**Quiz 2:** (50p max; 60p total = 10p each + 10p free)
Please write your answers in English and submit via Blackboard

1. Prove that $1^2 - 2^2 + 3^2 \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}n(n+1)/2$ for $n \in \mathbf{Z}^+$.

2. Solve the recurrence relation $T(n) = 8T(n / 2) - n^2$ by iterating it, where $n$ is a power of $2$ and $T(1) = 1$.

3. Use a combinatorial proof to show the identity holds:
$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r},$$ where $0 \le r \le k \le n$.

4. Find all solutions of the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + n$.

5. How many onto functions from a set with $5$ elements to a set with $3$ elements? (It is OK to write your answer with binomial coefficients. No need to calculate the exact number.)

# Solutions

- Q1. This is easy to prove by induction.
- Q2. Iterating the recurrence yields $T(n) = 8^{\log_2 n}T(1) - (n-1)n^2$. Note that $8^{\log_2 n} = n^3$ and $T(1) = 1$, we have $T(n) = n^2$.
- Q3. To perform a combinatorial proof, we need to construct a combinatorial problem and count the result in two ways. Let us choose $k$ players out of $n$ people and split them into two groups, with $r$ players in group 1 and $k$ - $r$ players in group 2.
  - It is easy to see there are $\binom{n}{k}\binom{k}{r}$ ways to form these two groups.
  - On the other hand, one can also get these groups by first choosing the $r$ players for group 1 and then choosing $k-r$ players from the remaining $n-r$ people for group 2, which yields $\binom{n}{r}\binom{n-r}{k-r}$ ways.

# Solutions

- Q4.
  - First, it is natural to guess a solution to the recurrence relation is of the form $a_n = p(n) = cn + d$. By plugging it into the recurrence relation, we can solve $c, d$ and get $p(n) = n + 4$.
  - Then, solve the associated homogeneous recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$. Its characteristic equation is $r^2 = 4r - 4$, which has only one root $2$ with multiplicity $2$. So, all solutions of $a_n = 4a_{n-1} - 4a_{n-2}$ are of the form $h(n) = \alpha_1 2^n + \alpha_2 n 2^n$.
  - Together, all solutions of the original recurrence relation are of the form $a_n = h(n) + p(n) = \alpha_1 2^n + \alpha_2 n 2^n + n + 4$.

- Q5. Please refer to page 39 of slides 07. The answer is

  $3^5 - \binom{3}{1} \cdot 2^5 + \binom{3}{2} \cdot 1^5$ * *no need to calculate the final answer*



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