

# PHASE DIAGRAM FOR INTERMITTENT MAPS

DANIEL CORONEL AND JUAN RIVERA-LETELIER

**ABSTRACT.** We explore the phase diagram for potentials in the space of Hölder continuous functions of a given exponent and for the dynamical system generated by a POMEAU–MANNEVILLE, or intermittent, map. There is always a phase where the unique GIBBS state exhibits intermittent behavior. It is the only phase for a specific range of values of the Hölder exponent. For the remaining values of the Hölder exponent, a second phase with stationary behavior emerges. In this case, a co-dimension 1 submanifold separates the intermittent and stationary phases. It coincides with the set of potentials at which the pressure function fails to be real-analytic. We also describe the relationship between the phase transition locus, (persistent) phase transitions in temperature, and ground states.

## 1. INTRODUCTION

Inspired by phase diagrams in statistical mechanics and bifurcation diagrams in dynamical systems, we propose studying regions of potentials where the corresponding GIBBS states exhibit qualitatively distinct behaviors for a fixed dynamical system. This paper focuses on one of the simplest cases of interest: The phase diagram for potentials in the space of Hölder continuous functions of a given exponent for the dynamical system generated by a POMEAU–MANNEVILLE, or intermittent, map. The simplicity of this setting allows for a detailed analysis of the associated phase diagram. There is always a phase where the unique GIBBS state exhibits intermittent behavior in the sense of POMEAU and MANNEVILLE [PM80]. It is the only phase for a specific range of values of the Hölder exponent. For the remaining values of the Hölder exponent, a second phase of stationary behavior emerges. In this case, a co-dimension 1 (topological) submanifold separates the intermittent and stationary phases. It coincides with the set of potentials at which the pressure function fails to be real-analytic. We also describe the relationship between the phase transition locus, (persistent) phase transitions in temperature, and ground states.

To state our results more precisely, we introduce some notation and terminology. For concreteness, we fix throughout the rest of the paper  $\alpha$  in  $(0, +\infty)$  and the POMEAU–MANNEVILLE or *intermittent* map  $f: [0, 1] \rightarrow [0, 1]$ , defined by

$$(1.1) \quad f(x) := \begin{cases} x(1 + x^\alpha) & \text{if } x(1 + x^\alpha) \leq 1; \\ x(1 + x^\alpha) - 1 & \text{otherwise.} \end{cases}$$

It is a prototypical example of an interval map with a neutral periodic point. It has been extensively studied in smooth ergodic theory, see, for example, [BC23, CV13, CCRV09, Hu04, GIR22, GIR24, Gou04, Klo19, Klo20, LRL14b, LSV99, Lop93, MT12, Pia80, PM80, PS92, PW99, PS92, Tha80, Sar01, Sar02, You99]. Our arguments apply with minor changes to variants of this map, like those studied in [You99, LSV99]. For expository purposes, the map  $f$  given

---

*Date:* January 1, 2026.

by (1.1) has the advantage that its geometric potential  $-\log Df$  is HÖLDER continuous and therefore belongs to (some of) the BANACH spaces of potentials considered here. However, we do not use this fact in our arguments. It is also possible to extend our results to maps with several neutral periodic points, as those considered by THALER [Tha80] and by LIVERANI, SAUSSOL, and VAIENTI [LSV99, §5].

Denote by  $C(\mathbb{R})$  the space of continuous real-valued functions defined on  $[0, 1]$  endowed with the uniform norm  $\|\cdot\|$ . For each  $\gamma$  in  $(0, 1]$ , denote by  $C^\gamma(\mathbb{R})$  the space of those functions in  $C(\mathbb{R})$  that are HÖLDER continuous of exponent  $\gamma$ . For each  $\varphi$  in  $C^\gamma(\mathbb{R})$ , put

$$(1.2) \quad |\varphi|_\gamma := \sup \left\{ \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\gamma} : x, y \in [0, 1] \text{ distinct} \right\} \text{ and } \|\varphi\|_\gamma := \|\varphi\| + |\varphi|_\gamma.$$

Denote by  $\mathcal{M}$  the space of BOREL probability measures on  $[0, 1]$  that are invariant by  $f$ . For each  $\nu$  in  $\mathcal{M}$ , denote by  $h_\nu$  the measure-theoretic entropy of  $\nu$ . For each  $\varphi$  in  $C(\mathbb{R})$ , the *pressure*  $P(\varphi)$  of  $f$  for the potential  $\varphi$  is defined by

$$(1.3) \quad P(\varphi) := \sup \left\{ h_\nu + \int \varphi \, d\nu : \nu \in \mathcal{M} \right\}.$$

A measure at which this supremum is attained is a GIBBS or *equilibrium state of  $f$  for the potential  $\varphi$* . There is at least 1 such measure (Lemma 2.4(1)). The pressure function  $P: C(\mathbb{R}) \rightarrow \mathbb{R}$  so defined is convex and LIPSCHITZ continuous.

A measure  $\nu$  in  $\mathcal{M}$  is *exponentially mixing for  $f$* , if there are  $C$  in  $(0, +\infty)$  and  $\rho$  in  $(0, 1)$  such that for every measurable bounded function  $\psi: [0, 1] \rightarrow \mathbb{R}$ , every LIPSCHITZ continuous function  $\tilde{\psi}: [0, 1] \rightarrow \mathbb{R}$ , and every  $n$  in  $\mathbb{N}$ , we have

$$(1.4) \quad \left| \int \psi \circ f^n \cdot \tilde{\psi} \, d\nu - \int \psi \, d\nu \int \tilde{\psi} \, d\nu \right| \leq C \|\psi\| \cdot |\tilde{\psi}|_1 \rho^n.$$

**1.1. Intermittent and stationary phases.** Denote by  $\mathbf{0}$  the function defined on  $[0, 1]$  that vanishes identically. Together with its unique GIBBS state for the potential  $\mathbf{0}$ , the map  $f$  is measurably isomorphic to a BERNoulli process (Lemma 2.4(2)). Due to the neutral fixed point at  $0$ , typical orbits exhibit intermittent behavior in the sense of POMEAU and MANNEVILLE [PM80]: They alternate irregularly between time intervals spent near  $0$  and random-looking excursions away from it. This intermittent behavior persists for HÖLDER continuous potentials near  $0$ . More precisely, for such a potential  $\varphi$ , there is a GIBBS state of  $f$  that is exponentially mixing and has strictly positive measure-theoretic entropy. See [IRRL25, Corollary A.4] and [Klo19, Theorems 1.2 and 3.3] for KLOECKNER's effective estimate of the size of this neighborhood of  $0$  for a map similar to  $f$ . These strong stochastic properties of the GIBBS state guarantee that typical orbits with respect to this measure also exhibit intermittent behavior. This motivates the following definition.

