The Existence of Solutions to an Even Order Right Focal Boundary Value Problem

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Problem

Consider the even order differential equation,

$$u^{(2n)} = \lambda h\left(t, u, u', u'', \dots, u^{(2n-1)}\right) \tag{1}$$

for $t \in (0,1)$ and $n \geq 2$, satisfying the right focal boundary conditions

$$u^{(2k)}(0) = 0, \tag{2}$$

$$u^{(2k+1)}(1) = (-1)^k a_k, \tag{3}$$

for $k = 0, 1, 2, \dots, n-1$, where

$$h : [0,1] \times \prod_{j=0}^{n-1} (-1)^j [0,\infty)^2 \to (-1)^n [0,\infty)$$

is continuous, $\lambda, a_0, \dots, a_{n-1} \geq 0$, and $\sum_{k=0}^{n-1} a_k > 0$. We outline a method for proving the existence of at least three positive solutions to (1)-(3).

Substitutions

Let

$$u_k = (-1)^k u^{(2k)}, \quad k = 0, \dots, n-1,$$

$$u_{k+1} = g_k\left(t, u_0, u_0', \dots, u_{n-1}, u_{n-1}'\right), \quad k = 0, \dots, n-2,$$

$$f\left(t, u_0, u_0', \dots, u_{n-1}, u_{n-1}'\right) = (-1)^n h\left(t, u_0, u_0', \dots, (-1)^{n-1} u_{n-1}, (-1)^{n-1} u_{n-1}'\right).$$

Then (1)-(3) becomes

$$-u_{n-1}'' = \lambda f\left(t, u_0, u_0', \dots, u_{n-1}, u_{n-1}'\right), \tag{4}$$

$$-u_k'' = g_k\left(t, u_0, u_0', \dots, u_{n-1}, u_{n-1}'\right), \tag{5}$$

where $k = 0, \dots, n-2$ and $t \in (0,1)$, satisfying

$$u_k(0) = 0, \quad k = 0, \dots, n-1, \tag{6}$$

$$u_k'(1) = a_k, \quad k = 0, \dots, n-1. \tag{7}$$

Transformation

We transform (4)-(7) into the system of second order differential equations

$$-u_{n-1}'' = \lambda f\left(t, u_0 + ta_0, u_0' + a_0, \dots, u_{n-1} + ta_{n-1}, u_{n-1}' + a_{n-1}\right), \tag{8}$$

$$-u_k'' = g_k\left(t, u_0 + ta_0, u_0' + a_0, \dots, u_{n-1} + ta_{n-1}, u_{n-1}' + a_{n-1}\right), \tag{9}$$

for $k = 0, \dots, n-2$ and $t \in (0,1)$, satisfying

$$u_k(0) = u_k'(1) = 0, \quad k = 0, \dots, n-1. \tag{10}$$

Solutions to (8)-(10) are of the form

$$u_{n-1} = \lambda \int_0^1 G(t,s) f(s, u_0 + sa_0, \dots, u_{n-1}' + a_{n-1}) ds,$$

$$u_k = \int_0^1 G(t,s) g_k(s, u_0 + sa_0, \dots, u_{n-1}' + a_{n-1}) ds,$$

for $k = 0, \dots, n-2$, where $G(t,s)$ denotes the Green's function

$$G(t,s) = \begin{cases} s & , 0 \leq s \leq t \leq 1 \\ t & , 0 \leq t \leq s \leq 1. \end{cases}$$

Hypotheses

(H0) $f, g : [0,1] \times [0,\infty)^{2n} \to [0,\infty)$ are continuous functions which are nondecreasing in the $(2j)^{th}$ variables and nonincreasing in the $(2j+1)^{th}$ variables, where $j = 1, 2, \dots, n$.

(H1) There exists $\alpha, \beta \in (0,1)$, $\alpha < \beta$, such that, given $(x_0, \dots, x_{2n-1}) \in [0,\infty)^{2n}$ with $\sum_{i=0}^{2n-1} x_i \neq 0$, there exists $k > 0$ such that $f(t, x_0, \dots, x_{2n-1}) > k$ for $t \in [\alpha, \beta]$.

(H2) Let $z = \sum_{i=0}^{2n-1} x_i$. Then $\lim_{z \to 0^+} \frac{f(t, x_0, \dots, x_{2n-1})}{z} = 0$ and $\lim_{z \to 0^+} \frac{g(t, x_0, \dots, x_{2n-1})}{z} = 0$ uniformly for $t \in [0,1]$.

