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Online Trajectory Optimization Based on Successive Convex Optimization

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Abstract: The traditional trajectory optimization methods are often difficult to achieve the rapid requirements of online planning. Therefore, the convex optimization method is applied to trajectory optimization of a missile to reduce the computational cost in this paper. With the convex optimization theory, the missile nonlinear motion model is firstly convexified based on the small perturbation linearization theory and then discretized using the Euler method, which is then solved by the successive convex optimization method. Simulation results show that the convex optimization technique can greatly reduce the computational time, while yields comparable optimal solutions compared to the well-known Gauss pseudospectral method.

Key Words: successive convex optimization, trajectory optimization, online

1 Introduction

The traditional trajectory planning and guidance methods for missiles generally include two steps: to generate a reference trajectory satisfying a variety of boundary conditions and path constraints; to design a guidance law to track the reference trajectory. The off-line trajectory planning can offer sufficient time for off-line debugging and simulation verification, but cannot address the varying missions and various types of uncertainties. On the contrary, the online trajectory planning can be adapted to different missions and deal with uncertainties, which is getting more and more extensive research.

The numerical algorithm for solving trajectory optimization can be generally divided into the indirect method and direct method[1] . Indirect methods, such as the minimum principle, are sensitive to the state initial value and has small convergence domain, which are difficult to deal with the problem with nonlinear complex constraints. At present, various direct approaches of trajectory planning have been proposed, such as the direct shooting method[1] , direct collocation method and the pseudo-spectral method[2] , of which the general idea is to transfer the optimal control problem into a finite-dimensional nonlinear programming problem (NLP) that is easy to be handled by many optimization algorithms. These methods can deal with multiple linear/nonlinear path and boundary constraints, exhibiting good performances in trajectory optimization. Due to its high robustness and efficiency, the pseudospectral method has been widely applied to aircraft guidance problems in recent years[2-8]. The reentry trajectory optimization of the reusable launch vehicle is solved by the Radau pseudospectral method[3] . An optimal on-board predictor-corrector guidance algorithm is proposed based on the Gauss pseudospectral method, in which the adaptive guidance is obtained using the on-line segmented optimization between routes[4] . A serial segmentation optimization strategy based on Gauss pseudospectral method is proposed to improve the optimization precision and computational efficiency for the gliding reentry trajectory optimization under remote multi-constraint conditions[5] . The good performances of the pseudospectral method have been illustrated in literatures. However, they can only be applied to off-line trajectory planning limited by its computational cost, which has been found in many references and the works of authors.

In recent years, practical methods and tools have been developed to solve the convex optimization problem quickly and reliably[9-13]. The convex optimization theory is being increasingly applied to rapid and online trajectory optimization problems. The convex optimization theory has been applied to the model predictive control to implement fast and accurate trajectory tracking[9] and the trajectory optimization [10] for the hypersonic vehicle reentry problem through employing the CVXGEN tool. To address the non-convex constraints during the minimum fuel dynamic descent guidance problem and the Mars fixed-point landing trajectory optimization, the trajectory optimization problem is firstly transformed into a subclass of convex optimization, *i.e.* a second-order cone programming problem, which is then solved by the convex optimization theory [11] .

In this paper, the convex optimization theory is applied to the trajectory optimization of a missile, which attacks a stationary target with the least control effort under the condition of multiple constraints. With the convex optimization theory, the missile nonlinear motion model is firstly convexified based on the small perturbation linearization theory and discretized using the Euler method, which is then solved by the successive convex optimization method[12] . The generated solutions by the convex optimization are compared to those from the Gauss pseudospectral method. The comparison shows that the two approaches produce comparable results that are very close to each other, which demonstrates the effectiveness of the convex optimization.

