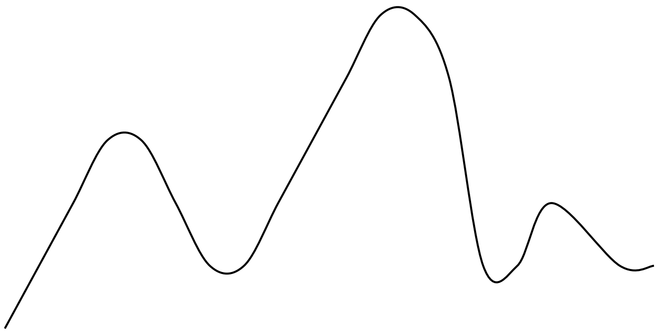
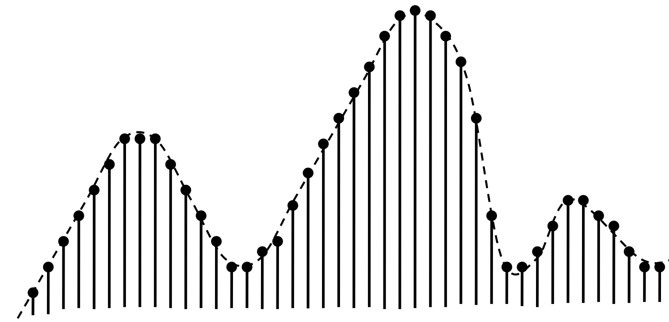
Signal compression: Quantization

To store, transmit or manipulate signals, they first must be digitized.

