IN3190 - Exam Preparation Questions H23
University of Oslo

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Innhold

1   Essential Topics

1.1   Discrete time

1.1.1   Sine, cosine and exponential functions

Mathematical formula that establishes the fundamental relationship between the trigonometric functions and the complex exponential function.
Euler's formula states that for any real number x:

$$e^{ix} = \cos(x) + i\sin(x) \tag{1}$$

When $x = \pi$, Euler's formula yields Euler's identity:

$$e^{i\pi} + 1 = 0 \tag{2}$$

1.1.2   Elementary discrete signals

Unit impulse   Also known as the dirac delta function. Shifted to the left when its negative and to the right when its positive. Can be used to find H(n) in a system. If the input $x(n) = \delta(n)$, then the output $y(n) = h(n)$.

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \qquad \text{may also be written as:} \qquad \delta[n] = u[n] - u[n-1] \tag{3}$$
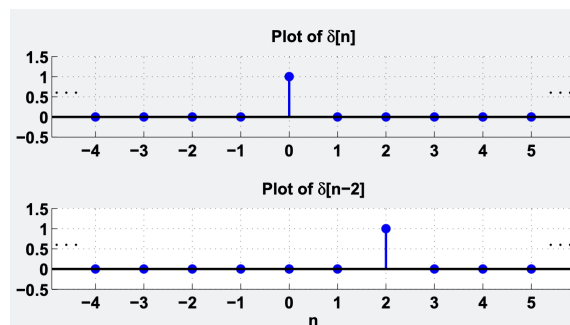
Figur 1: Unit impulse

Step function   Also known as unit step, unit step function or heaviside function. The value which is zero for negative arguments and one for positive arguments.

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \qquad \text{may also be written as:} \qquad u[n] = \sum_{k=0}^{\infty} \delta[n-k] \tag{4}$$
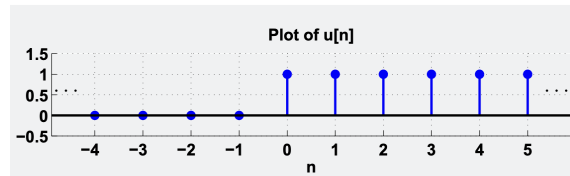
Figur 2: Unit step function

Ramp function Also known as the unit-ramp or unit ramp function. Graph shaped like a ramp.

$$u_r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{5}$$
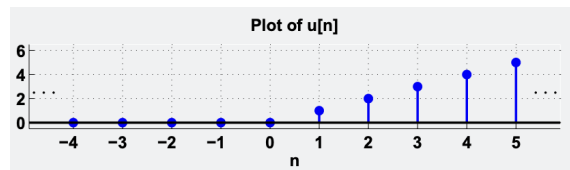
Figur 3: Unit ramp function

Periodic sequences $x[n]$ is periodic if and only if $x[n] = x[n + N]$

- **Fundamental period**:
  Smallest positive integer N which fulfills the relation above

- **Sinusoidal sequences**:
  $x[n] = A\cos(\omega_0 n + \phi)$, where

  - A is amplitude,
  - ω_0 is the angular frequency
  - and ϕ is the phase of $x[n]$.

$x[n]$ is periodic if and only if $\omega_0 N = 2\pi k$, for N and k as positive integers.
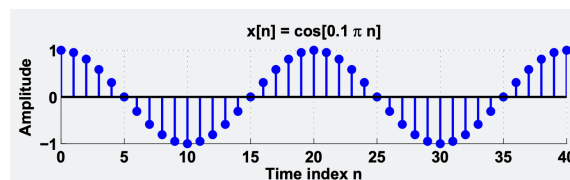
Figur 4: Sinusoidal sequences

1.1.3 LTI systems, including characteristics via the transformation between input and output

Linearity: The relationship between the input $x(t)$ and the output $y(t)$, both being regarded as functions. If a is a constant then the system output to $ax(t)$ is $ay(t)$. Linear system if and only if it $H\{\cdot\}$ is both additive and homogeneous, in other words: If it fulfills the superposition principle. That is:

$$H\{ax_1(t) + bx_x(t)\} = aH\{x_1(t)\} + bH\{x_2(t)\} \tag{6}$$

Time-invariance: A linear system is time-invariant or shift-invariant means that wether we apply an input to the system now or T seconds from now, the output will be identical except for a time delay of S seconds. In other words, if $y(t)$ is the output of a system with a input $x(t)$, then the output of the system with input $x(t - T)$ is $y(t - T)$. The system is invariant becasue the output does not depend on the particular time at which the input is applied.

The fundamental result in LTI system theory is that any LTI system can be characterized entirely by a single function called the system's impulse response $h(t)$. The output of the system $y(t)$ is simply the convolution of the input to the system $x(t)$ with the systems's impulse response $h(t)$.

