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Directed topological complexity

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Abstract
It has been observed that the motion planning problem of robotics reduces mathematically to the problem of finding a section of the path-space fibration, leading to the notion of topological complexity, as introduced by M. Farber. In this approach one imposes no limitations on motion of the system assuming that any continuous motion is admissible. In many applications, however, a physical apparatus may have constrained controls, leading to constraints on its potential dynamics. In the present paper we adapt the notion of topological complexity to the case of directed topological spaces, which encompass such controlled systems, and also systems which appear in concurrency theory. We study properties of this new notion and make calculations for some interesting classes of examples.

Keywords Directed topology · Robot motion planning · Topological complexity · Controlled systems · Concurrent systems · Homotopy theory

Mathematics Subject Classification 55P99 · 68Q10 · 93C99

1 Introduction

A mechanical system may function autonomously provided it is supplied with a motion planning algorithm. Such an algorithm takes an ordered pair of states (A, B) of the system as input and produces a continuous motion of the system starting at state A and

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ending at state B, as output. The notion of topological complexity $\mathsf{TC}(X)$ (introduced in Farber 2003) reflects the structure of motion planning algorithms for systems having X as their configuration space. To define $\mathsf{TC}(X)$ one considers the path space fibration

$$\chi : X^I \to X \times X, \tag{1}$$

where $\chi(p) = (p(0), p(1))$. The symbol X^I denotes the space of all continuous paths $p : I \to X$ equipped with the compact-open topology, where $I = [0, 1]$. A motion planning algorithm is a section of this fibration and it is easy to see that a continuous section exists if and only if the configuration space X is contractible. $\mathsf{TC}(X)$ is defined as the minimal number of continuous "local rules" that are needed to describe a section. The notion of topological complexity is well understood both algorithmically and topologically Farber (2008). One of the key facts is that for a topological space X the integer $\mathsf{TC}(X)$ depends only on the homotopy type of X and in particular $\mathsf{TC}(X) = 1$ is equivalent for X to be contractible. More information on the concept $\mathsf{TC}(X)$ and its relevance to the robot motion planning problem can be found in Farber (2008).

The goal of this paper is to analyse an analogue of $\mathsf{TC}(X)$ in the realm of directed topological spaces. To motivate our interest in "the directed version" of topological complexity we note that the above definition of $\mathsf{TC}(X)$ describes the motion planning problem of robotics when we ignore constraints on the actual controls that can be applied to the physical apparatus, i.e. when we assume that any continuous motion of the system is achievable. In many applications, however, a physical apparatus may have dynamics that can be described by an ordinary differential equation in the state variables $x \in \mathbb{R}^n$ in time t, and parameterised by the control parameters $u \in \mathbb{R}^p$,

$$\dot{x}(t) = f(t, x(t), u(t)). \tag{2}$$

The control parameters $u(t)$ are usually restricted to lie within a set $U \subset \mathbb{R}^p$. One may thus describe the variety of trajectories of the control system (2) by using the language of differential inclusions,

$$\dot{x}(t) \in F(t, x(t)), \tag{3}$$

where $F(t, x(t))$ is the set of all vectors $f(t, x(t), u)$ with $u \in U$. Under some well-known conditions this differential inclusion has solutions, at least locally. Under these conditions, the set of solutions of the differential inclusion (3) naturally forms a *directed space* (or a d-space for short) in the sense of Grandis (2009), see also Sect. 2 below. We observe in this paper that the motion planning problem of robotics in the presence of control constraints reduces to the problem of finding a section to a "directed analogue" of the path space fibration (1). The latter is the map taking a directed path to the pair of its end points; note that this map is typically not a fibration. This material is developed in the following sections. In particular, we introduce the notion of directed homotopy equivalence which helps to describe properties of the directed version of topological complexity.

2 Definitions

The context of a d-space was introduced in Grandis (2009); we will restrict ourselves later (Sect. 6) to a more convenient category of d-spaces, that ought to be thought of as some kind of cofibrant replacement of more general (but sometimes pathological) d-spaces.

Let $I = [0, 1]$ denote the unit segment with the topology inherited from $\mathbb{R}$.

Definition 1 (Grandis 2009) A directed topological space, or a d-space is a pair (X, dX) consisting of a topological space X equipped with a subset $dX \subset X^I$ of continuous paths $p : I \to X$, called directed paths or d-paths, satisfying three axioms:

- every constant map $I \to X$ is directed;
- dX is closed under composition with continuous non-decreasing maps $I \to I$;
- dX is closed under concatenation.

We shall abbreviate the notation (X, dX) to X.

Note that for a d-space X, the space of d-paths $dX \subset X^I$ is a topological space, it is equipped with the compact-open topology.

A map $f : X \to Y$ between d-spaces (X, dX) and (Y, dY) is said to be *a d-map* if it is continuous and for any d-path $p \in dX$ the composition $f \circ p : I \to Y$ belongs to dY. In other words we require that f preserves d-paths. We write $df\ : dX \to dY$ for the induced map between directed paths spaces.

Remark 1 Given a topological space X equipped with a set D of paths $p : I \to X$, closed under concatenation and such that the union of the images $p(I)$, for $p \in D$ is X, we call *saturation* $\overline{D}$ of D the smallest set of paths containing D that forms a d-space structure on X. The saturation $\overline{D}$ is made of all composites of path of D with continuous and non-decreasing maps $I \to I$.

Directed spaces in control theory Consider a differential inclusion

$$\dot{x} \in F(x) \tag{4}$$

where F is a map from $\mathbb{R}^n$ to $\wp(\mathbb{R}^n)$, the set of all subsets of $\mathbb{R}^n$. A function $x : [0, \infty) \to \mathbb{R}^n$ is a *solution* of inclusion (4) if x is absolutely continuous and for almost all $t \in \mathbb{R}$ one has $\dot{x}(t) \in F(x(t))$, see Aubin and Cellina (1984). In general, there can be many solutions to a differential inclusion.