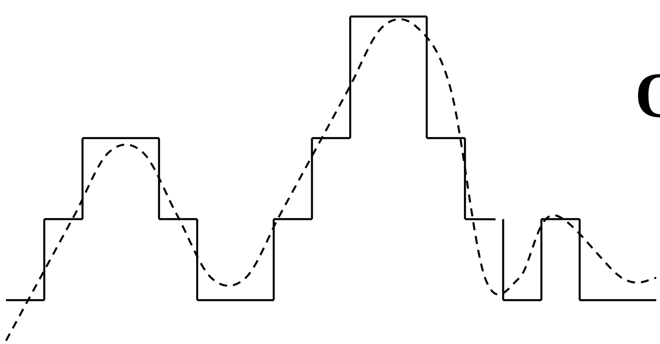


Sampling

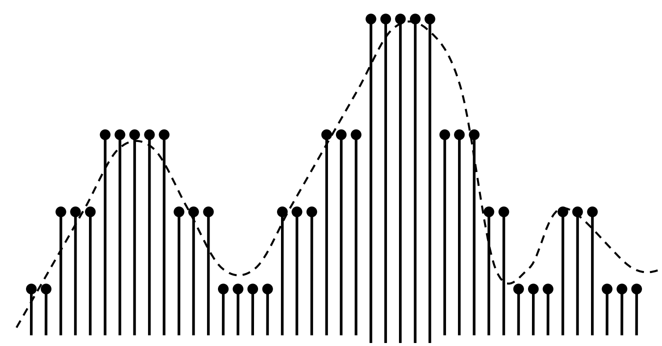


Amplitude-continuous
Time-continues

Amplitude-continuous,
Time-discrete

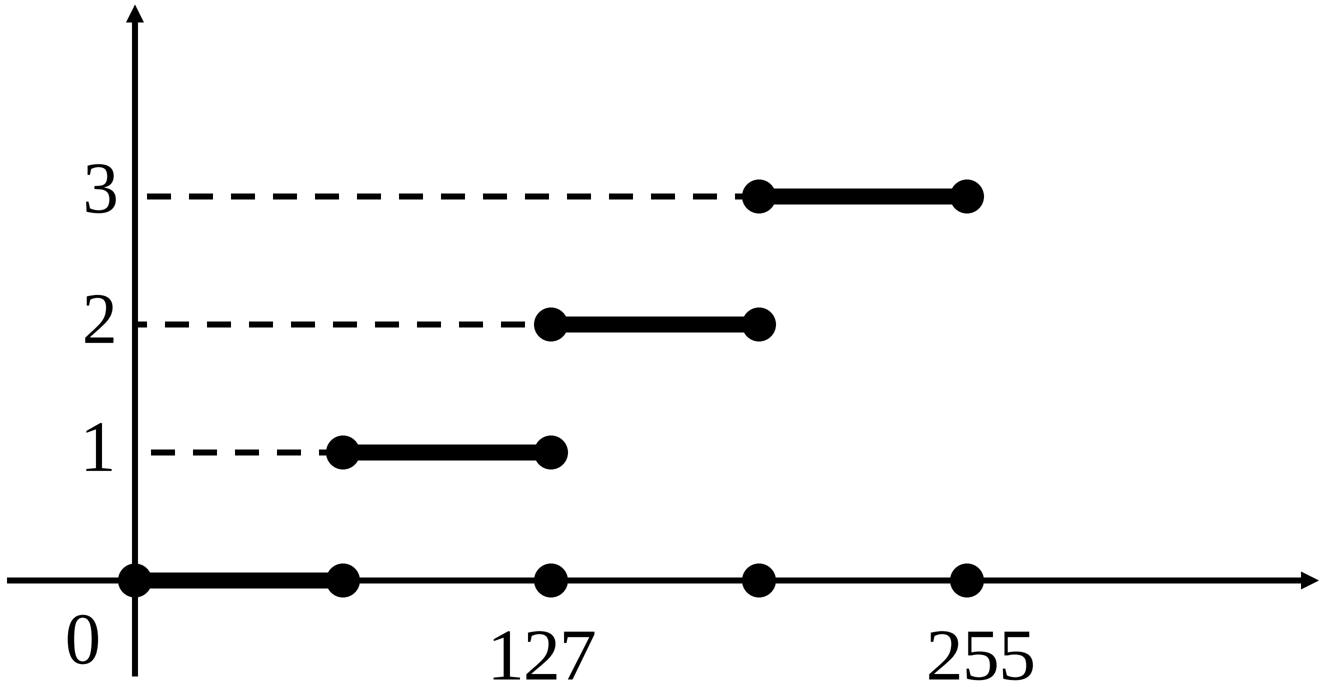


Quantization



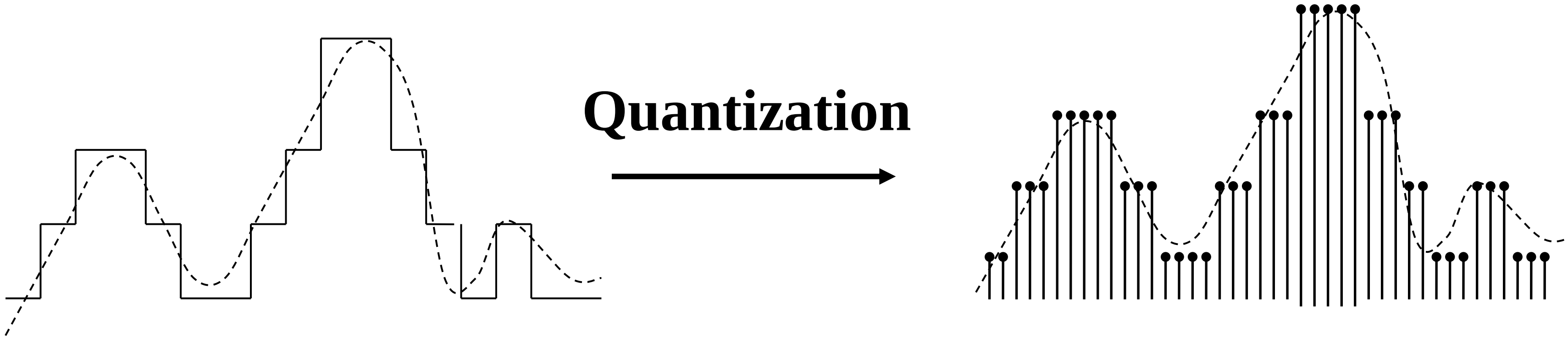
Amplitude-discrete,
Time-discrete

Image quantization



Matlab demo

A mathematical model of a quantizer



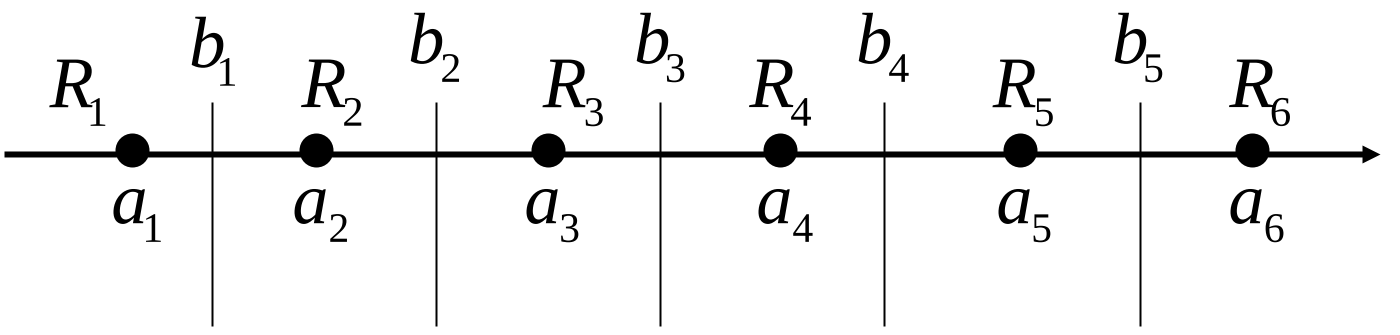
In coming sequence U_1, U_2, … is mapped into a sequence of discrete random variables V_1, V_2, … . The objective is that, for each m, V_{m} should represent U_{m} with as little distortion as possible.

When an analog random variable U is quantized into a discrete random variable V=V(U), the mean-squared distortion (MSE, mean-squared error) is defined by

$$E\left[\ U - V\ ^{2}\right]$$