LTI system can also be characterized in the frequency domain by the system's transfer function $H(s)$, which is the Laplace transform of the system's impulse response $h(t)$. Or Z-transform in the case of discrete time systems.

Figur 5: Relationship between the time domain and the frequency domain

1.1.4   Difference equations for system description / LTI systems written on general summation notation with coefficients and time offsets

General description of relationship between input x[n] and output y[n] of a LTI system in the time domain (n domain): (M amount of zeroes) (N amount of poles)

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_x x[n-k] \tag{7}$$

- FIR (finite impulse response): all $a_k = 0$
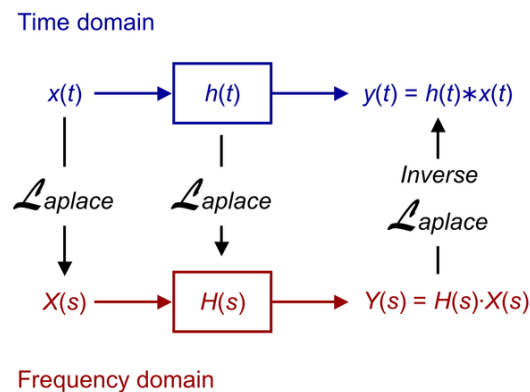
- IIR (infinite impulse response): at least one $a_k \neq 0$

The value N represent the order of the difference equation and corresponds to the memory of the system being represented. Because this equations relies on past values of the output, in order to compute a numerical solution, certain past outputs, referred to as the initial conditions, must be known.
a_k and b_k are the coefficients of the difference equation. Time offset is represented by the index k.

1.1.5   Casuality

In control theory, a causal system is a system where the output depends on past and current inputs, but not future inputs. The idea that the output of function at any time depends only on past and present values of input is defined by the property commonly referred to as casusality.

1.1.6   Stability and the related convergence for the DTFT

A linear system is called BIBO (Bounded-input bounded output) stable if its output will stay bounded for any bounded input.

Discrete-time signals: For a rational and discrete time system, the conditon for BIBO stability is that the ROC (region of convergence) of the z-transform includes the unit circle. When the system is causal, the ROC is the open region outside a circle whose radius is the magnitude of the pole with largest magnitude. Therefore, all poles of the system must be inside the unit circle in the z-plane for BIBO stability. For stable systems, the unit circle is part of ROC.

Causal stable filters have all poles inside $|z| = 1$.

1.1.7   FIR / IIR

FIR: Finite impulse response. The output from a FIR filter is calculated by the basis of the input samples and the proceding input. A FIR filter is always stable because the values from the output does not effect calculations. (Can be thought of as an op-amp without feedback loop)

IIR: Infinite impulse response. The output from a IIR filter is calculated by the basis of the input samples, the proceding input and the previous output. A IIR filter can be unstable because the values from the output effects the calculations. (Can be thought of as an op-amp with feedback loop)

1.1.8 Convolution

The resulting signal y(n) represents the amount of overlap between x and h at each point in time, and can be used for a variety of purposes, such as filtering, smoothing, deconvolution, and image processing. For example, if we convolve a signal with a smoothing kernel, we can reduce noise and make the signal smoother.
The summation:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{8}$$

is the convolution of $x(n)$ and $h(n)$

Properties of convolution:

- **Commutative:** $x(n) * h(n) = h(n) * x(n)$

- **Associative:** $x(n) * \{h_1(n) * h_2(n)\} = \{x(n) * h_1(n)\} * h_2(n)$

- **Distributive:** $x(n) * \{h_1(n) + h_2(n)\} = x(n) * h_1(n) + x(n) * h_2(n)$
  where $*$ is the convolution operator

- Can also use the matrix method to calculate convolution

1.1.9 Sampling frequency and sampling interval. Digital frequency / physical frequency, as well as corresponding digital angular frequency / physical angular frequency and how to map between these given a sampling interval T = 1/FT

Sampling frequency: The sampling frequency or sampling rate, f_s is the average number of samples obtained in one second (samples per second), thus $f_s = 1/T_s$ given in Hz or kHz. To avoid aliasing, the sampling frequency must be greater than twice the maximum frequency of the signal being sampled, according to the Nyquist–Shannon sampling theorem.

Sampling interval: The sampling interval or sampling period, T_s is the time between samples, and is equal to the inverse of the sampling frequency $T_s = 1/f_s$. The sampling interval should be chosen small enough to avoid aliasing, but large enough to get a reasonable signal-to-noise ratio.

