Logic and Proof: Supervision 1

Daniel Chatfield

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Lecture 2 Exercise Questions

3. Each of the following formulas is satisfiable but not valid. Exhibit an interpretation that makes the formula true and another that makes the formula false.

   (a) $P \to Q$

   > The formula is true under the interpretation that maps P and Q to 1 but false for the interpretation that maps P to 1 and Q to 0.

   (b) $P \lor Q \to P \land Q$

   > The formula is true under the interpretation that maps P and Q to 1 but false for the interpretation that maps P to 1 and Q to 0.

   (c) $\neg(P \lor Q \lor R)$

   > The formula is true under the interpretation that maps P, Q and R to 0 but false under all other interpretations.

   (d) $\neg(P \land Q) \lor \neg(Q \lor R) \land (P \lor R)$

   > The formula is true under the interpretation that maps P, Q and R to 0 but false under the interpretation that maps P and Q to 0 and R to 1.

4. Convert each of the following propositional formulas into Conjunctive Normal Form and also into Disjunctive Normal Form. For each formula, state whether it is valid, satisfiable, or unsatisfiable; justify each answer.

(a) $(P \to Q) \land (Q \to P)$

Eliminating $\to$ gives:

$$(\neg P \lor Q) \land (\neg Q \lor P)$$

Conjunctive Normal Form *Already in Conjunctive Normal Form after first step*

Disjunctive Normal Form Applying distributive law gives:

$$((\neg P \lor Q) \land \neg Q) \lor ((\neg P \lor Q) \land P)$$

Simplifying gives:

$$(\neg P \land \neg Q) \lor (Q \land P)$$

Valid The formula is not valid as there are interpretations for which it is inconsistent. For example, the interpretation that maps P and Q to 1 and 0 respectively.

Satisfiable The formula is *satisfiable* as there are interpretations for which it is consistent. For example, the interpretation that maps P and Q both to 1.

(b) $((P \land Q) \lor R) \land \neg(P \lor R)$

No $\to$ to eliminate

Pushing negations in until they apply only to atoms, giving:

$$((P \land Q) \lor R) \land \neg P \land \neg R$$

Conjunctive Normal Form Push disjunctions in until they apply only to literals, giving:

$$(P \lor R) \land (Q \lor R) \land \neg P \land \neg R$$

Which simplifies to false as P and R must both be 0 to satisfy the third and fourth conjunct which means that the first conjunct cannot be satisfied and thus the formula is equivalent to false.

Disjunctive Normal Form Applying other distributive law, pushing the conjuncts in gives:

$$((P \land Q) \land \neg P \land \neg R) \lor (R \land \neg P \land \neg R)$$

Simplifying gives just f since the first disjunct contains both P and $\neg P$ and the second contains both R and $\neg R$.

Unsatisfiable The formula is unsatisfiable as I have shown that it is equivalent to false.

(c) $\neg(P \vee Q \vee R) \vee (P \wedge Q) \vee R$

Pushing negations in until they apply only to atoms, giving:

$$\neg P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q) \vee R$$

Conjunctive Normal Form Push disjunctions in until they apply only to literals, using:

$$A \vee (B \wedge C) \simeq (A \vee B) \wedge (A \vee C) \tag{1}$$

$$(B \wedge C) \vee A \simeq (B \vee A) \wedge (C \vee A) \tag{2}$$

$$(\neg P \wedge \neg Q \wedge \neg R) \vee ((P \wedge Q) \vee R)$$

Using (2) with $B = \neg P$, $C = \neg Q \wedge \neg R$ and $A = (P \wedge Q) \vee R$.

$$\simeq (\neg P \vee (P \wedge Q) \vee R) \wedge ((\neg Q \wedge \neg R) \vee (P \wedge Q) \vee R)$$

Using (2) with $B = P$, $C = Q$ and $A = R \vee \neg P$.

$$\simeq ((P \vee R \vee \neg P) \wedge (Q \vee R \vee \neg P)) \wedge ((\neg Q \wedge \neg R) \vee (P \wedge Q) \vee R)$$

$$\simeq (Q \vee R \vee \neg P) \wedge ((\neg Q \wedge \neg R) \vee (P \wedge Q) \vee R)$$

Using (2) with $B = \neg Q$, $C = \neg R$ and $A = (P \wedge Q) \vee R$ gives:

$$\simeq (Q \vee R \vee \neg P) \wedge ((\neg Q \vee ((P \wedge Q) \vee R)) \wedge (\neg R \vee (P \wedge Q) \vee R))$$

$$\simeq (Q \vee R \vee \neg P) \wedge (\neg Q \vee R \vee (P \wedge Q))$$

Using (1) with $A = \neg Q \vee R$, $B = P$ and $C = Q$ gives:

$$\simeq (Q \vee R \vee \neg P) \wedge (\neg Q \vee R \vee P) \wedge (\neg Q \vee R \vee Q)$$

Disjunctive Normal Form Applying other distributive law, pushing the conjuncts in gives:

$$(\neg P \land \neg Q \land \neg R) \lor (P \land Q) \lor R$$

Satisfiable The formula is satisfiable as it is consistent under the interpretation that maps P, Q, and R to 0, 1, and 0 respectively.

