TTK4115 – Linear System Theory

Helicopter lab preparation

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3.7, August 2007, JM
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3.0, June 2001, TAJ

This document contains derivations and other preparatory work for the helicopter-lab. The preparation should be done individually and delivered on black-board before your lab-day. These tasks are not graded, but all tasks must be done to pass the course and to manage the lab. Discussing the task and helping each other is encouraged, as long as everyone understands the task and delivers their own work.

Your time at the lab is limited, and the lab is run at full capacity. You will not have enough time to complete the lab if you do not do the preparations ahead of time. Your time at the lab should not be wasted doing calculations and other things that could have been done ahead of time!

The derivations do not need to be included in the lab-report, unless they are relevant for the discussion. In which case, it is important to include all equations and derivations to support the discussion.

1 Part I – Monovariable control

The helicopter is modeled as three point-masses: two point-masses represent the two motors that are connected to the propellers, and one point mass represents the counterweight. The model of the helicopter is depicted in Fig. 1. The cubes in the figure represent the point masses whilst the cylinders represent the helicopter joints. The rotations of the joints are defined in the following manner: p denotes the pitch angle of the helicopter head, e denotes its elevation angle, and λ denotes the travel angle of the helicopter. In Fig. 1, all joint angles are zero. This means that $p = 0$ if the helicopter head is horizontal, and that $e = 0$ if the arm between the elevation axis and the helicopter head is horizontal.

The propeller forces of the front and back propeller are given by F_f and F_b, respectively. It is assumed that there is a linear relation between the voltages V_f and V_b supplied to the motors and the forces generated by the propellers:

$$F_f = K_f V_f \tag{1a}$$
$$F_b = K_f V_b \tag{1b}$$

Here, K_f is a motor force constant. The two propellers are placed symmetrically about the pitch axis. The sum of the forces $F_b + F_f$ developed by the two propellers determines the net lift of the helicopter. The difference between the forces $F_b - F_f$ is proportional to the torque about the pitch axis.

The gravitational forces for the front and the back motor are denoted by $F_{g,f}$ and $F_{g,b}$, while the gravitational force of the counterweight is denoted by $F_{g,c}$. Note that the gravitational forces always point in a vertical direction, whilst the direction of the propeller forces is dependent on the joint angles.

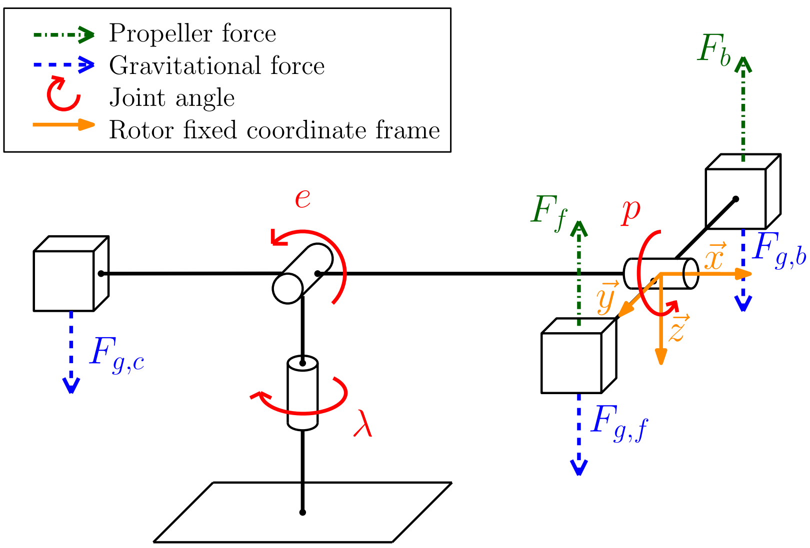

Fig. 1: Helicopter model

The masses of the front and the back motors are assumed to be equal and given by m_p. The mass of the counterweight is given by m_c. The distance from the elevation axis to the head of the helicopter is given by l_h, whilst the distance from the elevation axis to the counterweight is given by l_c. Because the two propellers are placed symmetrically about the pitch axis, the distance from the pitch axis to both motors is the same and is given by l_p. The masses and distances are depicted in Fig. 2. Other forces, such as friction and centripetal forces, are neglected.