Digital frequency: Digital frequency

Physical frequency:

Digital angular frequency:

Physical angular frequency:
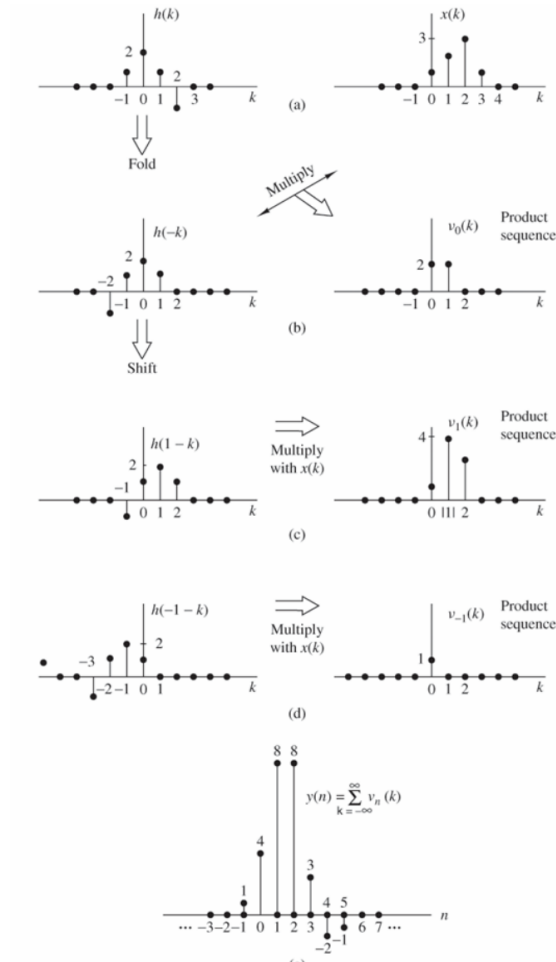
Figur 6: Convolution

1.2 Z-transform

1.2.1 Definition and how to derive the various z-transform properties

Definition: The Z-transform(z-domain) is a discrete-time equivalent of the Laplace transform(s-domain). It is a linear transformation of a discrete-time signal which maps a complex-valued discrete-time signal $x[n]$ to a complex-valued function of a complex variable $X(z)$.

$$X(z) = \mathcal{Z}[x[n]] = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{9}$$

Propterties

Linearity: If $x[n] \xleftrightarrow{Z.T} X(z)$ and $y[n] \xleftrightarrow{Z.T} Y(z)$, then linearity property states that:

$$ax[n] + by[n] \xleftrightarrow{Z.T} aX(z) + bY(z) \tag{10}$$

Time shifting: If $x[n] \xleftrightarrow{Z.T} X(z)$, then time shifting property states that:

$$x[n - m] \xleftrightarrow{Z.T} z^{-m} X(z) \tag{11}$$

Time reversal: If $x[n] \xleftrightarrow{Z.T} X(z)$, then time reversal property states that:

$$x[-n] \xleftrightarrow{Z.T} X(z^{-1}) \tag{12}$$

Scaling in the z-domain: If $x[n] \xleftrightarrow{Z.T} X(z)$, then scaling property states that:

$$a^n x[n] \xleftrightarrow{Z.T} X(a^{-1}z) \tag{13}$$

for any constant a, real or complex.

Differentiation in the z-domain: If $x[n] \xleftrightarrow{Z.T} X(z)$, then differentiation property states that:

$$x[n] \xleftrightarrow{Z.T} \left[-z \frac{dX(z)}{dz} \right]^k \tag{14}$$

Convolution of two sequences: If $x[n] \xleftrightarrow{Z.T} X(z)$ and $y[n] \xleftrightarrow{Z.T} Y(z)$, then convolution property states that:

$$x[n] * y[n] \xleftrightarrow{Z.T} X(z)Y(z) \tag{15}$$

1.2.2   Region of convergence (ROC)

The region of convergence, known is ROC, is important to understand because it defines the region where the z-transform exist.

$$ROC = \left\{ z : \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| < \infty \right\} \tag{16}$$

- ROC cannot contain any poles

- From ROC, a system is causal if goes to ∞

- If the ROC contains the unit circle, then the system is stable.