A scalar quantizer

A scalar quantizer partitions the set of real numbers into M quantizer regions (intervals): $R_1, R_2, \ldots, R_M$ containing representation points a_j and separated by boundaries $b_1, \ldots, b_{M-1}$

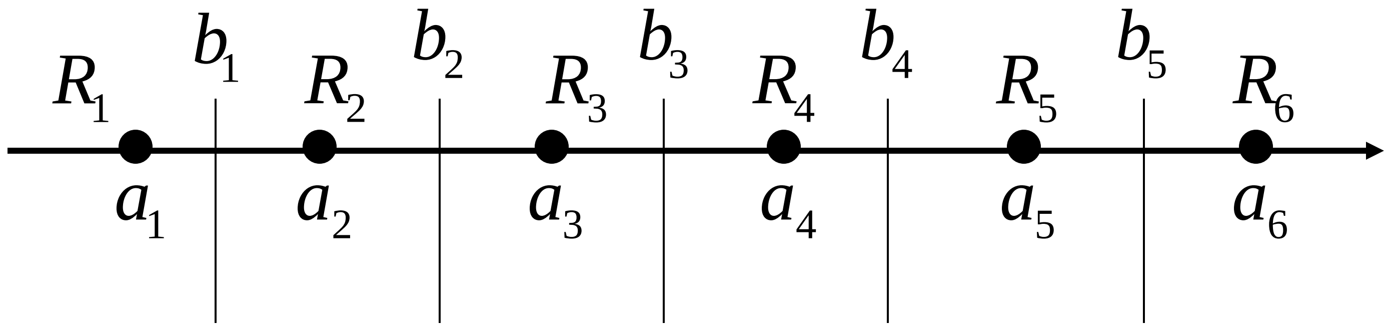


An analog sequence $u_1, u_2, \ldots$. These real numbers are sample values of random variables $U_1, U_2, \ldots$.

- Given a set of representation points $\{a_j\}$, how to choose $\{R_j\}$?
- Given a set of intervals $\{R_j\}$, how to choose representation points $\{a_j\}$?

Choice of intervals for given representation points

Given representation points $a_j : 1 \le j \le M$, determine b_j



Given u, the squared error to a_j is $|u - a_j|^2$

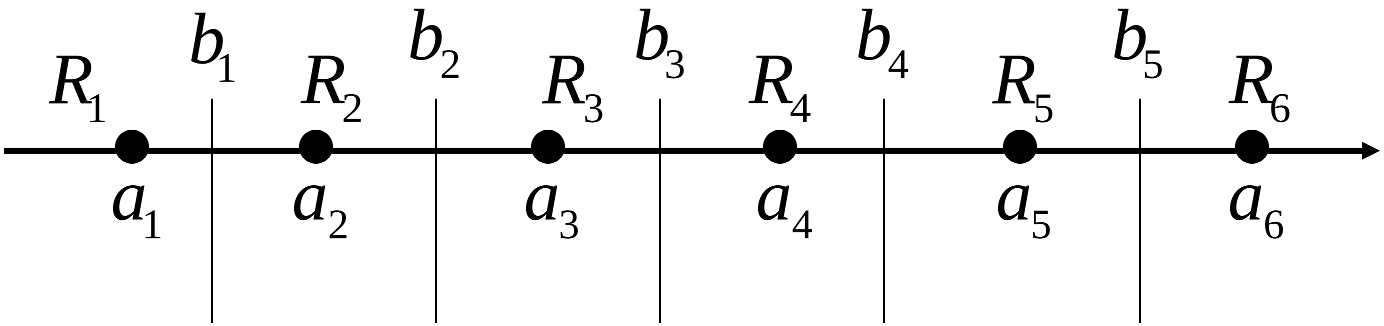
If u is between a_j and a_{j+1} then u is mapped to the closest one.

