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Should we teach linear algebra through geometry?

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Abstract

Can geometry help students learn linear algebra? I study this question and demonstrate that there is no obvious clear answer: geometry can be an obstacle to learning linear algebra; or it can be helpful. Geometry is helpful only under certain conditions and with a specific use of drawings. These special requirements for using geometry are apparently not much recognized in our teaching of linear algebra courses, at least in France, where my educational studies have taken place.

Keywords: Mathematics education; Linear algebra; Geometry; Intuition; Didactics

1. Introduction

Linear algebra is introduced during the first or second university year in many countries. It is well-known that the students find this subject difficult. It seems very abstract and disconnected from all previous math knowledge. Teachers are often convinced that one remedy is to base at least the elementary notions and results of linear algebra on geometry.

In what follows, we use the term “Geometry” to mean a mathematical model for our physical space. Thus geometry appears to us as concrete because of its link with physical space [4].

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Geometry allows the use of figures that associate a mathematical concept with a drawing.

Thus using geometry as a basis of a linear algebra course is appealing. The aim of my study is to determine whether this helps students to learn linear algebra or not.

The paper is divided into four sections:

- Section 2 presents the results of previous educational research that addresses the use of geometry in teaching linear algebra. The quoted research shows student difficulties that are connected with this use of geometry. These previous studies are the starting point of my own research, because the literature seems to confirm that the title question deserves to be thoroughly studied.
- In Section 3, I briefly outline a historical analysis on how geometry helped in the historical genesis of linear algebra as a field of mathematics.
- Section 4 is based on a study of several textbooks. This allows us to observe the potential and the limits of using geometry when teaching elementary linear algebra. I specifically look at textbook examples for orthogonality and dot products.
- The last section is devoted to student perceptions. I analyze the consequences of a teacher's particular choice how to teach linear algebra on the ensuing students' perceptions.

2. Difficulties with geometry as noted in previous research

2.1. The modes of description in linear algebra

Joel Hillel identifies three modes of description in linear algebra: the *abstract mode*, the *algebraic mode*, and the *geometric mode* [14]. The *abstract mode* uses the language of a formal theory: vectors spaces, subspaces, homomorphisms, their kernels and images, etc.; the *algebraic mode* uses the language of n-tuples and matrices; the *geometric mode* uses the language of two and three dimensional space: points, planes, lines, direction, etc. Hillel's description of student difficulties with the geometric mode seriously questions using a geometrical approach to teaching linear algebra.

The difficulties associated with the geometric "point and arrow" depiction of a vector are most striking. A vector (here the word "vector" means any element of a vector space) can be pictured as a point, or as an arrow. Hillel observes that most teachers use both depictions indiscriminantly. Vectors are sometimes represented as points, for example in the context of illustrating the distance from a vector to a subspace; but arrows are preferred when illustrating operations such as vector addition. Often, there are diagrams which use both representations at the same time. Hillel describes the subsequent misinterpretations of these different representations by students in detail. For example, when the students are asked to draw the set of vectors $D = \{(x, y) \in \mathbb{R}^2, x + y = 1\}$, they draw axes in $\mathbb{R}^2$ and the corresponding line

$x + y = 1$. But when asked to picture an element of D, most of them draw an arrow on the line, instead of an arrow joining the origin and a point of the line.

Such difficulties limit the benefits of drawings and even of mental pictures in learning linear algebra. More appropriate drawings can be used by teachers, for example by only using vectors with one fixed origin. But this is rarely done, and even in that case, the more "natural" affine representation is likely to remain. This constitutes a well-documented obstacle to student understanding.

2.2. *Using Cabri-geometry in linear algebra*

Sierpinska, Dreyfus, and Hillel have designed a learning environment for the notions of vector space, linear transformation, and eigenvector using Cabri-geometry II software [17]. An analysis of this teaching design can be found in [18]. Sierpinska identifies several difficulties encountered by the students, and especially a phenomenon that she describes as: "*Thinking of mathematical concepts in terms of their prototypical examples rather than the definitions*". For example, some students, given a pair of basis vectors and asked to construct a linear transformation with given values on the basis, look for a well-known geometrical transformation (dilation, rotation, etc.) or for some combination of these transformations. The students are using a geometric model of these well-known transformations of the plane—which they are likely remembering from their previous education—in order to find the corresponding linear transformation. But for the task proposed here, this cannot help the students. Instead it creates an obstacle.

