CS201: Discrete Mathematics (Fall 2023)
Written Assignment #5
(100 points maximum but 110 points in total)
Deadline: 11:59pm on Dec 25 (please submit via Blackboard)
PLAGIARISM WILL BE PUNISHED SEVERELY

Q.1 (**15p**) Let S be the set of all strings of English letters. Determine whether the following relations are *reflexive, irreflexive, symmetric, antisymmetric*, and/or *transitive*. No proof is required.

(a) (**3p**) $R_1 = \{(a, b)|a$ and b have no letters in common$\}$

(b) (**3p**) $R_2 = \{(a, b)|a$ and b are of the same length$\}$

(c) (**3p**) $R_3 = \{(a, b)|a$ is shorter than $b\}$

(d) (**3p**) $R_4 = \{(a, b)|a$ and b have exactly one letter in common$\}$

(e) (**3p**) $R_5 = \{(a, b)|a$ contains b as a substring$\}$

Q.2 (**15p**) Consider relations on a set A. **Prove or disprove** the following statements:

(a) (**5p**) If R is reflexive and symmetric, then R is transitive.

(b) (**5p**) If R_1, R_2 are reflexive, then $R_1 \cup R_2$ is reflexive.

(c) (**5p**) If R_1, R_2 are antisymmetric, then $R_1 \cup R_2$ is antisymmetric.

Q.3 (**10p**) Prove the following statements about n-ary relations:

(a) (**5p**) If C_1 and C_2 are conditions that elements of the n-ary relation $R : A_1, \ldots, A_n$ may satisfy, then $s_{C_1 \wedge C_2}(R) = s_{C_1}(s_{C_2}(R))$.

(b) (**5p**) If R and S are n-ary relations, then $P_{i_1,i_2,\ldots,i_m}(R \cup S) = P_{i_1,i_2,\ldots,i_m}(R) \cup P_{i_1,i_2,\ldots,i_m}(S)$.

Q.4 (**10p**) Suppose that a relation R on a set A is symmetric.

(a) (**7p**) Prove that, for any positive integer $n \geq 1$, R^n is symmetric.

(b) (**3p**) Prove that R^* is symmetric.

Q.5 (**5p**) Prove that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

Q.6 (**10p**) Use the Floyd-Warshall algorithm to find the transitive closures of the relation $R = \{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (c, b), (d, a), (e, d)\}$ on set $\{a, b, c, d, e\}$.

Q.7 (**10p**) Consider the relation $R = \{(x, y) \mid x - y \in \mathbf{Z}\}$.

(a) (**7p**) Prove that R is an equivalence relation on the set of real numbers $\mathbf{R}$.

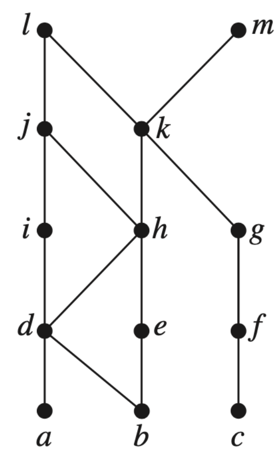
(b) (**3p**) Describe what elements the following equivalence classes consist of: $[1]$, $[\frac{1}{2}]$, and $[\pi]$.

Q.8 (**10p**) For any functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$. We say f is *dominated* by g, denoted by $f \preceq g$, if and only if $\forall x \in \mathbf{R}, f(x) \leq g(x)$ holds. Prove or disprove the following:

(a) (**7p**) The relation $\preceq$ is a partial ordering.

(b) (**3p**) The relation $\preceq$ is a total ordering.

Q.9 (**20p**) Answer questions about the partial order represented by the Hasse diagram below:



(a) (**3p**) Find the maximal elements.

(b) (**3p**) Find the minimal elements.

(c) (**2p**) Is there a greatest element?

(d) (**2p**) Is there a least element?

(e) (**3p**) Find all upper bounds of $\{a, e, f\}$.

(f) (**2p**) Find the least upper bound of $\{a, e, f\}$, if it exists.

(g) (**3p**) Find all lower bounds of $\{h, i, j\}$.

(h) (**2p**) Find the greatest lower bound of $\{h, i, j\}$, if it exists.

Q.10 (**5p**) Topological sorting. Find **all** compatible total orderings for the poset $(\{2, 3, 4, 6, 12\}, |)$.