1.2.3 Poles/zeroes as well as the connection to stability/causality/symmetry/real signals/frequency response/filter type/linear phase etc

1.3 Frequency analysis and DFT

1.3.1 DTFT and its central properties

Discrete time Fourier transform (DTFT): The DTFT of a discrete-time signal $x[n]$ is defined as:

$$\text{Synthesis equation:} \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \tag{17}$$

$$\text{Analysis equation:} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{18}$$

- $X(e^{j\omega})$ unique ocer the frequency interval $(-\pi, \pi)$, or equivalently $(0, 2\pi)$

- $X(e^{j\omega})$ is periodic with period 2π

Property	Time domain	Frequency domain
Notation	$x[n]$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(\omega) + bX_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x(n)y^*(n)$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)Y^*(\omega)d\omega$

Figur 7: Properties of DTFT

1.3.2 DFT

Discrete Fourier transform (DFT): The DFT of a discrete-time signal $x[n]$ is defined as:

$$\text{DFT of } N\text{-length signal:} \quad X(k) = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} \tag{19}$$

$$\text{Inverse DFT of } N\text{-length signal:} \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}kn} \tag{20}$$

1.3.3   Relationship between DFT and DTFT

1.3.4   The four Fourier transforms

1.3.5   Ideal filters

1.3.6   Example of simple filters

1.3.7   Min/max/mixed phase and inverse system

1.3.8   Circular and linear convolution

1.3.9   Advantages of FFT. The computational complexity to DFT to definition

1.4   Filter design

1.4.1   Advantages/Disadvantages of FIR and IIR filters

Advantages, FIR filters:

- Can be designed to have exact linear phase, which makes h(n) zero points symmetric about $|z| = 1$

- Filter structures are always stable for quantized coefficients.

    - Stable IIR filters can become unstable due to rounding errors in coefficients.

- Design methods are generally linear.

- Start transient has finite length.

- Can be realized efficently in HardWare (but not as efficient as IIR filters)

Disadvantages, FIR filters: Generally require a higher order than IIR filters to meet the same specifications. This often ensure the FIR filter having greater calculation burden/complexity and larger group delay than the corresponding IIR filter.

Advantages, IIR filters:

- In IIR filter design, the possibility of converting a analog filter to a digital filet gives them several advantages:

    - Analog approximation techniques are highly advanced.
    - They usually yield closed-form solutions.
    - Extensice tables are available for analog filters.
    - Many applications require digital simulations of analog system.

Disadvantages, IIR filters: We have no control over the phase characteristics of a IIR filter, we only know it cannot have linear phase). I.e. IIR filters have only magnitude-only deigns.

1.4.2   Common ways to design IIR filters

- Specify the filter as a analog/continuous time (CT) equivalent to the dicrete IIR filter you want to design.

- Design the analog lowpass filter in continuous time.

- Convert the analog lowpass filter to a digital filter using filter transformation.

- Convert the digital lowpass filter to the desired filter using a frequency band transformation.

- Convert the digital filter to a discrete time (DT) filter (From s-domain to z-domain).

- You could also just use an optimization algorithm (e.g. least-square method)

1.4.3   Typical IIR filters

When using the approximation method, you have the desired response given as:

$$|H_d(j\Omega)|^2 = \begin{cases} 1, & 0 \leq |\Omega| \leq \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases} \tag{21}$$

And the approximation as:

$$|H_c(j\Omega)|^2 = \frac{1}{1 + V^2(\Omega)} \tag{22}$$

Where $V^2(\Omega) << 1$ for $|\Omega| \leq \Omega_c$ and $V^2(\Omega) >> 1$ for $|\Omega| > \Omega_c$.

The $V^2(\Omega)$ is called the **approximation function** and is used to characterize the approximation. Different $V^2(\Omega)$ gives different approximations types.

Butterworth filter:

- $|H(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$, where N is the filter order.

- Is flat in the passband, and rolls off towards zero in the stopband.

- In a logarithmic plot (bode plot), the response slopes off linearly towards $-\infty$.

- Butterworth filters have a monotonically changing magnitude response with frequency ω.

Chebyshev filter:

- $|H(\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_N^2(\Omega/\Omega_c)}$, where T_N is the Nth order Chebyshev polynomial ().

- The Chebyshev filter has a ripple in either the passband or stopband, but a steeper roll-off than the Butterworth filter.

- Type I Chebyshev filter is an all pole filter and has equiripple (the ripples are of same height) in the passband and a monotonically decreasing stopband.

- Type II Chebyshev filter consists of both poles and zeros, with a monotonically decreasing passband and equiripple in the stopband.

Elliptic filter:

- $|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 U_N^2(\Omega/\Omega_c)}$, where $U_N(\Omega/\Omega_c)$ is a Jacobian elliptic function.

- The elliptic filter has a ripple in both the passband and stopband.

- The elliptic filter has both poles and zeros.

- Compared to the two other types, the elliptic filter has the narrowest transition band for a given N, cutoff frequency Ω_p and pass/stop-band ripples.

A small summary so far:　Of the prototype analog filters showcased here, **Elliptic gives best performance in the magnitude-squared response**, but have a highly nonlinear phase response in the passband (Often bad in application).