2.3. *The concreteness principle in linear algebra*

The *concreteness principle* is stated by Guershon Harel as follows:

> For students to abstract a mathematical structure from a given model of that structure, the elements of that model must be conceptual entities in the students' eyes; that is to say, the student has mental procedures that can take these objects as inputs [10].

Is it possible for students to learn linear algebra by abstracting from geometry? Can geometrical objects serve as conceptual entities?

Harel has experimented with teaching a linear algebra course following the concreteness principle and using a geometrical presentation of vector spaces. That approach has had positive effects on the students' performances in linear algebra, giving them even some ability to prove general linear algebra results. Students belonging to a group that followed the experiment seem to have gained a better control over the correctness of their answers than another group that was taught linear algebra without geometrical underpinnings.

But difficulties were observed that are associated with using a geometric model: some students could not learn the general theory as they remained entrapped in the world of geometric vectors. Harel concludes that geometry must be used carefully and not too early, for example only after a detailed presentation of $\mathbb{R}^n$ has been given.

Examining these findings demonstrates that our title question "*Should we Teach Linear Algebra through Geometry?*" has no obvious answer.

I will now present some ideas how geometry can benefit students when learning linear algebra, and how it can be effectively used by teachers and students. To determine the potential and the limits of using geometry for this purpose, I first examine the role of geometry in the historical process that has led us to modern linear algebra.

3. The role of geometry in the development of linear algebra

Linear algebra has several origins [3]. Some of those have no special link with geometry, like the study of systems of linear equations. The most "geometric" origin of linear algebra seems to be in the work of Gottfried Leibniz [11]. Leibniz criticized the analytical method of Descartes and Fermat. He tried to elaborate a geometrical calculus that would allow to compute directly on geometric objects. He did not reach that goal, but he opened a new direction of research.

More than a century later, Hermann Günther Graßmann developed a theory in his book: *Die lineale Ausdehnungslehre* [5] that came from his own search for a geometric calculus. His theory is very general and abstract, it provides much more than a geometric calculus. But in spite of its generality, the geometric origin is very present. Many results are introduced from within geometry and are then generalized. At the end of each chapter, applications to physical space are presented (I refer to these as "geometric applications" of Graßmann's theory). In fact, geometry is omnipresent and it is very difficult for the reader of the book to form a concrete idea of the theory outside of its geometric applications. Graßmann himself fails when he tries to use a different vocabulary in the general case and in the geometric case. After explaining how important it is to use different words for the general theory and for its geometric applications, he uses vocabulary in the general context that should be restricted to the geometrical case such as "direction" (Richtung) instead of "change" (Änderung).

A reader of the *Ausdehnungslehre* is likely to stay within a geometric context, instead of advancing to the general theory, just like some students of Harel did (see Section 2.3). This happened to a famous reader of the *Ausdehnungslehre*: Giuseppe Peano [15]. Graßmann's *Ausdehnungslehre* was misunderstood, if not ignored, in its time. Peano, a defender of Graßmann, attempted to present the theory of Graßmann in a book entitled "Calcolo geometrico" (1888). But again he restricted himself to its geometric applications.

Graßmann elaborated a very general theory that includes most of modern linear algebra. But the importance of geometry in his book and the lack of other concrete examples prevented the development of the general theory.

The determining works in developing modern linear algebra belong to functional analysis. From around 1880 to the 1930s, many mathematicians like Maurice Fréchet, Frédéric Riesz, Erhard Schmidt, Norbert Wiener, Hans Hahn, and Stefan Banach studied infinite dimensional linear spaces. Their research led to the concept of vector spaces of functions, and eventually to the modern axiomatic presentation of vector spaces. All of these authors introduce general theories that apply to geometry as well. Geometry plays a special part, because geometrical vocabulary such as orthogonality, distance, etc. is used in a general setting. Readers of Riesz' or Schmidt's papers will probably be helped by the geometric analogies. But the most important point is perhaps that the readers will not remain riveted to a geometric model because many other important concepts such as functions and sequences play an equally fundamental role.

