Complex and Social Networks
Assignment 6 - Network Dynamics

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Contents

1 Introduction

Today, scale-free networks are of great importance. Many studies found them in the wild, regardless of natural or human-made networks, such as the World Wide Web [4], citation networks [11], and protein-protein interaction [8]. See an example of a scale-free network in Figure 1

Many studies use random, scale-free networks to better understand the underlying model of their own problem. The most widely used random model for scale-free networks was introduced by Albert-László Barabási and Réka Albert in 1999 [3]. Their model uses two mechanics to achieve a scale-free network:

I **Vertex Growth**. Growth means that the number of nodes in the network increases over time.

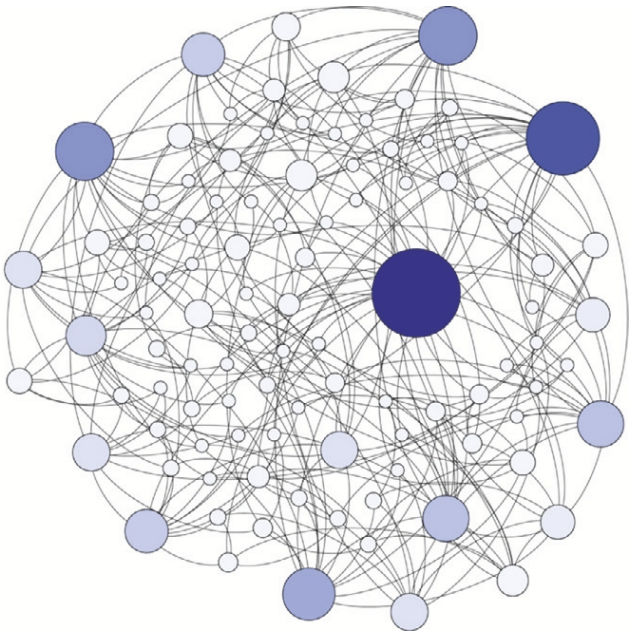

Figure 1: A scale-free network in which very different nodes coexists with 100 nodes and average degree of 2.5. The degree of each node is visually stressed by color and size. Figure extracted from [5].

II **Preferential attachment**. This mechanism causes nodes with a higher degree to be more likely to acquire new links. This is because they have a greater capacity to attract links that are added to the network.

During this report, we will examine how the different dynamic principles affect the random model. To do so, we will generate three random models: the classical Barabási-Albert model, the same model without preferential attachment, and finally, maintaining the preferential attachment and skipping the vertex growth. Then, we will analyse the degree distribution of the different models and the growth of vertex degree over time.

This document is structured in four different sections. Next, results are presented in Section 2. Discussion and conclusions are covered in Section 3, and finally some aspects of the methodology are presented in Section 4.

2 Results

2.1 Scaling of vertex degree

Tables 1, 2 and 3 show the raw AIC values for the different functions modeling the growth of vertex degree over time. A lower value is better. Since it can be hard to infer the best model given the raw data, Tables 4, 5 and 6 show the values relative to that of the best model, which will be 0.

Note that further details on the functions defining each model are given in Sections 4.2.1 and 4.3.

	Vertex arrival time					
Growth Model	10^0	10^1	10^2	10^3	10^4	10^5
0	13466729	12388632.8	10292925	7905469.7	5717260.3	3094972.14
1	1344240	330512.7	-572703.1	-892840.9	-1552647.95	-1896011.78
2	1343844	253638.8	-581026.6	-1890439.4	-2883683.82	-3795156.32
3	12264735	11193616	9080055	6643933.5	4410896.82	931830.53
4	14025181	12949158	10849575.1	8422310.1	6263055.27	2856641.09
0+	11270382	10200296.9	8081035	5645414.5	3336546.62	-30605.42
1+	1344242	118238.2	-620373.2	-2261837.1	-2534522.94	-3434708.24
2+	1340790	-128503.3	-748752.6	-2281540.1	-2927399.01	-3961792.37
3+	11283488	10212518.4	8092345.7	5655615.9	3347444.22	-14376.05
4+	12284239	7026026.6	5005728.1	2552066.2	-15020.58	-4541211.64

Table 1: Raw values of AIC of the different fits for the growth of vertex degree and for vertices with different arrival times for the **Barabási-Albert** model.

	Vertex arrival time					
Growth Model	10^0	10^1	10^2	10^3	10^4	10^5
0	8250251.9	8107313.2	7719411.6	7065672.5	5976657.7	3558414.4
1	7197981	7029556.2	6581725.4	5797791.99	4369493.7	489488.6
2	46022.3	180638.5	397381.9	660619.34	838752.6	-491206.3
3	3682816.7	3726539	3711418.8	3610671.16	3304714.6	1609452.5
4	181267.3	1391763.9	2516319.7	3322646.23	3713818.2	2893321.2
0+	3600713.5	3637001.1	3604182.6	3466481.52	3066513.5	1070307.4
1+	2787542	2815714.9	2763706.2	2579871.29	2070775.1	-193309.4
2+	-449654.6	-458499.9	-537092.5	13400.48	-126333.7	-1845014.7
3+	3601329.9	3637607.1	3604838.6	-	3067402.8	1076270.7
4+	-4195831.6	-4171181.4	-3494283.4	-3292412.9	-3977152	-3851501.3

Table 2: Raw values of AIC of the different fits for the growth of vertex degree and for vertices with different arrival times for the **Growth + Random Attachment** model.

	Vertex arrival time		
Growth Model	10^0	10^1	10^2
0	2665444	4562882	3078313
1	15919141	15917584	15920137
2	2383116	2994873	2359965
3	14887015	14886800	14888863
4	17505263	17506635	17506867
0+	2487147	2853452	2557081
1+	14532070	14531893	14533158
2+	2379677	2569593	2357503
3+	8059022	8072553	8073492
4+	16354605	16077101	16356210

Table 3: Raw values of AIC of the different fits for the growth of vertex degree and for vertices with different arrival times for the **No Growth + Preferential Attachment** model.

	Vertex arrival time					
Growth Model	10^0	10^1	10^2	10^3	10^4	10^5
0	12125939.427	12517136.1	11041677.6	10187009.83	8644659.31	7636183.8
1	3450.533	459016.1	176049.5	1388699.19	1374751.06	2645199.9
2	3054.056	382142.1	167725.9	391100.68	43715.19	746055.3
3	10923945.669	11322119.3	9828807.6	8925473.58	7338295.83	5473042.2
4	12684391.264	13077661.3	11598327.7	10703850.23	9190454.28	7397852.7
0+	9929591.812	10328800.3	8829787.6	7926954.64	6263945.63	4510606.2
1+	3452.368	246741.5	128379.4	19702.98	392876.07	1106503.4
2+	0	0	0	0	0	579419.3
3+	9942698.539	10341021.7	8841098.3	7937155.98	6274843.23	4526835.6
4+	10943448.824	7154529.9	5754480.6	4833606.26	2912378.43	0

Table 4: Difference of AIC from the best model, for the different fits for the growth of vertex degree and for vertices with different arrival times for the **Barabási-Albert** model.

