Topological Methods in Path Planning

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I Introduction

One of the crucial tasks for every autonomous robot is finding a collision free path through free space. Due to it's importance this problem has been studied for decades and is know as the path planning or the "piano movers" problem [32, 33]. Abstract reasoning is a key part of solving complex problems in an intelligent way [41]. Since topology studies the properties of spaces, incorporating it into solving the path planning problem leads to algorithms "making sense" of it's solutions.

The review is structured in the following way: Chapter II introduces basic preliminaries. Chapter III gives an overview of the most influential path planning methods. Chapter IV is the main part where we review different methods from topology used in the path planning problem. Finally Chapter V discusses possible future research directions.

II Problem Formulation

Preliminaries

This section introduces some fundamental concepts of topology and path planning [26, 24, 16].

Homeomorphism If there exists a bijective function $f : X \mapsto Y$ such that f and it's inverse f^{-1} are continuous functions, then the topological spaces X and Y are homeomorphic. Homeomorphism implies that both X and Y share the same topological properties.

Configuration Space Given a metric space $\mathcal{W}$ with metric d and a rigid body $\mathcal{A}$, a rigid body transformation is a function $f : \mathcal{A} \mapsto \mathcal{W}$ such that $d(a_1, a_2) = d(f(a_1), f(a_2))$ and no reflection occurs. This holds for rotations and translations. Given $GL(n)$ by the set of all invertible $n \times n$ matrices, $O(n)$ is a subgroup of $GL(n)$ such that $QQ^T = Q^T Q = I$ for all $Q \in O(n)$. The subgroup $SO(n)$ of $O(n)$ which contains all rotations matrices meaning $det(P) = 1$ for all $P \in SE(n)$. Combining arbitrary rotations and translations gives the special euclidean group $SE(n)$ which is homeomorphic to $\mathbb{R}^n \times SO(n)$.

The configuration space for a rigid robot is therefore $\mathcal{C} \cong \mathbb{R}^2 \times \mathbb{S}^1$ and $\mathcal{C} \cong \mathbb{R}^3 \times \mathbb{RP}^3$ in 2D space and 3D space, respectively.

Path Given two points $x_0, x_1 \in X$, a path is a continous function $f : [0, 1] \mapsto X$ such that $f(0) = x_0$ and $f(1) = x_1$. Paths for which $f(0) = f(1) = x_0$ are loops with basepoint x_0.

Homotopy Two paths f and g with the same endpoints are homotopic if one can be continuously deformed into the other. Formally, there exists a familiy $f_t : [0, 1] \mapsto X, 0 \leq t \leq 1$ such that $f_t(0) = x_0$, $f_t(1) = x_1$ and $f_0(s) = f(s)$, $f_1(s) = g(s)$ and the function $F(s, t) : [0, 1] \times$

$[0,1] \mapsto X$ defined by $F(s,t) = f_t(s)$ is continuous (Figure 1).

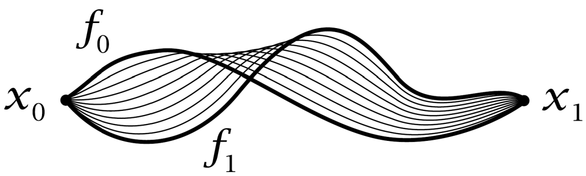

Figure 1: Homotopy [16].

Fiber Bundle A fiber bundle is a structure on a space E with fiber F and a projection $p : E \mapsto B$ such that a homeomorphism $h : p^{-1}(U) \mapsto U \times F$ exists for a neighborhood of every point of B. The map h therefore maps each fiber $F_b = p^{-1}(b)$ homeomorphically onto the copy $\{b\} \times F$ of F. The space E is called total space and B is called the base space of the bundle.

The Path Planning Problem

Given the set of all possible transformations of a rigid robot $\mathcal{C}$ and the transformations that lead to a collision with obstacles $\mathcal{C}_{obs}$, the free space is $\mathcal{C}_{free} = \mathcal{C} \setminus \mathcal{C}_{obs}$ (Figure 2). Let q_I be the initial configuration and q_G the goal configuration. The solution to the path planning problem is a continuous path $\pi : [0,1] \mapsto \mathcal{C}_{free}$ where $\pi(0) = q_I$ and $\pi(1) = q_G$.

Related Work

Path Planning

Due to the computational complexity of the path planning problem, there are different approaches to solve this problem in polynomial time.

