We split this again into the change from the coupled step, Δ_η and the change from the front tracking Δ_μ, noting that $\Delta_h \leq \Delta_\eta + \Delta_\mu$. Define $\Gamma = \sup_{x,y \in [0,1]} S(x,y)$, then we have the bound $\Delta_\eta \leq l\Gamma h$. Next we bound the change from the front tracking step. This can easily be done by applying (10), to get the bound

$$\Delta_\mu \leq \|f\|_{Lip} h M(T).$$

We can then combine these to obtain the bound

$$\left| w_i^j(x, t+s) - w_i^j(x,t) \right| dx \leq s(l\Gamma + \|f\|_{Lip} M(T)).$$

From this bound it follows easily that (ii) from (2.6) will also be satisfied. □

Thus the there is some subsequence of h_j such that $w_i^j \to w_i$, where $w_i \in C([0,T], L^1(\mathbb{R}))$. We wish to show that this holds not only for any time T, but also for a time interval going to infinity. This can be done quite simply be utilizing for example a diagonal argument which we give a short outline of here. Start by taking $T = 1$. Then we have the subsequence h_{j_1}, which gives convergence in $C([0,1], L^1(\mathbb{R}))$. Then let $T = 2$, giving a subsequence of h_{j_1}, which we denote h_{j_2}, giving convergence in $C([0,2], L^1(\mathbb{R}))$. This can be done inductively, and we then use the diagonal sequence h_{j_j}. Then for any natural number, n this gives convergence in $C([0,n], L^1(\mathbb{R}))$. Thus we can conclude that we also have convergence in $C([0,\infty), L^1(\mathbb{R}))$.

The next step is to show that w_i indeed is a solution of (1).

Theorem 5.2. *Suppose h_j is a sequence of step sizes, with an accompanying approximation as described previously, such that w_i^j converges to u_i for $i = 1, 2$. Then u_i is a weak solution satisfies* (3)*.*

Proof. Let w_1 denote the linear interpolated approximation based on h, noting that the proof is similar for lane 2. Let also φ be a test function. We now aim to prove,

$$\int_0^\infty \int_{\mathbb{R}} (w_1 \varphi_t + f_1(w_1)\varphi_x - S(w_1, w_2)\varphi) dx dt + \int_{\mathbb{R}} u_{1,0} \varphi|_{t=0} dx \to 0 \quad \text{as } h \to 0 \tag{18}$$

We note that since φ has compact support we can limit the domain we integrate over to $[0, b]$. Choosing m such that $mh > b$, and splitting the integral up along the time axis we get

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} (w_1\varphi_t + f_1(w_1)\varphi_x - S(w_1, w_2)\varphi)dxdt + \int_{\mathbb{R}} u_{1,0}\varphi|_{t=0}dx \\
&= \sum_{k=0}^m \int_{hk}^{h(k+1)} \int_{\mathbb{R}} (w_1\varphi_t + f_1(w_1)\varphi_x - S(w_1, w_2)\varphi)dxdt + \int_{\mathbb{R}} u_{1,0}\varphi|_{t=0}dx \\
&= \int_{\mathbb{R}} u_{1,0}\varphi|_{t=0}dx + \sum_{k=0}^m \int_{\mathbb{R}} (w_1^{k+1}\varphi|_{t=h(k+1)}) - (w_1^k\varphi|_{t=hk})dx \qquad (19) \\
&+ \sum_{k=0}^m \int_{hk}^{h(k+1)} \int_{\mathbb{R}} f_1(w_1)\varphi_x - \varphi\left(\frac{u_1^{k+1} - u_1^k}{h} + S(w_1, w_2)\right)dxdt \qquad (20)
\end{aligned}$$

where the second equality comes from performing a partial integration, and noting that the derivative of w_1 on $[hk, h(k+1)]$ is equal to $\frac{u_1^{k+1}-u_1^k}{h}$. The terms from (19) form a telescoping series, and the last term can be removed since φ is not supported at $t = mh$. Thus, (19) cancels out and can be ignored. We then move on to focusing on (20), focusing first on a single interval.

