Extreme Points on Circumconics Induced by Isogonal Conjugates in a Triangle

Daniel Hu
daniel.b.hu@gmail.com
Los Altos High School
Los Altos, CA

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Abstract

We first introduce a configuration of arbitrary isogonal conjugates related to a known property concerning the spiral center of two pairs of isogonal conjugates. We then consider a special case where two conics are tangent at exactly two points. Finally, we apply the discoveries made in both configurations to state a general result concerning the extreme points (those lying on either the major or minor axis) of certain circumconics of a triangle.

1 Introducing the Configuration

1.1 Conventions

The reader should be familiar with the relationship between isogonal conjugation and circumconics of a triangle, as well as the characterization of a conic by cross ratio properties.

All angles used here are directed angles mod $180°$. We use (XYZ) to denote the circumcircle of three points X, Y, Z. We use $(VWXYZ)$ to denote the circumconic of V, W, X, Y, Z. The symbol ∞_{XY} denotes the point of infinity along line XY. The expression $\mathcal{C}(AB; CD)$ for conic $\mathcal{C}$ denotes the cross ratio $(AB; CD)$ on $\mathcal{C}$.

This paper exclusively employs synthetic and projective techniques in order to provide a purely geometric perspective on the configuration.

1.2 The Configuration

The first configuration is as follows. In ABC with circumcircle Ω, let:

- P and P' be isogonal conjugates,
- Q lie on line PP',
- Ω meet $(ABCPP')$ at $D \neq A, B, C$,
- and DP meet Ω at $X \neq D$.

The following is the main result of our paper:

Theorem 1.1

Given the aforementioned configuration, we have that:

(a) $(PQX), (ABCPQ)$ are tangent at P.

(b) if (PQX) and $(ABCPQ)$ meet at $L \neq P, Q$, then PL and $(ABCPX)$ intersect on the radical axis of Ω and (PQX).

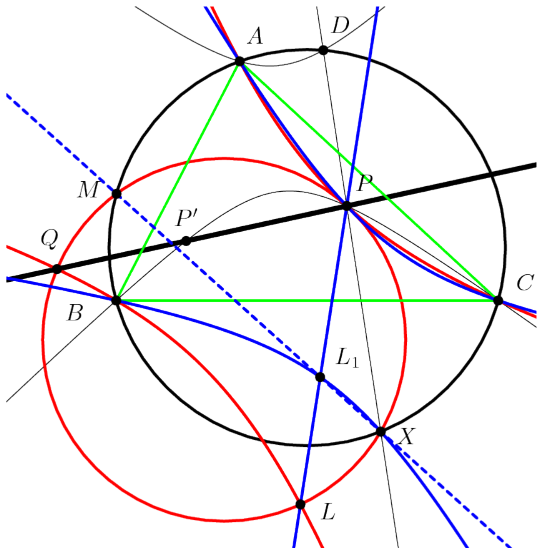

Figure 1: Main Configuration 1

From this main result, we are able to deduce the following two results as well:

Theorem 1.2

For ABC with circumcircle Ω and P with isogonal conjugate P', let $D = \Omega \cap (ABCPP')$. Let $X = DP \cap \Omega, Y = DP' \cap \Omega$. Let (XYZ) meet PP' at Z.

(a) (PXZ) and $(ABCPZ)$ are tangent at P and Z.

(b) Let (PXY) meet $(ABCPX)$ at Z_1, Z_2. Let Ω meet PZ_1, PZ_2 at Z_1', Z_2'. Then $Z_1'Z_2'$ is tangent to $(ABCPP')$ at P'.

Theorem 1.3

Let ABC have isogonal conjugates P, P' with $D = (ABC) \cap (ABCPP')$. Let $D' \in (ABC)$ such that $DD' \parallel PP'$. Then the following are equivalent:

- either PP' is tangent to $(ABCD'P)$ or PP' passes through the center of $(ABCD'P)$
- P lies on either the major or minor axis of $(ABCD'P)$.

2 Preliminary Lemmas

To prove this property, we first start with a very simple fact.

Fact 2.1

Fix conic $\mathcal{H}$ containing points A, B, C and line ℓ containing D. Vary point E on $\mathcal{H}$; let the circumconic $\mathcal{E}$ of $ABCDE$ meet ℓ at $F \neq D$. Then EF passes through a fixed point G on $\mathcal{H}$.

Proof. Let EF meet $\mathcal{H}$ at $G \neq E$. Then

$$\mathcal{H}(BC;AG) \stackrel{E}{=} \mathcal{E}(BC;AF) \stackrel{D}{=} (DB,DC;DA,\ell).$$

Since $(DB,DC;DA,\ell)$ is fixed, by [4], so is G, as desired. $\square$

We now provide two related lemmas. Labeling is distinct from that of the original configuration.