　　Butterworth filters are on the other end with low performance in the magnitude-squared response, but have **flat magnitude response in the passband and a fairly linear phase in the passband**. They do however require more poles (N) to achieve the same stopband specification.

1.4.4　Common ways to design FIR filters

FIR filter design is often based on a approximation of a specified magnitude response. The design of a FIR filter of order $M-1$ can be found by either the length M impulse response samples in $\{h(n)\}$ or the M samples of the frequency response in $\{H(e^{j\omega})\}$.

- **Window method:** Multiply the ideal impulse response with a window function to get the desired impulse response.

- **Frequency sampling method:** Sample the frequeny response (DTFT) of the desired filter by assuming it to be the DFT. Then use the inverse DFT to get the desired coefficients. This can work well for in the points sampled, but between the points the frequency response can not be controlled. Transition samples from tables can improve this.
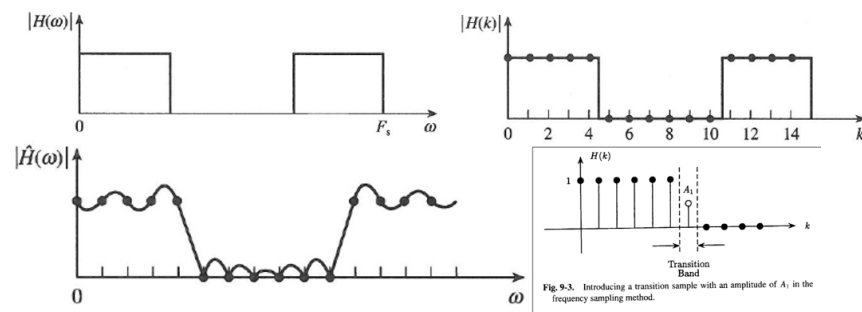

Figur 8: Visual representation of the frequency sampling method

- **Optimization method:** Use algorithms (e.g. Parks-McClellan) or optimization techniques like least squares to iteratively refine the filter coefficients by minimizing the error between the desired and actual frequency response.

1.4.5   Window functions and the consequences of applying them

Window functions $w(n)$ are of finite lengths and symmetric around their midpoints.

$$\underbrace{h(n)}_{\text{Desired filter}} = \underbrace{h_d(n)}_{\text{Ideal filter}} * w(n) \tag{23}$$

Or in the frequency domain:

$$H(e^{j\omega}) = H_d(e^{j\omega}) \circledast W(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\nu})W(e^{j(\omega-\nu)})d\nu \tag{24}$$

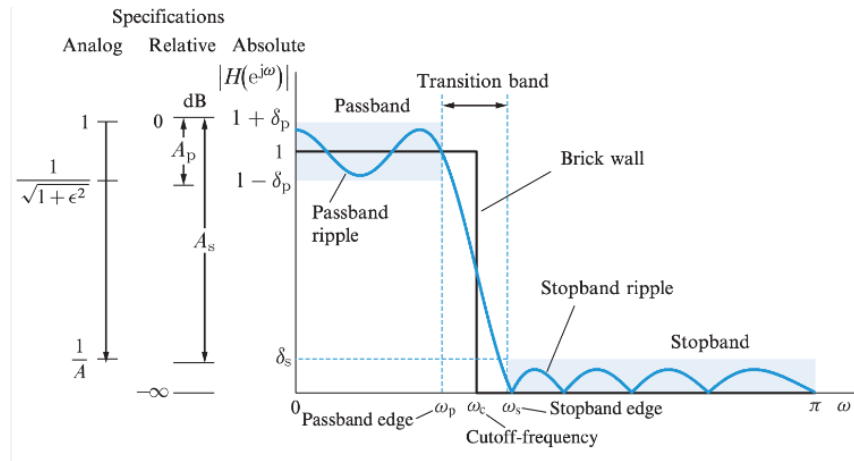Figur 9: Example of a lowpass filter tolerance diagram

How well $H(e^{j\omega})$ approximates $H_d(e^{j\omega})$ is determined by:

- The width of the main lobe of $W(e^{j\omega})$

- Maximum side lobe level of $W(e^{j\omega})$

This method also lacks precise control of w_p and w_s.

Some characteristics of window spectrum:

- Pass and stopband ripples are not constant

- Stopband level (filter) typically less than peak sidelobe level (PSL) to window.

- PSL, peak passband ripple and passband attenuation $\approx$ independent of N.

- Width of transition band $\approx$ given by main lobe width window, $F_{WS} \approx C/N$ where C is a constant for each window.

1.4.6　Linear phase and linear-phase filters

A LTI system has a **linear phase** if:

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\alpha\omega} \tag{25}$$

Or a **generalized linear phase** if:

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j(\alpha\omega-\beta)} \tag{26}$$

Where $A(e^{j\omega})$ is a real-valued function of ω and $\beta \in \Re$

If your system needs to both be **causal** and have **linear phase**, you can only use **FIR**.

