TREES

Chapter 6

Chapter Objectives

- To learn how to use a tree to represent a hierarchical organization of information
- To learn how to use recursion to process trees
- To understand the different ways of traversing a tree
- To understand the differences between binary trees, binary search trees, and heaps
- To learn how to implement binary trees, binary search trees, and heaps using linked data structures and arrays

Chapter Objectives (cont.)

- To learn how to use a binary search tree to store information so that it can be retrieved in an efficient manner
- To learn how to use a Huffman tree to encode characters using fewer bytes than ASCII or Unicode, resulting in smaller files and reduced storage requirements

Trees - Introduction

- All previous data organizations we've studied are linear—each element can have only one predecessor and successor
- Accessing all elements in a linear sequence is O(n)
- Trees are nonlinear and hierarchical
- Tree nodes can have multiple successors (but only one predecessor)

Trees - Introduction (cont.)

- Trees can represent hierarchical organizations of information:
  - class hierarchy
  - disk directory and subdirectories
  - family tree
- Trees are recursive data structures because they can be defined recursively
- Many methods to process trees are written recursively

Trees - Introduction (cont.)

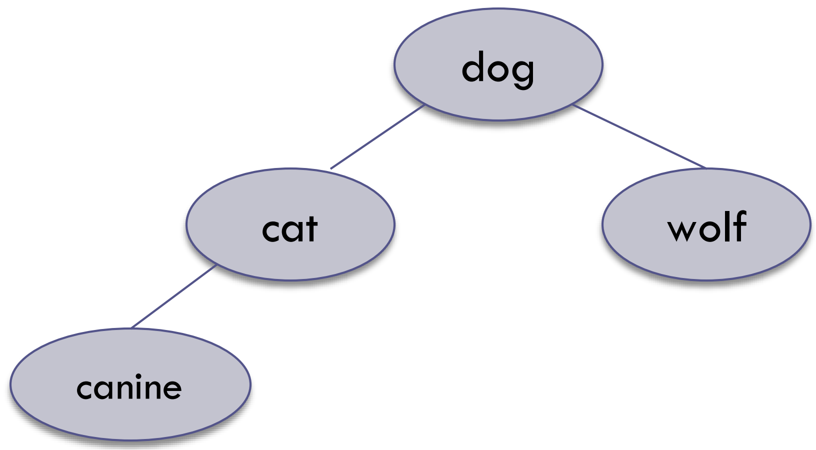
- This chapter focuses on the *binary tree*
- In a binary tree each element has two successors
- Binary trees can be represented by arrays and by linked data structures
- Searching a binary search tree, an ordered tree, is generally more efficient than searching an ordered list—O(log *n*) versus O(n)

Tree Terminology and Applications

Section 6.1

Tree Terminology

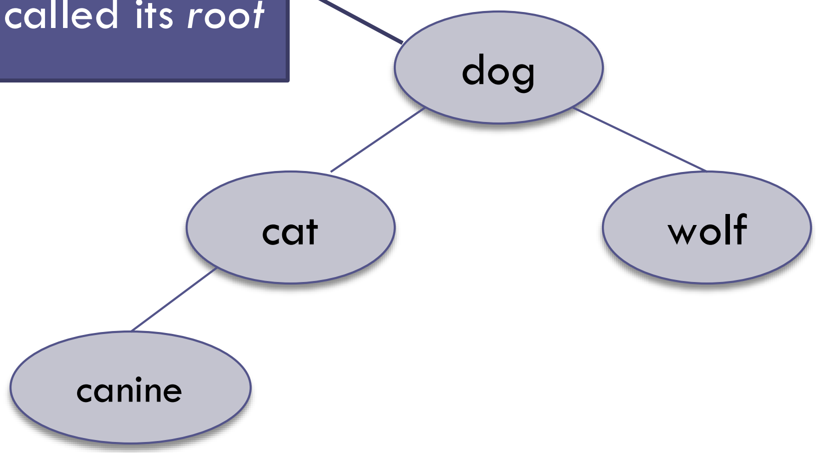
A tree consists of a collection of elements or nodes, with each node linked to its successors



Tree Terminology (cont.)

A tree consists of a collection of elements or nodes, with each node linked to its successors

The node at the top of a tree is called its *root*



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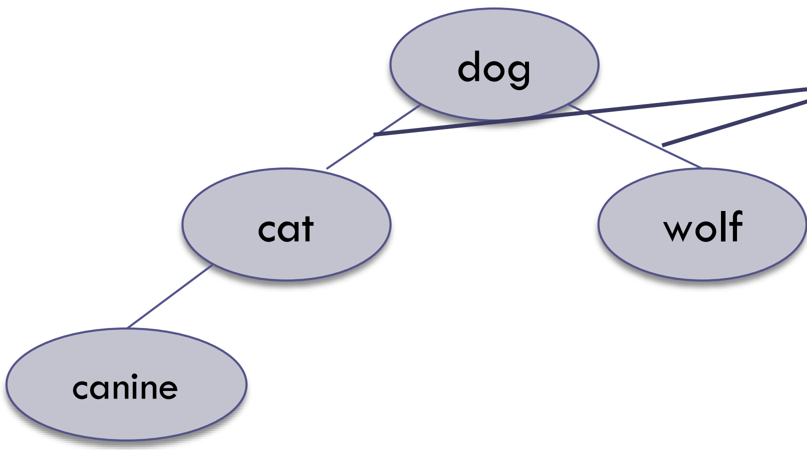
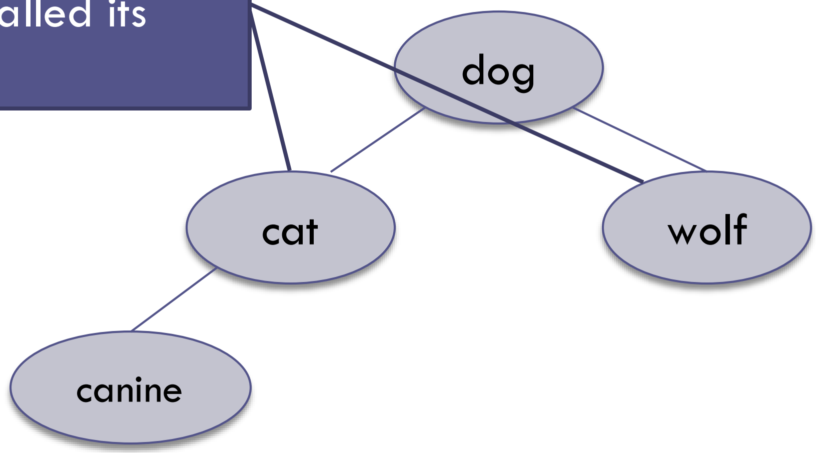


The links from a node to its successors are called *branches*

Tree Terminology (cont.)

A tree consists of a collection of elements or nodes, with each node linked to its successors

The successors of a node are called its *children*



Tree Terminology (cont.)

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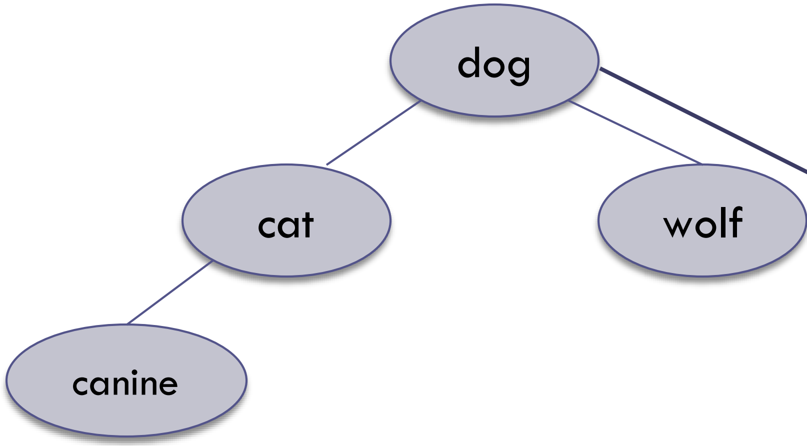
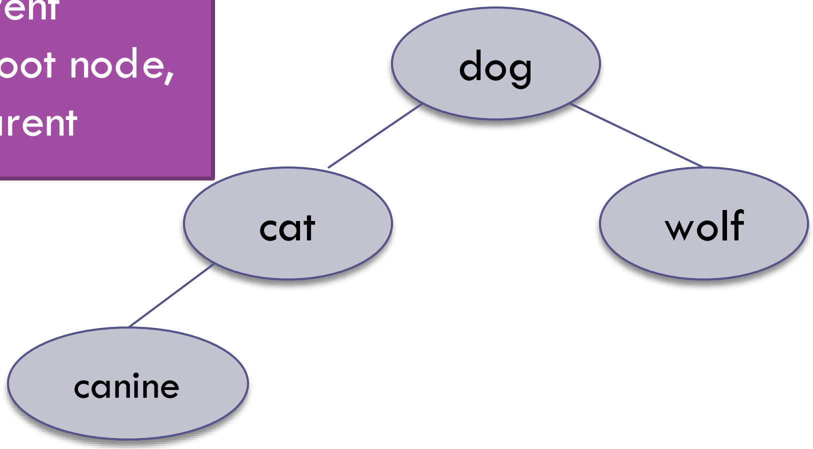


The predecessor of a node is called its *parent*

Tree Terminology (cont.)

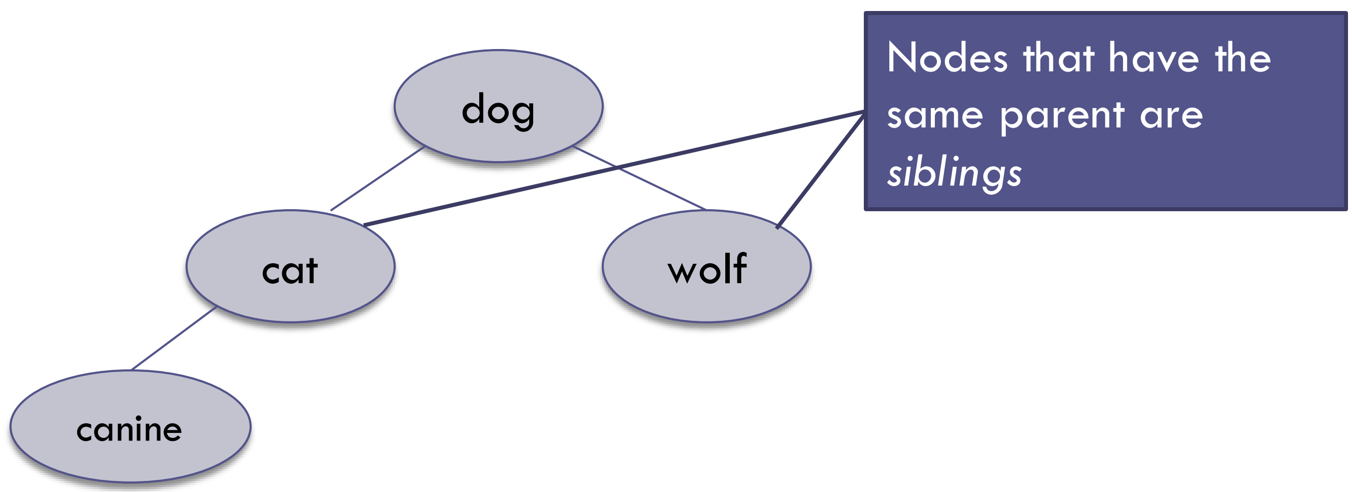
A tree consists of a collection of elements or nodes, with each node linked to its successors

Each node in a tree has exactly one parent except for the root node, which has no parent



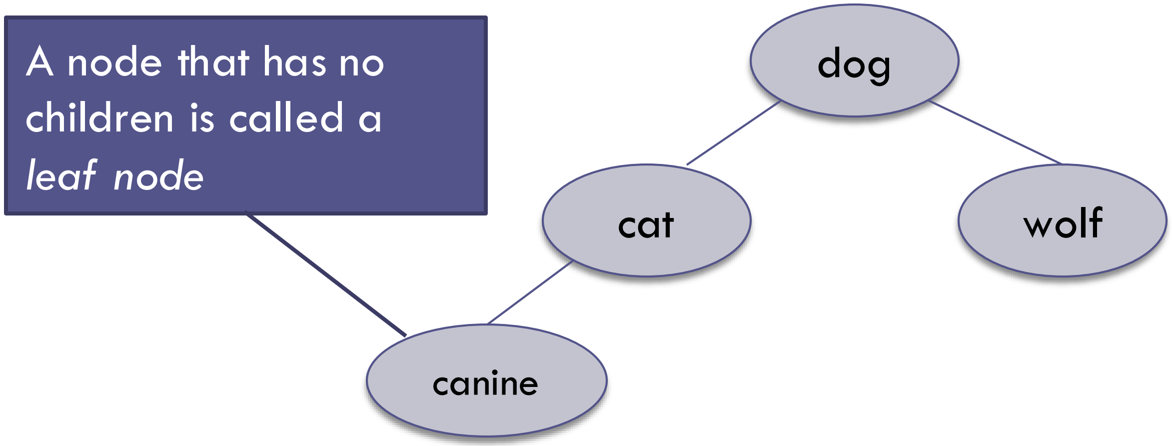
Tree Terminology (cont.)

A tree consists of a collection of elements or nodes, with each node linked to its successors



Tree Terminology (cont.)

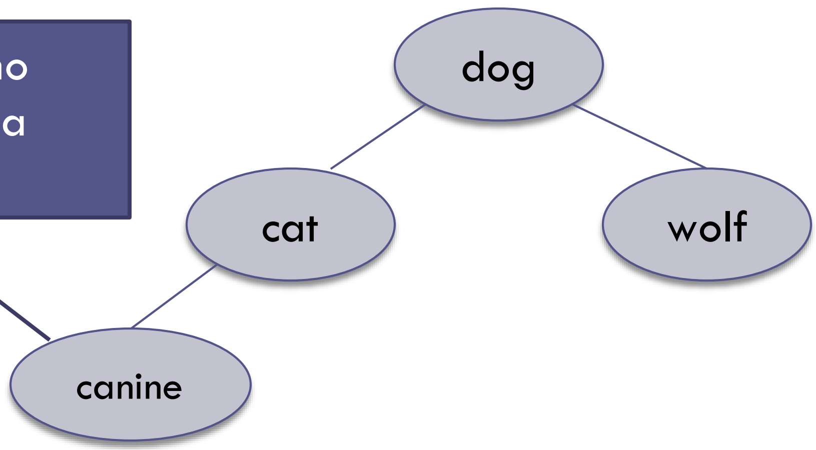
A tree consists of a collection of elements or nodes, with each node linked to its successors



Tree Terminology (cont.)

A tree consists of a collection of elements or nodes, with each node linked to its successors

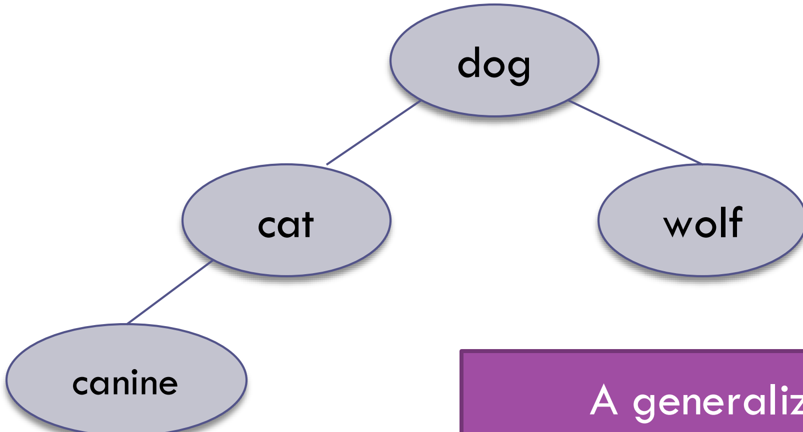
A node that has no children is called a *leaf node*



Leaf nodes also are known as *external nodes*, and nonleaf nodes are known as *internal nodes*

Tree Terminology (cont.)

A tree consists of a collection of elements or nodes, with each node linked to its successors

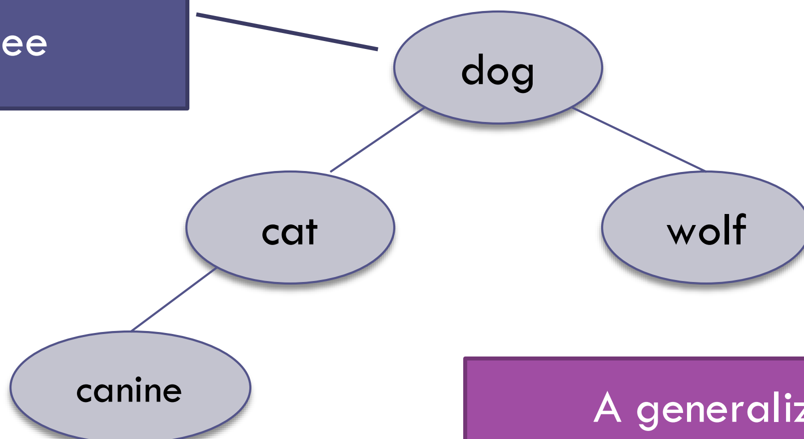


A generalization of the parent-child relationship is the *ancestor-descendant relationship*

Tree Terminology (cont.)

A tree consists of a collection of elements or nodes, with each node linked to its successors

dog is the parent of cat in this tree

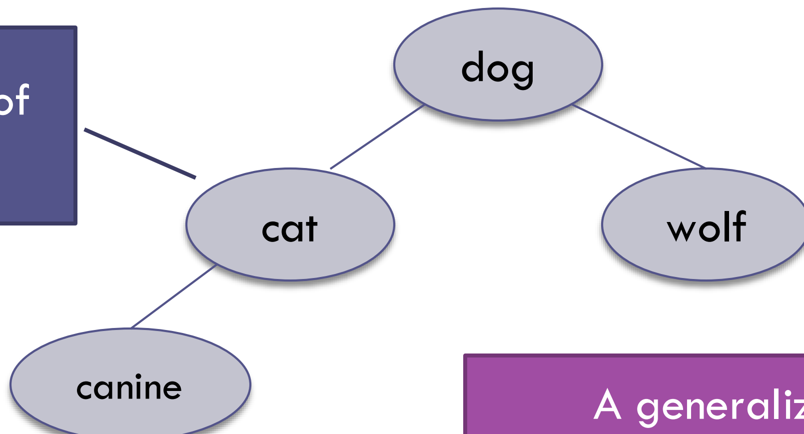


A generalization of the parent-child relationship is the *ancestor-descendant relationship*

Tree Terminology (cont.)

A tree consists of a collection of elements or nodes, with each node linked to its successors

cat is the parent of canine in this tree

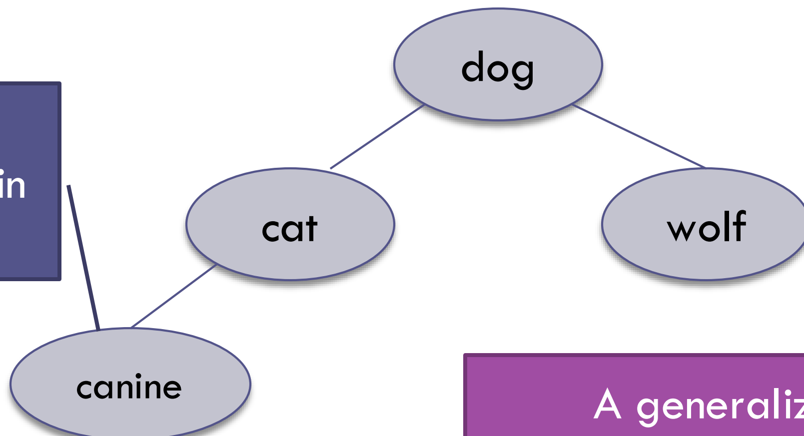


A generalization of the parent-child relationship is the *ancestor-descendant relationship*

Tree Terminology (cont.)

A tree consists of a collection of elements or nodes, with each node linked to its successors

canine is a *descendant* of cat in this tree

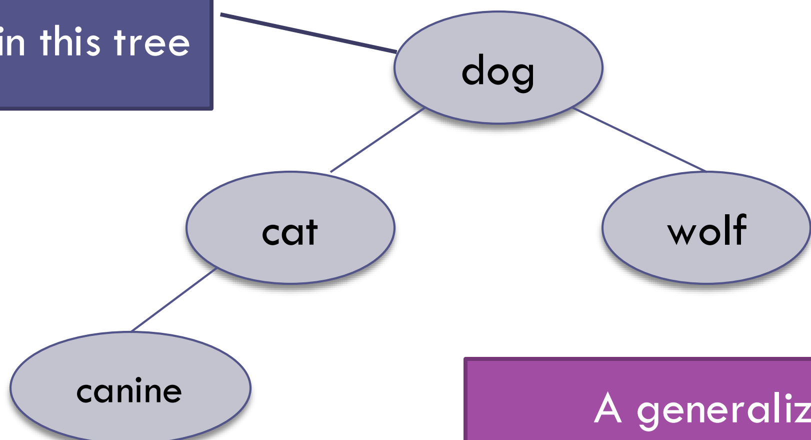


A generalization of the parent-child relationship is the *ancestor-descendant relationship*

Tree Terminology (cont.)

A tree consists of a collection of elements or nodes, with each node linked to its successors

dog is an *ancestor* of canine in this tree



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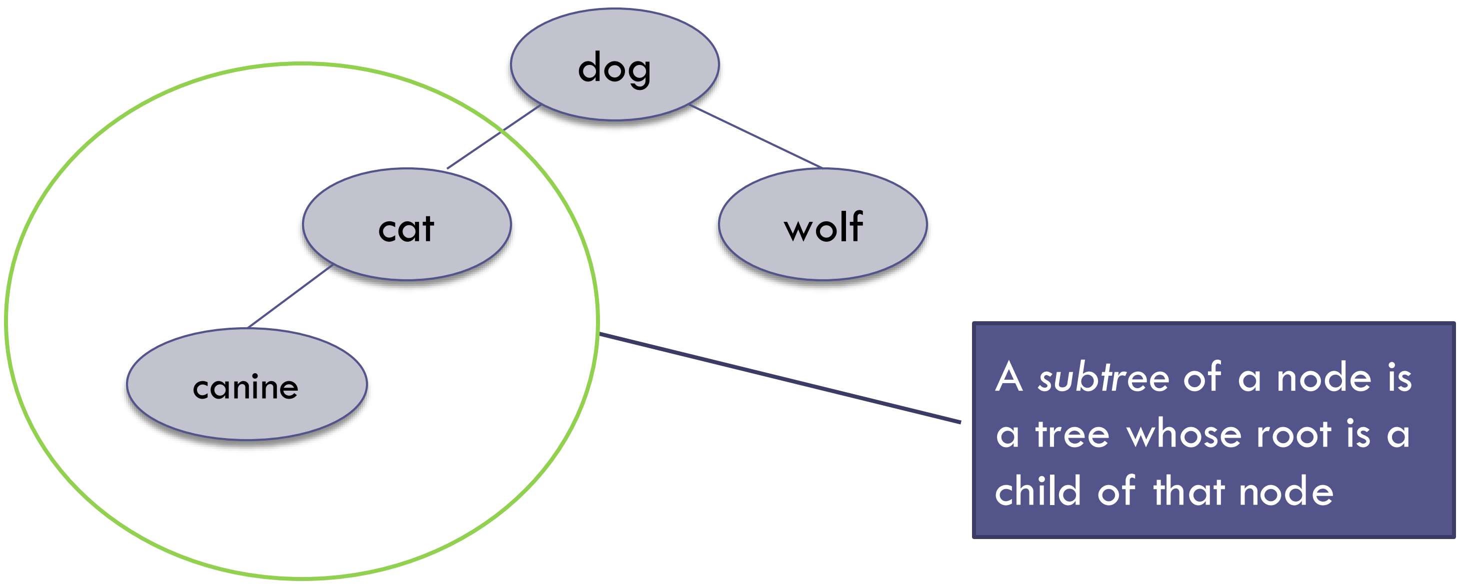

Tree Terminology (cont.)

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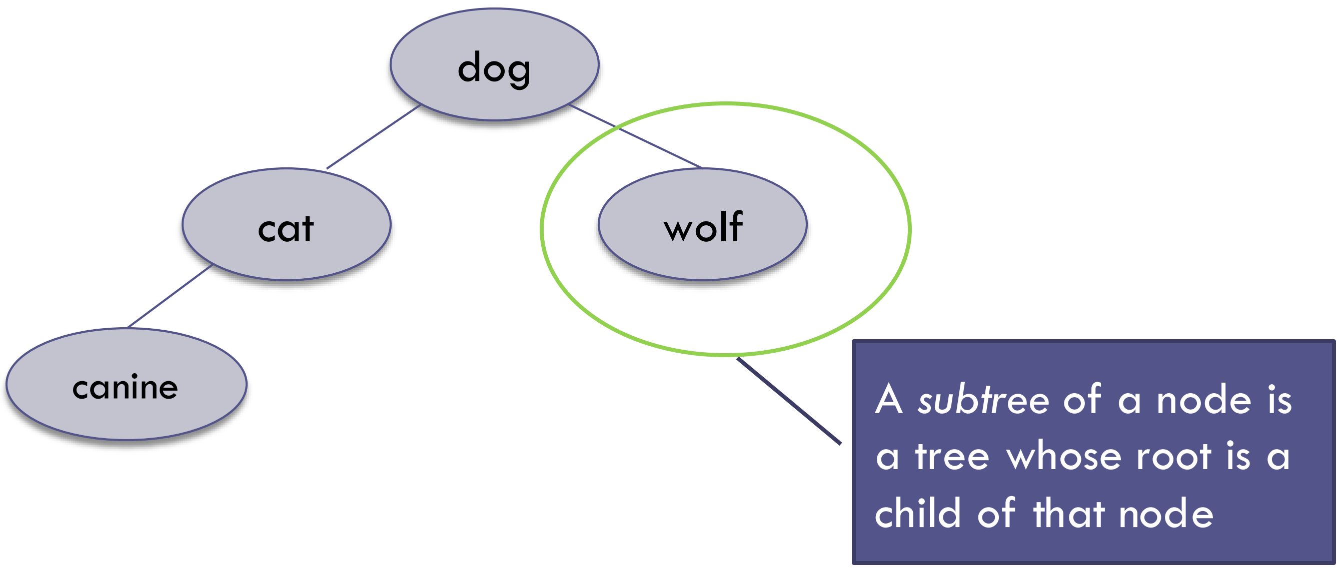

Tree Terminology (cont.)

