

West & Harrison

Chapter 10, Problem 7

The Dans

Thursday, March 16, 2017

The Data

The quarterly sales and cost index of a confectionary product are given as

Year	SALES				COST			
	Quarter 1	Quarter 2	Quarter 3	Quarter 4	Quarter 1	Quarter 2	Quarter 3	Quarter 4
1975	157	227	240	191	10.6	8.5	6.7	4.1
1976	157	232	254	198	1.9	0.4	1.1	1.9
1977	169	234	241	167	0.2	2.9	4.1	-1.4
1978	163	227	252	185	-4.0	-4.5	-5.3	-8.4
1979	179	261	264	196	-12.8	-13.2	-10.1	-4.6
1980	179	248	256	193	-1.1	-0.1	0.0	-2.5
1981	186	260	270	210	-5.1	-6.4	-8.0	-6.5
1982	171	227	241	170	-3.7	-1.3	6.1	16.5
1983	140	218	208	193	22.9	23.9	18.0	8.3
1984	184	235	245	209	2.9	0.7	-2.4	-7.0
1985	206	260	264	227	-9.8	-10.6	-12.3	-13.2

The Goal

Fit a DLM to the sales with first-order polynomial, regression on cost, and full seasonal components.
Based on previous information:

- ▶ the underlying level of the series when cost is zero is 220 with a nominal standard error of 15
- ▶ the regression coefficient of cost is estimated as -1.5 with a standard error of about 0.7
- ▶ the seasonal factors for the four quarters of the first year are expected to be -50, 25, 50, -25, with nominal standard errors of 25, 15, 25, and 15, respectively
- ▶ the trend, regression, and seasonal components are initially assumed to be uncorrelated
- ▶ the observational variance is estimated as 100, with initial degrees of freedom of 12

Part A

Derive the appropriate initial prior that satisfies the zero-sum constraint.

Theorem 8.2: Imposing the constraint $\mathbf{1}'\phi = 0$ on the prior $(\phi_0|\mathcal{D}_0^) \sim N(\mathbf{m}_0^*, \mathbf{C}_0^*)$ and writing $U = \mathbf{1}'\mathbf{C}_0^*\mathbf{1}$ and $\mathbf{A} = \mathbf{C}_0^*\mathbf{1}/U$ gives the revised joint prior*

$$\begin{aligned}(\phi_0|\mathcal{D}_0) &\sim N(\mathbf{m}_0, \mathbf{C}_0), \\ \mathbf{m}_0 &= \mathbf{m}_0^* - \mathbf{A}\mathbf{1}'\mathbf{m}_0^*, \\ \mathbf{C}_0 &= \mathbf{C}_0^* - \mathbf{A}\mathbf{A}'U.\end{aligned}$$

Part A

The given information on the seasonal components is

$$\mathbf{m}_0^* = (-25, -50, 25, 50)' \quad \mathbf{C}_0^* = \begin{pmatrix} 225 & 0 & 0 & 0 \\ 0 & 625 & 0 & 0 \\ 0 & 0 & 225 & 0 \\ 0 & 0 & 0 & 625 \end{pmatrix}$$

Remember to shift the seasonal components back one quarter, as time $t = 0$ is the fourth quarter of 1974!

Part A

Applying Theorem 8.2 yields

$$U = 1700$$

$$\mathbf{A} = (0.1323529, 0.3676471, 0.1323529, 0.3676471)'$$

$$\mathbf{m}_0^s = (-25, -50, 25, 50)'$$

$$\mathbf{C}_0^s = \begin{pmatrix} 195.22 & -82.72 & -29.78 & -82.72 \\ -82.72 & 395.22 & -82.72 & -229.78 \\ -29.78 & -82.72 & 195.22 & -82.72 \\ -82.72 & -229.78 & -82.72 & 395.22 \end{pmatrix}.$$

The superscript s denotes that this is for the seasonal component of the model.

Part A

Therefore, the entire initial prior is

$$\mathbf{m}_0 = (220, -1.5, -25, -50, 25, 50)'$$

$$\mathbf{C}_0 = \begin{pmatrix} 225.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.49 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 195.22 & -82.72 & -29.78 & -82.72 \\ 0.00 & 0.00 & -82.72 & 395.22 & -82.72 & -229.78 \\ 0.00 & 0.00 & -29.78 & -82.72 & 195.22 & -82.72 \\ 0.00 & 0.00 & -82.72 & -229.78 & -82.72 & 395.22 \end{pmatrix}.$$

Part B

Identify the full initial prior quantities for the 6-dimensional state vector θ_1 and the observational variance V . Write down the defining quantities a_1 , R_1 , n_0 , and S_0 based on the above initial information.