2 The Establishment of Missile Motion Model

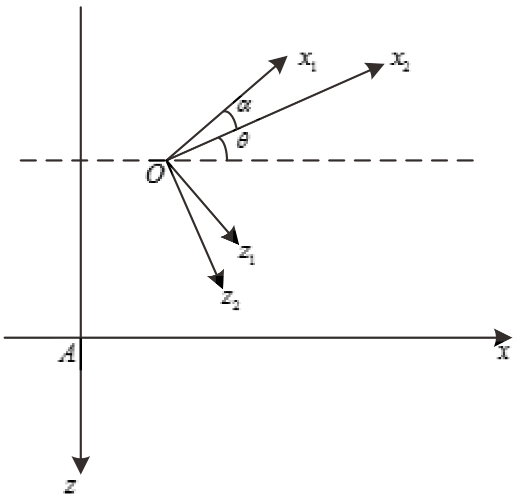

Fig. 1: Coordinate diagram

The coordinate system is established as shown in Figure 1, where Axz is the earth fixed coordinate system with the origin at a fixed point on the ground; Ox_1z_1 is the missile body coordinate system with the origin at the mass center of the missile body; Ox_2z_2 is the ballistic coordinate system with the origin at the center of the missile body.

Only considering the motion of a missile in the longitudinal plane, the dynamic equations are expressed as:

$$\begin{cases} m\dfrac{dV}{dt} = F_A \cdot \cos\alpha + F_N \cdot \sin\alpha - mg\sin\theta \\ mV\dfrac{d\theta}{dt} = F_A \cdot \sin\alpha - F_N \cdot \cos\alpha - mg\cos\theta \\ \dfrac{dx}{dt} = V\cos\theta \\ \dfrac{dz}{dt} = -V\sin\theta \end{cases} \tag{1}$$

where V is the speed of the missile, θ is flight path angle, α is angle of attack.

It is assumed that the positive directions of Ox_1 and Oz_1 of the missile body coordinate system are the positive directions of the axial force F_A and the normal force F_N, respectively. F_A and F_N of the missile are represented as:

$$\begin{aligned} F_A &= -\frac{1}{2}\rho V^2 S \cdot C_A \\ F_N &= -\frac{1}{2}\rho V^2 S \cdot C_N \end{aligned} \tag{2}$$

where the atmospheric density ρ is expressed as $\rho = \rho_0 \cdot e^{\beta z}$.

In Eqs (2), C_A and C_N are respectively the coefficients of axial force and normal force, which can approximately be computed as follows：

$$\begin{aligned} C_A &= C_{A0} \\ C_N &= C_N^{\ \alpha} \cdot \alpha \end{aligned} \tag{3}$$

where C_{A0} and $C_N^{\ \alpha}$ are calculated through linear interpolation according to the given table of aerodynamic data.

3 Description of Trajectory Optimization Problem

The trajectory optimization problem in this work aims to find an optimal law of the control variable, *i.e.* the angle of attack, which yields the least total control effort and satisfies relevant constrains. The trajectory optimization model is described as follows:

1) **Performance index**: the total control effort is minimized, so the performance index is:

$$\min\ J = \int_{t_0}^{t_f} \alpha^2 dt \tag{4}$$

2) **State constraints**: Eq. (1);

3) **Terminal constraints**: to ensure the accuracy of attack and improve the damage effect of the missile, the terminal states should satisfy the following terminal constraints:

$$
\begin{cases} V_f \in [V_{f\min}, V_{f\max}] \\ \theta_f \in [\theta_{f\min}, \theta_{f\max}] \\ \mathrm{x}_f = \mathrm{x}_t \\ z_f = z_t \end{cases} \tag{5}
$$

where $V_{f\min}$ and $V_{f\max}$ are the minimum and maximum values of the allowed impact velocity; $\theta_{f\min}$ and $\theta_{f\max}$ are the minimum and maximum values of the allowed impact angle; and (x_t, z_t) is the coordinates of the target position to be attacked by the missile.

4) **Control variable constraint**: the control variable to be designed is the angle of attack and its amplitude is restrained within the following range considering the control ability of the missile.

$$
|\alpha| < \alpha_{\max} \tag{6}
$$

where $\alpha_{\max}$ is the maximum value of the allowed angle of attack.

5) **Path constraint**: in order to ensure the structural safety of the missile, it is necessary to consider the constraint of the overload:

$$
a_n = \left| V \frac{d\theta}{dt} \right| < a_{n\max} \tag{7}
$$

4 Convexification and Discretization of the Problem

In order to solve the problem of trajectory optimization described above by convex optimization theory, convexification should be made on the original trajectory optimization problem firstly. In addition, the solution of the problem is a numerical solution, thus the model of the convex optimization problem needs to be discretized.