Lemma 1 Aubin and Cellina (1984) *Suppose a set-valued map $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is an upper semicontinuous function of x and such that the set $F(x)$ is closed and convex for all x. Then there exists a solution to Eq.* (4) *defined on an open interval of time.*

Consider a smooth manifold X and an upper semicontinuous set-valued mapping $x \mapsto F(x)$ where for $x \in X$ the image $F(x)$ is a convex cone contained in the tangent space to X at point x, i.e. $F(x) \subset T_x X$. Let dX denote the saturation of the set of all solutions to the differential inclusion $\dot{x} \in F(x)$. Then the pair (X, dX) is a d-space.
Directed spaces in concurrency and distributed systems theory The semantics of concurrent and distributed systems can be given in terms of d-spaces, more specifically in

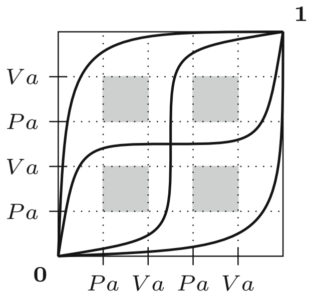


Fig. 1 The semantics of $Pa.Va.Pb.Vb|Pa.Va.Pb.Vb$

terms of geometric realizations Fajstrup et al. (2016) of certain pre-cubical sets. As an example, consider the following concurrent program, made of two processes T_1, T_2, and two binary semaphores a, b, i.e. resources, that can only be locked by one of the two processes at a time, using notation of Fajstrup et al. (2016) : $T_1 = Pa.Va.Pb.Vb$, $T_2 = Pa.Va.Pb.Vb$. This means that process T_1 is locking a (Pa), then relinquishing the lock on a (Va), then locking b (Pb), and finally relinquishing the lock of b (Vb). Process T_2 does the same sequence of actions. The semantics of this concurrent program is depicted in Fig. 1 : it is a partially ordered space X, i.e. a topological space with a global order $\leq$, closed in $X \times X$. Its d-space structure is given by choosing d-paths to be paths $p : I \to X$ such that p is non-decreasing. A few of such d-paths are depicted in Fig. 1.
The d-paths map In what follows, we will be particularly concerned with the following map :

Definition 2 Let (X, dX) be a d-space. Define the *d-paths map*

$$\chi : dX \to X \times X$$

by $\chi(p) = (p(0), p(1))$ where $p \in dX$.

This map is analogous to the classical path-space fibration (1); the essential difference is that in the directed setting χ, as defined above, it is not necessarily a fibration.
The image of χ is a subset of $X \times X$, denoted

$$\Gamma_X = \{(x, y) \in X \times X \mid \exists p \in dX,\ p(0) = x,\ p(1) = y\}.$$

Notations For $a, b \in X$, the symbol $dX(a, b)$ will denote the subspace of dX consisting of all d-paths starting at $a \in X$ and ending at $b \in X$. We denote by $*$ the concatenation map

$$dX(a, b) \times dX(b, c) \to dX(a, c).$$

Note that $dX(a, b)$ is nonempty if and only if $(a, b) \in \Gamma_X$.

Any d-map $f : X \to Y$ induces continuous maps $\Gamma f : \Gamma_X \to \Gamma_Y$ and $df : dX \to dY$ such that the diagram

$$\begin{array}{ccc} dX & \overset{df}{\to} & dY \\ \downarrow \chi_X & & \downarrow \chi_Y \\ \Gamma_X & \overset{\Gamma f}{\to} & \Gamma_Y \end{array}$$

commutes.

3 Directed topological complexity

Let (X, dX) be a d-space such that X is a Euclidean Neighbourhood Retract (ENR).

Recall that a topological space X is said to be an ENR if it can be embedded into a Euclidean space $X \subset \mathbb{R}^N$ such that for an open neighbourhood $X \subset U \subset \mathbb{R}^N$ there exists a retraction $r : U \to X$, $r|X = 1_X$. It is well-known that a subset $X \subset \mathbb{R}^N$ is an ENR iff it is locally compact and locally contractible. Thus, any finite dimensional simplicial complex is an ENR.

Definition 3 The directed topological complexity $\overrightarrow{TC}(X, dX)$ of a d-space (X, dX) is the minimum number n (or ∞ if no such n exists) such that there exists a map $s : \Gamma_X \to dX$ (not necessarily continuous) and Γ_X can be partitioned into n ENRs

$$\Gamma_X = F_1 \cup F_2 \cup \cdots \cup F_n, \quad F_i \cap F_j = \emptyset, \quad i \neq j,$$

such that

- $\chi \circ s = Id$, i.e. s is a (non-necessarily continuous) section of χ;
- $s|_{F_i} : F_i \to dX$ is continuous.

A collection of such ENRs, $F_1, \ldots, F_n$, with n equal the directed topological complexity of X is called *a patchwork*.

Example in control theory As in Farber (2008), a motion planner, for the dynamics described by the differential inclusion (4) is a section of the d-paths map produced by the differential inclusion. A section $s : \Gamma_X \to dX$ associates to any pair of points $(x, y) \in \Gamma_X$ an "admissible" path $s(x, y) = \gamma \in dX$ with $\gamma(0) = x$ and $\gamma(1) = y$. Such a path $\gamma \in dX$ will automatically be realisable as a solution of (2).
Example in concurrency and distributed systems theory Examine again Fig. 1; a section of χ is just a scheduler for the actions of the processes T_1 and T_2.