	Vertex arrival time					
Growth Model	10^0	10^1	10^2	10^3	10^4	10^5
0	12446084	12278495	11213695	10358085	9953810	7409916
1	11393813	11200738	10076009	9090205	8346646	4340990
2	4241854	4351820	3891665	3953032	4815905	3360295
3	7878648	7897720	7205702	6903084	7281867	5460954
4	4377099	5562945	6010603	6615059	7690970	6744822
0+	7796545	7808182	7098466	6758894	7043666	4921809
1+	6983374	6986896	6257990	5872284	6047927	3658192
2+	3746177	3712682	2957191	3305813	3850818	2006487
3+	7797162	7808789	7099122		7044555	4927772
4+	0	0	0	0	0	0

Table 5: Difference of AIC from the best model, for the different fits for the growth of vertex degree and for vertices with different arrival times for the **Growth + Random Attachment** model.

	Vertex arrival time		
Growth Model	10^0	10^1	10^2
0	285767.322	1993288.8	720810.214
1	13539464.645	13347990.9	13562634.04
2	3438.945	425279.9	2462.349
3	12507338.737	12317206.2	12531360.486
4	15125586.033	14937041.8	15149363.626
0+	107470.591	283858.3	199577.718
1+	12152393.14	11962299.5	12175655.439
2+	0	0	0
3+	5679345.288	5502960	5715989.132
4+	13974927.943	13507507.3	13998706.756

Table 6: Difference of AIC from the best model, for the different fits for the growth of vertex degree and for vertices with different arrival times for the **No Growth + Preferential Attachment** model.

2.2 Degree distribution

Table 7 presents the raw AIC values of the different distributions for the three graph models under consideration. Likewise, Table 8 presents the difference with respect to the best model. Note that in both tables, *BA* refers to tbe *Barabási-Albert* model, *G+RA* to the *Growth + Random Attachment* model, and *NG + PA* to the *No Growth + Preferential Attachment* model.

	Graph Model		
Distribution	BA	G+RA	NG+PA
Displaced Poisson	400226.7	3864871	15009.96
Displaced Geometric	270338.7	4306365	
Fixed Zeta	473815	8019202	38073.58
Zeta	328183.8	5586216	23845.98
Truncated Zeta	882413	6307312	17965.17
Altmann	270340.9	4306371	13913.82

Table 7: Raw AIC values for the different fits of the degree distribution of the graphs constructed with the different models.

	Graph Model		
Distribution	BA	G+RA	NG+PA
Displaced Poisson	129888.01	0	1096.136
Displaced Geometric	0.02539979	441493.1	
Fixed Zeta	203476.32	4154331	24159.753
Zeta	57845.08	1721344.5	9932.157
Truncated Zeta	612074.26	2442440.7	4051.349
Altmann	0	441499.7	0

Table 8: Difference of AIC from the best model, for the different fits of the degree distribution of the graphs constructed with the different models.

2.3 Conclusion

The results show that Model 2+ is the most suited to represent the scaling of vertex degree with the *Barabási-Albert* model and the *No Growth + Preferential Attachment*, while Model 4+ is the best fit for the *Growth + Random Attachment*.

Regarding the degree distribution, Altmann is the best for the first and third cases, while the second graph model has a degree distribution that could be best modeled with a Poisson function.

Note, however, that we think these values are not correct and are a result of a mistake in our code that we have not been able to fix. We give more details on this in the next section.

3 Discussion

3.1 Scaling of vertex degree

The traces of the evolution of the degree for different vertices, show that the evolution curve has the same shape independently of their arrival time. Indeed, looking at the plots from Appendix A, we can observe that the growth function of a vertex does not depend on its arrival time.

Different growth models, however, present very different scaling curves, as noted in Section 2. Table 9 shows the specific parameters of all fitted curves. We only consider the case for the vertex arriving at time 1, since all other curves present an identical shape once they reach the same lifespan of any other vertex.

	Growth Model							
	0	1	2		3		4	
Graph Model	a	a	a	b	a	c	a	d_1
Barabási-Albert	0	1.44	1.44	0.5	524.83	0	76.24	-1.95
Growth + Random Attachment	0	0.05	10.43	0.09	30.11	0	2.67	-1.56
No Growth + Preferential Attachment	0.01	4.8	0.01	1	1046.12	0	241.88	-1.96

0+		1+		2+			3+			4+		
a	d	a	d	a	b	d	a	c	d	a	d_1	d_2
0	383.23	1.44	0	0.69	0.23	-6.21	848.49	0	-844.52	3.23	26430.96	-34.85
0	29.8	0.01	26.38	18.01	0.07	-10.21	4815.89	0	-4786.09	2.93	4.54	-3.4
0.01	0.74	7.2	-1799.99	0.01	1	-0.19	98626.47	0	-98596.02	1506.69	60.81	-16308.43

Table 9: Values of fitted parameters per curve and graph model. Models 0 to 4 are shown above, and models 0+ to 4+ below.

Note that the mathematical expressions for each of the growth models are defined in Table 13.

Through visual comparison against the raw data, most functions seem to fit the data well, as shown in the figures of Appendix B, with the fitted function (green line) almost completely overlapping the data (black line) in all cases.

However, some of the results regarding which is the best model go against our intuition. It is especially the case for the *No Growth + Preferential Attachment* model, whose scaling of vertex degree clearly follows a straight line. Models 0 and 0+ should be able to fit this data, and indeed, they visually do, so they should be preferred over the more complex Models 2 and 2+, when measuring the AIC. However, this is not the case and, aside from a possible mistake in our implementation which we have not been able to identify, our reasoning for this behaviour considers two possibilities:

1. Although the data looks like a straight line, there might be some non-zero curvature that models 0 and 0+ are not able to capture, thus making their error be higher than it appears to be in plain sight.

2. Since the number of data points is very large, the RSS values tend to be very large (and n too), and even though the logarithm is considered for the computation of the AIC, RSS could still be the dominating factor, rendering p (the sole advantage of models 0 and 0+ over 2 and 2+) insignificant, and making the decision of the best model be guided by random noise in the data.