Lemma 2.2

For A, B, C, P, P', Q, Q' such that $(P,P'), (Q,Q')$ are pairs of isogonal conjugates, let R, R' respectively lie on $PQ, P'Q'$. Then $(ABCPR)$ intersects $(ABCQ'R')$ on RR'.

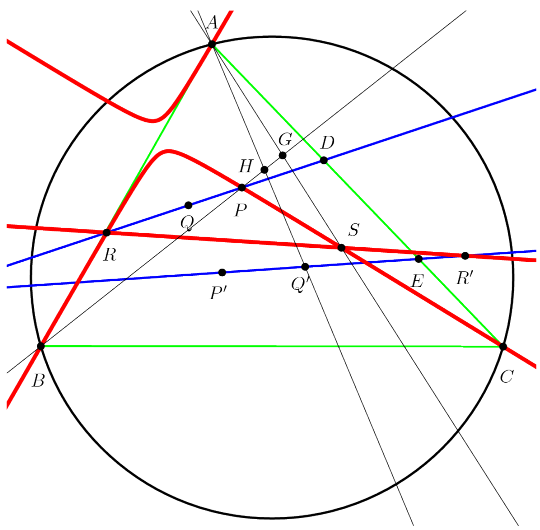

Figure 2: Lemma 2.2

Proof. Let AC meet PQ at D and $P'Q'$ at E. Let RR' meet BC at F and $(ABCPR)$ at $S \neq R$. Let AS meet BP at G. By Pascal's Theorem ([6]) on $ASRPBC$, D, F, G are collinear.

Let AQ' meet BP at H; $P'Q'$ meet BC at I; and AP' meet BQ at J. Let HJ meet PQ at D_0 and $P'Q'$ at I_0. By the Dual of Desargues' Involution Theorem (DDIT) ([1], 133), $(AB, AD_0), (AP, AJ), (AQ, AH)$ are pairs of an involution, so D_0 lies on AC. DDIT also gives $(BA, BI_0), (BP', BH), (BJ, BQ')$ to be pairs of an involution, so I_0 lies on BC. Thus $D \equiv D_0$ and $I \equiv I_0$, so D, H, I are collinear.

By Desargues' Theorem ([7]) on triangles $AQ'E$ and GBF, AG, BQ', EF concur. By the converse of Pascal on $Q'R'SACB$, S lies on $(ABCQ'R')$ as desired. ■

Lemma 2.3

Let P, P' and Q, Q' be two pairs of isogonal conjugates in ABC. Then the spiral center from PQ to $Q'P'$ lies on (ABC).

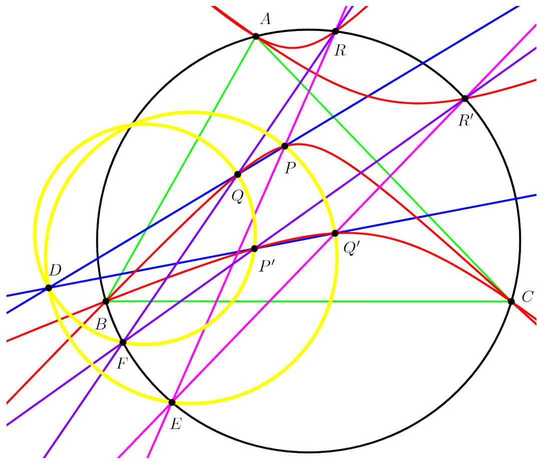

Figure 3: Noncollinear Isogonal Conjugates

Proof. Let S, S' lie on (ABC) such that $AS' \parallel PQ$ and $AS \parallel P'Q'$; let Let R, R' lie on (ABC) such that $BC \parallel RS \parallel R'S'$. Then ([2], Delta 7.1) R and R' are the isogonal conjugates of the respective points of infinity along $P'Q'$ and PQ, and thus respectively lie on $(ABCPQ)$ and $(ABCP'Q')$.

By applying isogonal conjugation on Lemma 2.2, if RP meets $Q'R'$ at E and RQ meets $P'R'$ at F, then E, F lie on (ABC).

Now, we take cases on whether or not three of P, P', Q, Q' are collinear.