**Definition 1.1.** Let  $\gamma$  in  $(0, 1]$  be given. The *intermittent phase  $\mathcal{I}(\gamma)$  of  $f$  in  $C^\gamma(\mathbb{R})$* , is the set of potentials in  $C^\gamma(\mathbb{R})$  for which there is an exponentially mixing GIBBS state of  $f$  of strictly positive measure-theoretic entropy.

For every  $\gamma$  in  $(0, 1]$  the intermittent phase  $\mathcal{I}(\gamma)$  is nonempty, because it contains (a neighborhood of)  $\mathbf{0}$ . Moreover, the set  $\mathcal{I}(\gamma)$  is open, the pressure function  $P$  is real-analytic on  $\mathcal{I}(\gamma)$ , and for every  $\varphi$  in  $\mathcal{I}(\gamma)$  there is a unique GIBBS state of  $f$  for the potential  $\varphi$  [IRRL25, Theorem A.2]. See §2.1 for the notion of real-analyticity used in this paper.

In the case where  $\alpha < 1$  and  $\gamma > \alpha$ , the intermittent phase  $\mathcal{I}(\gamma)$  is all of  $C^\gamma(\mathbb{R})$  and the pressure function is real-analytic on all of  $C^\gamma(\mathbb{R})$ , see [LRL14a, LRL14b] and, in the case where  $\varphi(0) = \varphi(1)$ , see also [GIR24, Proposition 11], KLOECKNER's proof based on ideas from transport theory [Klo20, Theorem A], or the Key Lemma in §2.2.

In the remaining case where  $\gamma \leq \alpha$ , the space  $C^\gamma(\mathbb{R})$  contains the geometric potential  $-\log Df$ . It is the quintessential example of a potential for which every GIBBS state of  $f$  different from  $\delta_0^*$  is at most subexponentially mixing [Hu04, Gou04, Sar02]. That is,  $-\log Df$  is outside the intermittent phase  $\mathcal{I}(\gamma)$  and therefore  $\mathcal{I}(\gamma)$  is strictly smaller than  $C^\gamma(\mathbb{R})$ . There is in fact an open set of potentials in  $C^\gamma(\mathbb{R})$  outside  $\mathcal{I}(\gamma)$ . For a concrete example, consider the function  $\omega_\gamma: [0, 1] \rightarrow \mathbb{R}$  given by  $\omega_\gamma(x) := -x^\gamma$ . For every sufficiently large constant  $\beta$  and every potential  $\varphi$  in a neighborhood of  $\beta\omega_\gamma$  in  $C^\gamma(\mathbb{R})$ , the measure  $\delta_0$  is the unique GIBBS state of  $f$  for the potential  $\varphi$  (Proposition 3.2(2)). Such a potential is thus outside  $\mathcal{I}(\gamma)$  and for it the system has a stationary behavior localized at the neutral fixed point 0. This motivates the following definition.

**Definition 1.2.** Let  $\gamma$  in  $(0, 1]$  be given. The *stationary phase  $S(\gamma)$  of  $f$  in  $C^\gamma(\mathbb{R})$*  is the subset of  $C^\gamma(\mathbb{R})$  defined by

$$(1.5) \quad S(\gamma) := \text{int}(\{\varphi \in C^\gamma(\mathbb{R}): \delta_0 \text{ is the unique GIBBS state of } f \text{ for the potential } \varphi\}).$$

By definition, the stationary phase  $S(\gamma)$  is open and disjoint from  $\mathcal{I}(\gamma)$ . In the case where  $\gamma \leq \alpha$ , the stationary phase  $S(\gamma)$  is nonempty and in fact unbounded because for some  $\beta_0$  in  $(0, +\infty)$  it contains  $(\beta\omega_\gamma)_{c \in (\beta_0, +\infty)}$ . We also show that in this case,  $\mathcal{I}(\gamma)$  is star-convex at 0 and that  $S(\gamma)$  is convex (Proposition 3.1(1, 2)). In particular,  $\mathcal{I}(\gamma)$  and  $S(\gamma)$  are both connected.

**1.2. Phase transition locus.** Fix  $\gamma$  in  $(0, \min\{\alpha, 1\}]$ , so that the stationary phase  $S(\gamma)$  is nonempty. We show that the union of intermittent phase  $\mathcal{I}(\gamma)$  and the stationary phase  $S(\gamma)$  is dense in  $C^\gamma(\mathbb{R})$  and that the boundaries of these sets coincide (Proposition 3.1(3)). So, the following definition arises naturally.

**Definition 1.3.** Let  $\gamma$  in  $(0, \min\{\alpha, 1\}]$  be given. The *phase transition locus  $\mathcal{PT}(\gamma)$  of  $f$  in  $C^\gamma(\mathbb{R})$* , is the common boundary of the intermittent  $\mathcal{I}(\gamma)$  and stationary  $S(\gamma)$  phases.

The phase transition locus  $\mathcal{PT}(\gamma)$  is thus closed, has empty interior, and satisfies

$$(1.6) \quad \mathcal{PT}(\gamma) = C^\gamma(\mathbb{R}) \setminus (\mathcal{I}(\gamma) \cup S(\gamma)).$$

Our first main result characterizes the phase transition locus  $\mathcal{PT}(\gamma)$  as the singular set of the pressure function  $P$ .