3.2 Degree distribution

The results of the fitted parameters can be seen in Table 10.

	Probability Model						
	D. Poisson	D. Geom	Zeta	Truncated Z		Altmann	
Graph Model	λ	q	γ_1	γ_2	k_{max}	γ_3	δ
Barabási-Albert	5.98	0.17	1.49	0	7120	0	0.18
Growth + Random Attachment	6.99	0.14	1.42	0	70	0	0.15
No Growth + Preferential Attachment	6008	0	1.11	0	8401	0	0

Table 10: Values of fitted parameters per distribution and graph model. Note that the model of a Zeta distribution with $\gamma = 3$ is not shown because no parameters were estimated.

With these values, we are able to plot the probability distribution function of each probability model and compare it to the raw data.

However, through a visual representation of the fitted curves, as shown in Appendix C, one can see that the curves do not model the data well. Note that the magnitude of the error is amplified by the representation in log-log scale (otherwise it is visually imperceptible), but it still shows that there has probably been a problem in the optimization procedure.

Indeed, for the *Barabási-Albert* graph, one would expect to see the Zeta distribution model the data almost perfectly, since the curve with $\gamma = 3$ is visually identical to our data (and is also what the theoretical knowledge of the *Barabási-Albert* model indicates). Instead, we see that it is somewhat displaced from the data and that the Zeta distribution with a variable γ parameter does not converge to $\gamma = 3$. Moreover, the geometric distribution, considered the best by AIC, is does not resemble the shape of the underlying data.

The distributions for the other two models show similar problems. For the *Growth + Random Attachment*, the Poisson distribution is considered best. While it does show a downwards curve, as the data, it does not visually fit the underlying distribution. The *No Growth + Preferential Attachment* model has a completely different shape, that our distributions are not able to fit; we would have expected the Poisson distribution to model this shape, but it is not the case.

4 Methods

4.1 Model simulation

We wanted to understand the dynamic mechanics of the Barabási-Albert model. Therefore, we performed three simulations of the models to obtain its degree sequence and degree evolution over time.

4.1.1 Barabási-Albert Model

This Barbásii-Albert model is the baseline model. To simulate it, instead of fully simulating the graph, we kept a list of stubs, S and when a node was added, it was chosen u.a.r from S the other node. We initialised the list of stubs by iterating over all the vertices and adding one random edge with some other vertex. This is done to ensure that all the nodes appear in the stub list, otherwise it would be impossible for them to appear ever due to the preferential attachment. This allows to properly implement the preferential attachment. A sketch of our simulation can be seen in Algorithm 1.

Algorithm 1 Barbási-Albert model

Require: $t_{max}, n_0, m_0 \in \mathbb{N}, n_0 \geq 2$
  for $u \in [0, n_0)$ **do** ▷ Initialise randomly the graph
    $v \leftarrow$ Choose u.a.r from $[0, n_0) \setminus \{u\}$
    $S \leftarrow S \cup \{u, v\}$
  end for
  for $t \in [n_0, n_0 + t_{max})$ **do**
    $K \leftarrow$ Sample u.a.r m_0 elements from S
    $L \leftarrow$ Repeat m_0 times t
    $S \leftarrow S \cup K \cup L$
  end for

4.1.2 Growth + Random Attachment

This model aims to eliminate the preferential attachment mechanism. We kept the random initialisation. Then, the row that samples u.a.r from the list of stubs (see line 6 is replaced by random samples of all the previous nodes. A sketch of our simulation can be seen in Algorithm 2.

Algorithm 2 Growth + Random Attachment model

Require: $t_{max}, n_0, m_0 \in \mathbb{N}, n_0 \geq 2$
  for $u \in [0, n_0)$ **do** ▷ Initialise randomly the graph
    $v \leftarrow$ Choose u.a.r from $[0, n_0) \setminus \{u\}$
    $S \leftarrow S \cup \{u, v\}$
  end for
  for $t \in [n_0, n_0 + t_{max})$ **do**
    $K \leftarrow$ Sample u.a.r m_0 elements from $[0, t)$
    $L \leftarrow$ Repeat m_0 times t
    $S \leftarrow S \cup K \cup L$
  end for

4.1.3 No Growth + Preferential Attachment

In this case, we remove the growth of the vertex, but retain the preferential attachment. Therefore, the initial value of n_0 will be the maximum number of vertices in our graph. We also initialised the graph randomly to ensure that all the vertices have at least one edge. Then, at every time step, we choose one vertex u.a.r. and some other vertices by sampling over the list of stubs. A sketch of our simulation can be seen in Algorithm 3.

Algorithm 3 No Growth + Preferential Attachment model

Require: $t_{max}, n_0, m_0 \in \mathbb{N}, n_0 \geq 2$
  for $u \in [0, n_0)$ **do** ▷ Initialise randomly the graph
    $v \leftarrow$ Choose u.a.r from $[0, n_0) \setminus \{u\}$
    $S \leftarrow S \cup \{u, v\}$
  end for
  for t times **do**
    $u \leftarrow$ Choose u.a.r a vertex
    $K \leftarrow$ Sample u.a.r m_0 elements from S
    $L \leftarrow$ Repeat m_0 times u
    $S \leftarrow S \cup K \cup L$
  end for

4.1.4 Experimental stability

Our models are generated pseudorandomly. In order to have stable results, we repeated our experiments 100 times.

4.2 Analysis of the degree distribution

In order to analyse the degree distribution of the different models, we will perform a maximum likelihood estimation of the different models. Maximum likelihood estimation (MLE) involves the process of determining the parameters of an assumed probability distribution based on observed data. It entails maximising a likelihood function such that, within the context of the assumed statistical model, the observed data have the highest probability.

It is defined as follow:

$$\mathcal{L}(\text{params}|\mathbf{x}, \text{model}) = \sum_{i=1} \log p(x_i; \text{params})$$

4.2.1 Models for degree distribution

The degree distribution can be defined in many ways. One possibility is the cumulative degree distribution, but instead we chose the probability mass function. There are several possible probability-mass functions that can model the degree distribution. We chose the geometric distribution, the Poisson distribution because it is a simple approximation for the binomial distribution that characterises Erdös-Rényi graphs [9]; and the Zeta distribution and some variants because they exhibit a so-called heavy tail, and the Altmann distribution (also known as Menzerath's law) [2]. A detailed description of the models used and their log-likelihood is shown in Table 11.