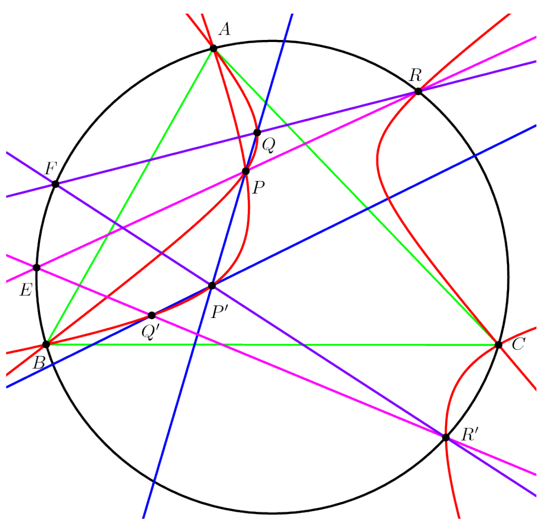

Figure 4: Three Collinear Points

If no three are collinear, then let $D = PQ \cap P'Q'$:

$$\measuredangle PEQ' = \measuredangle RER' = \measuredangle S'AS = \measuredangle PDQ',$$

so E lies on (DPQ'). Similarly, F lies on $(DP'Q)$. Let (DPQ') meet $(DP'Q)$ at $X \neq D$; then

$$\measuredangle EXF = \measuredangle EXD + \measuredangle DXF = \measuredangle EPD + \measuredangle DQF = \measuredangle RPQ + \measuredangle PQR = \measuredangle PRQ = \measuredangle ERQ,$$

so by ([3], Lemma 10.1), X lies on (ABC) as desired.

Now, if three of the four points are collinear, then WLOG assume P, P', Q are collinear; then $Q' \in (ABCPP')$, so Q cannot lie on PP'. Thus

$$\measuredangle PEQ' = \measuredangle RER' = \measuredangle S'AS = \measuredangle PP'Q',$$

implying that $PP'Q'E$ is cyclic. We also have

$$\measuredangle QFP' = \measuredangle RFR' = \measuredangle S'AS = \measuredangle QP'Q',$$

implying that $P'Q'$ is tangent to $(P'QF)$. Let $(PP'Q'E)$ meet $(P'QF)$ at X; then

$$\measuredangle XPQ = \measuredangle XQ'P', \quad XQP = \measuredangle XP'Q',$$

implying that X is the desired spiral center. Finally,

$$\measuredangle EXF = \measuredangle EXP' + \measuredangle P'XF = \measuredangle EPP' + \measuredangle P'QF = \measuredangle RPQ + \measuredangle PQR = \measuredangle ERF$$

as desired. ■

We may now proceed to prove the main theorem.

3 The Main Proof

Theorem 3.1 (Part (a))

In ABC with circumcircle Ω, let P, P' be isogonal conjugates. Let Q lie on PP'. Let Ω meet $(ABCPP')$ at $D \neq A, B, C$ and DP at $X \neq D$. Then $(PQX), (ABCPQ)$ are tangent at P.

Proof. Denote (ABC) by Ω. Let Q have isogonal conjugate Q'. Let M be the spiral center from PQ to $Q'P'$; by Lemma 2.3 $M \in \Omega$. By definition of spiral center, PQ' is tangent to (PQM), and $(PP'Q'M)$ is cyclic. Let (PQM) meet Ω at $X' \neq M$. By properties of Lemma 2.3, PX' passes through the intersection of Ω and $(ABCPP')$, which is precisely D, proving that $X \equiv X'$.

This implies that PQ' is tangent to (PQX), so it suffices to prove that PQ' is tangent to $(ABCPQ)$. To prove this, let $\mathcal{H}$ denote the circumconic of $ABCPP'Q'$ and $\mathcal{E}$ the circumconic of $ABCPQ$; then

$$\begin{aligned}
\mathcal{E}(BC; Q, PQ' \cap \mathcal{E}) &\overset{P}{=} \mathcal{H}(BC; P'Q') \\
&= (AB, AC; AP', AQ') \\
&= (AC, AB; AP, AQ) \\
&= (AB, AC; AQ, AP) \\
&= \mathcal{E}(BC; QP).
\end{aligned}$$

This is enough to imply that PQ' is tangent to $\mathcal{E}$, proving (a). □

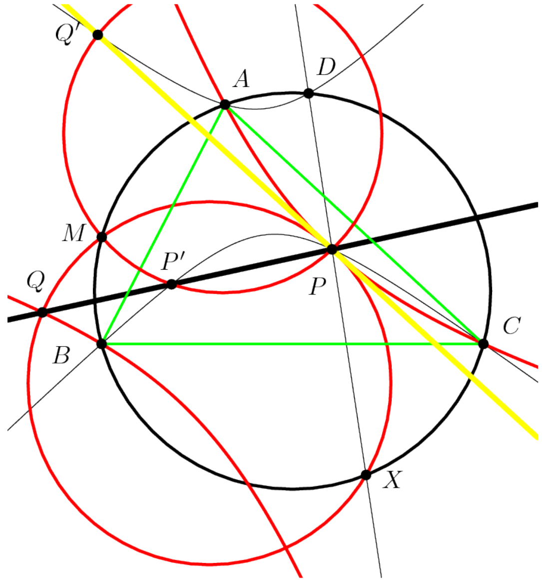

Figure 5: Proving part (a)

Theorem 3.2 (Part (b))

In ABC with circumcircle Ω, let P, P' be isogonal conjugates. Let Q lie on PP'. Let Ω meet $(ABCPP')$ at $D \neq A, B, C$. Let DP meet Ω at X. Let $(PQX), (ABCPQ)$ meet at $L \neq P, Q$. Then PL and $(ABCPX)$ intersect on the radical axis of Ω and (PQX).