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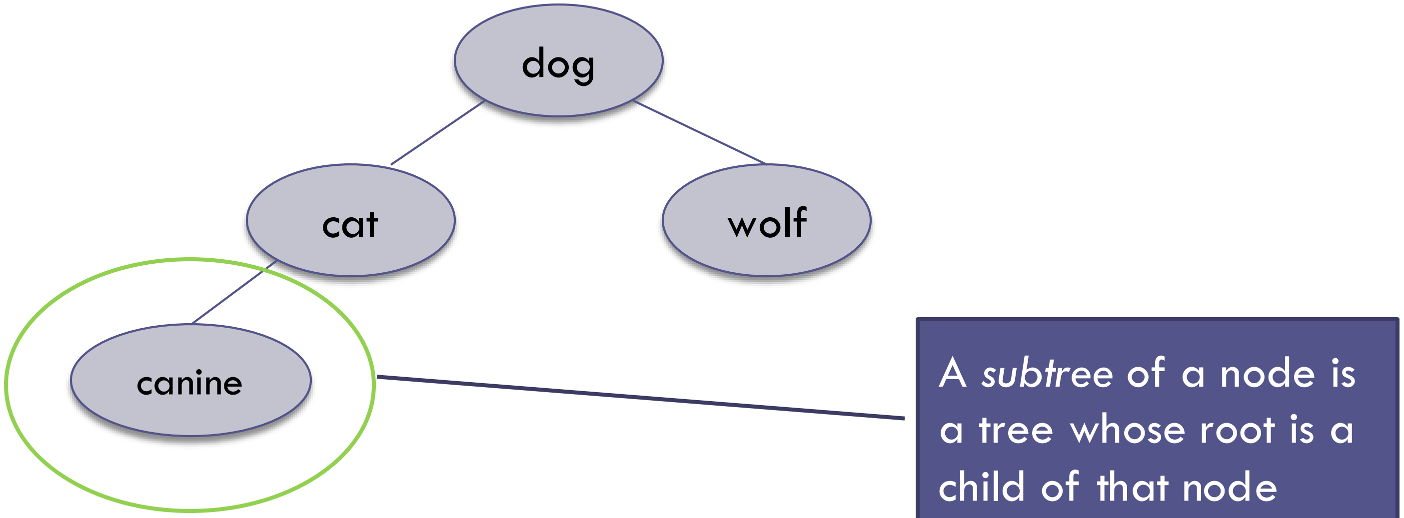

Tree Terminology (cont.)

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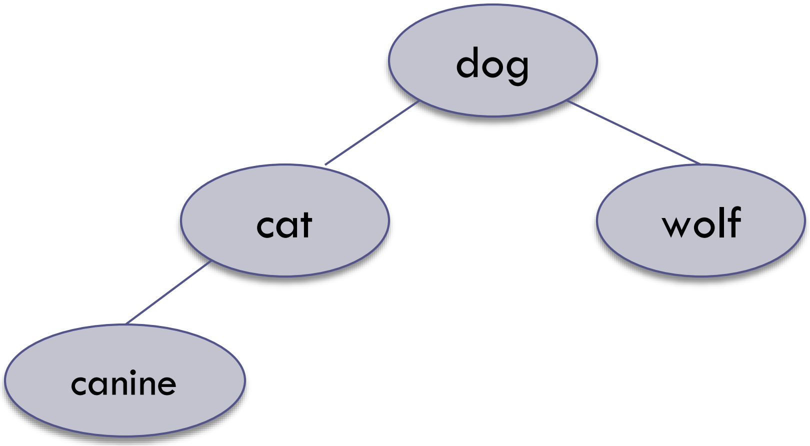
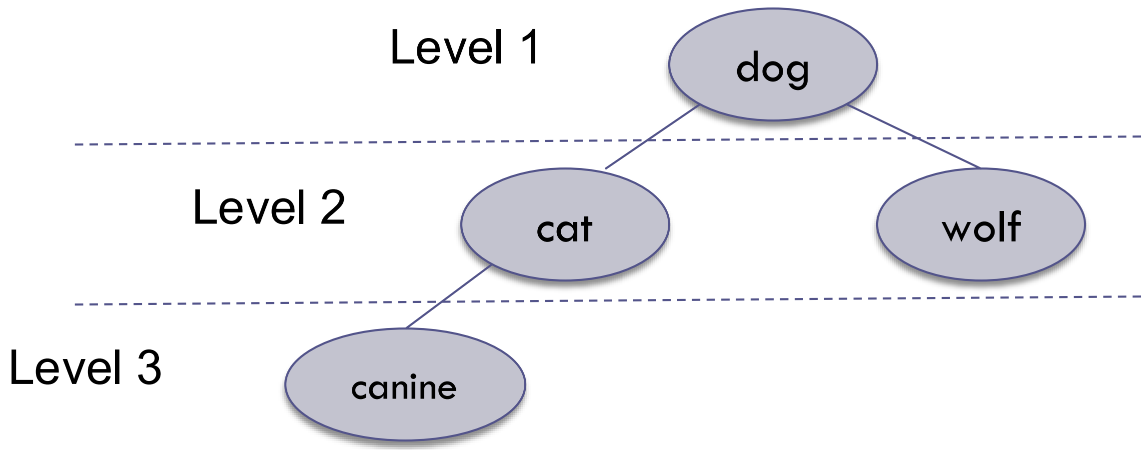


The *level of a node* is determined by its distance from the root

Tree Terminology (cont.)

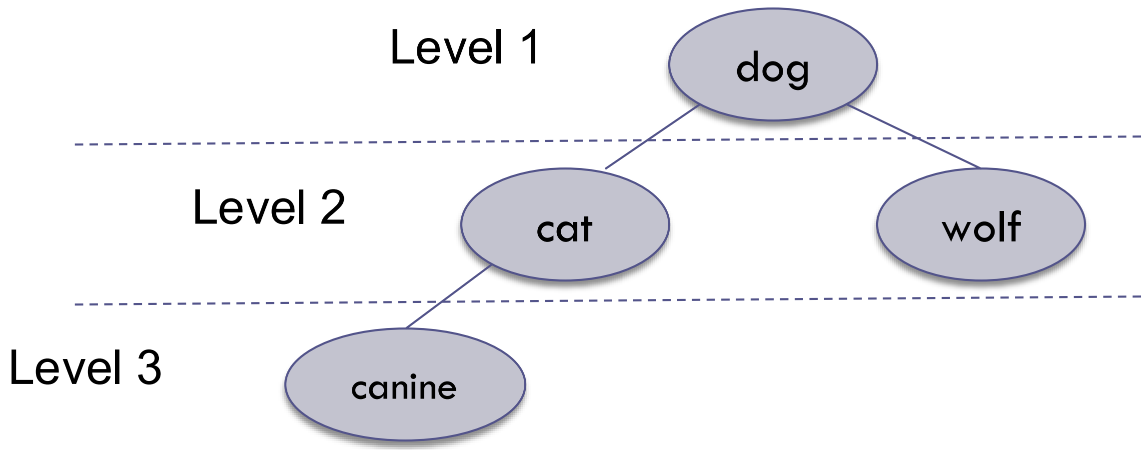
A tree consists of a collection of elements or nodes, with each node linked to its successors



The *level of a node* is its distance from the root plus 1

Tree Terminology (cont.)

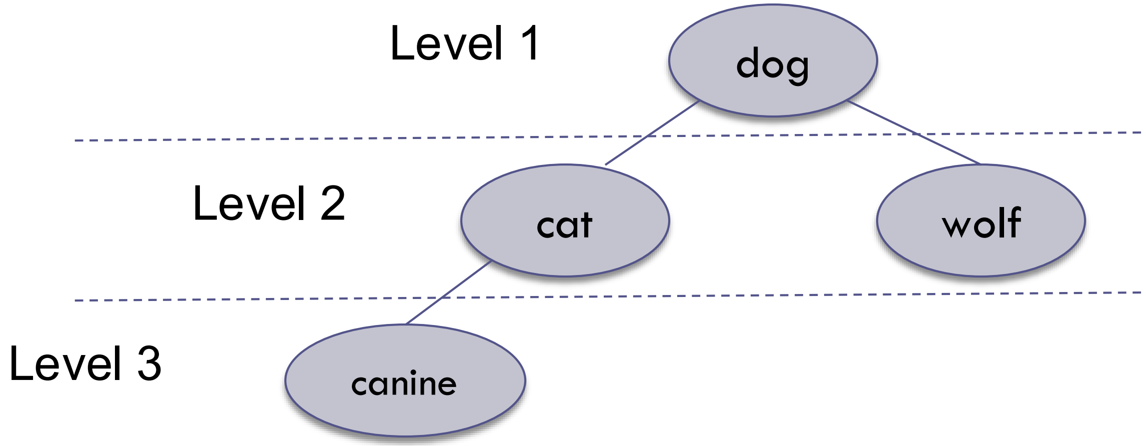
A tree consists of a collection of elements or nodes, with each node linked to its successors



The *level of a node* is defined recursively

Tree Terminology (cont.)

A tree consists of a collection of elements or nodes, with each node linked to its successors



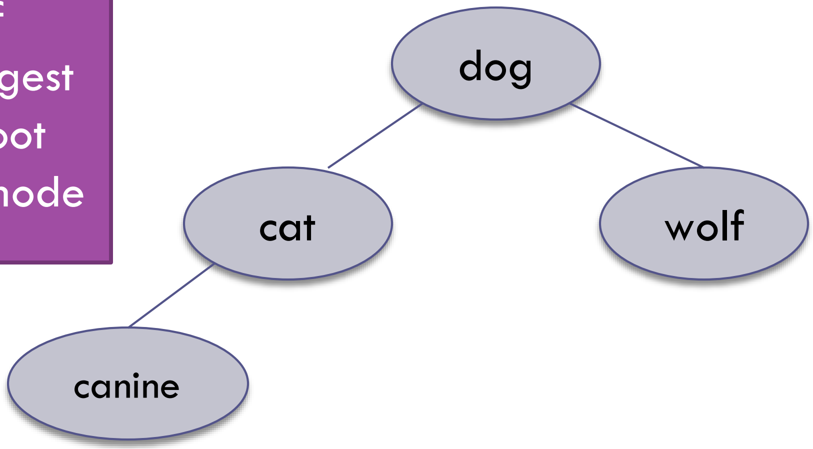
The *level of a node* is defined recursively

- If node *n* is the root of tree T, its level is 1
- If node *n* is not the root of tree T, its level is 1 + the level of its parent

Tree Terminology (cont.)

A tree consists of a collection of elements or nodes, with each node linked to its successors

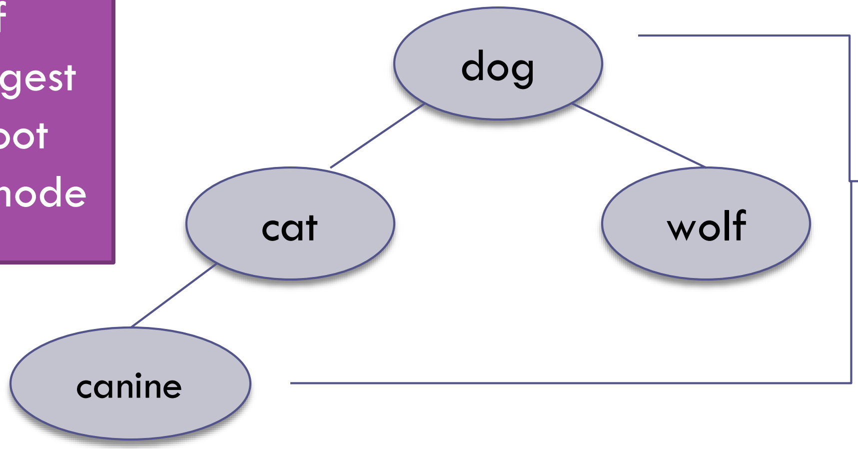
The *height of a tree* is the number of nodes in the longest path from the root node to a leaf node



Tree Terminology (cont.)

A tree consists of a collection of elements or nodes, with each node linked to its successors

The *height of a tree* is the number of nodes in the longest path from the root node to a leaf node



The height of this tree is 3

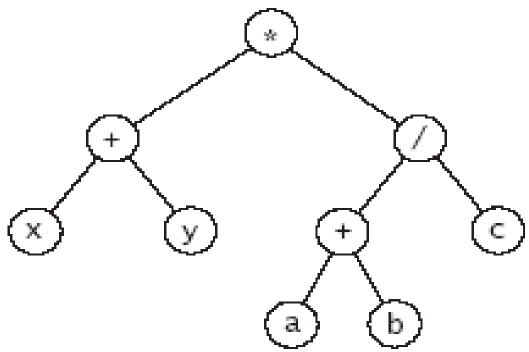
Binary Trees

- In a binary tree, each node has two subtrees
- A set of nodes T is a binary tree if either of the following is true
  - T is empty
  - Its root node has two subtrees, T_L and T_R, such that T_L and T_R are binary trees

  (T_L = left subtree; T_R = right subtree)

Expression Tree

- Each node contains an operator or an operand
- Operands are stored in leaf nodes



- Parentheses are not stored in the tree because the tree structure dictates the order of operand evaluation

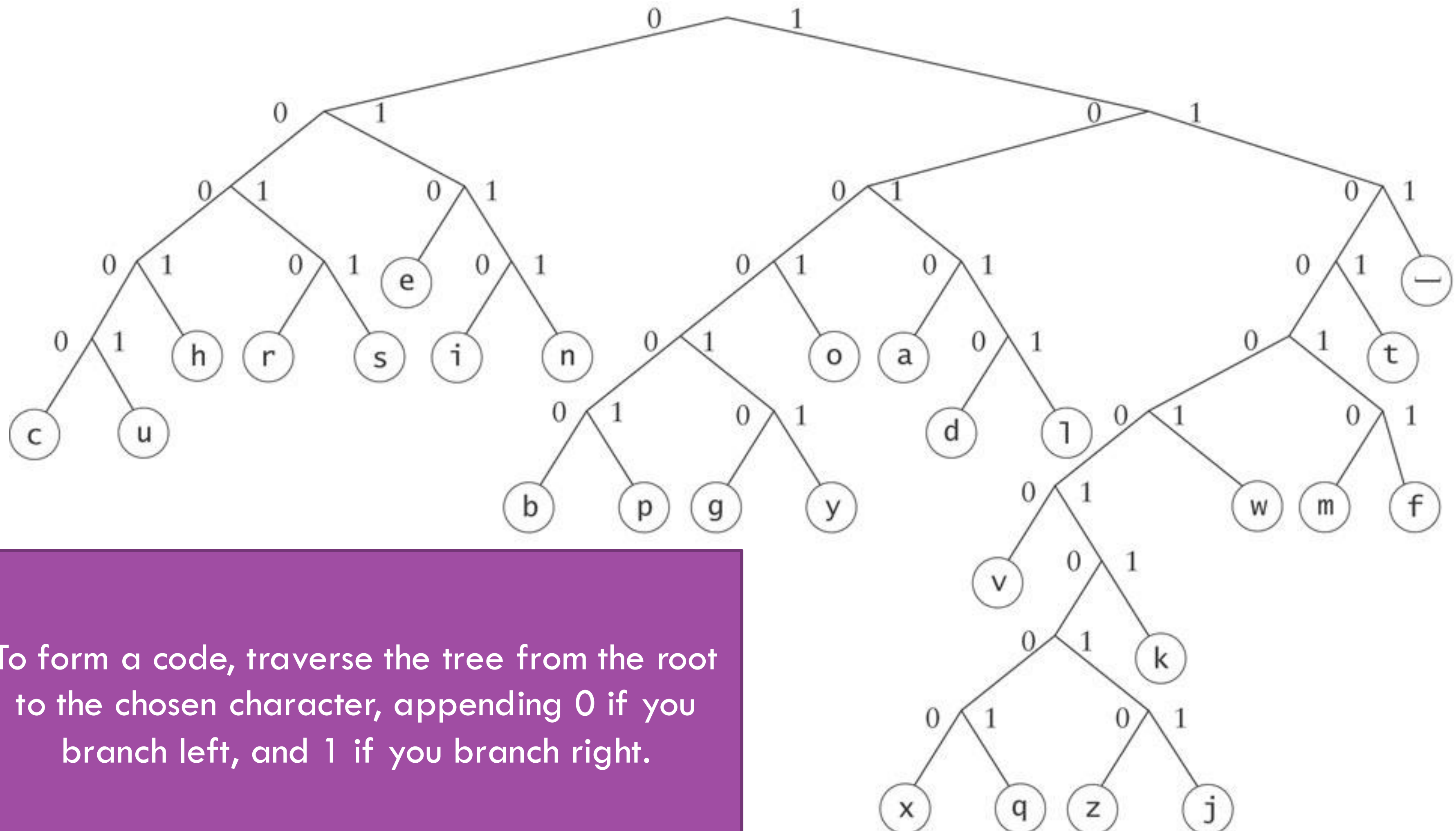
`(x + y) * ((a + b) / c)`

- Operators in nodes at higher tree levels are evaluated after operators in nodes at lower tree levels

Huffman Tree

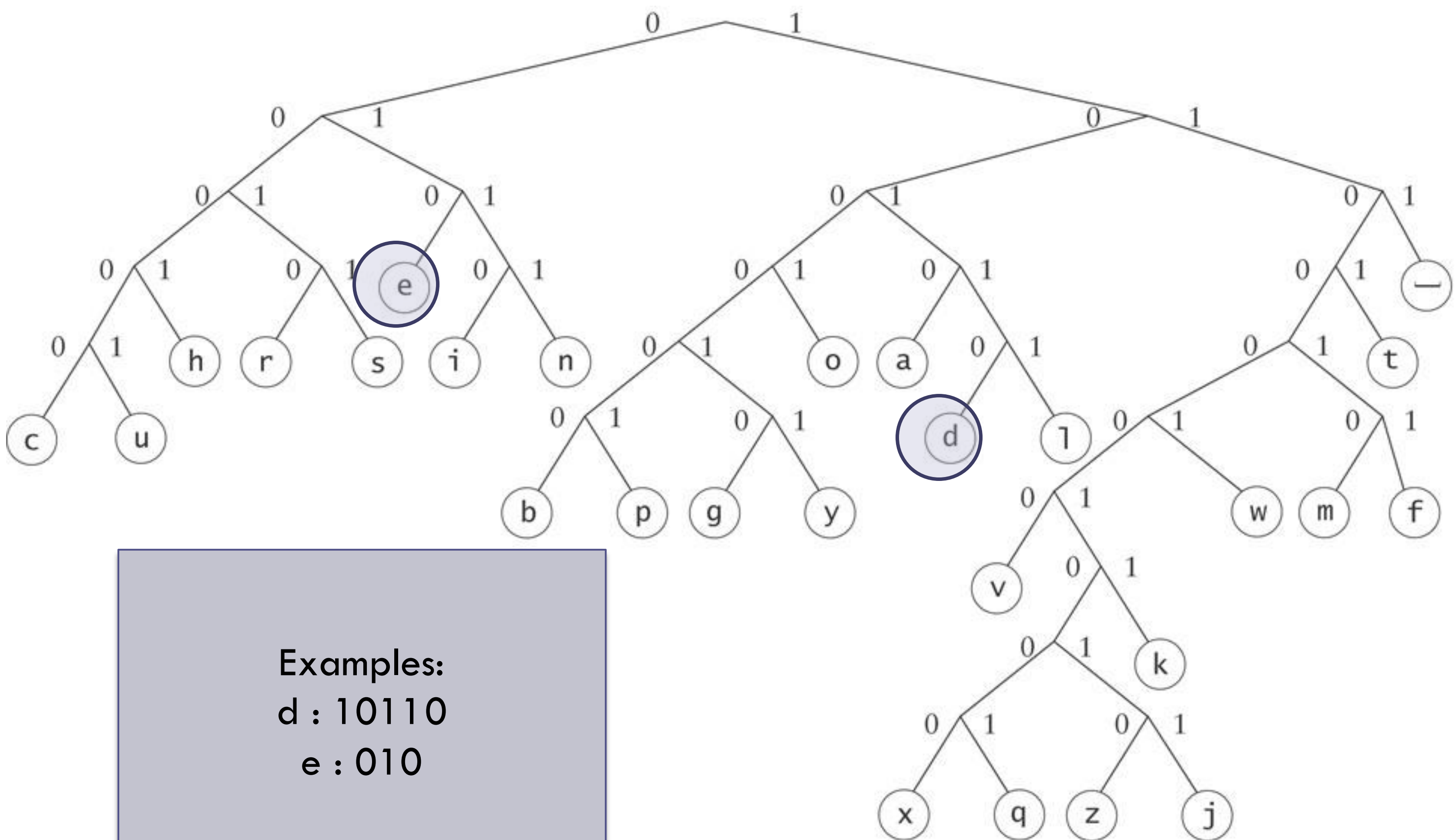
- A *Huffman tree* represents *Huffman codes* for characters that might appear in a text file
- As opposed to ASCII or Unicode, Huffman code uses different numbers of bits to encode letters; more common characters use fewer bits
- Many programs that compress files use Huffman codes

Huffman Tree (cont.)



To form a code, traverse the tree from the root to the chosen character, appending 0 if you branch left, and 1 if you branch right.

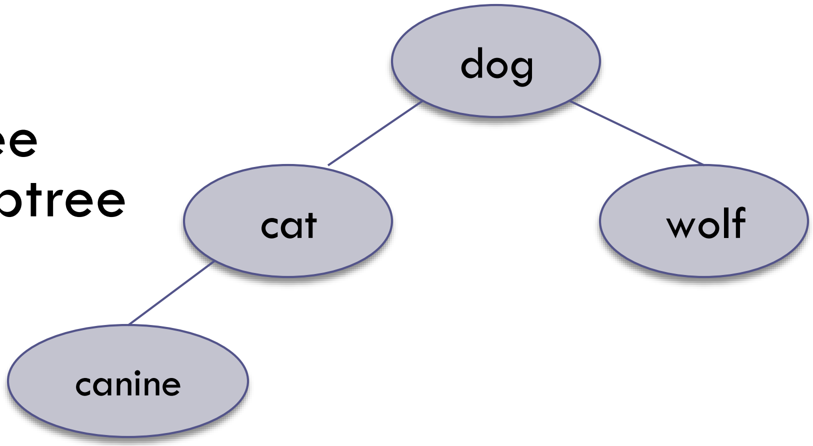
Huffman Tree (cont.)



Examples:
d : 10110
e : 010

Binary Search Tree

- Binary search trees
  - All elements in the left subtree precede those in the right subtree
- A formal definition:



A set of nodes T is a binary search tree if either of the following is true:

- T is empty
- If T is not empty, its root node has two subtrees, T_L and T_R, such that T_L and T_R are binary search trees and the value in the root node of T is greater than all values in T_L and is less than all values in T_R

Binary Search Tree (cont.)

- A binary search tree never has to be sorted because its elements always satisfy the required order relationships
- When new elements are inserted (or removed) properly, the binary search tree maintains its order
- In contrast, a sorted array must be expanded whenever new elements are added, and compacted whenever elements are removed—expanding and contracting are both O(n)

Binary Search Tree (cont.)

- When searching a BST, each probe has the potential to eliminate half the elements in the tree, so searching can be $O(\log n)$
- In the worst case, searching is $O(n)$

Recursive Algorithm for Searching a Binary Tree

1. `if` the tree is empty
2. return null (*target is not found*)

   `else if` the target matches the root node's data

3. return the data stored at the root node

   `else if` the target is less than the root node's data

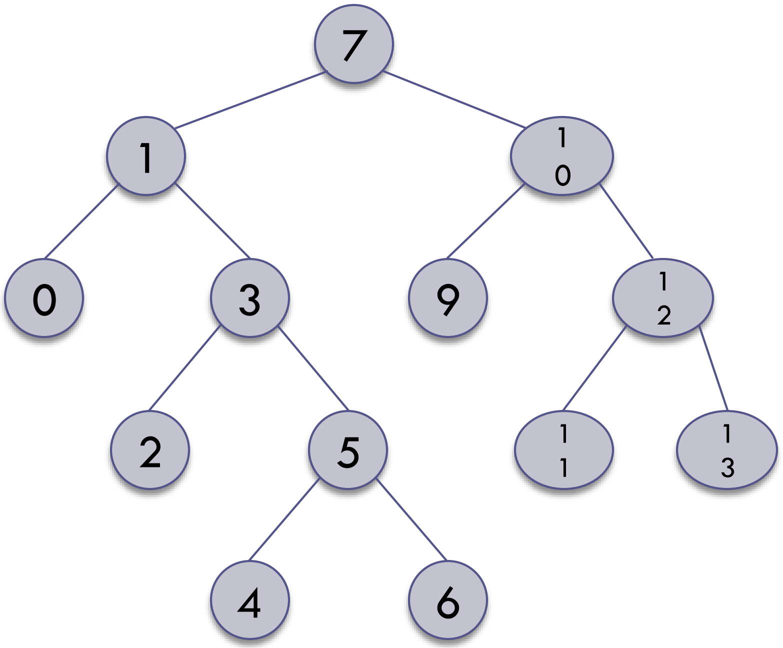
4. return the result of searching the left subtree of the root

   `else`

5. return the result of searching the right subtree of the root

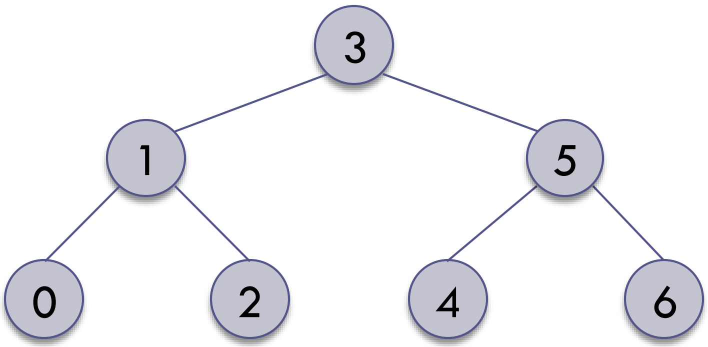
Full, Perfect, and Complete Binary Trees

- A full binary tree is a binary tree where all nodes have either 2 children or 0 children (the leaf nodes)



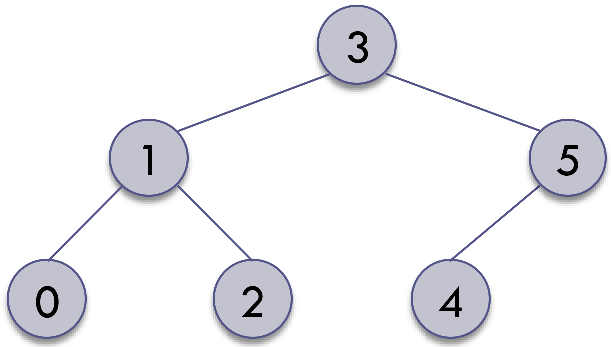
Full, Perfect, and Complete Binary Trees (cont.)