5. Using ML, define datatypes for representing propositions and interpretations. Write a function to test whether or not a proposition holds under an interpretation (both supplied as arguments). Write a function to convert a proposition to Negation Normal Form.

`sml` has stopped working on my machine and I haven't managed to fix it so I haven't been able to check this code.

```
datatype prop
  = Atom of Atom
  | Neg of prop
  | Conj of prop * prop
  | Disj of prop * prop

infix 3 Disj
infix 4 Conj

fun nnf(Neg(p1 And p2)) = nnf(Neg p1) Or nnf(Neg p2)
  | nnf(Neg(p1 Or p2))  = nnf(Neg p1) And nnf(Neg p2)
  | nnf(Neg (Neg p1))   = nnf p1
  | nnf(p1 And p2)      = nnf p1 Conj nnf p2
  | nnf(p1 Or p2)       = nnf p1 Disj nnf P2
  | nnf p               = p

type Atom = string
type interpretation = (Atom * bool) list
```

Lecture 3 Exercise Questions

6. Prove the following sequents:

   (a) $\neg\neg A \to A$

(b) $A \wedge B \to B \wedge A$

(c) $A \vee B \to B \vee A$

7. Prove the following sequents:

   (a) $(A \land B) \land C \to A \land (B \land C)$

   (b) $(A \lor B) \land (A \lor C) \to A \lor (B \land C)$

(c) $\neg(A \vee B) \to \neg A \wedge \neg B$

Lecture 4 Exercise Questions

9. (a) Describe a formula that is true in precisely those domains that contain at least m elements (We sat it *characterises* those domains.)

Formula:

$$\forall i, j \in \{0, 1, \ldots, m\} \quad \forall x_0, x_1, \ldots, x_m.\, x_i \neq x_j$$

(b) Describe a formula that characterises the domains containing fewer than m elements.

The negation of the first part:

$$\neg(\forall i, j \in \{0, 1, \ldots, m\}\, \forall x_0, x_1, \ldots, x_m.\, x_i \neq x_j)$$

10. Let $\approx$ be a 2-place predicate symbol, which we write using infix notation as $x \approx y$ instead of $\approx(x, y)$. Consider the axioms:

$$\forall x.\, x \approx x \tag{1}$$

$$\forall x, y.\, x \approx y \to y \approx x \tag{2}$$

$$\forall x, y, z.\, x \approx y \land y \approx z \to x \approx z \tag{3}$$

Let the universe be the set of natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$. Which axioms hold if $I[\approx]$ is:

(a) the empty relation, $\emptyset$?

Axioms (2) and (3) hold.

(b) the universal relation, $\{(x, y) \mid x, y \in \mathbb{N}\}$?

Axioms (1), (2) and (3) hold.

(c) the equality relation, $\{(x, x) \mid x \in \mathbb{N}\}$?

Axioms (1), (2) and (3) hold.

(d) the relation $\{(x, y) \mid x, y \in \mathbb{N} \land (x + y \text{ is even})\}$?

Axioms (1), (2) and (3) hold.

(e) the relation $\{(x, y) \mid x, y \in \mathbb{N} \land (x + y = 100)\}$?

Axioms (2) and (3) hold.

(f) the relation $\{(x, y) \mid x, y \in \mathbb{N} \land (x \leq y)\}$?

Axioms (1) and (3) hold.

11. Taking $=$ and R as 2-place relation symbols, consider the following axioms:

$$\forall x.\, \neg R(x, x) \tag{1}$$

$$\forall x, y.\, \neg(R(x, y) \land R(y, x)) \tag{2}$$

$$\forall x, y, z.\, R(x, y) \land R(y, z) \to R(x, z) \tag{3}$$

$$\forall x, y.\, R(x,y) \lor (x = y) \lor R(y,x) \tag{4}$$

$$\forall x, z.\, R(x,z) \to \exists y. R(x,y) \land R(y,z) \tag{5}$$

Consider only interpretations that make = denote the equality relation.

This exercise asks whether you can make the connection between the axioms and typical mathematical objects satisfying them. A start is to say that $R(x,y)$ means $x < y$, but on what domain?

(a) Exhibit two interpretations that satisfy axioms 1-5.

> Axioms 1-5 are satisfied under the interpretation that $R(x,y)$ means $x < y$ with the domain of real numbers. Similarly the interpretation that $R(x,y)$ means $x > y$.

(b) Exhibit two interpretations that satisfy axioms 1-4 and falsify axiom 5.

> Axioms 1-4 are satisfied and 5 falsified under the same interpretations as part (a) but with the domain of natural numbers.

(c) Exhibit two interpretations that satisfy axioms 1-3 and falsify 4 and 5.

> Axioms 1-3 are satisfied with 4 and 5 falsified under the interpretation that $R(x,y)$ means $x > 2y$ over the natural numbers.