Lecture 3:
Sorting Algorithms

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Sorting Problem

- Sorting problem
  - Input: an array $A[1..n]$ with n integers
  - Output: a sorted array A (in ascending order)

- Problem is: sort $A[1..n]$
- Input: | 8 | 6 | 1 | 3 | 7 | 2 | 5 | 4 |
- Output: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

Our Roadmap

- Comparison-based Sorting
  - Quadratic Cost
    - Selection Sort, Insertion Sort, Bubble Sort
  - O(n log n) Cost
    - Merge Sort, Heap Sort (we will skip here)
  - Quick Sort
- Other sorting algorithms
  - Counting sort, radix sort, bucket sort

Selection Sort

Selection Sort

- Idea of a selection sort method
  - Start with empty hand, all cards on table
  - Pick the smallest card from table
  - Insert the card into the hand

	8
	5
	2
	6
	9
	3
	1
	4
	0
	7

Selection Sort Algorithm

8 | 6 | 1 | 3 | 7 | 2 | 5 | 4

- SelectionSort
  - Input: an **array** A of n numbers
  - Output : an **array** A of n numbers in the <u>ascending order</u>
  - Selection-Sort ($A[1..n]$)

1. for integer $i \leftarrow 1$ to $n-1$
2. $\quad k \leftarrow i$
3. $\quad$ for integer $j \leftarrow i+1$ to n
4. $\quad\quad$ if $A[k] > A[j]$ then
5. $\quad\quad\quad k \leftarrow j$
6. $\quad$ swap $A[i]$ and $A[k]$

1 | 6 | 8 | 3 | 7 | 2 | 5 | 4

sorted *unsorted*

1 | 2 | 8 | 3 | 7 | 6 | 5 | 4

sorted *unsorted*

Selection Sort Time Complexity

- Selection Sort
  - Input: an **array** A of n numbers
  - Output : an **array** A of n numbers in the <u>ascending order</u>
  - Selection-Sort ($A[1..n]$)

1. for integer $i \leftarrow 1$ to $n-1$ — Cost: n-1=O(n)
2. $k \leftarrow i$ — Cost: n-1=O(n)
3. for integer $j \leftarrow i+1$ to n — Cost: n-1+n-2+…+1=O(n^2)
4. if $A[k] > A[j]$ then — Cost: O(n^2)
5. $k \leftarrow j$ — Cost: O(n^2)
6. swap $A[i]$ and $A[k]$ — Cost: O(n)

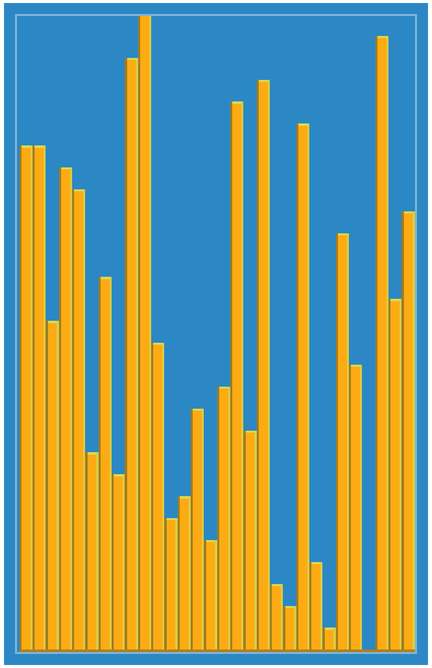
- Selection sort total cost:
- $O(n)+O(n)+O(n^2) +O(n^2) +O(n^2) +O(n)=O(n^2)$

Insertion Sort

Insertion Sort

- Idea of a insertion sort method
  - One input each iteration, growing a sorted output list
  - Remove one element from input data
  - Find the location it belongs within the sorted list
  - Repeat until no input elements remain

6 5 3 1 8 7 2 4



Insertion Sort Algorithm

- InsertionSort
  - Input: an **array** *A* of *n* numbers
  - Output : an **array** *A* of *n* numbers in the ascending order
  - Insertion-Sort (*A*[1..*n*])

```
for integer i ← 1 to n
    for integer j ← i to 1 with j > 1
        if  A[j-1] > A[j]  then
            swap A[j-1] and A[j]
        else  break
```