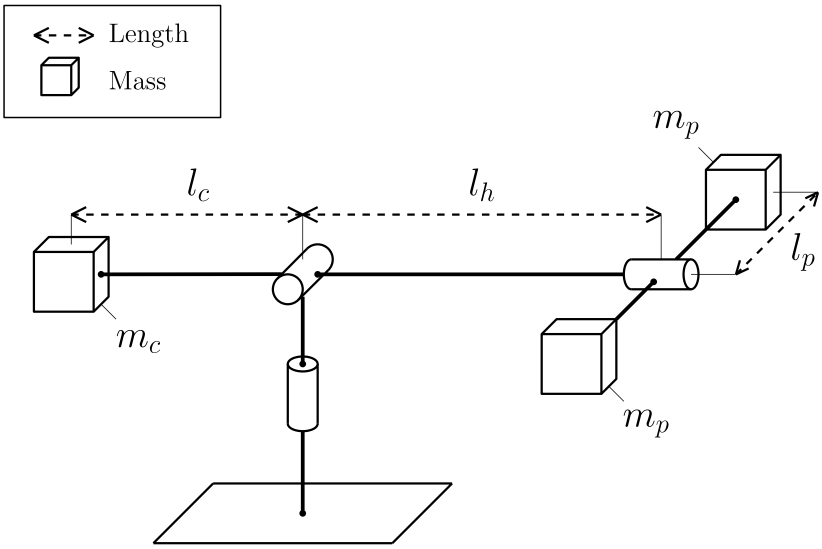

Fig. 2: Masses and distances

Note: The actual values for the masses and distances differ from helicopter to helicopter and are therefore not given here. These values can be found in an initialization file that can be downloaded from "Blackboard". Due to the mass disparities, some helicopters may be more docile than others. The motor force constant K_f also differs quite a bit from helicopter to helicopter. This constant is to be determined experimentally at the lab.

1.1 Problem 1 - Equations of motion

Compute the equations of motion (differential equations) for the pitch angle p, the elevation angle e, and the travel angle λ. Let the moments of inertia about the pitch, elevation, and the travel axes be denoted by J_p, J_e, and J_λ, respectively. Show that the equations of motion can be stated in the form

$$J_p \ddot{p} = L_1 V_d \tag{2a}$$

$$J_e \ddot{e} = L_2 \cos(e) + L_3 V_s \cos(p) \tag{2b}$$

$$J_\lambda \ddot{\lambda} = L_4 V_s \cos(e) \sin(p) \tag{2c}$$

where L_i, $i = 1, 2, 3, 4$ are constants, and where the sum of the motor voltages V_s and the difference of the motor voltages V_d are given by

$$V_s = V_b + V_f \tag{3a}$$

$$V_d = V_b - V_f \tag{3b}$$

1.2 Problem 2 - Linearization

Linearize the system (2a)-(2c) around the equilibrium point shown in Fig. 1. Note that a constant voltage, $V_{s,0}$, is required to keep the helicopter stationary at this equilibrium point. As we want to work with origo as our linearization point, do the following variable transformation: $\tilde{V}_s = V_s - V_{s,0}$. $V_{s,0}$ should be chosen such that $\tilde{V}_s = 0$ results in $e = 0$ being an equilibrium point of the system.

