ASTR 600 Pset #3

1 a. $\Omega_{m,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$, $h = 0.7$

At $z = 0.5$, $\Omega_m = \dfrac{\Omega_{m,0}(1+z)^3}{\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}$

$= 0.591$.

$\Omega_\Lambda = 1 - \Omega_m = 0.409$.

$\boxed{0.591; 0.409}$

b. Submitted via Jupyter .ipynb.

c. '' , calculated numerically, $z = 1.85$

d. $\left(\dfrac{\dot{a}}{a}\right)^2 = H_0^2 \Omega_\Lambda$

$\dfrac{\dot{a}}{a} = H_0 \sqrt{\Omega_\Lambda}$

$\dfrac{da}{a} = H_0 \sqrt{\Omega_\Lambda}\, dt$

$a = e^{H_0 \sqrt{\Omega_\Lambda}\, t}$.

The age in this universe is infinite and can only be described relatively.

2\. a. $\rho = \dfrac{g}{2\pi^2} T^4 \displaystyle\int_0^\infty d\xi \dfrac{\xi^2 \sqrt{\xi^2 + x^2}}{\exp(\sqrt{\xi^2 + x^2}) \pm 1}$, Eqn 3.14 from Baumann, $x \equiv \dfrac{m_\nu}{T_\nu}$

$g = 2$ as neutrinos have 2 spin states. Also, assuming $\xi^2 \gg x^2$, $\exp(\sqrt{\xi^2 + x^2}) \approx \exp(\xi)$, and we take the $+$ in the $\pm$ as neutrinos are fermions.

Then, $\rho = \dfrac{T^4}{\pi^2} \displaystyle\int d\xi \dfrac{\xi^2 \sqrt{\xi^2 + m_\nu^2/T_\nu^2}}{e^\xi + 1}$.

b. Expanding around $x = 0$, $\sqrt{\xi^2 + x^2} = \xi + \frac{x^2}{2\xi} + O(x^4)$

$$\rho \approx \frac{T^4}{\pi^2} \int_0^\infty d\xi \, \frac{\xi^2}{e^{\xi}+1} (\xi + x^2/2\xi)$$

$$= \frac{T^4}{\pi^2} \int_0^\infty d\xi \, \frac{\xi^3}{e^{\xi}+1} + \frac{T^4 x^2}{2\pi^2} \int_0^\infty d\xi \, \frac{\xi}{e^{\xi}+1}$$

$$= \frac{T^4}{\pi^2} \frac{7\pi^4}{120} + \frac{T^4 x^2}{2\pi^2} \frac{\pi^2}{12}.$$

Let $\rho_{\nu,0} = \frac{7\pi^2}{120} T^4$,

$$\rho_\nu \approx \rho_{\nu,0} \left(1 + \frac{T^4 x^2}{2\pi^2} \frac{\pi^2}{12} \left(\frac{120\pi^2}{7\pi^4 T^4}\right)\right).$$

$$= \rho_{\nu,0} \left(1 + \frac{5}{7\pi^2} x^2\right) = \rho_{\nu,0} \left(1 + \frac{5}{7\pi^2} \frac{m_\nu^2}{T_\nu^2}\right).$$

c. We assume we can detect neutrinos in the CMB if the first error term is as big as the leading term, s.t.

$$\frac{5}{7\pi^2} \frac{m_\nu^2}{T_\nu^2} = 1.$$

$T_\nu \propto a^{-1}$, $T_\nu = T_{\nu,0} (1+z) = T_{CMB} \left(\frac{4}{11}\right)^{1/3} (1+z)$, due to annihilation of e^- and e^+.
$T_{CMB} = 0.235$ meV, $z \sim 1000$

$\boxed{m_\nu = 624 \text{ meV}.}$

d. Neutrinos go nonrelativistic when $m_\nu = T_\nu = T_{CMB}\left(\frac{4}{11}\right)^{1/3}(1+z)$.

$$1+z = \frac{m_\nu}{T_{CMB}}, \quad \boxed{z = \frac{m_\nu}{(4/11)^{1/3}\, T_{CMB}} - 1 = \frac{m_\nu}{0.168\,meV} - 1.}$$

e. We find the number density at neutrino decoupling, then evolve with time, $n \propto a^{-3}$.

