ECSE 509 Project Report

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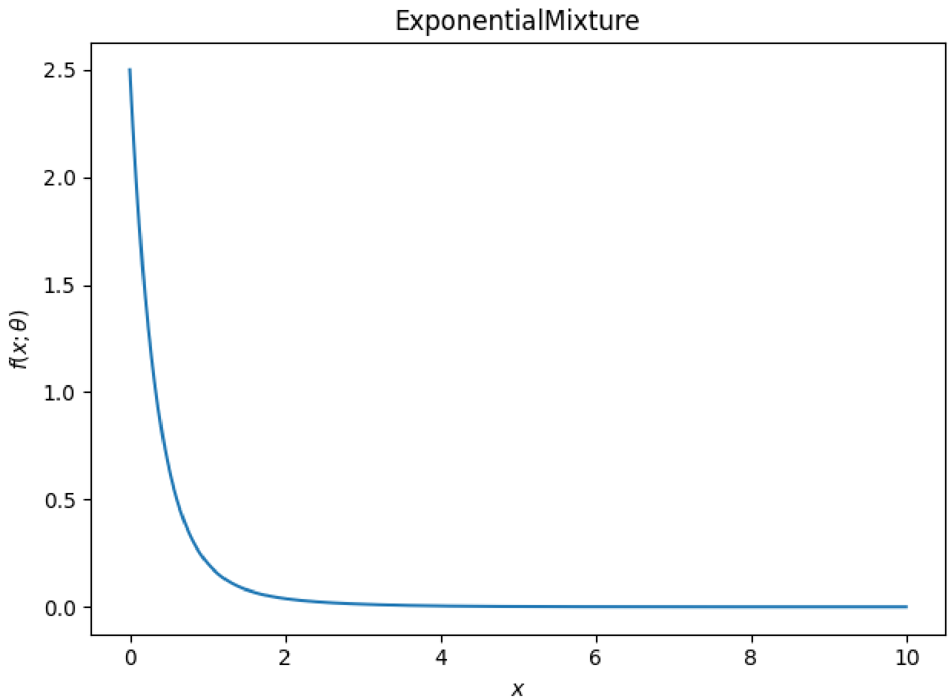
a) Pdf of the exponential mixture.



b) E and M update equations.

Let N be the number of observation we see, and J the number of exponential Pdf's we're mixing.

E step

$$P_{Y|X}(j|x_i) = \frac{\pi_j \lambda_j e^{-\lambda_j x_i}}{\sum_{j=1}^{J} \pi_j \lambda_j e^{-\lambda_j x_i}}$$

$Q(\theta|\theta^t) = \sum_{i=1}^{N}\sum_{j=1}^{J} P_{Y|X}(j|x_i)\log[P_{X|Y}(x_i,\theta)\pi_j]$

$\boxed{Q(\theta|\theta^t) = \sum_{i=1}^{N}\sum_{j=1}^{J} P_{Y|X}(j|x_i)\log[\pi_j\lambda_j e^{-\lambda_j x_i}]}$

M step

$\theta^{t+1} = \text{argmax} \sum_{i=1}^{N}\sum_{j=1}^{J} P_{Y|X}(j|x_i)\log[\pi_j\lambda_j e^{-\lambda_j x_i}]$

$\theta^{t+1} = \text{argmax} \sum_{i=1}^{N}\sum_{j=1}^{J} P_{Y|X}(j|x_i)(\log[\pi_j\lambda_j] + \log[e^{-\lambda_j x_i}])$

$\theta^{t+1} = \text{argmax} \sum_{i=1}^{N}\sum_{j=1}^{J} P_{Y|X}(j|x_i)(\log[\pi_j\lambda_j] + [-\lambda_j x_i])$

$\theta^{t+1} = \text{argmin} -\sum_{i=1}^{N}\sum_{j=1}^{J} P_{Y|X}(j|x_i)(\log[\pi_j\lambda_j] + [-\lambda_j x_i])$

$\boxed{\theta^{t+1} = \text{argmin} \sum_{i=1}^{N}\sum_{j=1}^{J} P_{Y|X}(j|x_i)(\lambda_j x_i - \log[\pi_j\lambda_j])}$

We have the constraint that $\sum_{j=1}^{J}\pi_j = 1$, therefor to enforce the condition, we have to write the following Lagrangian:

$\text{L} = \sum_{i=1}^{N}\sum_{j=1}^{J} P_{Y|X}(j|x_i)(\lambda_j x_i - \log[\pi_j\lambda_j]) + C(\sum_{j=1}^{J}\pi_j - 1)$

The update rule of our parameters are then found by:

$$\frac{dL}{d\lambda_j} = 0$$
$$\frac{dL}{d\pi_j} = 0$$

By solving the derivative with respect to both our terms, we get that the udpdate rule is then:

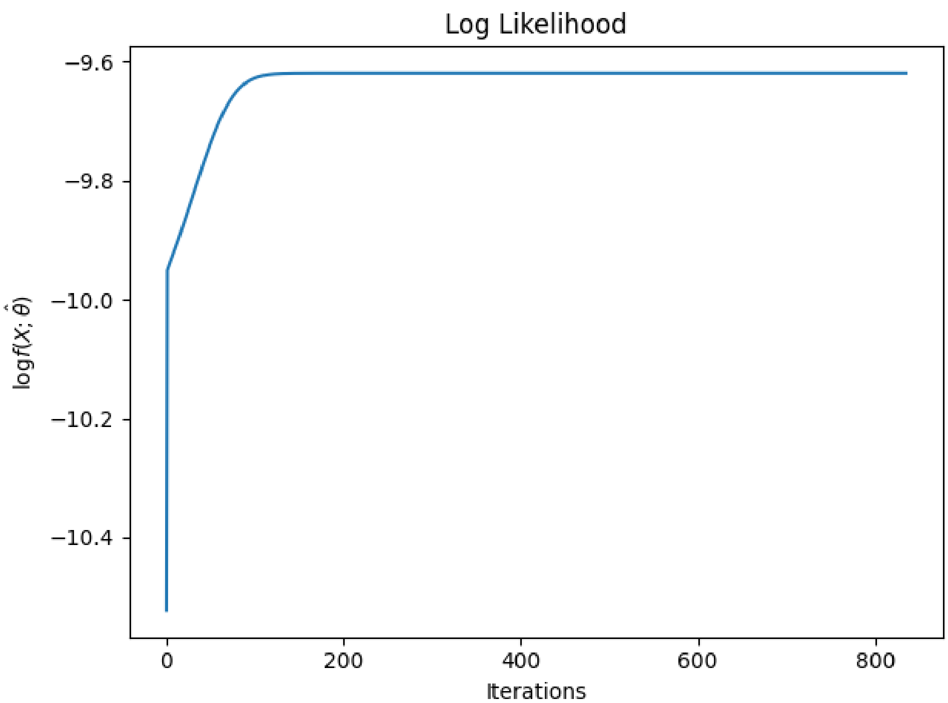
$$\boxed{\lambda_j = \frac{\sum_{i=1}^{N} P_{Y|X}(j|x_i)}{\sum_{i=1}^{N} P_{Y|X}(j|x_i)x_i}}$$

$$\boxed{\pi_j = \frac{\sum_{i=1}^{N} P_{Y|X}(j|x_i)}{\sum_{i=1}^{N}\sum_{j=1}^{J} P_{Y|X}(j|x_i)}}$$

c) EM Algorithm.

