SUPPLEMENTARY MATERIAL
Joint estimation of relative risk of dengue and Zika infections

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Let us assume the observed counts O_{ij} of dengue or Zika virus disease (ZVD) are Poisson distributed with mean parameter (μ_{ij}) where i is the aggregation area ($i = 1, \ldots, n$, and $n = 87$ municipalities at departmental level; or $n = 293$ census section for the municipal level), and j is the disease ($j = 1, \ldots, p$, and $p = 1$ for ZVD, or $p = 2$ for dengue).

$$O_{ij} \sim \text{Poisson}(\mu_{ij})$$

$$\mu_{ij} = E_{ij} \times \mathrm{r}_{ij}$$

$$\mathrm{r}_{ij} = \exp(\lambda_{ij})$$

Then, the mean parameter μ_{ij} is equal to the product of the expected values E_{ij} and the relative risk r_{ij}, with linear predictor λ_{ij}. The relative risk is the additive effect of spatially structured and unstructured random effects, and covariates. Spatially structured random effects are unobserved variables recovering a clustered risk pattern, or the fact that the risk in one area is highly associated with the neighboring areas. The lack of spatial association is accounted by the spatially unstructured random effects [1].

Model 1

Model 1 contains independent and identically distributed (IID) Normal spatially unstructured random effects for every disease (dengue and ZVD).

$$
\begin{aligned}
O_{ij} &\sim \text{Poisson}(\mu_{ij}) \\
\log(\mu_{ij}) &= \log(E_{ij}) + \alpha_j + \phi_{ij} \\
\boldsymbol{\phi}_j &\sim \text{Normal}(\mathbf{0}, \sigma^2_{\phi_j}\mathbf{I}) \\
\alpha_j &\sim \text{Normal}(0, 1000) \\
1/\sigma^2_{\phi_j} &\sim \text{Gamma}(0.01, 0.01)
\end{aligned}
$$

where $\boldsymbol{\phi}_j$ are spatially unstructured random effects, α_j are intercepts, $\sigma^2_{\phi_j}$ are variance parameters of the $\boldsymbol{\phi}_j$, and $\mathbf{I}$ is $n \times n$ identity matrix.

Model 2

Model 2 presents IID Normal spatially unstructured random effects linearly correlated for both diseases.

$$
\begin{aligned}
O_{ij} &\sim \text{Poisson}(\mu_{ij}) \\
\log(\mu_{ij}) &= \log(E_{ij}) + \alpha_j + \phi_{ij} \\
\begin{bmatrix} \boldsymbol{\phi}_1 \\ \boldsymbol{\phi}_2 \end{bmatrix} &\sim \text{Normal}(\mathbf{0}, \boldsymbol{\Sigma} \otimes \mathbf{I}) \\
\boldsymbol{\Sigma} &= \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \\
\boldsymbol{\Sigma^{-1}} &\sim \text{Wishart}(\mathbf{R_{2\times 2}}, 2) \\
\boldsymbol{R} &= \begin{bmatrix} 1/5 & 0 \\ 0 & 1/5 \end{bmatrix} \\
\alpha_j &\sim \text{Normal}(0, 1000)
\end{aligned}
$$

$\boldsymbol{\Sigma}$ is an $n \times n$ variance-covariance matrix accounting for the association of the spatially unstructured random effects $\boldsymbol{\phi}_j$, $\otimes$ corresponds to the Kronecker product, and the rest of the parameters similar to model 1.

Model 3

Model 3 accommodates conditionally autoregressive (CAR) [4] Normal spatially structured random effects for every disease.

$$
\begin{aligned}
O_{ij} &\sim \text{Poisson}(\mu_{ij}) \\
\log(\mu_{ij}) &= \log(E_{ij}) + \alpha_j + \phi_{ij} \\
\boldsymbol{\phi_j} &\sim \text{Normal}(\mathbf{0}, \sigma^2_{\phi_j}(\boldsymbol{D} - \boldsymbol{W})^{-1}) \\
\alpha_j &\sim \text{Normal}(0, 1000) \\
1/\sigma^2_{\phi_j} &\sim \text{Gamma}(0.01, 0.01) \\
\boldsymbol{W}_{n \times n} &= \begin{cases} w_{ij} = 1 & \text{if } i \sim j \\ w_{ij} = 0 & \text{if } i = j \\ w_{ij} = 0 & \text{otherwise} \end{cases} \\
w_{i+} &= \sum_j w_{ij} \\
\boldsymbol{D}_{n \times n} &= \text{diagonal}(w_{1+}, \dots, w_{n+})
\end{aligned}
$$

where $\boldsymbol{\phi_j}$ are spatially structured random effects, with structure given by the proximity matrix $\boldsymbol{W}$, and variance parameters $\sigma^2_{\phi_j}$