A sufficient condition for a real-valued FIR filter to have generalized linear phase is that its impulse response $h(n)$ is symmetric or antisymmetric.

- **Symmetric:** $h(n) = h(M - 1 - n), \quad n = 0, 1, \ldots, M - 1$
  Then $\alpha = (M - 1)/2$ and $\beta = 0 \ \lor \ \pi$

- **Antisymmetric:** $h(n) = -h(M - 1 - n), \quad n = 0, 1, \ldots, M - 1$
  Then $\alpha = (M - 1)/2$ and $\beta = \pi/2 \ \lor \ 3\pi/2$

For a real linear phase FIR filter with a zero z_1, you also have zeros at $z_1^*, \frac{1}{z_1}$ and $\frac{1}{z_1^*}$
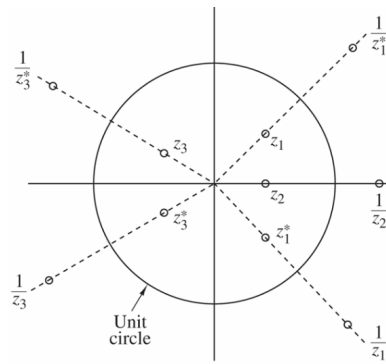
Figur 10: Example showing zero symmetry in a FIR filter

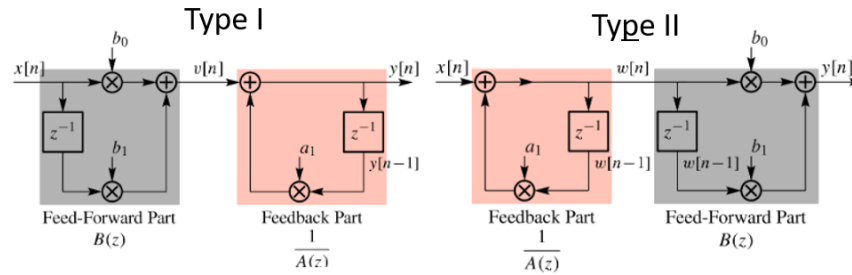### 1.4.7 Graphical representation of filter structures and operations

Figur 11: Example of a filter structure

Operator notation: The z^{-1} blocks in fig. 11 are called **delay operators** and are used to represent the delay of one sample. I.e. $x[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-1}X(z)$

How is it made Given a transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = (b_0 + b_1 z^{-1} \dots) \left(\frac{1}{1 - a_1 z^{-1} \dots} \right) = B(z) \frac{1}{A(z)}$$

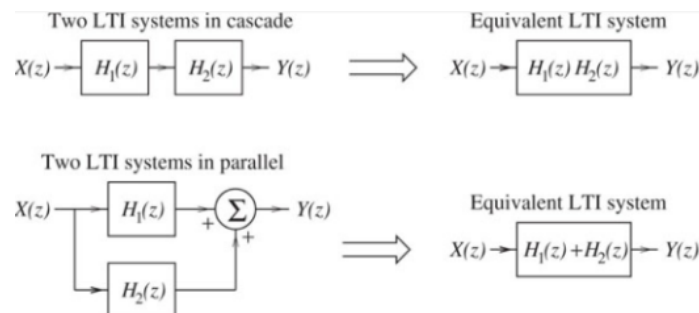Note we have two forms, **direct form I** and **direct form II**. The later saves one unit delay.

Figur 12: Illustration of connected systems

Figure 12 shows how a equivalent transfer function of systems in cascade is the product of the individual transfer functions. The equivalent transfer function of systems in parallel is the sum of the individual transfer functions.

1.4.8 Gibb's phenomenon

- Oscillatory behavior in the magnitude response of causal FIR filters found by truncating the filter coefficients of an ideal filter.

- If you increase the length of the filter, the number of ripple-peaks will increase in both pass-band and stopband and the width of these peaks decrease equivalent.

