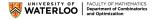
Convex relaxation of the binary quadratic knapsack problem

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Introduction

The Quadratic Knapsack Problem

$$\begin{array}{ll} \textbf{(QKP)} & \text{maximize} & \sum\limits_{i \in N} \sum\limits_{j \in N} q_{ij}x_ix_j & \left(=x^TQx\right) \\ & \text{subject to} & \sum\limits_{j \in N} w_jx_j \leq c, \\ & x_j \in \{0,1\}, & j \in N. \end{array}$$

where
$$0 < w_j \le c$$
 for all $j \in N$

- Application in capital budget investments, where c represents the
 capital budget to invest in n possible investments. The quadratic
 formulation model the case where the return of the investments
 considers the dependence between pair of investments.
- Subproblem on some solutions approaches for more general binary quadratic problems.
- We are interested here in finding tight upper bounds for (QKP).

Standard approach to obtain upper bounds: Lift the problem to the matrix space defined by $X := xx^T$

$$egin{aligned} (\mathit{QKP}_{\mathit{lifted}}) & \mathsf{maximize} & \langle \mathit{Q}, \mathit{X}
angle \ & \mathsf{subject} \; \mathsf{to} & \sum\limits_{j \in \mathit{N}} \mathit{w}_{j} \mathit{x}_{j} \leq c, \ & \mathit{x}_{j} \in \{0, 1\}, \;\; j \in \mathit{N}. \ & \mathit{X} = \mathit{xx}^{\mathsf{T}} \end{aligned}$$

Standard approach to obtain upper bounds: Lift the problem to the matrix space defined by $X := xx^T$

Relax the nonconvex constraints $X = xx^T, x_j \in \{0, 1\}, j \in \mathbb{N}$.

where \mathcal{P} is a compact convex subset of $[0,1]^n \times \mathbb{S}^n$ such that

$$\{(x,X) : w^T x \le c, X = xx^T, x \in \{0,1\}^n\} \subset \mathcal{P}.$$

As $x^T Qx = x^T (Q - Q_p)x + x^T Q_p x$, consider also the relaxation

$$(\mathbf{CQP}_{Q_p}) \quad \begin{array}{ll} p_{\mathrm{CQP}}^*(Q_p) := \max & x^{\mathsf{T}}(Q - Q_p)x + \langle Q_p, X \rangle \\ & \mathrm{s.t.} \quad (x, X) \in \mathcal{P}, \end{array}$$

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and where

$$Q-Q_p \leq 0$$
,

Searching for:

- a matrix Q_p that leads to a good decomposition of the objective function.
- ullet a tight relaxation $\mathcal P$ of the feasible set, that leads to a relaxation of $\mathbf Q \mathbf K \mathbf P$ that can be efficiently solved.

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Searching for:

- a matrix Q_p that leads to a good decomposition of the objective function.
- a tight relaxation \mathcal{P} of the feasible set, that leads to a relaxation of **QKP** that can be efficiently solved. \leftarrow new valid inequalities!

How to optimize the selection of Q_n ?

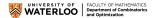
 Search for the decomposition that minimizes the upper bound of (QKP) given by our convex quadratic relaxation:

$$\begin{aligned} (\mathbf{CQP}_{\ Q_p}) \quad & p_{\text{\tiny CQP}}^*(Q_p) := \max_{\text{s.t.}} \quad & x^T(Q - Q_p)x + \langle Q_p, X \rangle \\ & \text{s.t.} \quad & (x, X) \in \mathcal{P}, \end{aligned}$$

where
$$Q - Q_p \leq 0$$

 To minimize the upper bound, consider the parametric problem

$$\mathit{param}^*_{\scriptscriptstyle \mathrm{QKP}} := \min_{Q-Q_p \preceq 0} \, p^*_{\scriptscriptstyle \mathrm{CQP}}(Q_p).$$



A primal-dual IPM to solve the parametric problem

• Minimize a log-barrier function $B_{\mu}(Q_p, Z)$

$$\begin{array}{ll} \text{min} & B_{\mu}(Q_p,Z) := p_{\scriptscriptstyle \mathrm{CQP}}^*(Q_p) - \mu \log \det Z \\ \text{s.t.} & Q - Q_p + Z = 0 \\ & Z \succ 0, \end{array} \tag{: Λ}$$

where Z, Λ are the slack and dual symmetric matrix variables, and $p_{\text{CQP}}^*(Q_p) := \max\{x^T(Q-Q_p)x + \langle Q_p, X \rangle | (x, X) \in P\}$

Consider the Lagrangian function:

$$L_{\mu}(Q_p, Z, \Lambda) := p_{\text{CQP}}^*(Q_p) - \mu \log \det Z + \langle Q - Q_p + Z, \Lambda \rangle.$$

 At each iteration of the IPM, we need to differentiate the Lagrangian function.