Model	Probability function	Likelihood function
Displaced Poisson	$p(k) = \lambda^k \frac{e^{-\lambda}}{k!(1-e^{-\lambda})}$	$\mathcal{L}(\lambda) = M \log \lambda - N(\lambda + \log(1 - e^{-\lambda})) - C$
Displaced Geometric	$p(k) = (1-q)^k q$	$\mathcal{L}(q) = (M - N)\log(1-q) + N \log q$
Fixed Zeta	$p(k) = \frac{k^{-3}}{\zeta(3)}$	-
Zeta	$p(k) = \frac{k^{-\gamma}}{\zeta(\gamma)}$	$\mathcal{L}(\gamma) = -\gamma M' - N \log \zeta(\gamma)$
Truncated Zeta	$p(k) = \begin{cases} \frac{1}{\sum_{x=1}^{k_{max}} x^{-\gamma}} k^{-\gamma} & \text{if } k \leq k_{max} \\ 0 & \text{otherwise} \end{cases}$	$\mathcal{L}(\gamma, k_{max}) = -\gamma M' - N \log \sum_{x=1}^{k_{max}} x^{-\gamma}$
Altmann	$p(k) = k^{-\gamma} e^{-\delta k} \frac{1}{\sum_{x=1}^{N} x^{-\gamma} e^{-\delta x}}$	$\mathcal{L}(\gamma, \delta) = -\gamma M' - \delta M - N \log(d)$

Table 11: Models used to analyse the degree distribution.

Finding the values of the parameters that maximize the likelihood is an optimization process that requires some initial guess. Table 12 shows the initial values for our implementation.

Model	Parameters	Initial values
Displaced Poisson	λ	$\lambda = N/M$
Displaced Geometric	q	$q = N/M$
Fixed Zeta	-	-
Zeta	γ	$\gamma = 3$
Truncated Zeta	γ, k_{max}	$k_{max} = N$, $\gamma = 3$
Altmann	γ, δ	$\gamma = 3$, $\delta = 2$

Table 12: Initial values for the optimization process. The values of M and N refer to the sum of degree values for all nodes in the data and the number of nodes, respectively.

In order to maximise this likelihood, we used the `mle` method of the R package `stats4`. We configured it to use a limited-memory Broyden-Fletcher-Goldfarb-Shannon bounded method (L-BFGS-B) [6]. We opted to use L-BFGS-B because it allowed us to establish bounds which is not possible with other methods such as BFGS, CG (more fragile than the BFGS methods), SANN (only allows one-dimensional parameters so we could not use it for Altmann) and also showed good performance.

4.2.2 Model selection

Once the models were fitted, we had to perform a model selection., by the Akaike information criterion (AIC)[1] with a correction for sample size:

$$AIC_c = -2\mathcal{L} - 2K\frac{N}{N-K-1} \tag{1}$$

Herein, we define AIC_{best} as the minimum AIC across all models. Consequently, for each probability mass function m, we calculate $\Delta AIC_m = AIC_m - AIC_{best}$, to identify the model that best approximates our data. In Table 7 in Section 2, we present the ΔAIC values for each probability model in each random graph model, where values closer to 0 indicate a stronger alignment with our optimal approximation.

We want to emphasise that although it is known that the probability that a vertex has degree k in the Barabási-Albert model is approximately defined by Equation 2[7]; it is not suitable for model selection because it is not warranted that the sum of all probabilities is 1.

$$p(k) \approx \frac{2m_0^2 t}{n_0 + t} k^{-3} \tag{2}$$

4.3 Analysis of the growth of vertex degree over time

We want to model the evolution of the vertex degree of the vertices that arrived at time 10^n, with $n \in [0, 5]$ [1]. We proposed different functions that could be used: some linear models, polynomial, exponential, and logarithmic. An ensemble of our models can be seen in Table 13.

Model	Parameters	Function
Model 0	a	$f(t) = at$
Model 1	a	$f(t) = at^{1/2}$
Model 2	a, b	$f(t) = at^b$
Model 3	a, c	$f(t) = ae^{ct}$
Model 4	a, d_1	$f(t) = a \log t + d_1$
Model 0+	a, d	$f(t) = at + d$
Model 1+	a, d	$f(t) = at^{1/2} + d$
Model 2+	a, b, d	$f(t) = at^b + d$
Model 3+	a, c, d	$f(t) = ae^{ct} + d$
Model 4+	a, d_1, d_2	$f(t) = a \log t + d_1 + d_2$

Table 13: Models used to analyse the evolution of the degree of a vertex at time t

4.3.1 Model selection

The different models have been fit on the raw data through non-linear regression. The best model has been chosen as the one with the lowest AIC. For non-linear regression models, we follow the definition of AIC from [10], that computes it as:

$$AIC = n \cdot 2\pi + n \cdot \log(RSS/n) + n + 2 \cdot (p + 1) \tag{3}$$

where n is the number of points in the data, p the number of parameters of the model and RSS the residual sum of squares of the regression fit.

[1] Note that for the *no-growth + preferential attachment* model, n was limited to $[0, 2]$, because our initial graph had 1k vertices, and in this model, no new vertices are added, so there is no vertex arriving at time > 1000.

References

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A Evolution of vertex degree for different arrival times