**Theorem A.** For every  $\gamma$  in  $(0, 1]$ , the phase transition locus  $\mathcal{PT}(\gamma)$  coincides with the set of potentials at which the pressure function  $P$  fails to be real-analytic.

This result has natural applications to 1-parameter families of potentials. Let  $I$  be an open interval of  $\mathbb{R}$  and let  $(\varphi_\tau)_{\tau \in I}$  be a real-analytic family of potentials in  $C^\gamma(\mathbb{R})$ . Theorem A implies that for every parameter  $\tau_0$  at which the function  $\tau \mapsto P(\varphi_\tau)$  fails to be real-analytic, the potential  $\varphi_{\tau_0}$  must be in  $\mathcal{PT}(\gamma)$ . In other words, every phase transition parameter for the pressure function occurs at a potential in the phase transition locus. In the opposite direction, we show that if  $\tau_0$  is a parameter such that  $\varphi_\tau$  is in  $\mathcal{I}(\gamma)$  if  $\tau < \tau_0$  and outside  $\mathcal{I}(\gamma)$

---

\*Strictly speaking,  $\delta_0$  is exponentially mixing for trivial reasons.

if  $\tau \geq \tau_0$ , then the function  $\tau \mapsto P(\varphi_\tau)$  fails to be real-analytic at  $\tau_0$  (Lemma 3.6). Together with the fact that the intermittent phase  $\mathcal{I}(\gamma)$  is star-convex at 0 (Proposition 3.1(1)) and equal to  $\{\varphi \in C^\gamma(\mathbb{R}): P(\varphi) > \varphi(0)\}$  [IRRL25, Theorem A.2], this implies the following corollary as an immediate consequence.

**Corollary 1** (Phase transitions in temperature). *For every Hölder continuous potential  $\varphi$ , precisely one of the following properties holds.*

1. *The family  $(\beta\varphi)_{\beta \in (0, +\infty)}$  is contained in  $\mathcal{I}(\gamma)$  and the function  $\beta \mapsto P(\beta\varphi)$  is real-analytic and strictly larger than  $\beta \mapsto \beta\varphi(0)$  on  $(0, +\infty)$ .*
2. *There is  $\beta_*$  in  $(0, +\infty)$  such that for every  $\beta$  in  $(0, +\infty)$ , the potential  $\beta\varphi$  is in  $\mathcal{I}(\gamma)$  if  $\beta$  is in  $(0, \beta_*)$  and outside  $\mathcal{I}(\gamma)$  if  $\beta$  is in  $[\beta_*, +\infty)$ . Thus,  $\beta_*\varphi$  is in  $\mathcal{PT}(\gamma)$  and the function  $\beta \mapsto P(\beta\varphi)$  is real-analytic and strictly larger than  $\beta \mapsto \beta\varphi(0)$  on  $(0, \beta_*)$ , it coincides with  $\beta \mapsto \beta\varphi(0)$  on  $[\beta_*, +\infty)$ , and it only fails to be real-analytic on  $(0, +\infty)$  at  $\beta_*$ .*

In the case where the latter property in Corollary 1 holds, the potential  $\varphi$  undergoes a *phase transition in temperature* and  $\beta_*$  is the *phase transition parameter of  $\varphi$* . This terminology comes from statistical mechanics, where  $\beta$  is interpreted as the inverse temperature. Thus, a Hölder continuous potential  $\varphi$  undergoes a phase transition in temperature if and only if the function  $\beta \mapsto P(\beta\varphi)$  fails to be real-analytic on  $(0, +\infty)$ . The corollary above implies that these equivalent conditions hold if and only if at low temperatures  $\delta_0$  is a GIBBS state of  $f$  for  $\varphi$ .

The geometric potential  $-\log Df$  is a well-known example of a potential having a phase transition in temperature. See, for example, Proposition 4.5, [PS92], [CR25, Proposition 4.2], as well as [BC23, Theorem B] for an analogous result for local diffeomorphisms of the circle. For potentials that have a precise asymptotic behavior near 0 and are sufficiently regular, phase transitions in temperature were studied by SARIG [Sar01, Proposition 1] and in the companion paper [CR25].

Our next result describes fundamental properties of the phase transition locus.

**Theorem B.** *For every  $\gamma$  in  $(0, \min\{\alpha, 1\}]$ , the phase transition locus  $\mathcal{PT}(\gamma)$  is a topological submanifold of  $C^\gamma(\mathbb{R})$  of co-dimension 1. Furthermore, this submanifold is homeomorphic to a vector subspace of  $C^\gamma(\mathbb{R})$  of co-dimension 1, but it is not an affine subspace of  $C^\gamma(\mathbb{R})$ .*

In fact, we show that the phase transition locus is linear homeomorphic to the graph of a real-valued convex function and that every affine subspace contained in the phase transition locus is of infinite co-dimension (Proposition 3.8).

**1.3. On the persistence of phase transitions in temperature.** Our next result is reminiscent of the PEIERLS condition for contour models, which ensures that ground states persist as GIBBS states at low temperatures [Pei36, Sin82]. In various situations of interest, like the ISING model on the square lattice [Ons44], this leads to the existence of phase transitions [Geo11, Chapter 6]. For intermittent maps and a Hölder continuous potential  $\varphi$ , we show that the condition of  $\delta_0$  being robustly a ground state implies that  $\delta_0$  is also a GIBBS state at low temperatures. As a consequence, such a potential  $\varphi$  undergoes a phase transition in temperature (Theorem C).

To state this result more precisely, we introduce some terminology. A *ground state* of  $f$  for a continuous potential  $\varphi$ , is a measure  $\nu$  in  $\mathcal{M}$  satisfying

$$(1.7) \quad \int \varphi \, d\nu = \sup_{\nu' \in \mathcal{M}} \int \varphi \, d\nu'.$$

Recall from §1.2 that for a Hölder continuous potential  $\varphi$  having a phase transition in temperature, the phase transition parameter of  $\varphi$  is the unique parameter in  $(0, +\infty)$  at which the function  $\beta \mapsto P(\beta\varphi)$  fails to be real-analytic.