4.1 Time Mapping Transformation

The flight time t_f of the missile considered here is free. The time variable should be first mapped to the interval [0,1], then transformations are correspondingly made on the performance index and the constraints.

The time variable t ($t \in [t_0, t_f]$) is transferred to τ ($\tau \in [0,1]$):

$$
t = t_0 + (t_f - t_0)\tau \tag{8}
$$

Then accordingly a new control variable u_2 and a new state variable x_5 are introduced:

$$
x_5 = t, u_2 = t_f - t_0 \tag{9}
$$

Clearly, one has:

$$
\frac{dt}{d\tau} = u_2 \tag{10}
$$

The Eq. (1) can be rewritten as:

$$
\dot{x}(t) = f\left(x(t), u_1(t)\right) \tag{11}
$$

where $x(t) = [V(t)\ \theta(t)\ \mathrm{x}(t)\ \mathrm{z}(t)]^T$, $u_1(t) = \alpha(t)$;

Eq. (11) can be expressed with τ as the new independent variable:

$$
\dot{x}(\tau) = \frac{dx}{dt} \cdot \frac{dt}{d\tau} \tag{12}
$$

Since a new control variable and a new state variable are introduced, the new state space expression obtained from Eqs. (8) - (12) can be represented as:

$$
\dot{x}(\tau) = [\dot{V}(t)\ \ \dot{\theta}(t)\ \ \dot{\mathrm{x}}(t)\ \ \dot{z}(t)\ \ 1]^T \cdot u_2 \tag{13}
$$

Eq. (13) can be further expressed as below for convenience:

$$
\dot{x} = f = [f_1\ f_2\ f_3\ f_4\ f_5]^T \tag{14}
$$

4.2 Convexification of the Problem

During the process of convex optimization, the trajectory optimization problem should be convexified firstly. The reference points on the trajectory are recorded as $x_{ref}(k) = x_k = [V_k\ \ \theta_k\ \ \mathrm{x}_k\ \ \mathrm{z}_k]^T$ and $\alpha(k) = \alpha_k$ ($k = 0,1,2,\ldots$), on behalf of the general time reference points. At the reference point x_k , the state equation (14) is linearized, through ignoring the higher-order term and considering only the first-order term:

$$
\dot{x} = A_1(k)x + B_1(k)u_1 + B_2(k)u_2 + C_1(k) \tag{15}
$$

where the coefficients in the equation above are expressed as follows:

$$
A_1(k) = \left.\begin{bmatrix} \frac{\partial f_1}{\partial V} & \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial \mathrm{x}} & \frac{\partial f_1}{\partial z} & \\ \frac{\partial f_2}{\partial V} & \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial \mathrm{x}} & \frac{\partial f_2}{\partial z} & \\ \frac{\partial f_3}{\partial V} & \frac{\partial f_3}{\partial \theta} & \frac{\partial f_3}{\partial \mathrm{x}} & \frac{\partial f_3}{\partial z} & 0 \\ \frac{\partial f_4}{\partial V} & \frac{\partial f_4}{\partial \theta} & \frac{\partial f_4}{\partial \mathrm{x}} & \frac{\partial f_4}{\partial z} & \\ & & 0 & & 0 \end{bmatrix}\right|_{x_k, u_{2k}}
$$

$$B_1(k)=\begin{bmatrix}\frac{\partial f_1}{\partial\alpha}\\ \frac{\partial f_2}{\partial\alpha}\\ \frac{\partial f_3}{\partial\alpha}\\ \frac{\partial f_4}{\partial\alpha}\\ 0\end{bmatrix}\Bigg|_{x_k,\alpha_k,u_{2k}}$$

$$B_2(k)=\begin{bmatrix}\frac{\partial f_1}{\partial u_2}\\ \frac{\partial f_2}{\partial u_2}\\ \frac{\partial f_3}{\partial u_2}\\ \frac{\partial f_4}{\partial u_2}\\ \frac{\partial f_5}{\partial u_2}\end{bmatrix}\Bigg|_{x_k,\alpha_k,u_{2k}}=\begin{bmatrix}\frac{\partial f_1}{\partial u_2}\\ \frac{\partial f_2}{\partial u_2}\\ \frac{\partial f_3}{\partial u_2}\\ \frac{\partial f_4}{\partial u_2}\\ 1\end{bmatrix}\Bigg|_{x_k,\alpha_k,u_{2k}}$$