In the theory of usual (i.e. undirected) topological complexity Farber (2003), Farber (2008), there are several other equivalent definitions of $\mathsf{TC}(X)$, for example the topological complexity $\mathsf{TC}(X)$ is also the minimal cardinality of the covering of $X \times X$ by open (resp. closed) sets admitting continuous sections; moreover, the book Farber (2008) contains four different definitions of $\mathsf{TC}(X)$ leading to the equivalent notions of $\mathsf{TC}(X)$. In the directed case, however, the definitions with open or closed covers lead to notions which can be distinct between themselves as well as distinct from the notion with the ENR partitions given above.

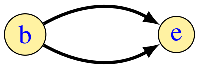
Example 1 Consider the interval $I = [0, 1]$ with the d-structure given by the set of all non-decreasing paths, i.e. $p : [0, 1] \to [0, 1]$ such that $p(t) \leq p(t')$ for any $t \leq t'$. The space Γ_I is $\{(x, y)|x \leq y\}$ and the map $\chi : dI \to \Gamma_I$ admits a continuous section

$$s(x, y)(t) = (1 - t)x + ty$$

where $t \in [0, 1]$. Hence $\overrightarrow{TC}(I) = 1$.

Note that in this example the space Γ_I is contractible and the map χ is a fibration with a contractible fibre.

Example 2 Let us consider the directed circle $\overrightarrow{\mathbb{S}^1}$ shown on the figure below:



It is a directed graph homeomorphic to the circle S^1 which is the union of two directed intervals $I_+ \cup I_-$; the d-paths of $\overrightarrow{\mathbb{S}^1}$ are the d-paths lying in one of the intervals $I_\pm$. We see that $P(\overrightarrow{\mathbb{S}^1}) = P(I_+) \cup P(I_-)$ and $P(I_+) \cap P(I_-)$ is a 2-point set containing the two constant paths $p_b(t) \equiv b$ and $p_e(t) \equiv e$. Similarly, one has $\Gamma_{\overrightarrow{\mathbb{S}^1}} = \Gamma_{I_+} \cup \Gamma_{I_-}$ and the intersection $\Gamma_{I_+} \cap \Gamma_{I_-}$ is a 3 point set $\{(b, b), (b, e), (e, e)\}$. Since each of the sets $\Gamma_{I_\pm}$ is contractible we obtain that $\Gamma_{\overrightarrow{\mathbb{S}^1}}$ is homotopy equivalent to the wedge $S^1 \vee S^1$.

Next we observe that the map $\chi : d\overrightarrow{\mathbb{S}^1} \to \Gamma_{\overrightarrow{\mathbb{S}^1}}$ *admits no continuous section over any neighbourhood U of the point* $(b, e) \in \Gamma_{\overrightarrow{\mathbb{S}^1}}$. To show this, one notes that the preimage $\chi^{-1}(b, e)$ has two connected components, one of which consists of the d-paths lying in I_+ and the other of the d-paths lying in I_-. Any open set $U \subset \Gamma_{\overrightarrow{\mathbb{S}^1}}$ containing (b, e) must contain a pair $(x^+, y^+) \in \Gamma_{I_+}$ and a pair $(x^-, y^-) \in \Gamma_{I_-}$, arbitrarily close to (b, e). Moreover, we may find two sequences $(x_n^\pm, y_n^\pm) \in \Gamma_{I_\pm}$ of points converging to (b, e) and the limits of any section over U along these sequences would land in different connected component of $\chi^{-1}(b, e)$. Hence, we obtain $\overrightarrow{TC}(\overrightarrow{\mathbb{S}^1}) \geq 2$. On the other hand, we may represent $\Gamma_{\overrightarrow{\mathbb{S}^1}}$ as the union

$$\Gamma_{\overrightarrow{\mathbb{S}^1}} = F_1 \cup F_2$$

where $F_1 = \Gamma_{I_+}$ and $F_2 = \Gamma_{I_-} - \{(b, e), (b, b), (e, e)\}$ and using the previous example we see that over each of the sets F_1, F_2 there exists a continuous section of χ. Hence we obtain

$$\overrightarrow{TC}(\overrightarrow{\mathbb{S}^1}) = 2. \tag{5}$$

4 Regular d-spaces

Definition 4 A d-space (X, dX) will be called regular if one can find a partition

$$\Gamma_X = F_1 \cup F_2 \cup \cdots \cup F_n, \quad n = \overrightarrow{TC}(X)$$

into ENRs such that the map χ admits a continuous section over each F_i and, additionally, the sets $\bigcup_{i=1}^r F_i$ are closed for any $r = 1, \ldots, n$.

Note the following property of the sets which appear in Definition 4:

$$\overline{F}_i \cap F_{i'} = \emptyset \quad \text{for} \quad i < i'. \tag{6}$$

In the "undirected" theory of $\mathsf{TC}(X)$ this property is automatically satisfied, see Proposition 4.12 of Farber (2008).

All examples of d-spaces which appear in this paper are regular. At present we know of no examples of d-spaces which are not regular; we plan to address this question in more detail elsewhere.

Example 3 The directed circle $\overrightarrow{\mathbb{S}^1}$ is regular as follows from the construction of Example 2.

The Cartesian product of d-spaces (X, dX) and (Y, dY) has a natural d-space structure. Any path $\gamma : [0, 1] \to X \times Y$ has the form $\gamma(t) = (\gamma_X(t), \gamma_Y(t))$ and we declare γ to be directed if both its coordinates are directed, i.e. $\gamma_X \in dX$ and $\gamma_Y \in dY$. Note that $\Gamma_{X\times Y} = \Gamma_X \times \Gamma_Y$.

