Probabilistic and Statistical Model Checking: Exam

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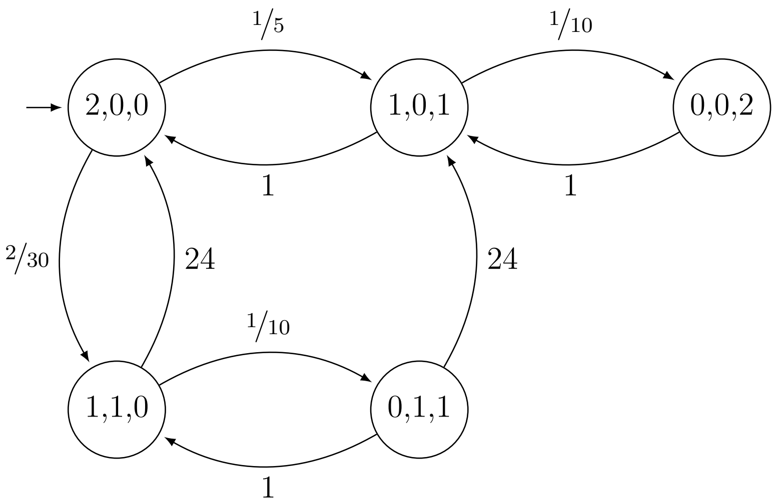
1.a

The states of the CTMC C are a triplet indicating the number of machines operational, the number of machines under maintenance and the number of machine that has failed. It has been assumed that a machine under maintenance can not fail. The CTMC model of the factory is the following:



we will refer to the states of the CTMC C as follows:

- $s_0 = (2, 0, 0)$
- $s_1 = (1, 0, 1)$
- $s_2 = (0, 0, 2)$

- $s_3 = (1, 1, 0)$
- $s_4 = (0, 1, 1)$

and the rate transition matrix of the CTMC C is:

$$\mathbf{R} = \begin{bmatrix} 0 & 1/5 & 0 & 1/30 & 0 \\ 1 & 0 & 1/10 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 24 & 0 & 0 & 0 & 1/10 \\ 0 & 24 & 0 & 1 & 0 \end{bmatrix}$$

1.b

Since all states of the CTMC C belong to a single BSCC, C is irreducible. In an irreducible CTMC the steady-state probabilities are independent of the initial state; thus there will be a unique solution for the steady-state probability vector, denoted by $\underline{\pi}$, which can be computed solving the following system of linear equations:

$$\begin{cases} \underline{\pi} \cdot \mathbf{Q} = \underline{0} \\ \sum_{s \in S} \pi_s = 1 \end{cases}$$

where $\mathbf{Q}$ is the infinitesimal generator matrix of the CTMC C which is:

$$\mathbf{Q} = \begin{bmatrix} -7/30 & 1/5 & 0 & 1/30 & 0 \\ 1 & -11/10 & 1/10 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 24 & 0 & 0 & -241/10 & 1/10 \\ 0 & 24 & 0 & 1 & -25 \end{bmatrix}$$

and

$$\underline{\pi} = \begin{bmatrix} \pi_{s_0} & \pi_{s_1} & \pi_{s_2} & \pi_{s_3} & \pi_{s_4} \end{bmatrix}$$

thus we need to solve:

$$\begin{array}{rcrcrcrcrcl}
-\frac{7}{30}\pi_{s_0} & + & \pi_{s_1} & & & + & 24\pi_{s_3} & & & = & 0 \\
\frac{1}{5}\pi_{s_0} & - & \frac{11}{10}\pi_{s_1} & + & \pi_{s_2} & & & + & 24\pi_{s_4} & = & 0 \\
& & \frac{1}{10}\pi_{s_1} & - & \pi_{s_2} & & & & & = & 0 \\
\frac{1}{30}\pi_{s_0} & & & & & - & \frac{241}{10}\pi_{s_3} & + & \pi_{s_4} & = & 0 \\
& & & & & & \frac{1}{10}\pi_{s_3} & - & 25\pi_{s_4} & = & 0 \\
\pi_{s_0} & + & \pi_{s_1} & + & \pi_{s_2} & + & \pi_{s_3} & + & \pi_{s_4} & = & 1
\end{array}$$

which gives the solution:

$$\underline{\pi} \simeq \begin{bmatrix} 0.818642 & 0.163837 & 0.016384 & 0.001132 & 0.000005 \end{bmatrix}$$

1.c

Let $L : S \to \mathscr{P}(AP)$ be the labeling function of the CTMC C, where:

$L(s_0) = L(s_1) = \emptyset$
$L(s_2) = \{both\_fail\}$ (will refer as f for short)
$L(s_3) = L(s_4) = \{maintenance\}$ (will refer as m for short)

then the property can be expressed as the CSL formula:

$$\varphi = \mathbf{P}_{<0.1}[\neg m \mathbf{U}^{[0,1]} f]$$

For model checking the property φ over the CTMC C we have to compute

$$Sat(\varphi) = \{s \in S \mid Prob(s, \neg m \mathbf{U}^{[0,1]} f) < 0.1\}$$

In order to do so the problem is reduced to transient analysis by making states in $Sat(f)$ and in $Sat(m \wedge \neg f)$ absorbing constructing the CTMC $C[f][m \wedge \neg f]$ and exploiting uniformisation to compute the vector of probabilities $\underline{Prob}(\neg m \mathbf{U}^{[0,1]} f)$.

Thus we need to compute

$$\underline{Prob}(\neg m \mathbf{U}^{[0,1]} f) = \sum_{i=0}^{\infty} \left(\gamma_{qt,i} \cdot (\mathbf{P}^{unif(C[f][m \wedge \neg f])})^i \cdot \underline{f}\right)$$

Note that the infinite summation can be truncated safely approximating the result and that $(\mathbf{P}^{unif(C[f][m \wedge \neg f])})^i \cdot \underline{f}$ can be computed iteratively avoiding matrix powers.

1.d

Probability that the first machine failure occurs on the second day

Let $L : S \to \mathscr{P}(AP)$ be:

- $L(s_0) = L(s_3) = \emptyset$
- $L(s_1) = L(s_2) = L(s_4) = \{fail\}$

then the property can be expressed as the CSL formula:

$$\mathbb{P}_{=?}[\neg fail \mathbf{U}^{[1,2]} fail]$$

The property can be evaluated by computing

$$Prob(s_0, \neg fail \mathbf{U}^{[1,2]} fail) = \sum_{i=0}^{\infty} \left(\gamma_{qt,i} \cdot \left(\mathbf{P}^{unif(C[fail])} \right)^i \cdot \underline{Prob}_{\neg fail}(\neg fail \mathbf{U}^{[0,t'-t]} fail) \right)$$

Where $\underline{Prob}_{\neg fail}(\neg fail \mathbf{U}^{[0,t'-t]} fail) = Prob(s, \neg fail \mathbf{U}^{[0,t'-t]} fail)$ if $s \in Sat(\neg fail)$ and 0 otherwise.