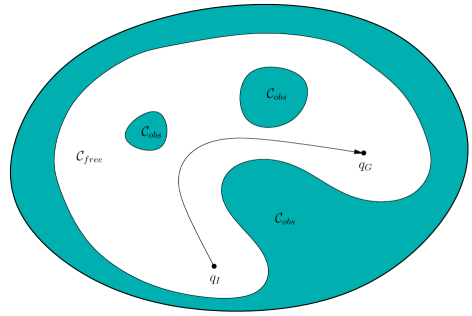

Figure 2: The path planning problem [24].

Graph search. These approaches assume that $\mathcal{C}$ is in the form of a graph. They then apply graph search techniques to find the path π. A widely used algorithm to compute shortest paths in a graph is Dijkstra's algorithm [9]. It is a greedy algorithm that computes the shortest path from every node to every other node in the graph. The next node for expansion is selected based on the lowest cost-to-come $\hat{g}(x)$. It is *complete*, meaning it finds a solution if one exists and reports failure otherwise. It is also *optimal* in the sense that the computed path is the shortest possible path.

A* [15] improves Dijkstra's algorithm by reducing the number of expanded nodes in the graph. This is achieved by using a different function $\hat{f}(x) = \hat{g}(x) + \hat{h}(x)$ to select the next node for expansion. The term $\hat{h}(x)$ is a heuristic for the least possible cost-to-go in a metric space. Many path planning algorithms in the current literature build upon A* (Figure 4).

Stentz et al. developed D*, an incremental algorithm based on A* that is able to find shortest paths on maps with changing costs, i.e. due to a robot moving trough the environment and updating the map. D* computes a path backwards from the goal to the robot. Koenig et

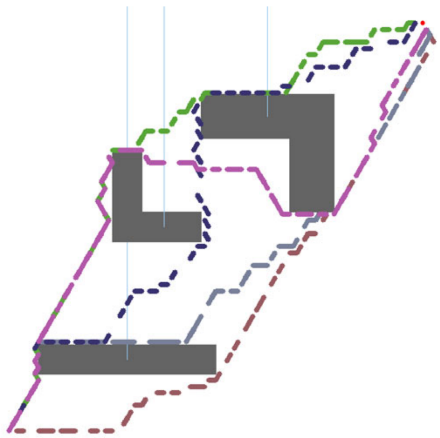

Figure 3: Paths belonging to different homotopy classes [3].

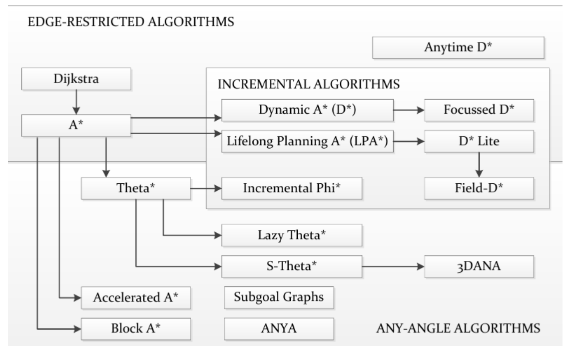

Figure 4: Evolution of graph search algorithms for the path planning problem [31].

al. [20] also proposed an incremental version of A* (LPA*), where successive runs only recalculate locally inconsistent nodes. They combined it with the backward search of D* and created D*-Lite [21], a less complex version of D* with at least the same performance therefore making D* obsolete.

Daniel et al. [8] showed an approach specifically for grid-maps that is not limited to predefined angles for the transition to other grid cells. It is based on A* and uses line of sight to determine successor nodes. Another any-angle approach proposed by Ferguson et al. [12] uses linear interpolation to find the least cost path trough a cell and produces globally smooth paths.

A comprehensive review of graph search algorithms for the path planning problem can be found in the literature [1, 31, 38, 27].

Sampling. Due to the curse of dimensionality [2], graph search in high dimensional configuration spaces can quickly become computationally intractable. Sampling-based approaches discretize the high-dimensional configuration space by generating random samples in $\mathcal{C}_{free}$. One highly influential contribution from Kavraki et. al. [19] is called Probabilistic Roadmap Method (PRM). It is a solution to the multi-query path planning problem in high dimensional configuration spaces. The method consists of a learning and a query phase. The former repeatedly generates a random configuration in $\mathcal{C}_{free}$ and connects this sample to neighboring nodes in a given radius with a fast local planner. In addition, a heuristic is used to generate extra nodes in "difficult" regions of $\mathcal{C}_{free}$. The method is therefore influenced by the topological properties of $\mathcal{C}_{free}$. The result is a graph representation of $\mathcal{C}_{free}$ in the form of a forest of trees.