Assume that the moments of inertia are constant and given by

$$J_p = 2m_p l_p^2 \tag{4a}$$

$$J_e = m_c l_c^2 + 2m_p l_h^2 \tag{4b}$$

$$J_\lambda = m_c l_c^2 + 2m_p(l_h^2 + l_p^2) \tag{4c}$$

Show that the linearized equations of motion can be written as

$$\ddot{p} = K_1 V_d \tag{5a}$$

$$\ddot{e} = K_2 \tilde{V}_s \tag{5b}$$

$$\ddot{\lambda} = K_3 p \tag{5c}$$

where K_i, $i = 1, 2, 3$ are constants.

1.3 Problem 3 - PD Control

A PD controller is to be implemented to control the pitch angle, p. This controller is given as

$$V_d = K_{pp}(p_c - p) - K_{pd}\dot{p} \tag{6}$$

where p_c is the desired reference for the pitch angle p. Substitute (6) in the equation of motion for the pitch angle in (5a). Apply the Laplace transform to the resulting differential equation to find the transfer function $G(s) = (p/p_c)(s)$.

1.4 Problem 4 - Pole Placement

We wish to experiment with different (complex) pole values for the closed loop pitch-system. Develop an expression for K_{pp} and K_{pd} as a function of the desired poles λ_1 and λ_2.

2 Part II – Multivariable control

2.1 Problem 1 - State space formulation

Put the system of equations given by the relations for pitch and elevation in (5a)-(5b) in a state-space formulation of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \tag{7}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices of suitable dimension. The state vector and the input vector are defined by

$$\boldsymbol{x} = \begin{bmatrix} p \\ \dot{p} \\ \dot{e} \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{u} = \begin{bmatrix} \tilde{V}_s \\ V_\mathrm{d} \end{bmatrix} \tag{8}$$

2.2 Problem 2 - Controllability

Examine the controllability of the system.

2.3 Problem 3 - Feedback and feedforward

We aim to track the reference $\mathbf{r} = [p_c,\ \dot{e}_c]^T$ for the pitch angle p and elevation rate $\dot{e}$, which will be given by the joystick output.

Consider a state-feedback controller with reference-feed-forward of the following form:

$$\boldsymbol{u} = \boldsymbol{F}\boldsymbol{r} - \boldsymbol{K}\boldsymbol{x} \tag{9}$$

where

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix} \tag{10}$$

Derive an expression for matrix $\boldsymbol{F}$ as a function of $k_{11}\ k_{12}\ k_{13}\ k_{21}\ k_{22}\ k_{23}$ such that (in theory) $\lim_{t\to\infty} p(t) = p_c$ and $\lim_{t\to\infty} \dot{e}(t) = \dot{e}_c$ for fixed values of p_c and $\dot{e}_c$.

2.4 Integral action

Modify the controller from the previous problems to include an integral effect for the elevation rate and the pitch angle. Note that this results in two additional states γ and ζ, for which the differential equations are given by

$$\begin{aligned} \dot{\gamma} &= p - p_c \\ \dot{\zeta} &= \dot{e} - \dot{e}_c \end{aligned} \tag{11}$$

The augmented state vector shown below should be used.

$$\boldsymbol{x} = \begin{bmatrix} p \\ \dot{p} \\ \dot{e} \\ \gamma \\ \zeta \end{bmatrix} \tag{12}$$

The system should take on the following form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{G}\boldsymbol{r} \tag{13}$$

Find the augmented $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{G}$, and $\boldsymbol{F}$ matrices.

3 Part III – Luenberger observer

In the previous tasks we have used the encoder measurements as a direct measurement of the state which we have used for feedback. Unfortunately encoders require the helicopter to be stuck to ground such that the angles of the joints can be measured. As we aspire towards controlling a free-flying helicopter one day, we have to use a different technology than encoders to measure the state of the system. An inertial measurement unit (IMU) called MPU-9250 produced by InvenSense will instead be used. This is a cheap IMU that is often used in hobby projects and light-weight drones.