Long-run probability of at least one machine being operational

Let $L : S \to \mathscr{P}(AP)$ be:

- $L(s_0) = L(s_1) = L(s_3) = \{op\}$
- $L(s_2) = L(s_4) = \emptyset$

then the property can be expressed as the CSL formula:

$$S_{=?}[op]$$

To evaluate the property we need to compute the steady-state probabilities $\underline{\pi}^C$ (remind that C is irreducible) by solving a system of linear equations similarly to what have been done in 1.b. Then we can compute the value of the quantitative property as

$$\sum_{s \models op} \underline{\pi}^C(s)$$

Expected number of operational machines after exactly 30 days

Let $\iota : S \times S \to \mathbb{R}_{\geq 0} = \mathbf{0}$ (the 0 matrix) and $\underline{\rho} : S \to \mathbb{R}_{\geq 0} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ then the property can be expressed as the CSL extended with rewards formula:

$$R_{=?}[\![\mathbf{I}^{=30}]\!]$$

In order to evaluate it we need to compute

$$\sum_{s \in S} Exp(s, \mathbf{X}_{I=30})$$

Expected amount of time spent repairing machines over the first year

Let $\iota : S \times S \to \mathbb{R}_{\geq 0} = \mathbf{0}$ and $\underline{\rho} : S \to \mathbb{R}_{\geq 0}$ be:

$$\underline{\rho} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

then the property can be expressed as the CSL extended with rewards formula:

$$R_{=?}[\mathbf{C}^{\leq 365}]$$

and it can be evaluated by computing

$$\sum_{s \in S} Exp(s, \mathbf{X}_{C \leq 365})$$

Maximum expected time, computed from any state where at least one machine is not operational, before both machines become operational

Let $L : S \to \mathscr{P}(AP)$ be:

- $L(s_0) = \{bop\}$
- $L(s_1) = L(s_2) = L(s_3) = L(s_4) = \emptyset$

and let $\iota : S \times S \to \mathbb{R}_{\geq 0} = \mathbf{0}$, $\underline{\rho} : S \to \mathbb{R}_{\geq 0}$ be:

$$\underline{\rho} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

then the property can be expressed as the CSL extended with rewards formula:

$$R_{max=?}[\mathbf{F} bop]$$

and it can be evaluated by computing

$$\max\{Exp(s, \mathbf{X}_{Fbop}) \mid s \in S\}$$

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2.a

$$Steps = \left[\begin{array}{cccc} 0 & 0 & 4/5 & 1/5 \\ 0 & 1/2 & 1/2 & 0 \\ \hline 1/2 & 0 & 0 & 1/2 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

2.a.i

$\mathbf{P}_{>0.4}[\mathbf{X}(a \vee b)]$**:**

The formula involves a lower probability bound thus it has to be computed the minimum probability of the formula for each state, namely:

$$\begin{aligned} Sat(\mathbf{P}_{>0.4}[\mathbf{X}(a \vee b)]) &= \{s \in S \mid p_{\min}(s, \mathbf{X}(a \vee b)) > 0.4\} \\ &= \{s \in S \mid \min \left\{ \sum_{s' \in Sat(a \vee b)} \mu(s') \mid (a, \mu) \in Steps(s) \right\} > 0.4\} \end{aligned}$$

where

$$Sat(a \vee b) = Sat(a) \cup Sat(b) = \{s_2, s_3\}$$

let $\underline{a \vee b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ be the indicator function of $Sat(a \vee b)$, then the minimum probabilities can be computed at once by:

- calculating $Steps \cdot \underline{a \vee b}$
- extracting for each state the miminum probability

$$Steps \cdot \underline{a \vee b} = \left[\begin{array}{cccc} 0 & 0 & 4/5 & 1/5 \\ 0 & 1/2 & 1/2 & 0 \\ \hline 1/2 & 0 & 0 & 1/2 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \left[\begin{array}{c} 1 \\ 1/2 \\ \hline 1/2 \\ \hline 1 \\ \hline 1 \end{array}\right] \rightarrow \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \\ 1 \end{bmatrix}$$

Thus

$$\begin{aligned} Sat(\mathbf{P}_{>0.4}[\mathbf{X}(a \vee b)]) &= \{s \in S \mid p_{\min}(s, \mathbf{X}(a \vee b)) > 0.4\} \\ &= \{s_0, s_1, s_2, s_3, s_4\} = S \end{aligned}$$

hence the MDP satisfies the property.

$\mathbf{P}_{\leq 0.4}[\mathbf{G}(\neg b)] \wedge \mathbf{P}_{\leq 0.4}[\mathbf{F}a]$:

Notice that:

$$\begin{aligned} \mathbf{P}_{\leq 0.4}[\mathbf{G}\neg b] \wedge \mathbf{P}_{\leq 0.4}[\mathbf{F}a] &\equiv \mathbf{P}_{\leq 0.4}[\neg \mathbf{F}b] \wedge \mathbf{P}_{\leq 0.4}[\mathbf{F}a] \\ &\equiv \mathbf{P}_{\geq 0.6}[\mathbf{F}b] \wedge \mathbf{P}_{\leq 0.4}[\mathbf{F}a] \end{aligned}$$

$$Sat(\mathbf{P}_{\geq 0.6}[\mathbf{F}b] \wedge \mathbf{P}_{\leq 0.4}[\mathbf{F}a]) = Sat(\mathbf{P}_{\geq 0.6}[\mathbf{F}b]) \cap Sat(\mathbf{P}_{\leq 0.4}[\mathbf{F}a])$$

$Sat(\mathbf{P}_{\geq 0.6}[\mathbf{F}b])$:

$$Sat(\mathbf{P}_{\geq 0.6}[\mathbf{F}b]) = \{s \in S \mid p_{\min}(s, \mathbf{F}b) \geq 0.6\}$$

Thus it has to be applied the algorithm for computing reachability probabilities.