$$\begin{aligned}
&\int_{hk}^{h(k+1)} \int_{\mathbb{R}} f_1(w_1)\varphi_x - \varphi\left(\frac{u_1^{k+1} - u_1^k}{h} + S(w_1, w_2)\right)dxdt \\
&= \int_{hk}^{h(k+1)} \int_{\mathbb{R}} f_1(w_1)\varphi_x - \varphi\left(\frac{\eta_1^k(x,h) - \mu_1^k(x,h) + \mu_1^k(x,h) - \mu_1^k(x,0)}{h} + S(w_1, w_2)\right)dxdt
\end{aligned}$$

We can now use the definition from (14), and simply integrate to obtain

$$\eta_1^k(x,h) = \mu_1^k(x,h) - \int_0^h S(\eta_1^k, \eta_2^k)dt$$

We can insert these into (20), to obtain

$$\begin{aligned}
&\int_{hk}^{h(k+1)} \int_{\mathbb{R}} f_1(w_1)\varphi_x - \varphi\left(\frac{\mu_1^k(x,h) - \mu_1^k(x,0)}{h}\right)dxdt \qquad (21) \\
&- \int_{hk}^{h(k+1)} \int_{\mathbb{R}} \varphi\left(S(w_1, w_2) - \frac{\int_0^h S(\eta_1^k, \eta_2^k)d\tau}{h}\right)dxdt \qquad (22)
\end{aligned}$$

It makes sense to handle these individually. We start by looking at (22). We can Taylor expand $S(w_1, w_2)$ about $t = h(k+1)$. This gives the term

$$\begin{aligned}
&S(w_1, w_2) - \frac{\int_0^h S(\eta_1^n, \eta_2^n)d\tau}{h} \\
&= S(\eta_1^n(x,h), \eta_2^n(x,h)) + \frac{d}{dt}(S(w_1(x,\zeta), w_2^n(x,\zeta)))(t - h(n+1)) \\
&- \frac{\int_0^h S(\eta_1^n(x,h), \eta_2^n(x,h)) + \frac{d}{dt}S(\eta_1^n(x,\xi(\tau)), \eta_2^n(x,\xi(\tau))(\tau - h))d\tau}{h} \\
&= \frac{d}{dt}(S(w_1(x,\zeta), w_2(x,\zeta)))(t - h(k+1)) - \frac{\int_0^h \frac{d}{dt}S(\eta_1^n(x,\xi(\tau)), \eta_2^n(x,\xi(\tau))(\tau - h))d\tau}{h}
\end{aligned}$$

Where we can take the explicit derivative by applying the chain rule, where we assume z_i can be either w_i or η_i, and v_i is shorthand for $v_i(z_i)$.

$$\frac{d}{dt}S(z_1, z_2) = (v_2 - v_1)([v_1 \le v_2]\partial_t z_1 + [v_1 > v_2]\partial_t z_2) + (\partial_t z_2 v_2' - \partial_t z_1 v_1')\begin{cases} z_1 & v_2 \ge v_1 \\ z_2 & v_2 < v_1 \end{cases} \qquad (23)$$

We note that this might be discontinuous at $v_1 = v_2$. However, in that case the solution would simply be constant, and so trivial to handle. Since both velocity functions are continuously differentiable on a compact domain, this term will be bounded given that $\partial_t z_i$ is bounded. We know that $\partial_t w_i$ is bounded by a similar argument as used when showing (ii) for existence of a convergent subsequence. Let then B_w denote this bound on the absolute value of $\frac{d}{dt}S(w_1, w_2)$, and let B_η denote the bound on the absolute value of $\frac{d}{dt}S(\eta_1, \eta_2)$. Then we can conclude with the bound

$$\left| S(w_1, w_2) - \frac{\int_0^h S(\eta_1^n, \eta_2^n) d\tau}{h} \right| \leq h(B_w + B_\eta) \tag{24}$$