$$C_1(k)=f-A_1(k)x-B_1(k)u_1-B_2(k)u_2\big|_{x_k,\alpha_k,u_{2k}}$$

$$=\begin{bmatrix}C_{11}\\C_{12}\\C_{13}\\C_{14}\\C_{15}\end{bmatrix}=\begin{bmatrix}-\frac{\partial f_1}{\partial V}V-\frac{\partial f_1}{\partial\theta}\theta-\frac{\partial f_1}{\partial \mathrm{x}}\mathrm{x}-\frac{\partial f_1}{\partial z}z-\frac{\partial f_1}{\partial\alpha}\alpha\\ -\frac{\partial f_2}{\partial V}V-\frac{\partial f_2}{\partial\theta}\theta-\frac{\partial f_2}{\partial \mathrm{x}}\mathrm{x}-\frac{\partial f_2}{\partial z}z-\frac{\partial f_2}{\partial\alpha}\alpha\\ -\frac{\partial f_3}{\partial V}V-\frac{\partial f_3}{\partial\theta}\theta-\frac{\partial f_3}{\partial \mathrm{x}}\mathrm{x}-\frac{\partial f_3}{\partial z}z-\frac{\partial f_3}{\partial\alpha}\alpha\\ -\frac{\partial f_4}{\partial V}V-\frac{\partial f_4}{\partial\theta}\theta-\frac{\partial f_4}{\partial \mathrm{x}}\mathrm{x}-\frac{\partial f_4}{\partial z}z-\frac{\partial f_4}{\partial\alpha}\alpha\\ 0\end{bmatrix}$$

4.3 Discretization of the Problem

Since the convex optimization is a numerical method, the state equation needs to be discretized. Supposing the sampling period is *T*, Eq. (15) can be discretized as:

$$x(k+1)=A(k)x(k)+B(k)u(k)+C(k) \tag{16}$$

where $A(k)=TA_1(k)+I_5$, $B(k)=T[B_1(k)\ B_2(k)]$, $C(k)=TC_1(k)$ and $u(k)=[u_1(k)\ u_2(k)]^T$.

The constraints on the control variable are:

$$|u_1(k)|\le u_{\max} \tag{17}$$

$$x_5(k)=u_2\frac{k-1}{N-1} \tag{18}$$

where *N* represents the number of time intervals of discretization; $k=1,2,\dots,N$; and $T=\frac{1}{N}$.

5 Successive Convex Optimization Algorithm

For the optimization problem obtained above, the successive convex optimization algorithm[12] is employed to solve it. The detailed procedure is shown as follows:

1) Set $n=0$, and provide a reference trajectory $x^{(0)}$;

2) In the $(n+1)$ *th* iteration, the optimal solution is obtained based on the parameters of the Eqs.(15) - (18) and the reference trajectory $x^{(n)}$ to obtain the current optimal solution $\{\boldsymbol{x}^{(n+1)};u^{(n+1)}\}$;

3) Check whether the following convergence criteria is satisfied:

$$\max\left|x^{(n+1)}(\tau)-x^{(n)}(\tau)\right|\le\delta \tag{19}$$

where $\delta\in R^4$ is a sufficiently small vector.

If the convergence criterion is satisfied, proceed to step 4); otherwise set $n=n+1$ and return to step 2).

4) Once the convergent optimal solution $\{\boldsymbol{x}^{(n+1)};u^{(n+1)}\}$ is found, it is considered as the optimal solution of the original problem, *i.e.*, $u^*=u^{n+1}$.

6 Numerical Simulation

The atmospheric model and the fitted aerodynamic data of the missile are $\rho_0=1.272$, $\beta=1.208e-4$, $C_{A0}=0.53$, $C_N{}^\alpha=11.78$.