First compute the sets $S^{\min=0} = \{s_3\}$, $Sat(b) = \{s_2\}$ and $S^? = \{s_0, s_1\}$. Then we can compute the reachability probabilities via a value iteration approach as follows:

$$x_s^{(n)} = \begin{cases} 1 & \text{if } s \in Sat(b) \\ 0 & \text{if } s \in S^{\min=0} \\ 0 & \text{if } s \in S^?, n = 0 \\ \min\left\{ \sum_{s' \in S} \mu(s') \cdot x_{s'}^{(n-1)} \mid (a, \mu) \in Steps(s) \right\} & \text{if } s \in S^?, n > 0 \end{cases}$$

Obtaining:

$$\underline{x}^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned}
\underline{x}^k &= Steps \cdot (\underline{x}^{k-1})^T \\
&= \begin{bmatrix} 0 & 0 & 4/5 & 1/5 \\ 0 & 1/2 & 1/2 & 0 \\ \hline 1/2 & 0 & 0 & 1/2 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_0^{(k-1)} \\ x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} \\
&= \begin{bmatrix} \min\{\frac{4}{5}x_2^{k-1} + \frac{1}{5}x_3^{k-1}, \frac{1}{2}x_1^{(k-1)} + \frac{1}{2}x_2^{(k-1)}\} & \frac{1}{2}x_0^{(k-1)} + \frac{1}{2}x_3^{(k-1)} & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \min\{\frac{4}{5}, \frac{1}{2}x_1^{(k-1)} + \frac{1}{2}\} & \frac{1}{2}x_0^{(k-1)} & 1 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\underline{x}^1 &= Steps \cdot (\underline{x}^0)^T \\
&= \begin{bmatrix} \min\{\frac{4}{5}, \frac{1}{2}x_1^{(k-1)} + \frac{1}{2}\} & \frac{1}{2}x_0^{(k-1)} & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 & 1 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\underline{x}^2 &= Steps \cdot (\underline{x}^1)^T \\
&= \begin{bmatrix} \min\{\frac{4}{5}, \frac{1}{2}x_1^{(k-1)} + \frac{1}{2}\} & \frac{1}{2}x_0^{(k-1)} & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 1 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\underline{x}^3 &= Steps \cdot (\underline{x}^2)^T \\
&= \begin{bmatrix} \min\{\frac{4}{5}, \frac{1}{2}x_1^{(k-1)} + \frac{1}{2}\} & \frac{1}{2}x_0^{(k-1)} & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{5}{8} & \frac{1}{4} & 1 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\underline{x}^4 &= Steps \cdot (\underline{x}^3)^T \\
&= \begin{bmatrix} \min\{\frac{4}{5}, \frac{1}{2}x_1^{(k-1)} + \frac{1}{2}\} & \frac{1}{2}x_0^{(k-1)} & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{5}{8} & \frac{5}{16} & 1 & 0 \end{bmatrix}
\end{aligned}$$

$$\cdots$$

$$\underline{x}^{17} \simeq \begin{bmatrix} 0.666664 & 0.333328 & 1 & 0 \end{bmatrix}$$

Hence $Sat(\mathbf{P}_{\geq 0.6}[\mathbf{F}b]) = \{s_0, s_2\}$

$Sat(\mathbf{P}_{\leq 0.4}[\mathbf{F}a])$**:**

In this case we have to compute

$$Sat(\mathbf{P}_{\leq 0.4}[\mathbf{F}a]) = \{s \in S \mid p_{\max}(s, \mathbf{F}a) \leq 0.4\}$$

First of all precompute $S^{\max=0} = \{s_2\}$, $Sat(a) = \{s_3\}$ and $S^? = \{s_0, s_1\}$. Then compute the reachability probabilities via value iteration as follows:

$$x_s^{(n)} = \begin{cases} 1 & \text{if } s \in Sat(a) \\ 0 & \text{if } s \in S^{\max=0} \\ 0 & \text{if } s \in S^?, n = 0 \\ \max\left\{\sum_{s' \in S} \mu(s') \cdot x_{s'}^{(n-1)} \mid (a, \mu) \in Steps(s)\right\} & \text{if } s \in S^?, n > 0 \end{cases}$$

Obtaining:

$$\underline{x}^0 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
\underline{x}^k &= Steps \cdot (\underline{x}^{k-1})^T \\
&= \left[\begin{array}{cccc} 0 & 0 & 4/5 & 1/5 \\ 0 & 1/2 & 1/2 & 0 \\ \hline 1/2 & 0 & 0 & 1/2 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \begin{bmatrix} x_0^{(k-1)} \\ x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} \\
&= \begin{bmatrix} \max\{\frac{4}{5}x_2^{k-1} + \frac{1}{5}x_3^{k-1}, \frac{1}{2}x_1^{(k-1)} + \frac{1}{2}x_2^{(k-1)}\} & \frac{1}{2}x_0^{(k-1)} + \frac{1}{2}x_3^{(k-1)} & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \max\{\frac{1}{5}, \frac{1}{2}x_1^{(k-1)}\} & \frac{1}{2}x_0^{(k-1)} + \frac{1}{2} & 0 & 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
x^{(1)} &= \begin{bmatrix} \frac{1}{5} & \frac{1}{2} & 0 & 1 \end{bmatrix} \\
x^{(2)} &= \begin{bmatrix} \frac{1}{4} & \frac{3}{5} & 0 & 1 \end{bmatrix} \\
x^{(3)} &= \begin{bmatrix} \frac{3}{10} & \frac{5}{8} & 0 & 1 \end{bmatrix} \\
&\dots \\
x^{(12)} &\simeq \begin{bmatrix} 0.33327 & 0.666602 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Hence $Sat(\mathbf{P}_{\leq 0.4}[\mathbf{F}a]) = \{s_0, s_2\}$, thus

$$\begin{aligned}
Sat(\mathbf{P}_{\geq 0.6}[\mathbf{F}b] \wedge \mathbf{P}_{\leq 0.4}[\mathbf{F}a]) &= Sat(\mathbf{P}_{\geq 0.6}[\mathbf{F}b]) \cap Sat(\mathbf{P}_{\leq 0.4}[\mathbf{F}a]) \\
&= \{s_0, s_2\} \cap \{s_0, s_2\} \\
&= \{s_0, s_2\}
\end{aligned}$$

2.a.ii

For computing $p_{\max}(s_0, \mathbf{F}^{\leq 3}a)$ we can simply use the value iteration at step $k = 3$ previously calculated in the case of $Sat(\mathbf{P}_{\leq 0.4}[\mathbf{F}a])$, obtaining:

$$x^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x^{(1)} = \begin{bmatrix} \frac{1}{5} & \frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} \frac{1}{4} & \frac{3}{5} & 0 & 1 \end{bmatrix}$$

$$x^{(3)} = \begin{bmatrix} \frac{3}{10} & \frac{5}{8} & 0 & 1 \end{bmatrix}$$

Hence $p_{\max}(s_0, \mathbf{F}^{\leq 3}a) = \frac{3}{10}$.