Proof. Let $(PP'Q')$ meet Ω at $N \neq M$ and MD at E. By properties of Lemma 2.3, NQ' passes through the intersection of (ABC) and $(ABCPP')$, which is precisely D. Let Ω meet NP, NP', XQ, MP at G, F, F', K respectively. By Reim's Theorem, PP' is parallel to both FK and $F'K$, implying that $F \equiv F'$, i.e. QX, NP' concur at a point F on Ω.

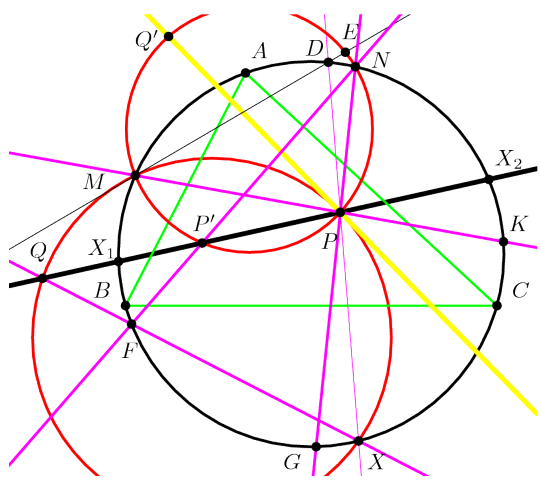

Figure 6: Proving part (b)

Let PP' meet Ω at (possibly complex) points X_1 and X_2, and MN at J. The key claim is that

$$(NQ, ND) \quad (NP, NP') \quad (NM, N\infty_{PP'}) \quad (NX_1, NX_2)$$

are pairs of an involution. To prove this, Desargues' Involution Theorem ([1], 125) on $NDFX$ yields reciprocal pairs

$$(Q, ND \cap PP') \quad (P, P') \quad (X_1, X_2)$$

and applying Desargues' Involution Theorem on $NMKF$ yields reciprocal pairs

$$(J, \infty_{PP'}) \quad (P, P') \quad (X_1, X_2).$$

Hence the aforementioned four reciprocal pairs indeed comprise a single involution. Let Ω meet NQ at R and $N\infty_{PP'}$ at S; then by projecting the involution from N, MS, DR, FG concur at a point T. Denote $(ABCPP'), (PP'Q'), (PP'DQ'E)$ by $\mathcal{H}, \gamma, \mathcal{H}'$ respectively; then

$$\begin{aligned}
\mathcal{H}(PP'; DQ') &\overset{A}{=} (AP', AP; A\infty_{PP'}, AQ) \\
&\overset{A}{=} (P'P; \infty_{PP'}Q) \\
&\overset{N}{=} \Omega(FG; SR) \\
&\overset{T}{=} \Omega(GF; MD) \\
&\overset{N}{=} \gamma(PP'; MQ') \\
&\overset{E}{=} \gamma(EP, EP'; EM, EQ') \\
&\overset{E}{=} \mathcal{H}'(PP'; DQ')
\end{aligned}$$

In other words, $\mathcal{H}(PP'; DQ') = \mathcal{H}'(PP'; DQ')$. Thus by [4], $E \in \mathcal{H}$.

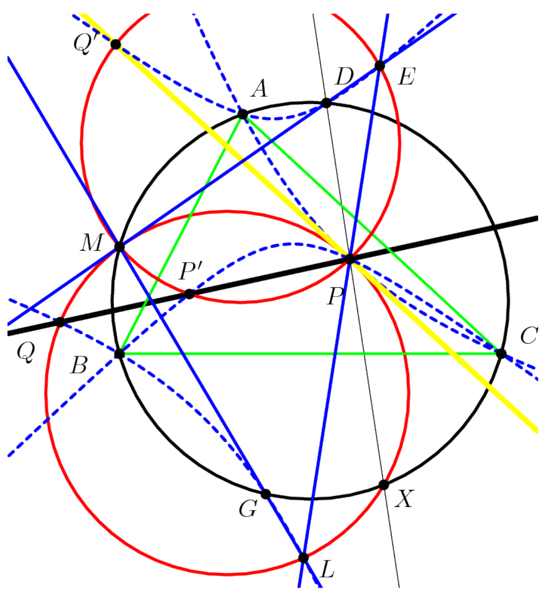

Figure 7: Second diagram for part (b)