- A *perfect binary tree* is a full binary tree of height n with exactly $2^n - 1$ nodes
- In this case, $n = 3$ and $2^n - 1 = 7$



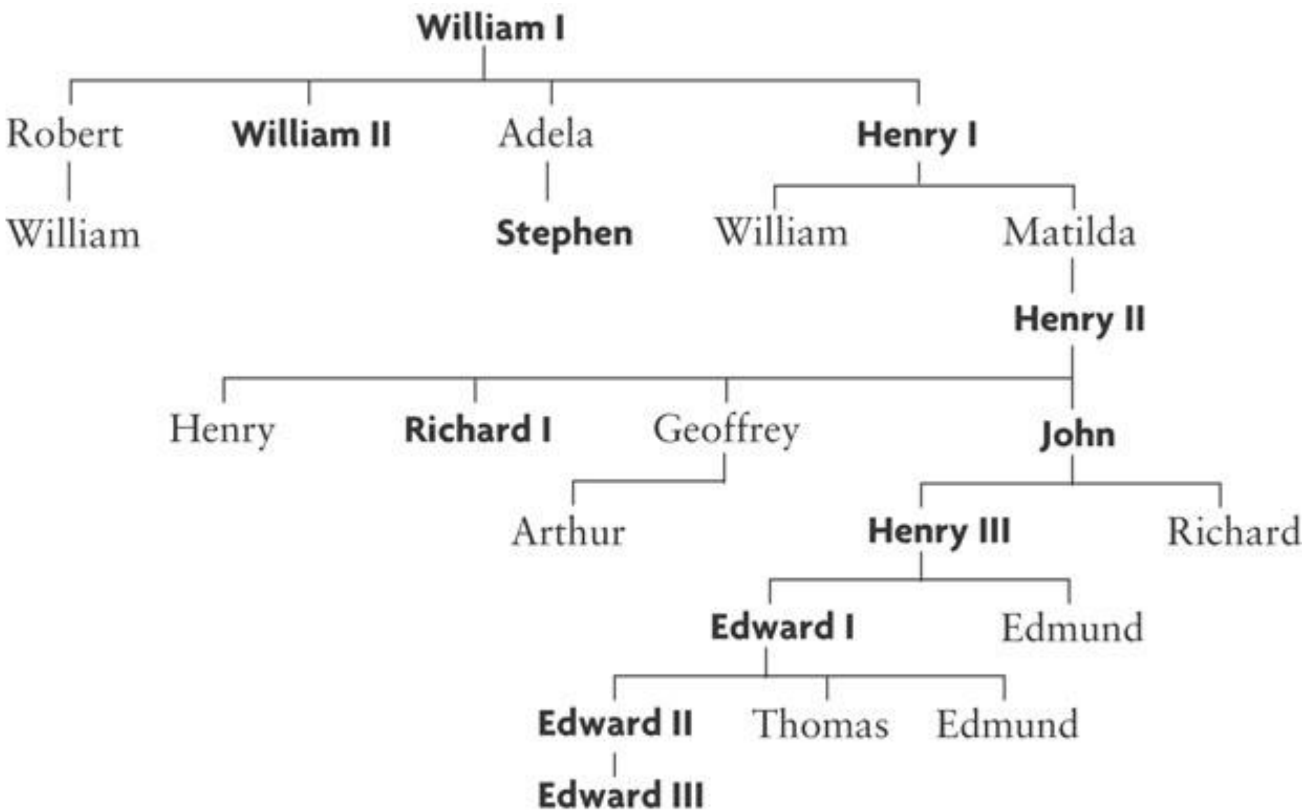
Full, Perfect, and Complete Binary Trees (cont.)

- A *complete binary tree* is a perfect binary tree through level n - 1 with some extra leaf nodes at level n (the tree height), all toward the left



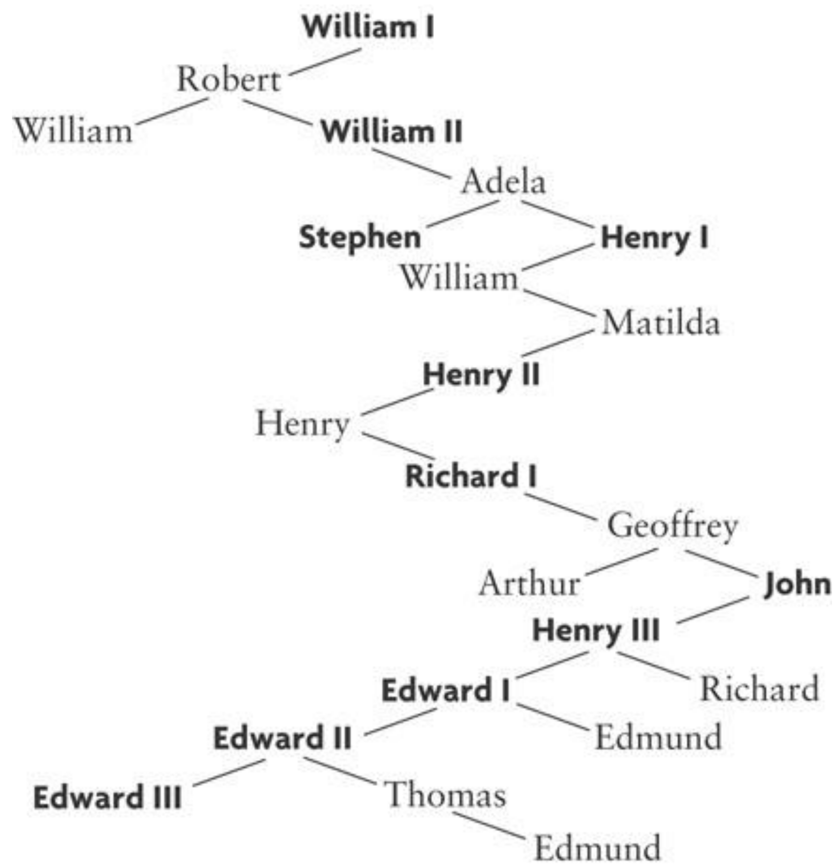
General Trees

- We do not discuss general trees in this chapter, but nodes of a general tree can have any number of subtrees



General Trees (cont.)

- A general tree can be represented using a binary tree
- The left branch of a node is the oldest child, and each right branch is connected to the next younger sibling (if any)



Tree Traversals

Section 6.2

Tree Traversals

- Often we want to determine the nodes of a tree and their relationship
  - We can do this by walking through the tree in a prescribed order and visiting the nodes as they are encountered
  - This process is called *tree traversal*
- Three common kinds of tree traversal
  - Inorder
  - Preorder
  - Postorder

Tree Traversals (cont.)

- Preorder: visit root node, traverse T_L, traverse T_R
- Inorder: traverse T_L, visit root node, traverse T_R
- Postorder: traverse T_L, traverse T_R, visit root node

Algorithm for Preorder Traversal

1. `if` the tree is empty
2. Return.

   `else`

3. Visit the root.
4. Preorder traverse the left subtree.
5. Preorder traverse the right subtree.

Algorithm for Inorder Traversal

1. `if` the tree is empty
2. Return.

   `else`

3. Inorder traverse the left subtree.
4. Visit the root.
5. Inorder traverse the right subtree.

Algorithm for Postorder Traversal

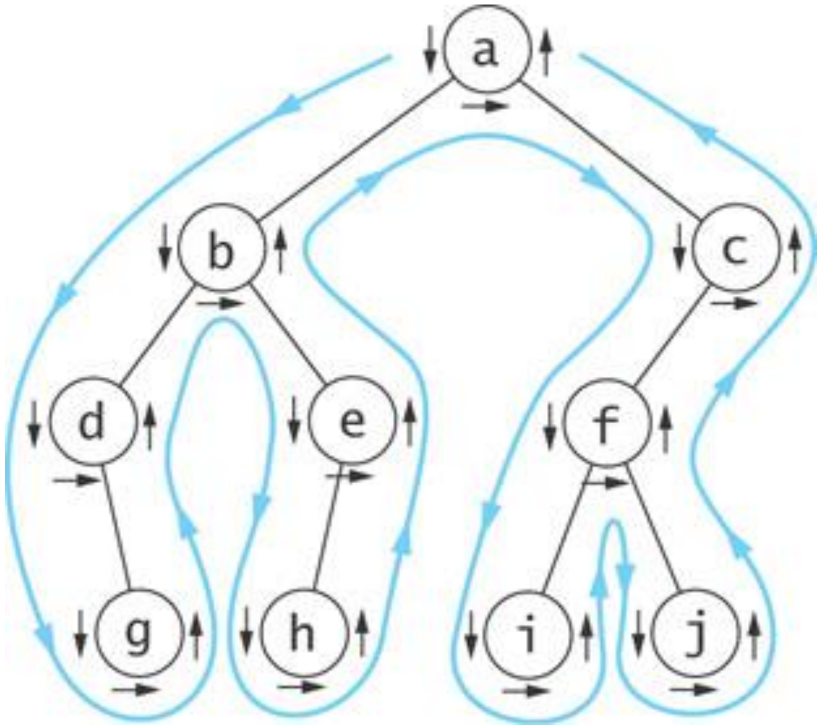
1. `if` the tree is empty
2. Return.

   `else`

3. Postorder traverse the left subtree.
4. Postorder traverse the right subtree.
5. Visit the root.

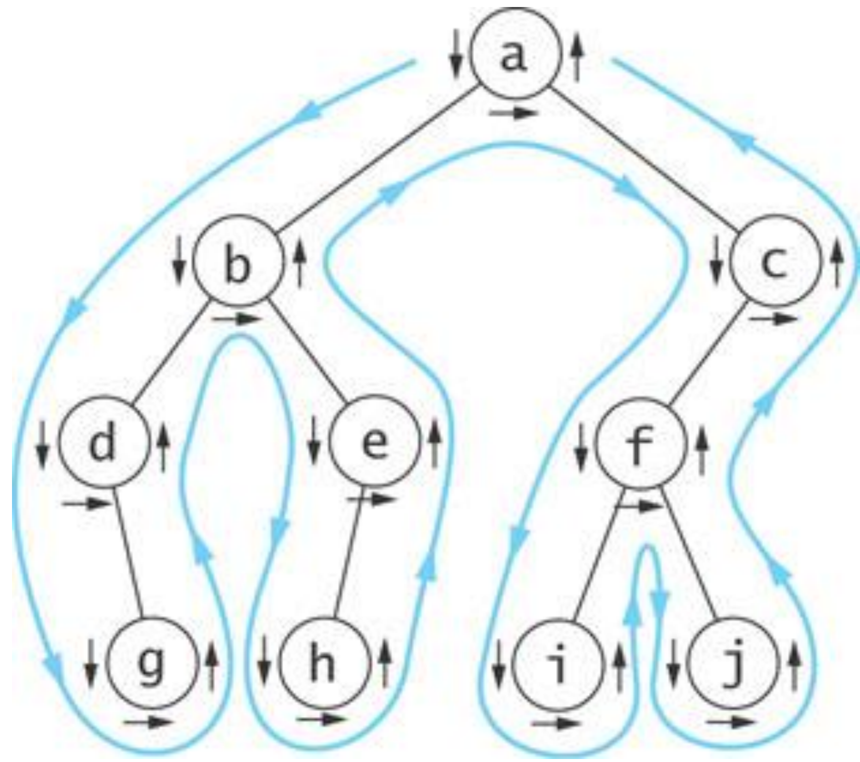
Visualizing Tree Traversals

- You can visualize a tree traversal by imagining a mouse that walks along the edge of the tree
- If the mouse always keeps the tree to the left, it will trace a route known as the *Euler tour*
- The Euler tour is the path traced in blue in the figure on the right



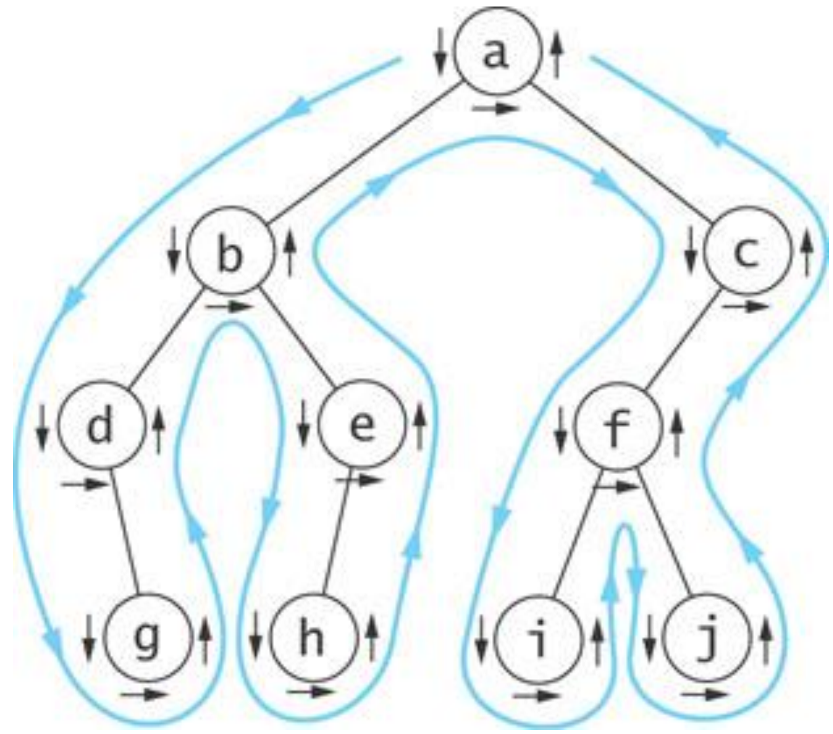
Visualizing Tree Traversals (cont.)

- A Euler tour (blue path) is a preorder traversal
- The sequence in this example is
  a b d g e h c f i j
- The mouse visits each node before traversing its subtrees (shown by the downward pointing arrows)



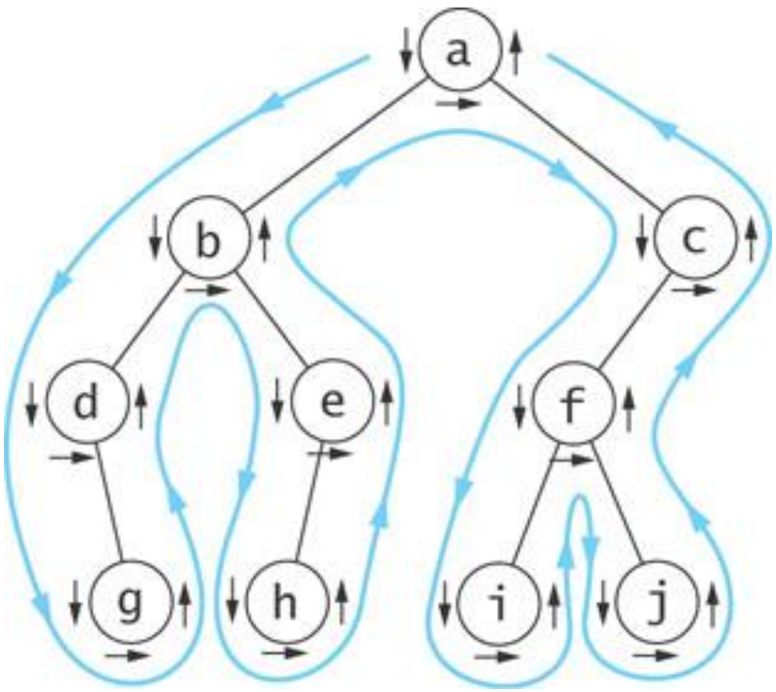
Visualizing Tree Traversals (cont.)

- If we record a node as the mouse returns from traversing its left subtree (shown by horizontal black arrows in the figure) we get an inorder traversal
- The sequence is
  d g b h e a i f j c



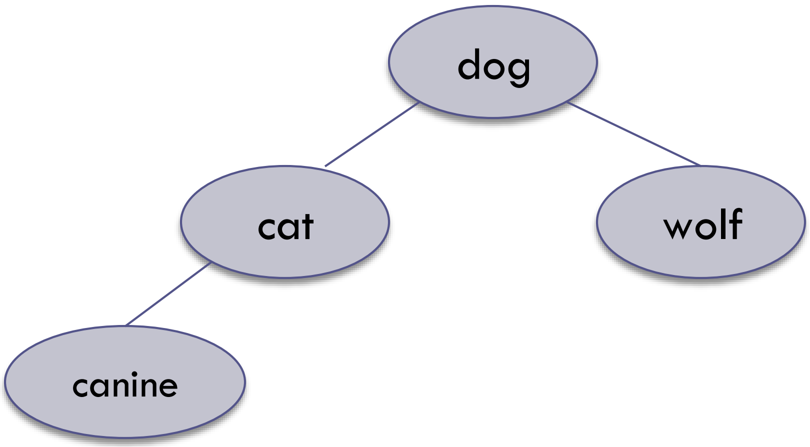
Visualizing Tree Traversals (cont.)

- If we record each node as the mouse last encounters it, we get a postorder traversal (shown by the upward pointing arrows)
- The sequence is g d h e b i j f c a



Traversals of Binary Search Trees and Expression Trees

- An inorder traversal of a binary search tree results in the nodes being visited in sequence by increasing data value

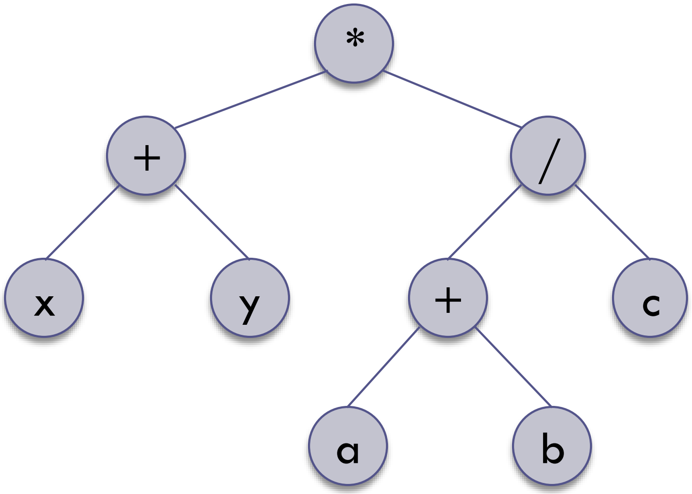


canine, cat, dog, wolf

Traversals of Binary Search Trees and Expression Trees (cont.)

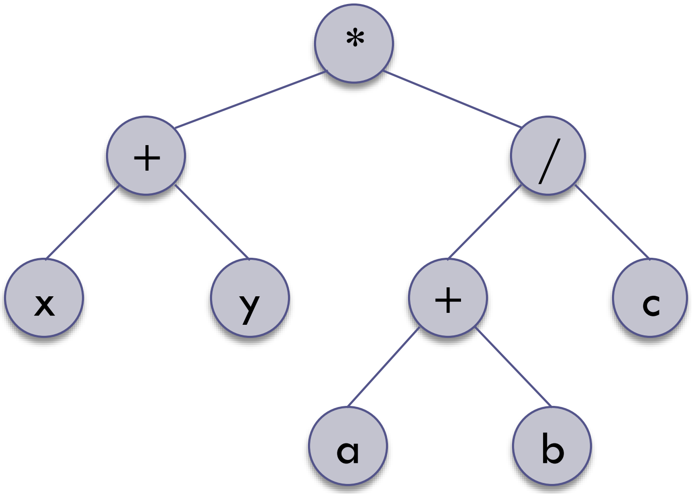
- An inorder traversal of an expression tree results in the sequence
  x + y * a + b / c
- If we insert parentheses where they belong, we get the infix form:

  `(x + y) * ((a + b) / c)`



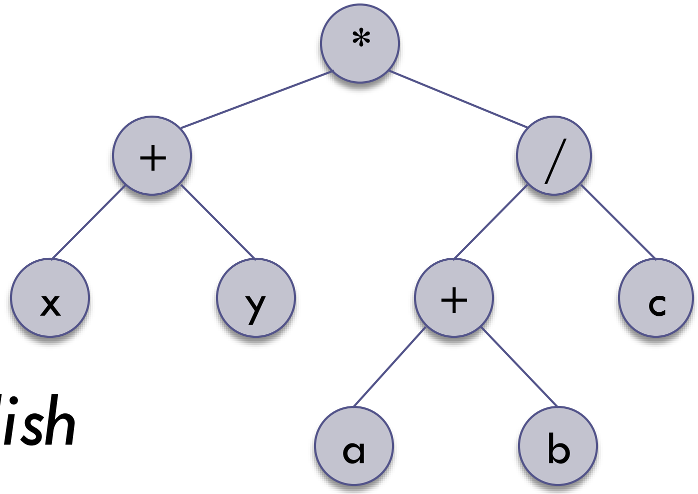
Traversals of Binary Search Trees and Expression Trees (cont.)

- A postorder traversal of an expression tree results in the sequence
  x y + a b + c / *
- This is the *postfix* or *reverse polish* form of the expression
- Operators follow operands



Traversals of Binary Search Trees and Expression Trees (cont.)

- A preorder traversal of an expression tree results in the sequence
  * + x y / + a b c
- This is the *prefix* or *forward polish* form of the expression
- Operators precede operands

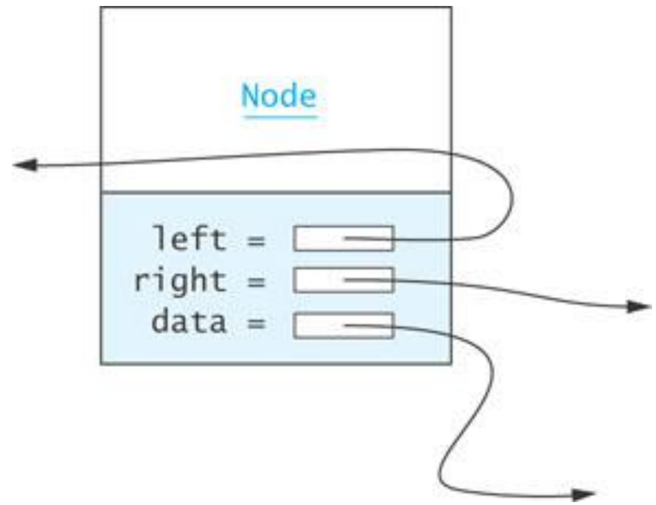


Implementing a `BinaryTree` Class

Section 6.3

Node<E> **Class**

- Just as for a linked list, a node consists of a data part and links to successor nodes
- The data part is a reference to type E
- A binary tree node must have links to both its left and right subtrees

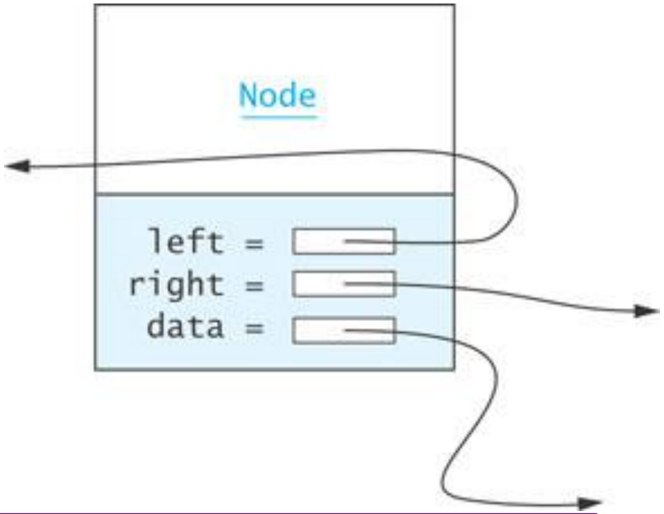


Node<E> Class (cont.)

```
protected static class Node<E>
         implements Serializable {
   protected E data;
   protected Node<E> left;
   protected Node<E> right;

   public Node(E data) {
     this.data = data;
     left = null;
     right = null;
   }

   public String toString() {
      return data.toString();
   }
}
```



`Node<E>` is declared as an inner class within `BinaryTree<E>`

Node<E> **Class (cont.)**

```
protected static class Node<E>
         implements Serializable {
   protected E data;
   protected Node<E> left;
   protected Node<E> right;

   public Node(E data) {
     this.data = data;
     left = null;
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   }

   public String toString() {
      return data.toString();
   }
}
```
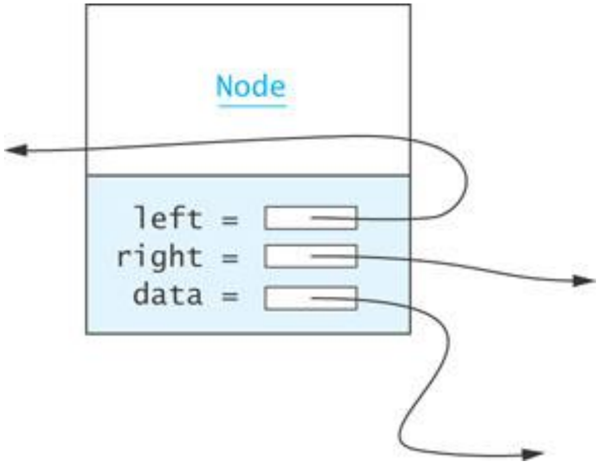
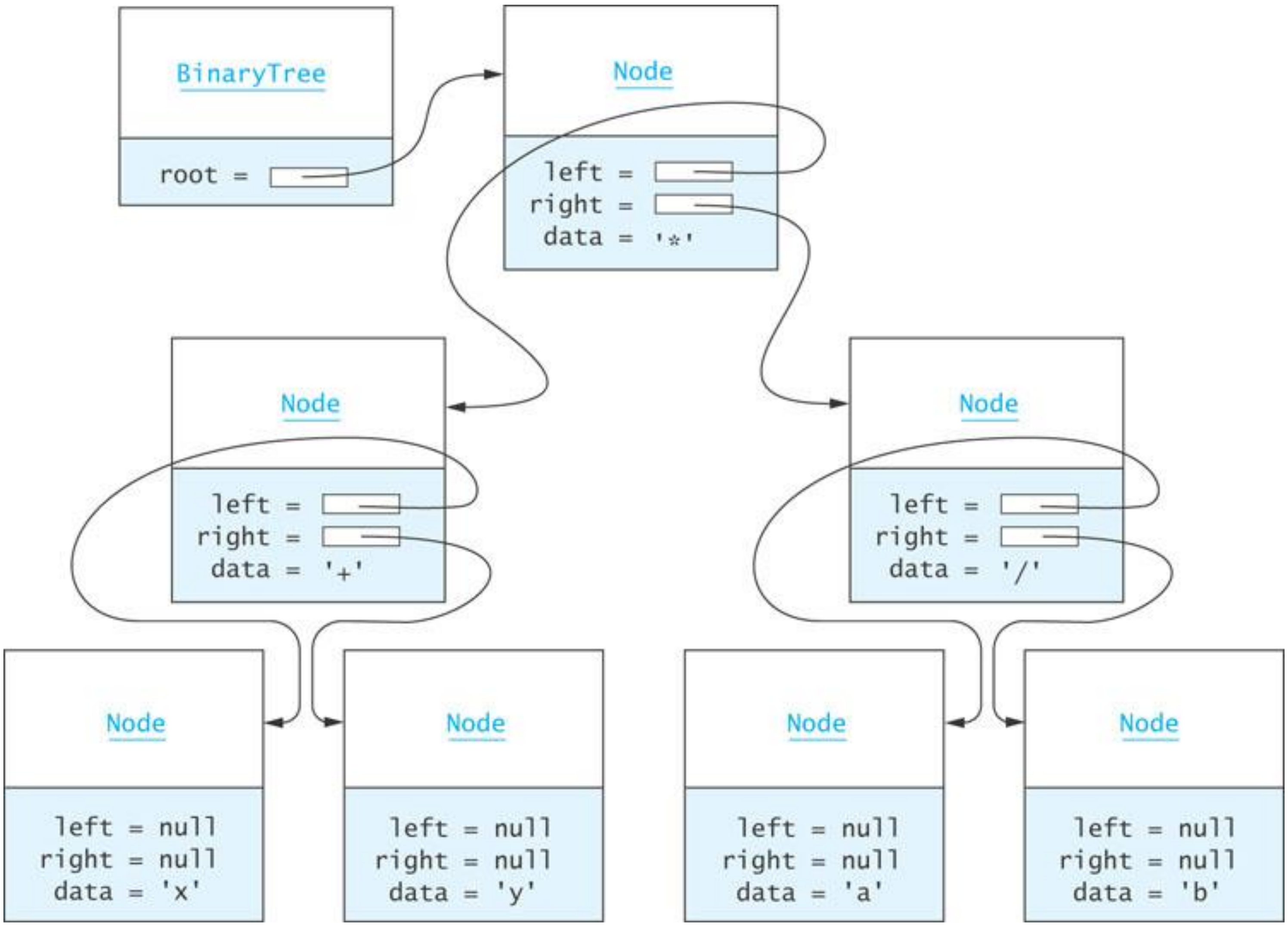



