Solution of the Problem of Covering Solid Bodies by Spheres using the Hyperbolic Smoothing Technique

Daniela Cristina Lubke
Helder Venceslau
Adilson Elias Xavier

Systems Engineering and Computer Sciences Department
Federal University of Rio de Janeiro

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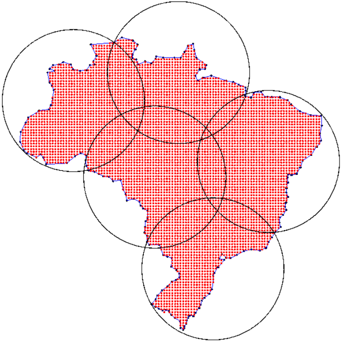
Objetives

- Presentation of a Hiperbolyc Smoothing (HS) based method to solve the covering of three dimensional bodies by equal (same radius) spheres;
- As a classical covering, all points pertaining to the body must be covered by at least one sphere (*order* 1 *covering*);
- As an optimization problem, the common radius of the spheres used in the covering process must be minimized;

Related problem in $\mathbb{R}^2$

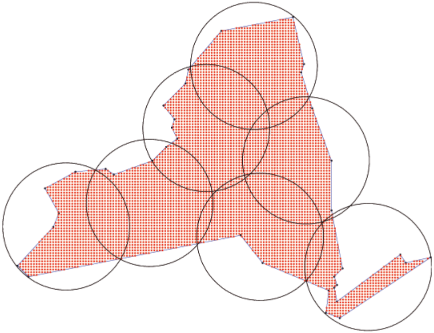
Covering of Brazil with 5 circles



Original figure from XAVIER e OLIVEIRA, 2005

Related problem in $\mathbb{R}^2$

Covering of New York with 7 circles



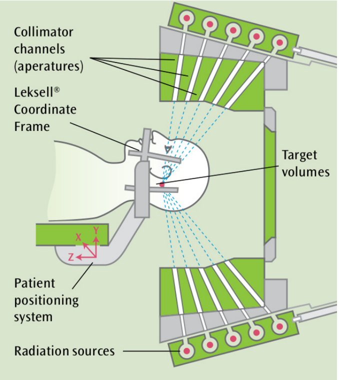
Original figure from XAVIER e OLIVEIRA, 2005

Applications of the covering problem

- Solving some crystalography models;
- Placing service centers;
- Locating and dimensioning telecommunications centers.

A motivating application: Gamma Knife radiosurgery

Radiosurgery is an important application of the covering problem.



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Problem Description

Let V be a bounded solid body that has to be covered by a set of q spheres with common radius d.

We consider the problem of finding such a covering with the smallest radius d (iow, the problem of finding the centers of q spheres that lead to a covering which has the smallest radius d).

For computational purposes the body V is discretized into a finite set of m elementary volumes called voxels.

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General Problem Formulation

Basic definitions:

$v_j,\ j = 1, \ldots, m$	Set of elementary volumes (voxels)
$x_i,\ i = 1, \ldots, q$	Centers of the spheres
$d(v, X) = \min\limits_{i=1,\ldots,q} \|v - x_i\|_2$	Distance from $v \in V$ to the nearest center in $X = \{x_1, \ldots, x_q\}$
$D(X) = \max\limits_{j=1,\ldots,m} d(v_j, X)$	The most critical covering of the voxels

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$D(X) = \max_{j=1,\ldots,m} \quad d(v_j, X)$	The most critical covering of the voxels

The optimal location of the centers should minimize the most critical covering to provide the best quality coverage of V:

$$\min_{X \in \mathbb{R}^{3q}} \quad D(X)$$

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General Problem Formulation

Therefore we formulate the *min-max-min* version of the problem:

$$\min_{X \in \mathbb{R}^{3q}} \max_{j=1,\ldots,m} \min_{i=1,\ldots,q} \|v_j - x_i\|_2.$$

If for fixed j we let $z_j(X)$ denote the innermost minimum then:

$$z_j(X) = \min_{i=1,\ldots,q} \|v_j - x_i\|_2,$$

then $z_j(X)$ must necessarily satisfy the following set of inequalities:

$$z_j(X) - \|v_j - x_i\|_2 \leq 0, \qquad i = 1, \ldots, q.$$

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Problem Transformation

The general covering problem is then equivalent to the following mathematical programming problem:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & z_j = \min_{i=1,\dots,q} \|v_j - x_i\|_2, \qquad j = 1, \dots, m \qquad \text{(P1)} \\ & z \geq z_j, \qquad j = 1, \dots, m. \end{aligned}$$