The IMU is mounted in the middle of the bar connecting the rotors. The IMU will measure acceleration and rotations in its local frame that rotates when the helicopter rotates, this frame is shown in orange in figure Fig. 1. This frame is defined such that the $\vec{y}$ axis always aligns with the bar between the rotors, and the $\vec{x}$ axis always aligns with the arm out to the rotors, even if the helicopter is rotated.

The gyroscope measures the rotational velocity in units rad/s, whilst the accelerometer measures the proper acceleration in units $\mathrm{m/s^2}$. In contrast to regular acceleration which measures the second derivative of linear position, proper acceleration measures the acceleration in relation to free-fall. When the system is in free-fall the proper acceleration evaluates to $0\mathrm{m/s^2}$. When the system is laying stationary on a table it will be affected by the normal force of the table pushing up, and will therefore measure $9.81\mathrm{m/s^2}$ in the upwards direction. This information will be used to indirectly measure the orientation of the IMU.

3.1 Problem 1 - Extended state-space formulation

Consider the following state-space formulation of the system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{14}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrices. The state vector and the output vector should be

$$\mathbf{x} = \begin{bmatrix} p \\ \dot{p} \\ e \\ \dot{e} \\ \dot{\lambda} \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} \tilde{V}_s \\ V_d \end{bmatrix} \tag{15}$$

Use the state-space formulation of the system in (5a)-(5c) to evaluate $\boldsymbol{A}$ and $\boldsymbol{B}$.

3.2 Problem 2 - Observability

Evaluate the observability matrix by hand when only the rates are measured. Is the system then observable?

Assume we are free to measure any of the states in $\mathbf{x}$. Which are the minimum set of states that if measured makes the system (15) observable? You may use the Matlab $obsv(A, C)$ command.

3.3 Problem 3 - Angle measurement

In the previous task you should have identified that it is not sufficient to only use the gyro measurements. Additionally some angle measurements are needed. The IMU does not have a sensor that lets us measure its angle. Instead we will utilize the fact that the accelerometer measures the force counteracting gravity when the helicopter is stationary. This force will always point straight upwards with a magnitude equal to the gravitational constant g.

1) Derive equations for the measured acceleration $[a_x, a_y, a_z]$ by the IMU given the elevation and pitch angle assuming that the helicopter is standing still. It is easier to first consider elevation alone assuming pitch equal to zero, and then consider the pitch afterward.

2) Use the equations for the measured acceleration to derive the following equations for pitch and elevation.

$$p = \arctan\left(\frac{a_y}{a_z}\right) \tag{16a}$$

$$e = \arctan\left(\frac{a_x}{\sqrt{a_y^2 + a_z^2}}\right) \tag{16b}$$

4 Part IV – Kalman filter

We wish to use a discrete-time Kalman Filter to estimate the state of the system. The Kalman Filter can be described with the following equations.

Correction with new data:

$$\boldsymbol{K}[k] = \bar{\boldsymbol{P}}[k]\boldsymbol{C}_{\mathrm{d}}^{\top}(\boldsymbol{C}_{\mathrm{d}}\bar{\boldsymbol{P}}[k]\boldsymbol{C}_{\mathrm{d}}^{\top} + \boldsymbol{R}_{\mathrm{d}})^{-1} \tag{17a}$$

$$\hat{\boldsymbol{x}}[k] = \bar{\boldsymbol{x}}[k] + \boldsymbol{K}[k](\boldsymbol{y}[k] - \boldsymbol{C}_{\mathrm{d}}\bar{\boldsymbol{x}}[k]) \tag{17b}$$

$$\hat{\boldsymbol{P}}[k] = (\boldsymbol{I} - \boldsymbol{K}[k]\boldsymbol{C}_{\mathrm{d}})\bar{\boldsymbol{P}}[k](\boldsymbol{I} - \boldsymbol{K}[k]\boldsymbol{C}_{\mathrm{d}})^{\top} + \boldsymbol{K}\boldsymbol{R}_{\mathrm{d}}\boldsymbol{K}^{\intercal} \tag{17c}$$