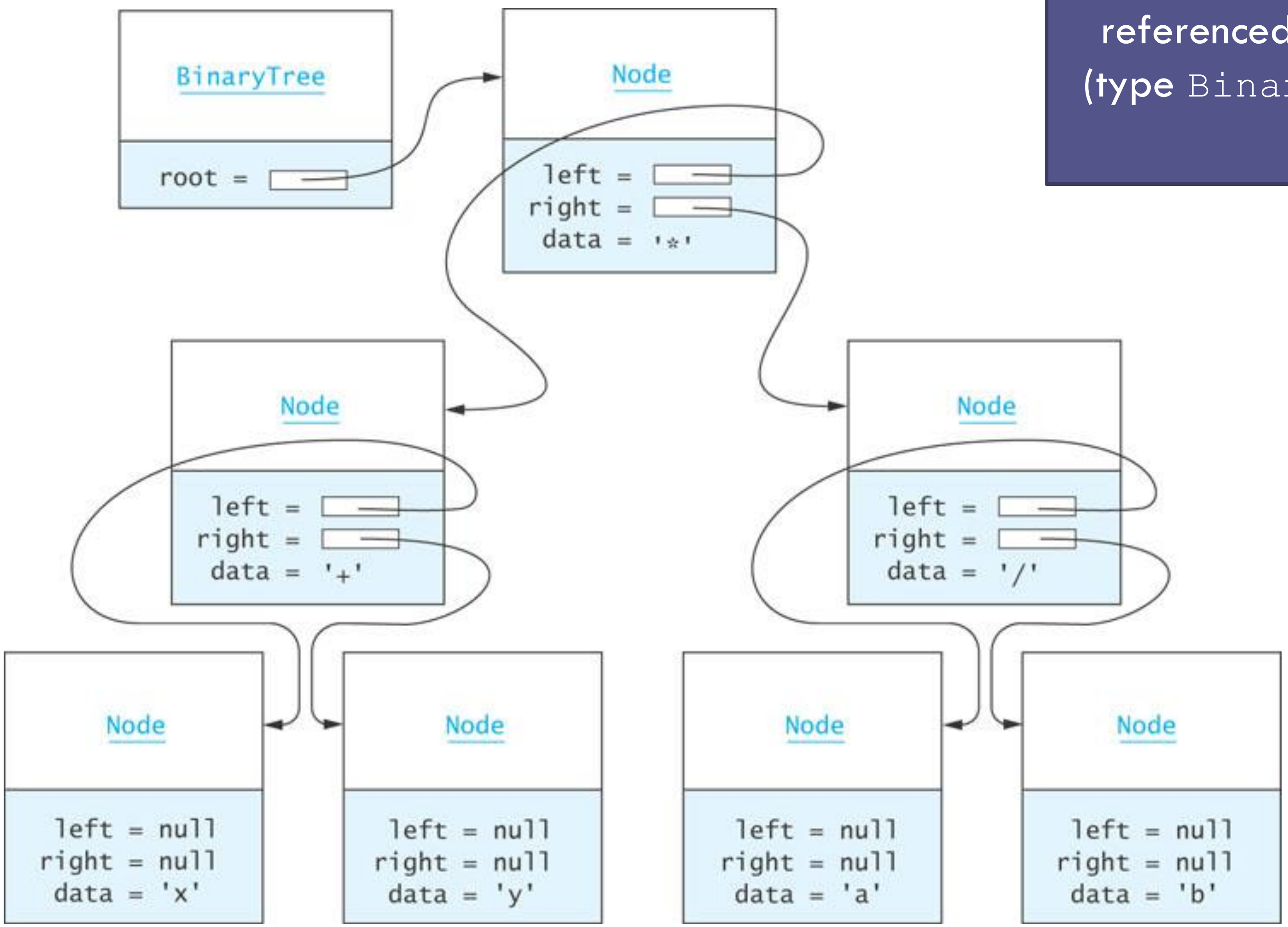
`Node<E>` is declared `protected`. This way we can use it as a superclass.

BinaryTree<E> **Class (cont.)**



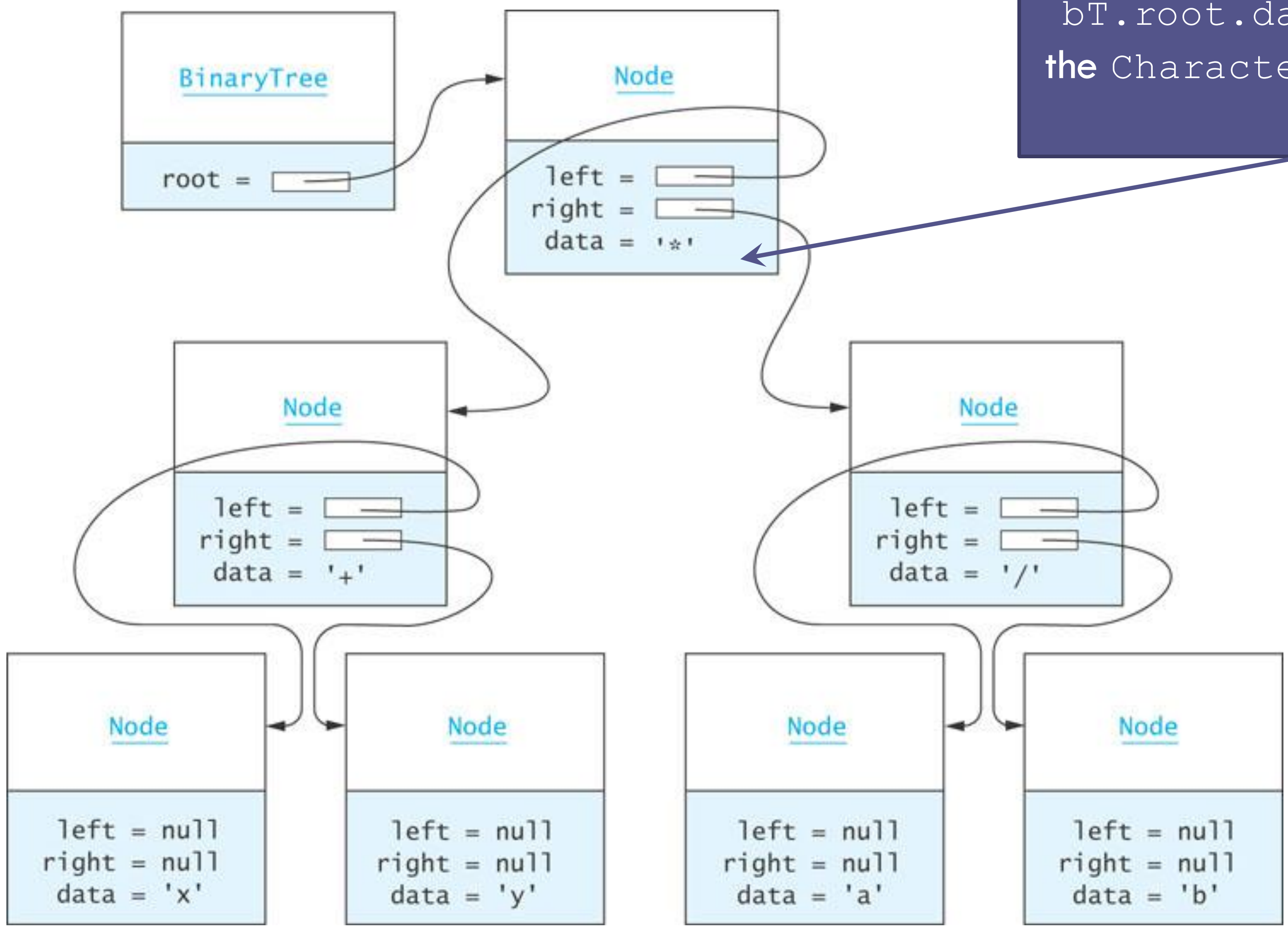
BinaryTree<E> **Class (cont.)**

Assuming the tree is referenced by variable `bT` (type `BinaryTree`) then . . .



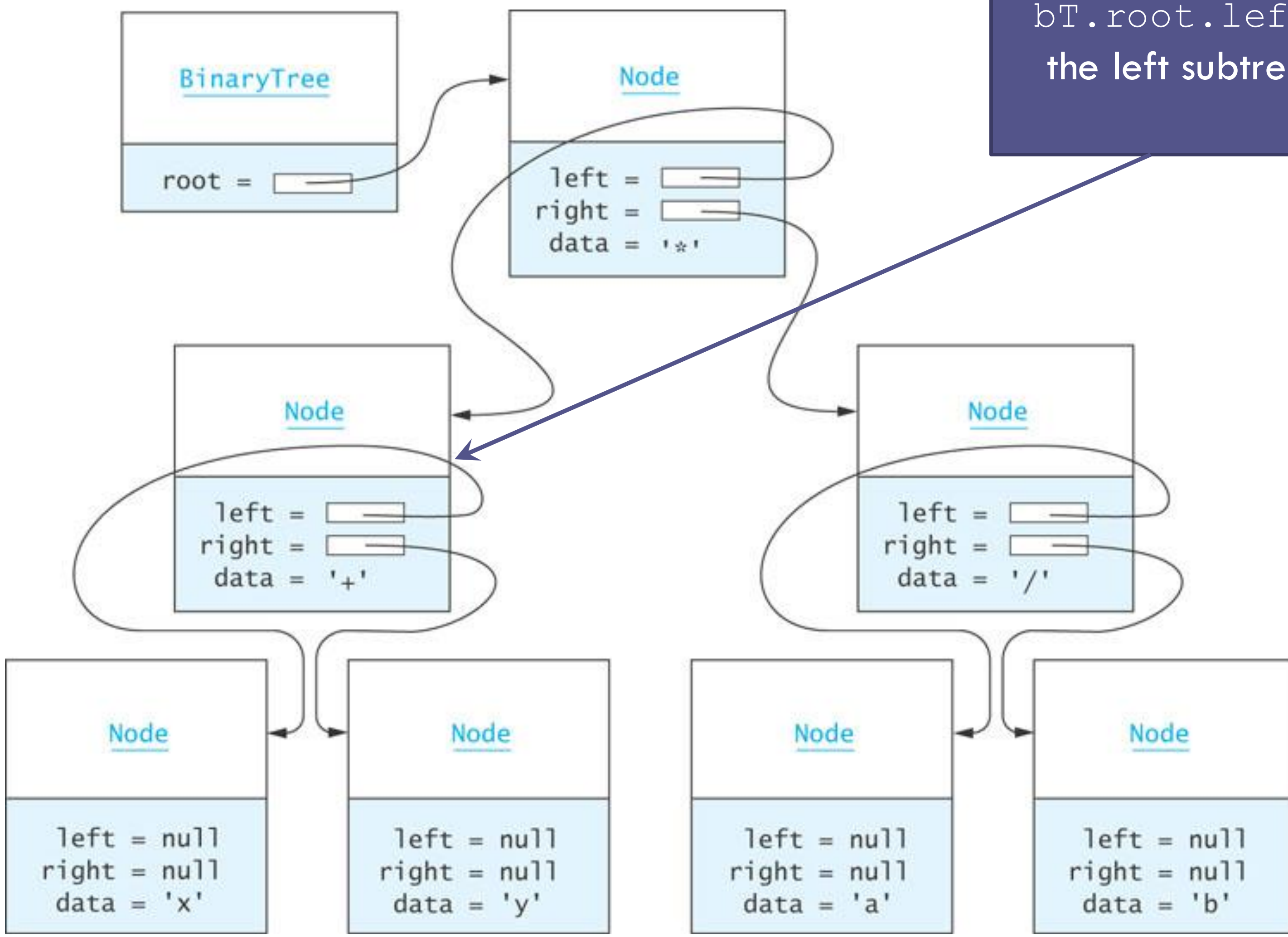
BinaryTree<E> Class (cont.)

`bT.root.data` **references the** `Character` **object storing** `'*'`



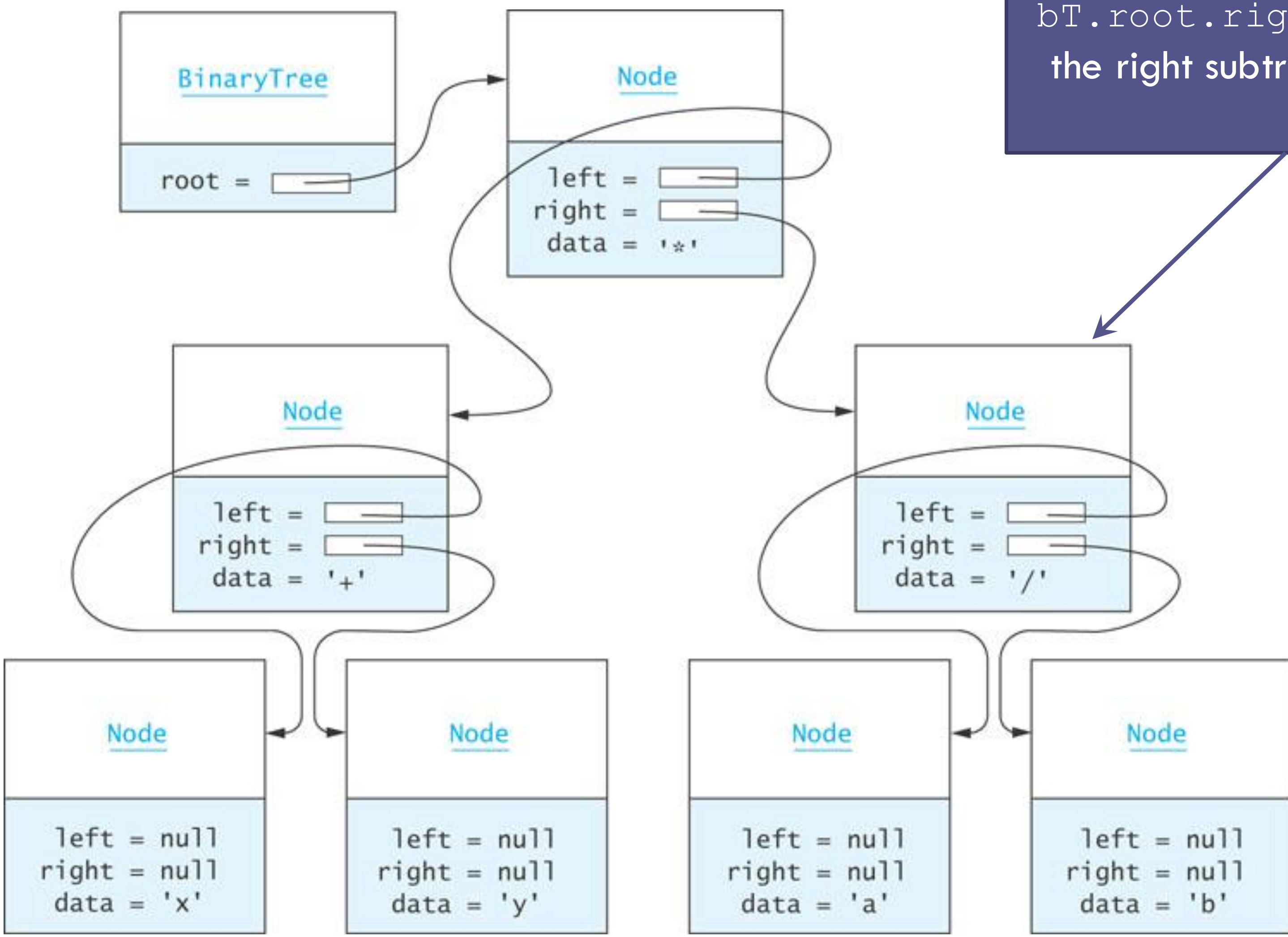
`BinaryTree<E>` Class (cont.)

bT.root.left references the left subtree of the root

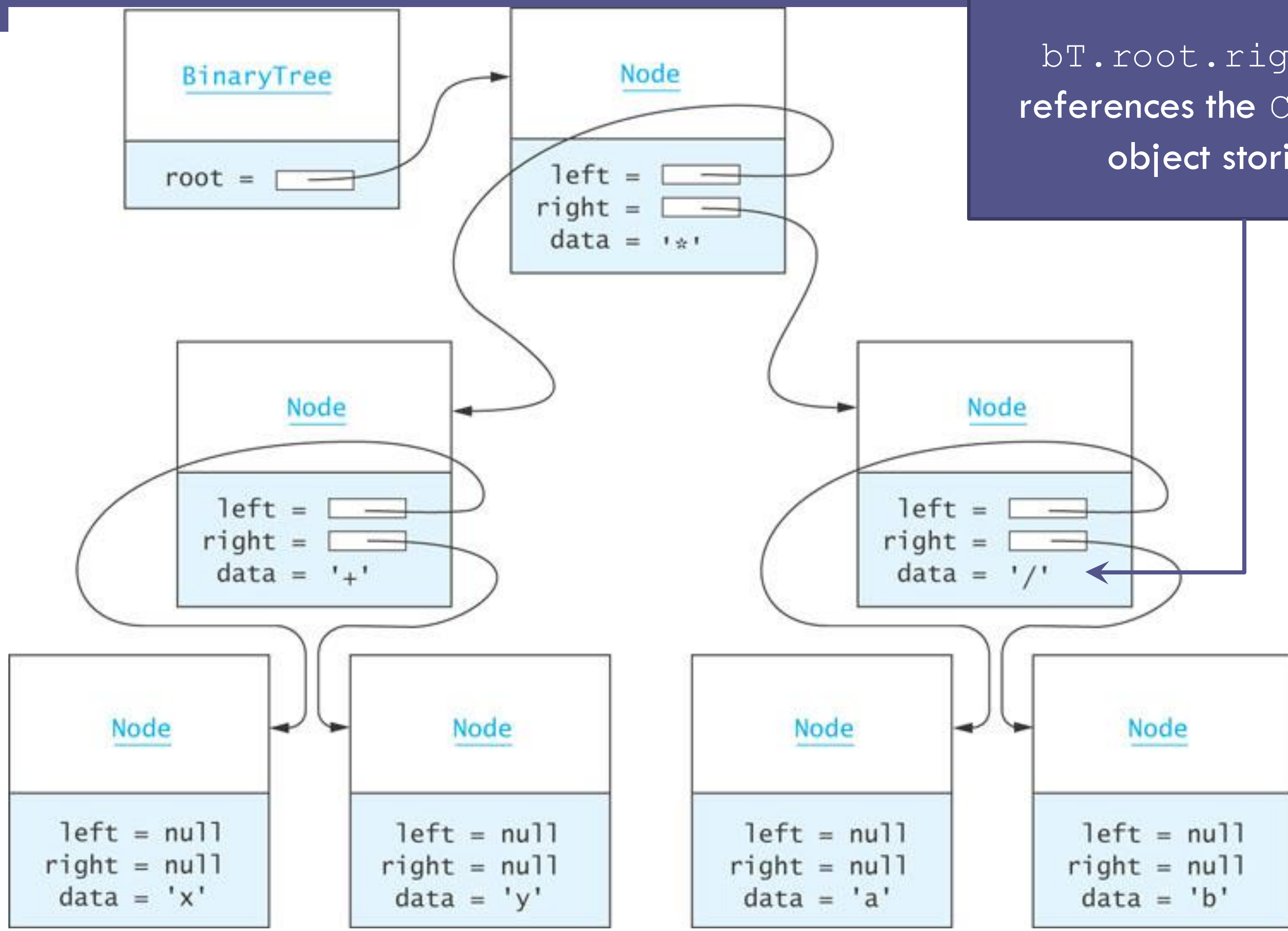


BinaryTree<E> **Class (cont.)**

`bT.root.right` **references the right subtree of the root**



BinaryTree<E> Class (cont.)



bT.root.right.data references the Character object storing '/'

BinaryTree<E> **Class (cont.)**

Data Field	Attribute
`protected Node<E> root`	Reference to the root of the tree.
Constructor	**Behavior**
`public BinaryTree()`	Constructs an empty binary tree.
`protected BinaryTree(Node<E> root)`	Constructs a binary tree with the given node as the root.
`public BinaryTree(E data, BinaryTree<E> leftTree, BinaryTree<E> rightTree)`	Constructs a binary tree with the given data at the root and the two given subtrees.
Method	**Behavior**
`public BinaryTree<E> getLeftSubtree()`	Returns the left subtree.
`public BinaryTree<E> getRightSubtree()`	Returns the right subtree.
`public E getData()`	Returns the data in the root.
`public boolean isLeaf()`	Returns **`true`** if this tree is a leaf, **`false`** otherwise.
`public String toString()`	Returns a `String` representation of the tree.
`private void preOrderTraverse(Node<E> node, int depth, StringBuilder sb)`	Performs a preorder traversal of the subtree whose root is `node`. Appends the representation to the `StringBuilder`. Increments the value of `depth` (the current tree level).
`public static BinaryTree<E> readBinaryTree(Scanner scan)`	Constructs a binary tree by reading its data using `Scanner scan`.

`BinaryTree<E>` Class (cont.)

- Class heading and data field declarations:

```
import java.io.*;

public class BinaryTree<E> implements Serializable {
   // Insert inner class Node<E> here

   protected Node<E> root;

   . . .
}
```

`BinaryTree<E>` Class (cont.)

- The `Serializable` interface defines no methods
- It provides a marker for classes that can be written to a binary file using the `ObjectOutputStream` and read using the `ObjectInputStream`

Constructors

- The no-parameter constructor:

```
public BinaryTree() {
    root = null;
}
```

- The constructor that creates a tree with a given node at the root:

```
protected BinaryTree(Node<E> root) {
    this.root = root;
}
```

Constructors (cont.)

- The constructor that builds a tree from a data value and two trees:

```
public BinaryTree(E data, BinaryTree<E> leftTree,
                  BinaryTree<E> rightTree) {

    root = new Node<E>(data);
    if (leftTree != null) {
        root.left = leftTree.root;
    } else {
        root.left = null;
    }
    if (rightTree != null) {
        root.right = rightTree.root;
    } else {
        root.right = null;
    }
}
```

`getLeftSubtree` and `getRightSubtree` Methods

```
public BinaryTree<E> getLeftSubtree() {
    if (root != null && root.left != null) {
        return new BinaryTree<E>(root.left);
  }   else {
        return null;
  }
}
```

- `getRightSubtree` method is symmetric

`isLeaf` **Method**

```
public boolean isLeaf() {
     return (root.left == null && root.right == null);
}
```

`toString` Method

- The toString method generates a string representing a preorder traversal in which each local root is indented a distance proportional to its depth

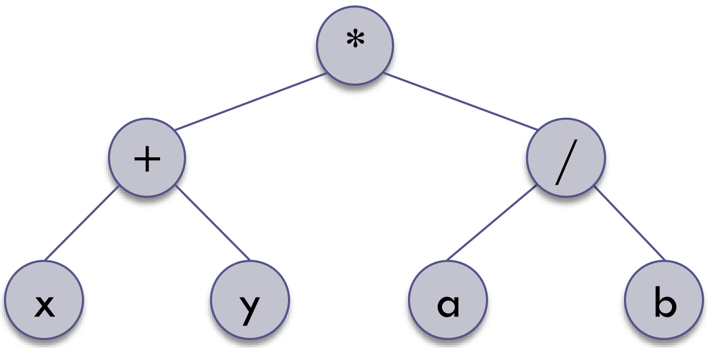
```
public String toString() {
    StringBuilder sb = new StringBuilder();
    preOrderTraverse(root, 1, sb);
    return sb.toString();
}
```

preOrderTraverse Method

```
private void preOrderTraverse(Node<E> node, int depth,
                              StringBuilder sb) {

    for (int i = 1; i < depth; i++) {
        sb.append("  "); // indentation
    }
    if (node == null) {
        sb.append("null\n");
    } else {
        sb.append(node.toString());
        sb.append("\n");
        preOrderTraverse(node.left, depth + 1, sb);
        preOrderTraverse(node.right, depth + 1, sb);
    }
}
```

`preOrderTraverse` Method (cont.)

```
*
  +
    x
      null
      null
    y
      null
      null
  /
    a
      null
      null
    b
      null
      null
```



`(x + y) * (a / b)`

Reading a Binary Tree

- If we use a Scanner to read the individual lines created by the `toString` and `preOrderTraverse` methods, we can reconstruct the tree

1. Read a line that represents information at the root
2. Remove the leading and trailing spaces using `String.trim`
3. **`if`** it is "`null`"
4. return **`null`**

   `else`

5. recursively read the left child
6. recursively read the right child
7. return a tree consisting of the root and the two children

Reading a Binary Tree (cont.)

```
public static BinaryTree<String>
                        readBinaryTree(Scanner scan) {
  String data = scan.next();
  if (data.equals("null")) {
    return null;
  } else {
     BinaryTree<String> leftTree = readBinaryTree(scan);
     BinaryTree<String> rightTree = readBinaryTree(scan);
     return new BinaryTree<String>(data, leftTree,
                                  rightTree);
  }
}
```

Using `ObjectOutputStream` and `ObjectInputStream`

- The Java API includes the class `ObjectOutputStream` that will write to an external file any object that is declared to be `Serializable`
- To declare an object `Serializable`, add

  `implements Serializable`

  to the class declaration
- The `Serializable` interface contains no methods, but it serves to mark the class and gives you control over whether or not you want your object written to an external file

Using `ObjectOutputStream` and `ObjectInputStream` (cont.)