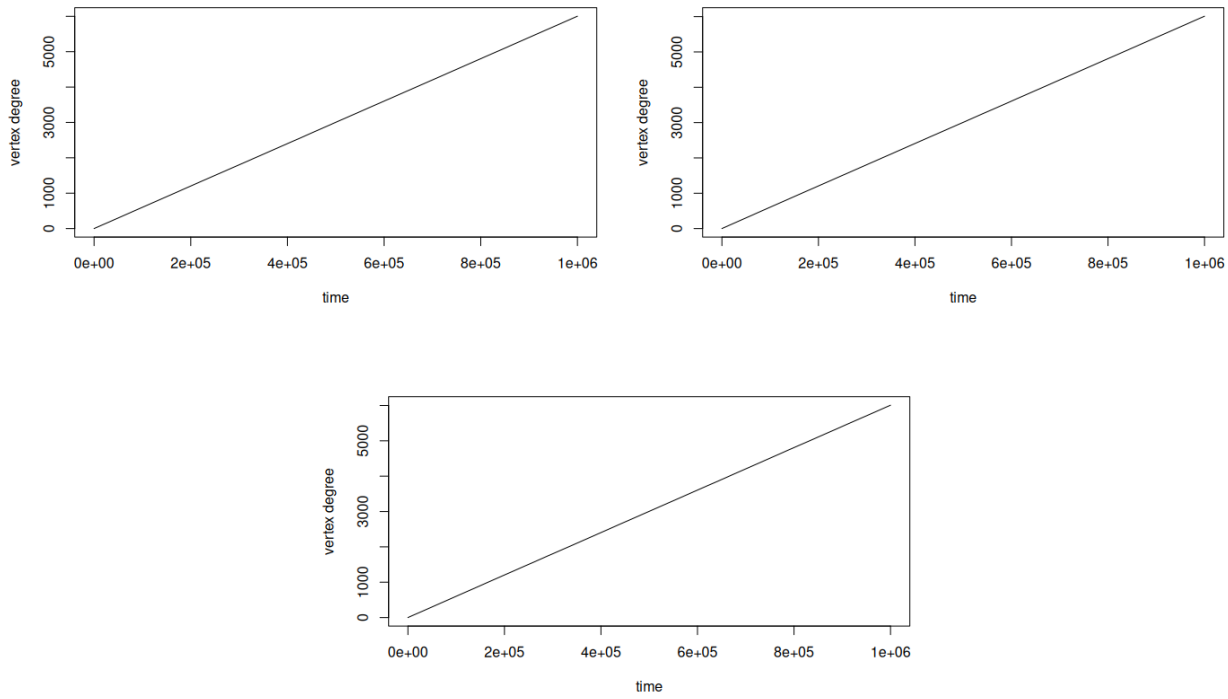

Figure 2: Vertex degree at every time step of a *No Growth + Preferential Attachment* model for vertices with arrival times (from left to right, top to bottom): 1, 10, 100.

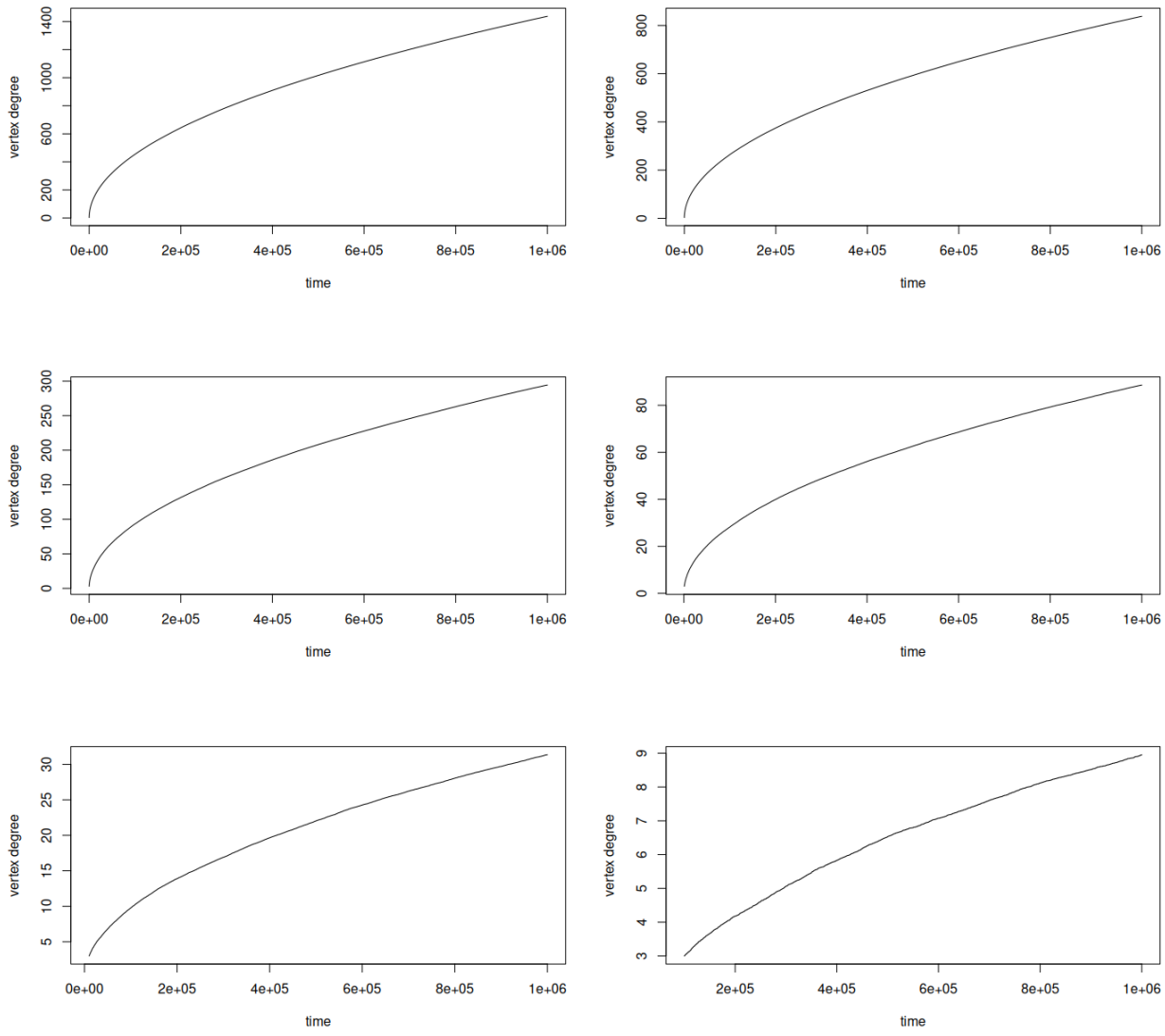

Figure 3: Vertex degree at every time step of a *Barabási-Albert* model for vertices with arrival times (from left to right, top to bottom): 1, 10, 100, 1000, 10000, 100000.