8 | 6 | 1 | 3 | 7 | 2 | 5 | 4

sorted *unsorted*

8 | 6 | 1 | 3 | 7 | 2 | 5 | 4

6 | 8 | 1 | 3 | 7 | 2 | 5 | 4

sorted *unsorted*

1 | 6 | 8 | 3 | 7 | 2 | 5 | 4

Insertion Sort Time Complexity

- Insertion Sort
  - Input: an **array** *A* of *n* numbers
  - Output : an **array** *A* of *n* numbers in the ascending order
  - Insertion-Sort (*A*[1..*n*])

1. for integer *i* ← 1 to *n* — Cost: $n-1=O(n)$
2. for integer *j* ← *i to 1 with j > 1* — Cost: $1+\ldots+n-2=O(n^2)$
3. if *A*[*j-1*] > *A*[*j*] then — Cost: $O(n^2)$
4. swap *A*[*j-1*] and *A*[*j*] — Cost: $O(n^2)$
5. else break

- Insertion sort total cost:
  - $O(n)+O(n^2)+O(n^2)+O(n^2)=O(n^2)$

Bubble Sort

Bubble Sort

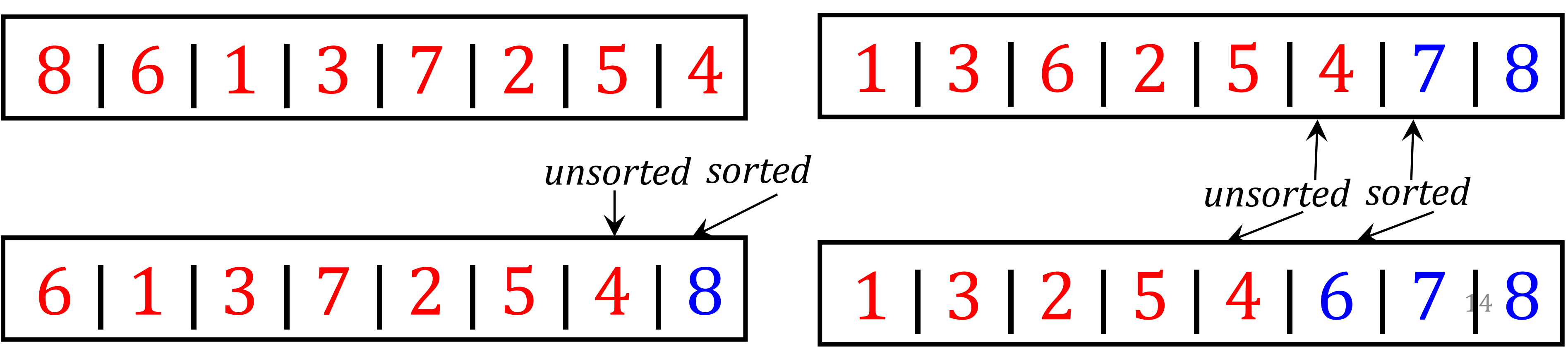
- Idea of a bubble sort method
  - For each pass
    - Compare the pair of adjacent item
    - Swap them if they are in the wrong order
  - Repeat the pass through until no swaps are needed

6 5 3 1 8 7 2 4

Bubble Sort Algorithm

- BubbleSort (optimized version)
  - Input: an **array** *A* of *n* numbers
  - Output : an **array** *A* of *n* numbers in the ascending order
  - Bubble-Sort ($A[1..n]$)

1. for integer $i \leftarrow 1$ to $n-1$
2. for integer $j \leftarrow 2$ to n
3. if $A[j-1] > A[j]$ then
4. swap $A[j-1]$ and $A[j]$



Bubble Sort Time Complexity

- Bubble Sort
  - Input: an **array** *A* of *n* numbers
  - Output : an **array** *A* of *n* numbers in the ascending order
  - Bubble-Sort ($A[1..n]$)

1. for integer $i \leftarrow 1$ to $n-1$ — Cost: n-1=O(n)
2. for integer $j \leftarrow 2$ to n — Cost: n-1+…+n-1=O(n^2)
3. if $A[j-1] > A[j]$ then — Cost: O(n^2)
4. swap $A[j-1]$ and $A[j]$ — Cost: O(n^2)