Proposition 1 *If the d-spaces (X_i, dX_i) are regular, where $i = 1, 2, \ldots, k$, then*

$$\overrightarrow{TC}(X_1 \times X_2 \times \cdots \times X_k) - 1 \le \sum_{i=1}^{k} \left[\overrightarrow{TC}(X_i) - 1\right]. \tag{7}$$

Proof Denote $\overrightarrow{TC}(X_i) = n_i + 1$ and let

$$\Gamma_{X_i} = F_0^i \cup F_1^i \cup \cdots \cup F_{n_i}^i$$

be a partition as in the Definition 4, i.e. each set F_j^i is an ENR, the map χ admits a continuous section over F_j^i and each union $F_0^i \cup \cdots \cup F_r^i$ is closed, $r = 0, \ldots, n_i$. Denoting $X = \prod_{i=1}^k X_i$ and identifying the space Γ_X with the product $\prod_{i=1}^k \Gamma_{X_i}$, we see that the sets

$$F_{j_1}^1 \times F_{j_2}^2 \times \cdots \times F_{j_k}^k$$

form an ENR partition of Γ_X, where each index j_s runs through $0, 1, \ldots, n_s$. The continuous sections $F_{j_s}^s \to dX_s$, where $s = 1, \ldots, k$, obviously produce continuous sections

$$\sigma_{j_1 j_2 \cdots j_k} : F^1_{j_1} \times F^2_{j_2} \times \cdots \times F^k_{j_k} \to dX.$$

Consider the sets

$$\bigcup_{j_1+\cdots+j_k=j} F^1_{j_1} \times F^2_{j_2} \times \cdots \times F^k_{j_k} = G_j \subset \Gamma_X, \tag{8}$$

with $j = 0, 1, \ldots, N$, where $N = n_1 + n_2 + \cdots + n_k$. We observe that the terms of the union (8) are pairwise disjoint and open in G_j (due to (6)) and hence the collection of continuous maps $\sigma_{j_1 j_2 \ldots j_k}$ defines a continuous section $G_j \to dX$. This proves that $\overrightarrow{TC}(X) \leq N + 1$ as claimed. □

Corollary 1 *The directed torus* $(\overrightarrow{\mathbb{S}^1})^n$ *satisfies* $\overrightarrow{TC}((\overrightarrow{\mathbb{S}^1})^n) \leq n + 1$.

Proof This follows from Proposition 1 and Example 2. □

Definition 5 We say that a d-space X is strongly connected if $\Gamma_X = X \times X$.

In other words, in a strongly connected d-space X for any pair (x, y) in $X \times X$ there exists a directed path $\gamma \in dX$ with $\gamma(0) = x$, $\gamma(1) = y$.

Proposition 2 *For any strongly connected d-space X one has* $\mathsf{TC}(X) \leq \overrightarrow{TC}(X)$.

Proof Let X be strongly connected and let $\Gamma_X = X \times X = F_1 \cup F_2 \cup \cdots \cup F_n$ be a partition into the ENRs as in Definition 3 with $n = \overrightarrow{TC}(X)$. Then the same partition can serve for the path space fibration $X^I \to X \times X$ which implies our result. □

Example 4 Consider *the directed loop* $\mathbb{O}^1$ which can be defined as the unit circle

$$S^1 = \{z \in \mathbb{C}; |z| = 1\} \subset \mathbb{C}$$

with the d-structure described below. Any continuous path $\gamma : [0, 1] \to S^1$ can be presented in the form $\gamma(t) = \exp(i\phi(t))$ where the function $\phi : [0, 1] \to \mathbb{R}$ is defined uniquely up to adding an integer multiple of $\pm 2\pi$. We declare a path γ to be positive if the function $\phi(t)$ is nondecreasing. It is obvious that the d-space thus obtained is strongly connected. Hence, using Proposition 2, we obtain $\overrightarrow{TC}(\mathbb{O}^1) \geq \mathsf{TC}(S^1) = 2$, . On the other hand, we can partition $\mathbb{O}^1 \times \mathbb{O}^1 = F_1 \cup F_2$ where $F_1 = \{(z_1, z_2) \in \mathbb{O}^1 \times \mathbb{O}^1; z_1 = z_2\}$ and $F_2 = \{(z_1, z_2) \in \mathbb{O}^1 \times \mathbb{O}^1; z_1 \neq z_2\}$. It is clear that we obtain a section of χ over F_1 by assigning the constant path at z for any pair $(z, z) \in F_1$. A continuous section of χ over F_2 can be defined as follows by moving z_1 along the circle in the positive direction towards z_2 with constant velocity. We conclude that

$$\overrightarrow{TC}(\mathbb{O}^1) = 2. \tag{9}$$

In addition, we see that the directed loop $\mathbb{O}^1$ is regular.

Corollary 2 *One has,*

$$\overrightarrow{TC}((\mathbb{O}^1)^n) = n + 1,$$

i.e. the directed topological complexity of the directed n-dimensional torus $(\mathbb{O}^1)^n$ *equals* $n + 1$.

Proof First we apply (9) and Proposition 1 to obtain the inequality $\overrightarrow{TC}((\mathbb{O}^1)^n) \leq n+1$. Next we observe that $(\mathbb{O}^1)^n$ is strongly connected and, by Proposition 2, $\overrightarrow{TC}((\mathbb{O}^1)^n) \geq \mathsf{TC}((S^1)^n) = n + 1$. □

5 Directed graphs

Let G be a directed connected graph, i.e. each edge of G has a specified orientation. One naturally defines a d-structure on G as follows. Each edge of G can be identified either with the directed interval I (see Example 1) or with the loop $\mathbb{O}^1$ (see Example 4); these are "small directed paths", i.e. the paths lying on a single edge. In general, the directed paths of G are concatenations of small directed paths.

For a directed graph G the set Γ_G has the following property: if a pair (x, y) belongs to Γ_G where x is an internal point of an edge e and $y \notin e$ then all pairs (x', y) also belong to Γ_G where $x' \in \mathrm{Int}(e)$.

Proposition 3 $\overrightarrow{TC}(G) \leq 3$.