At decoupling, neutrinos are relativistic and in thermal equil. and thus follow the Fermi-Dirac distribution.

$$n = \frac{\zeta(3)}{\pi^2} g T^3 \left(\frac{3}{4}\right) \quad \text{Eqn 3.22, Baumann.}$$

$$n \propto a^{-3}, \quad n_{\nu,0} = \frac{\zeta(3)}{\pi^2} g T_\nu^3 \left(\frac{3}{4}\right) \frac{1}{(1+z)^3}$$

$g = 2$, 2 spin states / flavor

$T_\nu = T_{CMB}\left(\frac{4}{11}\right)^{1/3}(1+z)$.

$$n_{\nu,0} = \frac{\zeta(3)}{\pi^2} 2 T_{CMB}^3 \left(\frac{4}{11}\right)\left(\frac{3}{4}\right)$$

$$= 8.604 \times 10^{-13}\, eV^3/\text{flavor} = 112\ cm^{-3}/\text{flavor}.$$

f. Assuming the neutrinos are nonrelativistic at $z=0$,

$$\rho_{\nu,0} = m_\nu n_{\nu,0}.$$

$$\Omega_{\nu,0} = \frac{m_\nu n_{\nu,0}}{\rho_{crit,0}}, \quad \rho_{crit,0} \approx 8.09 \times 10^{-11}\, eV^4\, h^2$$

$$\Omega_{\nu,0} h^2 = \frac{m_\nu}{94\,eV}.$$

3 a. $\chi = \int_0^z dz' \frac{c}{H_0 \sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\Lambda,0} + (1-\Omega_{m,0}-\Omega_{\Lambda,0})(1+z')^2}}$

Taylor expand around $z=0$ the integrand:

Wolfram Alpha gives

$$\chi \approx \int_0^z dz \left[\frac{c}{H_0} + z'\left(-\frac{\Omega_{m,0}}{2} + \Omega_{\Lambda,0} - 1\right) \right.$$

$$\left. + \frac{z'^2}{8}\left(3\Omega_{m,0}^2 - 12\Omega_{m,0}\Omega_{\Lambda,0} + 4\Omega_{m,0} + 12\Omega_{\Lambda,0}^2 - 20\Omega_{\Lambda,0} + 8\right) + O(z^3)\right].$$

$$\approx \frac{cz}{H_0} + \frac{cz^2}{2H_0}\left(\frac{-\Omega_{m,0}}{2} + \Omega_{\Lambda,0} - 1\right)$$

$$+ \frac{cz^3}{24H_0}\left(3\Omega_{m,0}^2 - 12\Omega_{m,0}\Omega_{\Lambda,0} + 4\Omega_{m,0} + 12\Omega_{\Lambda,0}^2 - 20\Omega_{\Lambda,0} + 8\right) + O(z^4)$$

b. Computing both quantities with $\Omega_{m,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$, the expansion is accurate to 10% for $z < 1.2$ (Shown in notebook)

c. For very low redshift, the leading term in z dominates. Thus, $\chi \approx \frac{cz}{H_0}$, and the only parameter that can be measured is H_0.

d. As we go to higher redshift, we introduce the z^2 and z^3 terms. However, the coefficients are a combination of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$, namely $\left(-\frac{\Omega_{m,0}}{2} + \Omega_{\Lambda,0} - 1\right)$ and $(3\Omega_{m,0}^2 - 12\Omega_{m,0}\Omega_{\Lambda,0} + 4\Omega_{m,0} + 12\Omega_{\Lambda,0}^2 - 20\Omega_{\Lambda,0} + 8)$.

This is because both $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ depend on z, and thus cannot be measured independently.

e Assuming the values are estimated using an unbiased estimator, its variance is bounded by the Cramér-Rao bound, given by

$$\sigma_{ii}^2 \geq (F^{-1})_{ii} = (F_{ii})^{-1}$$

$$F_{ii} = \mu_{,j}^T C^{-1} \mu_{,j} + \mathrm{Tr}[C^{-1} C_{,i} C^{-1} C_{,j}]$$, elements of Fisher matrix.