- To write a `Serializable` object to a file:

```
try {
    ObjectOutputStream out =
        new ObjectOutputStream(new FileOutputStream(filename));
    out.writeObject(nameOfObject);
} catch (Exception ex) {
    ex.printStackTrace();
    System.exit(1);
}
```

- A deep copy of all the nodes of the binary tree will be written to the file

Using `ObjectOutputStream` and `ObjectInputStream` (cont.)

- To read a `Serializable` object from a file:

```
try {
   ObjectInputStream in =
       new ObjectInputStream(new FileInputStream(filename));

   objectName = (objectClass)in.readObject();
} catch (Exception ex) {
    ex.printStackTrace();
    System.exit(1);
}
```

Using `ObjectOutputStream` and `ObjectInputStream` (cont.)

- Do not recompile the Java source file for a class after an object of that class has been serialized
- Even if you didn't make any changes to the class, the resulting `.class` file associated with the serialized object will have a different class signature
- When you attempt to read the object, the class signatures will not match, and you will get an exception

Binary Search Trees

Section 6.4

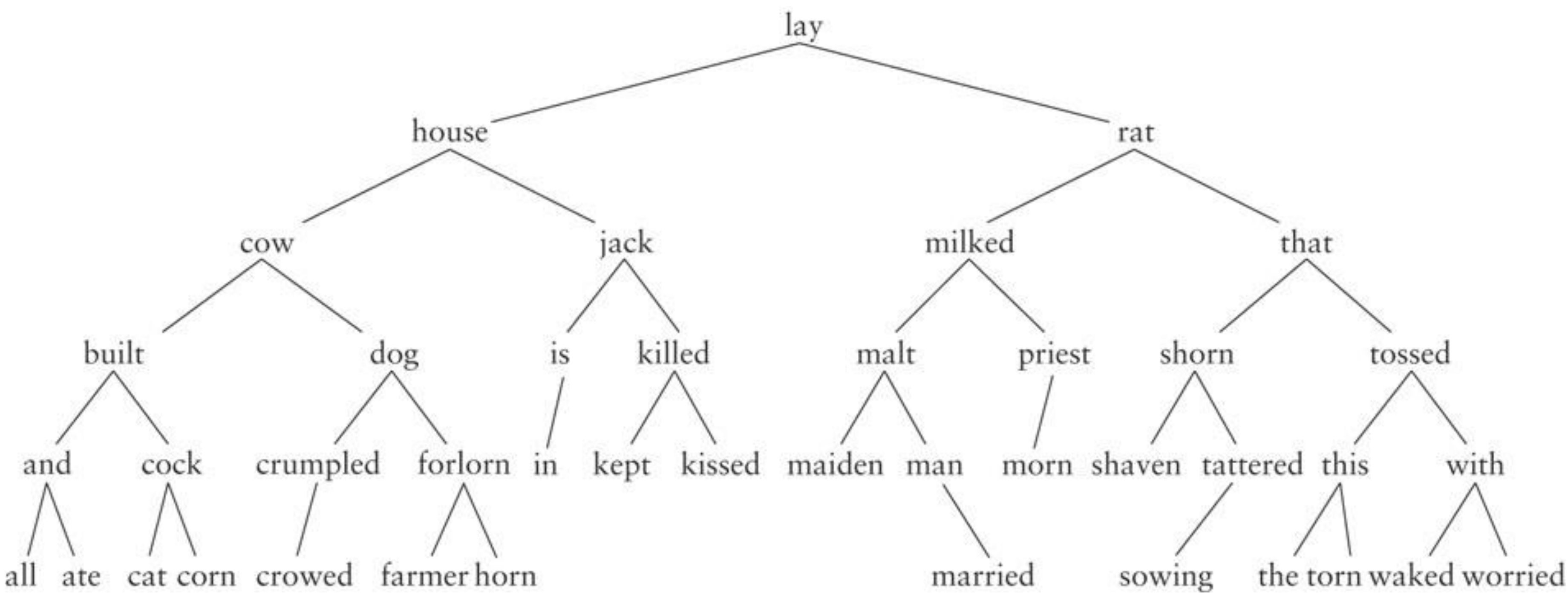
Overview of a Binary Search Tree

- Recall the definition of a binary search tree:

  A set of nodes T is a binary search tree if either of the following is true

  - T is empty
  - If T is not empty, its root node has two subtrees, T_L and T_R, such that T_L and T_R are binary search trees and the value in the root node of T is greater than all values in T_L and less than all values in T_R

Overview of a Binary Search Tree (cont.)



Recursive Algorithm for Searching a Binary Search Tree

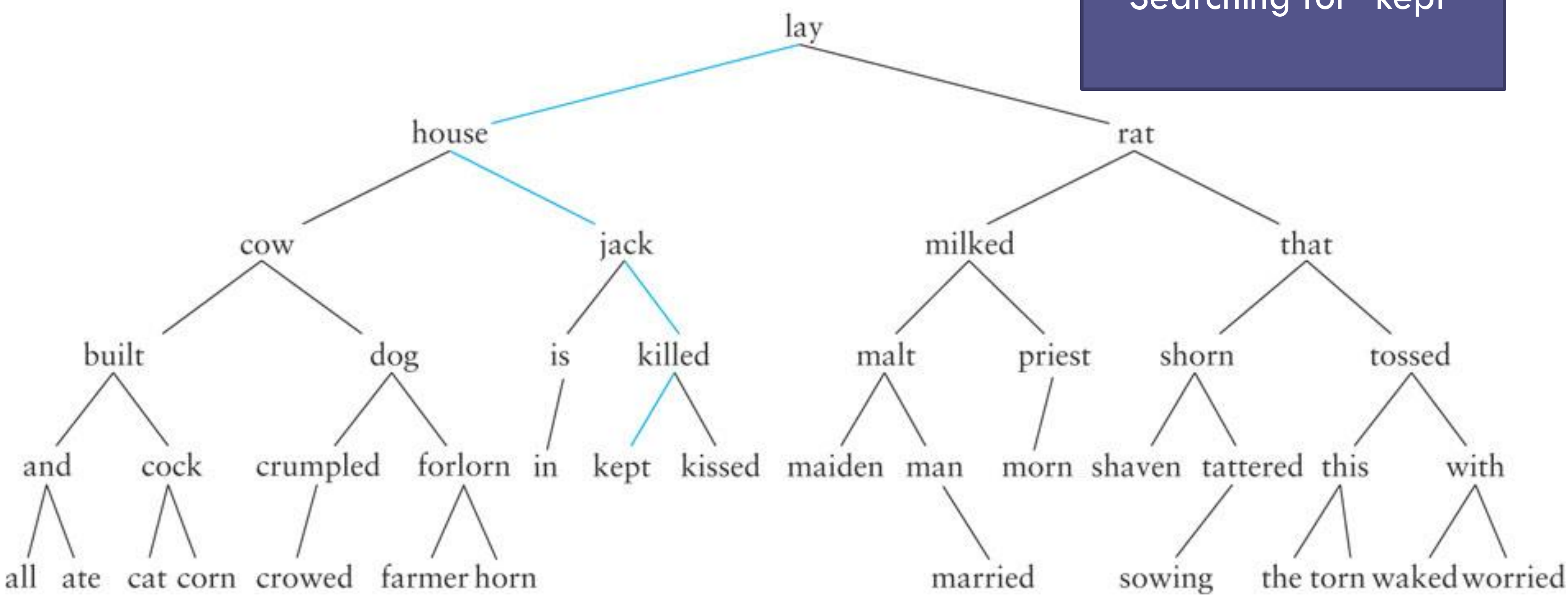
1. `if` the root is `null`
2. the item is not in the tree; return `null`
3. Compare the value of `target` with `root.data`
4. `if` they are equal
5. the target has been found; return the data at the root

   `else if` the target is less than `root.data`
6. return the result of searching the left subtree

   `else`
7. return the result of searching the right subtree

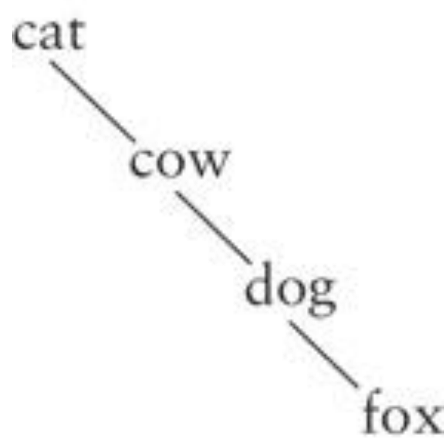
Searching a Binary Tree

Searching for "kept"



Performance

- Search a tree is generally O(log *n*)
- If a tree is not very full, performance will be worse
- Searching a tree with only right subtrees, for example, is O(*n*)

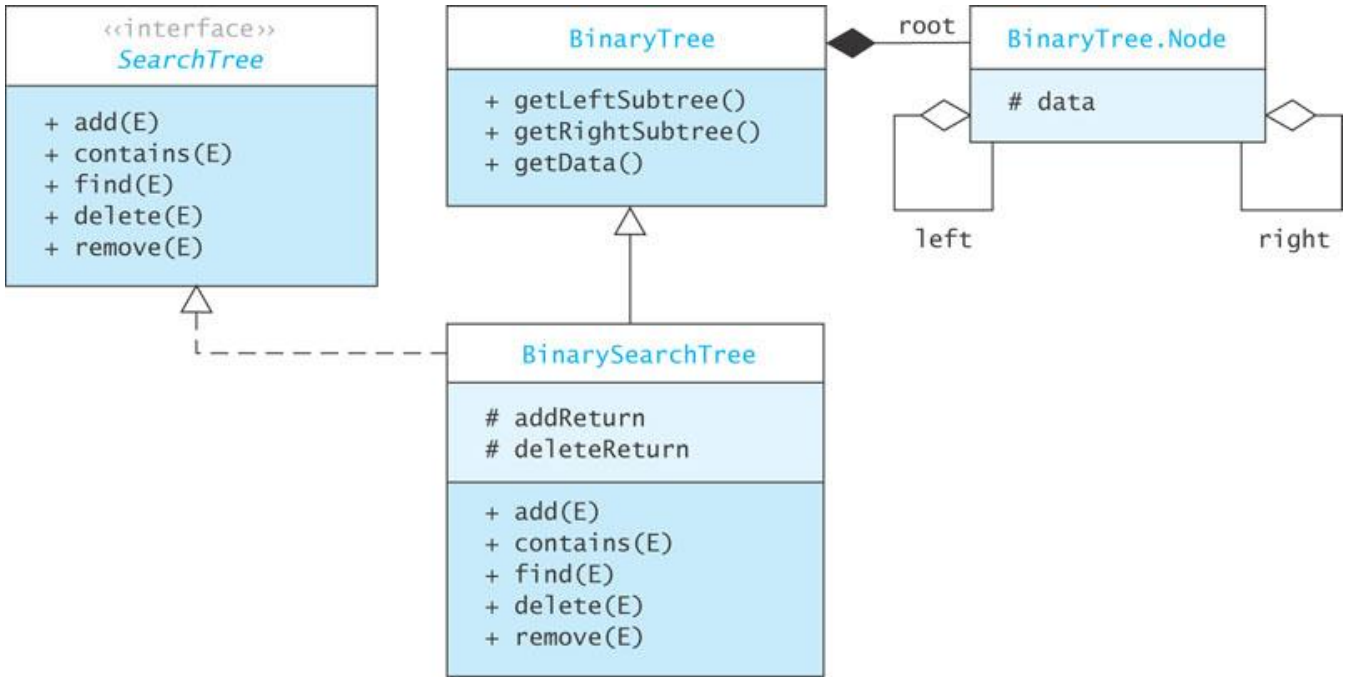


Interface SearchTree<E>

Method	Behavior
`boolean add(E item)`	Inserts `item` where it belongs in the tree. Returns **`true`** if item is inserted; **`false`** if it isn't (already in tree).
`boolean contains(E target)`	Returns **`true`** if `target` is found in the tree.
`E find(E target)`	Returns a reference to the data in the node that is equal to `target`. If no such node is found, returns **`null`**.
`E delete(E target)`	Removes `target` (if found) from tree and returns it; otherwise, returns **`null`**.
`boolean remove(E target)`	Removes `target` (if found) from tree and returns **`true`**; otherwise, returns **`false`**.

`BinarySearchTree<E>` Class

Data Field	Attribute
`protected boolean addReturn`	Stores a second return value from the recursive `add` method that indicates whether the item has been inserted.
`protected E deleteReturn`	Stores a second return value from the recursive `delete` method that references the item that was stored in the tree.



Implementing `find` Methods

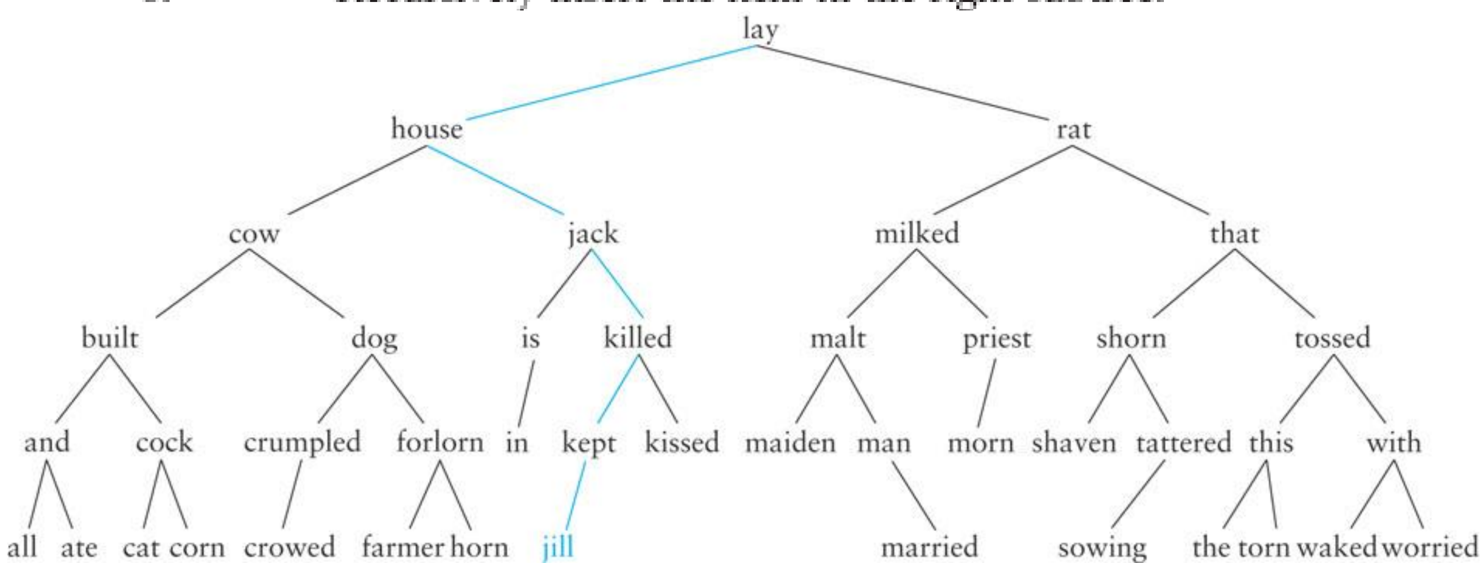
BinarySearchTree find Method

```java
/** Starter method find.
    pre: The target object must implement
         the Comparable interface.
    @param target The Comparable object being sought
    @return The object, if found, otherwise null
*/
public E find(E target) {
    return find(root, target);
}

/** Recursive find method.
    @param localRoot The local subtree's root
    @param target The object being sought
    @return The object, if found, otherwise null
*/
private E find(Node<E> localRoot, E target) {
    if (localRoot == null)
        return null;

    // Compare the target with the data field at the root.
    int compResult = target.compareTo(localRoot.data);
    if (compResult == 0)
        return localRoot.data;
    else if (compResult < 0)
        return find(localRoot.left, target);
    else
        return find(localRoot.right, target);
}
```

Insertion into a Binary Search Tree

Recursive Algorithm for Insertion in a Binary Search Tree

```
if the root is null
    Replace empty tree with a new tree with the item at the root and
    return true.
else if the item is equal to root.data
    The item is already in the tree; return false.
else if the item is less than root.data
    Recursively insert the item in the left subtree.
else
    Recursively insert the item in the right subtree.
```



Implementing the `add` Methods

```
/** Starter method add.
    pre: The object to insert must implement the
         Comparable interface.
    @param item The object being inserted
    @return true if the object is inserted, false
            if the object already exists in the tree
*/
public boolean add(E item) {
    root = add(root, item);
    return addReturn;
}
```

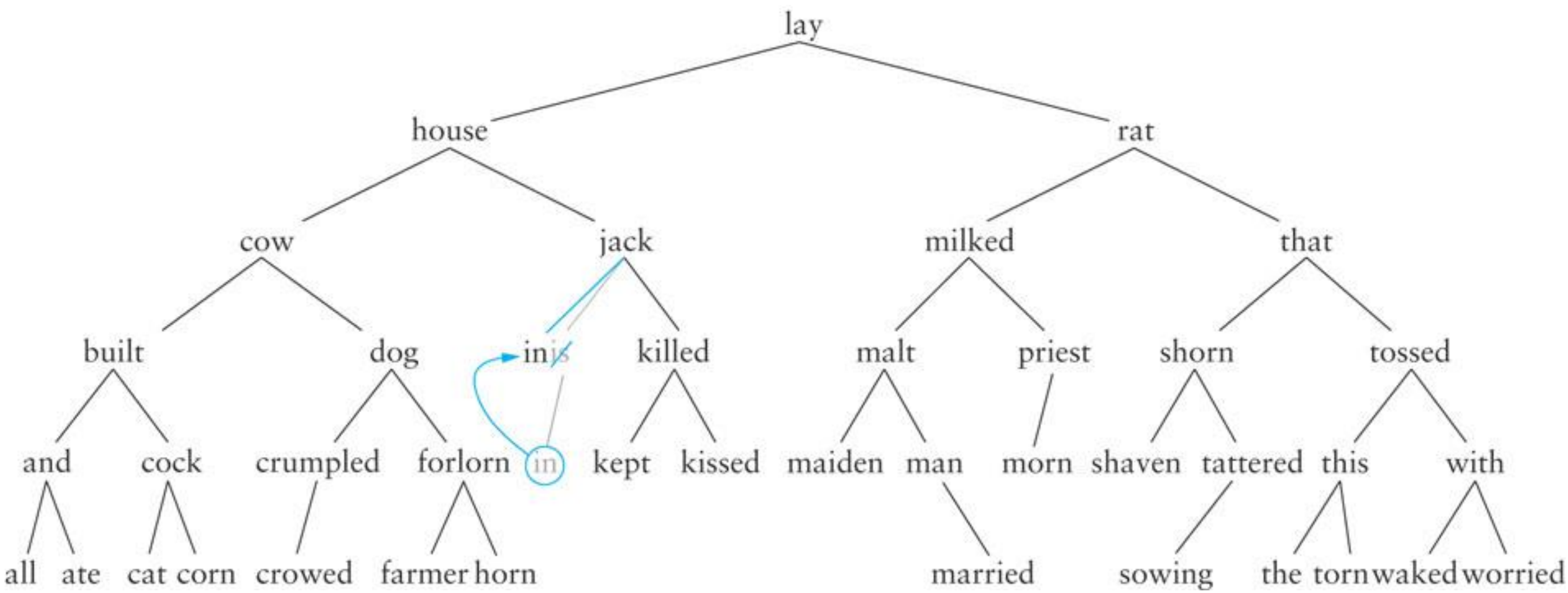
Implementing the `add` Methods (cont.)

```
/** Recursive add method.
    post: The data field addReturn is set true if the item is added to
          the tree, false if the item is already in the tree.
    @param localRoot The local root of the subtree
    @param item The object to be inserted
    @return The new local root that now contains the
          inserted item
*/
private Node<E> add(Node<E> localRoot, E item) {
    if (localRoot == null) {
          // item is not in the tree — insert it.
          addReturn = true;
          return new Node<E>(item);
    } else if (item.compareTo(localRoot.data) == 0) {
          // item is equal to localRoot.data
          addReturn = false;
          return localRoot;
    } else if (item.compareTo(localRoot.data) < 0) {
          // item is less than localRoot.data
          localRoot.left = add(localRoot.left, item);
          return localRoot;
    } else {
          // item is greater than localRoot.data
          localRoot.right = add(localRoot.right, item);
          return localRoot;
    }
}
```

Removal from a Binary Search Tree

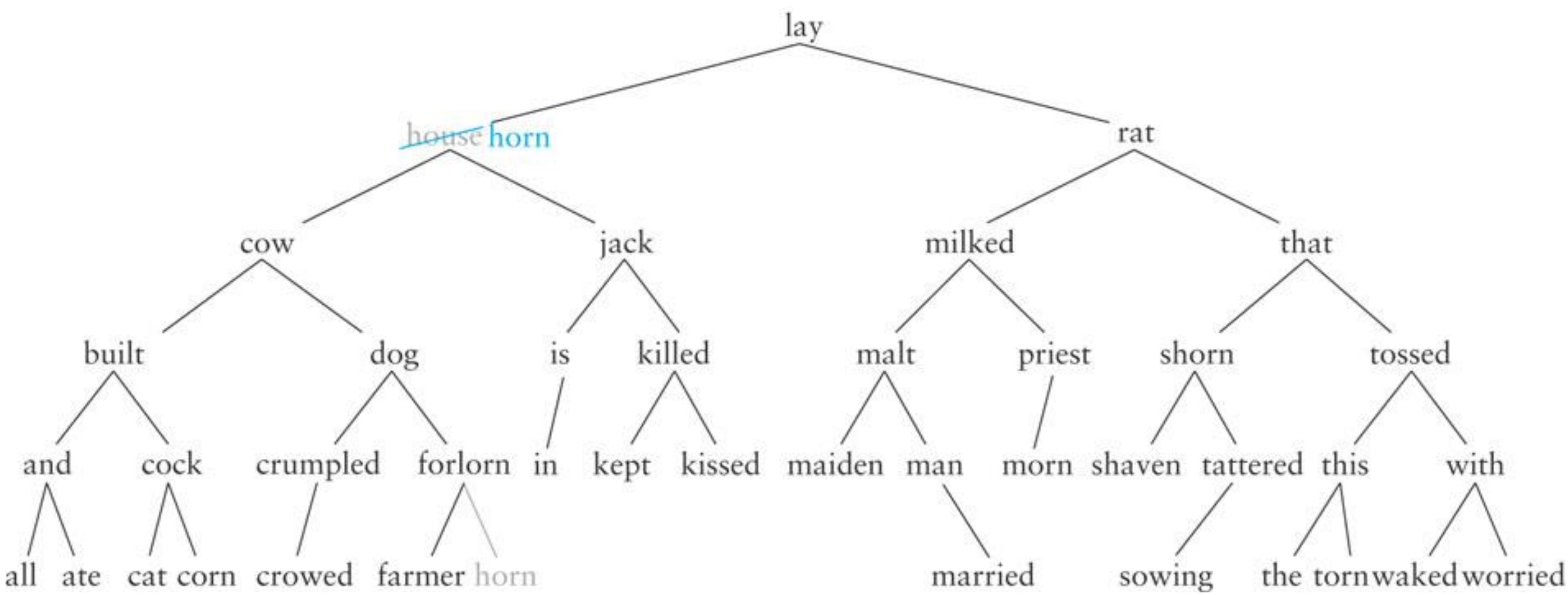
- If the item to be removed has no children, simply delete the reference to the item
- If the item to be removed has only one child, change the reference to the item so that it references the item's only child

Removal from a Binary Search Tree (cont.)

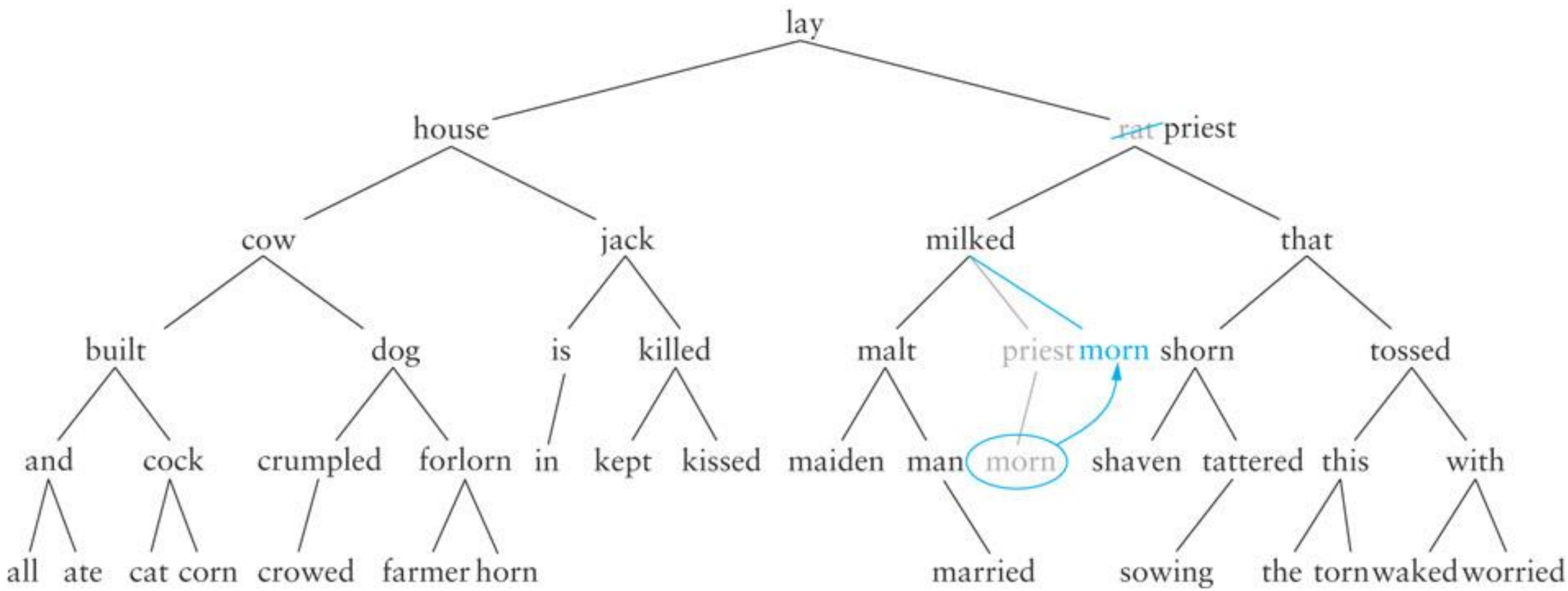


Removing from a Binary Search Tree (cont.)

