West & Harrison Chapter 10, Problem 7

The Dans

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The quarterly sales and cost index of a confectionary product are given in figure 1. The goal is to fit a DLM to the sales

	SALES				COST			
	Quarter				Quarter			
Year	1	2	3	4	1	2	3	4
1975	157	227	240	191	10.6	8.5	6.7	4.1
1976	157	232	254	198	1.9	0.4	1.1	1.9
1977	169	234	241	167	0.2	2.9	4.1	−1.4
1978	163	227	252	185	−4.0	−4.5	−5.3	−8.4
1979	179	261	264	196	−12.8	−13.2	−10.1	−4.6
1980	179	248	256	193	−1.1	−0.1	0.0	−2.5
1981	186	260	270	210	−5.1	−6.4	−8.0	−6.5
1982	171	227	241	170	−3.7	−1.3	6.1	16.5
1983	140	218	208	193	22.9	23.9	18.0	8.3
1984	184	235	245	209	2.9	0.7	−2.4	−7.0
1985	206	260	264	227	−9.8	−10.6	−12.3	−13.2

Figure 1: The data associated with the problem

with first-order polynomial, regression on cost, and full seasonal components. Based on previous information,

- the underlying level of the series when cost is zero is 220 with a nominal standard error of 15
- the regression coefficient of cost is estimated as -1.5 with a standard error of about 0.7
- the seasonal factors for the four quarters of the first year are expected to be -50, 25, 50, -25, with nominal standard errors of 25, 15, 25, and 15, respectively
- the trend, regression, and seasonal components are initially assumed to be uncorrelated
- the observational variance is estimated as 100, with initial degrees of freedom of 12

Using this information, we are to analyze the series along the following lines:

(a) **Using the information provided for the seasonal factors above, apply Theorem 8.2 to derive the appropriate initial prior that satisfies the zero-sum constraint.**

Theorem 8.2: Imposing the constraint $\mathbf{1}'\boldsymbol{\phi} = 0$ *on the prior* $(\boldsymbol{\phi}_0|\mathcal{D}_0^*) \sim N(\mathbf{m}_0^*, \mathbf{C}_0^*)$ *and writing* $U = \mathbf{1}'\mathbf{C}_0^*\mathbf{1}$ *and* $\mathbf{A} = \mathbf{C}_0^*\mathbf{1}/U$ *gives the revised joint prior*

$$(\boldsymbol{\phi}_0|\mathcal{D}_0) \sim \mathrm{N}(\mathbf{m}_0, \mathbf{C}_0),$$
$$\mathbf{m}_0 = \mathbf{m}_0^* - \mathbf{A}\mathbf{1}'\mathbf{m}_0^*,$$
$$\mathbf{C}_0 = \mathbf{C}_0^* - \mathbf{A}\mathbf{A}'U.$$

Using the given information for the seasonal components,

$$\mathbf{m}_0^* = (-50, 25, 50, -25)' \qquad \mathbf{C}_0^* = \begin{pmatrix} 625 & 0 & 0 & 0 \\ 0 & 225 & 0 & 0 \\ 0 & 0 & 625 & 0 \\ 0 & 0 & 0 & 225 \end{pmatrix}.$$

Applying Theorem 8.2 yields

$$
\begin{aligned}
U &= 1700 \\
A &= (0.3676471, 0.1323529, 0.3676471, 0.1323529)' \\
\mathbf{m}_0^s &= (-50, 25, 50, -25)' \\
\mathbf{C}_0^s &= \begin{pmatrix} 395.22 & -82.72 & -229.78 & -82.72 \\ -82.72 & 195.22 & -82.72 & -29.78 \\ -229.78 & -82.72 & 395.22 & -82.72 \\ -82.72 & -29.78 & -82.72 & 195.22 \end{pmatrix}.
\end{aligned}
$$

This leads to the initial prior satisfying the zero-sum constraint for the seasonal components to be $(\boldsymbol{\phi}_0|\mathcal{D}_0) \sim \mathrm{N}(\mathbf{m}_0^s, \mathbf{C}_0^s)$. Given a state vector $\boldsymbol{\theta}_t = (\theta_{1,t}, \theta_{2,t}, \phi_{1,t}, \ldots, \phi_{4,t})'$, the initial prior for the DLM is of the form $(\boldsymbol{\theta}_0|\mathcal{D}_0) \sim \mathrm{N}(\mathbf{m}_0, \mathbf{C}_0)$, where

$$
\begin{aligned}
\mathbf{m}_0 &= (220, -1.5, -50, 25, 50, -25)' \\
\mathbf{C}_0 &= \begin{pmatrix} 225.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.49 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 395.22 & -82.72 & -229.78 & -82.72 \\ 0.00 & 0.00 & -82.72 & 195.22 & -82.72 & -29.78 \\ 0.00 & 0.00 & -229.78 & -82.72 & 395.22 & -82.72 \\ 0.00 & 0.00 & -82.72 & -29.78 & -82.72 & 195.22 \end{pmatrix}.
\end{aligned}
$$

(b) **Identify the full initial prior quantities for the 6-dimensional state vector $\boldsymbol{\theta}_1$ and the observational variance V. Write down the defining quantities $\mathbf{a}_1, \mathbf{R}_1, n_0$, and S_0 based on the above initial information.**

The DLM model in this case is defined as $\{\mathbf{F}_t, \mathbf{G}, V, \mathbf{W}_t\}$, with

$$
\mathbf{F}_t = (1, \mathrm{Cost}_t, 1, 0, 0, 0)' \qquad \mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.
$$

Using this information,

$$
\begin{aligned}
\mathbf{a}_1 &= \mathbf{G}\mathbf{m}_0 = (220, -1.5, 25, 50, -25, -50)' \\
\mathbf{R}_1 &= \mathbf{G}\mathbf{C}_0\mathbf{G}' + \mathbf{W}_t = \begin{pmatrix} 225.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.49 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 195.22 & -82.72 & -29.78 & -82.72 \\ 0.00 & 0.00 & -82.72 & 395.22 & -82.72 & -229.78 \\ 0.00 & 0.00 & -29.78 & -82.72 & 195.22 & -82.72 \\ 0.00 & 0.00 & -82.72 & -229.78 & -82.72 & 395.22 \end{pmatrix} + \mathbf{W}_t \\
n_0 &= 12 \\
S_0 &= 100.
\end{aligned}
$$

Therefore, $(\boldsymbol{\theta}_1|\mathcal{D}_0) \sim \mathrm{T}_{12}(\mathbf{a}_1, \mathbf{R}_1)$ and $(V|\mathcal{D}_0) \sim$ Inverse Gamma$(n_0/2, d_0/2)$, where $d_0 = n_0 S_0 = 1200$.

(c) **Use three discount factors to structure the evolution matrices of the model: δ_T for the constant intercept term, δ_R for the regression coefficient, and δ_S for the seasonal factors. Consider initially the values $\delta_T = \delta_S = 0.9$ and $\delta_R = 0.98$. Fit the model and perform the retrospective, filtering calculations to obtain filtered estimates of the state vector and all model components over time.**