In order to provide the corresponding optimal policy we can use the following equation:

$$\sigma^k_{\max}(s) = \arg\max \left\{ \sum_{s' \in S} \mu(s') \cdot p_{\max}(s', \mathbf{F}^{\leq k-1} \mid (\alpha, \mu) \in Steps(s) \right\}$$

Thus obtaining $\sigma^3_{\max}(s_0) = \sigma_{\max}(s_0) = \beta$ and $\sigma^1_{\max}(s_0) = \sigma_{\max}(s_0 s_1 s_0)\alpha$. The scheduler is't sationary though since there's a step-bounded reachability property involved and some memory is needed to keep track of the history of the execution.

2.a.iii

Notice that

$$\mathbf{P}_{>0}[\mathbf{G}(\neg a \land \neg b)] \equiv \mathbf{P}_{<1}[\mathbf{F}(a \lor b)]$$

Applying the precomputations of DTMC model checking we obtain $S^{no} = \emptyset$, $S^{yes} = S$ hence

$$\forall s. Prob(s, \mathbf{F}(a \lor b) = 1$$

thus

$$Sat(\mathbf{P}_{<1}[\mathbf{F}(a \lor b)]) = \emptyset$$

so the property isn't satisfied by the resulting Markov chain.

Moreover the PCTL formula can't be expressed as an equivalent CTL formula, since the existence of a path satisfying a formula doesn't imply it's probability mass being greater than zero. For example consider the path $(s_0 s_1)^\omega$, it would satisfy the CTL formula $\exists \mathbf{G}(\neg a \land \neg b)$ but it doesn't satisfy the PCTL formula $\mathbf{P}_{>0}[\mathbf{G}(\neg a \land \neg b)]$ since that path has a zero probability mass associated with it.

2.b

An IMDP satisfies a PCTL property if for every induced MDP, the property holds. The induced MDP is obtained by fixing the probability distributions for each action. Thus the induced MDP is a regular MDP, and the PCTL property can be checked using the model checking algorithm for MDPs.

We will define $p_{\min}$ and $p_{\max}$ for IMDPs as follows:

- $p^I_{\min}(s, \psi) = \inf_{\sigma \in Adv, Steps^\star \in Steps} Prob^\sigma_{Steps*}(s, \psi)$
- $p^I_{\max}(s, \psi) = \sup_{\sigma \in Adv, Steps^\star \in Steps} Prob^\sigma_{Steps*}(s, \psi)$

where $Steps^\star$ is a valuation of the transition probability functions of the IMDP and $Prob^\sigma_{Steps*}$ is the probability of satisfying the property ψ with the transition probability functions $Steps^\star$.

Moreover we will define the satisfaction relation for PCTL properties for IMDPs:

$$s \vDash_I \mathbf{P}_{\sim p}[\phi] \Leftrightarrow p^I_{\min}(s, \phi) \sim p \text{ if } \sim \in \{>, \geq\}$$

$$s \vDash_I \mathbf{P}_{\sim p}[\phi] \Leftrightarrow p^I_{\max}(s, \phi) \sim p \text{ if } \sim \in \{<, \leq\}$$

2.b.i

$$Sat(\mathbf{P}_{>0.4}[\mathbf{X}(a \vee b)]) = \{s \in S \mid p^I_{\min}(s, \mathbf{X}(a \vee b)) > 0.4\}$$

and

$$p^I_{\min}(s, \mathbf{X}(a \vee b)) = \min \left\{ \sum_{s' \in Sat(a \vee b)} \mu(s') \mid (\alpha, \mathcal{M}) \in Steps(s), \mu \in \mathcal{M} \right\}$$

$$Sat(a \vee b) = \{s_2, s_3\} \Rightarrow \underline{a \vee b} = \begin{bmatrix} 0; & 0; & 1; & 1 \end{bmatrix}$$

Since the probability distributions intervals involve two transition per action we will introduce a variable for each action denoting the probability of performing one transition and with the complement the probability of performing the other. Namely:

$$Steps^\star = \left[\begin{array}{cccc} 0 & 0 & \mu^\alpha_0 & 1-\mu^\alpha_0 \\ 0 & \mu^\beta_0 & 1-\mu^\beta_0 & 0 \\ \hline \mu^\alpha_1 & 0 & 0 & 1-\mu^\alpha_1 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

with $\mu_0^\alpha \in [0.7; 0.9]$, $\mu_0^\beta \in [0.3; 0.6]$ and $\mu_1^\alpha \in [0.3; 0.7]$. hence we can compute the minimum probability of satisfying the property:

$$
\begin{aligned}
\underline{p^I_{\min}}(\mathbf{X}(a \vee b)) &= \mathit{Steps}^\star \cdot \underline{a \vee b} \\
&= \left[\begin{array}{cccc}
0 & 0 & \mu_0^\alpha & 1-\mu_0^\alpha \\
0 & \mu_0^\beta & 1-\mu_0^\beta & 0 \\
\hline
\mu_1^\alpha & 0 & 0 & 1-\mu_1^\alpha \\
\hline
0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 1
\end{array}\right] \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \left[\begin{array}{c}
1 \\
1-\mu_0^\beta \\
\hline
1-\mu_1^\alpha \\
\hline
1 \\
\hline
1
\end{array}\right]
\end{aligned}
$$

Extracting the minimum probability we get:

$$
\underline{p^I_{\min}}(\mathbf{X}(a \vee b)) = \begin{bmatrix} 1-\mu_0^\beta \\ 1-\mu_1^\alpha \\ 1 \\ 1 \end{bmatrix}
$$

and yielding the minimum of the probabilities induced by the variables we get

$$
\underline{p^I_{\min}}(\mathbf{X}(a \vee b)) = \begin{bmatrix} [0.4; 0.7] \\ [0.3; 0.7] \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.3 \\ 1 \\ 1 \end{bmatrix}
$$

thus $\mathit{Sat}(\mathbf{P}_{>0.4}[\mathbf{X}(a \vee b)]) = \{s \in S \mid \underline{p^I_{\min}}(s, \mathbf{X}(a \vee b)) > 0.4\} = \{s_2, s_3\}$

2.b.ii

In order to check whether the IMDP satisfies the property $\mathbf{P}_{\leq 0.5}[\mathbf{F}^{\leq 3} a]$ we will compute

$$
\mathit{Sat}(\mathbf{P}_{\leq 0.5}[\mathbf{F}^{\leq 3} a]) = \{s \in S \mid p^I_{\max}(s, \mathbf{F}^{\leq 3} a) \leq 0.5\}
$$

hence we will compute $p^I_{\max}(s, \mathbf{F}^{\leq 3} a)$ for each state s by setting up a recursive equation, similarly to the MDP case. The intervals will be handled as in the previous question but this time the maximum probabilities will be extracted only at $k = 3$ if not trivial to do so at every step as for MDPs.