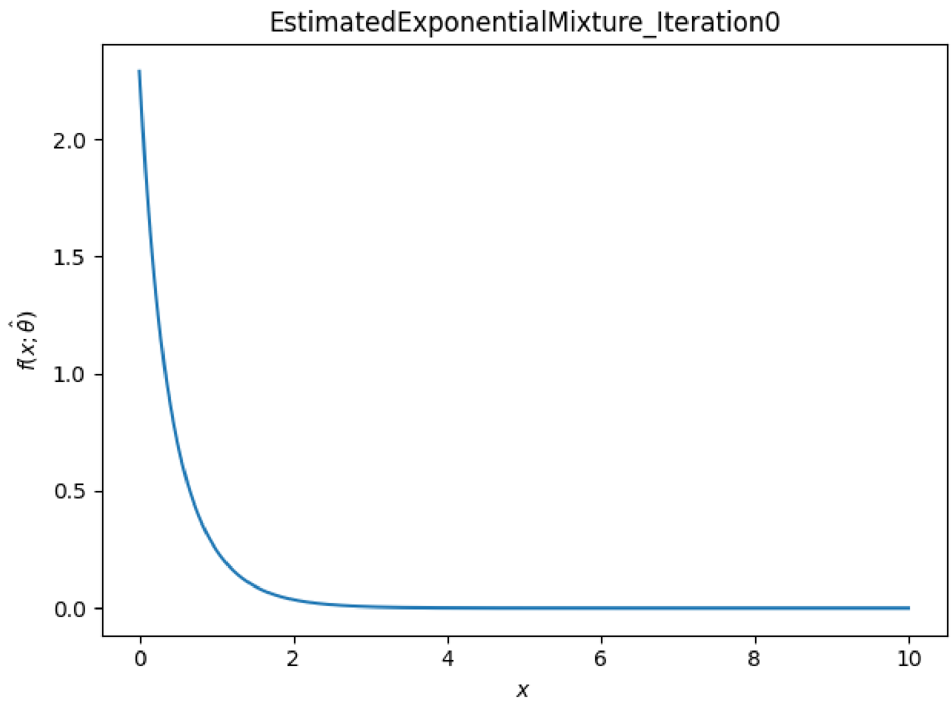
Please refer to the PDF containing the code uploaded separately on crowdmark.

d) Log Likelihood.



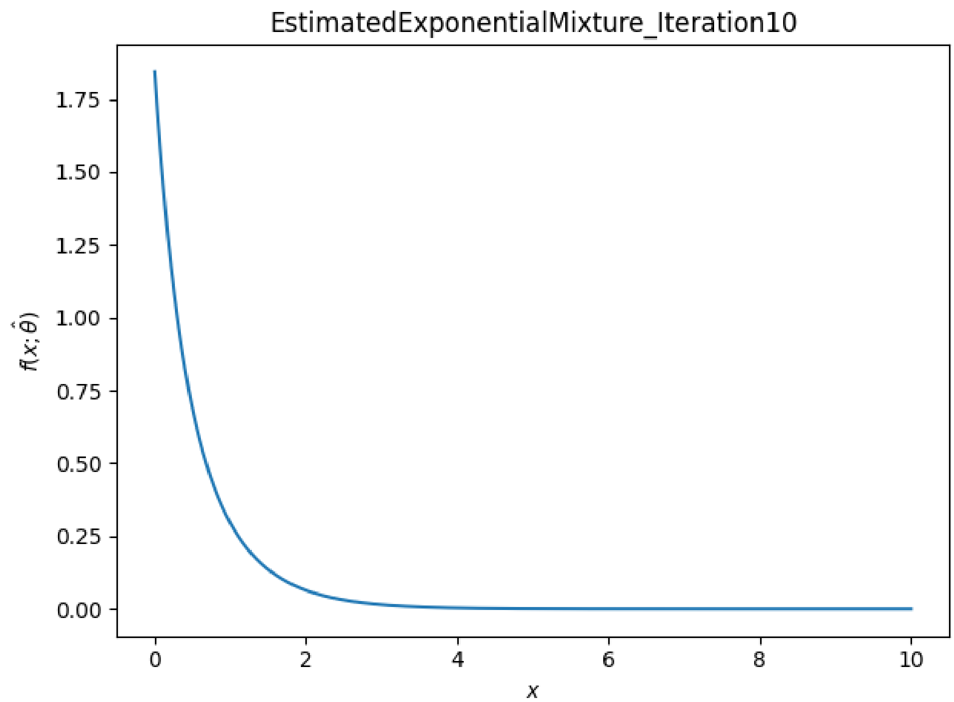
From the graph above, we can see that the log likelihood is monotonically increasing.

e) Estimated exponential mixture pdf.