Proof Consider the following partition $\Gamma_G = F_1 \cup F_2 \cup F_3$ where

- F_1 is the set of pairs of vertices (α_i, α_j) of G which are in Γ_G;
- F_2 is the set of pairs $(x, y) \in \Gamma_G$ where either x or y is a vertex and the other point lies in the interior of an arc;
- F_3 is the set of pairs $(x, y) \in \Gamma_G$ where x and y lie in the interiors of arcs.

For each pair of vertices $(\alpha_i, \alpha_j) \in \Gamma_G$ fix a directed path γ_{ij} from α_i to α_j. This defines a section of χ over F_1. Note that all pairs (α_i, α_i) belong to Γ_G and the path γ_{ii} can be chosen to be constant.

Consider now an oriented edge e and a vertex α_j such that $(x, \alpha_j) \in \Gamma_G$ for an internal point $x \in \mathrm{Int}(e)$. Let α_i be the end point of e and let γ_{x,α_i} denote the constant velocity path along e from x to α_i. A continuous section of χ over $\mathrm{Int}(e) \times \alpha_j$ can be defined as $(x, \alpha_j) \mapsto \gamma_{x,\alpha_i} \star \gamma_{ij}$ where $*$ stands for concatenation. A continuous section over $\alpha_j \times \mathrm{Int}(e)$ can be defined similarly, and hence we have a continuous section of χ over F_2.

Finally we describe a continuous section of χ over F_3. Consider two oriented edges e and e' where we shall first assume that $e \neq e'$. Let α denote the end point of e and β denote the initial point of e'. We define a section of χ by

$$(x, y) \mapsto \gamma_{x,\alpha} * \gamma_{\alpha\beta} * \gamma_{\beta y}$$

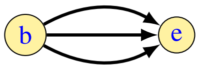
for $x \in \mathrm{Int}(e)$ and $y \in \mathrm{Int}(e')$. Here $\gamma_{x\alpha}$ denotes a constant velocity directed path along e connecting x to α; the path $\gamma_{\beta y}$ is defined similarly and $\gamma_{\alpha\beta}$ is a positive path from α to β.

Finally we consider the case when $e = e'$. For a pair $(x, y) \in \Gamma_G$ with $x, y \in \mathrm{Int}(e)$ we define the section by $(x, y) \mapsto \gamma_{xy}$ where γ_{xy} is a constant velocity path along e from x to y.

All the partial sections described above over various parts of F_3 obviously combine into a continuous section over F_3. □

The following example shows that the directed topological complexity can be smaller than the usual complexity.

Example 5 Consider the following graph :



A patchwork for Γ_G : $F_1 = \{(b, e)\}$ and $F_2 = \Gamma_G \backslash F_1$. We thus have $\overrightarrow{TC}(G) = 2$ (here again, it is easy to see that there is no global section). But $\mathsf{TC}(G) = 3$.

However in the special case of strongly connected graphs, the directed and classical topological complexity coincide:

Proposition 4 *Let G be a strongly connected directed graph. Then*

$$\overrightarrow{TC}(G) = \mathsf{TC}(G) = \min(b_1(G), 2) + 1.$$

Here $b_1(G)$ denotes the first Betti number of the graph G.

Proof By Farber (2008), we know that $\mathsf{TC}(G) = \min(b_1(G), 2) + 1$. As G is strongly connected, we have $\overrightarrow{TC}(G) \geq \mathsf{TC}(G) = \min(b_1(G), 2) + 1$, see Proposition 2. To prove that we have in fact an equality consider the following cases:

- $b_1(G) = 0$. Since G is contractible and strongly connected, G must be a single point. Then $\overrightarrow{TC}(G) = 1$ and the result follows.
- $b_1(G) = 1$. It is easy to see that in this case G must be a cycle, i.e. for some n the graph G must have n vertices $v_1, v_2, \ldots, v_n$ and n oriented edges $e_1, e_2, \ldots, e_n$ where e_i connects v_i with v_{i+1} for $i = 1, \ldots, n-1$ and e_n connects v_n and v_1. We see that $\overrightarrow{TC}(G) = 2$ similarly to Example 4.
- $b_1(G) \geq 2$. Then $\mathsf{TC}(G) = 3$ (see above) and hence $\overrightarrow{TC}(G) \geq 3$. On the other hand, $\overrightarrow{TC}(G) \leq 3$ by Proposition 3. Thus $\overrightarrow{TC}(G) = 3$.

□

6 Higher-dimensional directed spaces

We begin by recalling the definition of "geometric" precubical sets Fajstrup (2005). The interest Fajstrup et al. (2016) of such precubical sets is that the precubical semantics of most programs is a geometric precubical set. Also they are sufficiently tractable for us to compute, in some cases, their directed topological complexity, or more precisely, the directed topological complexity of their directed geometric realization, that we call, cubical complexes (see Definition 7).

Definition 6 A precubical set C is geometric when it satisfies the following conditions:

1. *no self-intersection*: two distinct iterated faces of a cube in C are distinct
2. *maximal common faces*: two cubes admitting a common face admit a maximal common face.

Definition 7 A cubical complex K is a topological space of the form

$$K = \left(\bigsqcup_{\lambda \in \Lambda} I^{n_\lambda} \right) / \approx$$

where Λ is a set, $(n_\lambda)_{\lambda\in\Lambda}$ is a family of integers, and $\approx$ is an equivalence relation, such that, writing $p_\lambda : I^{n_\lambda} \to K$ for the restriction of the quotient map $\bigsqcup_{\lambda\in\Lambda} I^{n_\lambda} \to K$, we have

1. for every $\lambda \in \Lambda$, the map p_λ is injective,
2. given $\lambda, \mu \in \Lambda$, if $p_\lambda(I^{n_\lambda}) \cap p_\mu(I^{n_\mu}) \neq \emptyset$ then there is an isometry $h_{\lambda,\mu}$ from a face J_λ of I^{n_λ} to a face J_μ of I^{n_μ} such that $p_\lambda(x) = p_\mu(y)$ if and only if $y = h_{\lambda,\mu}(x)$.