The position of the target is $[100km,0]$. The initial values of states of the missile are $V_0=1120m/s$, $\theta_0=40^\circ$, position $\mathrm{x}_0=0$ and $z_0=-10km$. The terminal constraints on the state variables of the missile are $V_{f\min}=300m/s$, $V_{f\max}=500m/s$, $\theta_{f\min}=-70^\circ$, $\theta_{f\max}=-60^\circ$, $\mathrm{x}_t=100km$ and $z_t=0$. $a_{n\max}$ is set as $a_{n\max}=10$. The tool MOSEK is employed to solve the convex optimization problem during each iteration to obtain a reference trajectory till the convergence criteria is satisfied.

In order to verify the effectiveness of the successive convex optimization method, the Gauss pseudospectral method(GPM) is also employed to solve the original trajectory optimization problem shown in Eq. (1) and Eqs.

(4) - (7). The optimal solutions of the proposed method in this work and the GPM are shown in Figures 2-5, in which the solid and dotted lines respectively denote the results from the proposed method and GPM.

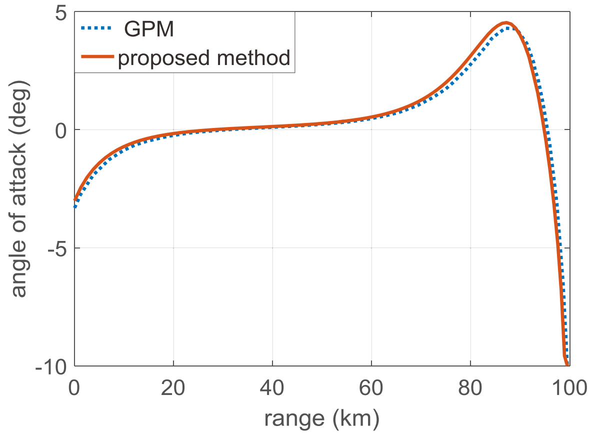

Fig. 2: Obtained angle of attack of successive convex optimization and GPM

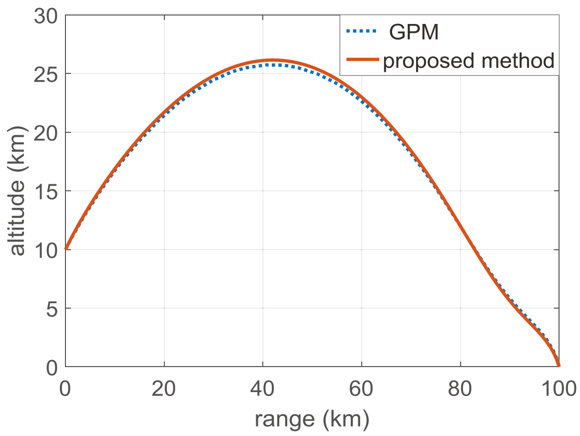

Fig. 3: Trajectory of successive convex optimization and GPM

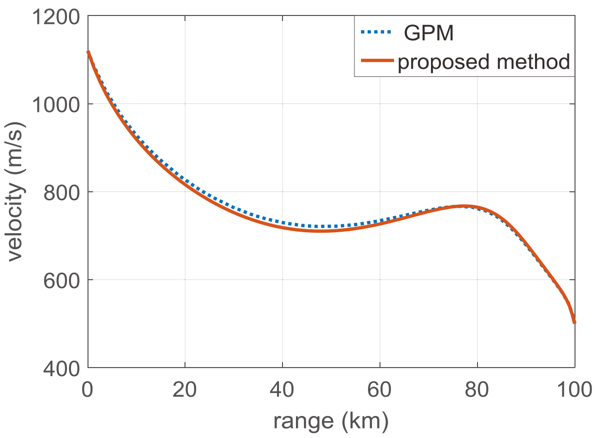

Fig. 4: Velocity of successive convex optimization and GPM

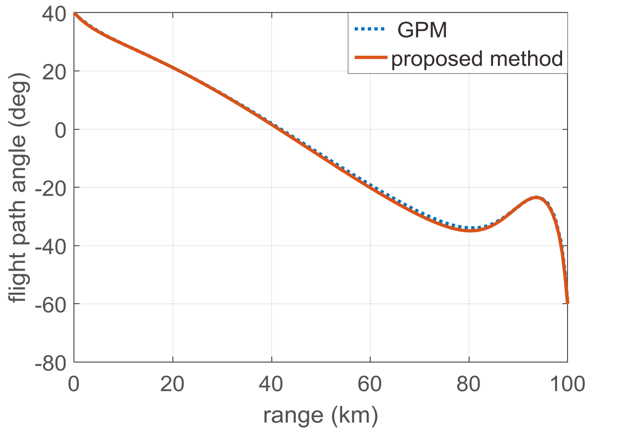

Fig. 5: Flight path angle of successive convex optimization and GPM

It can be seen from these figures that the optimal control variable $\alpha(t)$ and the state variables obtained by the successive convex optimization method and GPM are almost the same, which are very close to each other. Moreover, the missile can accurately hit the target with the obtained control $\alpha(t)$ of the two methods. Constraints on the impact speed and impact angle are satisfied. Therefore, the convex optimization problem with convexification can be considered as a good approximation to the original optimal control problem, and the effectiveness of the successive convex optimization method is well demonstrated.

GPM takes about 8.93s to find the optimal solution on a desktop computer with Intel Core i3 3.20 GHz, which is impossible to be applied to online optimization. In contrast, the successive convex optimization method gets converged in five iterations and takes about 0.405s to obtain the final solution with about 0.081s during each iteration. Hence, the successive convex optimization method can significantly improve the computational efficiency. Such an improvement on the computational cost potentially allows the proposed method to be implemented online.

7 Conclusions