Part B

The DLM model in this case is defined as $\{\mathbf{F}_t, \mathbf{G}, V, \mathbf{W}_t\}$, with

$$\mathbf{F}_t = (1, \text{Cost}_t, 1, 0, 0, 0)' \quad \mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Part B

$$\mathbf{a}_1 = \mathbf{G}\mathbf{m}_0 = (220, -1.5, 25, 50, -25, -50)'$$

$$\mathbf{R}_1 = \mathbf{G}\mathbf{C}_0\mathbf{G}' + \mathbf{W}_t = \begin{pmatrix} 225.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.49 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 395.22 & -82.72 & -229.78 & -82.72 \\ 0.00 & 0.00 & -82.72 & 195.22 & -82.72 & -29.78 \\ 0.00 & 0.00 & -229.78 & -82.72 & 395.22 & -82.72 \\ 0.00 & 0.00 & -82.72 & -29.78 & -82.72 & 195.22 \end{pmatrix} + \mathbf{W}_t$$

$$n_0 = 12$$

$$S_0 = 100.$$

Therefore, $(\boldsymbol{\theta}_1 | \mathcal{D}_0) \sim T_{12}(\mathbf{a}_1, \mathbf{R}_1)$ and $(V | \mathcal{D}_0) \sim \text{Inverse Gamma}(n_0/2, d_0/2)$, where $d_0 = n_0 S_0 = 1200$.

Part C

Use three discount factors to structure the evolution matrices of the model: δ_T for the constant intercept term, δ_R for the regression coefficient, and δ_S for the seasonal factors. Consider initially the values $\delta_T = \delta_S = 0.9$ and $\delta_R = 0.98$. Fit the model and perform the retrospective, filtering calculations to obtain filtered estimates of the state vector and all model components over time.

Part C

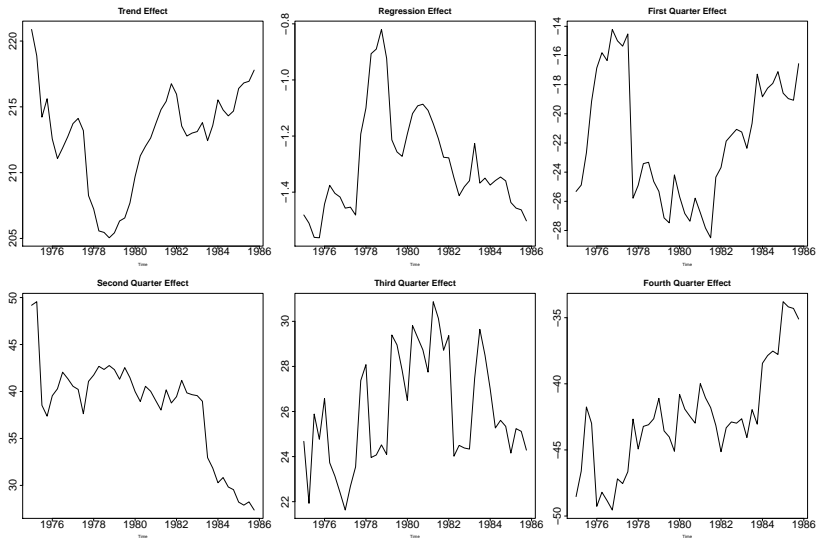


Figure: Filtered estimates for all components of the state vector

Part C

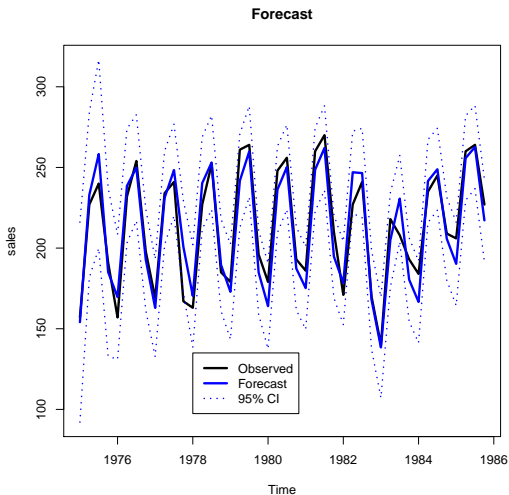
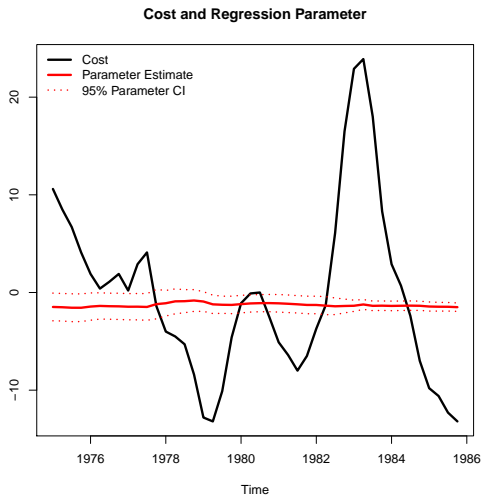


