Decomposition Methods and B&B

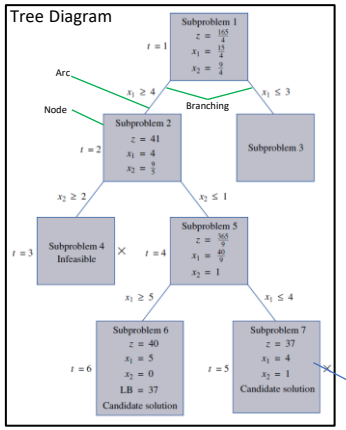
Given an Optimization Problem[4]

$$\min \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{s.t. } \boldsymbol{Ax} \le \boldsymbol{b}$$

B&B Example[2]

$$\max z = 8x_1 + 5x_2$$
$$s.t.\ x_1 + x_2 \le 6$$
$$9x_1 + 5x_2 \le 45$$
$$x_1, x_2 \in \mathbb{Z}_+$$

LP Relaxation can be applied



Can consider:

- Branch variable with highest objective importance
- Eliminate infeasible subproblems
- Eliminate subproblems with worse solution than identified bounds
- Subproblem may be selected by **backtracking** (LIFO) or **jumptracking**

Columns may be added at each node

Branch and Bound[1]

Branch and Price[3]

Branch Price and Cut[8]

Cutting plane used to tighten LP relaxation

For explanation, please refer to Example_Walmart_Dantzig_Wolfe.ppt

Dantzig-Wolfe Decomposition[6]

Column Generation[5] (*Inner Methods*)

Row Generation[9] (*Outer Methods*)

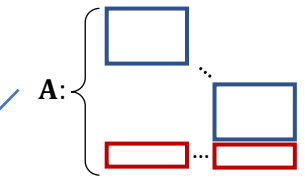
Cutting Plane[10]

Benders Decomposition[11]

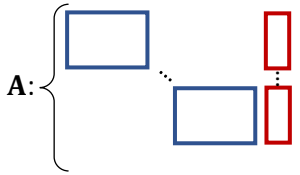
- MILP Benders Decomposition
- Nested Benders Decomposition

Column and Constraint Generation[12]

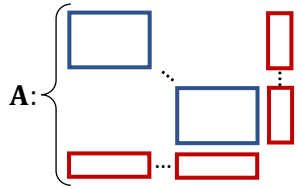
- C&CG Algorithm
- Nested C&CG Algorithm



Complicated Constraints



Complicated Variables

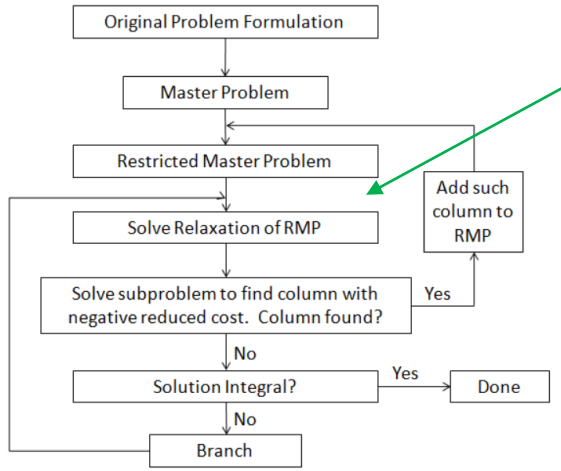


Complicated Variables and Constraints

Get creative and apply approaches:

- Two-stage stochastic optimization
- Multi-stage stochastic optimization
- Roust optimization
- Adaptive robust optimization
- Distributionally robust optimization

B&P Algorithm[7]



Reference: Akella, M.; S. Gupta; A. Sarkar. "Branch and Price: Column Generation for Solving Huge Integer Programs"

Consider the optimization problem

$$\min_{\boldsymbol{x}} \left[\boldsymbol{f}^\intercal\boldsymbol{x} + \min_{\boldsymbol{y}\ge\boldsymbol{0}} \{\boldsymbol{c}^\intercal\boldsymbol{y} | \boldsymbol{Ay} \le \boldsymbol{b} - \boldsymbol{Bx}\} \right]$$

The inner problem dual linear program is

$$\max_{\boldsymbol{\alpha}} \left[(\boldsymbol{b} - \boldsymbol{B}\bar{\boldsymbol{x}})^\intercal\boldsymbol{\alpha} | \boldsymbol{A}^\intercal\boldsymbol{\alpha} \le \boldsymbol{c}, \boldsymbol{\alpha} \ge \boldsymbol{0} \right]$$

Algorithm 1 MILO Benders Decomposition Algorithm.

Let $\boldsymbol{x}$ be an initial feasible integer solution
$LB := -\infty$, $UB := \infty$
while $UB - LB > gap$ **do**
  Solve subproblem $\max_{\bar{\boldsymbol{\alpha}}} \{\boldsymbol{f}^\intercal\bar{\boldsymbol{x}} + (\boldsymbol{b} - \boldsymbol{B}\bar{\boldsymbol{x}})^\intercal\boldsymbol{\alpha} | \boldsymbol{A}^\intercal\boldsymbol{\alpha} \le \boldsymbol{c}, \boldsymbol{\alpha} \ge \boldsymbol{0}\}$
  if Unbounded **then**
    Get unbounded ray $\bar{\alpha}$
    Add cut $(\boldsymbol{b} - \boldsymbol{Bx})^\intercal\bar{\alpha} \le 0$ to master problem
  else
    Get extreme point $\bar{\alpha}$
    Add cut $z \ge \boldsymbol{f}^\intercal\boldsymbol{x} + (\boldsymbol{b} - \boldsymbol{Bx})^\intercal\bar{\boldsymbol{\alpha}}$
    $UB := \min\{UB, \boldsymbol{f}^\intercal\bar{\boldsymbol{x}} + (\boldsymbol{b} - \boldsymbol{B}\bar{\boldsymbol{x}})^\intercal\bar{\boldsymbol{\alpha}}\}$
  end if
  Solve master problem $\min_{\boldsymbol{x}} \{z | cuts, \boldsymbol{x} \in \{0,1\}^n\}$
  $LB := \bar{z}$
end while