A primal-dual IPM to solve the parametric problem

• Minimize a log-barrier function $B_{\mu}(Q_p, Z)$

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Consider the Lagrangian function:

$$L_{\mu}(Q_p, Z, \Lambda) := p_{\text{CQP}}^*(Q_p) - \mu \log \det Z + \langle Q - Q_p + Z, \Lambda \rangle.$$

 At each iteration of the IPM, we need to differentiate the Lagrangian function.

Important observations for the IPM

$$\begin{aligned} & (\mathbf{CQP}_{Q_p}) \quad p_{\mathrm{CQP}}^*(Q_p) := \max & x^T(Q - Q_p)x + \langle Q_p, X \rangle \\ & \text{s.t.} & (x, X) \in \mathcal{P}, \end{aligned} \\ \text{or} \qquad & (\mathbf{CQP}_{Q_p}) \quad p_{\mathrm{CQP}}^*(Q_p) := \max & \langle Q_p, X - xx^T \rangle + x^T Qx \\ & \text{s.t.} & (x, X) \in \mathcal{P}, \end{aligned}$$

ullet $p^*_{ ext{CQP}}(Q_p)$ is differentiable at all Q_p surch that $Q-Q_p \preceq 0$ and,

$$\nabla p_{\text{CQP}}^*(Q_p) = X - xx^T.$$

- We use a quasi-Newton direction at each iteration of our IPM. To compute the direction, we need the optimal solution of CQP Qp.
- The relaxation **CQP** Q_p is solved at each iteration of the IPM method, each time for a new perturbation Q_p , such that $Q Q_p \leq 0$.
- Each time the relaxation is solved, valid cuts are added to strength the relaxation.

Valid linear inequalities in the lifted matrix space

• The identity derived from $x_i \in \{0,1\}$:

$$X_{ii} = x_i$$

• McCormick inequalities derived from $x_i \ge 0$ and $1 - x_i \ge 0$

$$X_{ij} \ge 0$$

 $X_{ii} - X_{ij} \ge 0$
 $1 - X_{ii} - X_{jj} + X_{ij} \ge 0$

• Inequalities derived from the product of the knapsack constraint $\sum_{i \in N} w_i x_i \le c$, by x_i and $(1 - x_i)$:

$$\sum_{j\in N} w_j \overset{x_i x_j}{X_{ij}} \leq c \overset{x_i}{X_{ii}}.$$

$$\sum_{j\in N} w_j(X_{jj}-X_{ij}) \leq c(1-X_{ii}).$$

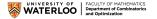
Cover Inequalities

The subset $C \subseteq N$ is a cover if it satisfies

$$\sum_{j\in C} w_j > c$$

The cover inequalities CI is defined as

$$\sum_{i\in C} x_j \le |C| - 1.$$



More valid inequalities in the lifted matrix space

Let us consider the cover inequality

$$\sum_{j\in\mathcal{C}}x_j\leq\beta. \tag{1}$$

where $\beta = |C| - 1$.

Definition: (Generalized cover inequality **GCI**)

Define

$$\sum_{j\in C} X_{ij} \leq \beta X_{ii}.$$

$$\sum_{i\in\mathcal{C}}X_{ij}-X_{ii}\leq\beta(1-X_{ii}).$$

for all $i \in N$, as the derived from (1)

New valid inequalities in the lifted space

Let us consider the cover inequality

$$\sum_{j \in C} x_j \le \beta \quad \text{where} \quad \beta = |C| - 1 \tag{1}$$

If no more than β items in C can be in the solution, then if we consider all 2-combinations of items in C, then the sum of the elements of X corresponding to there combinations cannot be greater than β choose 2.

New valid inequalities in the lifted space

Definition: (Cover inequality in the lifted space CILS)

Define

$$\sum_{i,j\in\mathcal{C},i< j} X_{ij} \le \binom{\beta}{2}. \tag{2}$$

as the **CILS** derived from (16)

Proposition:

If inequality (1) is valid for \mathbf{QKP} , then the \mathbf{CILS} (2) is a valid inequality for \mathbf{QKP}_{lifted}

Illustrating a CILS

Let

$$C := \{1, 2, 3\}$$

be a cover. So

$$\{1,2\},\{1,3\},\{2,3\}$$

are the 2-combinations of C.