Let MG meet (PQM) at $H \neq M$. By Reim ([5]), both PH and PE are parallel to DG, hence $P \in EH$. By properties of Lemma 2.3, NP passes through the intersection of Ω with $(ABCPQ)$, i.e. $G \in (ABCPQ)$. It is suddenly clear that $H \in (ABCPQ)$ by Fact 2.1, i.e. $H \equiv L$. Finally, by Fact 2.1 once again with line PE and Ω, the intersections of $(ABCPX)$ with PE and Ω are collinear with M - i.e., PE meets MX on $(ABCPX)$, completing (b).

□

The remainder of our paper will be dedicated to showing each of the aforementioned results.

4 Circles Tangent to Circumconics at Exactly Two Points

Having shown the main result in our paper, we will now consider a special case of Q.

Theorem 4.1

For ABC with circumcircle Ω and P with isogonal conjugate P', let $D = \Omega \cap (ABCPP')$. Let $X = DP \cap \Omega, Y = DP' \cap \Omega$. Let (XYZ) meet PP' at Z.

(a) (PXZ) and $(ABCPZ)$ are tangent at P and Z.

(b) Let (PXY) meet $(ABCPX)$ at Z_1, Z_2. Let Ω meet PZ_1, PZ_2 at Z_1', Z_2'. Then $Z_1'Z_2'$ is tangent to $(ABCPP')$ at P'.

To prove this, we once again start with a simple lemma.

Lemma 4.2

Two circles Ω, ω meet at a point X. Let A, B, C, D lie on ω such that $AB \parallel CD$. Let Ω meet XA, XB, XC, XD at E, F, G, H. Then $EF \parallel GH$.

Proof. $\measuredangle EFH = \measuredangle EXH = \measuredangle ACD = \measuredangle CDB = \measuredangle CXB = \measuredangle GHF$. ■

We proceed to prove part (a). Denote $(ABCPP'), (ABCPZ), (ABCPX), (PXY)$ by $\mathcal{H}, \mathcal{E}, \mathcal{C}_1, \gamma$ respectively.

Proof. By treating Z as Q as in Theorem 1.1, we deduce γ and $\mathcal{E}$ tangent at P; it suffices to prove the two tangent at Z.

Assume the contrary - that γ and $\mathcal{E}$ meet at a point $Z^* \neq P, Z$. By construction, Y is the spiral center from PZ to $Z'P'$, where Z' is the isogonal conjugate of Z. Then by Theorem 3.2, PZ^* and XY meet on $\mathcal{C}_1$. Let PZ meet $\mathcal{C}_1$ at I. We wish to show $I \in XY$. To prove this,

$$\mathcal{C}_1(BC; AI) \stackrel{P}{=} \mathcal{H}(BC; AP') \stackrel{D}{=} \Omega(BC; AY) \stackrel{X}{=} \mathcal{C}_1(B, C; A, XY \cap \mathcal{C}_1),$$

so I lies on XY. Thus I lies on both PZ and PZ^*, so we can characterize both Z and Z^* as the unique intersection of PI and $\mathcal{E}$ other than P. This is the desired contradiction. □

A corollary of (a) is that by Theorem 3.2, YZ passes through $\Omega \cap \mathcal{E}$, which we will denote as G. By Reim, we notably have $DG \parallel PZ$.

Before we proceed to part (b), we first state another lemma.

Lemma 4.3

For conic $\mathcal{C}$ containing two points A and B, let C, D vary on $\mathcal{C}$ such that $ABCD$ is cyclic. Then CD is parallel to a fixed line.

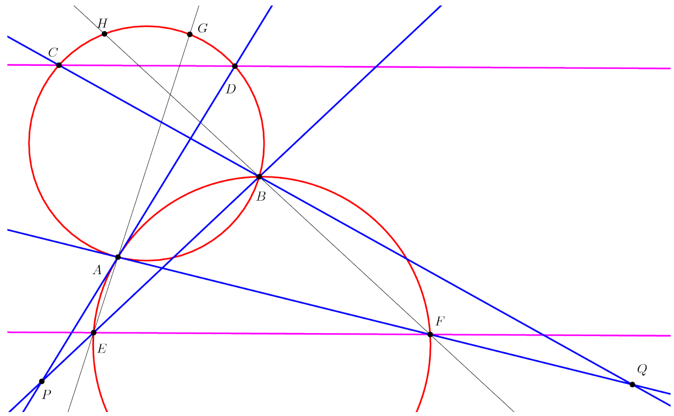

Figure 8: Circles Yielding a Family of Parallel Lines

Proof. It suffices to prove that, for two cyclic quadrilaterals $ABCD$ and $ABEF$ with $CD \parallel EF$, then $ABCDEF$ is circumscribed by a single conic. To prove this, let $(ABCD)$ meet AE, BF at G, H; AD meet BE at P; and AF meet BC at Q. Then $GH \parallel EF$ by Reim. From