Predicting ahead:

$$\bar{\boldsymbol{x}}[k+1] = \boldsymbol{A}_{\mathrm{d}}\hat{\boldsymbol{x}}[k] + \boldsymbol{B}_{\mathrm{d}}\boldsymbol{u}[k] \tag{17d}$$

$$\bar{\boldsymbol{P}}[k+1] = \boldsymbol{A}_{\mathrm{d}}\hat{\boldsymbol{P}}[k]\boldsymbol{A}_{\mathrm{d}}^{\intercal} + \boldsymbol{Q}_{\mathrm{d}} \tag{17e}$$

These equations might at first glance seem intimidating, but worry not. In the following, we will gradually develop one equation at the time, which will hopefully make the Kalman filter seem quite natural.

A difference between the Kalman filter and the observer considered in the last part of the lab, is that the Kalman filter models and considers the noise when choosing how much to trust the measurements and the prediction. The following discrete-time model will be used to describe the system.

$$\boldsymbol{x}[k+1] = \boldsymbol{A}_{\mathrm{d}}\boldsymbol{x}[k] + \boldsymbol{B}_{\mathrm{d}}\boldsymbol{u}[k] + \boldsymbol{w}_{\mathrm{d}}[k] \tag{18a}$$

$$\boldsymbol{y}[k] = \boldsymbol{C}_{\mathrm{d}}\boldsymbol{x}[k] + \boldsymbol{v}_{\mathrm{d}}[k] \tag{18b}$$

$$\boldsymbol{w}_{\mathrm{d}} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_{\mathrm{d}}), \qquad \boldsymbol{v}_{\mathrm{d}} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}_{\mathrm{d}}) \tag{18c}$$

The covariance of the stochastic disturbance $\boldsymbol{Q}_{\mathrm{d}}$ serves as a tuning variable and can be left undetermined. The measurement covariance $\boldsymbol{R}_{\mathrm{d}}$ will be experimentally estimated in the lab.

4.1 Task 1 - Prediction

In this task we will consider the state prediction step. We will use our previous estimate $\hat{\boldsymbol{x}}[k]$ as a basis for the prediction

$$\bar{\boldsymbol{x}}[k+1] = \boldsymbol{A}_{\mathrm{d}}\hat{\boldsymbol{x}}[k] + \boldsymbol{B}_{\mathrm{d}}\boldsymbol{u}[k] \tag{19}$$

Let the error in our prediction be defined by

$$\bar{\boldsymbol{\varepsilon}}[k] = \boldsymbol{x}[k] - \bar{\boldsymbol{x}}[k] \tag{20}$$

And the error in our current estimate

$$\hat{\boldsymbol{\varepsilon}}[k] = \boldsymbol{x}[k] - \hat{\boldsymbol{x}}[k] \tag{21}$$

We now wish to examine how the covariance of this error develops over time. Derive an expression for the covariance of $\bar{\boldsymbol{\varepsilon}}[k+1]$, denoted as $\bar{\boldsymbol{P}}[k+1]$, given the error covariance of our current estimate, $\hat{\boldsymbol{P}}[k]$. The definition of covariance and the equations 18a and 19 can be useful in finding this expression. The definition is as follows.

$$\bar{\boldsymbol{P}}[k+1] = \mathsf{E}[\bar{\boldsymbol{\varepsilon}}[k+1]\bar{\boldsymbol{\varepsilon}}^{\mathsf{T}}[k+1]] \tag{22}$$

Keep in mind that the covariance of uncorrelated terms is 0. You should end up with equation 17e.

