

Take Home Computer Exam Bayesian Learning (732A91/TDDE07/732A73), 6 hp

Time:	14-18
Allowable material:	All aids are permitted during the exam with the following exceptions: <ul style="list-style-type: none">• You may not communicate with anyone else except the responsible teacher• You may not look at the solutions of any other students.
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Exam scores:	Maximum number of credits on the exam: 40. Maximum number of credits on each exam question: 10.
Grades (732A91):	A: 36 points B: 32 points C: 24 points D: 20 points E: 16 points F: <16 points
Grades (TDDE07):	5: 34 points 4: 26 points 3: 18 points U: <18 points

INSTRUCTIONS:

For full information, see the document *Info take home exam.pdf*.

You should submit your solutions via the Submission in LISAM.

Your submitted solutions must contain the following two things:

- A PDF named *ComputerSol.pdf* containing your computer-based solutions
- Files named *PaperSol* containing your hand-written solutions (when asked to answer on [Paper](#))

Full score requires clear and well motivated answers.

1. PATIENTS

The number of visiting patients at a health center in a given month is assumed to be Poisson distributed with unknown parameter θ . A Bayesian analysis has resulted in a posterior distribution for θ based on historical data D such that $\theta|D \sim N(10000, 500^2)$. If simulation is used in your solutions, use at least 10000 samples.

- (a) *Credits: 2p.* Compute a 90% equal-tail credible interval for θ .
- (b) *Credits: 3p.* Compute a 90% equal-tail prediction interval for the number of patients in the next month.
- (c) *Credits: 5p.* The health centre is considering hiring more doctors to increase its capacity. Currently there are 20 doctors employed. Each doctor can treat 500 patients each month, so the current capacity is 10000 patients. If the number of patients exceeds the capacity in a given month, the health centre has to pay overtime salary which is expensive. The health centre has the following loss function

$$L(p, d) = (d + 0.04\max(p - 500d, 0))^2,$$

where d is the number of doctors employed, p is the number of patients in a month, and $\max(a, b)$ is the function that returns the largest value of a and b . Should the health centre hire more doctors, and if so, how many, according to the Bayesian decision? Assume that all doctors work full-time.

2. REGRESSION

The dataset `muscle` which is loaded by the code in `ExamData.R` contains experimental data on strips of heart tissue from rats. Each row of data contains the shortening of a strip (stored in the variable `Length`) after being dipped in a calcium chloride solution of a certain concentration (stored in the variable `Conc`). Type `?muscle` for more information about this data. Consider the following two linear regression models

$$M_1 : \text{Length} = \beta_1 \cdot \text{Conc} + \varepsilon$$

$$M_2 : \text{Length} = \beta_0 + \beta_1 \cdot \text{Conc} + \beta_2 \cdot \text{Conc}^2 + \varepsilon,$$

For each model ε is iid. Gaussian measurement noise with variance σ^2 , and $\beta_0, \beta_1, \beta_2$ and σ^2 are unknown parameters. For each model, use the function `BayesLinReg` to sample from the joint posterior distribution of the parameters, using 5000 draws. Analyze the dataset by simulating 5000 draws from the joint posterior. Use the prior with μ_0 as a zero vector, $\Omega_0 = 0.001 \cdot I$, $\nu_0 = 1$ and $\sigma_0^2 = 10$. [Hint: The input `X` to `BayesLinReg` must be a matrix, you can use `as.matrix()`.]

- (a) *Credits: 2p.* Compute the 95% equal tail credible intervals for all β -parameters for both models.
- (b) *Credits: 5p.* Produce two figures, one for each model, with a scatter plot of the data and overlay a curve for the posterior predictive mean for values of `Conc` on the grid given by `seq(0,5,0.01)`. Also overlay curves representing the 95% posterior prediction bands for the different values of `Conc`.
- (c) *Credits: 3p.* On [Paper](#), briefly discuss advantages and disadvantages with the two models for this dataset. In particular, discuss which model gives the best `Length` predictions for new measurements at previously unobserved `Conc` values and whether the models can be used for drawing conclusions about whether larger `Conc` values leads to larger `Length` values.

3. DELIVERY TIMES

The file `delivery` which is loaded by the code in `ExamData.R` contains data on delivery times (in days) for products from an online sales company. The data x_1, \dots, x_n are assumed to be independent and Weibull distributed with parameters k and λ according to the following density function

$$p(x_i|k, \lambda) = \frac{k}{\lambda} x_i^{k-1} \exp(-x_i^k/\lambda), \quad x_i > 0, k > 0, \lambda > 0.$$

Note that this parameterisation is different than the one in R and in the table of distributions. It is further assumed that the value of k is known and that the prior for λ is Inverse-Gamma(α, β).

- (a) *Credits: 3p.* On [Paper](#), show that the posterior for λ is Inverse-Gamma($\alpha + n, \beta + \sum_{i=1}^n x_i^k$).
- (b) *Credits: 3p.* Assume that $\alpha = 2$ and $\beta = 2$. Consider the following three values for k : 0.5, 1.5 and 2.5. For each value of k , draw samples from the posterior of λ and use the samples to compute the posterior mode of λ using the `density()` function. [Hint: To sample from the Inverse-Gamma distribution, one can use that if $Y \sim \text{Gamma}(a, b)$ then $1/Y \sim \text{Inverse-Gamma}(a, b)$.]
- (c) *Credits: 4p.* Make a figure that contains a histogram of the data and three Weibull densities for the data based on the three values for k and the corresponding posterior modes for λ . Based on the figure, which value for k would you say gives the best fit to the data? Motivate your answer.

4. MORE DELIVERY TIMES

Consider the same data and model as in Assignment 3. That is, the file `delivery` which is loaded by the code in `ExamData.R` contains data on delivery times (in days) for products from an online sales company. The data x_1, \dots, x_n are assumed to be independent and Weibull distributed with parameters k and λ according to the following density function

$$p(x_i|k, \lambda) = \frac{k}{\lambda} x_i^{k-1} \exp(-x_i^k/\lambda), \quad x_i > 0, k > 0, \lambda > 0.$$

Note that this parameterisation is different than the one in R and in the table of distributions. It is further assumed that the value of k is known and that the prior for λ is $\lambda \sim \text{Inverse-Gamma}(\alpha, \beta)$.

- (a) *Credits: 4p.* On [Paper](#), show that the marginal likelihood of the data can be expressed as

$$p(x_1, \dots, x_n) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \frac{\beta^\alpha k^n \prod_{i=1}^n x_i^{k-1}}{(\beta + \sum_{i=1}^n x_i^k)^{\alpha+n}}.$$

- (b) *Credits: 4p.* Now, instead view k as an unknown parameter, and assume the prior $k \sim \text{Exp}(1)$. Write a function in R that computes the log unnormalized posterior distribution of k . Use that function to plot the normalized posterior distribution of k on the interval (0,3). Assume that $\alpha = 2$ and $\beta = 2$.
- (c) *Credits: 2p.* Compute a 95% equal-tail credible interval for k based on the posterior in (b).

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