Toeplitz Embedding for Non-Cartesian Subspace Reconstruction - Preliminary Theory and Methods

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May 18, 2023

1 Purpose

This is a rough draft of the theory and methodology behind the Toeplitz subspace approach. We provide this document to assist in understanding the code.

2 Theory

2.1 Toeplitz Embedding

We first consider the non-Cartesian SENSE [1] forward model

$$\underbrace{\mathbf{F}\begin{bmatrix}\mathbf{S}_1 \\ \vdots \\ \mathbf{S}_C\end{bmatrix}}_{\mathbf{A}}\mathbf{x} = \mathbf{b} \tag{1}$$

where $\mathbf{F}$ is the Non-Uniform Fourier Transform (NUFFT) operator, $\mathbf{S}_i$ form a set of C sensitivity maps, $\mathbf{x}$ is the volume of interest, and $\mathbf{b}$ is the acquired data from the scanner. After acquiring the scanner data $\mathbf{b}$, the volume $\mathbf{x}$ is usually reconstructed by solving the regularized inverse problem:

$$\underset{\mathbf{x}}{\operatorname{argmin}} \, ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2^2 + R(\mathbf{x}) \tag{2}$$

Due to the problem scale of MRI, it is not computationally tractable to analytically solve (2). Thus, iterative methods [2] are used to solve for the reconstructed volume. These iterative methods will almost always involve the repeated application of the Gram matrix $\mathbf{A}^H\mathbf{A}$. The Gram matrix is costly to compute due to the NUFFT term. An NUFFT consists of a Fast Fourier Transform (FFT) followed by a Kaiser-Bessel interpolation operation, which can become computationally expensive. Hence, it is desirable to eliminate the need for an NUFFT, and only use FFTs.

Previous works [3], [4] on Toeplitz embedding show that the operator $\mathbf{F}^H\mathbf{F}$ is a shift-invariant operator and therefore has a block-Toeplitz structure that can be exploited:

$$\mathbf{F}^H\mathbf{F} = \tilde{R}^H\tilde{\mathcal{F}}^H\mathbf{diag}(\tilde{\mathbf{T}})\tilde{\mathcal{F}}\tilde{R} \tag{3}$$

Here, we use the $\tilde{}$ notation to denote an operation with a twice oversampled representation. $\tilde{\mathcal{F}}$ is a twice oversampled FFT, $\tilde{R}$ zero pads the input volume by $2\times$ in each spatial dimension, $\tilde{\mathbf{T}}$ is called the Toeplitz kernel, and the operator $\tilde{R}^H$ crops a $2\times$ padded volume down to the original dimensions of $\mathbf{x}$. According to [3], the Toeplitz kernel can be computed as

$$\tilde{\mathbf{T}} = \tilde{\mathcal{F}}\tilde{\mathbf{F}}^H\tilde{\mathbf{F}}\tilde{\delta} \tag{4}$$

where $\tilde{\mathbf{F}}$ is an NUFFT on the zero-padded input volume, and δ is the Dirac delta function. Avoiding Kaiser-Bessel interpolation by replacing $\mathbf{F}^H\mathbf{F}$ with $\tilde{R}^H\tilde{\mathcal{F}}^H\mathbf{diag}(\tilde{\mathbf{T}})\tilde{\mathcal{F}}\tilde{R}$ as in (3) leads to significant speedups, but comes at the expense of the memory required to store $\tilde{\mathbf{T}}$.

2.2 Subspace Reconstruction with Toeplitz Embedding

Many time-resolved MRI acquisitions [5], [6], [7] reconstruct a time series of T volumes $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_T)$. We will assume that each time-point t with corresponding volume $\mathbf{x}_t$ has a unique sampling trajectory, and hence will have a corresponding unique NUFFT $\mathbf{F}_t$ with Toeplitz embedding $\mathbf{diag}(\mathbf{T}_t)$. The idea of a subspace reconstruction [5], [6], [7] is to leverage a temporal subspace to reduce the dimensionality of the problem:

$$\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_T \end{bmatrix} = \Phi \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_K \end{bmatrix} \tag{5}$$

where Φ is the linear subspace with dimension $[T, K]$. The linear subspace represented by Φ is typically determined by taking the first K principal components of a matrix containing all possible signal evolutions. We can incorporate this into the forward model as

$$\begin{bmatrix} \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_T \end{bmatrix} \begin{bmatrix} \mathbf{S}_1 \\ \vdots \\ \mathbf{S}_C \end{bmatrix} \mathbf{\Phi}\alpha = \underbrace{\mathbf{F}_s\mathbf{S}\mathbf{\Phi}}_{\mathbf{A}_s}\mathbf{b} \tag{6}$$

where $\mathbf{A}_s$ is the subspace forward operator, and $\mathbf{F}_s$ contains the T NUFFTs. It can be shown that the subspace Gram matrix is then

$$\mathbf{A}_s^H\mathbf{A}_s\alpha = \sum_{i=1}^{C} \mathbf{S}_i^H\tilde{R}^H\tilde{\mathcal{F}}^H \underbrace{\begin{bmatrix} \mathbf{M}_{11} & \cdots & \mathbf{M}_{1K} \\ \vdots & & \vdots \\ \mathbf{M}_{K1} & \cdots & \mathbf{M}_{KK} \end{bmatrix}}_{\mathbf{M}} \tilde{\mathcal{F}}\tilde{R}\mathbf{S}_i \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_K \end{bmatrix} \tag{7}$$

