Quiz 1: (50p max; 60p total = 10p each + 10p free)
Please write your answers in English and submit via Blackboard

1. Determine if each statement is a tautology: (no proof required)

   (1) $p \to q \leftrightarrow \neg q \to \neg p$

   (2) $\forall x\, (P(x) \to Q(x)) \leftrightarrow \exists x\, P(x) \to \forall x\, Q(x)$

2. Given a finite set C and its subsets A, B, prove "$\big|\, |A| - |B| \,\big| \leq |C|$".

3. Prove $(B - A) \cup (C - A) = (B \cup C) - A$ with set identities.

4. List the first 5 terms of $\{a_n\}$, $n = 0, 1, 2, \ldots,$ where a_n equals:

   (1) $\lfloor -n / 3 \rfloor + \lceil n / 4 \rceil + n!$

   (2) $\sum_{k=0}^{n} 2^k$

5. Prove that "if A, B, C are sets such that $|A| \leq |B|$ and $|B| = |C|$, then $|A| \leq |C|$". (Note that A, B, C could be infinite sets.)

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Solutions

- Q1.
  - (1) Yes, implication is equivalent to its contrapositive.
  - (2) No, e.g., it is obviously not a tautology when *P(x)* = *Q(x)*.
- Q2. Use proof by cases, e.g., case 1: $|A| < |B|$, case 2: $|A| \geq |B|$.
- Q3. Proof with set identities (no need to write out their names):

$$
\begin{aligned}
&(B - A) \cup (C - A) \\
&= (B \cap \bar{A}) \cup (C \cap \bar{A}) \quad \text{* Definition} \\
&= (B \cup C) \cap \bar{A} \quad \text{* Distributive} \\
&= (B \cup C) - A \quad \text{* Definition}
\end{aligned}
$$

Solutions

- Q4. (1) *1, 1, 2, 6, 23* (2) *1, 3, 7, 15, 31*
- Q5. (key points: injective/bijective functions and composition)
  - By definition

    $|A| \leq |B|$ means that there is a injective function $f: A \to B$

    $|B| = |C|$ means that there is a bijective function $g: B \to C$
  - Then, by definition we need to show there is a injective function from A to C. It suffices to show the composition $g \circ f$ is injective:

    i.e., for any $x, y \in A$ such that $x \neq y$, we have $g \circ f\ (x) \neq g \circ f\ (y)$.
  - The above holds because f and g are both injective:

    f injective: for any $x, y \in A$ such that $x \neq y$, we have $f(x) \neq f(y)$

    g injective: for any $x, y \in B$ such that $x \neq y$, we have $g(x) \neq g(y)$
  - So, for $x, y \in A$ such that $x \neq y$, we have $f(x) \neq f(y)$ with $f(x), f(y) \in B$, and hence $g(f(x)) \neq g(f(y))$, i.e., $g \circ f\ (x) \neq g \circ f\ (y)$.