As shown in Goubault and Mimram (2019) :

Proposition 5 *The realization of a geometric precubical set is a cubical complex.*

Generalising Proposition 3, and as in the case of complexes in classical topology Farber (2008), one may show that

$$\overrightarrow{TC}(X) \leq 2\dim(X) + 1$$

for nice cubical complexes X. We shall address this claim elsewhere.
The directed spheres Let $\square^n$ be the cartesian product of n copies of the unit segment with the d-structure generated by the standard ordering on [0, 1]. Its d-space structure is generated by a partially-ordered space Fajstrup et al. (2016).

Definition 8 The directed sphere $\overrightarrow{\mathbb{S}^n}$ of dimension n is defined as the boundary $\partial\square^{n+1}$ of the hypercube $\square^{n+1}$. Its d-structure is inherited from the one of $\square^{n+1}$.

Proposition 6 $\overrightarrow{TC}(\overrightarrow{\mathbb{S}^n}) = 2$ *for any* $n \geq 1$.

The case $n = 1$ is covered by Example 2; see Borat and Grant (2019) for the general case.

We finish this section by the following two examples.

Example 6 Consider the 2-disc $X = D^2 = \{z \in \mathbb{C}; |z| \leq 1\}$ with the following directed structure: any directed path $\gamma : [0,1] \to D^2$ starting at an internal point of the disc, i.e. with $|\gamma(0)| < 1$, is constant. The directed paths $\gamma : [0,1] \to D^2$ with $|\gamma(0)| = 1$ are of two types: either $\gamma(t) = e^{i\alpha(t)}$ or $\gamma(t) = e^{-i\alpha(t)}$ where $\alpha : [0,1] \to [0,\pi]$ is a nondecreasing continuous function. In this example the interior of the disc is in some sense disconnected from the boundary circle, more precisely one cannot reach the interior from the boundary moving along directed paths. The boundary circle as a d-space is isomorphic to the directed circle $\overrightarrow{\mathbb{S}^1}$, as described in Example 2. Using the result of that Example, we obtain that $\overrightarrow{TC}(D^2) = 2$. Since the disc D^2 is contractible we know that $\mathsf{TC}(D^2) = 1$.

Example 7 In this example we show the existence of a directed space X satisfying $\overrightarrow{TC}(X) = 1$ and $\mathsf{TC}(X) = 2$. Let X be the circle $S^1 = \{z; |z| = 1\}$ with the d-structure consisting of continuous paths $\gamma : [0,1] \to S^1$ satisfying the following properties: (1) if $\gamma(0) = 1$ then γ is constant; (2) If $\gamma(0) \neq 1$ then the quantity $|\gamma(t) + 1|$ is non-increasing. In other words, we require that the distance from $\gamma(t)$ to the point $-1 \in S^1 \subset \mathbb{C}$ is non-increasing. It is easy to see that $\overrightarrow{TC}(X) = 1$ in this example, while obviously $\mathsf{TC}(X) = 2$.

7 Directed homotopy equivalence and topological complexity

As of now, there is no uniquely well-established notion of directed homotopy equivalence between directed spaces, although there has been numeral proposals, among which one linked to our present problem Goubault (2017).

We take the view here that directed homotopy equivalences should at least induce equivalent trace categories, viewed with enough structure. We will show in the following sections that directed topological complexity is an invariant of basic equivalences that should be implied by any "reasonable" directed equivalences.

7.1 A basic dihomotopy equivalence, and dicontractibility

In Goubault (2017), the author introduced a notion of dihomotopy equivalence. The most important ingredients are that two equivalent d-spaces should be homotopy equivalent in some naive way, and their trace spaces should be homotopy equivalent as well.[1] First, we need to define what we call continuous gradings. The idea is these graded d-maps are such that they map path spaces between pair of points that vary continuously with respect to the (target) pair of points :

Definition 9 Let $q : U \to V$ be a d-map between two d-spaces U and V, $v, v' \in V$, and let $W \subseteq U \times U$ denote $W = (q \times q)^{-1}(v, v')$. Suppose we have continuous maps

$$h^{v,v'} : dV(v, v') \times W \to dU$$

[1] In Goubault (2017), an extra "bisimulation relation" was added to the definition. This is not necessary for the main aims of this section, Proposition 7 and Theorem 1.

such that for all $(u, u') \in W$, $h^{v,v'}(p, u, u') \in dU(u, u')$.

In this case, we say that $h = (h^{v,v'})$ is continuously graded, and by abuse of notation, we write this graded map as a $h : dV \multimap dU$ given by gradings $h_{u,u'} : dV(q(u), q(u')) \to dU(u, u')$, varying continuously for $(u, u') \in W$ in $dU^{dV(v,v')}$, with the compact-open topology.

Any reasonable dihomotopy equivalence should be in particular a d-map inducing a (classical) homotopy equivalence since we want that being dihomotopy equivalent implies being homotopy equivalent. It should also induce (classical) homotopy equivalences on the corresponding path spaces. We call this minimum requirement, a basic dihomotopy equivalence :

Definition 10 Let X and Y be two d-spaces. A basic dihomotopy equivalence between X and Y is a d-map $f : X \to Y$ such that :

- f is a d-homotopy equivalence between X and Y, i.e. a d-map which is a dihomotopy equivalence in the sense of Grandis (2009) with homotopy inverse a d-map $g : Y \to X$.
- There exists a map $F : dY \multimap dX$, continuously graded by $F_{x,x'} : dY(f(x), f(x')) \to dX(x, x')$ for $(x, x') \in \Gamma_X$, such that $(df_{x,x'}, F_{x,x'})$ is a homotopy equivalence between $dX(x, x')$ and $dY(f(x), f(x'))$
- There exists a map $G : dX \multimap dY$, continuously graded by $G_{y,y'} : dX(g(y), g(y')) \to dY(y, y')$ for $(y, y') \in \Gamma_Y$ such that $(dg_{y,y'}, G_{y,y'})$ is a homotopy equivalence between $dY(y, y')$ and $dX(g(y), g(y'))$.