$$\measuredangle APB = \measuredangle DAE + \measuredangle AEB = \measuredangle DAG + \measuredangle AFB = \measuredangle HBC + \measuredangle QFB = \measuredangle AQB,$$

follows $ABPQ$ cyclic, so $PQ \parallel CD$ by Reim. By converse Pascal ([6]) on $CDAFEB$, since $P, Q, CD \cap EF$ are collinear, $ABCDEF$ is circumscribed by a single conic as desired. ■

The above lemma implies the following:

Lemma 4.4

Let X_1, X_2 vary on $\mathcal{C}_1$ such that PXX_1X_2 is cyclic.
(a) $X_1X_2 \parallel PZ$.
(b) If Ω meets XX_1, XX_2 at X_1', X_2', then P', X_1', X_2' are collinear.

Proof. (a)

By Lemma 4.3, it suffices to verify this for one choice of X_1, X_2.

Let $G' = DG \cap \mathcal{E}$. Since $\mathcal{E}$ is tangent to (XPZ) at P and Z, both P and Z must be equidistant from the center of $\mathcal{E}$. Since $GG' \parallel PZ$, $PZGG'$ is an isosceles trapezoid. Let the tangent to $\mathcal{C}_1$ at P meet $\mathcal{E}$ at $G^* \neq P$; then

$$\mathcal{E}(BC; AG^*) \stackrel{P}{=} \mathcal{C}_1(BC; AP) \stackrel{X}{=} \Omega(BC; AD) \stackrel{G}{=} \mathcal{E}(BC; AG'),$$

implying that $G^* \equiv G'$, i.e. PG' is tangent to $\mathcal{C}_1$. Let PG' meet γ at Y'; then $YY' \parallel PZ$, so

$$\measuredangle Y'PI = \measuredangle PZY = \measuredangle PXY = \measuredangle PXI,$$

implying that PY' is tangent to (PXI), so (PXI) is tangent to $\mathcal{C}_1$ at P. With this tangency, by choosing $X_1 \equiv P, X_2 \equiv I$, we deduce that all X_1X_2 are parallel to PZ, as desired.

(b)

All such reciprocal pairs (X_1, X_2) comprise an involution, so projecting from X yields all reciprocal pairs (X_1', X_2') comprising a single involution. Therefore, it suffices to prove that P' lies on

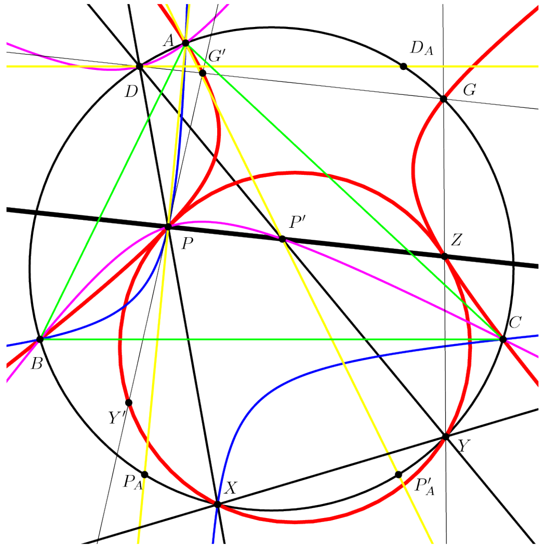

Figure 9: One Tangency Point Becomes Two

X_1', X_2' for two choices of the pair (X_1, X_2). Since D, P', Y are collinear, we already have one pair $(X_1, X_2) \equiv (P, I)$.

For our next pair, let D_A lie on Ω such that $AD_A \parallel PP'$; let Ω meet AP, AP' at AP_A, AP'_A. Then $BC \parallel DD_A \parallel P_AP'_A$. Let $\mathcal{C}_1$ meet AD_A at A_1 and XP'_A at A_1'. Then

$$\mathcal{C}_1(BC; PA_1) \stackrel{A}{=} \Omega(BC; P_AD_A) = \Omega(BC; DP'_A) \stackrel{X}{=} \mathcal{C}_1(BC; PA_1'),$$

so $A_1 \equiv A_1'$, implying that AD_A, XP'_A meet on $\mathcal{C}_1$. Since $AA_1 \parallel PP'$, by part (a),(AA_1PX) is cyclic. Since A, P', P'_A are collinear, our desired second pair is (A, P'_A). This completes this lemma. ∎

Now we may finally approach part (b) of the main result of this section.