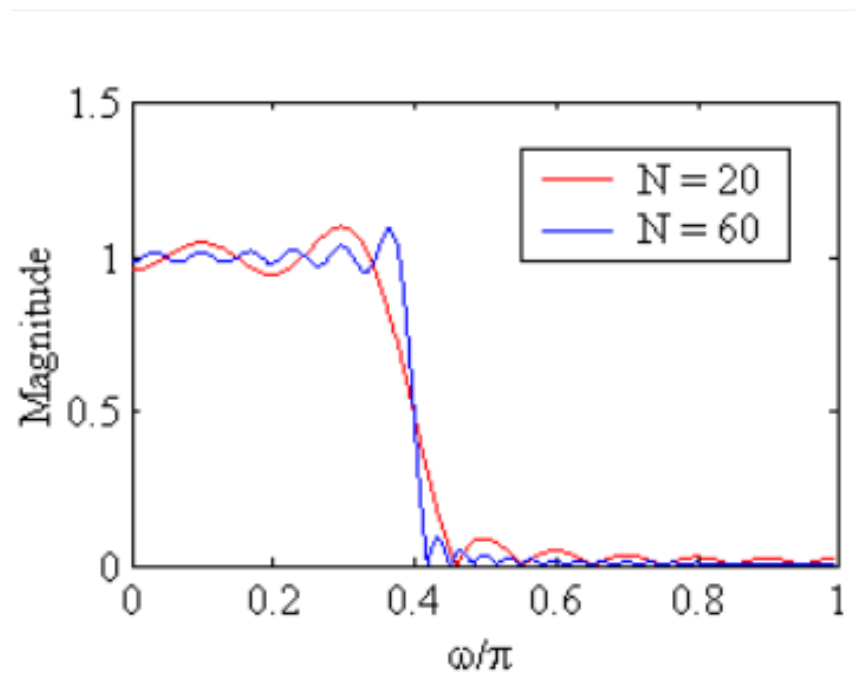- Given the same weighting function, the highest ripple-peak will be unchanged.


Figur 13: Example of Gibb's effect in a lowpass filter

1.5 Sampling and reconstruction

1.5.1 Sampling theorem

The sampling frequency $\Omega_T = 2\Omega_H$ is called the Nyquist frequency. If this term is satisfied, then the original signal can be recovered from the sampled signal. In practice we often use oversampling (sampling at a higher rate than the Nyquist frequency) to get an appropiate reconstruction.

### 1.5.2 Synthesizing an arbitrary signal as a sum of unit impulses

Sampling corresponds to multiplying with a Dirac comb with distance T between the unit impulses. In the Fourier domain, this corresponds to convolving with a Dirac comb with distance F_T between scaled unit impulses. (The distance is Ω_T if we look at the angular frequency axis)
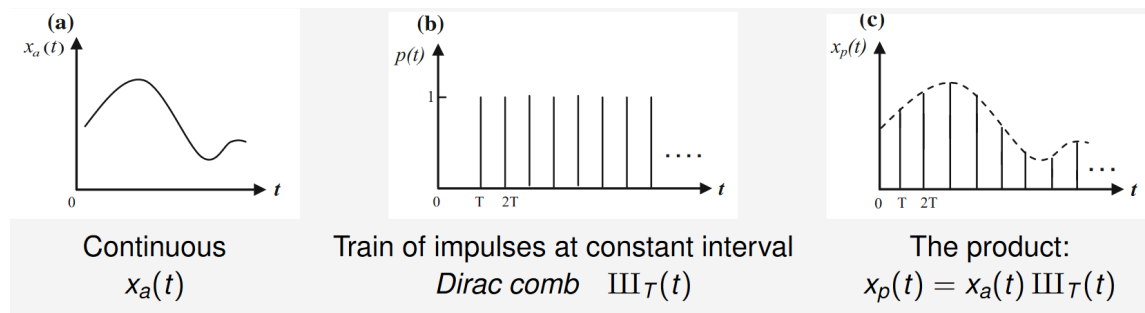
Figur 14: Sampling with dirac impulses

$$x_p(t) = x_a(t)III_T(t) = \sum_{n=-\infty}^{\infty} x_a(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t - nT) \triangleq x(n) \qquad (27)$$

1.5.3 Sampling of band-limited signals
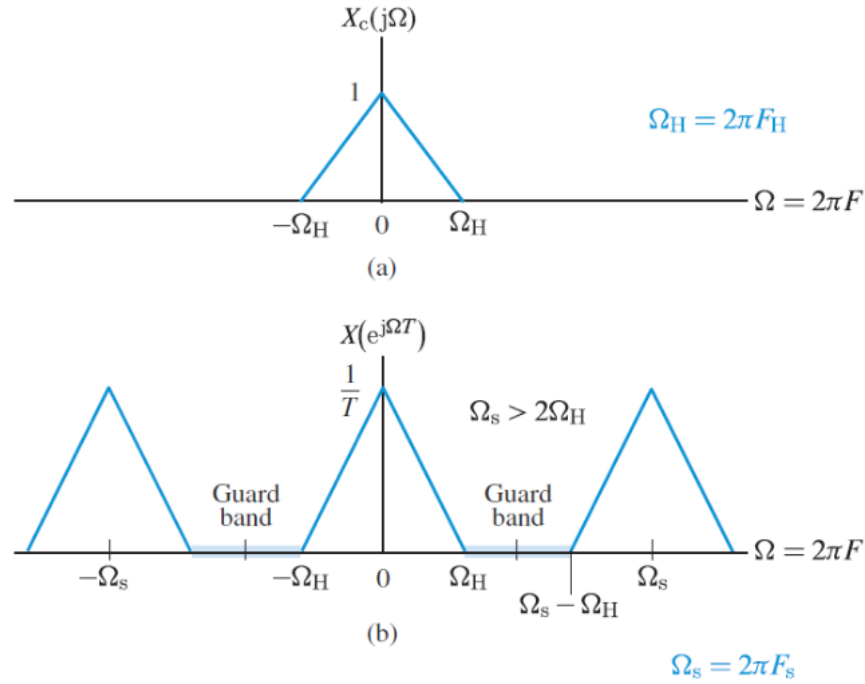
Figur 15: Periodic sampling in freq. domain