- If the item to be removed has two children, replace it with the largest item in its left subtree – the inorder predecessor



Removing from a Binary Search Tree (cont.)



Algorithm for Removing from a Binary Search Tree

Recursive Algorithm for Removal from a Binary Search Tree

```
1.  if the root is null
2.      The item is not in tree – return null.
3.  Compare the item to the data at the local root.
4.  if the item is less than the data at the local root
5.      Return the result of deleting from the left subtree.
6.  else if the item is greater than the local root
7.      Return the result of deleting from the right subtree.
8.  else // The item is in the local root
9.      Store the data in the local root in deletedReturn.
10.     if the local root has no children
11.         Set the parent of the local root to reference null.
12.     else if the local root has one child
13.         Set the parent of the local root to reference that child.
14.     else // Find the inorder predecessor
15.         if the left child has no right child it is the inorder predecessor
16.             Set the parent of the local root to reference the left child.
17.         else
18.             Find the rightmost node in the right subtree of the left
                child.
19.             Copy its data into the local root's data and remove it by
                setting its parent to reference its left child.
```

Implementing the `delete` Method

- Listing 6.5 (`BinarySearchTree delete` Methods; pages 325-326)

Method `findLargestChild`

BinarySearchTree `findLargestChild` Method

```java
/** Find the node that is the
    inorder predecessor and replace it
    with its left child (if any).
    post: The inorder predecessor is removed from the tree.
    @param parent The parent of possible inorder
                  predecessor (ip)
    @return The data in the ip
*/
private E findLargestChild(Node<E> parent) {
    // If the right child has no right child, it is
    // the inorder predecessor.
    if (parent.right.right == null) {
        E returnValue = parent.right.data;
        parent.right = parent.right.left;
        return returnValue;
    } else {
        return findLargestChild(parent.right);
    }
}
```

Testing a Binary Search Tree

- To test a binary search tree, verify that an inorder traversal will display the tree contents in ascending order after a series of insertions and deletions are performed

Writing an Index for a Term Paper

- Problem: write an index for a term paper
  - The index should show each word in the paper followed by the line number on which it occurred
  - The words should be displayed in alphabetical order
  - If a word occurs on multiple lines, the line numbers should be listed in ascending order:

```
a, 003
a, 013
are, 003
```

Writing an Index for a Term Paper (cont.)

- Analysis
  - Store each word and its line number as a string in a tree node
  - For example, two occurences of "`java`": "`java, 005`" and "`java, 010`"
  - Display the words in ascending order by performing an inorder traversal

Writing an Index for a Term Paper (cont.)

- Design
  - Use `TreeSet<E>`, a class based on a binary search tree, provided in the Java API
  - Write a class `IndexGenerator` with a `TreeSet<String>` data fields

Data Field	Attribute
`private TreeSet<String> index`	The search tree used to store the index.
Method	**Behavior**
`public void buildIndex(Scanner scan)`	Reads each word from the file scanned by `scan` and stores it in tree `index`.
`public void showIndex()`	Performs an inorder traversal of tree `index`.

Writing an Index for a Term Paper (cont.)

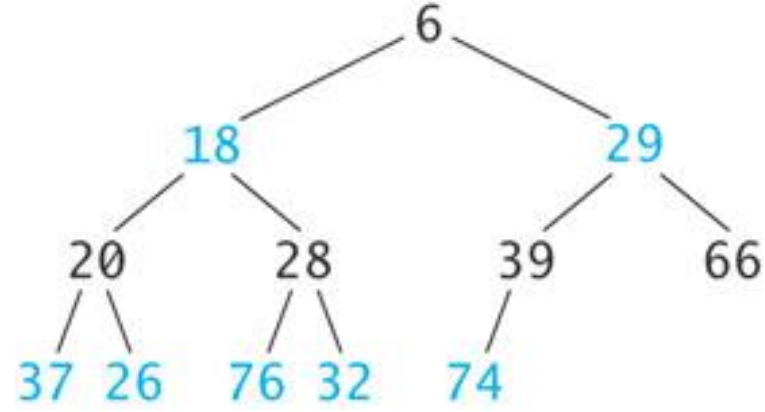
- Listing 6.7 (Class `IndexGenerator.java`; pages 330-331)

Heaps and Priority Queues

Section 6.5

Heaps and Priority Queues

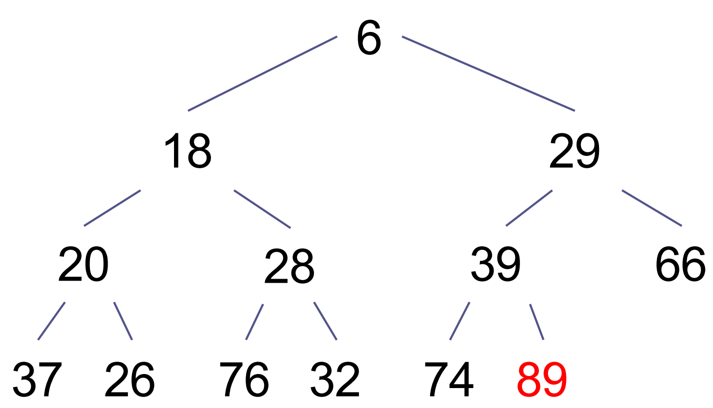
- A heap is a complete binary tree with the following properties
  - The value in the root is a smallest item in the tree
  - Every nonempty subtree is a heap



Inserting an Item into a Heap

Algorithm for Inserting in a Heap

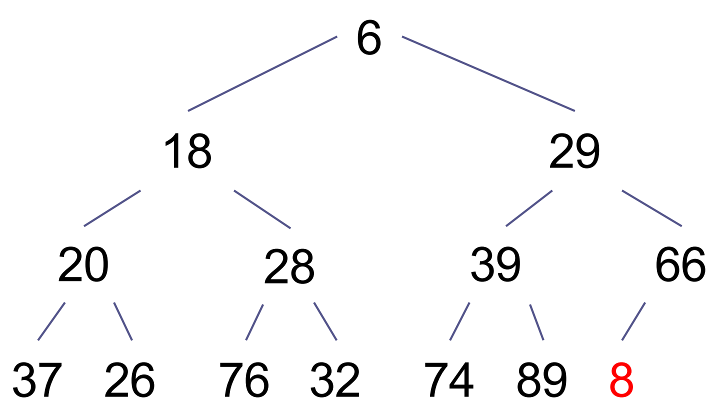
1. Insert the new item in the next position at the bottom of the heap.
2. `while` new item is not at the root and new item is smaller than its parent
3. Swap the new item with its parent, moving the new item up the heap.



Inserting an Item into a Heap (cont.)

Algorithm for Inserting in a Heap

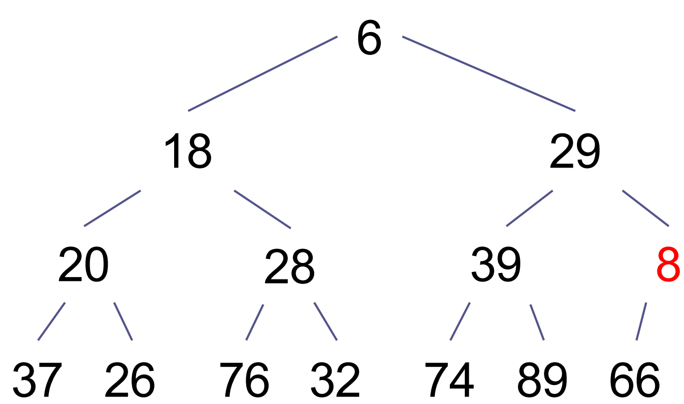
1. Insert the new item in the next position at the bottom of the heap.
2. `while` new item is not at the root and new item is smaller than its parent
3. Swap the new item with its parent, moving the new item up the heap.



Inserting an Item into a Heap (cont.)

Algorithm for Inserting in a Heap

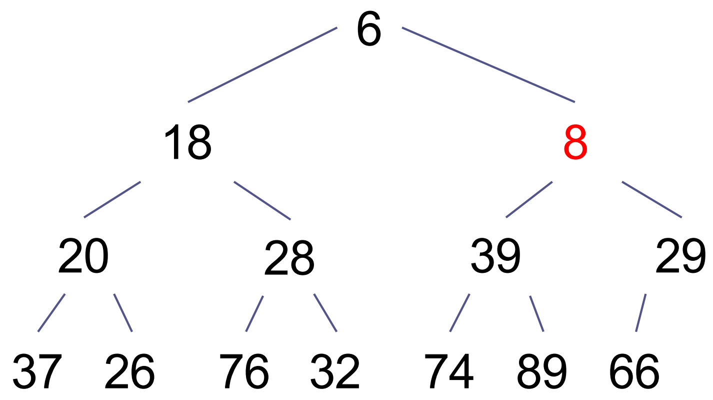
1. Insert the new item in the next position at the bottom of the heap.
2. `while` new item is not at the root and new item is smaller than its parent
3. Swap the new item with its parent, moving the new item up the heap.



Inserting an Item into a Heap (cont.)

Algorithm for Inserting in a Heap

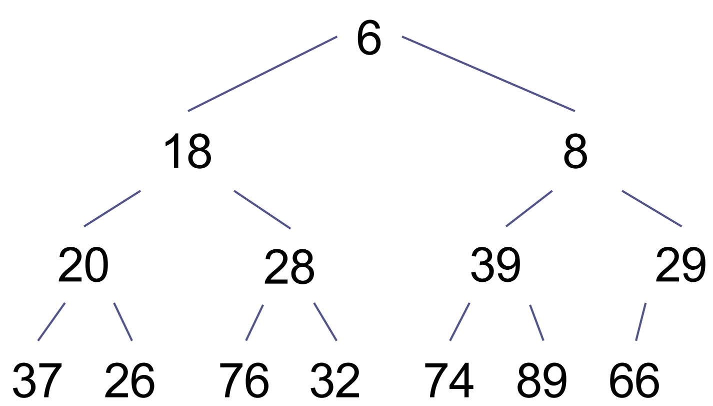
1. Insert the new item in the next position at the bottom of the heap.
2. `while` new item is not at the root and new item is smaller than its parent
3.   Swap the new item with its parent, moving the new item up the heap.



Inserting an Item into a Heap (cont.)

Algorithm for Inserting in a Heap

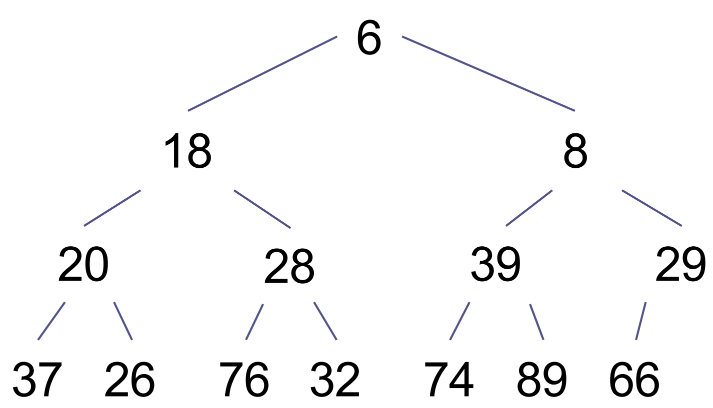
1. Insert the new item in the next position at the bottom of the heap.
2. `while` new item is not at the root and new item is smaller than its parent
3. Swap the new item with its parent, moving the new item up the heap.



Removing an Item from a Heap

Algorithm for Removal from a Heap

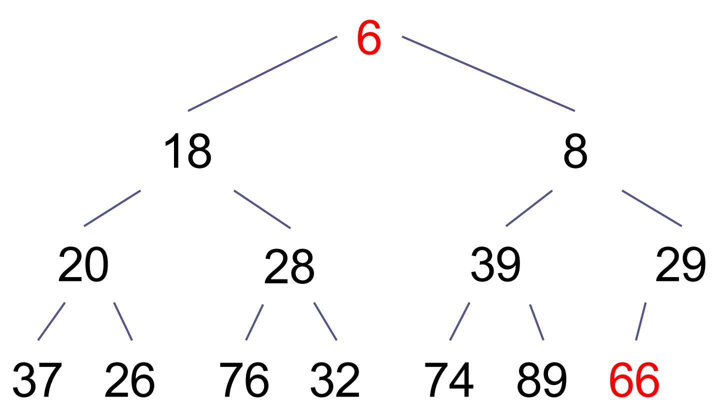
1. Remove the item in the root node by replacing it with the last item in the heap (LIH).
2. `while` item LIH has children and item LIH is larger than either of its children
3. Swap item LIH with its smaller child, moving LIH down the heap.



Removing an Item from a Heap (cont.)

Algorithm for Removal from a Heap

1. Remove the item in the root node by replacing it with the last item in the heap (LIH).
2. `while` item LIH has children and item LIH is larger than either of its children
3.   Swap item LIH with its smaller child, moving LIH down the heap.



Removing an Item from a Heap (cont.)

Algorithm for Removal from a Heap

1. Remove the item in the root node by replacing it with the last item in the heap (LIH).
2. `while` item LIH has children and item LIH is larger than either of its children
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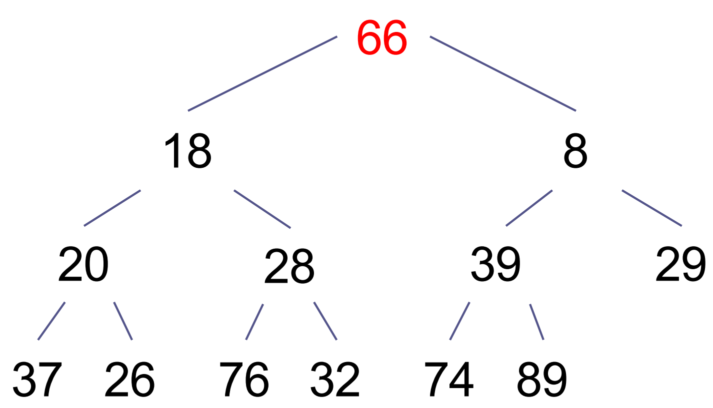
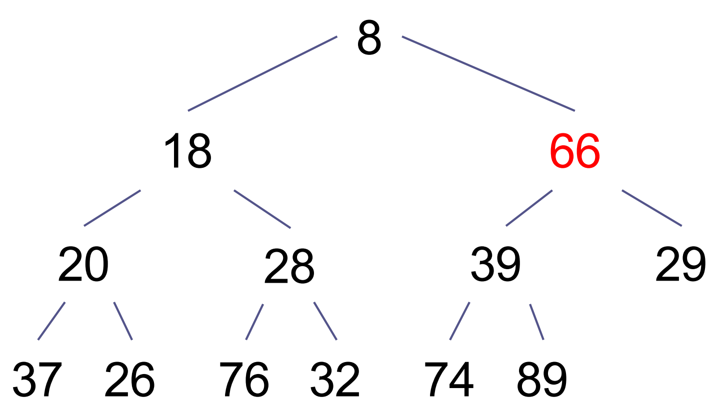


Removing an Item from a Heap (cont.)

Algorithm for Removal from a Heap

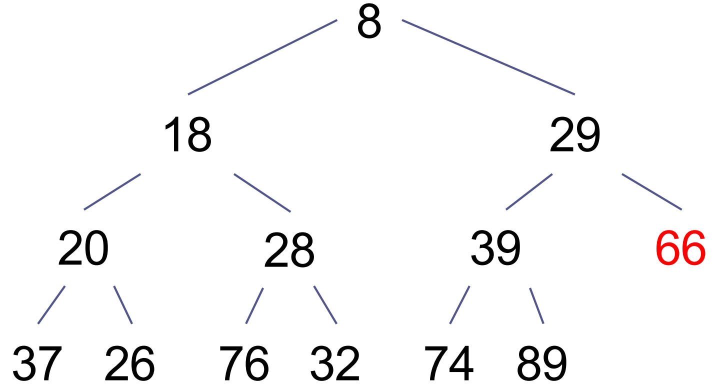
1. Remove the item in the root node by replacing it with the last item in the heap (LIH).
2. `while` item LIH has children and item LIH is larger than either of its children
3. Swap item LIH with its smaller child, moving LIH down the heap.



Removing an Item from a Heap (cont.)

Algorithm for Removal from a Heap

1. Remove the item in the root node by replacing it with the last item in the heap (LIH).
2. `while` item LIH has children and item LIH is larger than either of its children
3. Swap item LIH with its smaller child, moving LIH down the heap.



Removing an Item from a Heap (cont.)

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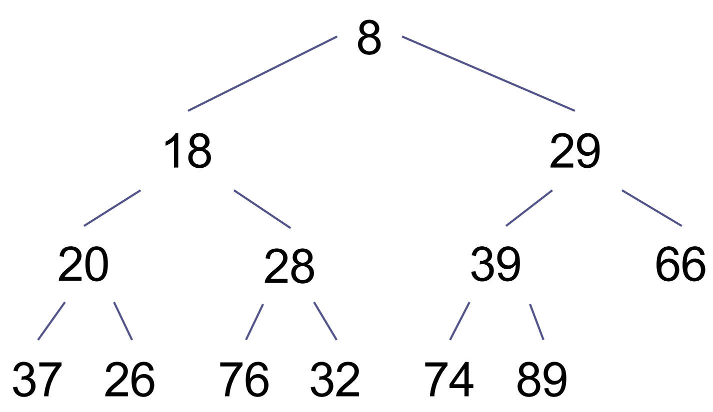
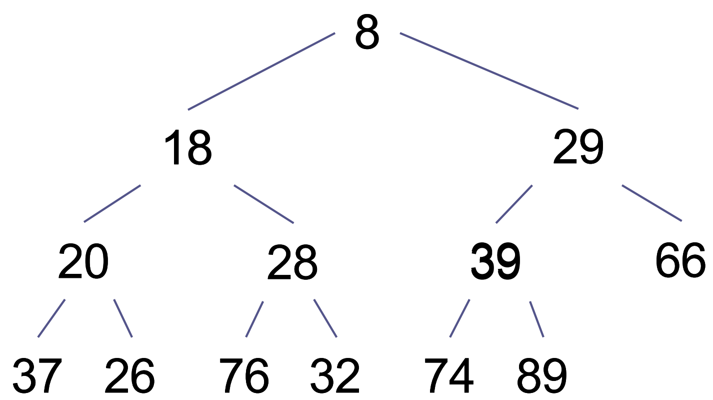


Implementing a Heap

- Because a heap is a complete binary tree, it can be implemented efficiently using an array rather than a linked data structure

Implementing a Heap (cont.)

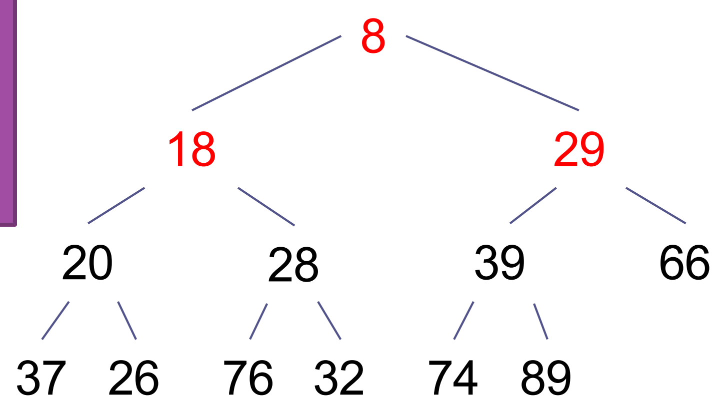


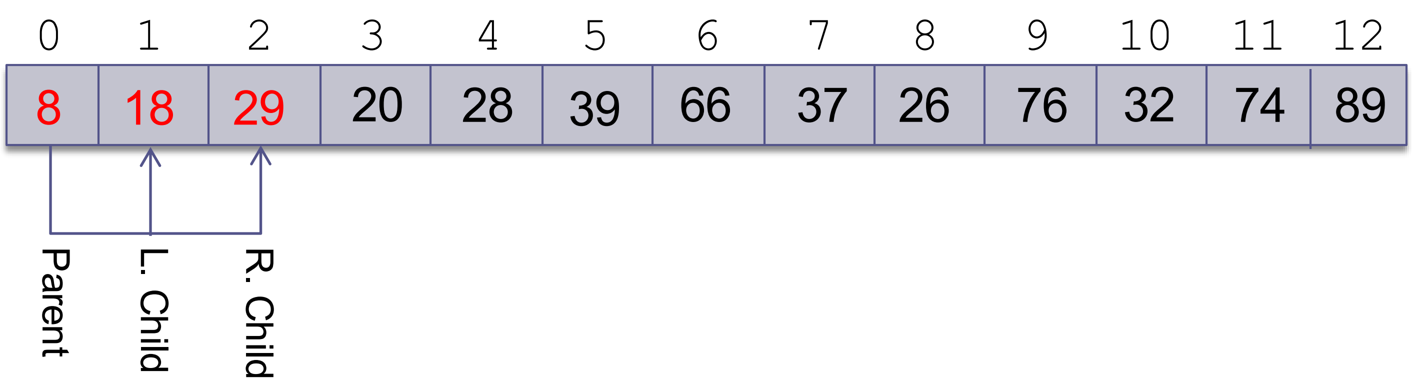
0	1	2	3	4	5	6	7	8	9	10	11	12
8	18	29	20	28	39	66	37	26	76	32	74	89

Implementing a Heap (cont.)

For a node at position p,

L. child position: $2p + 1$
R. child position: $2p + 2$



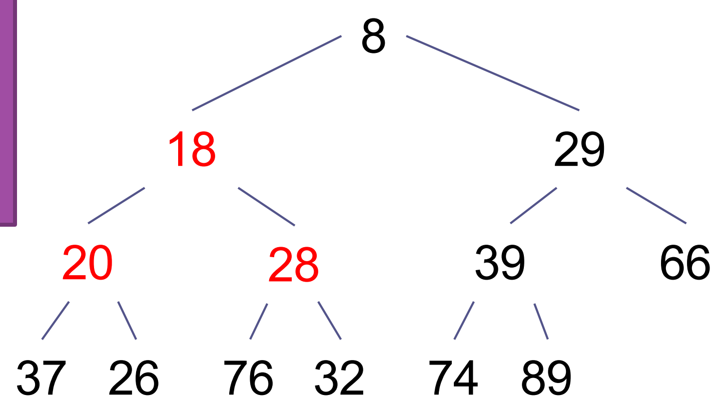


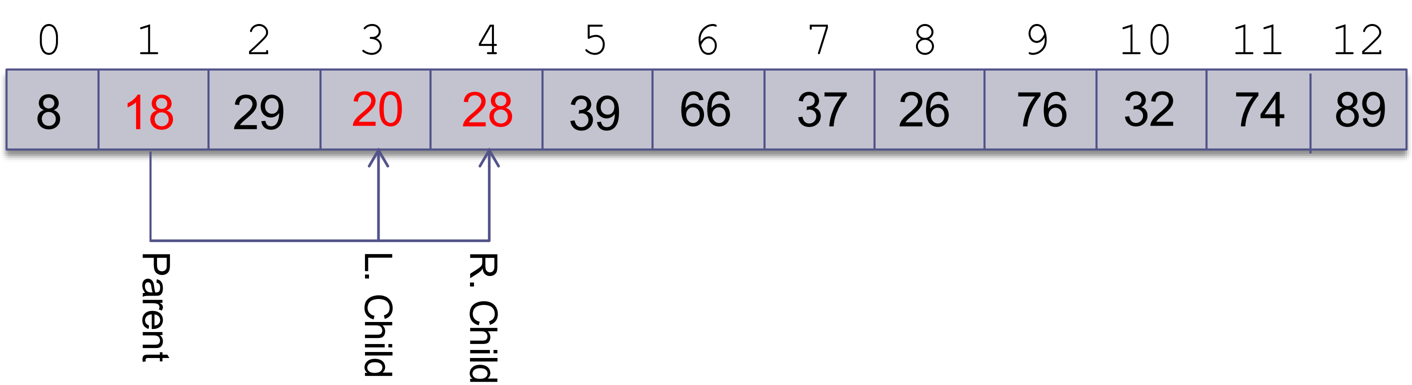
Implementing a Heap (cont.)

For a node at position *p*,

L. child position: $2p + 1$

R. child position: $2p + 2$





Implementing a Heap (cont.)