- Bubble sort total cost:
  - $O(n)+O(n^2)+O(n^2)+O(n^2)=O(n^2)$

Pop Quiz

- We say a sorting algorithm is "stable" if it does not change the relative order of elements with equal keys, which of the following is/are stable ()

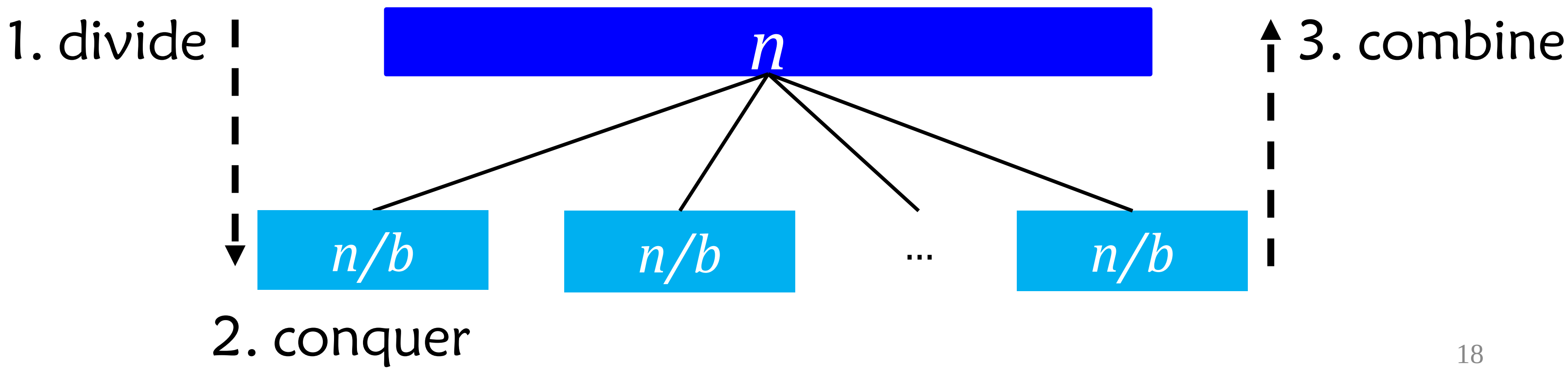
  A: Selection sort B: Insertion Sort C: Bubble Sort

- Watch a video:
  - 1) Which sorting algorithm is used in that video?
  - 2) TB is No. x, so x = ?

Merge Sort
(Divide-and-Conquer)

Divide and Conquer

- Divide and Conquer: an algorithmic technique
  - **Divide:** divide the problem into smaller subproblems
  - **Conquer:** solve each subproblem recursively
  - **Combine:** combine the solution of subproblems into the solution of the original problem



Example: Merge Sort

- Sorting problem
  - Input: an array $A[1..n]$ with n integers
  - Output: a sorted array A (in ascending order)