4.2 Task 2 - Correction

Denote a measurement-corrected estimate at time-step k by $\hat{\boldsymbol{x}}[k]$. Correction is achieved by taking the weighted average of a measurement from the IMU and what we predict that the measurement would be. Let $\bar{\boldsymbol{y}}[k]$ denote the predicted measurement so that $\boldsymbol{y}[k] - \bar{\boldsymbol{y}}[k]$ represents the disagreement between prediction and measurement.

$$\hat{\boldsymbol{x}}[k] = \bar{\boldsymbol{x}}[k] + \boldsymbol{K}[k](\boldsymbol{y}[k] - \bar{\boldsymbol{y}}[k]) \tag{23a}$$
$$\bar{\boldsymbol{y}}[k] = \boldsymbol{C}_{\mathrm{d}}\bar{\boldsymbol{x}}[k] \tag{23b}$$

Here, $\boldsymbol{K}[k]$ serves as a weighting matrix. Moving around the terms of equations 23a - 23b gives us the following equation where it's easier to see that $\boldsymbol{K}[k]$ decides how much to weight the measurement in relation to the predicted state.

$$\hat{\boldsymbol{x}}[k] = (\boldsymbol{I} - \boldsymbol{K}[k]\boldsymbol{C}_{\mathrm{d}})\bar{\boldsymbol{x}}[k] + \boldsymbol{K}[k]\boldsymbol{y}[k] \tag{24}$$

As before we want to know the covariance of the error in our estimate. The covariance of the error can be written as

$$\hat{\boldsymbol{P}}[k] = \mathsf{E}[\hat{\boldsymbol{\varepsilon}}[k]\hat{\boldsymbol{\varepsilon}}^{\mathsf{T}}[k]] \tag{25}$$

Use equation 24 together with equations 18b to find an expression of $\hat{\boldsymbol{P}}[k]$ given $\bar{\boldsymbol{P}}[k]$. You should end up with equation 17c.

4.3 Task 3 - Weighting

There are multiple ways of choosing $\boldsymbol{K}[k]$, for example through pole placement which we did for the continuous time estimator in the last part of the lab. What separates the Kalman filter from other linear observers is that it minimizes the covariance estimate to find an optimal time-dependent $\boldsymbol{K}[k]$. The diagonal elements in the estimate error covariance $\hat{\boldsymbol{P}}[k]$ contains the error variance of each state, and the off-diagonal elements contain the error covariance between states. We need to decide which elements of $\hat{\boldsymbol{P}}[k]$ we want to minimize and how to weight them in relation to each other. The Kalman filter only considers the variances (the diagonal elements) and weights them equally. We therefore want to minimize:

$$\mathrm{var}(\hat{\varepsilon}_p[k]) + \mathrm{var}(\hat{\varepsilon}_{\dot{p}}[k]) + \mathrm{var}(\hat{\varepsilon}_e[k]) + \mathrm{var}(\hat{\varepsilon}_{\dot{e}}[k]) + \mathrm{var}(\hat{\varepsilon}_\lambda[k]) + \mathrm{var}(\hat{\varepsilon}_{\dot{\lambda}}[k]) = \mathrm{tr}(\hat{\boldsymbol{P}}[k]) \tag{26}$$

The trace function returns the sum of the diagonal elements of a matrix. Find the time-varying $\boldsymbol{K}[k]$ that minimizes $\mathrm{tr}(\hat{\boldsymbol{P}}[k])$. The following properties of trace will be useful in the derivations.

$$\frac{\partial \mathrm{tr}(\boldsymbol{AB})}{\partial \boldsymbol{A}} = \boldsymbol{B}^\intercal \quad , \quad \frac{\partial \mathrm{tr}(\boldsymbol{ACA}^\intercal)}{\partial \boldsymbol{A}} = 2\boldsymbol{AC} \tag{27a}$$

$$\mathrm{tr}(\boldsymbol{D}) = \mathrm{tr}(\boldsymbol{D}^\intercal) \quad , \quad \mathrm{tr}(\boldsymbol{A}+\boldsymbol{B}) = \mathrm{tr}(\boldsymbol{A}) + \mathrm{tr}(\boldsymbol{B}) \tag{27b}$$

You should end up with equation 17a.