Daniel Chang

P600 Pset #2

1. $\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$, $\dot\rho + 3H(\rho + P) = 0$

$$\frac{d(\dot a^2)}{dt} = \frac{d\left(\frac{8\pi G}{3}\rho a^2 - k\right)}{dt}$$

$$\Rightarrow 2\dot a \ddot a = \frac{8\pi G}{3}\dot\rho a^2 + \frac{8\pi G}{3}\rho(2a\dot a)$$

As $\dot\rho = -3H(\rho+P)$, substituting,

$$\Rightarrow \ddot a = \frac{4\pi G a^2(-3H(\rho+P))}{3\dot a} + \frac{8\pi G}{3}\rho a$$

$$\Rightarrow \frac{\ddot a}{a} = \frac{4\pi G(-3H(\rho+P))}{3H} + \frac{8\pi G}{3}\rho$$

$$= \frac{4\pi G}{3}(-3\rho - 3P + 2\rho)$$

$= -\frac{4\pi G}{3}(\rho + 3P)$, the 2nd Friedmann Eqn.

2. $I_0 = \frac{F_0}{\Omega} = \frac{F_0}{A/d_A^2} = \frac{F d_A^2}{A}$ $F_0 = \frac{L_e}{4\pi d_L^2}$, so.

$I_0 = \frac{L_e}{4\pi d_L^2}\frac{d_A^2}{A}$. As $d_L = d_A(1+z)^2$

$$I_0 = \frac{L_e}{4\pi A}\frac{1}{(1+z)^4} = \frac{I_e}{(1+z)^4}$$

3- $m = -2.5\log_{10}\left(\frac{f}{f_0}\right), = -2.5\log_{10}\left(\frac{L/4\pi d_L^2}{f_0}\right)$.

$= -2.5\log_{10}(L) + 2.5\log_{10}(4\pi d_L^2 f_0)$.

As M is determined by the flux of object was 10 pc away,

$M = -2.5\log_{10}(L) + 2.5\log_{10}(4\pi(10pc)^2 f_0)$.

Then, $m = M - 2.5\log_{10}(10pc)^2 + 2.5\log_{10}(d_L)^2$

$= M + 5\log_{10}\left(\frac{d_L}{10pc}\right)$

$S = \frac{L}{4\pi D_L^2}$

$\Rightarrow \int S_{\nu_0}\, d\nu_0 = \int \frac{L_{\nu_e}\, d\nu_e}{4\pi D_L^2}$ $\quad\quad$ $\nu_e = (1+z)\nu_0$
$d\nu_e = (1+z)\, d\nu_0$

$\Rightarrow \int S_{\nu_0}\, d\nu_0 = \int \frac{L_{\nu_e}}{4\pi D_L^2}(1+z)\, d\nu_0$

$S_{\nu_e} = \frac{(1+z)\, L_{\nu_e}}{4\pi D_L^2}$

$S_\nu = (1+z)\frac{L_{\nu(1+z)}}{L_\nu}\frac{L_\nu}{4\pi D_L^2}$

Then,

$m = -2.5\log_{10}\left(\frac{f}{f_0}\right)$. Substituting S_ν for f,

$$= -2.5\log_{10}\left[\frac{\left(\frac{(1+z)L_{\nu(1+z)}}{L_\nu}\frac{L_\nu}{4\pi D_L^2}\right)}{f_0}\right]$$

$$= -2.5\log_{10}\left(\frac{L_\nu/4\pi D_L^2}{f}\right) - 2.5\log_{10}\left[\frac{(1+z)L_{\nu(1+z)}}{L_\nu}\right].$$

where the first term is the previous $m = M + DM$, and the second term is the K-correction.

Thus, $m = M + DM + K$.