Proof. By part (a) of Lemma 4.4, $PZ \parallel Z_1Z_2$; by Lemma 4.2, this implies that $DF \parallel Z_1'Z_2'$, where $F = XZ \cap \Omega$. By part (b) of Lemma 4.4, P', Z_1', Z_2' are collinear. Therefore, it suffices to show that the DF is parallel to the tangent to $\mathcal{H}$ at P'. Let $U = \Omega \cap PY$; denote $(ABCPU)$ by $\mathcal{C}_2$. Then

$$\mathcal{C}_2(BC; AP) \stackrel{U}{=} (BC; AY) \stackrel{D}{=} \mathcal{H}(BC; AP') \stackrel{P}{=} \mathcal{C}_2(B, C; A, PP' \cap \mathcal{C}_2),$$

so PP' is tangent to $\mathcal{C}_2$. Therefore, U is the isogonal conjugate of the point of infinity along the tangent to $(ABCPP')$ at P'. It suffices to prove that AU is isogonal to the line through A parallel to DF. Since AD is isogonal to the line through A parallel to PP', if we let $W = DF \cap PZ$, it suffices to prove $\measuredangle DWP = \measuredangle DAU$. This follows from

$$\measuredangle DWP = \measuredangle DPW + \measuredangle WDP = \measuredangle XPZ + \measuredangle FDX = \measuredangle XYZ + \measuredangle FYX = \measuredangle FYG = \measuredangle DAU$$

where the final step follows from $DG \parallel UF \parallel PZ$ by Reim's Theorem, as desired. □

With slight modification, we may phrase this fact as the following:

Theorem 4.5

In triangle ABC with circumcircle Ω, let $\mathcal{C}$ be a circumconic of ABC tangent to a circle γ at two points P, Q. Suppose PQ passes through the isogonal conjugate P' of P. Let P_0 lie on Ω such that $P'P_0$ is tangent to $(ABCPP')$. Then one of the intersections X of γ and Ω has the property that P_0X meets γ on $(ABCPX)$ at a point distinct from X.

Proof. We will first revisit the labelling in Theorem 3.2. For any point P with isogonal conjugate P' and point Q on PP', there is exactly one circle through P, Q tangent to $(ABCPQ)$. By Theorem 3.1, this circle passes through X (using the labelling of Theorem 3.2).

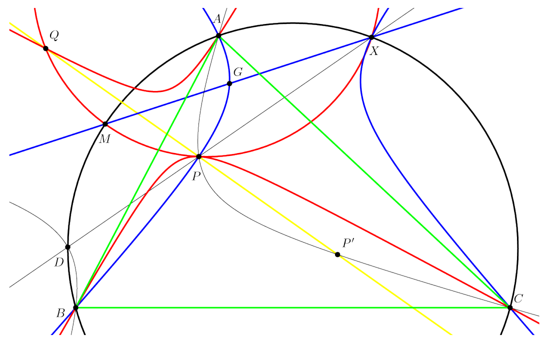

Figure 10: Proving Uniqueness

Now, by Theorem 3.2, we may redefine G to be the intersection of MX and $(ABCPX)$ other than X. Now, if (PQX) and $(ABCPQ)$ are not tangent at both P and Q, then their other intersection L lies on PG. In other words, if (PQX) and $(ABCPQ)$ are tangent at P and Q, then G must lie on PP', i.e. $G = PP' \cap (ABCPX)$. Such a point G is unique.

Note that X is fixed. Therefore, given any point G on $(ABCPX)$, we may reconstruct the corresponding Q as follows: we construct $M = XG \cap \Omega$ and $Q = (PMX) \cap PP'$ (if XG is tangent to Ω then we would set $M \equiv X$, and if PP' is tangent to (PMX) then we would set $Q \equiv P$. Now, when we consider the unique G lying on PP', there is exactly one corresponding Q. Therefore, there is exactly one $Q \in PP'$ for which $(ABCPQ)$ and (PQX) are tangent at both P and Q. This completes the proof. □

5 Applications to Extreme Points of Conics

We shall now show the third of our main results.

Theorem 5.1

Let ABC have isogonal conjugates P, P' with $D = (ABC) \cap (ABCPP')$. Let $D' \in (ABC)$ such that $DD' \parallel PP'$. Then the following are equivalent:

- either PP' is tangent to $(ABCD'P)$ or PP' passes through the center of $(ABCD'P)$
- P lies on either the major or minor axis of $(ABCD'P)$.