Model 4

Model 4 contains CAR Normal spatially structured random effects linearly correlated for both diseases.

$$
\begin{aligned}
O_{ij} &\sim \text{Poisson}(\mu_{ij}) \\
\log(\mu_{ij}) &= \log(E_{ij}) + \alpha_j + \phi_{ij} \\
\begin{bmatrix} \boldsymbol{\phi_1} \\ \boldsymbol{\phi_2} \end{bmatrix} &\sim \text{Normal}(\mathbf{0}, \boldsymbol{\Sigma} \otimes (\boldsymbol{D} - \boldsymbol{W})^{-1}) \\
\boldsymbol{\Sigma} &= \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \\
\boldsymbol{\Sigma^{-1}} &\sim \text{Wishart}(\mathbf{R_{2\times 2}}, 2) \\
\boldsymbol{R} &= \begin{bmatrix} 1/5 & 0 \\ 0 & 1/5 \end{bmatrix} \\
\alpha_j &\sim \text{Normal}(0, 1000) \\
\boldsymbol{W}_{n\times n} &= \begin{cases} w_{ij} = 1 & \text{if } i \sim j \\ w_{ij} = 0 & \text{if } i = j \\ w_{ij} = 0 & \text{otherwise} \end{cases} \\
w_{i+} &= \sum_j w_{ij} \\
\boldsymbol{D}_{n\times n} &= \text{diagonal}(w_{1+}, \ldots, w_{n+})
\end{aligned}
$$

where $\boldsymbol{\phi_j}$ are spatially structured random effects, $\boldsymbol{\Sigma}$ is the variance-covariance matrix accounting the spatial association of the $\boldsymbol{\phi_j}$, and proximity matrix $\boldsymbol{W}$.

Model 5

Model 5 includes IID Normal spatially unstructured random effects for every disease with a CAR shared-parameter.

$$
\begin{aligned}
O_{ij} &\sim \text{Poisson}(\mu_{ij}) \\
\log(\mu_{i,1}) &= \log(E_{i,1}) + \alpha_1 + \psi_i \times \gamma + \phi_{i,1} \\
\log(\mu_{i,2}) &= \log(E_{i,2}) + \alpha_2 + \psi_i / \gamma + \phi_{i,2} \\
\boldsymbol{\psi} &\sim \text{Normal}(\mathbf{0}, \sigma_\psi^2 (\boldsymbol{D} - \boldsymbol{W})^{-1}) \\
\boldsymbol{\phi_j} &\sim \text{Normal}(\mathbf{0}, \sigma_{\phi_j}^2 \mathbf{I}) \\
\alpha_j &\sim \text{Normal}(0, 1000) \\
\sigma_\psi &\sim \text{Uniform}(0, 10) \\
\sigma_{\phi_j} &\sim \text{Uniform}(0, 10) \\
\gamma &\sim \text{Normal}(0, 100) \\
\boldsymbol{W}_{n \times n} &= \begin{cases} w_{ij} = 1 & \text{if } i \sim j \\ w_{ij} = 0 & \text{if } i = j \\ w_{ij} = 0 & \text{otherwise} \end{cases} \\
w_{i+} &= \sum_j w_{ij} \\
\boldsymbol{D}_{n \times n} &= \text{diagonal}(w_{1+}, \ldots, w_{n+})
\end{aligned}
$$

$\boldsymbol{\psi}$ are the spatially structured shared-parameter components, γ is a scaling parameter, $\boldsymbol{\phi_j}$ are spatially unstructured random effects, $\boldsymbol{W}$ is the proximity matrix, and $\sigma_{\phi_j}^2$, σ_ψ^2 are variance parameters for $\boldsymbol{\phi_j}$ and $\boldsymbol{\psi}$.

Model 6

Model 6 includes CAR Normal spatially structured random effects for every disease with a CAR shared-parameter.

$$
\begin{aligned}
O_{ij} &\sim \text{Poisson}(\mu_{ij}) \\
\log(\mu_{i,1}) &= \log(E_{i,1}) + \alpha_1 + \psi_i \times \gamma + \phi_{i,1} \\
\log(\mu_{i,2}) &= \log(E_{i,2}) + \alpha_2 + \psi_i/\gamma + \phi_{i,2} \\
\boldsymbol{\psi} &\sim \text{Normal}(\mathbf{0}, \sigma^2_{\psi}(\boldsymbol{D} - \boldsymbol{W})^{-1}) \\
\boldsymbol{\phi_j} &\sim \text{Normal}(\mathbf{0}, \sigma^2_{\phi_j}(\boldsymbol{D} - \boldsymbol{W})^{-1}) \\
\alpha_j &\sim \text{Normal}(0, 1000) \\
\sigma_{\psi} &\sim \text{Uniform}(0, 10) \\
\sigma_{\phi_j} &\sim \text{Uniform}(0, 10) \\
\gamma &\sim \text{Normal}(0, 100) \\
\boldsymbol{W}_{n\times n} &= \begin{cases} w_{ij} = 1 & \text{if } i \sim j \\ w_{ij} = 0 & \text{if } i = j \\ w_{ij} = 0 & \text{otherwise} \end{cases} \\
w_{i+} &= \sum_j w_{ij} \\
\boldsymbol{D}_{n\times n} &= \text{diagonal}(w_{1+}, \dots, w_{n+})
\end{aligned}
$$