For a node at position *p*,

L. child position: $2p + 1$
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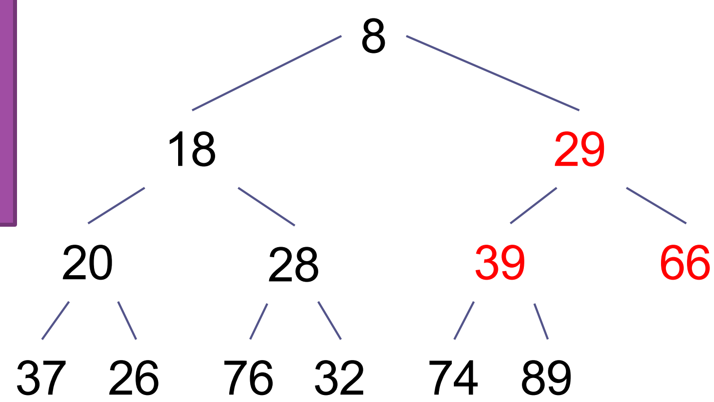
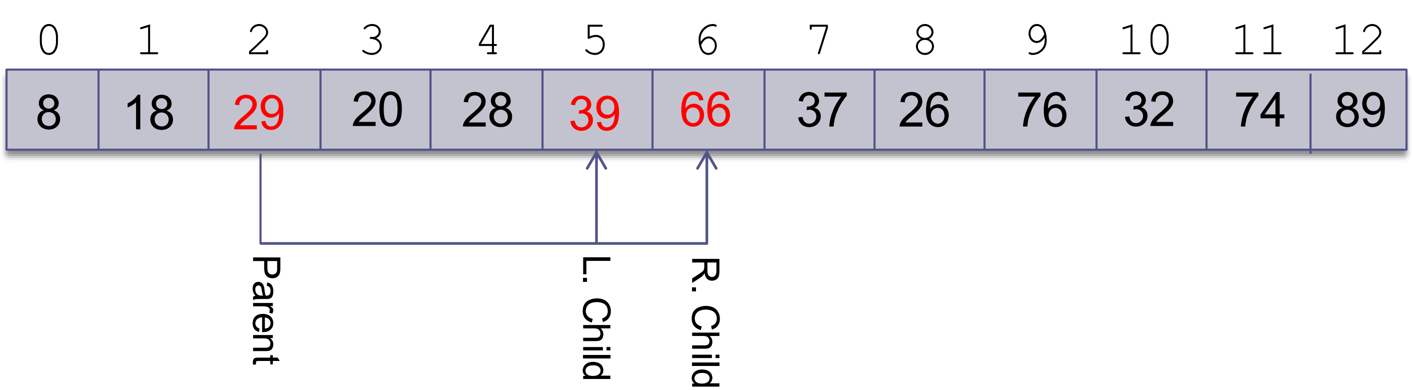
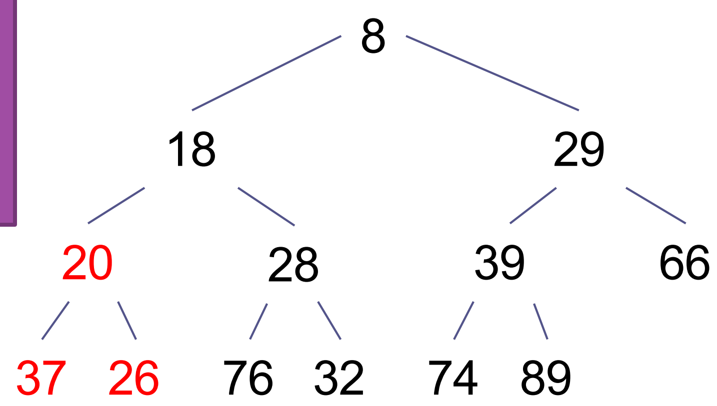




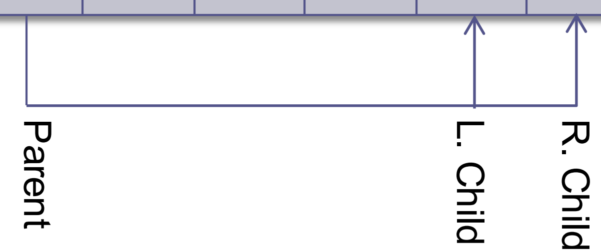
Implementing a Heap (cont.)

For a node at position *p*,

L. child position: $2p + 1$
R. child position: $2p + 2$



0	1	2	3	4	5	6	7	8	9	10	11	12
8	18	29	20	28	39	66	37	26	76	32	74	89



Implementing a Heap (cont.)

For a node at position *p*,

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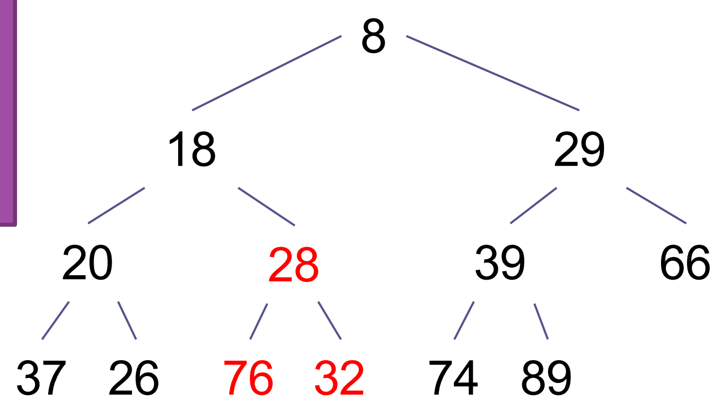
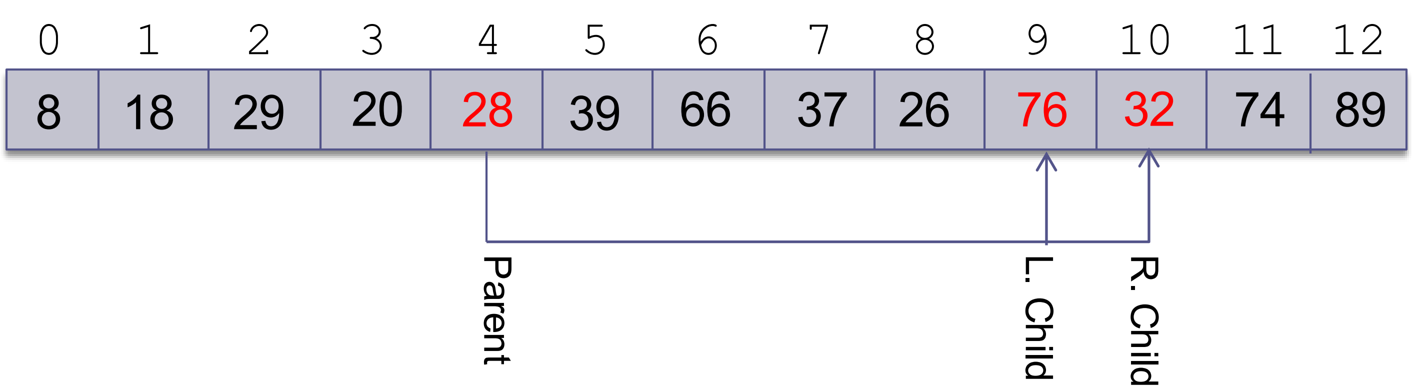
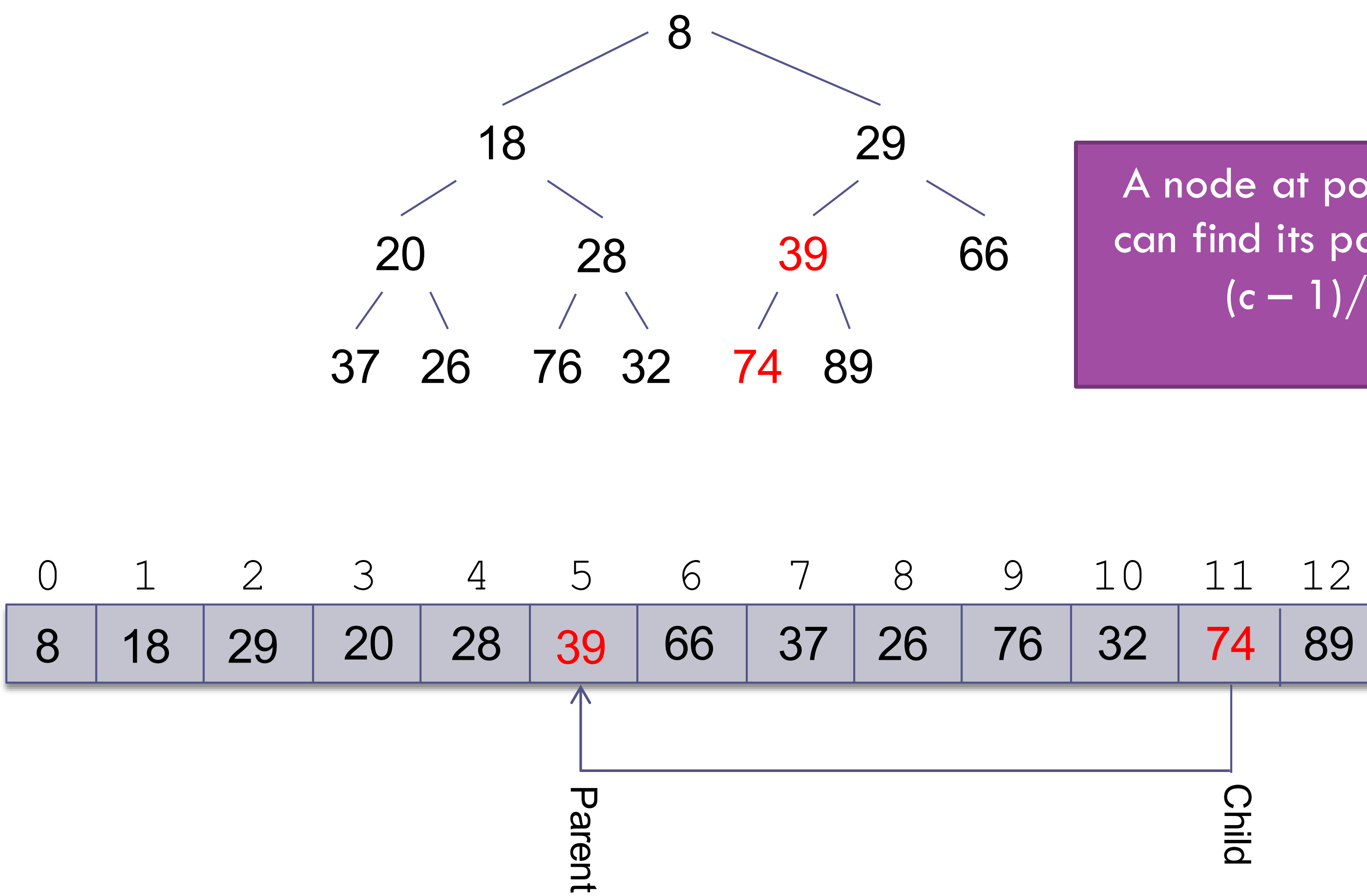


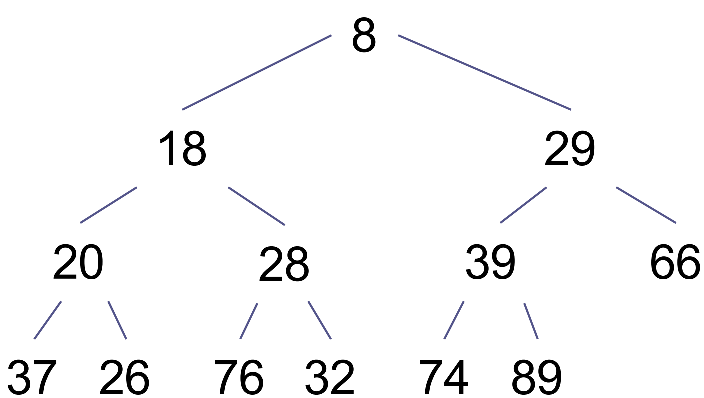


Implementing a Heap (cont.)



A node at position c can find its parent at $(c - 1)/2$

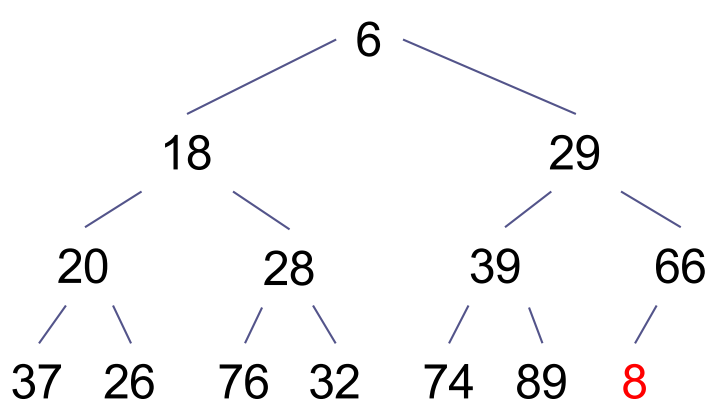
Inserting into a Heap Implemented as an `ArrayList`



1. Insert the new element at the end of the `ArrayList` and set `child` to `table.size() - 1`

0	1	2	3	4	5	6	7	8	9	10	11	12	13
8	18	29	20	28	39	66	37	26	76	32	74	89	

Inserting into a Heap Implemented as an `ArrayList` (cont.)



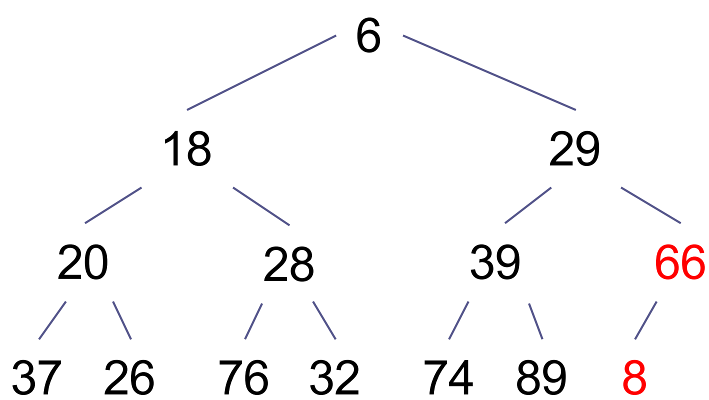
1. Insert the new element at the end of the `ArrayList` and set `child` to `table.size() - 1`

0	1	2	3	4	5	6	7	8	9	10	11	12	13
6	18	29	20	28	39	66	37	26	76	32	74	89	8

`Child`

Inserting into a Heap Implemented as an `ArrayList` (cont.)

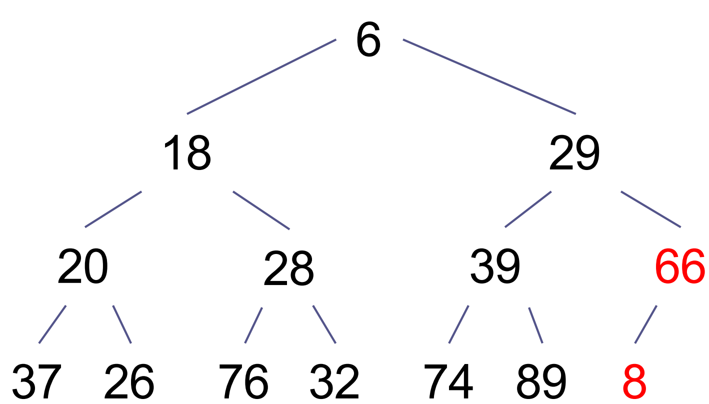
2. Set `parent` to `(child – 1)/ 2`



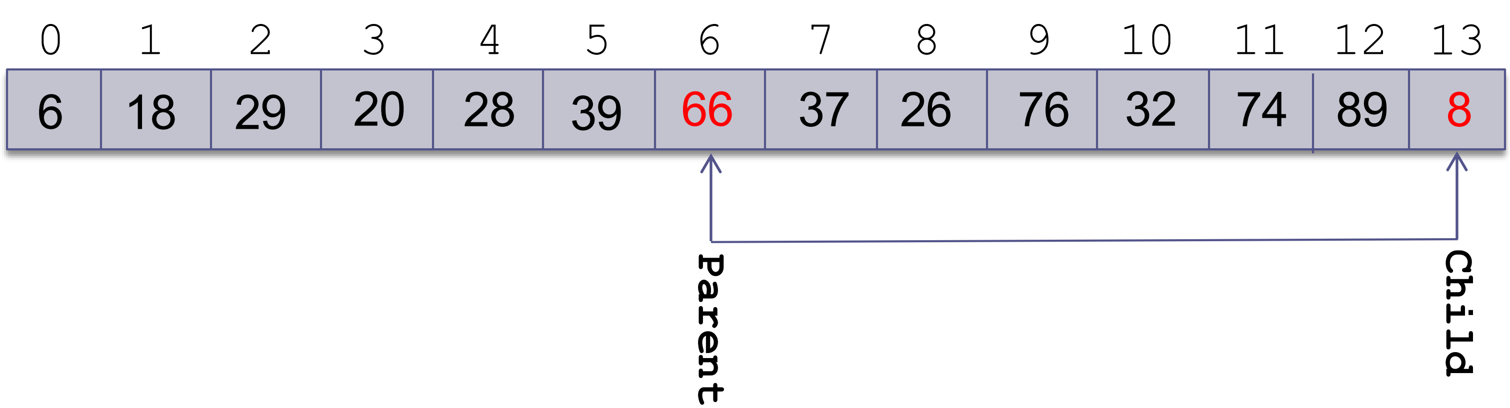
0	1	2	3	4	5	6	7	8	9	10	11	12	13
6	18	29	20	28	39	66	37	26	76	32	74	89	8

`Parent` `Child`

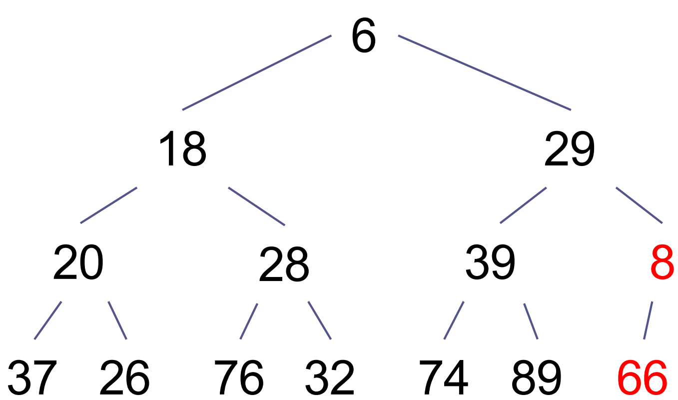
Inserting into a Heap Implemented as an `ArrayList` (cont.)



3. while (`parent >= 0`
   `and`
   `table[parent] > table[child]`)
4. Swap `table[parent]`
   and `table[child]`
5. Set `child` equal to `parent`
6. Set `parent` equal to `(child-1)/2`



Inserting into a Heap Implemented as an `ArrayList` (cont.)



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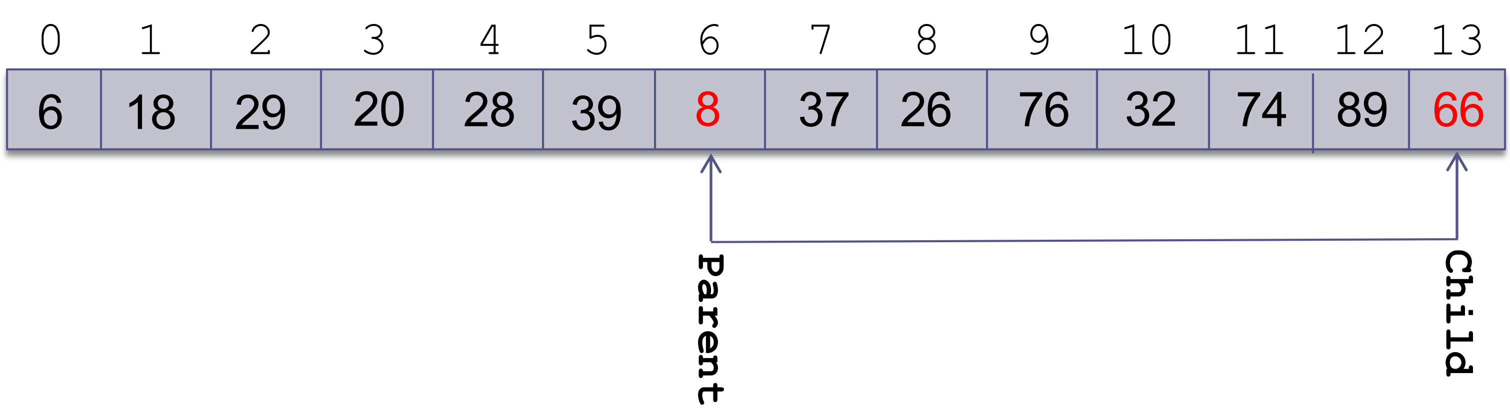
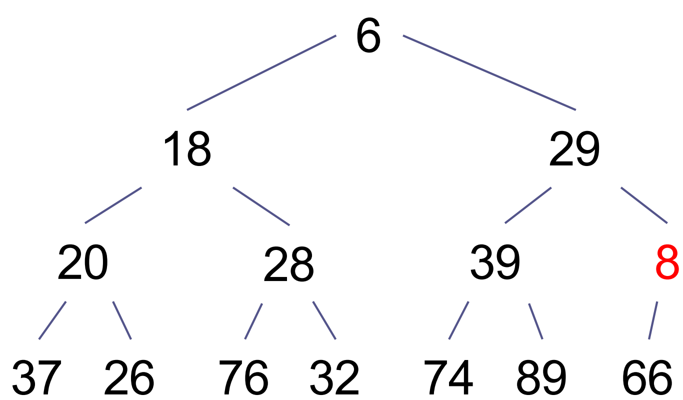


Inserting into a Heap Implemented as an `ArrayList` (cont.)



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4. Swap `table[parent]` and `table[child]`
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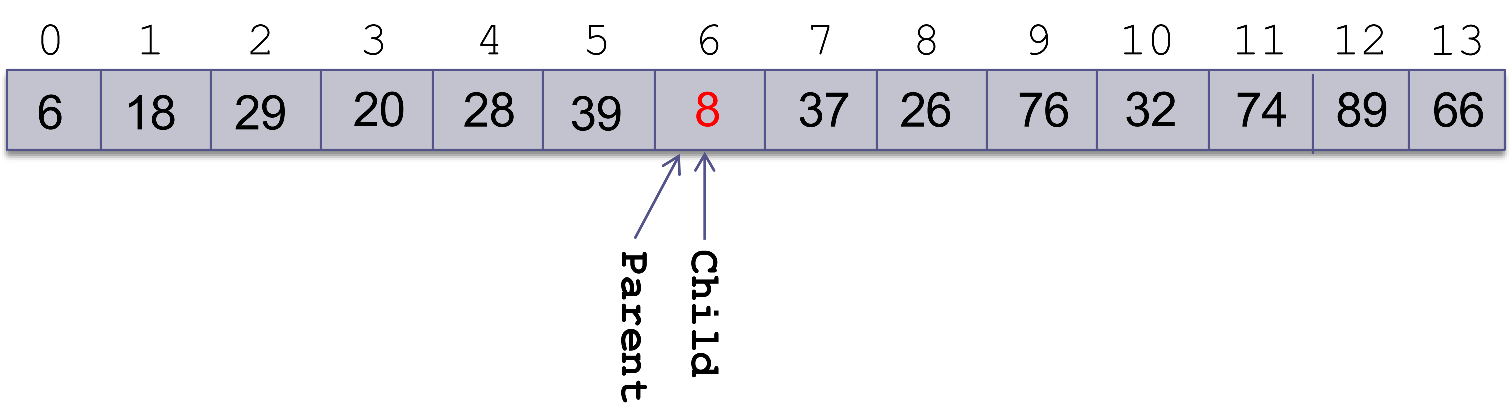
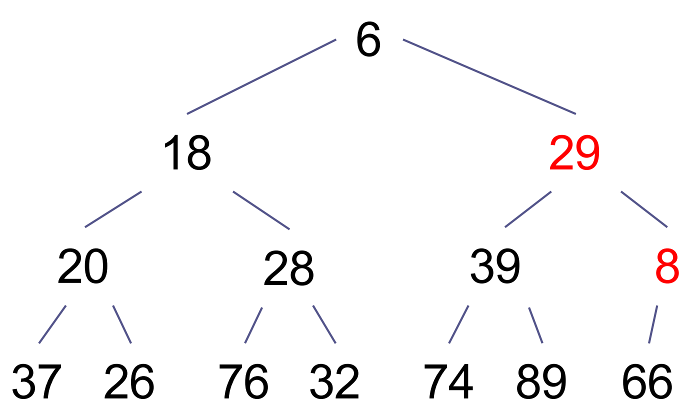


Inserting into a Heap Implemented as an `ArrayList` (cont.)



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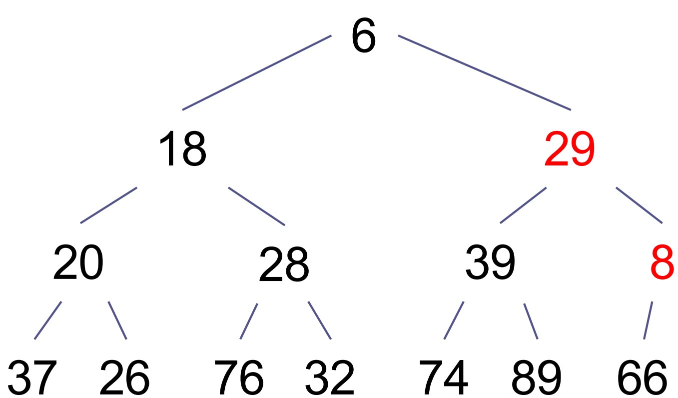
0	1	2	3	4	5	6	7	8	9	10	11	12	13
6	18	29	20	28	39	8	37	26	76	32	74	89	66

Inserting into a Heap Implemented as an `ArrayList` (cont.)



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4. Swap `table[parent]`
   and `table[child]`
5. Set `child` equal to `parent`
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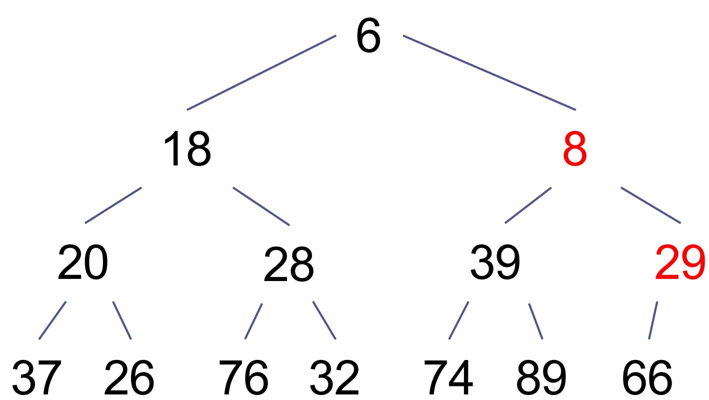
0	1	2	3	4	5	6	7	8	9	10	11	12	13
6	18	29	20	28	39	8	37	26	76	32	74	89	66

Parent **Child**

Inserting into a Heap Implemented as an `ArrayList` (cont.)