**Definition 1.4.** Let  $\gamma$  in  $(0, 1]$  be given. A potential  $\varphi$  in  $C^\gamma(\mathbb{R})$  undergoes a *persistent phase transition in temperature in  $C^\gamma(\mathbb{R})$* , if it undergoes a phase transition in temperature and if every potential close to  $\varphi$  in  $C^\gamma(\mathbb{R})$  undergoes a phase transition in temperature whose phase transition parameter is close to that of  $\varphi$ .

By definition, for a potential  $\varphi$  undergoing a persistent phase transition in temperature in  $C^\gamma(\mathbb{R})$ , the phase transition parameter is defined on a neighborhood of  $\varphi$  in  $C^\gamma(\mathbb{R})$  and it is continuous at  $\varphi$ .

**Theorem C** (From zero to low temperatures). *For all  $\gamma$  in  $(0, 1]$  and  $\varphi$  in  $C^\gamma(\mathbb{R})$ , the following properties are equivalent.*

1. *The potential  $\varphi$  undergoes a persistent phase transition in temperature in  $C^\gamma(\mathbb{R})$ .*
2. *The measure  $\delta_0$  is a ground state of  $f$  for every potential in  $C^\gamma(\mathbb{R})$  close to  $\varphi$ .*

When these equivalent conditions hold,  $\delta_0$  is the unique ground state of  $f$  for the potential  $\varphi$ .

The following corollary is a direct consequence of Theorem C. For each  $\gamma$  in  $(0, 1]$ , put

$$(1.8) \quad \mathcal{G}(\gamma) := \{\varphi \in C^\gamma(\mathbb{R}) : \delta_0 \text{ is a ground state of } f \text{ for the potential } \varphi\}$$

and note that  $\mathcal{G}(\gamma)$  is a closed convex cone. Moreover, let  $\beta_{\text{crit},\gamma} : C^\gamma(\mathbb{R}) \rightarrow (0, +\infty)$  be the function defined for  $\varphi$  in  $C^\gamma(\mathbb{R})$  as follows. If  $\varphi$  undergoes a phase transition in temperature, then  $\beta_{\text{crit},\gamma}(\varphi)$  is its phase transition parameter. Otherwise,  $\beta_{\text{crit},\gamma}(\varphi) := +\infty$ . Note that by Corollary 1 in §1.2 every potential in  $C^\gamma(\mathbb{R})$  undergoing a phase transition in temperature is in  $\mathcal{G}(\gamma)$  and we have

$$(1.9) \quad \beta_{\text{crit},\gamma}^{-1}((1, +\infty]) = \mathcal{G}(\gamma) \text{ and } \beta_{\text{crit},\gamma}^{-1}(1) \subseteq \mathcal{PT}(\gamma).$$

**Corollary 2.** *For every  $\gamma$  in  $(0, 1]$ , the function  $\beta_{\text{crit},\gamma}$  is constant equal to  $+\infty$  outside  $\mathcal{G}(\gamma)$  and it is finite and continuous on  $\text{int}(\mathcal{G}(\gamma))$ .*

To introduce our next result, recall that the geometric potential  $-\log Df$  undergoes a phase transition in temperature and that its phase transition parameter  $\beta_{\text{crit},\gamma}(-\log Df)$  is equal to 1 (Proposition 4.5). This function is Hölder continuous of exponent  $\min\{\alpha, 1\}$ , so for every  $\gamma$  in  $(0, \min\{\alpha, 1\}]$ , it belongs to  $C^\gamma(\mathbb{R})$ . In the case where  $\gamma < \alpha$ , we show that the phase transition in temperature of  $-\log Df$  is nonpersistent in  $C^\gamma(\mathbb{R})$ , and thus  $\beta_{\text{crit},\gamma}$  is discontinuous at this potential. Moreover, we show that the phase transition locus fails to be a co-dimension 1 real-analytic subset of  $C^\gamma(\mathbb{R})$  at  $-\log Df$ . See Proposition 4.4 for a precise statement of these results.

In the remaining case where  $\alpha$  is in  $(0, 1]$  and  $\gamma = \alpha$ , the following result shows that the situation is different.

**Theorem D.** *If  $\alpha$  is in  $(0, 1]$ , then every phase transition in temperature is persistent in  $C^\alpha(\mathbb{R})$ , the function  $\beta_{\text{crit}, \alpha}$  is continuous, and we have*

$$(1.10) \quad \beta_{\text{crit}, \alpha}^{-1}(1) = \mathcal{PT}(\alpha) \subseteq \text{int}(\mathcal{G}(\alpha)) \text{ and } \beta_{\text{crit}, \gamma}^{-1}((0, +\infty)) = \text{int}(\mathcal{G}(\alpha)).$$

The following corollary is a direct consequence of Theorems A and D.

**Corollary 3.** *Suppose  $\alpha$  is in  $(0, 1]$ , let  $I$  be an open subinterval of  $\mathbb{R}$  containing 0, and let  $(\varphi_\tau)_{\tau \in I}$  be a real-analytic family of potentials in  $C^\alpha(\mathbb{R})$  having a phase transition at 0. Then,  $\varphi_0$  undergoes a phase transition in temperature and its phase transition parameter  $\beta_{\text{crit}, \alpha}(\varphi_0)$  is equal to 1.*

See Example 4.8 for a concrete example showing that the analogous statement is false for every  $\gamma$  in  $(0, 1]$  satisfying  $\gamma < \alpha$ .

The main ingredient in the proof of Theorem D is a characterization of those potentials in  $C^\alpha(\mathbb{R})$  having a phase transition in temperature (Theorem D' in §4.3).