4. $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3} - \frac{k}{a^2}$. $P_m = 0$, $P_\Lambda = -\rho_\Lambda$

$$\rho_\Lambda = \frac{\Lambda}{8\pi G},$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda) - \frac{k}{a^2},$$

and the second Friedmann Eqn becomes

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}(\rho + 3P) = \frac{-4\pi G}{3}\left(\rho_m + \rho_\Lambda + 3(P_m + P_\Lambda)\right)$$

$$= \frac{-4\pi G}{3}(\rho_m + \rho_\Lambda - 3\rho_\Lambda) = \frac{-4\pi G}{3}\left(\rho_m - \frac{\Lambda}{4\pi G}\right) = \boxed{\frac{-4\pi G}{3}\rho_m + \frac{\Lambda}{3}}.$$

$\dot{a}=0, \quad 0 = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3} - \frac{k}{a^2}$, 1st Friedmann Eqn

$$k = a^2\left(\frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3}\right).$$

$\ddot{a}=0, \quad 0 = \frac{-4\pi G}{3}\rho_m + \frac{\Lambda}{3}$, 2nd Friedmann Eqn from above

$$\boxed{\Lambda = 4\pi G\rho_m, \quad k = 4\pi G\rho_m a^2}$$

As $\rho_m, a^2 > 0$, $k > 0$, and the universe is closed.

$a(t) = 1 + \delta a(t)$, causes $\rho_m(t) = \rho_m(1 - 3\delta a(t))$.

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}\rho_m(t) + \frac{\Lambda}{3}$$

$$\Rightarrow \ddot{\delta a}(t) = \frac{-4\pi G}{3}\rho_m + 4\pi G\rho_m\delta a(t) + \frac{4\pi G}{3}\rho_m.$$

$$= 4\pi G\rho_m\delta a(t) = \Lambda\delta a(t).$$

The general solution to $\ddot{\delta a}(t) = \Lambda\delta a(t)$ is

$\delta a(t) = C_1 e^{\sqrt{\Lambda}t} + C_2 e^{-\sqrt{\Lambda}t}$
$\delta a(t=0) = C_1 + C_2 = \delta a_0.$

$\dot{\delta a}(t) = C_1\sqrt{\Lambda}e^{\sqrt{\Lambda}t} - C_2\sqrt{\Lambda}e^{-\sqrt{\Lambda}t}$
$\dot{\delta a}(0) = C_1\sqrt{\Lambda} - C_2\sqrt{\Lambda} = 0$
$C_1 = C_2.$

With the initial conditions, $\boxed{\delta a(t) = \frac{\delta a_0}{2} e^{\sqrt{\lambda} t} + \frac{\delta a_0}{2} e^{-\sqrt{\lambda} t}}$

This system has a positive restoring force, so any perturbation will lead it away from the initial state, instead of restoring equilibrium.

5. $$1+z = \frac{a(t_0)}{a(t_1)}$$

$$\frac{d(1+z)}{dt_0} = \frac{d\left(a(t_0)/a(t_1)\right)}{dt_0}$$

$$\Rightarrow \frac{dz}{dt_0} = \frac{a(t_1)\frac{da(t_0)}{dt_0} - a(t_0)\frac{da(t_1)}{dt_0}}{a(t_1)^2}$$

$$= \frac{\dot{a}(t_0)}{a(t_1)} - \frac{a(t_0)}{a(t_1)^2}\frac{da(t_1)}{dt_1}\frac{dt_1}{dt_0}.$$

As with the frequency of light, $dt_0 = (1+z)\,dt_1$, $\frac{dt_1}{dt_0} = \frac{1}{1+z}$.

Then,

$$\frac{dz}{dt_0} = \frac{\dot{a}(t_0)}{a(t_0)}\frac{a(t_0)}{a(t_1)} - \frac{a(t_0)}{a(t_1)}\frac{\dot{a}(t_1)}{a(t_1)}\frac{1}{1+z}.$$

$$= H_0(1+z) - \frac{(1+z)}{(1+z)}H(t_1).$$

$$\boxed{\frac{dz}{dt_0} = (1+z)H_0 - H(t_1).}$$

$$H(z) = H_0 (1+z)^{3/2}$$

$$\frac{dz}{dt_0} = (1+z) H_0 - H_0 (1+z)^{3/2}$$

At $z=1$,

$$\frac{dz}{dt_0} = H_0 (2 - 2^{3/2}) = -82.84h \ km/s/Mpc$$

$$= -2.684 \times 10^{-18} \ h/s.$$