Proof. First, we revisit the configuration in Theorem 4.1. First, we may redefine $\gamma = (PXY)$, G to be the point on Ω for which $DG \parallel PP'$, and $\mathcal{E} = (ABCPG)$. Note that the two tangency points

P, Z of $\mathcal{E}$ and γ can be treated as two intersections, each of multiplicity two. Now, if P and Z coincide at a single point P, then our new definitions of $\mathcal{E}$ and γ would intersect at a single point P with multiplicity 4.

Now, for any given conic containing a given point P, there is a circle through P tangent to the conic with multiplicity 4 if and only if P is one of the extreme points of ABC. Furthermore, P and Z coincide if and only if γ is tangent to $\mathcal{E}$ with multiplicity 4, which implies that P is one of the extreme points of $\mathcal{E}$. On the other hand, P and Z coincide if and only if γ is tangent to PP' at P. By Reim ([5]), this occurs if and only if $YP \cap \Omega$ lies on the line through D parallel to PP', i.e. Y, P, G are collinear. This occurs if and only if DP' and GP meet on Ω, which using the labelling in the current configuration, is equivalent to DP' meeting $D'P$ on Ω.

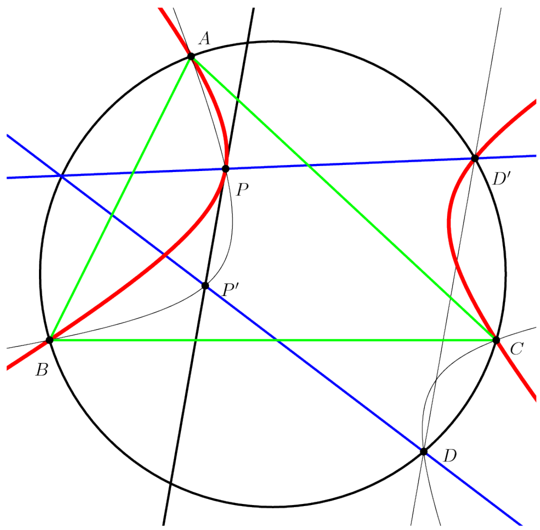

Figure 11: Case 1 - Multiplicity 4

Now, we claim that PP' is tangent to $(ABCD'P)$ if and only if DP' and $D'P$ intersect on (ABC). Denote $(ABC), (ABCPP'), (ABCD'P)$ by $\Omega, \mathcal{H}, \mathcal{C}$ respectively. Then

$$\mathcal{C}(BC; AP) \stackrel{D'}{=} \Omega(BC; A, D'P \cap \Omega), \quad \mathcal{C}(BC; A, PP' \cap \mathcal{C}) \stackrel{P}{=} \mathcal{H}(BC; AP') \stackrel{D}{=} \Omega(BC; A, DP' \cap \Omega)$$

so PP' is indeed tangent to $(ABCD'P)$ if and only if $D'P \cap DP' \in \Omega$. By the above, this occurs if and only if the corresponding γ is tangent to $\mathcal{C}$ with multiplicity 4, which would imply that P is one of the extremes of $\mathcal{C}$.

Now, if PP' passes through the center of $\mathcal{C}$, then P cannot be the center of $\mathcal{C}$ (since $P \in \mathcal{C}$), so P and the corresponding Z are distinct. Since the two tangency points of γ and $\mathcal{C}$ are collinear with the center of $\mathcal{C}$, both P and Z are extremes of $\mathcal{C}$. Conversely, if P and Z are both extremes of $\mathcal{C}$, then PP' must contain the center of $\mathcal{C}$, so PP' contains the center of $\mathcal{C}$ if and only if P and the corresponding Z are two distinct extremes of $\mathcal{C}$.

Finally, we note that if P is one of the extremes of $\mathcal{C}$, then either the corresponding γ to P is tangent to $\mathcal{C}$ at two extreme points of $\mathcal{C}$, or the corresponding γ is tangent to $\mathcal{C}$ with multiplicity 4. This completes the if and only if condition, completing the proof. $\square$

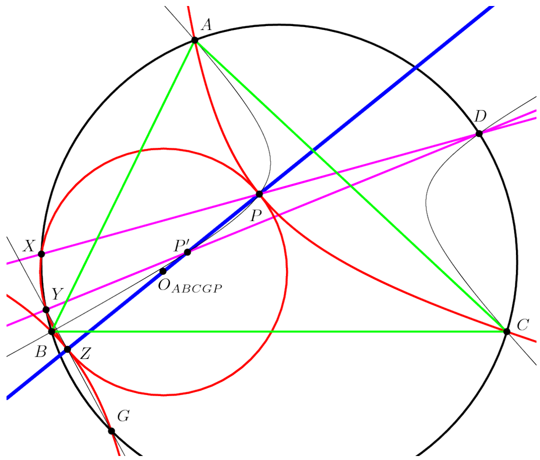

Figure 12: Case 2 - Both Tangency Points are Extreme Points

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