As mentioned above, for every  $\gamma$  in  $(0, 1]$  satisfying  $\gamma < \alpha$ , the phase transition locus  $\mathcal{PT}(\gamma)$  fails to be a co-dimension 1 real-analytic subset of  $C^\gamma(\mathbb{R})$ . In the case where  $\alpha$  is in  $(0, 1]$ , it is unclear to us whether  $\mathcal{PT}(\alpha)$  is a co-dimension 1 real-analytic subset of  $C^\alpha(\mathbb{R})$ .

**1.4. Notes and references.** The intermittent behavior of a positive entropy GIBBS state reflects the concentration near the neutral fixed point 0 of its density with respect to the conformal measure. See, for example, [LSV99, Tha80].

We show that for each HÖLDER continuous potential  $\varphi$  undergoing a phase transition in temperature,  $\delta_0$  is the unique ground state of  $f$  for the potential  $\varphi$  (Corollary 1 in §1.2 and Corollary 4 in §2.2). The following is a counterexample for the reverse implication. Choose  $\gamma$  in  $(\alpha, +\infty)$  and let  $\omega_\gamma: [0, 1] \rightarrow \mathbb{R}$  be the function defined by  $\omega_\gamma(x) := -x^\gamma$ . It is HÖLDER continuous of exponent  $\min\{1, \gamma\}$ . Clearly,  $\delta_0$  is the unique ground state of  $f$  for the potential  $\omega_\gamma$ . However,  $\omega_\gamma$  has no phase transition in temperature by Proposition 3.2.

In [CR25], we studied phase transitions in temperature for a class of HÖLDER continuous potentials introduced by SARIG in [Sar01]. Using the notion of “robust phase transition in temperature,” we proved results analogous to Theorems C and D [CR25, Theorem 3 and Corollary 1.3]. For  $\gamma$  in  $(0, 1]$ , this notion can be formulated in the space  $C^\gamma(\mathbb{R})$  as follows: A potential  $\varphi$  in  $C^\gamma(\mathbb{R})$  exhibits a *robust phase transition in temperature in  $C^\gamma(\mathbb{R})$*  if every potential sufficiently close to  $\varphi$  in  $C^\gamma(\mathbb{R})$  undergoes a phase transition in temperature. Every phase transition in temperature persistent in  $C^\gamma(\mathbb{R})$  is also robust in  $C^\gamma(\mathbb{R})$ . Corollary 2 in §1.3 shows that these notions coincide in  $C^\gamma(\mathbb{R})$ . However, we have not investigated if these notions coincide in the space of potentials in [CR25, Sar01].

The inducing scheme has been successful in establishing fundamental properties of intermittent maps. However, for  $\gamma$  in  $(0, 1]$  satisfying  $\gamma < \alpha/(\alpha + 1)$ , the inducing scheme approach breaks down for potentials in  $C^\gamma(\mathbb{R})$  because the distortion is, in general, unbounded. We have thus avoided the inducing scheme approach entirely.

**1.5. Organization.** After some preliminaries in §2, we explore the phase diagram in §3. In §3.1, we examine the intermittent and stationary phases. In §3.2, we prove that the phase transition locus coincides with the set where the pressure function fails to be real-analytic (Theorem A). In §3.3, we analyze the geometry at infinity of both the stationary phase and the phase transition locus. Finally, in §3.4, we use this analysis to establish fundamental properties of the phase transition locus and to derive Theorem B.

In §4 we study the persistence of phase transitions in temperature. In §4.1, we relate, for each  $\gamma$  in  $(0, \min\{\alpha, 1\}]$ , the stationary phase  $S(\gamma)$  to the interior of the cone  $\mathcal{G}(\gamma)$  and prove Theorem C. In §4.2, we state and prove our results in the case where  $\gamma < \alpha$ . In §4.3, we characterize the potentials that undergo a phase transition in temperature in the remaining case, where  $\alpha \leq 1$  and  $\gamma = \alpha$ . This characterization is the main ingredient in the proof of Theorem D, which is given in §4.4.

## 2. PRELIMINARIES

Let  $X$  be a topological space. For a subset  $S$  of  $X$ , denote by  $\text{int}(S)$  and  $\text{cl}(S)$  the interior and the closure of  $S$ , respectively. A subset of  $X$  is a *regular open set*, if it is equal to the interior of its closure.

Given a subset  $K$  of  $\mathbb{R}$ , denote by  $\text{HD}(K)$  its HAUSDORFF dimension.

For each  $v$  in  $\mathcal{M}$ , denote by  $\text{supp}(v)$  its support.

Denote by  $C(\mathbb{C})$  the space of continuous complex-valued functions defined on  $[0, 1]$  and for  $\varphi$  in  $C(\mathbb{C})$ , put

$$(2.1) \quad \|\varphi\| := \sup_{[0,1]} |\varphi|.$$

Furthermore, denote by  $C(\mathbb{R})$  the subspace of  $C(\mathbb{C})$  of real-valued functions and by  $\mathbf{0}$  the function in  $C(\mathbb{R})$  that is constant equal to 0.

For each  $\gamma$  in  $(0, 1]$ , denote by  $C^\gamma(\mathbb{C})$  the space of those functions in  $C(\mathbb{C})$  that are HÖLDER continuous of exponent  $\gamma$ . For each  $\varphi$  in  $C^\gamma(\mathbb{C})$ , put

$$(2.2) \quad |\varphi|_\gamma := \sup \left\{ \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\gamma} : x, y \in [0, 1] \text{ distinct} \right\} \text{ and } \|\varphi\|_\gamma := \|\varphi\| + |\varphi|_\gamma.$$

Furthermore, denote by  $C^\gamma(\mathbb{R})$  the subspace of  $C^\gamma(\mathbb{C})$  of real-valued functions. The normed spaces  $(C^\gamma(\mathbb{R}), \|\cdot\|_\gamma)$  and  $(C^\gamma(\mathbb{C}), \|\cdot\|_\gamma)$  are both BANACH.