This implies that $b_j = (a_j + a_{j+1})/2$ and the quantization regions are determined.

Note that, in this case, the quantization result does not depend on on the probabilistic model for U_1, U_2, …

Choice of representation points for given intervals 1

Given intervals (i.e., b_j) find representation points a_j



$$\mathrm{MSE} = \boldsymbol{E}\left[\left(U - V(U)\right)^2\right] = \int_{-\infty}^{\infty} f_U(u)\left(u - v(u)\right)^2 du$$

$$= \sum_{j=1}^{M} \int_{R_j} f_U(u)\left(u - a_j\right)^2 du \rightarrow \min \ \text{over} \ a_j$$

Let $f_j(u)$ be the conditional pdf of U given that $u \in R_j$

Choice of representation points for given intervals 2

$$\mathrm{MSE} = \boldsymbol{E}\left[\left(U - V(U)\right)^2\right] = \int_{-\infty}^{\infty} f_U(u)\left(u - v(u)\right)^2 du$$

$$= \sum_{j=1}^{M} \int_{R_j} f_U(u)\left(u - a_j\right)^2 du \to \min \text{ over } \{a_j\}$$

Let $f_j(u)$ be the conditional pdf of U given that $u \in R_j$

$$f_j(u) = \begin{cases} f_U(u)/Q_j & \text{if } u \in R_j \\ 0 & \text{otherwise} \end{cases} \quad \text{where } Q_j = \Pr(U \in R_j)$$

Thus $\int_{R_j} f_U(u)\left(u - a_j\right)^2 du = Q_j \int_{R_j} f_j(u)\left(u - a_j\right)^2 du$

We have to minimize $\int_{R_j} f_j(u)\left(u - a_j\right)^2 du$ w.r.t. a_j

Choice of representation points for given intervals 3

We have to minimize $\int_{R_j} f_j(u)\,(u-a_j)^2\,du$ over a_j

$$\int_R f(u)\,(u-a)^2\,du = \int_R f(u)\,u^2 du - 2a\int_R f(u)\,u\,du + a^2$$

$$\overline{(X-a)^2} = \overline{X^2} - 2a\overline{X} + a^2$$

a quadratic polynomial which is minimized over a when $a = \overline{X}$

We arrive at a set of conditions that the endpoints $\{b_j\}$ and the points $\{a_j\}$ must satisfy to achieve the MSE:

- Each b_j must be the midpoint between a_j and a_{j+1}
- Each a_j and must be the mean of random variable U_j with pdf $f_j(u)$

The Lloyd-Max algorithm 1

Necessary conditions that the endpoints $\{b_j\}$ and the points $\{a_j\}$ must satisfy to achieve the MSE:

- Each b_j must be the midpoint between a_j and a_{j+1}
- Each a_j and must be the mean of random variable U_j with pdf $f_j(u)$

The Lloyd-Max algorithm alternates between

1. optimising the endpoints $\{b_j\}$ for a given set $\{a_j\}$ of and then
2. optimising the points $\{a_j\}$ for the new endpoints $\{b_j\}$.

Assume that M and $f_U(u)$ are given.

1. Choose an arbitrary initial set of $a_1 < a_2 < \ldots < a_M$
2. Set $b_j = (a_{j+1} + a_j)$
3. Set a_j equal to the conditional mean of U given $U \in (b_{j-1} b_j]$
4. Repeat 2. and 3. until further improvement in MSE is negligible.