In fact, our historical analysis leads us to state that a geometric model alone seems insufficient to justify the need for a general theory. Especially when the general theory has emerged from a unification of many mathematical concepts.

Transferring our last statement into a teaching context requires further study. The statement itself is consistent with the results of educational studies like Harel's necessity principle [10], or Robert's observations on "unifying notions" [16]. Robert describes ways to introduce new notions at the university, depending on some characteristics of the mathematical content involved. Linear algebra is a "unifying" notion, and thus it cannot be presented as a natural development of previous knowledge and neither as a way to solve new problems.

Another important statement can be derived from our historical analysis. The links between linear algebra and geometry do not seem to have the same importance in all parts of linear algebra. The geometric vocabulary that is used by Schmidt, Riesz and others occurs mostly with terms like orthogonality, norm, dot product etc. Thus the historic link between linear algebra and geometry concerns primarily inner product spaces. For that reason, I have chosen to study that link in modern university textbooks next.

4. The potential and the limits of a geometrical model: orthogonality and dot products

In examining several textbooks [1,6,8,9,12,13] I note that the strongest links between linear algebra and geometry appear when studying inner product spaces. This is reminiscent of our earlier historical analysis. For this reason I focus on the questions of orthogonality.

The study of $\mathbb{R}^2$ and $\mathbb{R}^3$ as vector spaces with a dot product can serve as a geometrical model for the general notions of vector spaces with a dot product and of quadratic forms. Simple properties and results can be introduced in $\mathbb{R}^2$ and $\mathbb{R}^3$ first and then generalized to arbitrary inner product spaces. I just mention here two typical examples.

- The notion of orthogonal projection on a subspace F can be presented in $E = \mathbb{R}^2$ or $E = \mathbb{R}^3$ (and associated with drawings). Any vector u of E can be decomposed on a unique way as $u = u_F + u_{F^\perp}$, with $u_F \in F$ and $u_{F^\perp} \in F^\perp$; the orthogonal projection p on F is defined by $p(u) = u_F$. The same definition applies to any inner product space.
- By starting out with defining orthogonality and length via a dot product, the Pythagorean theorem for two vectors (for example in $\mathbb{R}^2$) is a direct consequence of bilinearity properties. And it can be generalized as $|e_1 + \cdots + e_k|^2 = |e_1|^2 + \cdots + |e_k|^2$ for any set $\{e_1, \ldots, e_k\}$ of orthogonal vectors. Thus, we immediately get a generalized Pythagorean theorem which is extendible to infinite dimensional Hilbert spaces.

But even if such possibilities exist, they remain limited by several factors.

A first difficulty stems from the way the related topics are taught at secondary school level (at least in France). For example, orthogonality is introduced through its intuitive notion of right angle when first encountered in primary school. Similarly, the notion of distance is introduced intuitively and it is used to define the notion of a vector. These intuitive notions of orthogonality and distance are then used to define orthonormal basis, coordinates, and the dot product in $\mathbb{R}^2$. Orthogonal projections are mentioned, but only points are projected, and the property: "*If $p(M)$ is the orthogonal projection of M onto a line D (or plane P), then the distance from M to D is $d(M, p(M))$*" cannot be established, because orthogonality and the Pythagorean theorem belong to right triangles.

In university linear algebra courses, these concepts are presented in the opposite order, and orthogonality of vectors is defined in terms of their dot product being zero. Hence there are no natural links between the university level inner product space definition of orthogonality and that of school geometry, links which would allow students to utilize their intuitive notions.