$\boldsymbol{\psi}$ are the spatially structured shared-parameter components, γ is a scaling parameter, $\boldsymbol{\phi_j}$ are spatially structured random effects, $\boldsymbol{W}$ is the proximity matrix, and $\sigma^2_{\phi_j}$, σ^2_{ψ} are variance parameters for $\boldsymbol{\phi_j}$ and $\boldsymbol{\psi}$.

Models 7 and 8

Models 7 accommodates the Generalized Multivariate CAR model [3], where the CAR Normal spatially structured random effects of ZVD by small area are conditioned by the CAR Normal spatially structured random effects of dengue. Model 8 presents the Generalized Multivariate CAR model [3], where the CAR Normal spatially structured random effects of dengue per area are conditioned by the CAR Normal spatially structured random effects of ZVD.

$$
\begin{aligned}
O_{ij} &\sim \text{Poisson}(\mu_{ij}) \\
\log(\mu_{ij}) &= \log(E_{ij}) + \phi_{ij} \\
\boldsymbol{\phi}_1 | \boldsymbol{\phi}_2 &\sim \text{Normal}(\boldsymbol{\delta}_1, \boldsymbol{\Xi}_1) \\
\boldsymbol{\delta}_1 &= \beta_1 \mathbf{1} + (\eta_0 \mathbf{I} + \eta_1 \boldsymbol{W})(\boldsymbol{\phi}_2 - \beta_2 \mathbf{1}) \\
\boldsymbol{\Xi}_1 &= \sigma_1^2 (\boldsymbol{D} - \kappa_1 \boldsymbol{W})^{-1} \\
\boldsymbol{\phi}_2 &\sim \text{Normal}(\boldsymbol{\delta}_2, \boldsymbol{\Xi}_2) \\
\boldsymbol{\delta_2} &= \beta_2 \mathbf{1} \\
\boldsymbol{\Xi}_2 &= \sigma_1^2 (\boldsymbol{D} - \kappa_2 \boldsymbol{W})^{-1} \\
\beta_j &\sim \text{Normal}(0, 100) \\
\eta_0, \eta_1 &\sim \text{Normal}(0, 10) \\
\sigma_j &\sim \text{Uniform}(0, 10) \\
\kappa_j &\sim \text{Uniform}(0, 0.99) \\
\boldsymbol{W}_{n \times n} &= \begin{cases} w_{ij} = 1 & \text{if } i \sim j \\ w_{ij} = 0 & \text{if } i = j \\ w_{ij} = 0 & \text{otherwise} \end{cases} \\
w_{i+} &= \sum_j w_{ij} \\
\boldsymbol{D}_{n \times n} &= \text{diagonal}(w_{1+}, \ldots, w_{n+})
\end{aligned}
$$

where $\boldsymbol{\phi}_j$ are spatially structured random effects, $\boldsymbol{W}$ is the proximity matrix, $\boldsymbol{\phi}_1 | \boldsymbol{\phi}_2$ is the conditional distribution of $\boldsymbol{\phi}_1$, and $\boldsymbol{\phi}_2$ is a marginal distribution.

REFERENCES

1. Banerjee S, Carlin BP, Gelfand AE (2014) Hierarchical Modeling and Analysis for Spatial Data, Second Edition Chapman & Hall/CRC Monographs on Statistics & Applied Probability. Boca Raton, Fl. 584 Pages.

2. Ma H and Carlin BP (2005) Bayesian Multivariate Areal Wombling for Multiple Disease Boundary Analysis. Technical Report. School of Public Health, University of Minnesota. url: http://www.biostat.umn.edu/∼brad/software/mc.pdf

3. Jin X, Bradley P. Carlin BP, and Banerjee S. Generalized Hierarchical Multivariate CAR Models for Areal Data. Biometrics. 2005; 61(4): 950-961.

4. Besag J, York J, Mollie A. Bayesian image restoration with two applications in spatial statistics. Annals of the Institute of Statistical Mathematics. 1991; 43(1):1–59.