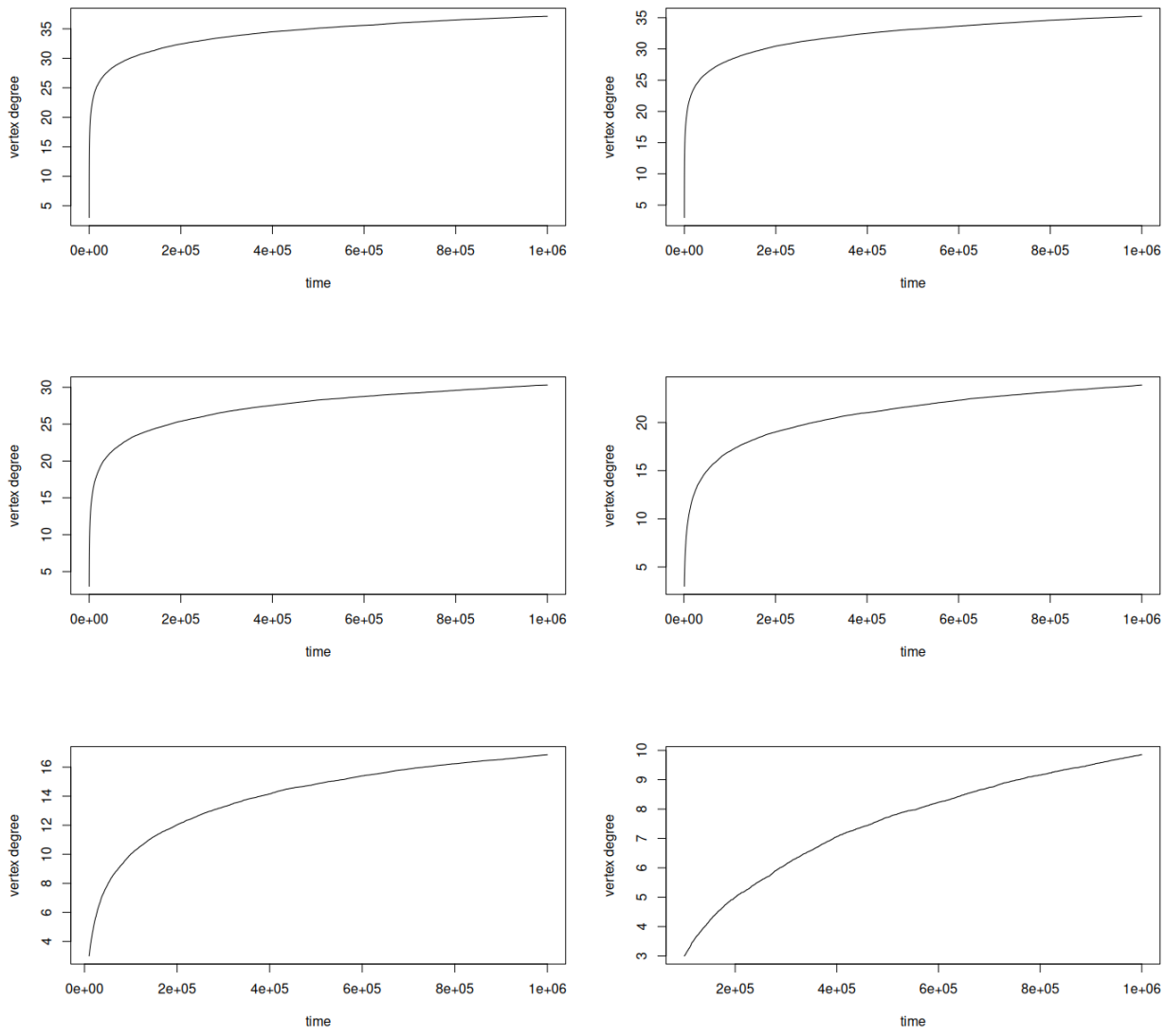

Figure 4: Vertex degree at every time step of a *Growth + Random Attachment* model for vertices with arrival times (from left to right, top to bottom): 1, 10, 100, 1000, 10000, 100000.

B Fit of the curves modeling the scaling of vertex degree over time

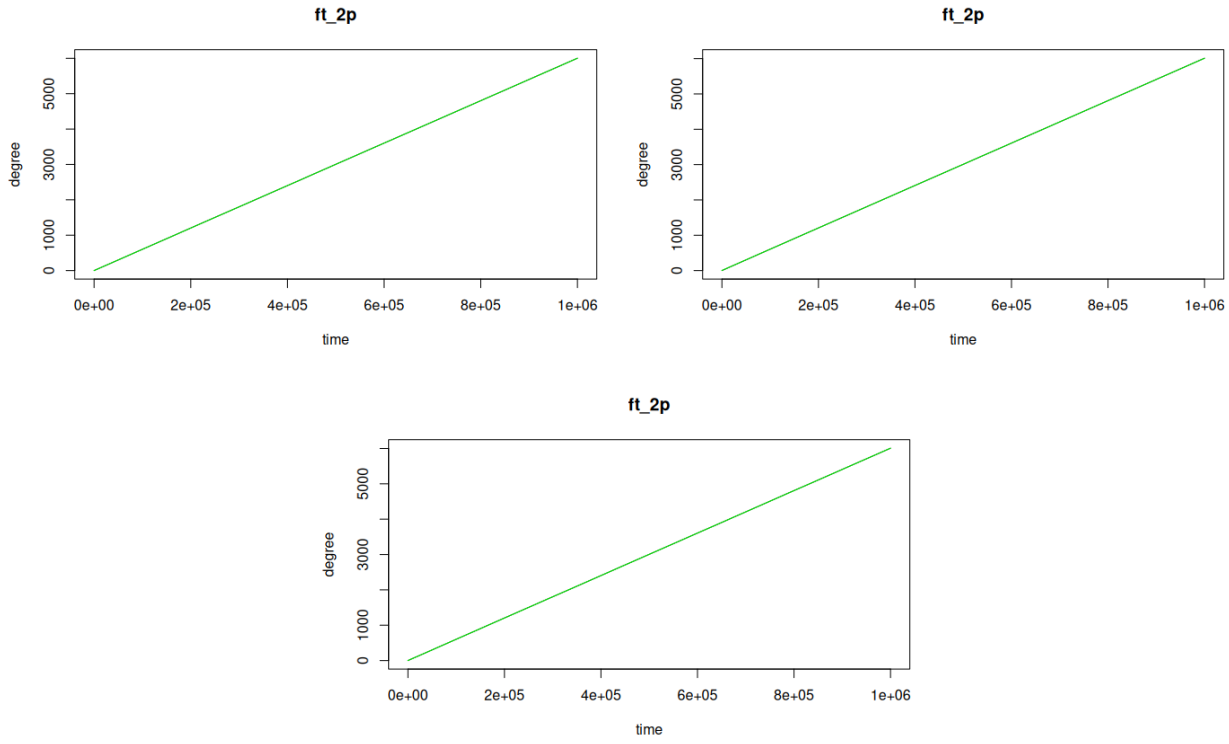

Figure 5: Fitted curve (green) on top of the curve showing vertex degree evolution (black) for the *No Growth + Preferential Attachment* model.