(H3) Let $z = \sum_{i=0}^{2n-1} x_i$. Then $\lim_{z \to \infty} \frac{f(t, x_0, \dots, x_{2n-1})}{z} = 0$ and $\lim_{z \to \infty} \frac{g(t, x_0, \dots, x_{2n-1})}{z} = 0$ uniformly for $t \in [0,1]$.

Preliminaries

Let $(X, \|\cdot\|)$ denote the Banach space $X = \prod_{i=0}^{n-1} C^1([0,1], \mathbb{R})$ endowed with the norm

$$\|(u_0, \dots, u_{n-1})\| = \|u_0\|_\infty + \|u_0'\|_\infty + \cdots + \|u_{n-1}\|_\infty + \|u_{n-1}'\|_\infty,$$

where $\|u\|_\infty = \sup_{t \in [0,1]} |u(t)|$. Define $C \subset X$ to be the cone

$$C = \left\{ \begin{array}{c} (u_0, \dots, u_{n-1}) \in X : (u_0, \dots, u_{n-1})(0) = (u_0', \dots, u_{n-1}')(1) = (0, \dots, 0) \\ \text{and } u_0, \dots, u_{n-1} \text{ are concave} \end{array} \right\}.$$

Moreover, let Ω_p denote the open set $\Omega_p = \{(u_0, \dots, u_{n-1}) \in X : \|(u_0, \dots, u_{n-1})\| < p\}$. Finally, define $T : X \to X$ to be the operator $T(u_0, \dots, u_{n-1}) = (A_0(u_0, \dots, u_{n-1}), \dots, A_{n-1}(u_0, \dots, u_{n-1}))$, where

$$A_k = \int_0^1 G(t,s) g_k(s, u_0 + sa_0, u_0' + a_0, \dots, u_{n-1} + sa_{n-1}, u_{n-1}' + a_{n-1}) ds$$

and

$$A_{n-1} = \lambda \int_0^1 G(t,s) f(s, u_0 + sa_0, u_0' + a_0, \dots, u_{n-1} + sa_{n-1}, u_{n-1}' + a_{n-1}) ds,$$

for $k = 0, \dots, n-2$ and $t \in (0,1)$. It can be shown T is a completely continuous operator and $T : C \to C$.

Lemmas

Lemma 1.1 Suppose (H0) and (H1) hold, and let $\rho^* > 0$. Then there exists $\Lambda > 0$ such that for every $\lambda \geq \Lambda$ and $(a_0, \dots, a_{n-1}) \in [0,\infty)^n$, we have

$$\|T(u_0, \dots, u_{n-1})\| \geq \|(u_0, \dots, u_{n-1})\|,$$

for $(u_0, \dots, u_{n-1}) \in C \cap \partial\Omega_{\rho^*}$.

Lemma 1.2 Fix $\Lambda > 0$ and suppose (H0) and (H1) hold. Then for every $\lambda \geq \Lambda$ and $(a_0, \dots, a_{n-1}) \in [0,\infty)^n$ with $\sum_{i=0}^{n-1} a_i > 0$, there is a $\rho_1 = \rho_1(\Lambda, a_0, \dots, a_{n-1})$ such that for every $\rho \leq \rho_1$, we have

$$\|T(u_0, \dots, u_{n-1})\| \geq \|(u_0, \dots, u_{n-1})\|,$$

for $(u_0, \dots, u_{n-1}) \in C \cap \partial\Omega_\rho$.

Lemma 1.3 Suppose (H0) and (H2) hold, and let $\rho^* > 0$ be fixed. Then given $\lambda > 0$, there is a $\rho_2 \in (0, \rho^*)$ and a $\delta > 0$ such that for every $(a_0, a_1, \dots, a_{n-1}) \in [0,\infty)^n$, with $0 < \sum_{i=0}^{n-1} a_i < \delta$, we have

$$\|T(u_0, \dots, u_{n-1})\| \leq \|(u_0, \dots, u_{n-1})\|,$$

for $(u_0, \dots, u_{n-1}) \in C \cap \partial\Omega_{\rho_2}$.

Lemma 1.4 Let $\delta > 0$. Suppose for all $(a_0, a_1, \dots, a_{n-1}) \in [0,\infty)^n$, we have $0 < \sum_{i=0}^{n-1} a_i < \delta$. Suppose further (H0) and (H3) hold. Then, for every $\lambda > 0$, there is a $\rho_3 = \rho_3(\delta, \lambda)$ such that for all $\rho \geq \rho_3$,

$$\|T(u_0, \dots, u_{n-1})\| \leq \|(u_0, \dots, u_{n-1})\|,$$

for $(u_0, \dots, u_{n-1}) \in C \cap \partial\Omega_\rho$.