Assume errors indep. of cosmological parameters, so.

$$F_{ij} = \mu_{,i}^T C^{-1} \mu_{,j}.$$

As we are working w/ relative error, $\mu = \ln X$.
$C^{-1} = 1/\sigma^2 I$

For H_0, $\mu_{,H_0} = \frac{\partial \ln X}{\partial H_0} = \frac{\partial}{\partial H_0}\left(\ln\left(\frac{c}{H_0}\right) + \ln\left(\int dz \ldots\right)\right)$

$$= \frac{\partial}{\partial H_0}\left(\ln\left(\frac{c}{H_0}\right)\right) = -\frac{1}{H_0}$$

$$F_{11} = \frac{1}{H_0^2 \sigma^2}, \quad \sigma_{H_0}^2 = H_0^2 \sigma^2$$

Thus $\frac{\sigma_{H_0}}{H_0} = \sigma$, and $\boxed{\sigma_{H_0} \text{ is also relative, with error } 1\% \text{ for all } z.}$

Note that it does not scale with redshift.

For $\Omega_{m,0}$, $\mu_{,\Omega_{m,0}} = \frac{\partial \ln \chi}{\partial \Omega_{m,0}}$

$$\approx \frac{\partial}{\partial \Omega_{m,0}} \left[\ln\left(\frac{c}{H_0}\right) + \ln\left(z + \frac{z^2}{2}\left(\frac{-\Omega_{m,0}}{2} + \Omega_{\Lambda,0} - 1\right) + \frac{z^3}{24}\left(3\Omega_{m,0}^2 - 12\Omega_{m,0}\Omega_{\Lambda,0} + 4\Omega_{m,0} + 12\Omega_{\Lambda,0}^2 - 20\Omega_{\Lambda,0} + 8\right)\right)\right]$$

using the Taylor expansion around $z=0$.

We know that for $z<0.3$, the error is $\sim 0.2\%$, which is a good approximation, as shown in the notebook.

$$\Rightarrow \frac{1}{\left(z + \frac{z^2}{2}\left(\frac{-\Omega_{m,0}}{2} + \ldots\right.\right.} \left(\frac{-z^2}{4} + \frac{z^3}{4}\Omega_{m,0} - \frac{z^3}{2}\Omega_{\Lambda,0} + \frac{z^3}{6}\right)$$

where the denominator is $\frac{H_0}{c}\chi$.

Then, $$\sigma^2_{\Omega_{m,0}} = \sigma^2 \frac{H_0^2 \chi^2}{c^2\left(-\frac{z^2}{4} + \frac{z^3}{4}\Omega_{m,0} - \frac{z^3}{2}\Omega_{\Lambda,0} + \frac{z^3}{6}\right)^2}.$$

Assuming $\Omega_{m,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$, the results are calculated via Jupyter notebook, and the errors are $\sigma = (3.97, 0.37, 0.18, 0.11)$ for redshifts $z = (0.01, 0.1, 0.2, 0.3)$ respectively.

Repeating for $\mu_{,\Omega_{\Lambda,0}} = \frac{\partial \ln \chi}{\partial \Omega_{\Lambda,0}} = \frac{c}{H_0 \chi}\left(\frac{z^2}{2} - \frac{z^3}{2}\Omega_{m,0} + z^3\Omega_{\Lambda,0} - \frac{5z^3}{6}\right)$

$$\sigma^2_{\Lambda,0} = \sigma^2 \frac{H_0^2 \chi^2}{c^2\left(\frac{z^2}{2} - \frac{z^3}{2}\Omega_{m,0} + z^3\Omega_{\Lambda,0} - \frac{5}{6}z^3\right)}$$

Again assuming $\Omega_{m,0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$, the errors are $\sigma = (2.00, 0.21, 0.11, 0.07)$ for redshift $z = (0.01, 0.1, 0.2, 0.3)$.

The errors for $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ are extremely large at low redshift, but decrease as redshift increases. By $z \approx 0.3$, decent observations can be made for both Ωs.