- Original problem is: sort $A[1..n]$

| 8 | 6 | 1 | 3 | 7 | 2 | 5 | 4 |

- What is a subproblem?
  - Sort a subarray $A[l..r]$

| 7 | 2 | 5 | 4 |

Merge Sort

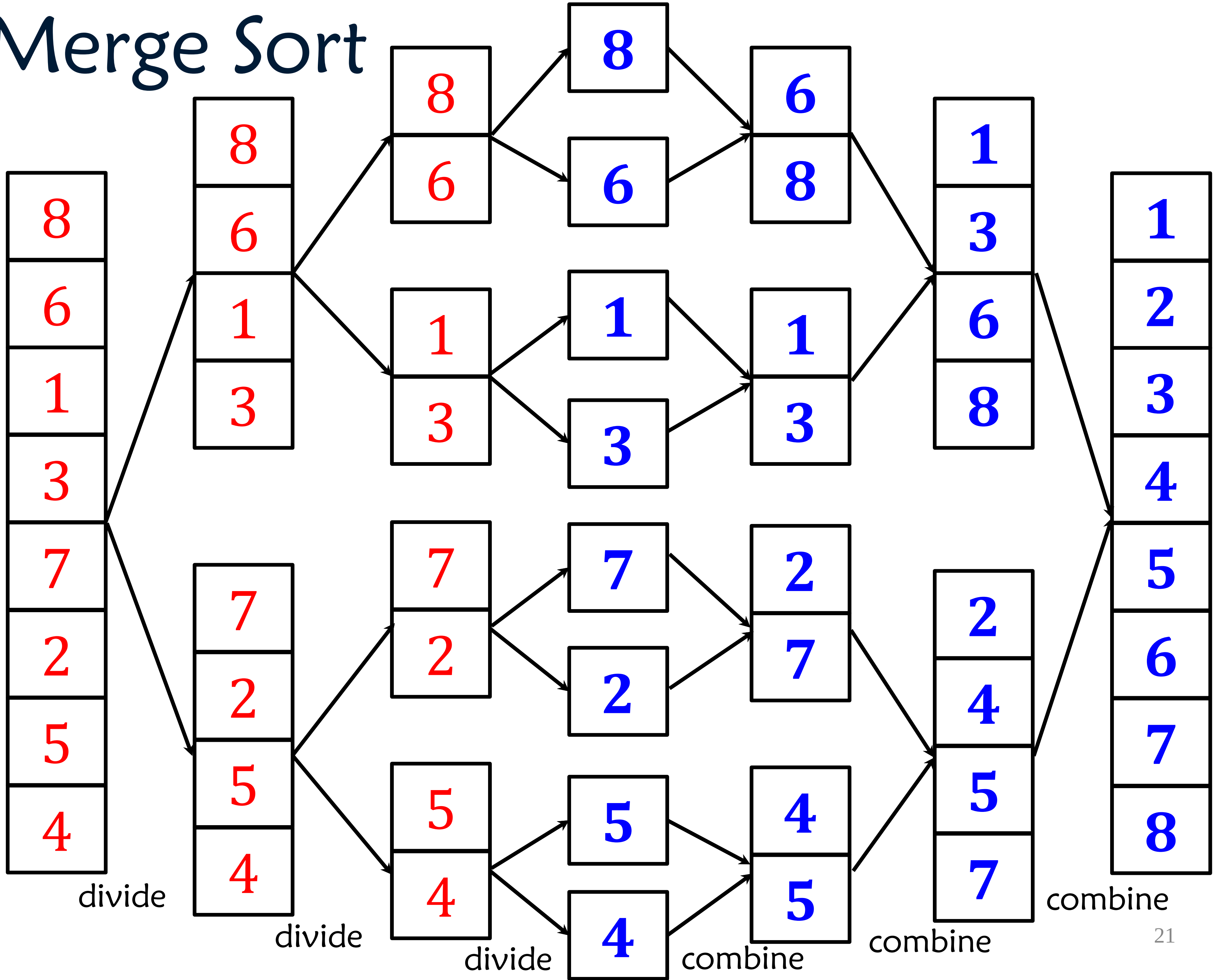
- Merge Sort
  - **Divide:** divide the array into two subarrays of n/2 numbers each
  - **Conquer:** sort two subarrays recursively
  - **Combine:** merge two sorted subarrays into a sorted array

Merge-Sort(A, n)

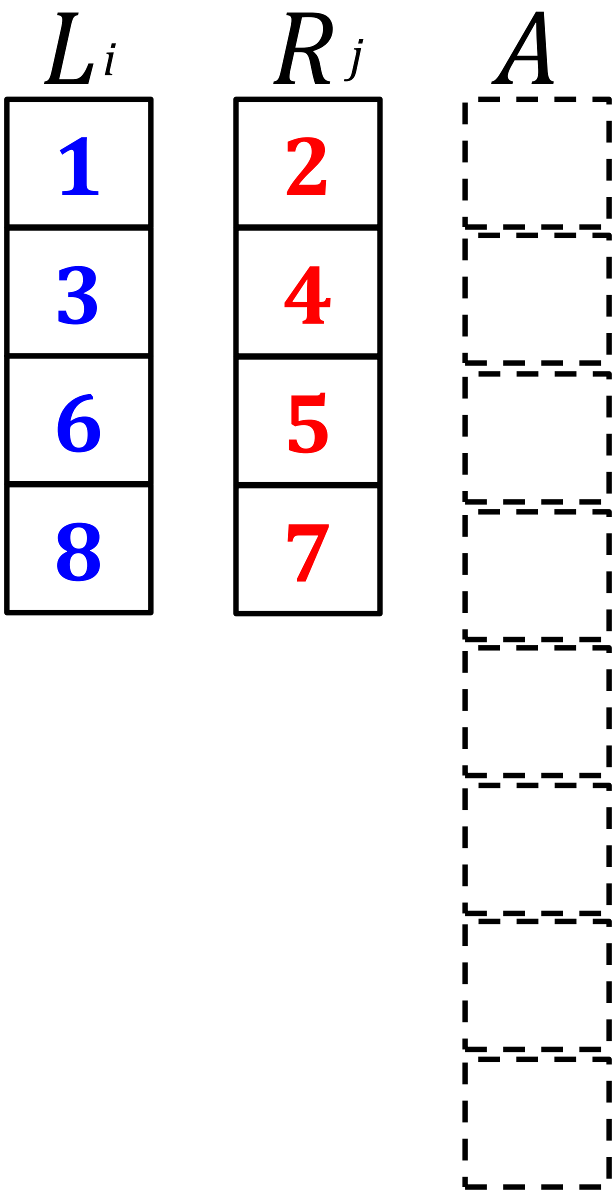
1. if $n > 1$
2. $p \leftarrow \lfloor n/2 \rfloor$
3. $B[1..p] \leftarrow A[1..p]$
4. $C[1..n\text{-}p] \leftarrow A[p\text{+}1..n]$
5. Merge-Sort(B, p)
6. Merge-Sort(C, n-p)
7. $A[1..n] \leftarrow$ Merge(B, p, C, n-p)