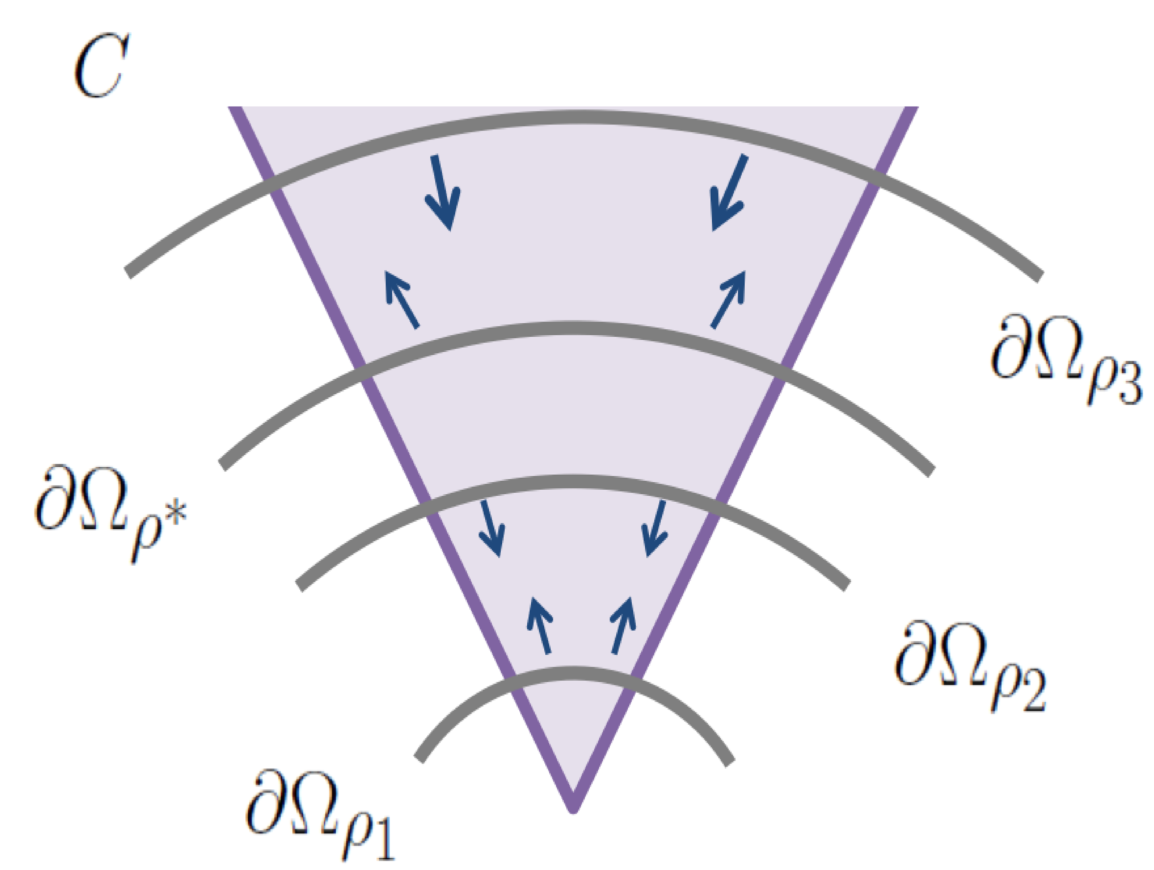

Figure 1: Picking ρ_1, ρ_2, ρ^*, and ρ_3 as we have allows for a triple application of the Guo-Krasnosel'skii Fixed Point Theorem.

Guo-Krasnosel'skii Fixed Point Theorem

Let $(X, \|\cdot\|)$ be a Banach Space and $C \subset X$ be a cone. Suppose Ω_1, Ω_2 are open subsets of X satisfying $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. If $T : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ is a completely continuous operator such that either

(1) $\|Tu\| \leq \|u\|$ for $u \in C \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in C \cap \partial\Omega_2$, or

(2) $\|Tu\| \geq \|u\|$ for $u \in C \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in C \cap \partial\Omega_2$,

then T has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Main Result

Theorem 1.5 Let continuous functions

$$f, g : [0,1] \times [0,\infty)^{2n} \to [0,\infty)$$

satisfy hypotheses (H0)-(H3). Then there exists $\Lambda > 0$ such that given $\lambda \geq \Lambda$, there exists $\delta > 0$ such that for every $a_0, \dots, a_{n-1} \geq 0$ satisfying $0 < \sum_{i=0}^{n-1} a_i < \delta$, the system (8)-(10) has at least three positive solutions.