First let's compute $S^{yes} = \mathit{Sat}(a) = \{s_3\}$, $S^{no} = \emptyset$ and $S^? = \{s_0, s_1, s_2\}$ and then setup the recursive equation

$$p^I_{\max}(s, \mathbf{F}^{\leq k} a) =$$

$$= \begin{cases} 1 & \text{if } s \in S^{yes} \\ 0 & \text{if } s \in S^{no} \\ 0 & \text{if } s \in S^{?}, k = 0 \\ \max\left\{ \sum\limits_{s' \in S} \mu(s') \cdot p^I_{\max}(s', \mathbf{F}^{\leq k-1} a) \mid (\alpha, \mathcal{M}) \in Steps(s), \mu \in \mathcal{M} \right\} & \text{otherwise} \end{cases}$$

Hence getting:

$$\underline{p^I_{\max}}(\mathbf{F}^{\leq 0} a) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
\underline{p^I_{\max}}(\mathbf{F}^{\leq 1} a) &= Steps^{\star} \cdot \underline{p^I_{\max}}(\mathbf{F}^{\leq 0} a) \\
&= \left[\begin{array}{cccc} 0 & 0 & \mu^\alpha_0 & 1-\mu^\alpha_0 \\ 0 & \mu^\beta_0 & 1-\mu^\beta_0 & 0 \\ \hline \mu^\alpha_1 & 0 & 0 & 1-\mu^\alpha_1 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \left[\begin{array}{c} 1-\mu^\alpha_0 \\ 0 \\ \hline 1-\mu^\alpha_1 \\ \hline 0 \\ \hline 1 \end{array}\right] \\
&= \begin{bmatrix} 1-\mu^\alpha_0 \\ 1-\mu^\alpha_1 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\underline{p^I_{\max}}(\mathbf{F}^{\leq 2} a) &= Steps^{\star} \cdot \underline{p^I_{\max}}(\mathbf{F}^{\leq 1} a) \\
&= \left[\begin{array}{cccc} 0 & 0 & \mu^\alpha_0 & 1-\mu^\alpha_0 \\ 0 & \mu^\beta_0 & 1-\mu^\beta_0 & 0 \\ \hline \mu^\alpha_1 & 0 & 0 & 1-\mu^\alpha_1 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \begin{bmatrix} 1-\mu^\alpha_0 \\ 1-\mu^\alpha_1 \\ 0 \\ 1 \end{bmatrix} \\
&= \left[\begin{array}{c} 1-\mu^\alpha_0 \\ \mu^\beta_0(1-\mu^\alpha_1) \\ \hline \mu^\alpha_1(1-\mu^\alpha_0) + 1 - \mu^\alpha_1 \\ \hline 0 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} 1-\mu^\alpha_0 \\ \mu^\beta_0(1-\mu^\alpha_1) \\ \hline 1-\mu^\alpha_0\mu^\alpha_1 \\ \hline 0 \\ \hline 1 \end{array}\right]
\end{aligned}$$

$$
\begin{aligned}
\underline{p^I_{\max}}(\mathbf{F}^{\leq 3}a) &= Steps^{\star} \cdot \underline{p^I_{\max}}(\mathbf{F}^{\leq 2}a) \\
&= \left[\begin{array}{cccc} 0 & 0 & \mu_0^\alpha & 1-\mu_0^\alpha \\ 0 & \mu_0^\beta & 1-\mu_0^\beta & 0 \\ \hline \mu_1^\alpha & 0 & 0 & 1-\mu_1^\alpha \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{c} 1-\mu_0^\alpha \\ \mu_0^\beta(1-\mu_1^\alpha) \\ \hline 1-\mu_0^\alpha\mu_1^\alpha \\ \hline 0 \\ \hline 1 \end{array}\right] \\
&= \left[\begin{array}{c} 1-\mu_0^\alpha \\ 1-\mu_0^\alpha \\ \mu_0^\beta(1-\mu_0^\alpha\mu_1^\alpha) \\ \mu_0^\beta(1-\mu_0^\alpha\mu_1^\alpha) \\ \hline \mu_1^\alpha(1-\mu_0^\alpha)+1-\mu_1^\alpha \\ \mu_1^\alpha\mu_0^\beta(1-\mu_1^\alpha)+1-\mu_1^\alpha \\ \hline 0 \\ \hline 1 \end{array}\right] \\
&= \left[\begin{array}{c} 1-\mu_0^\alpha \\ \mu_0^\beta(1-\mu_0^\alpha\mu_1^\alpha) \\ \hline 1-\mu_0^\alpha\mu_1^\alpha \\ \mu_1^\alpha\mu_0^\beta(1-\mu_1^\alpha)+1-\mu_1^\alpha \\ \hline 0 \\ \hline 1 \end{array}\right]
\end{aligned}
$$

Considering the maximum probabilities achievable we get:

$$
\underline{p^I_{\max}}(\mathbf{F}^{\leq 3}a) = \left[\begin{array}{c} 1-\mu_0^\alpha \\ \mu_0^\beta(1-\mu_0^\alpha\mu_1^\alpha) \\ \hline 1-\mu_0^\alpha\mu_1^\alpha \\ \mu_1^\alpha\mu_0^\beta(1-\mu_1^\alpha)+1-\mu_1^\alpha \\ \hline 0 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} 0.3 \\ 0.474 \\ \hline 0.79 \\ 0.79 \\ \hline 0 \\ \hline 1 \end{array}\right] = \begin{bmatrix} 0.474 \\ 0.79 \\ 0 \\ 1 \end{bmatrix}
$$

Thus $Sat(\mathbf{P}_{\leq 0.5}[\mathbf{F}^{\leq 3}a]) = \{s \in S \mid p^I_{\max}(s, \mathbf{F}^{\leq 3}a) \leq 0.5\} = \{s_0, s_2\}$.