Another highly cited approach using only a single tree is the Rapidly-exploring Random Tree (RRT) algorithm by Lavalle [23]. The algorithm grows a tree in $\mathcal{C}_{free}$ by generating a sample from a uniform distribution and growing the nearest node of the tree in the direction of the sample. Since the probability that a node is selected for expansion is proportional to the size of it's Voronoi cell (see Figure 5), the tree grows with a bias towards unexplored regions in $\mathcal{C}_{free}$.

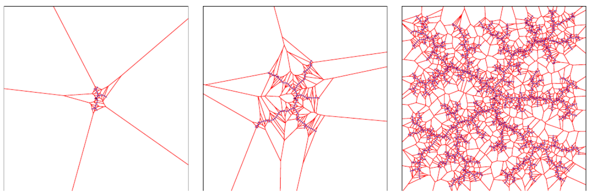

Figure 5: Voronoi diagram of a RRT tree.

Karaman et. al. [17] showed that both PRM and RRT algorithms are not asymptotically optimal. That means that with a growing number of nodes the probability that the algorithm finds an optimal solution converges to zero. In their paper they also proposed two new algorithms PRM* and RRT* which are shown to be asymptotically optimal. The former was improved by calculating the radius, in which a sample is connected to it's neighbors, based on the number of present samples. The latter was improved by selecting the node for expansion based on the cost-to-come in the neighborhood of a sample and rewiring the tree after insertion of a new node.

There are many algorithms improving and extending RRT and RRT* in the literature. With RRT-Connect, Kuffner et. al. [22] introduced a bidirectional RRT that grows one tree from q_I and another tree from q_G while trying to connect the two trees. Anytime RRT* [18] returns a fast initial solution and a robot commits to execute a subset of the full path $\pi_{com} : [0, t_{com}] \mapsto C_{free}$. While executing π_{com} the remaining path is further improved until the robot commits to the next subset of the path. Realizing that there is a subset of C_{free} that contains configurations that are guaranteed to improve the current solution, Informed RRT* [13] uses a ellipsoidal heuristic to sample from this set and improve the convergence rate of RRT*. BIT* [14] does not sample a single configuration, instead it samples a batch of configurations and grows a tree from this batch. After finding a solution or no further possible expansion, the next batch is sampled. If a solution has been found with the previous batch, this batch is sampled from the ellipsoidal subset introduced by Informed RRT* and the tree is updated by identifying locally inconsistent nodes.

A comprehensive review of RRT* variants has been published by Noreen et. al. [28] and a review of more sampling-based approaches for the path planning problem can be found in [10].

Topological Methods

Homotopy Invariants

Given a map that associates to every path the pair of start and endpoints $\pi : PX \mapsto X \times Y$, Farber et. al. [11] define the motion planning problem as finding a function $s : X \times X \mapsto PX$ such that $\pi \circ s = id$. They introduce the notion of continuous motion planning, where close initial-final pairs produce close paths. Based on this definition, they introduce a homotopy invariant called topological complexity ($\mathbf{TC}(X)$) which is the minimal number k such that

$$X \times X = U_1 \cup U_2 \cup ... \cup U_k$$

and there exists a continuous motion planning for each U_i.

Bhattachary et. al. [5, 6] construct an augmented graph that can then be searched with classic graph search algorithms. Given a graph $G = (V, E)$, the augmented graph is a lift of G into the covering space of X. In configuration spaces homeomorphic to $\mathbb{R}^2$ they use Cauchy's Integral Theorem and the Residue Theorem from complex analysis to calculate a L-value which is equal for paths in the same homtopy classes and not equal otherwise. In configuration spaces homeomorphic to $\mathbb{R}^3$ they use laws from electromagnetism to calculate a H-value. These values are then used to augment the respective graphs. They propose another method to augment a graph in [3]. Here homotopy invariants for a D-dimensional manifold X are constructed by introducing $(D-1)$-submanifolds. Words are then formed by tracing a path f and inserting a letter every time f intersects one of the submanifolds.

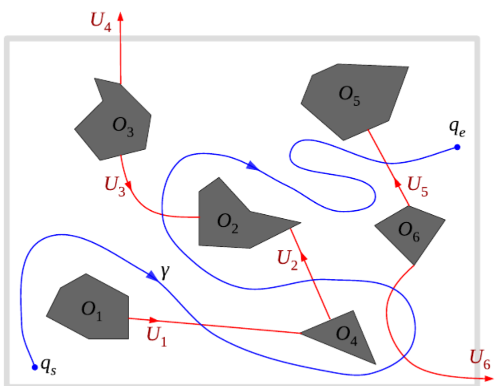

Figure 6: The blue path yields the constructed word $h(\gamma) = u_1^{-1}u_6u_6^{-1}u_2u_3u_5^{-1} = u_1u_2u_3u_5^{-1}$ [3].

