# West & Harrison
# Chapter 10, Problem 7

The Dans

Thursday, March 16, 2017

# The Data

The quarterly sales and cost index of a confectionary product are given as

| | SALES | | | | COST | | | |
|---|---|---|---|---|---|---|---|---|
| | Quarter | | | | Quarter | | | |
| Year | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 |
| 1975 | 157 | 227 | 240 | 191 | 10.6 | 8.5 | 6.7 | 4.1 |
| 1976 | 157 | 232 | 254 | 198 | 1.9 | 0.4 | 1.1 | 1.9 |
| 1977 | 169 | 234 | 241 | 167 | 0.2 | 2.9 | 4.1 | −1.4 |
| 1978 | 163 | 227 | 252 | 185 | −4.0 | −4.5 | −5.3 | −8.4 |
| 1979 | 179 | 261 | 264 | 196 | −12.8 | −13.2 | −10.1 | −4.6 |
| 1980 | 179 | 248 | 256 | 193 | −1.1 | −0.1 | 0.0 | −2.5 |
| 1981 | 186 | 260 | 270 | 210 | −5.1 | −6.4 | −8.0 | −6.5 |
| 1982 | 171 | 227 | 241 | 170 | −3.7 | −1.3 | 6.1 | 16.5 |
| 1983 | 140 | 218 | 208 | 193 | 22.9 | 23.9 | 18.0 | 8.3 |
| 1984 | 184 | 235 | 245 | 209 | 2.9 | 0.7 | −2.4 | −7.0 |
| 1985 | 206 | 260 | 264 | 227 | −9.8 | −10.6 | −12.3 | −13.2 |

# The Goal

Fit a DLM to the sales with first-order polynomial, regression on cost, and full seasonal components. Based on previous information:

- the underlying level of the series when cost is zero is 220 with a nominal standard error of 15
- the regression coefficient of cost is estimated as -1.5 with a standard error of about 0.7
- the seasonal factors for the four quarters of the first year are expected to be -50, 25, 50, -25, with nominal standard errors of 25, 15, 25, and 15, respectively
- the trend, regression, and seasonal components are initially assumed to be uncorrelated
- the observational variance is estimated as 100, with initial degrees of freedom of 12

# Part A

**Derive the appropriate initial prior that satisfies the zero-sum constraint.**

*Theorem 8.2: Imposing the constraint* $\mathbf{1}'\phi = 0$ *on the prior* $(\phi_0|\mathcal{D}_0^*) \sim N(\mathbf{m}_0^*, \mathbf{C}_0^*)$ *and writing* $U = \mathbf{1}'\mathbf{C}_0^*\mathbf{1}$ *and* $\mathbf{A} = \mathbf{C}_0^*\mathbf{1}/U$ *gives the revised joint prior*

$$
\begin{aligned}
(\phi_0|\mathcal{D}_0) &\sim \mathrm{N}(\mathbf{m}_0, \mathbf{C}_0), \\
\mathbf{m}_0 &= \mathbf{m}_0^* - \mathbf{A}\mathbf{1}'\mathbf{m}_0^*, \\
\mathbf{C}_0 &= \mathbf{C}_0^* - \mathbf{A}\mathbf{A}'U.
\end{aligned}
$$

# Part A

The given information on the seasonal components is

$$\mathbf{m}_0^* = (-25, -50, 25, 50)' \quad \mathbf{C}_0^* = \begin{pmatrix} 225 & 0 & 0 & 0 \\ 0 & 625 & 0 & 0 \\ 0 & 0 & 225 & 0 \\ 0 & 0 & 0 & 625 \end{pmatrix}$$

Remember to shift the seasonal components back one quarter, as time $t = 0$ is the fourth quarter of 1974!

# Part A

Applying Theorem 8.2 yields

$$
\begin{aligned}
U &= 1700 \\
\mathbf{A} &= (0.1323529, 0.3676471, 0.1323529, 0.3676471)' \\
\mathbf{m}_0^s &= (-25, -50, 25, 50)' \\
\mathbf{C}_0^s &= \begin{pmatrix} 195.22 & -82.72 & -29.78 & -82.72 \\ -82.72 & 395.22 & -82.72 & -229.78 \\ -29.78 & -82.72 & 195.22 & -82.72 \\ -82.72 & -229.78 & -82.72 & 395.22 \end{pmatrix}.
\end{aligned}
$$

The superscript $s$ denotes that this is for the seasonal component of the model.