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Problem Transformation

From problem (P1) we obtain the following relaxed problem:

$$
\begin{array}{lll}
\min & z & \\
\text{s.t.} & z_j - \|v_j - x_i\|_2 \leq 0, \quad j = 1, \ldots, m, \quad i = 1, \ldots, q & \text{(P2)} \\
 & z \geq z_j, \qquad j = 1, \ldots, m. &
\end{array}
$$

Problem Transformation

Let us use the auxiliary function

$$\varphi(y) = \max\{0, y\},$$

From the inequalities

$$z_j - \|v_j - x_i\|_2 \leq 0, \quad j = 1, \ldots, m, \quad i = 1, \ldots, q,$$

it follows:

$$\sum_{i=1}^{q} \varphi(z_j - \|v_j - x_i\|_2 \,) = 0, \qquad j = 1, \ldots, m.$$

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Problem Transformation

Using this set of equality constraints in place of the constraints in (P2), we obtain the following problem:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & \sum_{i=1}^{q} \varphi(z_j - \|v_j - x_i\|_2) = 0, \quad j = 1, \dots, m. \\ & z \geq z_j, \qquad j = 1, \dots, m. \end{aligned} \tag{P3}$$

This problem also corresponds to a relaxation of problem (P1)

Problem Transformation

Decreasing z implies decreasing z_j, which will tend to $-\infty$,

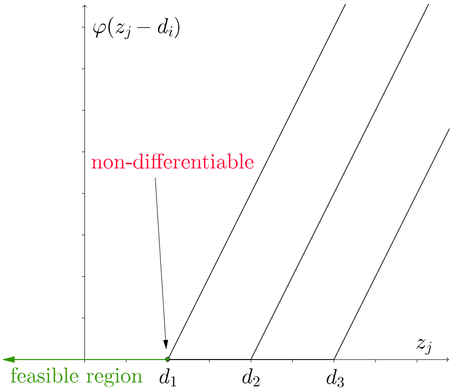

Figure 1 : Summands in (P3)

Problem Transformation

Lower bounding (P3) by adding a perturbation $\varepsilon > 0$ results in the following modified problem:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & \sum_{i=1}^{q} \varphi(z_j - \|v_j - x_i\|_2) \geq \varepsilon, \qquad j = 1, \ldots, m \qquad \text{(P4)} \\ & z \geq z_j, \qquad j = 1, \ldots, m. \end{aligned}$$

Problem Smoothing

- The definition of the function φ imposes a non differentiable structure on problem (P4).
- A smoothing approach is adopted to solve problem (P4).

Problem Smoothing

Substituting z for each z_j a feasible solution of (P4) is maintained, then

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & \sum_{i=1}^{q} \varphi(z - \|v_j - x_i\|_2\,) \geq \varepsilon\,, \qquad j = 1, \ldots, m \qquad \text{(P5)} \end{aligned}$$

Problem Smoothing

Let us define the auxiliary function

$$\phi(y,\tau) = \left(y + \sqrt{y^2 + \tau^2}\,\right)/2$$

for $y \in \mathbb{R}$ and $\tau > 0$.

Problem Smoothing

Function ϕ has the following properties:

$$(a) \quad \phi(y,\tau) > \varphi(y), \quad \forall \tau > 0;$$

$$(b) \quad \lim_{\tau \to 0} \phi(y,\tau) = \varphi(y);$$

(c) $\phi(y,\tau)$ is an increasing convex function of class C^{∞}.

Problem Smoothing

By substituting function φ by function ϕ in problem (P5) we obtain

$$
\begin{aligned}
\min \quad & z \\
\text{s.t.} \quad & \sum_{i=1}^{q} \phi(z - \|v_j - x_i\|_2, \tau) \geq \varepsilon, \qquad j = 1, \ldots, m. \qquad \text{(P6)}
\end{aligned}
$$

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Problem Smoothing

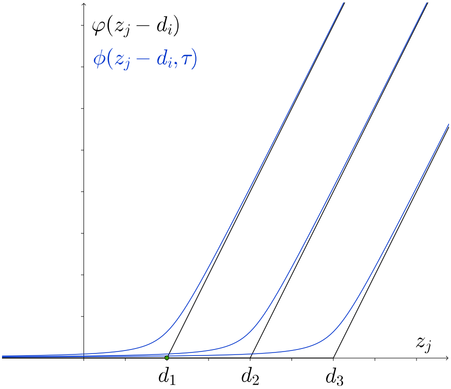


Figure 2 : Original and smoothed summands in (P3)

Smoothing the Euclidean distance

For the Euclidian distance, let us define the auxiliary function

$$\theta(v_j, x_i, \mu) = \sqrt{\sum_{i=1}^{q} (v_j - x_i)^2 + \mu^2}$$

for $\mu > 0$.

