Problem 3

a) $X_6 \mid X_{1:5} \sim N(\mu_n, \sigma^2 + \tau_n^2)$, $\sigma^2 = 25^2$

$n = 5$, $\bar{x} = 320.4$, $\sigma^2 = 25^2$, $\mu_0 = 200$, $\tau_0^2 = 50^2$

$\Rightarrow X_6 \mid X_{1:5} \sim N\left(\mu_n = 315,\ \sigma^2 + \tau_n^2 = 25^2 + 119 = 27.3^2\right)$

$w = 0.95$

b) $U(\text{no campaign}) = U\Big(\underbrace{(\overset{10}{p} - \overset{5}{Q})X_6}_{Y_1}\Big) = 1 - \exp\Big(\underbrace{-5 \cdot X_6 / 1000}_{S_1}\Big)$

$U(\text{campaign}) = U\Big(\underbrace{1.2 \cdot X_6 \cdot (p - Q) - 300}_{Y_2}\Big) = 1 - \exp\Big[\underbrace{-(6X_6 - 300)/1000}_{S_2}\Big]$

$E[\exp(S_1) \mid X_{1:5}] = \exp\Big[\underbrace{-5 \cdot \mu_n / 1000}_{\mu_1 = -1.575} + 0.5 \cdot \underbrace{\left(5 \cdot \sqrt{\tau_n^2 + \sigma^2} / 1000\right)^2}_{\sigma_1^2 = 0.1365^2}\Big]$

$= 0.2089$

$E[\exp(S_2) \mid X_{1:5}] = \exp\Big[\underbrace{-(6 \cdot \mu_n - 300)/1000}_{\mu_2 = -1.59} + 0.5 \cdot \underbrace{\left(6 \cdot \sqrt{\tau_n^2 + \sigma^2} / 1000\right)^2}_{\sigma_2^2 = 0.1638^2}\Big]$

$= 0.2067$

$\Rightarrow E[U(Y_1) \mid X_{1:5}] = 1 - 0.2089 = 0.7911$

$\wedge$

$E[U(Y_2) \mid X_{1:5}] = 1 - 0.2067 = 0.7933$

choose the campaign because $E[U(Y_2) \mid X_{1:5}] > E[U(Y_1) \mid X_{1:5}]$

Problem 4

Lecture 2 slide 15
$$\begin{cases} y_1, \dots, y_{10} \mid \theta \sim \text{Pois}(\theta) & ; \; n=10, \; \sum_{i=1}^{n} y_i = 238 \\ \theta \sim \text{Gamma}(\alpha=0, \beta=0) \\ \theta \mid y_1, \dots, y_{10} \sim \text{Gamma}(238, 10) \end{cases}$$

Normal approximation:

- $$E[y_{11} \mid y_{1:10}] = E\left[E[y_{11} \mid \theta, y_{1:10}] \mid y_{1:10}\right] = E[\theta \mid y_{1:10}] = \frac{238}{10} = 23.8$$
- $$\begin{aligned} Var[y_{11} \mid y_{1:10}] &= E\left[Var[y_{11} \mid \theta, y_{1:10}] \mid y_{1:10}\right] + Var\left[E[y_{11} \mid \theta, y_{1:10}] \mid y_{1:10}\right] \\ &= E[\theta \mid y_{1:10}] + Var[\theta \mid y_{1:10}] \\ &= 23.8 + \frac{238}{10^2} = 26.18 \end{aligned}$$

So,
$$y_{11} \mid y_{1:10} \approx N(23.8, 26.18)$$

- 95% posterior predictive interval for y_{11}:

$$23.8 \pm 1.96 \cdot \sqrt{26.18} \implies 13.77 < y_{11} < 33.83$$

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