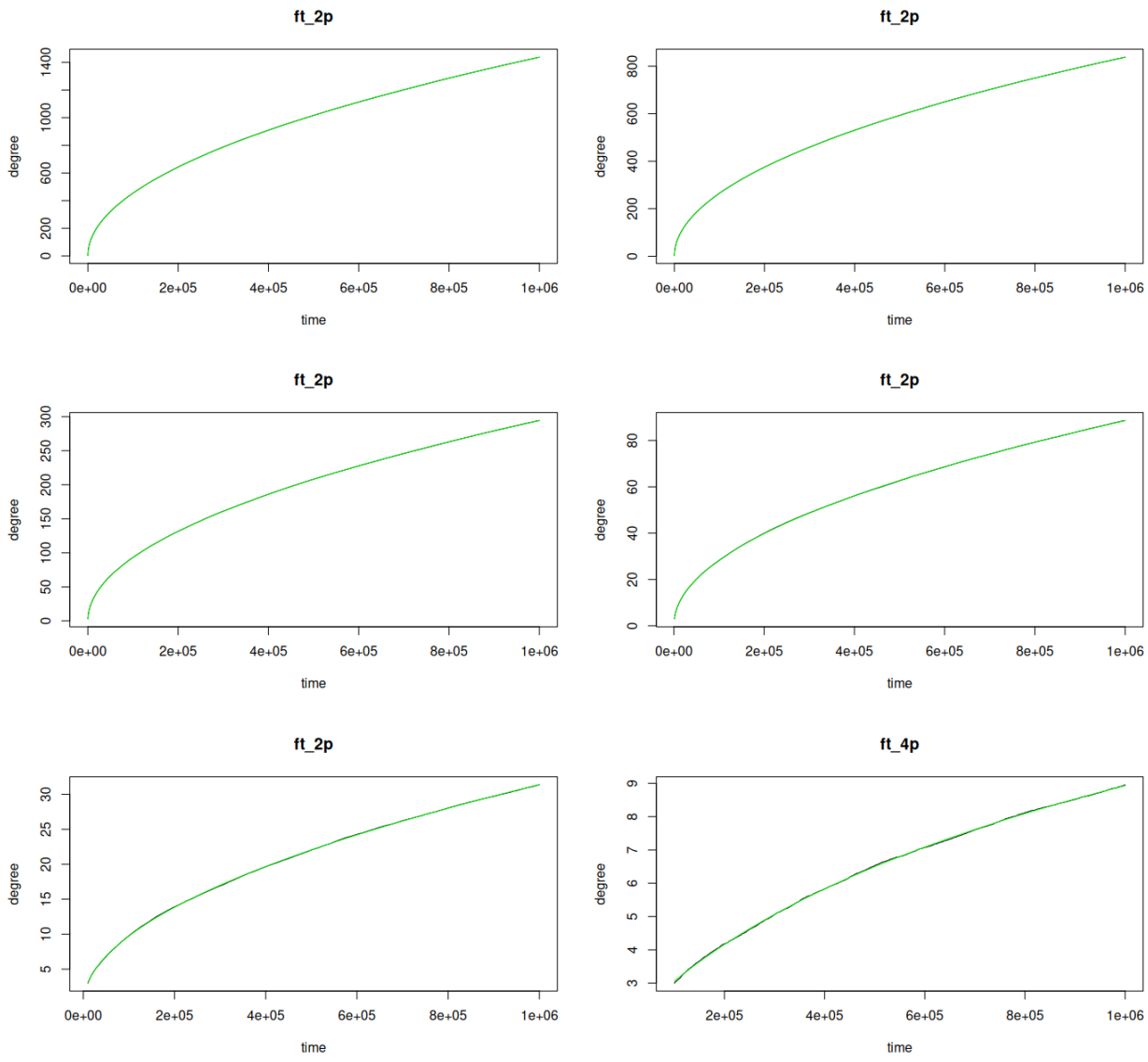

Figure 6: Fitted curve (green) on top of the curve showing vertex degree evolution (black) for the *Barabási-Albert* model.