We sometimes write (f, g, F, G) for the full data associated to the basic dihomotopy equivalence $f : X \to Y$.

Remark This definition clearly bears a lot of similarities with Dwyer-Kan weak equivalences in simplicial categories (see e.g. Bergner 2004). The main ingredient of Dwyer-Kan weak equivalences being exactly that df induces a homotopy equivalence. But our definition adds continuity and directedness requirements which are instrumental to our theorems and to the classification of the underlying directed geometry.

Remark As noted by one of the referees, another way to view this is akin to a fiber homotopy equivalence : the conditions for $dY(f(x), f(x')) \to dX(x, x')$ and $dX(g(y), g(y')) \to dY(y, y')$ to be homotopy equivalences mean that df and dg are fibre homotopy equivalences, for fibers given by χ_X and χ_Y ; but the usual theory does not apply because χ_X and χ_Y are not fibrations in general. Still, a notion of basic dihomotopy equivalence closer to fibre homotopy equivalence, where we ask for f to be a homotopy equivalence between Γ_X and Γ_Y, will be investigated elsewhere.

Example 8 – Let X, Y be two directed spaces. Suppose X and Y are isomorphic as d-spaces i.e. that there exists $f : X \to Y$ a d-map, which has an inverse, also a d-map. Then X and Y are basic dihomotopy equivalent. The proof goes as follows. Take $f = u$, $g = u^{-1}$, $dg = F$ the pointwise application of u^{-1} on paths in Y and $df = G$ the pointwise application of u on paths in X. This data obviously forms a directed homotopy equivalence.

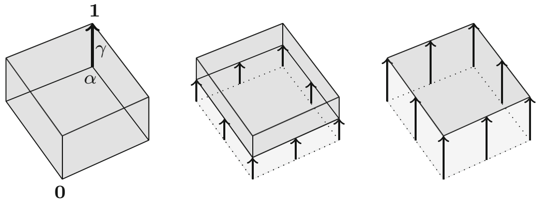


Fig. 2 Naive equivalence between the Fahrenberg's matchbox M and its upper face T

- The directed unit segment $\overrightarrow{I}$ is basic dihomotopy equivalent to a point. Consider the unique map $f : \overrightarrow{I} \to \{*\}$, and $g : \{*\} \to \overrightarrow{I}$ (the inclusion of the point as 0 in $\overrightarrow{I}$). Define $F : d\{*\} \to d\overrightarrow{I}$ by $F(*)$ being the constant map on 0 and $G : d\overrightarrow{I} \to d\{*\}$ to be the unique possible map (since $d\{*\}$ is a singleton $\{*\}$).
- Consider Dubut et al. (2015) the directed space ("Fahrenberg's matchbox") depicted on the left of Fig. 1, composed of the 5 upper faces of a directed cube $[0, 1]^3$, with the d-space structure induced by the componentwise partial order. Figure 2 also depicts a dihomotopy equivalence (in the sense of Grandis (2009)), between M and to its upper face (so to a point). More precisely, the d-map f, which maps any point of M to the point of T just above of it, is a dihomotopy equivalence, whose inverse modulo dihomotopy is the embedding g of T into M. Hence, $f \circ g = id_T$ and a dihomotopy from id_M to $g \circ f$ is depicted in Fig. 2. But this homotopy equivalence, (f, g), does not induce a basic dihomotopy equivalence in our sense. As a matter of fact, consider points 0 and α : $dX(0, \alpha)$ is homotopy equivalent to two points whereas $dX(f(0), f(\alpha))$ is homotopy equivalent to a point.

As expected, directed topological complexity is an invariant of basic dihomotopy equivalence :

Proposition 7 *Let X and Y be two simply dihomotopy equivalent d-spaces. Then $\overrightarrow{TC}(X) = \overrightarrow{TC}(Y)$.*

Proof As X and Y are basic dihomotopy equivalent, we have $f : X \to Y$ and $g : Y \to X$ d-maps, which form a d-homotopy equivalence between X and Y. We also get G a continuously graded map from dX to dY, which can be restricted to $G_{y,y'} : dX(g(y), g(y')) \to dY(y, y')$, inverse modulo homotopy to $dg_{y,y'}$; and F a continuously graded map from dY to dX such that its restriction to $dX(x, x')$, for $(x, x') \in \Gamma_X$, $F_{x,x'} : dX(x, x') \to dY(f(x), f(x'))$ is inverse modulo homotopy to $df_{x,x'}$.

Suppose first $k = \overrightarrow{TC}(X)$. Thus we can write $\Gamma_X = F_1^X \cup \ldots \cup F_k^X$ such that we have a map $s : \Gamma_X \to dX$ with $\chi \circ s = Id$ and $s_{|F_i^X}$ is continuous.

Define $F_i^Y = \{u \in \Gamma_Y \mid g(u) \in F_i^X\}$ (which is either empty or an ENR as F_i^X is ENR and g is continuous) and define $t_{|F_i^Y}(u) = G_u \circ s_{|F_i^X} \circ g(u) \in dY(u)$ for all $u \in F_i^Y \subseteq \Gamma_Y$. This is a continuous map in u since $s_{|F_i^X}$ is continuous, g is continuous, and G is continuous and graded. Therefore $\overrightarrow{TC}(Y) \leq \overrightarrow{TC}(X)$. Exchanging the roles of X and Y, we find using the same reasoning that $\overrightarrow{TC}(X) \leq \overrightarrow{TC}(Y)$. Hence we conclude that $\overrightarrow{TC}(X) = \overrightarrow{TC}(Y)$ and directed topological complexity is an invariant of basic dihomotopy equivalence. $\square$

A very simple application is that some spaces must have directed topological complexity of 1 :

Definition 11 A d-space X is dicontractible if it is basic dihomotopy equivalent to a point.