**2.1. Holomorphic and real-analytic functions and subsets.** Fix  $\gamma$  in  $(0, 1]$ .

Given an open subset  $\Delta$  of  $\mathbb{C}$ , a function  $H: \Delta \rightarrow C^\gamma(\mathbb{C})$  is *holomorphic* if for every  $z$  in  $\Delta$  there is  $\varphi_z$  in  $C^\gamma(\mathbb{C})$  such that

$$(2.3) \quad \lim_{h \rightarrow 0} \left\| \varphi_z - \frac{H(z+h) - H(z)}{h} \right\|_\gamma = 0.$$

On other hand, given an open subset  $\mathcal{U}$  of  $C^\gamma(\mathbb{C})$ , a function  $h: \mathcal{U} \rightarrow \mathbb{C}$  is *holomorphic* if for every open subset  $\Delta$  of  $\mathbb{C}$  and every holomorphic function  $H: \Delta \rightarrow \mathcal{U}$  the composition  $h \circ H$  is holomorphic.

A real-valued function defined on an open subset of  $C^\gamma(\mathbb{R})$  is *real-analytic*, if it extends to a complex-valued holomorphic function defined on an open subset of  $C^\gamma(\mathbb{C})$ . Similarly, a function defined on an open subset of  $\mathbb{R}$  and taking values in  $C^\gamma(\mathbb{R})$  is *real-analytic*, if it extends to a function defined on an open subset of  $\mathbb{C}$  and taking values in  $C^\gamma(\mathbb{C})$ .

A subset  $S$  of  $C^\gamma(\mathbb{R})$  is a *co-dimension 1 real-analytic subset* of  $C^\gamma(\mathbb{R})$ , if for every  $\varphi$  in  $S$  there is an open neighborhood  $\mathcal{U}$  of  $\varphi$  and a nonconstant real-analytic function  $\mathcal{U} \rightarrow \mathbb{R}$  vanishing precisely on  $\mathcal{U} \cap S$ .

**Lemma 2.1.** Fix  $\gamma$  in  $(0, 1]$  and let  $S$  be a co-dimension 1 real-analytic subset of  $C^\gamma(\mathbb{R})$  that is closed. Moreover, let  $I$  be an open interval of  $\mathbb{R}$  and let  $A: I \rightarrow C^\gamma(\mathbb{R})$  be a real-analytic function. Then, either  $A(I) \subseteq S$  or every point of  $A^{-1}(S)$  is isolated in  $I$ .

*Proof.* Put  $K := A^{-1}(S)$ . Since  $S$  is closed by hypothesis,  $K$  is closed in  $I$ . It is thus sufficient to show that every accumulation point  $\tau_0$  of  $K$  in  $I$  is in the interior of  $K$ . Let  $\mathcal{U}$  be an open neighborhood of  $A(\tau_0)$  in  $C^\gamma(\mathbb{R})$  and  $a: \mathcal{U} \rightarrow \mathbb{R}$  a real-analytic function vanishing precisely on  $\mathcal{U} \cap S$ . Furthermore, let  $I_0$  be an open subinterval of  $I$  such that  $A(I_0) \subseteq \mathcal{U}$ . Then, the composition  $a \circ A|_{I_0}$  vanishes on each point of  $K \cap I_0$  and it extends to a holomorphic function defined on a neighborhood of  $I_0$  in  $\mathbb{C}$ . Combined with our hypothesis that  $\tau_0$  is not isolated in  $K \cap I_0$  and the Identity Principle, it follows that  $a \circ A|_{I_0}$  vanishes on all of  $I_0$ . In particular,  $K$  contains  $I_0$  and therefore  $\tau_0$  is in the interior of  $K$ . The proof of the lemma is thus complete.  $\square$

**2.2. Dynamics and thermodynamic formalism of intermittent maps.** Note that the function  $\log Df$  is strictly increasing and Hölder continuous of exponent  $\min\{1, \alpha\}$ .

Denote by  $x_1$  the unique discontinuity of  $f$ . Then  $f(x_1) = 1$  and the map  $f: [0, x_1] \rightarrow [0, 1]$  is a diffeomorphism. Put  $x_0 := 1$  and for each integer  $j$  satisfying  $j \geq 2$ , put  $x_j := f|_{[0, x_1]}^{-(j-1)}(x_1)$ . Moreover, for every  $j$  in  $\mathbb{N}_0$  put  $J_j := (x_{j+1}, x_j]$  and note that  $f(x_{j+1}) = x_j$  and  $f(J_{j+1}) = J_j$ .

**Lemma 2.2** ([CR25, Lemma 3.1]). We have

$$(2.4) \quad \lim_{n \rightarrow +\infty} n \cdot x_n^\alpha = \frac{1}{\alpha}.$$

Moreover, there is  $\varepsilon_0$  in  $(0, 1)$  such that for all  $n$  in  $\mathbb{N}$  and  $x$  in  $J_n$  we have

$$(2.5) \quad (1 + \varepsilon_0 n)^{\frac{1}{\alpha} + 1} \leq Df^n(x) \leq (1 + \varepsilon_0^{-1} n)^{\frac{1}{\alpha} + 1}.$$

Let  $\mathcal{P}$  be the partition of  $[0, 1]$  defined by  $\mathcal{P} := \{[0, x_1], (x_1, 1]\}$  and for every  $n$  in  $\mathbb{N}$  put  $\mathcal{P}_n := \bigvee_{k=0}^{n-1} \mathcal{P}$ .