We move on to (21). Similar to above, we insert

$$\mu_1^k(x, h) = \mu_1^k(x, 0) - \int_0^h \partial_x g_1(\mu_1^k) d\tau$$

Which gives

$$\begin{aligned} &\int_{hk}^{h(k+1)} \int_{\mathbb{R}} (\varphi_x f_1(w_1) + \frac{\varphi}{h} \int_0^h \partial_x g_1(\mu_1^k) d\tau) dx dt \\ &= \int_{hk}^{h(k+1)} \int_{\mathbb{R}} \varphi_x \left(f_1(w_1) - \frac{\int_0^h g_1(\mu_1^k) d\tau dx dt}{h} \right) dx dt \end{aligned}$$

We do a Taylor expansion about $f_1(u_1^k)$,

$$\begin{aligned} &f_1(u_1^k) + \left(\frac{d}{dt} f_1(w_1(x, \zeta)) \right) (t - hk) - \frac{\int_0^h (g_1(\mu_1^k) - f_1(\mu_1^k) + f_1(\mu_1^k)) d\tau}{h} \\ &= \left(\frac{d}{dt} f_1(w_1(x, \zeta)) \right) (t - hk) - \frac{\int_0^h (g_1(\mu_1^k) - f_1(\mu_1^k)) d\tau}{h} \end{aligned}$$

By assumption, we have $|g_1(u) - f_1(u)| \leq h$. Furthermore, we have

$$\frac{d}{dt} f(w_1) = (v_1(w_1) + w_1 v_1'(w_1)) \partial_t w_1 \leq B_f$$

As with the previous part where we bounded (22), ∂w_1 has a bound independent of h, thus we can do the total bound

$$\left| \int_{hk}^{h(k+1)} \int_{\mathbb{R}} (\varphi_x f_1(w_1) + \frac{\varphi}{h} \int_0^h \partial_x g_1(\mu_i^k) d\tau) dx dt \right| \leq \int_{hk}^{h(k+1)} \int_{\mathbb{R}} |\varphi_x (B_f + 1) h| dx dt \tag{25}$$

Thus every expression can be bounded by some constant times h, and we conclude that if a sequence w_1^k converges to something, it must satisfy (3). $\square$

Theorem 5.3. *Let $\{h_j\}$ be a sequence of step sizes such that $h_j > 0$ and $h_j \to 0$ as $j \to \infty$, with $\{g_j\}$ an accompanying sequence of approximate flux functions as before. Then let $w^j = \{w_1^j, w_2^j\}$ be a sequence of elements from $C([0, \infty), L^1(\mathbb{R}))$ generated by our semi-discretization, such that $w^k \to u$. Then u is an entropy solution of* (1). *Meaning u satisfies* (4).

Proof. As for the previous theorem we do only the case for lane one, and note that the proof is similar for the other lane. Let us also pick one specific $h > 0$, along with an accompanying approximate flux function g_i. We now insert into the left hand side of (4), using the same arguments as from the start of the last proof to obtain,

$$\int_0^\infty \int_{\mathbb{R}} (\lambda(u_1)\varphi_t + q_1(u_1)\varphi_x) dxdt + \int_{\mathbb{R}} \lambda(u_{1,0})\varphi|_{t=0} dx$$
$$= \sum_{k=0}^{m} \int_{hk}^{h(k+1)} \int_{\mathbb{R}} (q_1(u_1)\varphi_x - \varphi \partial_t \lambda(w_1)) dxdt$$

where the index m, is so that $\operatorname{supp} \varphi \subset \mathbb{R} \times [0, mh]$. Again, using similar techniques as the last proof we get

$$\int_{hk}^{h(k+1)} \int_{\mathbb{R}} (-\varphi \partial_t \lambda(w_1)) dxdt = \int_{hk}^{h(k+1)} \int_{\mathbb{R}} \left(-\varphi \lambda'(w_1) \frac{u_1^{k+1} - u_1^k}{h} \right) dxdt$$
$$= \int_{hk}^{h(k+1)} \int_{\mathbb{R}} (-\varphi \lambda'(w_1) \frac{\int_0^h S(\eta_1^k(x,\tau), \eta_2^k(x,k\tau)) d\tau + \mu_1^k(x,h) - \mu_1^k(x,0)}{h}) dxdt$$

