1.) $\min_{w} \|Xw - y\|_2^2 + a^T w + b$

$X \in \mathbb{R}^{n \times d}$ $w \in \mathbb{R}^d$ $y \in \mathbb{R}^n$ $a \in \mathbb{R}^d$ $b \in \mathbb{R}$

col of X are lin ind $\therefore$ $X^T X$ is invertible

show $\|Xw - y\|_2^2 + a^T w + b = \|Xw - y + f\|_2^2 + g$

where $f \in \mathbb{R}^n$ $g \in \mathbb{R}$

expand left size:

$\|Xw - y\|_2^2 + a^T w + b$

$= (Xw)^T (Xw) - 2y^T Xw + y^T y + a^T w + b$

$= (Xw)^T (Xw) + (a^T - 2y^T X) w + y^T y + b$

expand right:

$\|Xw - y + f\|_2^2 + g$

$= (Xw)^T (Xw) - 2y^T Xw + 2f^T Xw - 2y^T f + y^T y$
$+ f^T f + g$

$= (Xw)^T (Xw) + (2f^T X - 2y^T X) w - 2y^T f + y^T y + f^T f + g$

set two sides equal, find f and g

$$\underbrace{(Xw)^T(Xw)} + \underbrace{(2f^TX - 2y^TX)w} + \underbrace{(-2y^Tf + y^Ty + f^Tf + g)}$$

$$= \overbrace{(Xw)^T(Xw)} + \overbrace{(a^T - 2y^TX)w} + \overbrace{(y^Ty + b)}$$

$$2f^TX - 2y^TX = a^T - 2y^TX$$

$$\Rightarrow 2f^TX = a^T$$

$$\Rightarrow f^TX = \frac{a^T}{2} \Rightarrow X^Tf = \frac{a}{2}$$

let $\boxed{f = X(X^TX)^{-1}\frac{a}{2}}$

$$X^T\left(X(X^TX)^{-1}\frac{a}{2}\right) = \frac{a}{2} \checkmark$$

$$-2y^Tf + y^Ty + f^Tf + g = y^Ty + b$$

$$\boxed{g = b - f^Tf + 2y^Tf}$$

---

solve $\underset{w}{\operatorname{argmin}} \|Xw - y + f\|_2^2 + g$

$$\frac{\partial}{\partial w}\left[\|Xw - y + f\|_2^2 + g\right] = 0$$

$$2X^T(Xw - y + f) = 0$$

$$2X^TXw - 2X^Ty + 2X^Tf = 0$$

$$X^TXw = X^Ty - X^Tf$$

$\boxed{w = (X^TX)^{-1}X^T(y - f)}$ where $f =$