In this paper, the trajectory optimization problem of a missile attacking a stationary target with the least control effort under multi-constraints is studied. The successive convex optimization method is proposed to solve this nonlinear optimal control problem through convexification and discretization. Simulation results show that the optimal solutions of the convex optimization method exhibit great agreements to those obtained by the traditional Gauss pseudospectral method, while the computational time of the former is significantly reduced compared to the latter. The successive convex optimization strategy can be considered as a new way to implement online trajectory planning.

References

[1] Betts J T. Survey of numerical methods for trajectory optimization. *Journal of guidance, control, and dynamics*, 1998, 21(2): 193-207.

[2] S. Hu, Y. Chen, Analysis of pseudospectral methods applied to aircraft trajectory optimization. *Journal of Rocket Propulsion,* 2014 (5): 61-68.

[3] Re-entry trajectory optimization using Radau pseudospectral method. *Control Theory & Applications*, 2013, 30(8).

[4] On-Line predictor-corrector reentry guidance law based on Gauss pseudospectral method. *Journal of Astronautics,* 2011, (06):1249-1255.

[5] E. Yong, G. Tang, L. Chen. Rapid trajectory optimization for hypersonic reentry vehicle via Gauss pseudospectral method. *Journal of Astronautics*, 2008, 6(29): 1766-1772.

[6] Jorris T, Schulz C, Friedl F, et al. Constrained trajectory optimization using pseudospectral methods. *AIAA Atmospheric Flight Mechanics Conference and Exhibit*. 2008: 6218.

[7] Benson D. A Gauss pseudospectral transcription for optimal control. *Massachusetts Institute of Technology*, 2005.

[8] Q. Chen, Z. Wang, S. Chang, et al. Multiphase trajectory optimization for gun-launched glide guided projectiles. *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering,* 2016, 230(6): 995-1010.

[9] F. Tan, H. Chen, M. Hou, et al. Model predictive trajectory tracking of a reentry hypersonic vehicle based on convex optimization. *Control Conference (CCC)*, 2013 32nd Chinese. IEEE, 2013: 4167-4171.

[10] Z. Lu, F. Tan. Online trajectory optimization for the terminal stage of reentry hypersonic vehicles. *Harbin institute of technology*, 2014.6.

[11] Acikmese B, Ploen S R. Convex programming approach to powered descent guidance for mars landing. *Journal of Guidance, Control, and Dynamics*, 2007, 30(5): 1353-1366.

[12] X. Liu, Z. Shen, P. Lu. Closed-Loop optimization of guidance gain for constrained impact. *Journal of Guidance, Control, and Dynamics*, 2016.

[13] Hanger M, Johansen T A, Mykland G K, et al. Dynamic model predictive control allocation using CVXGEN. *Control and Automation (ICCA)*, 2011 9th *IEEE International Conference on. IEEE*, 2011: 417-422.