Smoothing the Euclidean distance

Function θ has the following properties:

$$(a) \quad \lim_{\mu \to 0} \theta(v_j, x_i, \mu) = ||v_j - x_i||_2$$

(b) θ is a C^{∞} function

Suavização do problema

Therefore, it is now obtained a completely smooth problem:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & \sum_{i=1}^{q} \phi(z - \theta(v_j, x_i, \mu), \tau) \geq \varepsilon, \qquad j = 1, \ldots, m. \end{aligned} \tag{P7}$$

Simplified HS Algorithm

Initialization Step: choose decreasing rate $0 < \rho < 1$,

and the tolerance $\delta > 0$;

choose initial values $x_0,\ \mu^1,\ \tau^1,\ \varepsilon^1$;

Let $k = 1$;

Main step: Repeat while $|f(x^k) - f(x^{k-1})| < \delta$

Solve problem (P7) with $\mu = \mu^k$,

$\tau = \tau^k, \varepsilon = \varepsilon^k$ starting at the initial point x^{k-1},

calculating the solution x^k;

Let $\mu^{k+1} = \rho\mu^k,\ \tau^{k+1} = \rho\tau^k$,

$\varepsilon^{k+1} = \rho\varepsilon^k,\quad k = k+1$;

Torus covering: analytic approach

Summary of the analytic solution

(a) If $q < \pi / \arctan \sqrt{\gamma}$ then

$$\rho^* = (R+r)\cos(\pi/q) \quad \text{and} \quad d^* = (R+r)\sin(\pi/q)$$

(b) If $q \geq \pi / \arctan \sqrt{\gamma}$ then

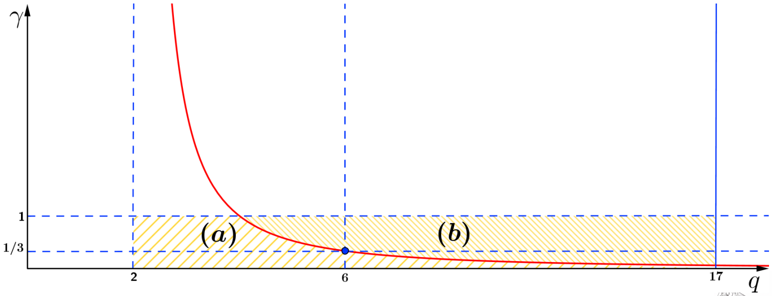
$$\rho^* = R\sec(\pi/q) \quad \text{and} \quad d^* = \sqrt{r^2 + R^2 \tan^2(\pi/q)}$$

(c) q-threshold for breaking symmetry constraint

$$\cos^2(2\pi/q) - (1-\gamma^2)\cos^2(\pi/q) > 0$$

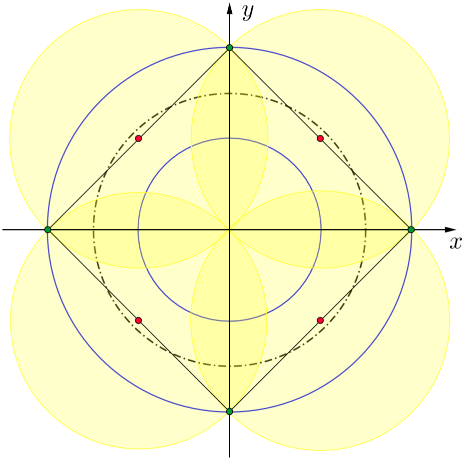
Torus covering: analytic approach

Graphic depiction of solutions (a) and (b) applicable regions in the plane $\gamma \times q$.