There are other obstacles to learning which are the result of representations that are limited to $\mathbb{R}^2$ or $\mathbb{R}^3$. Some important theorems can appear self evident in $\mathbb{R}^2$ or $\mathbb{R}^3$ because of our associated drawings or mental pictures. For example theorem such as "*A set of non-zero orthogonal vectors is linearly independent*" is often not considered as an interesting result by our students. Because of the associated picture in dimensions two or three, this theorem appears to be self-evident, namely that "any two non-zero orthogonal vectors are not on the same line".

Sometimes mental pictures can lead to intuitions that complicate seeing the general result. For example, when studying general quadratic forms it is very difficult for students to accept that a subspace can be its own orthogonal complement.

I draw several conclusions.

Some linear algebra notions and results can be based on a geometry. For example, inner product spaces can be studied well with such an approach.

But even within that specific topic the mental pictures associated with $\mathbb{R}^2$ or $\mathbb{R}^3$ can constitute an obstacle for understanding some of the results. Our historical analysis indicates that linear algebra cannot appear as a generalization of geometry alone; it rather must be grounded in several mathematical domains.

Our main conclusion regarding the usefulness of geometry in teaching linear algebra is that geometry can help, but not serve as the single topic leading to linear algebra. Instead linear algebra must be associated with other subjects such as polynomials, functions, sequences, etc. In a multi-based approach to linear algebra our notions would not only have to be grounded in geometry, but at least in the necessary analogies between geometry and several other subjects.

I will now examine some effective uses of geometry in teaching and learning linear algebra. I will especially observe the consequences on students' reasonings in linear algebra during a classroom presentation that included many geometric examples.

5. Use of geometry in linear algebra by teachers and students

5.1. Teachers' choices

The first stage of the practical aspects of this study was a questionnaire addressed to university teachers (in France; 31 teachers answered the questionnaire). I asked them about their use of geometry and drawings in their linear algebra courses.

I only present its main results briefly here because the questionnaire was only the first step. From the answers to the questionnaire I was led to distinguish two main tendencies among teachers:

Some praise a structural approach to linear algebra, with almost no drawings shown to the students. Geometry is then presented as a natural consequence of the general theory. That tendency was shared by around 30% of the teachers I asked. Others choose to present affine geometry with associated pictorial representations before introducing linear algebra (40% of the teachers I asked).

This is a clear sign of the influence, still very strong, of the discussions before and during the reform of the modern mathematics curriculum in France (1960–1970). Only a minority of teachers (15% of the teachers I asked[1]) propose to use specially designed drawings for teaching linear algebra. This may have negative consequences on the students' ability to use drawings themselves (in order to understand linear algebra properties, but also to solve linear algebra problems). If students need mental pictures to help their reasoning in linear algebra, they will probably use representations associated with affine geometry that are unsuitable for vector spaces (see Section 2 and Hillel's observations).

Regarding pictorial representations, teachers do not seem to develop specific drawings for linear algebra classes. In some classes drawings are only used when affine geometry enters the linear algebra course. A minority of teachers uses drawings in

[1] For the remaining 15%, it was not possible to identify a clear preference.

linear algebra, but only for $\mathbb{R}^2$ and $\mathbb{R}^3$. In that case linear algebra in $\mathbb{R}^2$ and $\mathbb{R}^3$ can provide a model on which to base the general theory. But there is no evidence that students will be able to use the geometric insights, especially if the teachers do not use drawings in general vector space settings.

In order to study these phenomena, I observed a course on inner product spaces and quadratic forms, given by a teacher who used many illustrative drawings, but only within the $\mathbb{R}^2$ and $\mathbb{R}^3$ context. I later met some of his students for an interview of which I present the results next.

5.2. Students and the Pythagorean theorem

During six weeks I observed a teacher giving a course on quadratic forms and inner product spaces to second-year students. He used many drawings and asked his students to draw themselves (32 exercises were assigned during the six weeks; the students were explicitly asked to draw in 10 of them). But the drawings were only for two- and three-dimensional vector spaces, notably $\mathbb{R}^2$ and $\mathbb{R}^3$. After the end of the semester I met with eight of the students for interviews. I have chosen to focus on the Pythagorean theorem, because of the potential of its generalization as mentioned in Section 4.