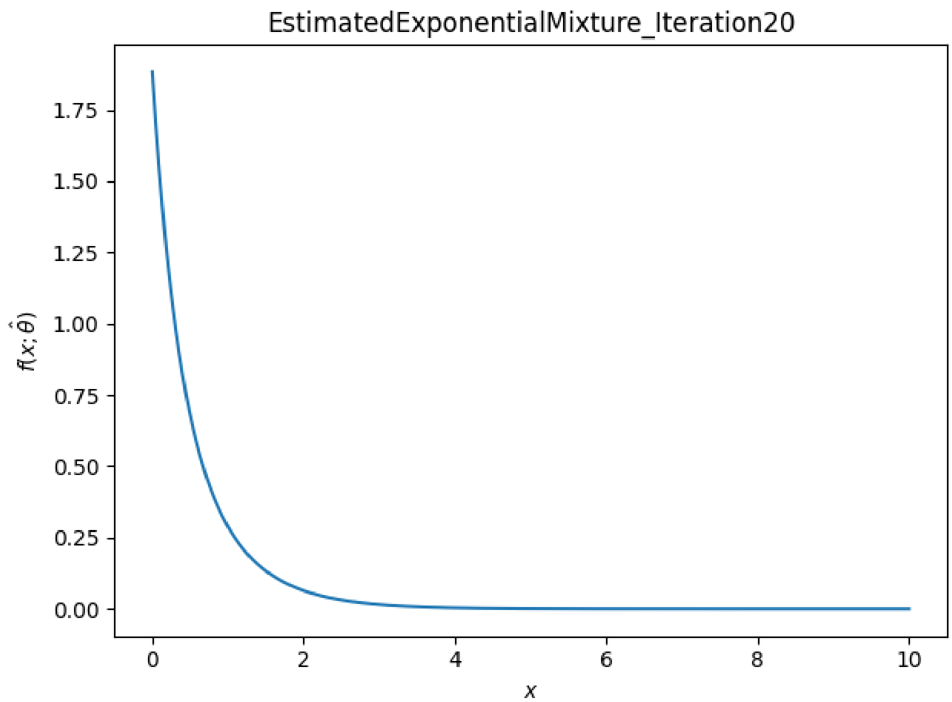
$\hat{\theta}$ estimate at iteration 0:

- $\hat{\lambda}_1 = 1.896595$
- $\hat{\lambda}_2 = 4.295177$
- $\hat{\pi}_1 = 0.8357424$
- $\hat{\pi}_2 = 0.1642576$



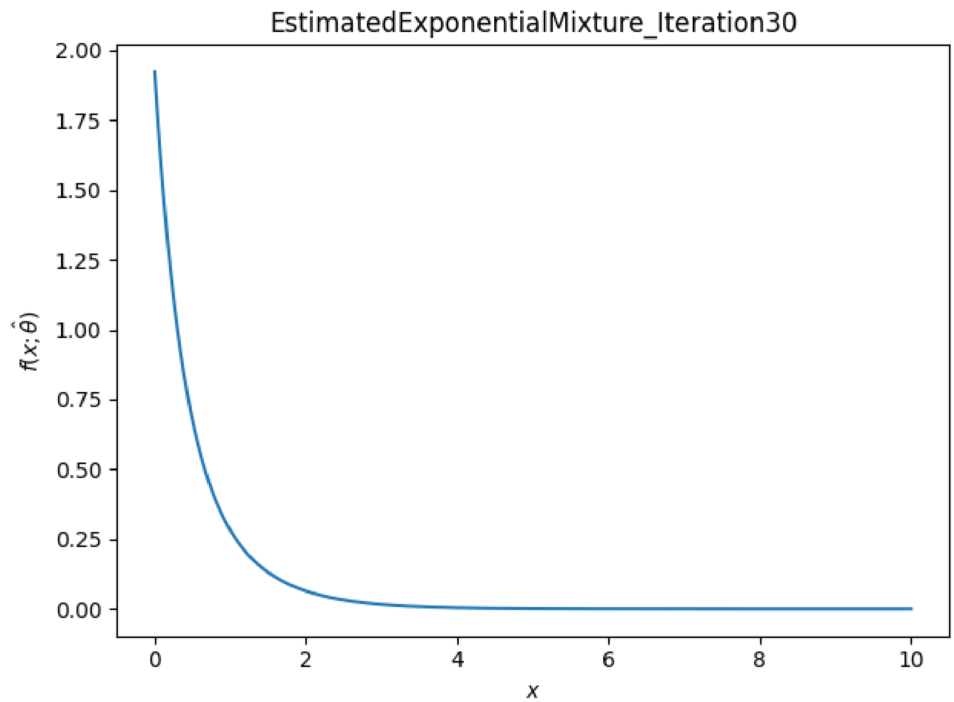
$\hat{\theta}$ estimate at iteration 10:

- $\hat{\lambda}_1 = 1.4575794$
- $\hat{\lambda}_2 = 3.436903$
- $\hat{\pi}_1 = 0.8049060$
- $\hat{\pi}_2 = 0.1950930$



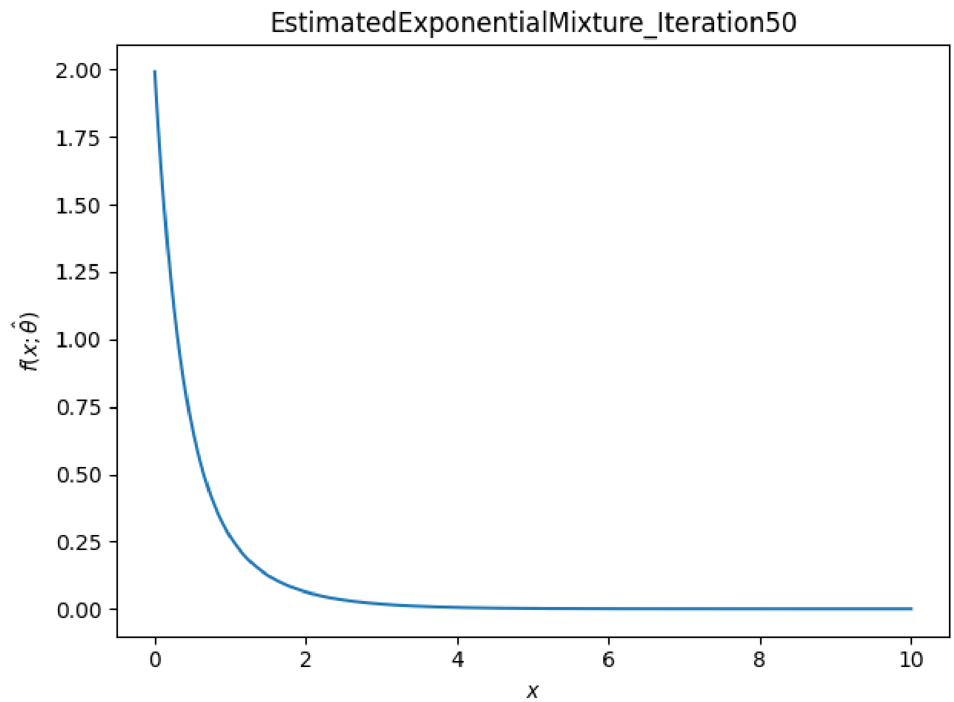
$\hat{\theta}$ estimate at iteration 20:

- $\hat{\lambda}_1 = 1.4025173$
- $\hat{\lambda}_2 = 3.295298$
- $\hat{\pi}_1 = 0.74601278$
- $\hat{\pi}_2 = 0.25398722$



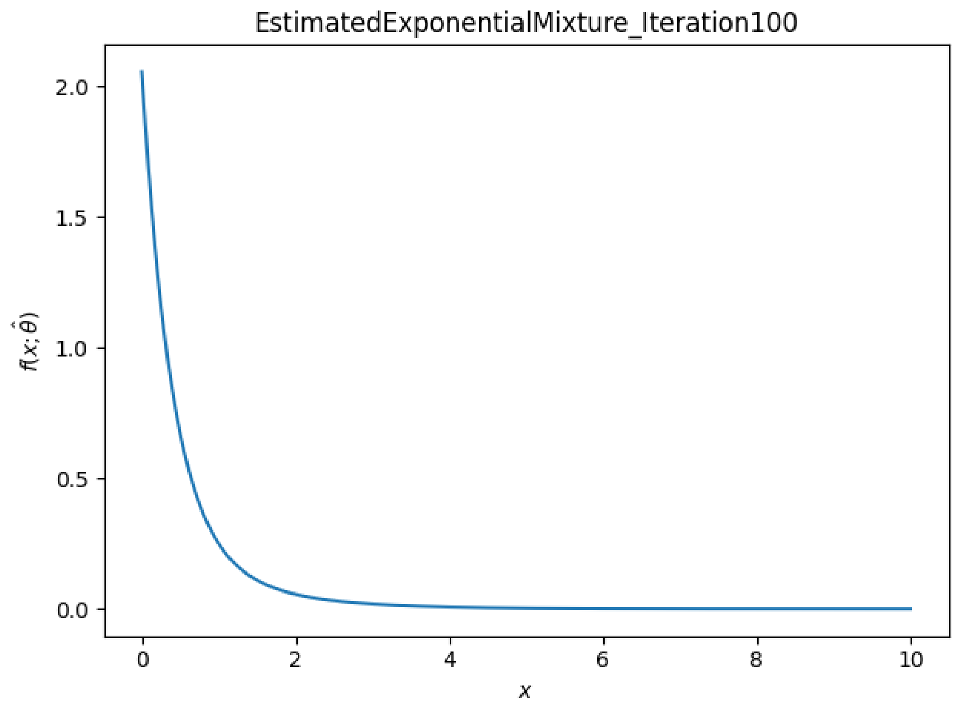
$\hat{\theta}$ estimate at iteration 30:

- $\hat{\lambda}_1 = 1.3397352$
- $\hat{\lambda}_2 = 3.1667288$
- $\hat{\pi}_1 = 0.680864$
- $\hat{\pi}_2 = 0.319136$



$\hat{\theta}$ estimate at iteration 50:

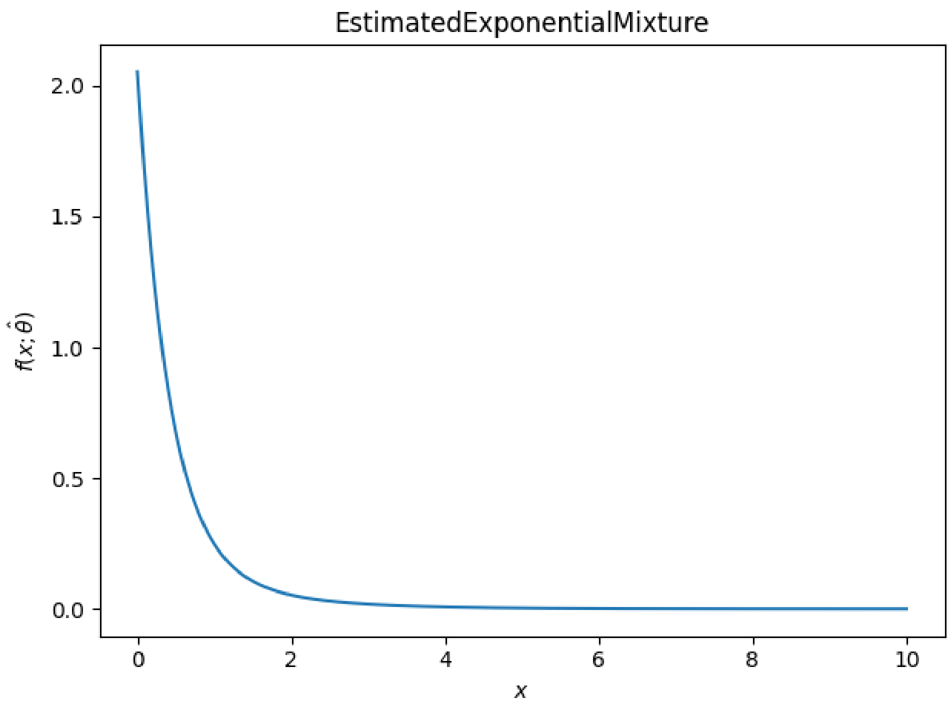
- $\hat{\lambda}_1 = 1.199646$
- $\hat{\lambda}_2 = 2.938034$
- $\hat{\pi}_1 = 0.544634$
- $\hat{\pi}_2 = 0.455369$



$\hat{\theta}$ estimate at iteration 100:

- $\hat{\lambda}_1 = 0.9189678$
- $\hat{\lambda}_2 = 2.5808147$
- $\hat{\pi}_1 = 0.316126$
- $\hat{\pi}_2 = 0.683873$

The final estimate is:



The final values found for $\hat{\theta}$ are:

- $\hat{\lambda}_1 = 0.8330837$
- $\hat{\lambda}_2 = 2.4772141$
- $\hat{\pi}_1 = 0.2577022$
- $\hat{\pi}_2 = 0.7422978$