Figure: One-step ahead forecast estimate and 95% confidence interval

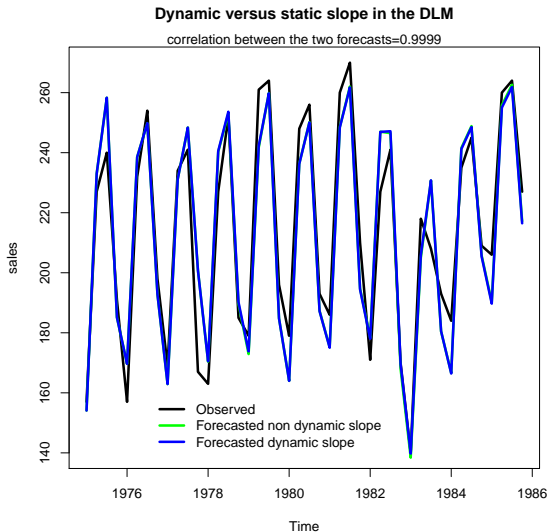
Part D

Based on this analysis, verify that the regression parameter on Cost is, in retrospect, rather stable over time.

Part D



Part D



Part E

Produce step ahead forecasts from the end of the data in the fourth quarter of 1985 for the next three years. The estimated values of Cost to be used in forecasting ahead are given by

Year	Quarter			
	1	2	3	4
1986	8.4	10.6	7.2	13.0
1987	-2.9	-0.7	-6.4	-7.0
1988	-14.9	-15.9	-18.0	-22.3

Part E

Appealing to Harrison and West [1999] page 199, and with $\mathbf{F}(k)$ as the \mathbf{F} vector at time k , the k -step forecast distributions are

$$(y(k)|\mathcal{D}_T) \sim T_{38}(F_T(k), Q_T(k))$$

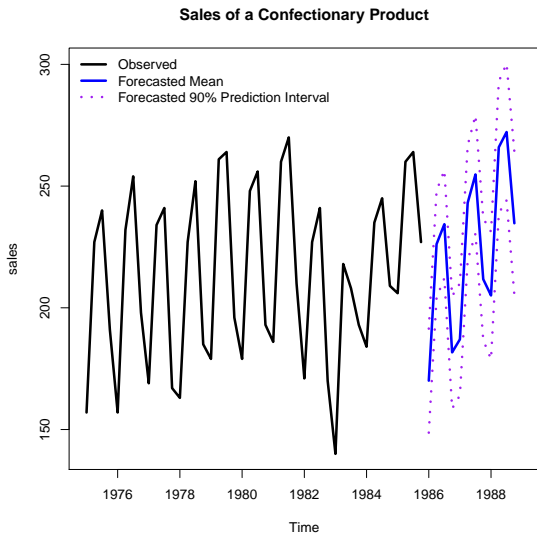
$$F_T(k) = \mathbf{F}(k)' \mathbf{a}_T(k)$$

$$Q_T(k) = \mathbf{F}(k)' \mathbf{R}_T(k) \mathbf{F}(k)$$

$$\mathbf{a}_T(k) = \mathbf{G} \mathbf{a}_T(k-1)$$

$$\mathbf{R}_T(k) = \mathbf{G} \mathbf{R}_T(k-1) \mathbf{G}' \text{diag} \left(\frac{1}{\delta_t^k}, \frac{1}{\delta_r^k}, \frac{1}{\delta_s^k}, \frac{1}{\delta_s^k}, \frac{1}{\delta_s^k}, \frac{1}{\delta_s^k} \right)$$

Part E



References

Jeff Harrison and Mike West. *Bayesian forecasting & dynamic models*. Springer New York, 1999.

Raquel Prado and Mike West. *Time series: modeling, computation, and inference*. CRC Press, 2010.