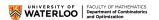
The inequality

$$x_1 + x_2 + x_3 \le 2$$

is a cover inequality, and

$$X_{12} + X_{13} + X_{23} < 1$$

is a CILS.



More valid inequalities in the lifted space

Let us consider the cover inequality

$$\sum_{j \in C} x_j \le \beta \quad \text{where} \quad \beta = |C| - 1 \tag{1}$$

Definition: (Set of cover inequalities in the lifted space **SCILS**)

Let $\bar{C}:=\{(i_1,j_1),\ldots,(i_p,j_p)\}$ be a partition of C, if |C| is even, or a partition of $C\setminus\{i_0\}$ for each $i_0\in C$, if |C| is odd.

Let $C_s := \overline{C}$ if |C| is even and $C_s := \{(i_0, j_0) \cup \overline{C}, \text{ if } |C| \text{ is odd.} \}$

Define the inequalities in **SCILS** derived from (1) as

$$\sum_{(i,j)\in C_s} X_{ij} \le \left\lfloor \frac{\beta}{2} \right\rfloor, \text{ for all partitions } C_s. \tag{3}$$

Proposition

If inequality (1) is valid for QKP, then the inequalities in the **SCILS** (3) are valid for QKP_{lifted} .

Illustrating a **SCILS**

Let

$$C := \{1, 2, 3\}$$

be a cover.

Consider the partition of C

$$\{\{1\},\{2,3\}\}, \{\{2\},\{1,3\}\}, \{\{3\},\{1,2\}\}$$

The inequality

$$x_1+x_2+x_3\leq 2$$

is a cover inequality, and

$$X_{11} + X_{23} \le 1$$

$$X_{22} + X_{13} \le 1$$

$$X_{33} + X_{12} \le 1$$

is a SCILS.

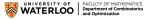


An algorithmic framework (QKP)

We propose an algorithm for **QKP**, that computes upper bounds for the problem by alternating between optimizing Q_n in the convex quadratic relaxation

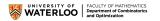
$$(\mathbf{CQP}_{Q_p}) \quad \begin{array}{ll} p_{\text{CQP}}^*(Q_p) := \max & x^T(Q - Q_p)x + \langle Q_p, X \rangle \\ \text{s.t.} & (x, X) \in \mathcal{P}, \end{array}$$

and applying cutting planes generated by the valid inequalities proposed.



The algorithmic framework for QKP

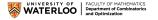
- Consists of an **IPM** to improve the matrix parameter Q_p on a convex quadratic relaxation for (QKP).
- At every 10 iterations of the IPM, separation problems are solved to generate cuts (n GCI, 5 CILS and 5 SCILS are added to the relaxation of (QKP)).
- When cuts are added to the relaxation, the computation of the search direction of the IPM changes accordingly.
- At very iteration of the **IPM** an upper bound is computed by solving the convex quadratic relaxation for the current matrix Q_p



Results

Inst	OptGap	Time	lter	TimeSep
	(%)	(seconds)		(seconds)
11	0.23	1013.50	100	641.98
12	0.00	632.5	64	411.67
13	0.00	392.55	44	205.70
14	0.00	289.97	31	160.37
15	0.21	1093.60	10	698.04

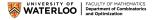
Table: Results for n = 50



Results

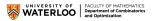
Inst	OptGap	Time	Iter	TimeSep
	(%)	(seconds)		(seconds)
l1	0.00	2035.30	20	737.86
12	0.25	2177.30	20	919.41
13	0.00	2007.10	20	773.00
14	0.12	1885.90	20	828.98
15	0.04	2309.50	20	970.49

Table: Results for n = 100



Conclusions

- We presented a cutting plane algorithm (CPA) to iteratively improve the upper bound for the quadratic knapsack problem (QKP).
- We presented a **IPM**, which update the perturbation Q_p at each iteration of the CPA aiming at reducing the upper bound given by the relaxation.
- We also presented new classes of cuts that are added during the execution of the CPA.
 - The cuts generated are effective in improving the upper bound for the QKP.
 - These procedures could be applied to more general binary indefinite quadratic problems.



Thank you!

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