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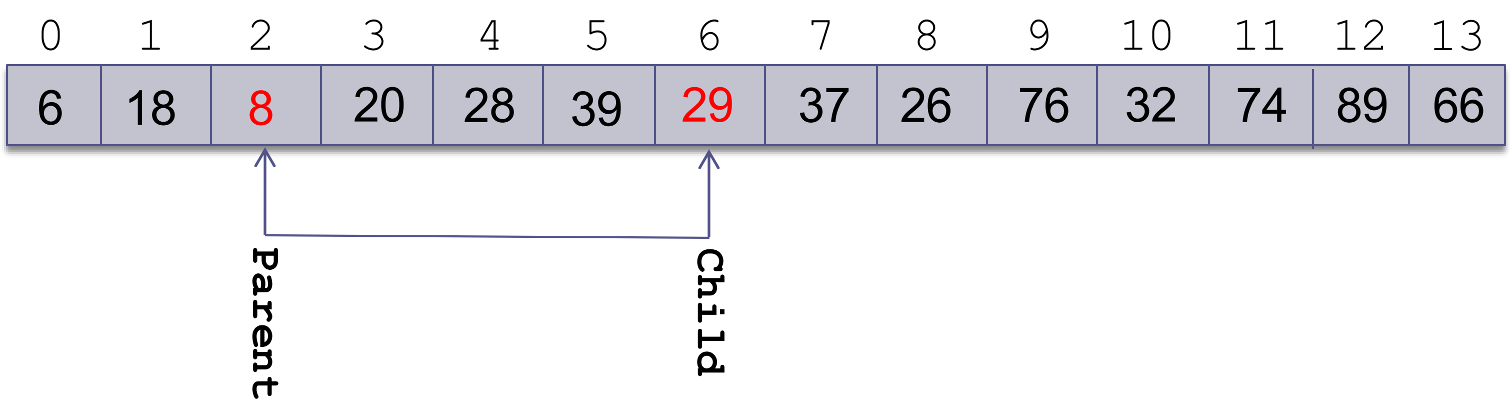
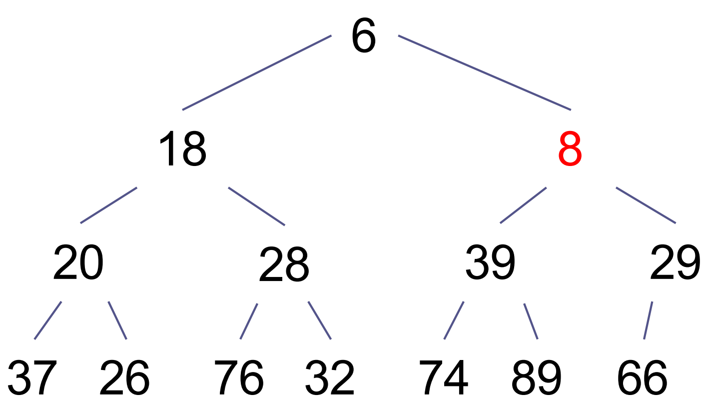


Inserting into a Heap Implemented as an `ArrayList` (cont.)



3. while (`parent >= 0`
   `and`
   `table[parent] > table[child]`)
4. Swap `table[parent]`
   and `table[child]`
5. Set `child` equal to `parent`
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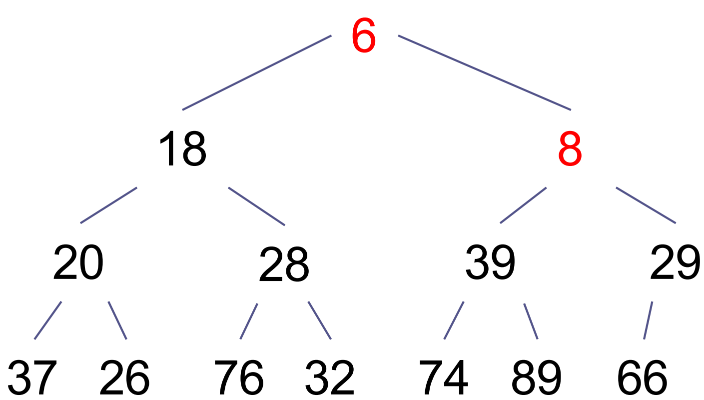
0	1	2	3	4	5	6	7	8	9	10	11	12	13
6	18	8	20	28	39	29	37	26	76	32	74	89	66

`Parent` `Child`

Inserting into a Heap Implemented as an `ArrayList` (cont.)



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   `table[parent] > table[child]`)
4. Swap `table[parent]` and `table[child]`
5. Set `child` equal to `parent`
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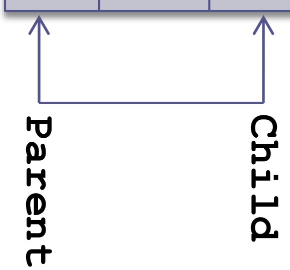
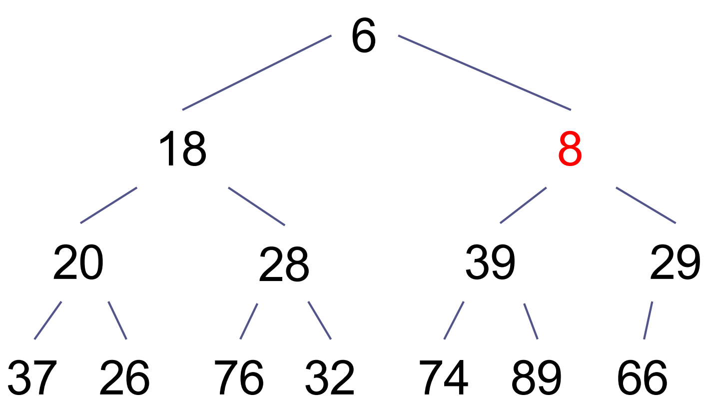
0	1	2	3	4	5	6	7	8	9	10	11	12	13
6	18	8	20	28	39	29	37	26	76	32	74	89	66



Inserting into a Heap Implemented as an `ArrayList` (cont.)



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0	1	2	3	4	5	6	7	8	9	10	11	12	13
6	18	8	20	28	39	29	37	26	76	32	74	89	66

Removal from a Heap Implemented as an `ArrayList`

Removing an Element from a Heap Implemented as an ArrayList

1. Remove the last element (i.e., the one at size() – 1) and set the item at 0 to this value.
2. Set `parent` to 0.
3. **`while (true)`**
4.   Set `leftChild` to (2 * `parent`) + 1 and `rightChild` to `leftChild` + 1.
5.   **`if leftChild >= table.size()`**
6.   Break out of loop.
7.   Assume `minChild` (the smaller child) is `leftChild`.
8.   **`if`** `rightChild < table.size() and table[rightChild] < table[leftChild]`
9.   Set `minChild` to `rightChild`.
10.   **`if table[parent] > table[minChild]`**
11.   Swap `table[parent]` and `table[minChild]`.
12.   Set `parent` to `minChild`.

        `else`

13.   Break out of loop.

Performance of the Heap

- `remove` traces a path from the root to a leaf
- `insert` traces a path from a leaf to the root
- This requires at most h steps where h is the height of the tree
- The largest *full* tree of height h has 2^h-1 nodes
- The smallest *complete* tree of height h has $2^{(h-1)}$ nodes
- Both `insert` and `remove` are $O(\log n)$

Priority Queues

- The heap is used to implement a special kind of queue called a priority queue
- The heap is not very useful as an ADT on its own
  - We will not create a `Heap` interface or code a class that implements it
  - Instead, we will incorporate its algorithms when we implement a priority queue class and heapsort
- Sometimes a FIFO queue may not be the best way to implement a waiting line
- A priority queue is a data structure in which only the highest-priority item is accessible

Priority Queues (cont.)

- In a print queue, sometimes it is more appropriate to print a short document that arrived after a very long document
- A *priority queue* is a data structure in which only the highest-priority item is accessible (as opposed to the first item entered)

Insertion into a Priority Queue

pages = 1 title = "web page 1"	pages = 4 title = "history paper"

After inserting document with 3 pages

pages = 1 title = "web page 1"	pages = 3 title = "Lab1"	pages = 4 title = "history paper"

After inserting document with 1 page

pages = 1 title = "web page 1"	pages = 1 title = "receipt"	pages = 3 title = "Lab1"	pages = 4 title = "history paper"

`PriorityQueue` Class

- Java provides a `PriorityQueue<E>` class that implements the `Queue<E>` interface given in Chapter 4.

Method	Behavior
`boolean offer(E item)`	Inserts an item into the queue. Returns **`true`** if successful; returns **`false`** if the item could not be inserted.
`E remove()`	Removes the smallest entry and returns it if the queue is not empty. If the queue is empty, throws a `NoSuchElementException`.
`E poll()`	Removes the smallest entry and returns it. If the queue is empty, returns **`null`**.
`E peek()`	Returns the smallest entry without removing it. If the queue is empty, returns **`null`**.
`E element()`	Returns the smallest entry without removing it. If the queue is empty, throws a `NoSuchElementException`.

Using a Heap as the Basis of a Priority Queue

- In a priority queue, just like a heap, the smallest item always is removed first
- Because heap insertion and removal is O(log *n*), a heap can be the basis of a very efficient implementation of a priority queue
- While the `java.util.PriorityQueue` uses an `Object[]` array, we will use an `ArrayList` for our custom priority queue, `KWPriorityQueue`

Design of a `KWPriorityQueue` Class

Data Field	Attribute
`ArrayList<E> theData`	An `ArrayList` to hold the data.
`Comparator<E> comparator`	An optional object that implements the `Comparator<E>` interface by providing a `compare` method.
Method	**Behavior**
`KWPriorityQueue()`	Constructs a heap-based priority queue that uses the elements' natural ordering.
`KWPriorityQueue (Comparator<E> comp)`	Constructs a heap-based priority queue that uses the `compare` method of `Comparator comp` to determine the ordering of the elements.
`private int compare(E left, E right)`	Compares two objects and returns a negative number if object `left` is less than object `right`, zero if they are equal, and a positive number if object `left` is greater than object `right`.
`private void swap(int i, int j)`	Exchanges the object references in `theData` at indexes `i` and `j`.

Design of a KWPriorityQueue Class (cont.)

```
import java.util.*;

/** The KWPriorityQueue implements the Queue interface
    by building a heap in an ArrayList. The heap is structured
    so that the "smallest" item is at the top.
*/
public class KWPriorityQueue<E> extends AbstractQueue<E>
                                implements Queue<E> {

// Data Fields
/** The ArrayList to hold the data. */
private ArrayList<E> theData;
/** An optional reference to a Comparator object. */
Comparator<E> comparator = null;

// Methods
// Constructor
public KWPriorityQueue() {
    theData = new ArrayList<E>();
}
. . .
```

offer Method

```
/** Insert an item into the priority queue.
    pre: The ArrayList theData is in heap order.
    post: The item is in the priority queue and
          theData is in heap order.
    @param item The item to be inserted
    @throws NullPointerException if the item to be inserted is null.
*/
@Override
public boolean offer(E item) {
    // Add the item to the heap.
    theData.add(item);
    // child is newly inserted item.
    int child = theData.size() - 1;
    int parent = (child - 1) / 2; // Find child's parent.
    // Reheap
    while (parent >= 0 && compare(theData.get(parent),
                               theData.get(child)) > 0) {
        swap(parent, child);
        child = parent;
        parent = (child - 1) / 2;
    }
    return true;
}
```

`poll` Method

```
/** Remove an item from the priority queue
    pre: The ArrayList theData is in heap order.
    post: Removed smallest item, theData is in heap order.
    @return The item with the smallest priority value or null if empty.
*/
@Override
public E poll() {
    if (isEmpty()) {
        return null;
    }
    // Save the top of the heap.
    E result = theData.get(0);
    // If only one item then remove it.
    if (theData.size() == 1) {
        theData.remove(0);
        return result;
}
```

`poll` Method (cont.)

```
    /* Remove the last item from the ArrayList and place it into
       the first position. */
    theData.set(0, theData.remove(theData.size() - 1));
    // The parent starts at the top.
    int parent = 0;
    while (true) {
        int leftChild = 2 * parent + 1;
        if (leftChild >= theData.size()) {
            break; // Out of heap.
        }
        int rightChild = leftChild + 1;
        int minChild = leftChild; // Assume leftChild is smaller.
        // See whether rightChild is smaller.
        if (rightChild < theData.size()
            && compare(theData.get(leftChild),
                       theData.get(rightChild)) > 0) {
            minChild = rightChild;
        }
        // assert: minChild is the index of the smaller child.
        // Move smaller child up heap if necessary.
        if (compare(theData.get(parent),
                    theData.get(minChild)) > 0) {
            swap(parent, minChild);
            parent = minChild;
        } else { // Heap property is restored.
            break;
        }
    }
    return result;
}
```

Other Methods

- The `iterator` and `size` methods are implemented via delegation to the corresponding `ArrayList` methods
- Method `isEmpty` tests whether the result of calling method `size` is `0` and is inherited from class `AbstractCollection`
- The implementations of methods `peek` and `remove` are left as exercises

Using a `Comparator`

- To use an ordering that is different from the natural ordering, provide a constructor that has a `Comparator<E>` parameter

```
/** Creates a heap-based priority queue with the specified initial
    capacity that orders its elements according to the specified
    comparator.
    @param cap The initial capacity for this priority queue
    @param comp The comparator used to order this priority queue
    @throws IllegalArgumentException if cap is less than 1
*/
public KWPriorityQueue(Comparator<E> comp) {
    if (cap < 1)
        throw new IllegalArgumentException();
    theData = new ArrayList<E>();
    comparator = comp;
}
```

`compare` Method

- If data field `comparator` references a `Comparator<E>` object, method `compare` delegates the task to the object's `compare` method
- If comparator is `null`, it will delegate to method `compareTo`

compare Method (cont.)

```
/** Compare two items using either a Comparator object's compare method
    or their natural ordering using method compareTo.
    pre: If comparator is null, left and right implement Comparable<E>.
    @param left One item
    @param right The other item
    @return Negative int if left less than right,
            0 if left equals right,
            positive int if left > right
    @throws ClassCastException if items are not Comparable
*/
private int compare(E left, E right) {
    if (comparator != null) { // A Comparator is defined.
        return comparator.compare(left, right);
    } else {                    // Use left's compareTo method.
        return ((Comparable<E>) left).compareTo(right);
    }
}
```

`PrintDocuments` Example

- The class `PrintDocument` is used to define documents to be printed on a printer
- We want to order documents by a value that is a function of both size and time submitted
- In the client program, use

```
Queue printQueue =
    new PriorityQueue(new ComparePrintDocuments());
```

PrintDocuments Example (cont.)

```
ComparePrintDocuments.java

import java.util.Comparator;

/** Class to compare PrintDocuments based on both
    their size and time stamp.
*/
public class ComparePrintDocuments implements Comparator<PrintDocument> {
    /** Weight factor for size. */
    private static final double P1 = 0.8;
    /** Weight factor for time. */
    private static final double P2 = 0.2;

    /** Compare two PrintDocuments.
        @param left The left-hand side of the comparison
        @param right The right-hand side of the comparison

        @return -1 if left < right; 0 if left == right;
                and +1 if left > right
     */
     public int compare(PrintDocument left, PrintDocument right) {
         return Double.compare(orderValue(left), orderValue(right));
     }

     /** Compute the order value for a print document.
         @param pd The PrintDocument
         @return The order value based on the size and time stamp
     */
     private double orderValue(PrintDocument pd) {
         return P1 * pd.getSize() + P2 * pd.getTimeStamp();
     }
 }
```

Huffman Trees

Section 6.6

Huffman Trees

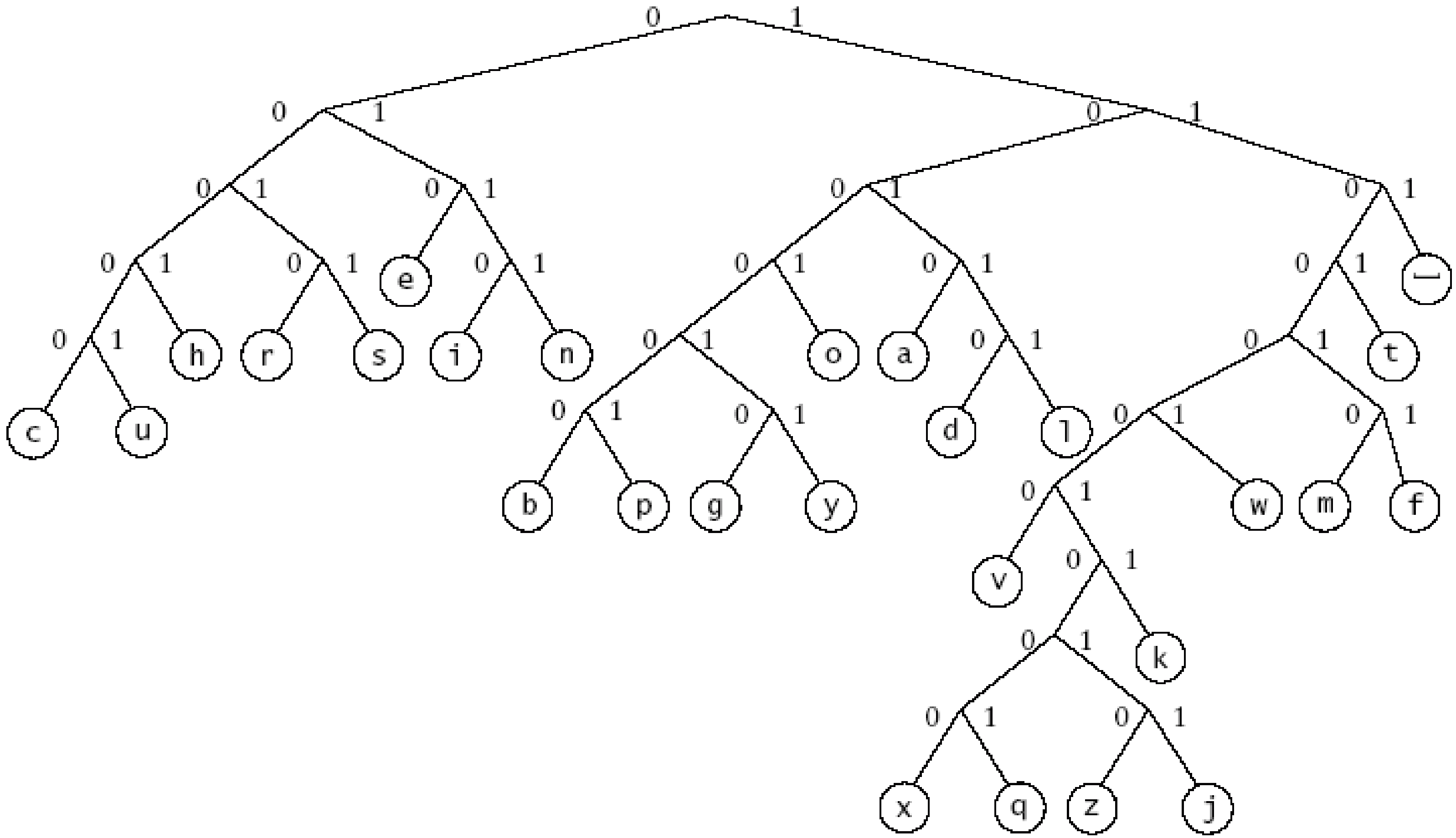
- A Huffman tree can be implemented using a binary tree and a PriorityQueue
- A straight binary encoding of an alphabet assigns a unique binary number to each symbol in the alphabet
  - Unicode is an example of such a coding
- The message “go eagles” requires 144 bits in Unicode but only 38 bits using Huffman coding

Huffman Trees (cont.)

Symbol	Frequency	Symbol	Frequency	Symbol	Frequency
␣	186	h	47	g	15
e	103	d	32	p	15
t	80	l	32	b	13
a	64	u	23	v	8
o	63	c	22	k	5
i	57	f	21	j	1
n	57	m	20	q	1
s	51	w	18	x	1
r	48	y	16	z	1

Huffman Trees (cont.)

Huffman Tree Based on Frequency of Letters in English Text



Building a Custom Huffman Tree

- Suppose we want to build a custom Huffman tree for a file
- **Input:** an array of objects such that each object contains a reference to a symbol occurring in that file and the frequency of occurrence (weight) for the symbol in that file

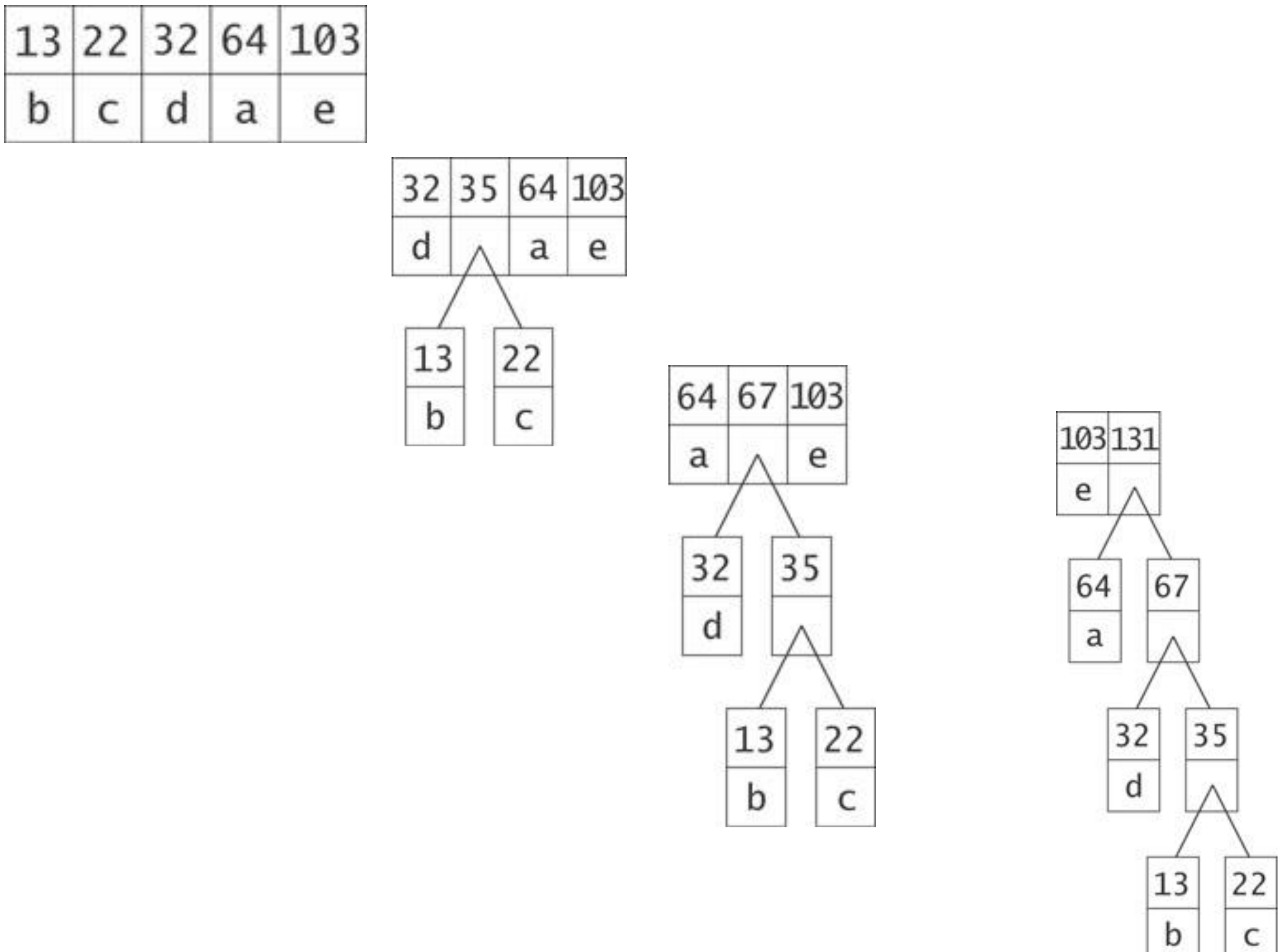
Building a Custom Huffman Tree (cont.)

- **Analysis:**
  - Each node will have storage for two data items:
    - the weight of the node and
    - the symbol associated with the node
  - All symbols will be stored in leaf nodes
  - For nodes that are not leaf nodes, the symbol part has no meaning
  - The weight of a leaf node will be the frequency of the symbol stored at that node
  - The weight of an interior node will be the sum of frequencies of all leaf nodes in the subtree rooted at the interior node

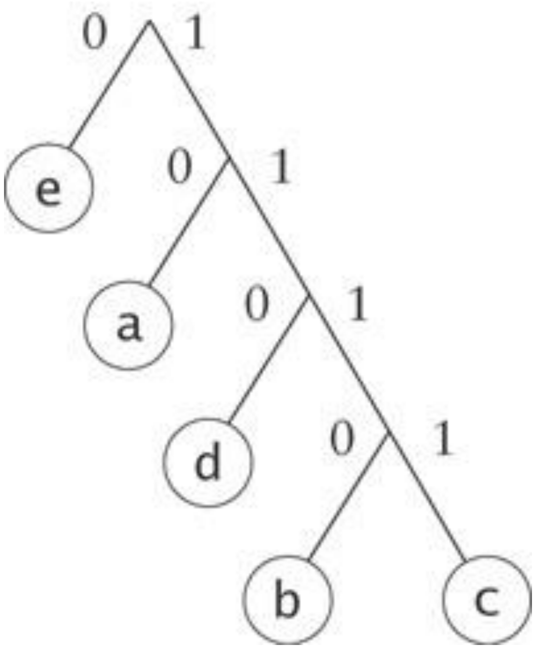
Building a Custom Huffman Tree (cont.)

- **Analysis:**
  - A priority queue will be the key data structure in our Huffman tree
  - We will store individual symbols and subtrees of multiple symbols in order by their priority (frequency of occurrence)

Building a Custom Huffman Tree (cont.)



Building a Custom Huffman Tree (cont.)



Symbol	Code
a	10
b	1110
c	1111
d	110
e	0

Design

Algorithm for Building a Huffman Tree

1. Construct a set of trees with root nodes that contain each of the individual symbols and their weights.
2. Place the set of trees into a priority queue.
3. **`while`** the priority queue has more than one item
4.   Remove the two trees with the smallest weights.
5.   Combine them into a new binary tree in which the weight of the tree root is the sum of the weights of its children.
6.   Insert the newly created tree back into the priority queue.

Design (cont.)

Data Field	Attribute
`BinaryTree<HuffData> huffTree`	A reference to the Huffman tree.
Method	**Behavior**
`buildTree(HuffData[] input)`	Builds the Huffman tree using the given alphabet and weights.
`String decode(String message)`	Decodes a message using the generated Huffman tree.
`printCode(PrintStream out)`	Outputs the resulting code.

Implementation

- Listing 6.9 (Class `HuffmanTree`; page 349)
- Listing 6.10 (The `buildTree` Method (`HuffmanTree.java`); pages 350-351)
- Listing 6.11 (The `decode` Method (`HuffmanTree.java`); page 352)