**Lemma 2.3.** [CR25, Lemma 3.6] For every continuous potential  $\varphi$ , we have

$$(2.6) \quad P(\varphi) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{Q \in \mathcal{P}_n} \sup_Q \exp(S_n(\varphi))$$

and that for every  $x$  in  $(0, 1]$  we have

$$(2.7) \quad P(\varphi) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{x' \in f^{-n}(x)} \exp(S_n(\varphi)(x')).$$

**Lemma 2.4.** The following properties hold.

1. The measure-theoretic entropy is upper-semicontinuous and for every continuous potential there is at least 1 GIBBS state.
2. We have

$$(2.8) \quad \sup_{\nu \in \mathcal{M}} h_\nu = \log 2$$

and there is a unique measure  $\nu_{\max}$  realizing this supremum. Furthermore,  $(f, \nu_{\max})$  is measurably isomorphic to a BERNoulli process. That is, there is a measurable map  $[0, 1] \rightarrow \{0, 1\}^{\mathbb{N}_0}$  mapping  $\nu_{\max}$  to the BERNoulli measure with weights  $\frac{1}{2}$  and  $\frac{1}{2}$  and conjugating  $f$  to the shift map on a set of full measure.

*Proof.* Denote by  $\sigma: \{0, 1\}^{\mathbb{N}_0} \rightarrow \{0, 1\}^{\mathbb{N}_0}$  the shift map, by  $\mathcal{M}_\sigma$  the space of Borel probability measures on  $\{0, 1\}^{\mathbb{N}_0}$  invariant by  $\sigma$ , and, for each  $\mu$  in  $\mathcal{M}_\sigma$ , denote by  $h_\mu(\sigma)$  the measure-theoretic entropy of  $\mu$ .

By [CR25, Lemma 3.3], there is a continuous surjective map  $\pi: \{0, 1\}^{\mathbb{N}_0} \rightarrow [0, 1]$  such that  $\pi \circ \sigma = f \circ \pi$ . Moreover, for every invariant measure  $\nu$  for  $f$ , there is a unique invariant measure  $\mu$  for  $\sigma$  such that  $\pi_*\mu = \nu$ , and the systems  $(\{0, 1\}^{\mathbb{N}_0}, \sigma, \mu)$  and  $([0, 1], f, \nu)$  are isomorphic in measure. In particular, the map  $\pi_*: \mathcal{M}_\sigma \rightarrow \mathcal{M}$  is one-to-one. Since  $\mathcal{M}_\sigma$  is compact and  $\pi$  is continuous, it follows that  $\pi_*$  is a homeomorphism. Together with the fact that the measure-theoretic entropy of  $\sigma$  is upper semicontinuous, see, for example, [Bow08, Proposition 2.19], this implies the same property for  $f$ . Since  $\mathcal{M}$  is compact, it follows that for every continuous potential there is at least 1 GIBBS state. This proves item 1.

To prove item 2, note that

$$(2.9) \quad \sup_{\mu \in \mathcal{M}_f} h_\mu = \sup_{\mu \in \mathcal{M}_\sigma} h_\mu(\sigma) = \log 2$$

and recall that the BERNoulli measure  $\rho$  with weights  $\frac{1}{2}$  and  $\frac{1}{2}$  on  $\{0, 1\}^{\mathbb{N}_0}$  is the unique measure of maximal entropy for  $\sigma$ . It follows that  $\pi_*\rho$  is the unique maximal entropy measure for  $f$ , and the systems  $(\sigma, \rho)$  and  $(f, \pi_*\rho)$  are isomorphic in measure.  $\square$

The following is a version for intermittent maps of a result originated in [IRRL12] in the context of complex rational maps, see also the version for multimodal maps in [Li15]. See [CR25, Appendix A] for a simplified proof for intermittent maps.

**Key Lemma.** *For every HÖLDER continuous potential  $\varphi$  and every  $\nu$  in  $\mathcal{M}$  distinct from  $\delta_0$ , we have*

$$(2.10) \quad P(\varphi) > \int \varphi \, d\nu.$$

**Corollary 4.** *Let  $\varphi$  be a HÖLDER continuous potential such that for some  $\beta_*$  in  $(0, +\infty)$  we have  $P(\beta_*\varphi) = \beta_*\varphi(0)$ . Then,  $\delta_0$  is the unique ground state of  $f$  for the potential  $\varphi$  and for every  $\beta$  in  $(\beta_*, +\infty)$  the measure  $\delta_0$  is the unique GIBBS state of  $f$  for the potential  $\varphi$ .*

*Proof.* To prove the first assertion, note that

$$(2.11) \quad \beta\varphi(0) = P(\beta\varphi) \geq \beta \sup_{\nu \in \mathcal{M}} \int \varphi \, d\nu \geq \beta\varphi(0).$$

Thus,  $\delta_0$  is a ground state of  $f$  for the potential  $\varphi$ . To prove uniqueness, note that for every ground state  $\nu_0$  of  $f$  for  $\varphi$  we have

$$(2.12) \quad \int \varphi \, d\nu_0 = \sup_{\nu \in \mathcal{M}} \int \varphi \, d\nu = \varphi(0) = P(\varphi) \geq h_{\nu_0} + \int \varphi \, d\nu_0$$

and therefore

$$(2.13) \quad h_{\nu_0} = 0 \text{ and } P(\varphi) = \int \varphi \, d\nu_0.$$

Together with the Key Lemma, this implies  $\nu_0 = \delta_0$ .

To prove the second assertion, let  $\beta$  in  $(\beta_*, +\infty)$  be given and let  $\nu$  be a GIBBS state of  $f$  for the potential  $\beta\varphi$  (Lemma 2.4(2)). Then we have

$$(2.14) \quad \beta\varphi(0) = \frac{\beta}{\beta_*} P(\beta_*\varphi) \geq \frac{\beta}{\beta_*} h_\nu + \beta \int \varphi \, d\nu = \frac{\beta - \beta_*}{\beta_*} h_\nu + P(\beta\varphi) \geq \frac{\beta - \beta_*}{\beta_*} h_\nu + \beta\varphi(0)$$

and therefore  $h_\nu = 0$  and  $P(\beta\varphi) = \beta\varphi(0)$ . The latter implies that  $\delta_0$  is a GIBBS state of  $f$  for the potential  $\varphi$ . Together with the Key Lemma, the former implies  $\chi_\nu(f) = 0$  and therefore  $\nu = \delta_0$ . That is,  $\delta_0$  is the unique GIBBS state of  $f$  for the potential  $\varphi$ .  $\square$

### 3. PHASE DIAGRAM

This section explores the phase diagram. In §3.1, we examine the intermittent and stationary phases (Proposition 3.1). In §3.2, we prove that the phase transition locus coincides with the set where the pressure function fails to be real-analytic (Theorem A). In §3.3, we analyze the geometry at infinity of both the stationary phase and the phase transition locus (Proposition 3.7). Building on this analysis, we establish fundamental facts about the structure of the phase transition locus (Proposition 3.8) and derive Theorem B in §3.4.