While Bhattachary et. al. proposed a homotopy invariant for graph search, Sakcak et. al. [30] propose a homotopy invariant for the use with kinodynamic path planning with RRT-based planners. ...

In [4] the authors try to solve the problem of how to threshold an occupancy grid map to get a binary map for the use with classic path planning techniques. They use persistent homology to get the homology class of paths that is persistent over the widest range of threshold values.

Wang et. al. [35] show a method for constructing path classes in the workspace which are finder but extend to homotopy classes in the configuration space. °°Fiberbundle°°

Yang et. al. [39] propose a method to calculate the k-shortest non-homotopic paths by simplifiying the topology and discarding $2^n - k$ of the 2^n possible paths in a two dimensional environment with n obstacles.

Liang et. al. [25] developed an algorithm for a robot to explore human-made space autonomously. Signs are used to dictate the general direction and if this direction is a dead end, backtracking is performed and a different homotopy class is explored.

An approach for exploration with a tethered robot is proposed by the authors of [34] . They use homotopy information to limit the length of the tether and ensure that entanglement of the tether is avoided (see Figure 7).

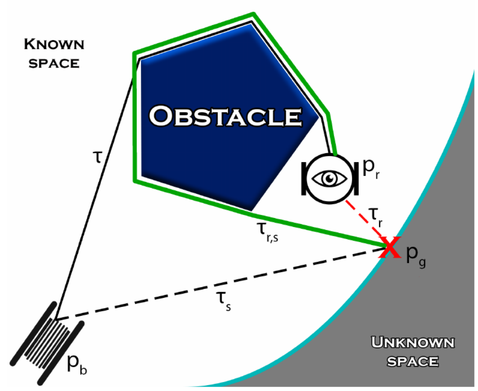

Figure 7: The robot at position p_r chooses path $\tau_{r,s}$ because $\tau + \tau_{r,s}$ is homotopic to the shortest path τ_s. Notice that $\tau + \tau_r$ is not null-homotopic and would entangle the tether [34].

Zhou et. al. [40] applied the concept of different homotopy classes to aggresive UAV flight. They use it to activate a more comprehensive exploration of the search space of gradient-based trajectory optimization which is used for trajectory replanning and is prone to getting stuck in local minima.

Orthey et. al. [29] created a visual explorer that shows different local minima given a start and endpoint. Local minima are defined to be classes of paths that converge to the same minimum under minimization of a chosen cost functional.

Wu et. al. [37] propose a method to deconflict paths of multiple robots in cluttered environments. Sequencially each robot computes it's path. The order is determined by the number of homology classes of paths of each robot.

A solution for a similar problem is shown by Wang et. al. [36]. They show a solution to multi-robot path planning in complex cluttered environments without inter-robot communication or coordination. The approach assigns robots stochastically to different topological classes and uses a potential field based controller to avoid local obstacles.

Conclusion

This review of topological methods in robot path planning shows that the topological properties of the configuration space play a fundamental role in solving the path planning problem. These properties can be used for example in exploration, multi-robot path planning or path optimization. We believe that making use of the additional knowledge of the structure of the configuration space allows for faster and more intelligent algorithms. Both of these traits are the motivation of more than a decade of research in robot path planning.

Possible future research directions include:

1. Can existing or new topological invariants be used to calculate admissible heuristics to speed up existing path planning algorithms like for example A*?

2. Can the topological properties be encoded [7] and fed into other machine learning models to increase their understanding and boost performance?

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Homotopy		
Bhattacharya et. al. [5]	2010	2-D homotopy invariant based on Cauchy's Integral Theorem and Residue Theorem
Bhattacharya et. al. [6]	2012	3-D homtopy invariant based on Biot-Severts Law and Ampere's Law
Bhattacharya et. al. [3]	2018	homotopy invariants using $(D-1)$-dimensional submanifolds and word construction
Sakcak et. al. [30]	2019	blabla
Shapovalov et. al. [34]	2020	entangle-free exploration of unknown environments with tethered robot
Zhou et. al. [40]	2020	improve gradient-based trajectory optimization prone to local minima with homtopy classes
Orthey et. al. [29]	2020	visualize local minima
Wu et. al. [37]	2020	deconfliction of multiple paths
Liang et. al. [25]	2021	human-made space exploration using signs and homotopy exploration strategy
Wang et. al. [35]	2022	finer than homotopy classes
Yang et. al. [39]	2022	k-shortest non-homotopic paths
Wang et. al. [36]	2022	mulit-agent path planning without communication and coordination
Fiber Bundle		

Table 1: Topological methods combined with the path planning problem.

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