We'll discuss the Combine phase ("Merge" function) later

Merge Sort


8
6
1
3
7
2
5
4
divide
divide
divide
combine
combine
combine
1
2
3
4
5
6
7
8

Merge Sort: Combine Phase



Sorted arrays

Merge(L, n_L, R, n_R)

1. n ← $n_L + n_R$
2. let A[1..n] be a new array
3. i ← 1; j ← 1
4. for k ← 1 to n
5.   if $i \le n_L$ and ($j > n_R$ or $L[i] \le R[j]$)
6.   A[k] ← $L[i]$; i ← $i + 1$
7.   else
8.   A[k] ← $R[j]$; j ← $j + 1$
9. return A

Running time of Merge

Sorted arrays

Merge(L, n_L, R, n_R)

```
1. n ← nL + nR
2. let A[1..n] be a new array
3. i ← 1; j ← 1
4. for k ← 1 to n
5.    if i ≤ nL and (j>nR or L[i] ≤ R[j])
6.       A[k] ← L[i] ;  i ← i + 1
7.    else
8.       A[k] ← R[j] ;  j ← j + 1
9.  return A
```

- Let $n = n_L + n_R$ be the total number of items
- Time of merge: $O(n)$ time
  - Line 1: $O(1)$
  - Line 2: $O(n)$
  - Line 3: $O(1)$
  - Lines 4-8: $O(n)$

Running time of Merge Sort

Merge-Sort(A, n)

1. if $n > 1$
2. $p \leftarrow \lfloor n/2 \rfloor$
3. $B[1..p] \leftarrow A[1..p]$
4. $C[1..n\text{-}p] \leftarrow A[p+1..n]$
5. Merge-Sort(B, p)
6. Merge-Sort(C, n-p)
7. $A[1..n] \leftarrow$ Merge(B, p, C, n-p)

- Let $T(n)$ be the running time of Merge Sort
  - Lines 3, 4 take $O(n)$ time
  - Line 5 takes $T(n/2)$ time
  - Line 6 takes $T(n/2)$ time
  - Line 7 takes $O(n)$ time
- Thus, we obtain the recurrence