2.b.iii

For computing $p^I_{\max}(s, \phi_1 \mathbf{U}^{\leq k} \phi_2])$ we would need to setup a recursive equation as follows:

$$p^I_{\max}(s, \phi_1 \mathbf{U}^{\leq k} \phi_2]) =$$

$$= \begin{cases} 1 & \text{if } s \in S^{yes} \\ 0 & \text{if } s \in S^{no} \\ 0 & \text{if } s \in S^{?}, k = 0 \\ \max\left\{ \sum_{s' \in S} \mu(s') \cdot p^I_{\max}(s', \phi_1 \mathbf{U}^{\leq k-1} \phi_2) \mid (\alpha, \mathcal{M}) \in \mathit{Steps}(s), \mu \in \mathcal{M} \right\} & \text{otherwise} \end{cases}$$

where $S^{yes} = \mathit{Sat}(\phi_2)$, $S^{no} = S \setminus (\mathit{Sat}(\phi_1) \cup \mathit{Sat}(\phi_2))$ and $S^{?} = S \setminus (\mathit{Sat}(\phi_1) \cup \mathit{Sat}(\phi_2))$. It can be noted the similarity to the recursive equation for the MDP case which is the following

$$p_{\max}(s, \phi_1 \mathbf{U}^{\leq k} \phi_2]) =$$

$$= \begin{cases} 1 & \text{if } s \in S^{yes} \\ 0 & \text{if } s \in S^{no} \\ 0 & \text{if } s \in S^{?}, k = 0 \\ \max\left\{ \sum_{s' \in S} \mu(s') \cdot p_{\max}(s', \phi_1 \mathbf{U}^{\leq k-1} \phi_2) \mid (\alpha, \mu) \in \mathit{Steps}(s) \right\} & \text{otherwise} \end{cases}$$

however in the IMDP case we need to consider all the possible intervals for the transition probability function hence they arise in the recursive case.

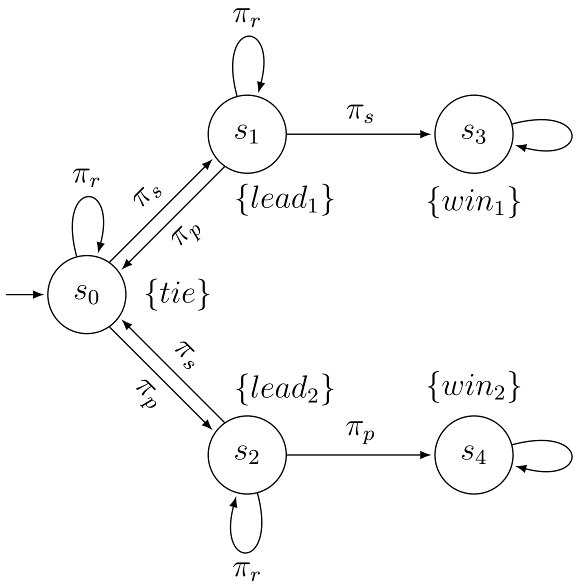
In order to incorporate the intervals for the computations we would need to define for eact action $n - 1$ variables representing the actual transition probability distribution of the action, where n is the number of transitions with an associated interval of probability distribution for the action. The computations should be performed parametrically and then evaluated at the last step in order to not prune out possible solutions.

3

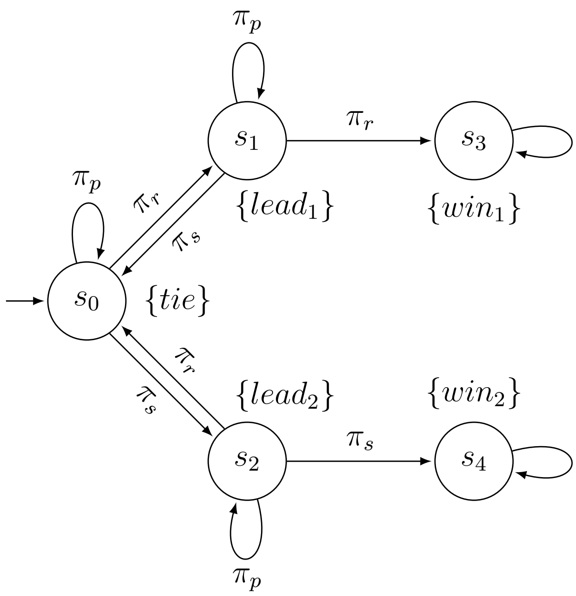
3.a

As suggested, for the sake of clarity, it will be presented a Markov chain for each of the possible actions of player 1, namely *rock*, *paper* and *scissor*.

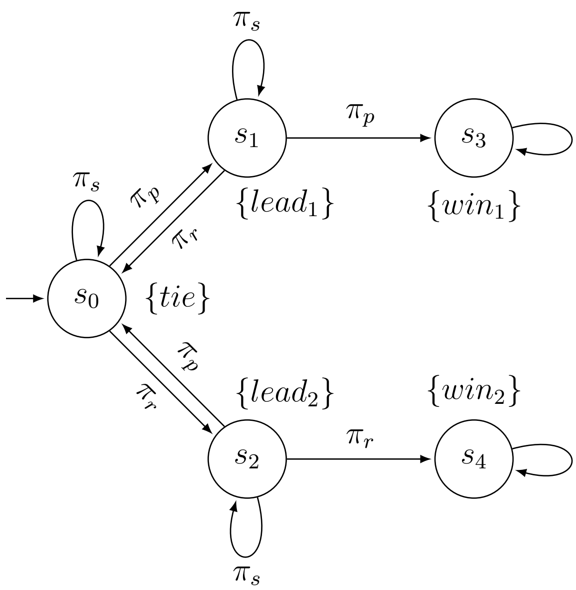
Rock:



Paper:



Scissor:



thus we obtain the following transition probability function (with actions in the order $rock, paper, scissor$)

$$Steps = \left[\begin{array}{ccccc}
\pi_r & \pi_s & \pi_p & 0 & 0 \\
\pi_p & \pi_r & \pi_s & 0 & 0 \\
\pi_s & \pi_p & \pi_r & 0 & 0 \\
\hline
\pi_p & \pi_r & 0 & \pi_s & 0 \\
\pi_s & \pi_p & 0 & \pi_r & 0 \\
\pi_r & \pi_s & 0 & \pi_p & 0 \\
\hline
\pi_s & 0 & \pi_r & 0 & \pi_p \\
\pi_r & 0 & \pi_p & 0 & \pi_s \\
\pi_p & 0 & \pi_s & 0 & \pi_r \\
\hline
0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 1
\end{array}\right]$$

3.b

3.b.i

The statement:

"What is the maximum probability of player 1 winning within 3 rounds?"