During his course, the teacher presented the Pythagorean theorem as follows:

- The general result was written, and called "Theorème de Pythagore".

  If q is a quadratic form, f its underlying bilinear form, and if x and y are two vectors that are orthogonal with respect to f; or if $f(x, y) = 0$, then $q(x + y) = q(x) + q(y)$.

- An example of $\mathbb{R}^2$ was given in the following drawing:

It carried the text:

> $\mathbb{R}^2$ with the usual dot product. Let a = distance from x to O, b = distance from y to O, and c = distance from $x + y$ to O, then $a^2 + b^2 = c^2$.

A proof was not given. This was a typical example of this teacher's choice: a general result involving quadratic forms is stated for a real vector space (finite dimensional in this case) and for an arbitrary quadratic form. A drawing in $\mathbb{R}^2$ illustrates the result. But this drawing represents the general result only for $\mathbb{R}^2$. Under these conditions, will it be possible for the students to recognize that the Pythagorean theorem can apply in a problem set outside of the $\mathbb{R}^2$ context? Are they likely to accept the use of the same drawing to illustrate the more general setting of the theorem?

In order to answer these questions, I proposed the following exercise to the students during their interviews.

Let $E = \mathbb{R}_3[X]$ *be the inner product space of degree* 3 *polynomials and let* P *and* Q *be two orthogonal elements of* E *whose length is* 1. *Can you determine the length of* $P + Q$?

Of the eight students I asked, only four found the result. One of them found the solution directly, using the Pythagorean theorem. He did not use that term, but he clearly applied the result. Three found the result after computing $q(P + Q)$ using the bilinear form f for q.

Among the four students who did not come up with the right answer, two declared that there must exist a formula, but they did not remember it; the two others declared it was not possible to compute the length of $P + Q$. None of them made a drawing.

When they were finished, I showed the eight students the following drawing:

I then asked them about the drawing, and especially whether they thought it illustrated the studied situation well, and whether they might be able to use it to solve the exercise.

The one student who solved the problem directly said he thought of his proof from an image like the picture I drew.

Five students said they understood the drawing, but would not have come up with it themselves. They all thought, afterwards, of the Pythagorean theorem.

The two others rejected representing polynomials as arrows; they were among the four who were not able to solve the exercise.

So, only one student was able to recognize that the Pythagorean theorem was applicable here. This particular student associated the exercise with a mental picture representing polynomials as arrows. He then observed that it corresponded to an earlier class drawing relating to $\mathbb{R}^2$, and thus he applied the Pythagorean theorem. Five students understood the transfer that is implied in the drawing when I proposed it. The pictorial representation of the question reminded them of the Pythagorean theorem and thus led them to accept the proposed solution.

Other parts of the students' interviews led to similar conclusions: When a general result is given and associated with a geometric example (that example can be given before stating the result as an introduction or afterwards, as an illustration), most students can not apply the result (the Pythagorean theorem of $\mathbb{R}^2$ or $\mathbb{R}^3$ in our case) in other contexts such as in the space of polynomials or of functions. Our study indicates that the use of many different examples stemming from various areas, such as geometry, polynomials, sequences, etc. would more likely help students to confer widely useful meaning to a general result.

6. Conclusions

Educational research that has examined the use of geometry for teaching linear algebra has proved both that this can help students and that it can generate specific difficulties, called "didactic obstacles" by [2].

A historical analysis of the genesis process for linear algebra, as well as my textbook study, demonstrate that linear algebra cannot be constructed as a mere generalization of geometry. Our first important conclusion, based on Sections 2–4, is that geometry must be used very carefully in linear algebra courses.

Potential for a geometric model exists: The examples of orthogonal projection and the Pythagorean theorem demonstrate it. However, the concreteness that seems to lack in linear algebra could be more efficiently provided by the use of drawings, especially drawings illustrating concepts and properties in abstract vector spaces. These drawings could foster the students' understanding of linear algebra, and their ability to solve linear algebra problems. The study of the teachers' choices in their linear algebra courses indicates that this potential for the use of drawings has not been sufficiently explored at this time.

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