The relationship between the symmetric operator $\mathbf{M}$ and the time-series NUFFTs is a grid structure of K^2 Toeplitz kernels:

$$\mathbf{M}_{ij} = \sum_{t=1}^{T} \bar{\mathbf{\Phi}}_{ti}\mathbf{diag}(\tilde{\mathbf{T}}_t)\mathbf{\Phi}_{tj} \tag{8}$$

Equations (7), (8), show that there exists a much more compact subspace Toeplitz matrix. This can be efficiently computed without the need to compute Toeplitz embeddings for each time point, as shown in Algorithm 1.

3 Methods

Algorithm 1 Computing Toeplitz Embedding Matrix M

Input: Over-sampled Time-Series NUFFTs $\tilde{F}_s$, Temporal Subspace Φ

1: **for** $j \in \{1, \cdots, K$ **do** ▷ The terms with ˜ are zero-padded by 2×
2: $\tilde{\alpha} \leftarrow \tilde{0}$ ▷ Initialize to zeros
3: $\tilde{\alpha_j} \leftarrow \tilde{\delta}$ ▷ Set the j^{th} coefficient to a twice-oversampled delta
4: $\{M_{1j} \cdots M_{Kj}\} \leftarrow \tilde{\mathcal{F}}\Phi^H \tilde{F_s}^H \tilde{F}_s \Phi \tilde{\alpha}$ ▷ Compute j^{th} column of M
5: **end for**
6: **return** M

From equations (4), (7), (8), we can formulate a simple, memory efficient algorithm for calculating $\mathbf{M}$. Instead of computing T Toeplitz kernels, we instead use a PSF approach similar to [3]. The details of this algorithm are shown in Algorithm 1. Note that we use the ˜ notation to denote image space zero-padding by a factor of 2. After completing Algorithm 1, one must store the Toeplitz embedding matrix $\mathbf{M}$ in memory. Since $\mathbf{M}$ is symmetric, we will need to store $K(K+1)/2$ embeddings. Each embedding has a twice oversampled volume size, so the total memory requirement will be $\frac{K(K+1)}{2}(2N)^d$, where d is the dimension of the problem (usually d=2 or d=3).

During image reconstruction, we will need to repeatedly apply $\mathbf{M}$ to α. In order to combat GPU memory constraints, we store the relevant terms of $\mathbf{M}$ in CPU memory, and only move the rows of $\mathbf{M}$ on and off the GPU to perform the dot product with α.

References

[1] Pruessmann KP, Weiger M, Börnert P, Boesiger P. Advances in sensitivity encoding with arbitrary k-space trajectories. Magn. Reson. Med. 2001; 46:638–651.

[2] Fessler JA, Noll C. Iterative reconstruction methods for non-cartesian mri. Iterative reconstruction methods for non-cartesian mri, 2007.

[3] Baron CA, Dwork N, Pauly JM, Nishimura DG. Rapid compressed sensing reconstruction of 3D non-cartesian MRI. Magn. Reson. Med. 2018; 79:2685–2692.

[4] Fessler JA, Lee S, Olafsson VT, Shi HR, Noll DC. Toeplitz-based iterative image reconstruction for MRI with correction for magnetic field inhomogeneity. IEEE Trans. Signal Process. 2005; 53:3393–3402.

[5] Tamir JI, Uecker M, Chen W, Lai P, Alley MT, Vasanawala SS, Lustig M. T 2 shuffling: Sharp, multicontrast, volumetric fast spin-echo imaging. Magn. Reson. Med. 2017; 77:180–195.

[6] Dong Z, Wang F, Reese TG, Bilgic B, Setsompop K. Echo planar time-resolved imaging with subspace reconstruction and optimized spatiotemporal encoding. Magn. Reson. Med. 2020; 84:2442–2455.

[7] Cao X, Liao C, Iyer SS, Wang Z, Zhou Z, Dai E, Liberman G, Dong Z, Gong T, He H, Zhong J, Bilgic B, Setsompop K. Optimized multi-axis spiral projection MR fingerprinting with subspace reconstruction for rapid whole-brain high-isotropic-resolution quantitative imaging. Magn. Reson. Med. 2022; 88:133–150.

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