can be expressed with the following PCTL formula:

$$\mathbf{P}_{\max=?}[\mathbf{F}^{\leq 3} win_1]$$

3.b.ii

Evaluating the formula $\mathbf{P}_{\max=?}[\mathbf{F}^{\leq 3} win_1]$ boils down to compute the quantity $p_{\max}(s_0, \mathbf{F}^{\leq 3} win_1)$ defined recursively as follows:

$$p_{\max}(s, \mathbf{F}^{\leq k}\varphi) =$$
$$= \begin{cases} 1 & \text{if } s \in S^{yes} \\ 0 & \text{if } s \in S^?, k = 0 \\ \max\left\{ \sum_{s' \in S} \mu(s') \cdot p_{\max}(s', \mathbf{F}^{\leq k-1}\varphi) \mid (\alpha, \mu) \in Steps(s) \right\} & \text{if } s \in S^?, k > 0 \end{cases}$$

where $S^{yes} = Sat(\varphi) = Sat(win_1) = \{s_3\}$ and $S^? = S \setminus S^{yes} = \{s_0, s_1, s_2, s_4\}$. In particular we will compute the vector $\underline{p_{\max}}(\mathbf{F}^{\leq 3} win_1)$ where

$$\underline{p_{\max}}(\mathbf{F}^{\leq 0} win_1) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\underline{p_{\max}}(\mathbf{F}^{\leq k} win_1) = Steps \cdot \underline{p_{\max}}(\mathbf{F}^{\leq k-1} win_1)^T$$
$$= \left[\begin{array}{ccccc} \pi_r & \pi_s & \pi_p & 0 & 0 \\ \pi_p & \pi_r & \pi_s & 0 & 0 \\ \pi_s & \pi_p & \pi_r & 0 & 0 \\ \hline \pi_p & \pi_r & 0 & \pi_s & 0 \\ \pi_s & \pi_p & 0 & \pi_r & 0 \\ \pi_r & \pi_s & 0 & \pi_p & 0 \\ \hline \pi_s & 0 & \pi_r & 0 & \pi_p \\ \pi_r & 0 & \pi_p & 0 & \pi_s \\ \pi_p & 0 & \pi_s & 0 & \pi_r \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \cdot \begin{bmatrix} p_{\max}(s_0, \mathbf{F}^{\leq k-1} win_1) \\ p_{\max}(s_1, \mathbf{F}^{\leq k-1} win_1) \\ p_{\max}(s_2, \mathbf{F}^{\leq k-1} win_1) \\ p_{\max}(s_3, \mathbf{F}^{\leq k-1} win_1) \\ p_{\max}(s_4, \mathbf{F}^{\leq k-1} win_1) \end{bmatrix}$$

extracting at each iteration step the maximum value yielded by each state. Hence we have:

$$\begin{aligned}\underline{p_{\max}}(\mathbf{F}^{\leq 1} win_1) &= Steps \cdot \underline{p_{\max}}(\mathbf{F}^{\leq 0} win_1)^T \\ &= \begin{bmatrix}\max\{0,0,0\} & \max\{\pi_s, \pi_r, \pi_p\} & \max\{0,0,0\} & 1 & 0\end{bmatrix} \\ &= \begin{bmatrix}0 & \pi_r & 0 & 1 & 0\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\underline{p_{\max}}(\mathbf{F}^{\leq 2} win_1) &= Steps \cdot \underline{p_{\max}}(\mathbf{F}^{\leq 1} win_1)^T \\ &= \begin{bmatrix}\max\{\pi_s\pi_r, \pi_r^2, \pi_p\pi_r\} & \max\{\pi_r^2 + \pi_s, \pi_p\pi_r + \pi_r, \pi_s\pi_r + \pi_p\} & 0 & 1 & 0\end{bmatrix} \\ &= \begin{bmatrix}\pi_r^2 & \pi_p\pi_r + \pi_r & 0 & 1 & 0\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\underline{p_{\max}}(\mathbf{F}^{\leq 3} win_1) &= Steps \cdot \underline{p_{\max}}(\mathbf{F}^{\leq 2} win_1)^T \\ &= \begin{bmatrix}\pi_r^2(2\pi_p + 1) & \pi_s\pi_r^2 + \pi_r(\pi_p^2 + \pi_p + 1) & \pi_r^3 & 1 & 0\end{bmatrix} \\ &= \begin{bmatrix}0.6 & 0.890\overline{6} & 0.216 & 1 & 0\end{bmatrix}\end{aligned}$$

Thus $\mathbf{P}_{\max=?}[\mathbf{F}^{\leq 3} win_1] = 0.6$.

Since the strategy requested is relative to a step-bounded reachability property in general it will need to be a finite-memory one. In order to compute it we will need to perform backward computations and selecting the argmax of $\underline{p_{\max}}$ (action that yields the maximum probability) at each step starting from the goal state, namely:

- $k = 1$:

$$\begin{aligned}\sigma^1_{\max}(s_0) &= \_ \\ \sigma^1_{\max}(s_1) &= paper \\ \sigma^1_{\max}(s_2) &= \_\end{aligned}$$

- $k = 2$:

$$\begin{aligned}\sigma^2_{\max}(s_0) &= paper \\ \sigma^2_{\max}(s_1) &= paper \\ \sigma^2_{\max}(s_2) &= \_\end{aligned}$$

- $k = 3$:

$$\begin{aligned}\sigma^3_{\max}(s_0) &= paper \\ \sigma^3_{\max}(s_1) &= paper \\ \sigma^3_{\max}(s_2) &= paper\end{aligned}$$

Thus obtaining $\sigma_{\max}(s) = paper$ for the property analyzed.

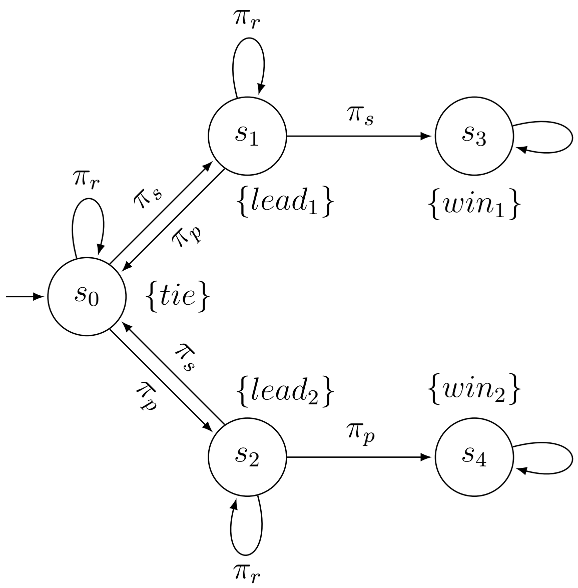
3.b.iii

In a setting where player 1 know player 2's strategy and plays optimally against it, in order to minimise the maximum probability of player 1 winning player 2 will need to minimise the maximum of mass of probability of player 1 winning that leaks out at every step. Hence player 2 will need to adopt a strategy where every action is be equiprobable, thus the optimal strategy for player 2 will be $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

