Paired Integers and Averages

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Combinatorics Problem

Problem Statement

This document discusses a mathematical problem involving paired integers. The goal is to show that in any set of 5 paired integers, there must exist two pairs whose averages also form another pair of integers.

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1 Restatement

In a set of 5 paired integers $(x_1, y_1), (x_2, y_2), \ldots, (x_5, y_5)$, prove that there exist two pairs whose averages also produce another pair of integers.

2 Mathematical Background

For any integers x and y, the residue classes modulo 2 are defined as:

$$x \mod 2 \in \{0, 1\}, \quad y \mod 2 \in \{0, 1\}.$$

The pigeonhole principle states that if n items are placed into $k < n$ categories, at least one category must contain more than one item.

3 Proof

We are given 5 paired integers $(x_1, y_1), (x_2, y_2), \ldots, (x_5, y_5)$. To prove the statement, consider the residues of x_i and y_i modulo 2.
1. Each pair (x_i, y_i) corresponds to a residue pair:

$$(x_i \mod 2, y_i \mod 2) \in \{(0,0), (0,1), (1,0), (1,1)\}.$$

2. Since there are only 4 possible residue pairs but 5 given pairs, the pigeonhole principle ensures that at least two pairs, say (x_1, y_1) and (x_2, y_2), share the same residue pair:

$$x_1 \mod 2 = x_2 \mod 2 \quad \text{and} \quad y_1 \mod 2 = y_2 \mod 2.$$

3. Consider the average of these two pairs:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

4. Since $x_1 + x_2$ and $y_1 + y_2$ are both even (because their residues modulo 2 are equal), their averages:

$$\frac{x_1 + x_2}{2} \quad \text{and} \quad \frac{y_1 + y_2}{2}$$

are integers.
Thus, the averages of (x_1, y_1) and (x_2, y_2) produce another pair of integers, completing the proof.

4 Discussion

This result illustrates the application of modular arithmetic and the pigeonhole principle in number theory. The proof relies heavily on recognizing patterns in residue classes and their interactions under addition. The pigeonhole principle ensures the overlap of residue classes, which guarantees the existence of pairs with the required property.
Future investigations could explore generalizations of this result, such as sets of paired integers modulo higher bases or extensions to multi-dimensional tuples.

5 References

References

[1] A. C. Melissinos and J. Napolitano, *Experiments in Modern Physics*, (Academic Press, New York, 2003).

[2] N. Cyr, M. Têtu, and M. Breton, IEEE Trans. Instrum. Meas. **42**, 640 (1993).