The Lloyd-Max algorithm 2

1. Start from an initial set of representation points $a_j \; {}_{j=1}^{M}$
2. Find $M-1$ interval endpoints $b_j = \; a_j + a_{j+1} \; \big/ 2$
3. Compute the mean-squared error (MSE)

$$\mathrm{MSE} = \sum_{k=1}^{M} \int_{b_{k-1}}^{b_k} \; x - a_k \;^2 \, f \;\; x \;\; dx$$

4. Stop if MSE changes little from last time.
5. Otherwise, update $a_j = \int_{b_{j-1}}^{b_j} x \; f \;\; x \;\; dx \Bigg/ \int_{b_{j-1}}^{b_j} f \;\; x \;\; dx$
6. Go to Step 2.

The Lloyd-Max algorithm: an example

$$b_j = \left(a_j + a_{j+1} \right) \Big/ 2 \qquad a_j = \int_{b_{j-1}}^{b_j} x\, f(x)\, dx \Big/ \int_{b_{j-1}}^{b_j} f(x)\, dx$$

Consider a random variable **uniformly distributed** between 0 and 2. Let us use the Lloyd-Max algorithm to find the 2-bit quantizer that minimizes the MSE.

$$f(x) = \text{const}$$

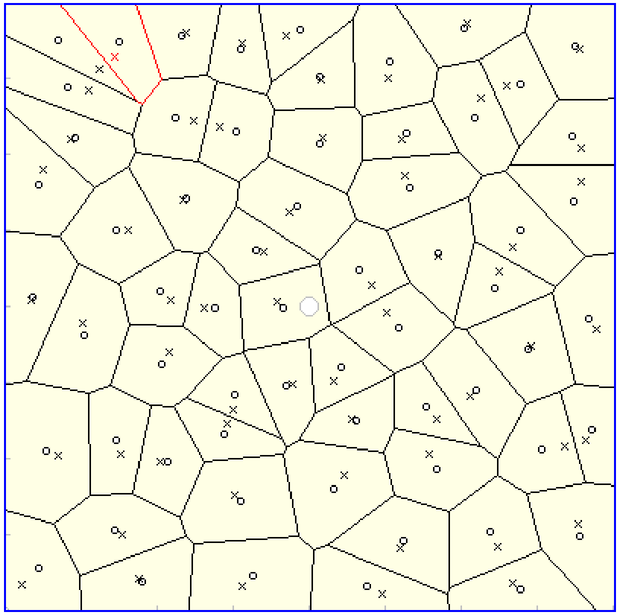
$$a_j = \int_{b_{j-1}}^{b_j} x\, f(x)\, dx \Big/ \int_{b_{j-1}}^{b_j} f(x)\, dx = \tfrac{1}{2}\left(b_j^2 - b_{j-1}^2 \right) \Big/ \left(b_j - b_{j-1} \right) = \frac{b_j + b_{j-1}}{2}$$

$$b_0 = 0, \quad b_1 = 0.5, \quad b_2 = 1, \quad b_3 = 1.5, \quad b_4 = 2$$

$$a_1 = 0.25, \quad a_2 = 0.75, \quad a_3 = 1.25, \quad a_4 = 1.75$$

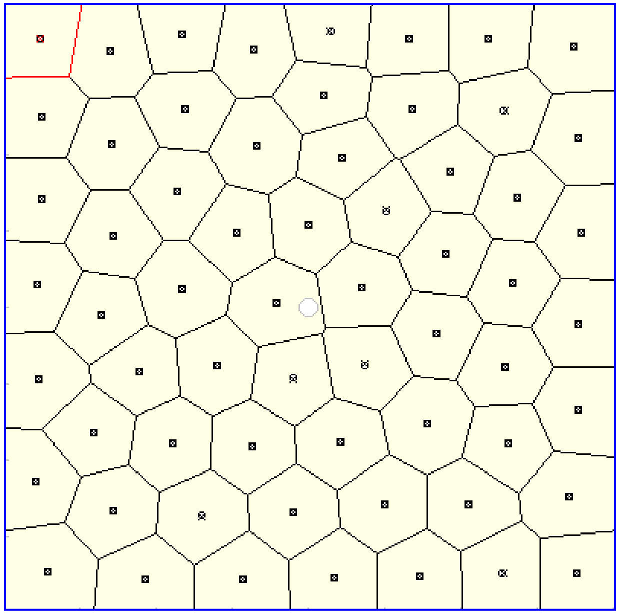
Vector quantization

The Lloyd-Max algorithm can be used in the multidimensional case.



Lloyd-Max

algorithm



Voronoi diagram / tesselation

Centroidal Voronoi diagram

A nice Java
applet: http://www.raskob.de/fun/d/index-en.php