$$T(n) = 2\,T(n/2) + O(n)$$

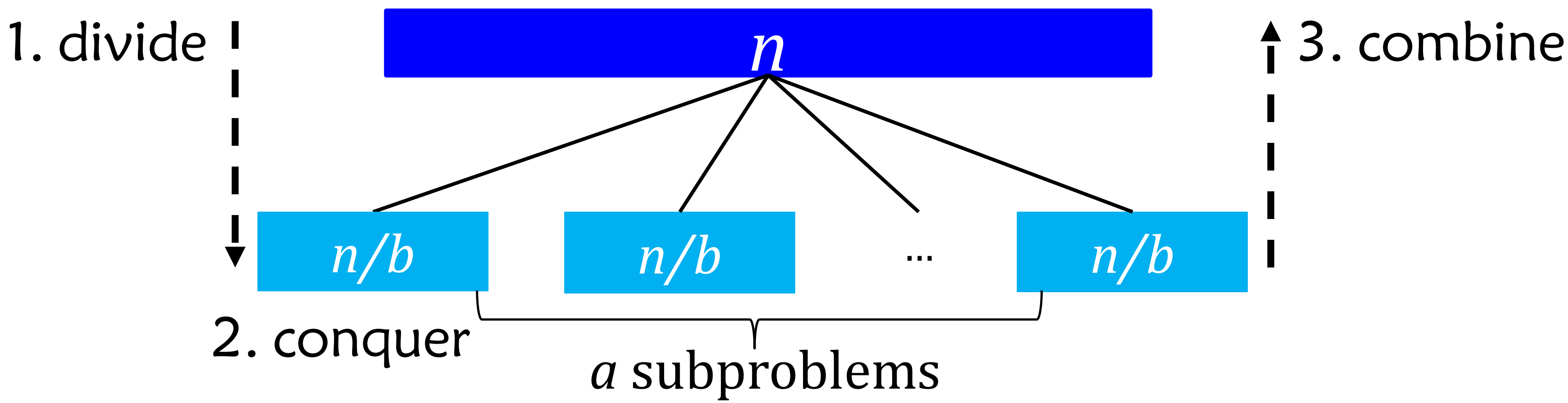
- Solving it, we get:

$$T(n) = O(n\log n)$$

Time Complexity

- T(n): time complexity of algorithm at input size n
  - Divide the problem into a subproblems
  - Size of each subproblem is n/b
  - Combine phase takes $f(n)$ time

Note: a and b can have different values



- Recurrence equation: $T(n) = a\,T(n/b) + f(n)$
- E.g., Merge Sort: $T(n) = 2\,T(n/2) + O(n)$

Methods for Solving Recurrences

- Recurrence equation: $T(n) = a\, T(n/b) + f(n)$
- Two methods for solving recurrences
  - Master theorem
  - Substitution method
- Master theorem
  - It could be proved by carefully applying the "expansion method", the details are tedious and omitted from this course
- Substitution method (we skip it here)
  - It is mathematical induction

Master Theorem

- Recurrence equation: T(n) = a T(n/b) + f(n)
- Let T(n) be a function that return a positive value for every integer n>0. We know that:
  - $T(1) = O(1)$
  - $T(n) = \alpha T\left(\left\lceil\frac{n}{\beta}\right\rceil\right) + O(n^{\gamma})$ for (n ≥ 2)

  where $\alpha \geq 1, \beta > 1,$ and $\gamma \geq 0$. Then:
  - If $\log_\beta \alpha < \gamma$, then $T(n) = O(n^{\gamma})$
  - If $\log_\beta \alpha = \gamma$, then $T(n) = O(n^{\gamma} \log n)$
  - If $\log_\beta \alpha > \gamma$, then $T(n) = O(n^{\log_\beta \alpha})$

Master Theorem

- Consider the recurrence of **binary search**:
  - T(1) ≤ c1
  - T(n) ≤ T(n/2) + c2 (for n ≥ 2)
  - Hence, α = 1, β = 2, and γ = 0. Since $\log_\beta \alpha = \log_2 1 = 0 = \gamma$, we know that $T(n) = O(n^0 \log n) = O(\log n)$.
- Consider the recurrence of **merge sort**:
  - T(1) ≤ c1
  - T(n) = 2 T($n/2$) + O(n) = 2 T($n/2$) + c2 n (for n ≥ 2)
  - Hence, α = 2, β = 2, and γ = 1. Since $\log_\beta \alpha = \log_2 2 = 1 = \gamma$, we know that $T(n) = O(n^1 \log n) = O(\mathrm{n}\log n)$.

Quick Sort
RAM with Randomization

Deterministic & Randomized

- So far in CS203, all our algorithms are deterministic, namely, they do not involve any randomization.
- We will introduce randomized algorithms, e.g., quick sort in the sorting problem.
- Randomized algorithms play an important role in computer science, they often simpler, and sometimes can be provably faster as well.
- Recall the core of the RAM model is a set of atomic operations, we extend this set with:
  - RANDOM(x, y): given integers x and y (x <= y), this operation returns an integer chosen uniformly at random in [x,y], i.e., x, x+1, ..., y has the same probability of being returned.

