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Bayesian Learning
Mathematical Exercises 3

Try to solve the problems before class and don't worry if you fail. The important thing is to try. You should not hand in any solutions.
This part of the course is not obligatory and is not graded.

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1. Gibbs step for an indicator variable
   - Show that the full conditional posterior of I_i on Lecture 7, Slide 21, is correct.

2. Frequentist vs Bayesian
   (a) Let $x_1, \ldots, x_n|\theta \sim Uniform\left(\theta - \frac{1}{2}, \theta + \frac{1}{2}\right)$. Let $\hat{\theta} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ be an estimator of θ. Derive an expression for the (repeated) sampling variance of $\hat{\theta}$.
   (b) Derive the posterior distribution for θ assuming a uniform prior distribution. [Hint: Here it is absolutely crucial to think about the support for the data distribution. Once you have observed some data, some values are no longer possible. I strongly suggest that you plot some imaginary data on the real line and plot the data distribution in the same graph for some made up values of θ. Just to make you think in the right direction.]
   (c) Assume that you have observed three data observations: $x_1 = 1.13$, $x_2 = 2.1$, $x_3 = 1.38$. What do we conclude from a frequentist perspective about θ? What do we conclude from a Bayesian perspective about θ? Discuss.

3. the posterior becomes more normal
   (a) Let $x_1, \ldots, x_n|\theta \overset{iid}{\sim} Bern\,(\theta)$ and $\theta \sim Beta\,(\alpha, \beta)$ a priori. Find the posterior mode of θ.
   (b) Approximate the posterior distribution of θ by a normal distribution.
   (c) Assume now that you have the data $n = 8$ and $s = 2$. Plot the true posterior distribution and the normal approximation in the same graph. Assume a uniform prior for θ.
   (d) Redo the previous exercise, but this time with twice the data size: $n = 16$ and $s = 4$. What do you conclude?

Good luck!

Best, Bertil