We leave the term containing the coupled function for later, and make the derivation

$$\int_{hk}^{h(k+1)} \int_{\mathbb{R}} \left(\varphi_x q_1(w_1) - \varphi \lambda'(w_1) \frac{\mu_1^k(x,h) - \mu_1^k(x,0)}{h} \right) dxdt$$
$$= \int_{hk}^{h(k+1)} \int_{\mathbb{R}} \left(\varphi_x q_1(w_1) + \varphi_t(\lambda(\mu_1^k) - \lambda(\mu_1^k)) - \varphi \lambda'(w_1) \left(\frac{\mu_1^k(x,h) - \mu_1^k(x,0)}{h} \right) \right) dxdt$$

Again, we split this into the following two integrals

$$\int_{hk}^{h(k+1)} \int_{\mathbb{R}} \left(\varphi_x q_1(w_1) + \varphi_t \lambda(\mu_1) \right) dxdt \tag{26}$$
$$\int_{hk}^{h(k+1)} \int_{\mathbb{R}} \left(-\lambda(\mu_1)\varphi_t - \varphi \lambda'(w_1) \left(\frac{\mu_1^k(x,h) - \mu_1^k(x,0)}{h} \right) \right) dxdt \tag{27}$$

We start with (26). By use of another Taylor expansion we obtain

$$q_1(w_1(x,t)) = q_1(\mu_1^k(x,t)) + q_1'(\zeta w_1(t,x) + (1-\zeta)\mu_1^k(x,t))(\zeta(w_1 - \mu_1^k)) \tag{28}$$

Here we use that $q_1(u)$ is bounded, $|\zeta| \leq 1$, and that $\int_{hk}^{h(k+1)} \int_{\mathbb{R}} |w_1 - \mu_1^k| dxdt$ goes to zero with the step size, to conclude that we can safely state that the term with q_1' will go to zero as h goes to zero. Then we note that since μ_1^k should be an entropy solution to the problem (13), we know that the following inequality will be satisfied

$$\sum_{k=0}^{m} \int_{hk}^{h(k+1)} \int_{\mathbb{R}} (\lambda(\mu_1^k)\varphi_t + q(\mu_1^k)\varphi_x) dxdt \geq \sum_{k=0}^{m} \int_R \lambda(\mu_1^k)\varphi|_{t=h(k+1)} dx - \int_R \lambda(\mu_1^k)\varphi|_{t=hk} dx \tag{29}$$

We know however that the right hand side of this equation will sum to zero. Thus we can gather up the approximations thus far into the inequality

$$\int_0^\infty \int_{\mathbb{R}} (\lambda(u_1)\varphi_t + q_1(u_1)\varphi_x) dxdt + \int_{\mathbb{R}} \lambda(u_{1,0})\varphi|_{t=0} dx$$
$$\geq \sum_{k=0}^{m} \int_{hk}^{h(k+1)} \int_{\mathbb{R}} -\varphi \lambda'(w_1) \frac{\int_0^h S(\eta_1^k(x,\tau), \eta_2^k(x,\tau)) d\tau}{h} dxdt \tag{30}$$
$$-\sum_{k=0}^{m} \int_{hk}^{h(k+1)} \int_{\mathbb{R}} \left(\varphi_t \lambda(\mu_1^k) + \varphi \lambda'(w_1) \left(\frac{\mu_1^k(x,h) - \mu_1^k(x,0)}{h} \right) \right) dxdt \tag{31}$$

We will now aim to show that (31) goes to zero when the step size goes to zero. Then we will show that (30) goes to the right hand side of (4).