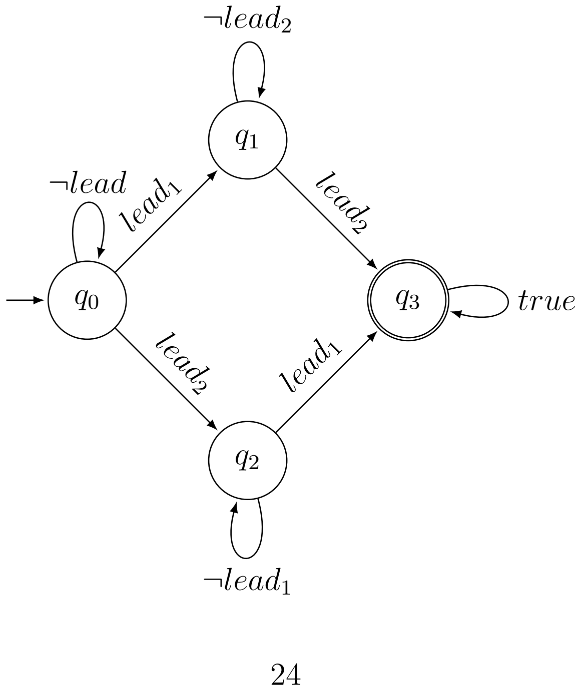
3.c

3.c.i

The DTMC D induced by the strategy of player 1 of always playing *rock* is the following:



The (total) DFA A representing the property *"Both players lead at least once during the game"* is instead the following:

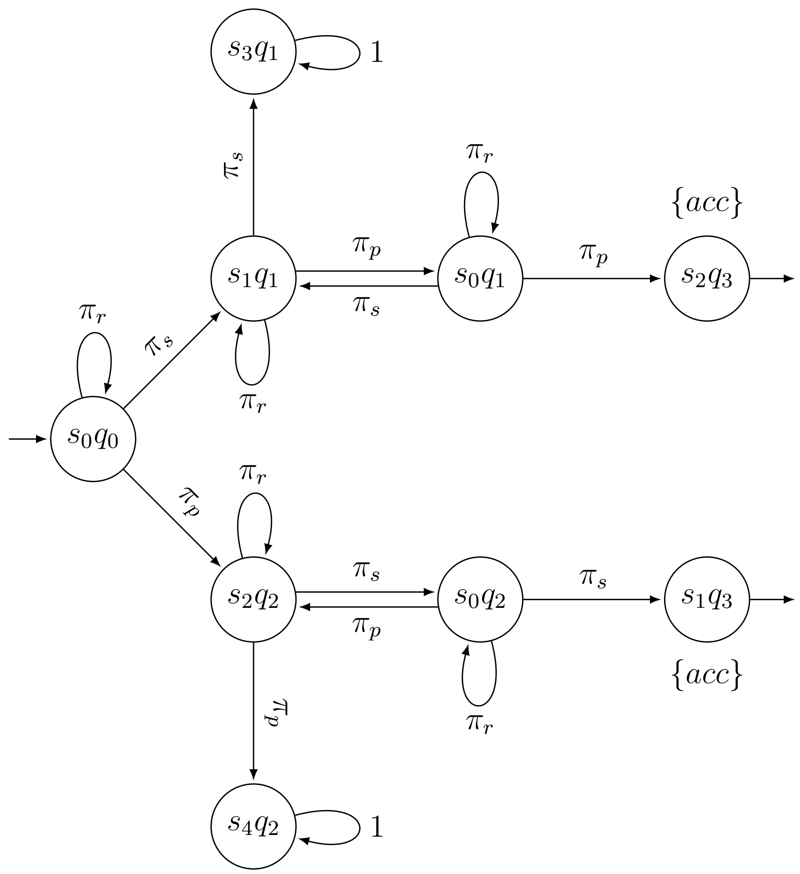


where:

$$\neg lead = \{tie, win_1, win_2\}$$
$$\neg lead_1 = \{tie, lead_2, win_1, win_2\}$$
$$\neg lead_2 = \{tie, lead_1, win_1, win_2\}$$
$$true = \{lead_1, lead_2, tie, win_1, win_2\}$$

3.c.ii

The product DTMC $D \otimes A$ is the following:



with every other state beyond s_1q_3 and s_2q_3 being accepting.

Now in order to compute the probability of the property being satisfied by the DTMC, we need to compute $Prob^D(s_0, P) = Prob^{D\otimes A}(s_0q_0, \mathbf{F}acc)$.

First we precompute the sets S^{nO} and S^{yes} for optimization purposes using the Prob0 and Prob1 algorithms, obtaining:

$$S^{no} = \{s_3q_1, s_4q2\}$$
$$S^{yes} = \{s_2q_3, s_1q3\}$$

then we setup the system of linear equations:

$$
\begin{aligned}
x_{s_3q_1} &= x_{s_4q2} = 0 \\
x_{s_2q_3} &= x_{s_1q3} = 1 \\
x_{s_0q_0} &= \pi_r x_{s_0q_0} + \pi_s x_{s_1q_1} + \pi_p x_{s_2q_2} \\
x_{s_1q_1} &= \pi_r x_{s_1q_1} + \pi_s x_{s_3q_1} + \pi_p x_{s_0q_1} \\
x_{s_0q_1} &= \pi_r x_{s_0q_1} + \pi_s x_{s_1q_1} + \pi_p x_{s_2q_3} \\
x_{s_2q_2} &= \pi_r x_{s_2q_2} + \pi_s x_{s_0q_2} + \pi_p x_{s_4q_2} \\
x_{s_0q_2} &= \pi_r x_{s_0q_2} + \pi_s x_{s_1q_3} + \pi_p x_{s_2q_2}
\end{aligned}
$$

and solving we obtain:

$$
\begin{aligned}
x_{s_3q_1} &= x_{s_4q2} = 0 \\
x_{s_2q_3} &= x_{s_1q3} = 1 \\
x_{s_0q_0} &= 0.1613 \\
x_{s_1q_1} &= 0.8064 \\
x_{s_0q_1} &= 0.9677 \\
x_{s_2q_2} &= 0.0323 \\
x_{s_0q_2} &= 0.1935
\end{aligned}
$$