Randomized Algorithm Example

- Find-a-Zero: Given an array of integers with size n, among which there is at least 0. Design an algorithm to report an arbitrary position of A that contains a 0
- Suppose A = (9,18,0,0,15,0), an algorithm can report 3,4 or 6, consider the following randomized algorithm
- 1. do
- 2. r ← RANDOM(1,n)
- 3. until A[r]=0
- 4. return r
- What is the cost of the algorithm? It depends
  - If all numbers in A are 0, O(1) time. If A has only one 0, O(n) expected time.
  - As before, we care about the worst expected time: O(n)

Quick Sort

- Idea of a quick sort method
  - Randomly pick an integer p in A, call it the pivot
  - Re-arrange the integers in an array A' such that
    - All the integers smaller than p are positioned before p in A'
    - All the integers larger than p are positioned after p in A'
  - Sort the part of A' before p recursively
  - Sort the part of A' after p recursively

Quicksort

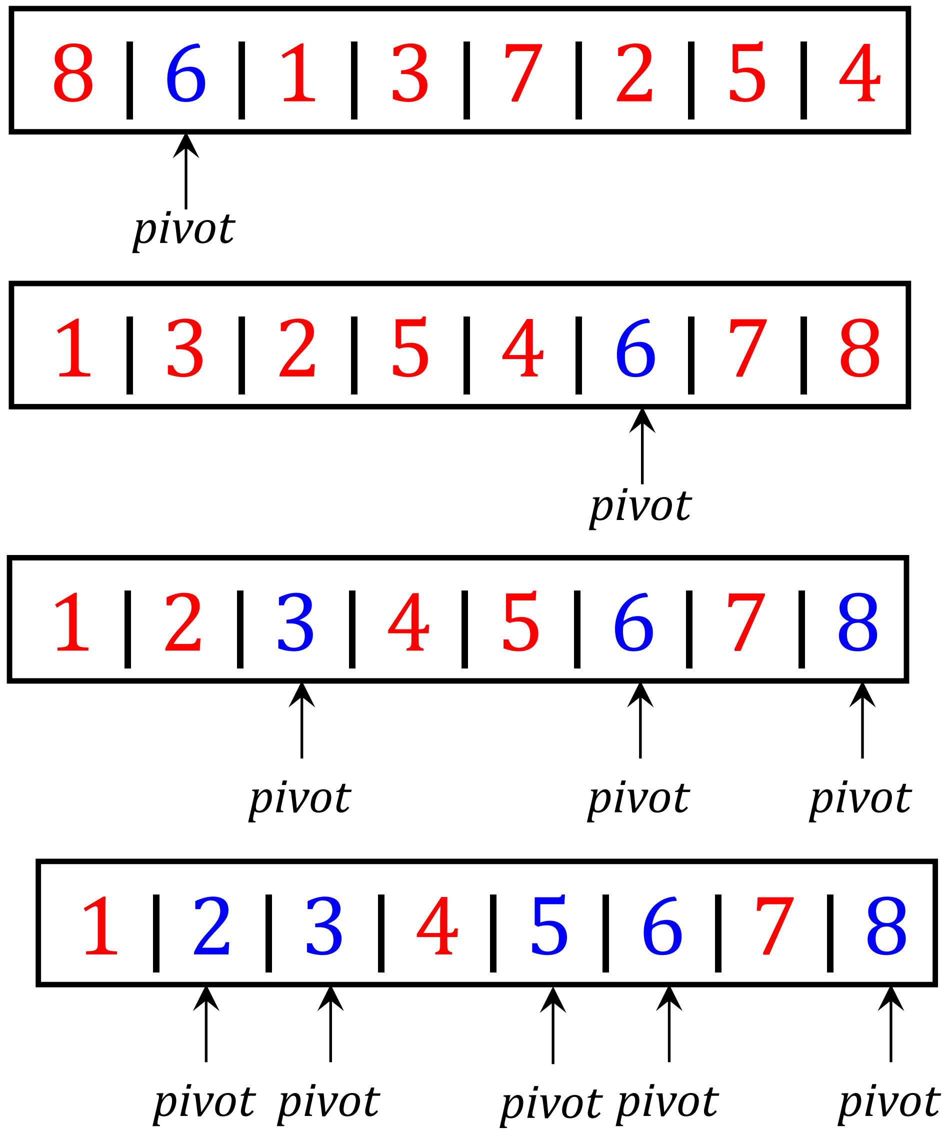
- Quick Sort
  - Input: an **array** *A* of *n* numbers
  - Output : an **array** *A* of *n* numbers in the ascending order
  - Quicksort ($A[1..n]$, lo=1, hi=n)
    1. p ← partition(A, lo, hi)
    2. Quicksort(A,lo,p-1)
    3. Quicksort(A,p+1,hi)