Important part: A signal that is limited to a band $F \in [F_L, F_H]$ is a bandpass with bandwidth $B = F_H - F_L$. This signal can be perfectly reconstructed if $2B \leq F_T \leq 4B$

Not important The reconstructed signal will repeat itself in the frequency domain. As long the sampling frequency (Ω_T) is equal or greater than the Nyquist rate $(2\Omega_H)$, we will be able to seperate the original signal from the aliases. The reconstruction in fig. 15 can be written as a convolution:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x_a(nT) g_r(t - nT) \tag{28}$$

Where $g_r(t)$ is the interpolation reconstruction function (I.e. impulse response).

1.5.4 Ideal reconstruction

The fundamental copy of the Fourier-domain representation of the sampled signal can be found by the 'ideal' filter for the convolution operation is:

$$G_r(j\Omega) = \begin{cases} T, & |\Omega| \leq \Omega_T/2 \\ 0, & |\Omega| > \Omega_T/2 \end{cases} \tag{29}$$

This gives the ideal: $X_r(j\Omega) = X_a(j\Omega)$
(Reconstructed Fourier domain representation equals the original Fourier domain representation).
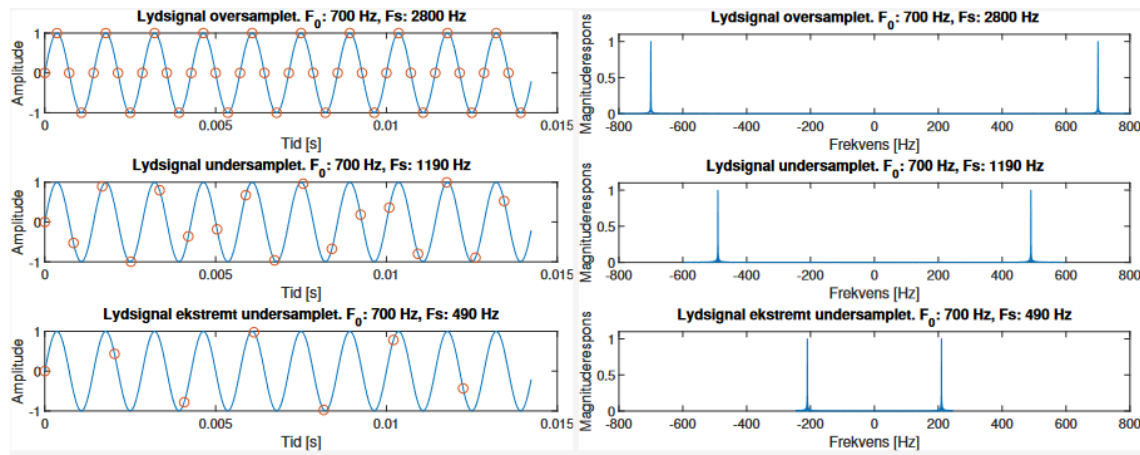
1.5.5   Upsampling and downsampling

Figur 16: Example of over/undersampling

2   Additional Topics

2.1   Discrete time

2.1.1   Symmetrical signals

2.2   LTI systems and characteristics

2.3   Convolution and correlation

2.4   Z-transform

2.5   Frequency analysis and DFT

2.6   Filter design

2.7   Sampling and reconstruction

Notes from group session:

- x(n) is input, y(n) is ouput and h(n) is impulse response.

- FIR filter has to have symmetrical coefficients to have linear phase.

- The placement of the zero can give us the magnitude plot. And from the magnitude plot we can see the frequency spectrum.

- Three ways to see if it is causal, ROC or difference equation

- From ROC, it is causal if it contains infinity

- For n<0, if h(n) (Impulse response) have a postive value, it it anti-causal.