By applying Proposition 7, as the directed topological complexity of a point is 1, all dicontractible spaces have complexity 1, as in the undirected case. Similarly to the undirected case again, the converse is also true.

Theorem 1 *Suppose X is a contractible d-space. Then, the dipath space map has a continuous section if and only if X is dicontractible.*

Proof As X is contractible, we have $f : X \to \{a_0\}$ (the constant map) and $g : \{a_0\} \to X$ (the inclusion) which form a (classical) homotopy equivalence. Trivially, f and g are d-maps, and form a d-homotopy equivalence.

Suppose that we have a continuous section s of χ. There is an obvious inclusion map $i : \{s(a, b)\} \to dX(a, b)$, which is graded in a and b. Define R to be this map. Now the constant map $r : dX(a, b) \to \{s(a, b)\}$ is a retraction map for i.

We define

$$\begin{aligned} H : dX \times [0,1] \to & \quad dX \\ (u,t) \quad \to v \text{ s.t. } & \begin{cases} v(x) = u(x) & \text{if } 0 \leq x \leq t \\ v(x) = s\left(u(t), b\right)\left(\frac{x-t}{1-t}\right) & \text{if } t \geq x \leq 1 \end{cases} \end{aligned}$$

($H(u, t)(x)$ is extended by continuity at $t = 1$, when $x = 1$, as being equal to $s(b, b)(1) = b$).[2]

As concatenation and evaluation are continuous and as s is continuous in both arguments H is continuous in $u \in dX$ and in t. H induces families $H_{a,b} : dX(a, b) \times [0, 1] \to dX(a, b)$, and because H is continuous in u in the compact-open topology, this family $H_{a,b}$ is continuous in a and b in X.

Finally, we note that $H(u, 1) = u$ and $H(u, 0) = s(u(0), b) = i \circ r(u)$. Hence r is a deformation retraction and $dX(a, b)$ is homotopy equivalent to $\{s(a, b)\}$ and has the homotopy type we expect (is contractible for all a and b), meaning that R is a (graded) homotopy equivalence.

Conversely, suppose X is dicontractible. We have in particular a continuous map $R : \{*\} \to dX$, which is graded in $(a, b) \in \Gamma_X$. Define $s(a, b) = R_{a,b}(*)$, this is a continuous section of χ. $\square$

Remark Sometimes, we do not know right away, in the theorem above, that X is contractible. But instead, there is an initial state in X, i.e. a state a_0 from which every

[2] Note that we do not need to ask for $s(b, b)$ to be the constant path on b. The section on the diagonal of Γ_X is in fact necessarily made of paths homotopic to b. We thank an anonymous referee and Jeremy Dubut for pointing out this fact to us.

point of X is reachable. Suppose then that, as in the Theorem above, χ has a continuous section $s : \Gamma_X \to dX$. Consider $s'(a, b) = s^{-1}(a_0, a) * s(a_0, b)$ the concatenation of the inverse dipath, going from a to a_0, with the dipath going from a_0 to b : this is a continuous path from a to b for all a, b in X. Now, s' is obviously continuous since concatenation, and s, are. By a classical theorem Farber (2008), this implies that X is contractible and the rest of the theorem holds.

Example 9 Direct applications of Proposition 7 show that :

- Directed n-tori $\mathbb{O}^{1^n}$ and $\mathbb{O}^{1^m}$ cannot be basic dihomotopy equivalent when $n \neq m$.
- Directed n-tori $\mathbb{O}^{1^n}$ cannot be basic dihomotopy equivalent to any directed graph for $n \geq 3$.

We end this section by sketching a first connection between directed topological complexity and some invariants (see e.g. Dubut et al. 2015) that have been introducted in directed topology, like natural homology, Dubut et al. (2016). We refer the reader to Dubut et al. (2016) for the precise definitions of natural homology and of bisimulation (in that context) :

Proposition 8 *Let X be a d-space. X has directed topological complexity of one (i.e. is dicontractible) implies that its natural homologies $\overrightarrow{H}_n(X)$ are all bisimulation equivalent to either, $1_{\mathbb{Z}} : \mathbf{1} \to \mathbb{Z}$ for $n = 1$, or to $1_0 : \mathbf{1} \to 0$ for $n > 1$, defined as :*

- *$\mathbf{1}$ is the terminal category, with one object 1 and one morphism (the identity on 1)*
- *$1_{\mathbb{Z}}(1) = \mathbb{Z}$, $1_0(1) = 0$.*

Proof Suppose that X has directed topological complexity of 1. Then by Theorem 1, all dipath spaces $dX(xy)$ are contractible, for all $(x, y) \in \Gamma_X$, hence $\overrightarrow{H}_1(X)(x, y) = \mathbb{Z}$ and $\overrightarrow{H}_n(X)(x, y) = 0$ for $n > 1$. Therefore the natural homology functors are all constant, either with value $\mathbb{Z}$ or with value 0, and it is a simple exercise to see that the relation between the induced trace category (see Dubut et al. 2015) and $\mathbf{1}$ which relates all its objects to the only object 1 of $\mathbf{1}$ is hereditary, hence is a bisimulation equivalence. □

Example 10 We get back to example $\overrightarrow{\mathbb{S}^1}$.

Its first homology functor was calculated in Dubut et al. (2015), and is not a constant functor (it contains $\mathbb{Z}^2$ and $\mathbb{Z}$ in its image). Therefore $\overrightarrow{\mathbb{S}^1}$ cannot have directed topological complexity of 1. It is also easy to see that the first natural homology functor of $\mathbb{O}^1$ is $\mathbb{Z}^{\mathbb{N}}$ between two equal points and hence cannot have directed topological complexity of 1.

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