# Part A

Therefore, the entire initial prior is

$$\mathbf{m}_0 = (220, -1.5, -25, -50, 25, 50)'$$

$$\mathbf{C}_0 = \begin{pmatrix} 225.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.49 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 195.22 & -82.72 & -29.78 & -82.72 \\ 0.00 & 0.00 & -82.72 & 395.22 & -82.72 & -229.78 \\ 0.00 & 0.00 & -29.78 & -82.72 & 195.22 & -82.72 \\ 0.00 & 0.00 & -82.72 & -229.78 & -82.72 & 395.22 \end{pmatrix}.$$

# Part B

**Identify the full initial prior quantities for the 6-dimensional state vector $\theta_1$ and the observational variance $V$. Write down the defining quantities $\mathbf{a}_1, \mathbf{R}_1, n_0$, and $S_0$ based on the above initial information.**

# Part B

The DLM model in this case is defined as $\{\mathbf{F}_t, \mathbf{G}, V, \mathbf{W}_t\}$, with

$$\mathbf{F}_t = (1, \text{Cost}_t, 1, 0, 0, 0)' \quad \mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

# Part B

$$\mathbf{a}_1 = \mathbf{G}\mathbf{m}_0 = (220, -1.5, 25, 50, -25, -50)'$$

$$\mathbf{R}_1 = \mathbf{G}\mathbf{C}_0\mathbf{G}' + \mathbf{W}_t = \begin{pmatrix} 225.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.49 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 395.22 & -82.72 & -229.78 & -82.72 \\ 0.00 & 0.00 & -82.72 & 195.22 & -82.72 & -29.78 \\ 0.00 & 0.00 & -229.78 & -82.72 & 395.22 & -82.72 \\ 0.00 & 0.00 & -82.72 & -29.78 & -82.72 & 195.22 \end{pmatrix} + \mathbf{W}_t$$

$$n_0 = 12$$

$$S_0 = 100.$$

Therefore, $(\boldsymbol{\theta}_1 | \mathcal{D}_0) \sim \mathrm{T}_6(\mathbf{a}_1, \mathbf{R}_1)$ and $(V | \mathcal{D}_0) \sim$ Inverse Gamma$(n_0/2, d_0/2)$, where $d_0 = n_0 S_0 = 1200.$

# Part C

**Use three discount factors to structure the evolution matrices of the model: $\delta_T$ for the constant intercept term, $\delta_R$ for the regression coefficient, and $\delta_S$ for the seasonal factors. Consider initially the values $\delta_T = \delta_S = 0.9$ and $\delta_R = 0.98$. Fit the model and perform the retrospective, filtering calculations to obtain filtered estimates of the state vector and all model components over time.**

# Part C

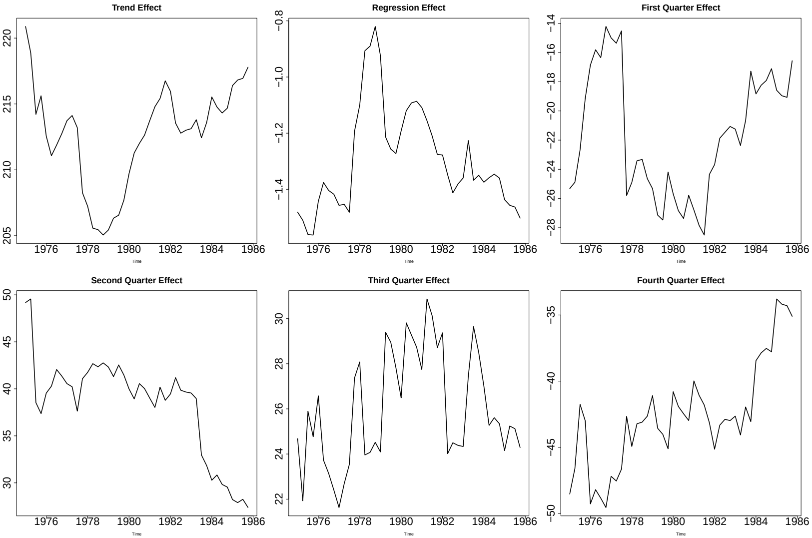



Figure: Filtered estimates for all components of the state vector

# Part C

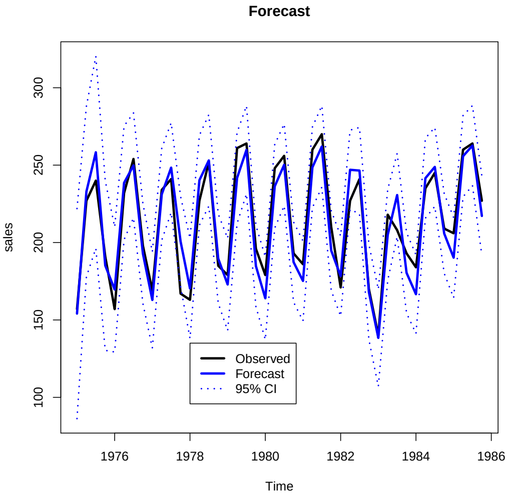
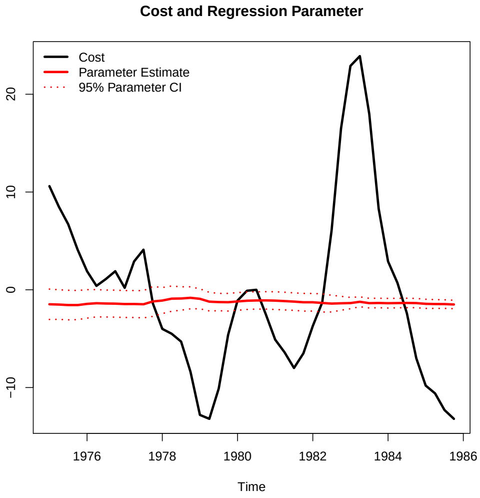



