Circumcenter of the Reflection Triangle

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Abstract

We investigate properties relating the circumcenter and Kosnita point of a triangle ABC with its reflection triangle $A'B'C'$. We then extend these results to a fact relating the reflection triangle of the intouch triangle, the incenter, and the Euler line of a triangle ABC.

1 Definitions

The following are the nontrivial definitions we will use throughout this paper.

- The **Kosnita Point** ([1]) of a triangle ABC is the isogonal conjugate of the nine-point center of ABC. It is alternatively defined as the concurrence of AO_A, BO_B, CO_C where O_A is the circumcenter of BOC and O is the circumcenter of ABC; O_B, O_C are defined similarly.
- The **Gergonne Point** of a triangle ABC is the concurrence of AA_1, BB_1, CC_1 where A_1 is the point of tangency of the incircle and BC, and B_1, C_1 are defined similarly.
- The **Nagel Point** of a triangle ABC is the concurrence of AA_2, BB_2, CC_2 where A_2 is the point of tangency of the excircle and BC, and B_2, C_2 are defined similarly.
- The **Jerabek Hyperbola** ([3]) of a triangle ABC is the conic that passes through the vertices, orthocenter, and incenter of a triangle ABC.

2 Main Result

All angles are directed mod 180°.

Theorem 2.1
In triangle ABC, let A' be the reflection of A over BC; define B' and C' similarly. Let $ABC, A'B'C'$ have circumcenters O, O'. Then the midpoint of OO' is the Kosnita point K of ABC.

Proof. Let T_A be the intersection of the tangents to (ABC) at B and C. Let the nine-point center N of ABC have pedal triangle $A_NB_NC_N$.

Lemma 2.2

$\triangle T_A BC' \overset{+}{\cong} \triangle T_A CB'$

Proof.

$$\angle T_A BC' = \angle BAC + 2\angle CBA = \angle CAB + 2\angle BCA = \angle T_A CB',$$

combined with $T_A B = T_A C$ and $BC' = BC = B'C$ yields the desired result. ∎

Lemma 2.3

$B'C' \parallel B_N C_N$

Proof. The parallels from A, B, C to BC, CA, AB form a triangle $A_1B_1C_1$ homothetic to ABC. Lines $A'A_1, B'B_1, C'C_1$ form a triangle $A_2B_2C_2$ homothetic to ABC. By construction, ABC is the medial triangle of and thus has the same centroid G as $A_1B_1C_1$; similarly, $A_1B_1C_1$ shares a centroid with $A_2B_2C_2$.

Therefore, ABC and $A_2B_2C_2$ share the same centroid G, with $A_2B_2C_2$ being the image of ABC under homothety at G with ratio $+4$. In this case, $A'B'C'$ is the pedal triangle of the orthocenter H of ABC with respect to $A_2B_2C_2$. Note that H is the image of N under homothety at G with ratio $+4$; thus, $B'C' \parallel B_N C_N$ as desired. ∎

Recall that K is the isogonal conjugate of N, and is collinear with A and the midpoint O_A of OT_A, so $B_N C_N \perp AO_A$. If we analogously define T_B, T_C, O_B, O_C, we conclude O' lies on lines through T_A, T_B, T_C parallel to AO_A, AO_B, AO_C. Since O_A, O_B, O_C are the midpoints of OT_A, OT_B, OT_C, by homothety ([2], Lemma 3.10) we conclude that K is the midpoint of OO' as desired.

□

3 Additional Results

Now, we aim to apply this fact to prove two beautiful results.

Theorem 3.1 (Fact 1)

Using the same point labelling, $(AB'C'), (A'BC'), (A'B'C)$ concur on OO'.

Proof. Let AO_A meet $A'O$ at D; define E and F analogously.

Lemma 3.2

D is the inverse of A' in (ABC).

Proof. It suffices to show that the inverse of O_A in (ABC) lies on the circumcircle of $AA'O$. But this just follows because the reflection of O over BC is the inverse of O_A in (ABC). ∎

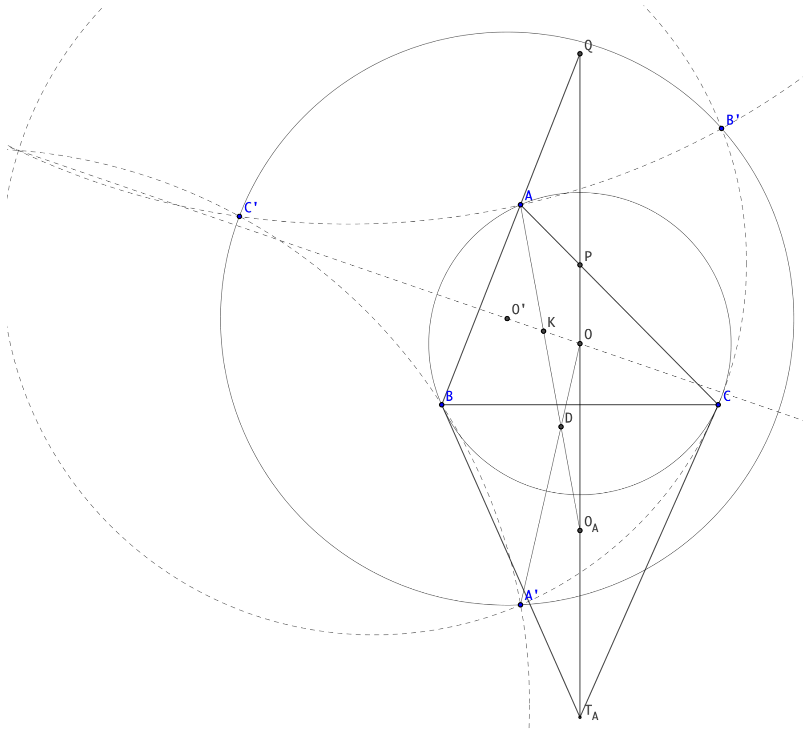

Figure 1: Fact 1

Lemma 3.3

$AEFK$ is cyclic.