Next we look closer at (31). Here we can first do a partial integration

$$\sum_{k=0}^{m}\int_{hk}^{h(k+1)}\int_{\mathbb{R}}\left(\varphi_t\lambda(\mu_1^k)\right)dxdt = \sum_{k=0}^{m}\int_{hk}^{h(k+1)}\int_{\mathbb{R}}\left(\varphi\lambda'(\mu_1^k)\partial_t\mu_1^k\right)$$

We can do a Taylor expansion about $\lambda'(w_1)$ as done before. This argument is really the same as previous Taylor expansions, and is therefore dropped here. Inserting this into (31), and focusing on a single integral gives the expression

$$\begin{aligned}&\int_{hk}^{h(k+1)}\int_{\mathbb{R}}\varphi\lambda'(w_1)\left(\partial_t\mu_1^k + \frac{\int_0^h \partial_x g_1(\mu_1^k)d\tau}{h}\right)dxdt\\ &= \int_{hk}^{h(k+1)}\int_{\mathbb{R}}\varphi\lambda'(w_1)\left(\frac{\int_0^h \partial_x g_1(\mu_1^k)d\tau}{h} - \partial_x g_1(\mu_1^k)\right)dxdt\\ &= \int_{hk}^{h(k+1)}\int_{\mathbb{R}}\varphi\lambda'(w_1)\left(\frac{\int_0^h \partial_x g_1(\mu_1^k(x,\tau)) - \partial_x g_1(\mu_1^k(x,t))d\tau}{h}\right)dxdt\end{aligned}$$

Where we know that the final term goes to zero as h goes to zero for almost every (x, h). Thus we conclude that (31) goes to zero. Finally we go on to handling (30).

$$\begin{aligned}&\int_{hk}^{h(k+1)}\int_{\mathbb{R}}\varphi\lambda'(w_1)\frac{\int_0^h S(\eta_1^k(x,\tau),\eta_2^k(x,\tau))d\tau}{h}dxdt\\ &= \int_{hk}^{h(k+1)}\int_{\mathbb{R}}\varphi\lambda'(w_1)\left(\frac{\int_0^h S(\eta_1^k(x,\tau),\eta_2^k(x,\tau))d\tau}{h} - S(w_1,w_2) + S(w_1,w_2)\right)dxdt\\ &= \int_{hk}^{h(k+1)}\int_{\mathbb{R}}\varphi\lambda'(w_1)S(w_1,w_2)dxdt\\ &+ \int_{hk}^{h(k+1)}\int_{\mathbb{R}}\varphi\lambda'(w_1)\left(\frac{\int_0^h S(\eta_1^k(x,\tau),\eta_2^k(x,\tau))d\tau}{h} - S(w_1,w_2)\right)dxdt\end{aligned}$$

We see that the third line is the expression we want. We can apply the bound from (24) to get the bound

$$\left|\int_{hk}^{h(k+1)}\int_{\mathbb{R}}\varphi\lambda'(w_1)\left(\frac{\int_0^h S(\eta_1^k(x,\tau),\eta_2^k(x,\tau))d\tau}{h} - S(w_1,w_2)\right)dxdt\right| \leq \int_{hk}^{h(k+1)}\int_{\mathbb{R}}|\varphi||\lambda'(w_1)|h(B_w+B_\eta)$$

Thus we have shown that

$$\int_0^\infty\int_{\mathbb{R}}(\lambda(u_1)\varphi_t+q_1(u_1)\varphi_x)dxdt+\int_{\mathbb{R}}\lambda(u_{1,0})\varphi|_{t=0}dx \geq -\int_{hk}^{h(k+1)}\int_{\mathbb{R}}\varphi\lambda'(w_1)S(w_1,w_2)dxdt+\xi(w_1,w_2) \tag{32}$$

where $\xi \to 0$ as $h \to 0$. Thus we can conclude that the point of convergence is indeed the entropy solution of (1). $\square$

We remark here that this proof is not quite finished currently. Due to time limitations some shortcuts were taken in the proof that (31) goes to zero. Here some arguments using smooth approximations are likely what is needed to make this argument more rigorous.

6 Conclusion and further work

With the introduction of the operator splitting scheme we now have a tool which can be used to analyze the asymptotic behavior of (1). The next step would be to first apply this scheme to the Riemann problem. Then combine different Riemann problems into the general case for a step function. The final step is then use step functions to approximate general L^1 initial conditions, for a general result on the asymptotic behavior. Due to time limitations this work must however be saved for a later work.

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