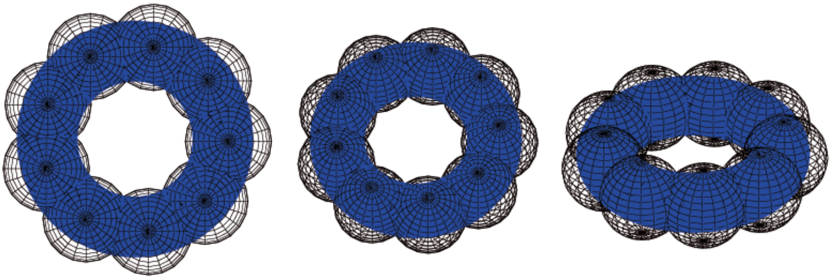
Torus covering: analytic approach

Geometric solution for $q = 4$ spheres



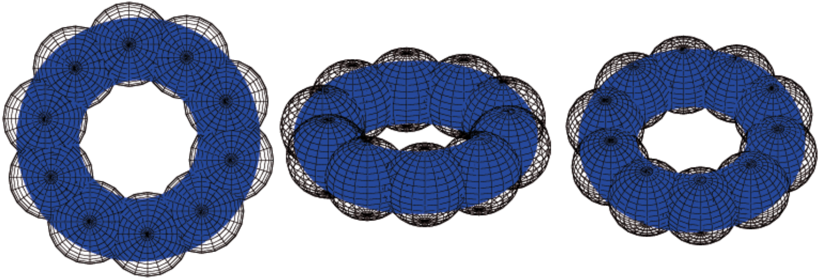
Torus covering: HS approach

Torus covering with 9 equal spheres



Torus covering: HS approach

Torus covering with 10 equal spheres



Torus covering: HS approach

Torus covering with 20 equal spheres

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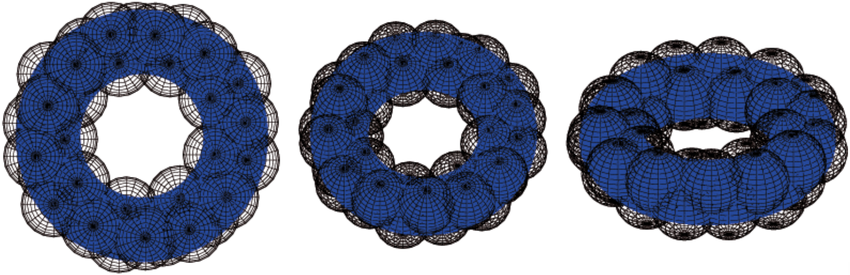
Torus covering: HS approach

Torus covering with 36 equal spheres



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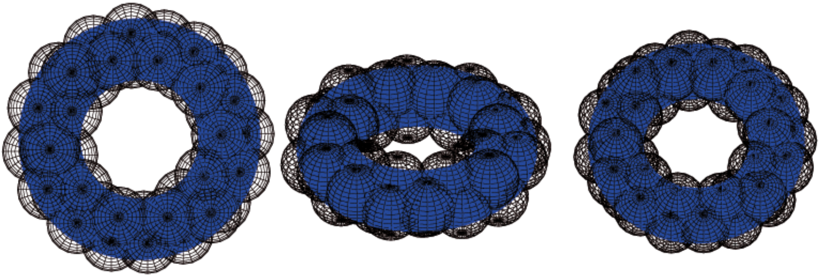
Torus covering: HS approach

Torus covering with 40 equal spheres



Torus covering: HS approach

Discretized Torus (244080 voxels)

q	d^*	$d^*_{AHSC-L2}$	E_{Mean}	T_{Mean}
2	0.100000E01	0.100000E01	0.00	11.21
3	0.866025E00	0.865590E00	0.00	19.03
4	0.707107E00	0.706491E00	0.00	26.94
5	0.587785E00	0.587081E00	0.00	35.56
6	0.500000E00	0.499247E00	0.00	45.61

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Torus covering: HS approach

Discretized Torus (244080 voxels)

q	d^*	$d^*_{AHSC-L2}$	E_{Mean}	T_{Mean}
7	0.439263E00	0.439064E00	0.03	62.34
8	0.398760E00	0.398166E00	0.10	83.77
9	0.370158E00	0.369614E00	0.05	75.08
10	0.349120E00	0.348708E00	0.06	87.26
12	0.320758E00	0.320322E00	0.06	103.40
16	0.291129E00	0.290933E00	0.07	253.29

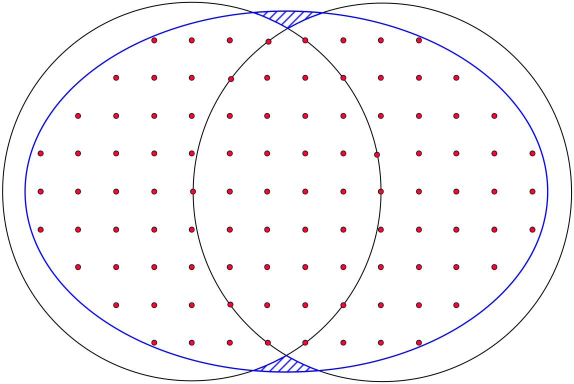
Torus covering: HS approach

Discretized Torus (244080 voxels)

q	d^*	$d^*_{AHSC-L2}$	E_{Mean}	T_{Mean}
20	—	0.276674E00	0.49	303.08
24	—	0.269039E00	4.56	538.10
30	—	0.266424E00	4.24	683.48
36	—	0.260362E00	0.66	878.66
40	—	0.252293E00	0.51	979.58

Torus covering: HS approach

Discretization side effect



Conclusions

- The method is robust and efficient when used with multistart, due to the global characteristic of *min-max-min* problems;
- The presented method can be applied to solve any problem of the *min-max-min* family;
- It is under investigation the possibility of using unequal spheres, which would be a better model for some applications (like Gamma-kniffe).

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