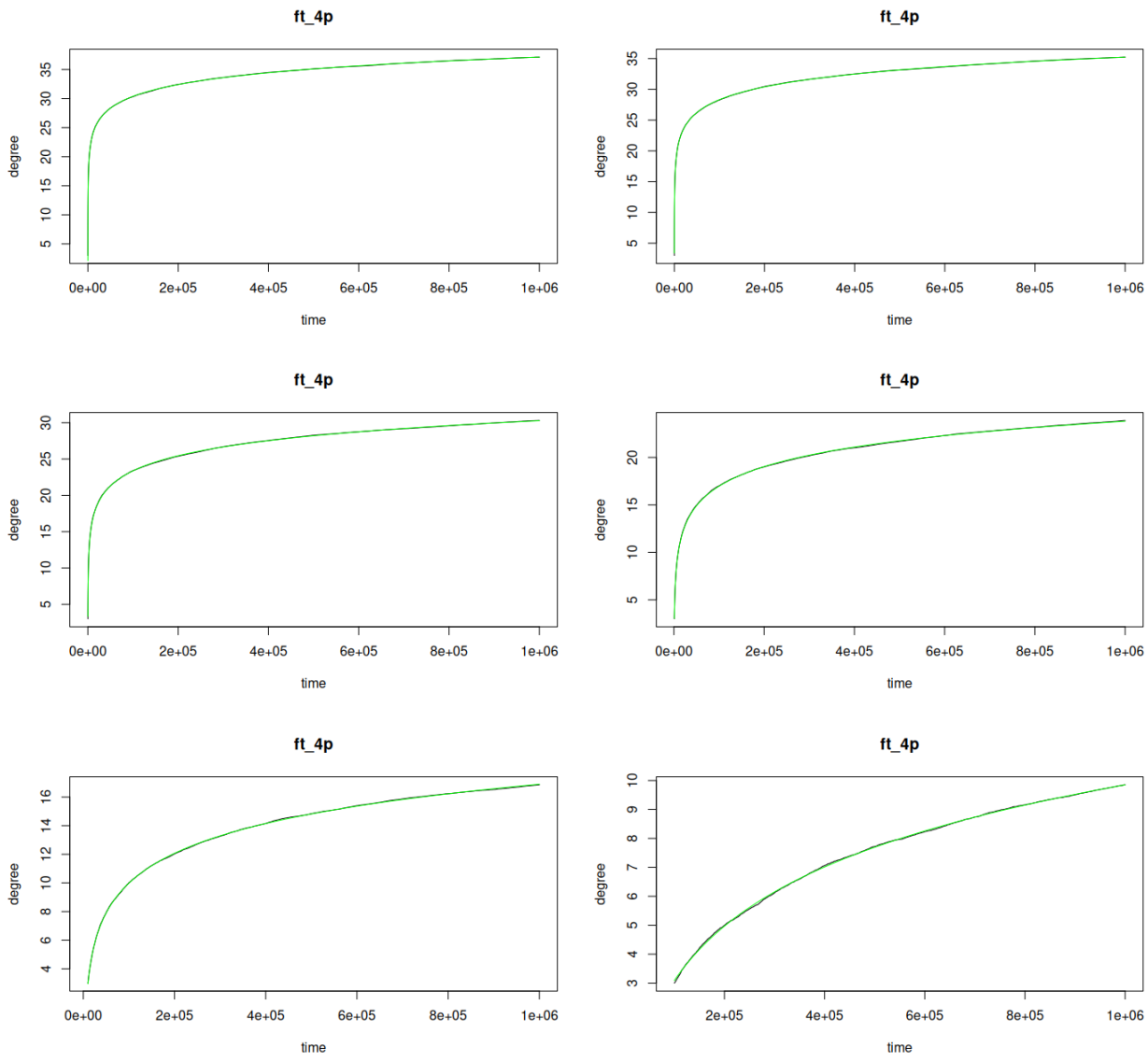

Figure 7: Fitted curve (green) on top of the curve showing vertex degree evolution (black) for the *Growth + Random Attachment* model.

C Curves of the fitted degree distributions over the data

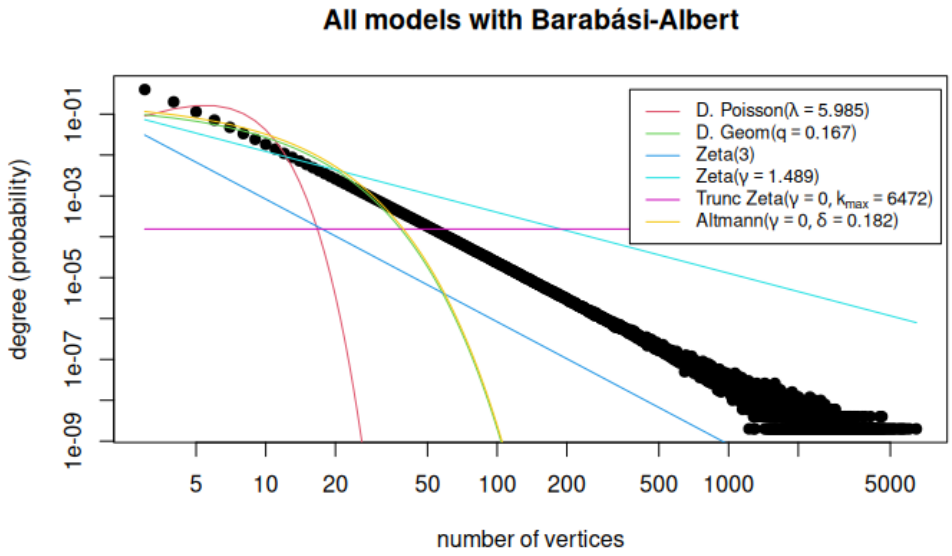

Figure 8: Plot of the degree distribution of the *Barabási-Albert* model in a log-log scale, with the curves of the fitted distributions overlaid.

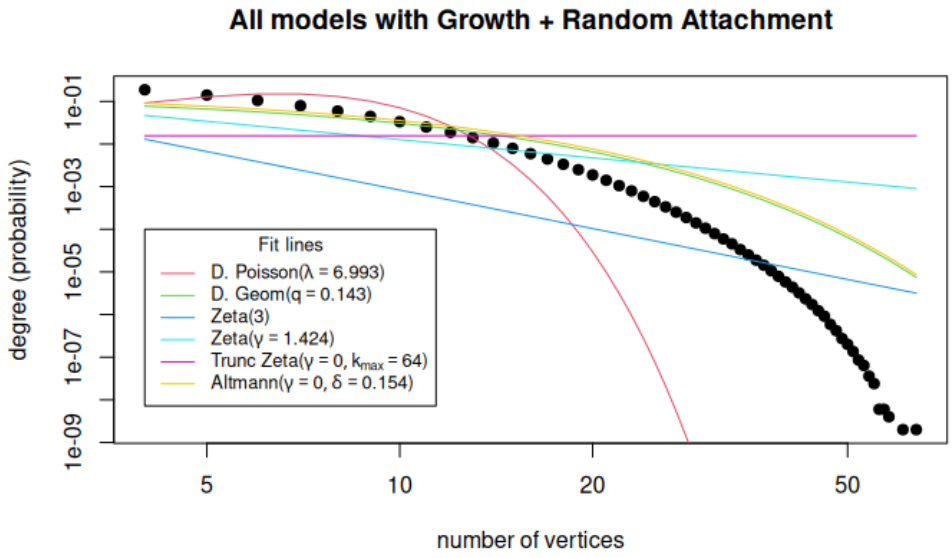

Figure 9: Plot of the degree distribution of the *Growth + Random Attachment* model in a log-log scale, with the curves of the fitted distributions overlaid.

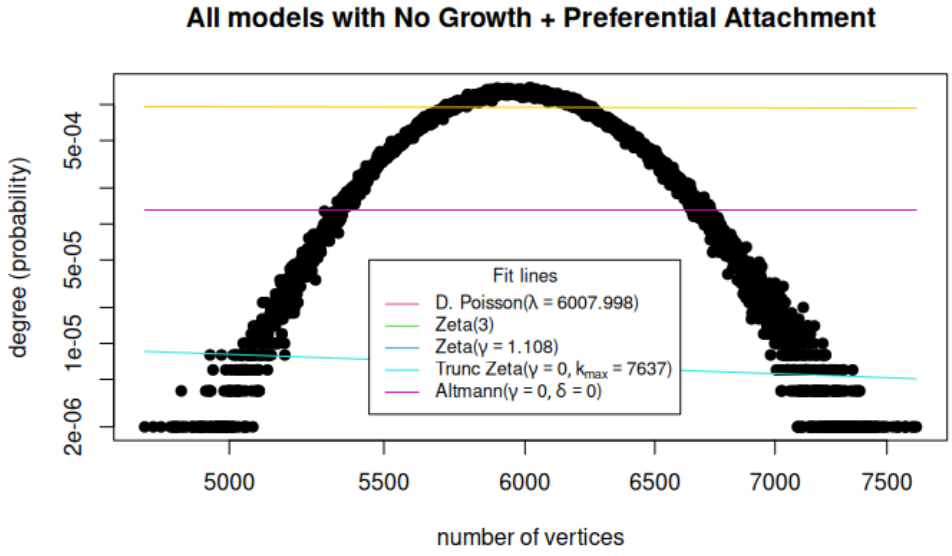

Figure 10: Plot of the degree distribution of the *No Growth + Preferencial Attachment* model in a log-log scale, with the curves of the fitted distributions overlaid.