Proof. Let OT_A meet AC at P and AB at Q. Let O_C' be the reflection of O over AB. Then

$$\angle AQO = 90^\circ - \angle CBA = \angle ACO,$$

so $(ACOQ)$ is cyclic, thus P and Q are inverses in (ABC), and $(AC'O_C'Q)$ is cyclic as well. By inversion around (ABC), this implies that $APFO_C$ is cyclic; similarly $AQEO_B$ is cyclic.

Notice that $\angle TOP = \angle ABC = \angle TAP$, so $AOPT_C$ is cyclic. Thus,

$$\angle AO_CO_B = \angle O_BO_CO = \angle AT_CO = \angle APQ,$$

and similarly $\angle AO_BO_C = \angle AQP$, implying $\triangle AO_CO_B \overset{+}{\sim} \triangle APQ$, so by spiral similarity $\triangle AO_CP \overset{+}{\sim} \triangle AO_BQ$. Thus $\angle AFK = \angle APO_C = \angle AQO_B = \angle AEK$ as desired. ■

Finish:

Since $BFDK$ and $CDEK$ are cyclic, inversion at (ABC) implies $(AB'C'), (A'BC'), (A'B'C)$ concur at the inverse of K in (ABC), which indeed lies on OO' as desired. □

Up until now, the most advanced tools we have used are isogonal conjugate properties and inversion. The next fact will require some knowledge in projective geometry.

Theorem 3.4 (Fact 2)

Let ABC have incenter I, circumcenter O, and orthocenter H. The incircle meets BC, CA, AB at D, E, F. The reflections of D, E, F over EF, FD, DE are D', E', F'. The circumcenter of $D'E'F'$ is X. Then $IX \parallel OH$.

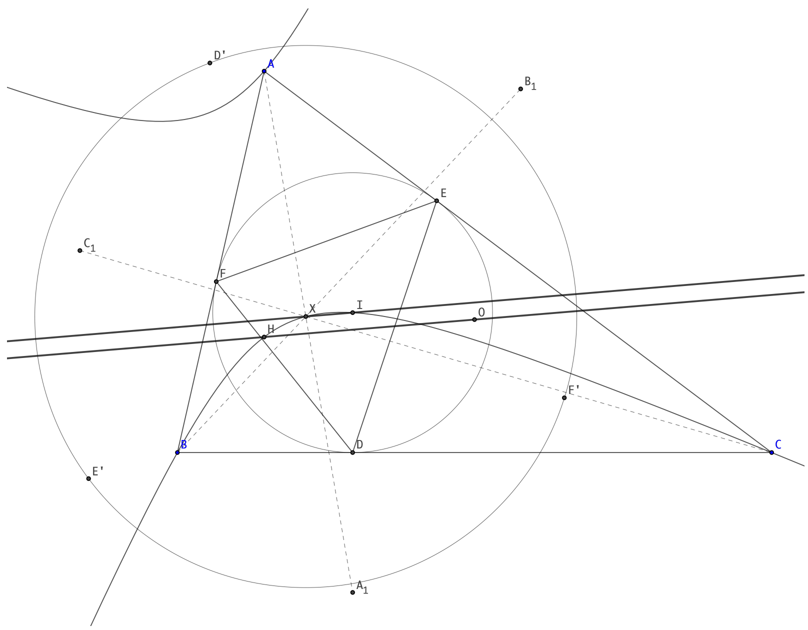

Figure 2: Fact 2

Proof. Let A_1, B_1, C_1 be the reflections of I over BC, CA, AB. Let AI, BI, CI have midpoints M_A, M_B, M_C. We know that X is the reflection of I over the Kosnita point K of DEF. Since K lies on DM_A, EM_B, FM_C, by homothety at I, X must lie on AA_1, BB_1, CC_1.

Let ABC have Gergonne point Ge, Nagel point Na, centroid G, and Jerabek hyperbola $\mathcal{H}$. Let AA_1 meet $\mathcal{H}$ at $X' \neq A$; let ∞_{AH} be the point of infinity on AH. Since D is the midpoint of A_1I ([4], Delta 10.2),

$$(IX'; HD) \stackrel{A}{=} (IA_1; \infty_{AH} D) = -1.$$

Thus X' is the unique point on $\mathcal{H}$ for which $(IX'; HD) = -1$, so X' lies on BB_1 and CC_1 as well, implying that $X \equiv X'$. Let OH meet IX at P and IGe at L; since IO is tangent to $\mathcal{H}$,

$$(PO; HL) \stackrel{I}{=} (XI; HGe) = -1.$$

Let the incircle and circumcircle of ABC have insimilicenter Y and exsimilicenter Z; these are harmonic conjugates with respect to O and H. Since INa passes through G, and Y, Z are respectively the isogonal conjugates of Na, Ge in ABC,

$$-1 = (AZ, AY; AH, AO) = (AGe, ANa; AO, AH) = (IGe, INa; IO, IH) = (LG; OH),$$

so O is in fact the midpoint of HL. Earlier we deduced $(PO; HL) = -1$; this implies that P is the point of infinity along OH, as desired. $\square$

4 Acknowledgements

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References

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