Quicksort

- Partition(A, lo, hi)
  1. p ← RANDOM(lo, hi); pivot ← A[p];
  2. L ← lo, R← hi
  3. for integer i from lo to hi
  4.   if(i!=p)
  5.   if(A[i] < pivot) A’[L++]←A[i]
  6.   else A’[R--]←A[i]
  7. A’[L] ← pivot
  8. A[lo, hi] ← A’
  9. return L;

- Question:
  - If we set p ← lo or hi in Line 1, quick sort is still correct ?
  - What are the difference between p ← lo/hi and p ← RANDOM(lo, hi)?

Quicksort Example



Quicksort Time Complexity

- Quicksort's running time is not attractive in the worst case: it is $O(n^2)$ (why?) However, quick sort is fast in expectation, i.e., O(nlogn), remember this holds for every input array A.
- Whether quicksort has any advantage over merge sort? which guarantees O(nlogn) in the worst case.
- No in theory, but there is an advantage in practice
- Quicksort permits a faster implementation that leads to a smaller hidden constant compared to merge sort. (why?)

Quicksort Time Complexity

- Let X be the number of comparisons in quicksort algorithm. The running is bounded by O(n+x).
- We prove that E[X]=O(n log n)
- Denote e_i be the i-th smallest integer in A, consider e_i and e_j for any i,j such that i!=j
- What is the probability that quicksort compares e_i and e_j?
  - Every element will be selected as pivot precisely once
  - e_i and e_j are not compared, if any element between them gets selected as a pivot before them.
  - Therefore, e_i and e_j are compared if and only if either one is the first among e_i , e_{i+1} ,..., e_j picked as a pivot
  - The probability is 2/(j-i+1) (random pivot selection)

Quicksort Time Complexity

- Define random variable X_{ij} to be 1, if e_i and e_j are compared. Otherwise, X_{ij} to be 0. Thus, we have
- $Pr[X_{ij} = 1] = 2/(j-i+1)$, that is $E[X_{ij}] = 2/(j-i+1)$
- Since $X = \sum_{i,j} X_{ij}$, hence:
- $E[X] = \sum_{i,j} E[X_{ij}] = \sum_{i,j} \frac{2}{j-i+1}$
- $= 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{j-i+1}$
- $= 2 \sum_{i=1}^{n-1} O(\log(j-i+1))$ $(1+1/2+\ldots+1/n = O(\log n))$
- $= 2 \sum_{i=1}^{n-1} O(\log n)$
- $= O(n \log n)$
- Harmonic series: $1+1/2+\ldots+1/n$, which is frequently encountered in computer science.

Summary

Sort	Average	Space
Selection	$O(n^2)$	O(1)
Insertion	$O(n^2)$	O(1)
Bubble	$O(n^2)$	O(1)
Heap	O(nlogn)	O(1)
Merge	O(nlogn)	Depends
Quick	O(nlogn)	O(1)

- Comparison lower bound of sorting algorithm: $\Omega(n \log n)$
- We omit the proof here.

Other Sorting Methods

Other Sorting Algorithms

- Counting sort (Chapter 8.2)
  - it is applicable when each input is known to belong to a particular set, S, of possibilities. The algorithm runs in $O(|S| + n)$ time and $O(|S|)$ memory where n is the length of the input.
- Radix sort (Chapter 8.3)
  - radix sort is an algorithm that sorts numbers by processing individual digits. n numbers consisting of k digits each are sorted in $O(n \cdot k)$ time
- Bucket sort (Chapter 8.4)
  - Bucket sort is a divide and conquer sorting algorithm that generalizes counting sort by partitioning an array into a finite number of buckets.

Thank You!