Figure: One-step ahead forecast estimate and 95% confidence interval

# Part D

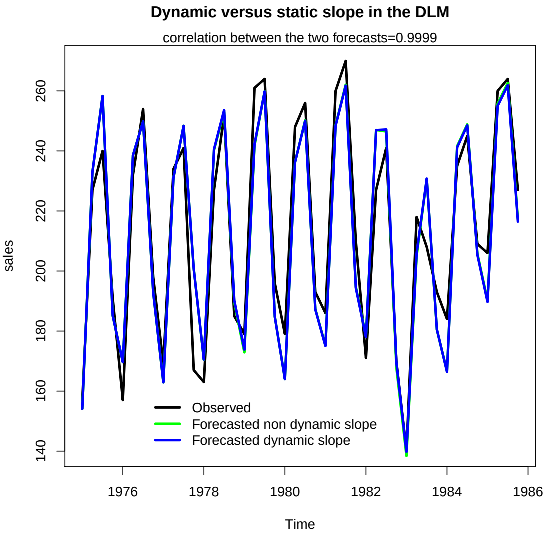
**Based on this analysis, verify that the regression parameter on Cost is, in retrospect, rather stable over time.**

# Part D



Cost and Regression Parameter
Cost
Parameter Estimate
95% Parameter CI
20
10
0
−10
1976
1978
1980
1982
1984
1986
Time

# Part D


Dynamic versus static slope in the DLM
correlation between the two forecasts=0.9999
sales
260
240
220
200
180
160
140
1976
1978
1980
1982
1984
1986
Time
Observed
Forecasted non dynamic slope
Forecasted dynamic slope

# Part E

**Produce step ahead forecasts from the end of the data in the fourth quarter of 1985 for the next three years. The estimated values of Cost to be used in forecasting ahead are given by**

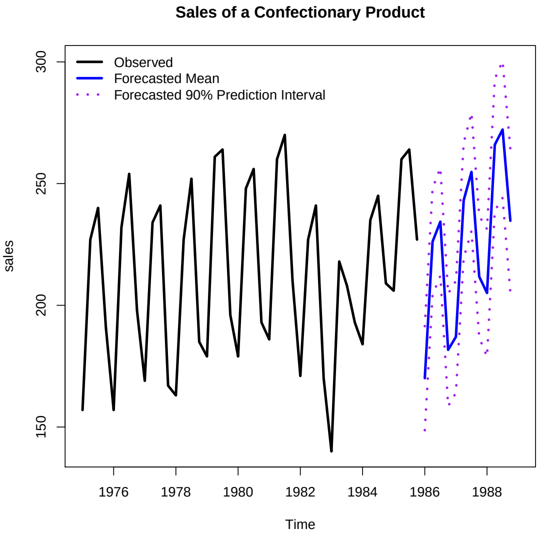
| Year | Quarter | | | |
|---|---|---|---|---|
| | 1 | 2 | 3 | 4 |
| 1986 | 8.4 | 10.6 | 7.2 | 13.0 |
| 1987 | −2.9 | −0.7 | −6.4 | −7.0 |
| 1988 | −14.9 | −15.9 | −18.0 | −22.3 |

# Part E

Appealing to Harrison and West [1999] page 199, and with $\mathbf{F}(k)$ as the $\mathbf{F}$ vector at time $k$, the $k$-step forecast distributions are

$$
\begin{aligned}
(y(k)|\mathcal{D}_T) &\sim T_{38}(F_T(k), Q_T(k)) \\
F_T(k) &= \mathbf{F}(k)'\mathbf{a}_T(k) \\
Q_T(k) &= \mathbf{F}(k)'\mathbf{R}_T(k)\mathbf{F}(k) \\
\mathbf{a}_T(k) &= \mathbf{G}\mathbf{a}_T(k-1) \\
\mathbf{R}_T(k) &= \mathbf{G}\mathbf{R}_T(k-1)\mathbf{G}'\mathrm{diag}\left(\frac{1}{\delta_t^k}, \frac{1}{\delta_r^k}, \frac{1}{\delta_s^k}, \frac{1}{\delta_s^k}, \frac{1}{\delta_s^k}, \frac{1}{\delta_s^k}\right)
\end{aligned}
$$

# Part E


Sales of a Confectionary Product
Observed
Forecasted Mean
Forecasted 90% Prediction Interval
sales
150
200
250
300
1976
1978
1980
1982
1984
1986
1988
Time

# References


Jeff Harrison and Mike West. *Bayesian forecasting & dynamic models*. Springer New York, 1999.

Raquel Prado and Mike West. *Time series: modeling, computation, and inference*. CRC Press, 2010.