**3.1. Intermittent and stationary phases.** The goal of this section is to prove the following.

**Proposition 3.1.** *Let  $\gamma$  in  $(0, 1]$  be given. The intermittent phase  $\mathcal{I}(\gamma)$  contains 0 and it is therefore nonempty. In the case where  $\alpha < 1$  and  $\gamma > \alpha$ , we have*

$$(3.1) \quad \mathcal{I}(\gamma) = C^\gamma(\mathbb{R}) \text{ and } \mathcal{S}(\gamma) = \emptyset.$$

*In the remaining case where  $\gamma \leq \alpha$ , the stationary phase  $\mathcal{S}(\gamma)$  is nonempty and the following properties hold.*

1. *The intermittent phase  $\mathcal{I}(\gamma)$  is star-convex at 0. In particular,  $\mathcal{I}(\gamma)$  is connected.*
2. *The stationary phase  $\mathcal{S}(\gamma)$  is unbounded, convex, and coincides with the interior of*

$$(3.2) \quad \{\varphi \in C^\gamma(\mathbb{R}): \delta_0 \text{ is a GIBBS state of } f \text{ for the potential } \varphi\}.$$

*In particular,  $\mathcal{S}(\gamma)$  is connected.*

3. *Each of the phases  $\mathcal{I}(\gamma)$  and  $\mathcal{S}(\gamma)$  is a regular open set. Furthermore, these sets are disjoint and their union is dense in  $C^\gamma(\mathbb{R})$ . In particular, we have*

$$(3.3) \quad \partial\mathcal{I}(\gamma) = \partial\mathcal{S}(\gamma) = C^\gamma(\mathbb{R}) \setminus (\mathcal{I}(\gamma) \cup \mathcal{S}(\gamma)).$$

The first assertion of the proposition, that in the case where  $\alpha < 1$  and  $\gamma > \alpha$  we have  $\mathcal{I}(\gamma) = C^\gamma(\mathbb{R})$ , was shown in [LRL14a, LRL14b] as a consequence of a more general result. In the case where  $\varphi(0) = \varphi(1)$ , see also [GIR24, Proposition 11] or [Klo20, Theorem A]. We give a proof specialized for intermittent maps that is significantly shorter.

The proof of Proposition 3.1 is at the end of this section. It relies on the following direct consequence of [IRRL25, Theorem A.2], which is also used several times in the rest of the paper.

**Theorem 1.** *For every  $\gamma$  in  $(0, 1]$ , we have*

$$(3.4) \quad \mathcal{I}(\gamma) = \{\varphi \in C^\gamma(\mathbb{R}): P(\varphi) > \varphi(0)\},$$

*the set  $\mathcal{I}(\gamma)$  is open in  $C^\gamma(\mathbb{R})$ , and the pressure function  $P$  is real-analytic on  $\mathcal{I}(\gamma)$ .*

For each  $\gamma$  in  $(0, +\infty)$ , let  $\omega_\gamma$  be the function in  $C^\gamma(\mathbb{R})$  defined by  $\omega_\gamma(x) := -x^\gamma$ . In the case where  $\gamma$  is in  $(0, 1]$ , this function is in  $C^\gamma(\mathbb{R})$  and for every  $\psi$  in  $C^\gamma(\mathbb{R})$  we have

$$(3.5) \quad \psi \geq \psi(0) + |\psi|_\gamma \omega_\gamma.$$

Recall from §1.2 that a HÖLDER continuous potential  $\varphi$  undergoes a phase transition in temperature, if the function  $\beta \mapsto P(\beta\varphi)$  fails to be real-analytic on  $(0, +\infty)$ .

**Proposition 3.2** ([CR25, Proposition 2.8]). *For every  $\gamma$  in  $(0, +\infty)$ , the following properties hold.*

1. *If  $\gamma > \alpha$ , then for every  $\beta$  in  $(0, +\infty)$  we have  $P(\beta\omega_\gamma) > 0$  and  $\beta\omega_\gamma$  is in  $\mathcal{I}(\gamma)$ .*
2. *If  $\gamma \leq \alpha$ , then for every sufficiently large  $\beta$  in  $(0, +\infty)$  we have  $P(\beta\omega_\gamma) = 0$  and  $\beta\omega_\gamma$  is outside  $\mathcal{I}(\gamma)$ .*

In particular, the potential  $\omega_\gamma$  undergoes a phase transition in temperature if and only if  $\gamma \leq \alpha$ .

We give a different proof of this result. To do this, we introduce some notation. For each  $j$  in  $\mathbb{N}$ , let  $y_j$  be the unique point in  $J_0$  such that  $f(y_j) = x_{j-1}$ , put  $I_j := (y_{j+1}, y_j]$ , and let  $m: J_0 \rightarrow \mathbb{N}$  be the function so that for each  $j$  in  $\mathbb{N}$  we have  $m^{-1}(j) = I_j$ . It is the *first return time to  $J_0$* . The *first return map of  $f$  to  $J_0$*  is the function  $F: J_0 \rightarrow J_0$  defined by  $F(x) := f^{m(x)}(x)$ . Note that for each  $j$  in  $\mathbb{N}$ , the map  $F$  maps  $I_j$  diffeomorphically onto  $J_0$ .

*Proof of Proposition 3.2.* To prove item 1, put

$$(3.6) \quad \zeta := \beta \sum_{j=0}^{+\infty} x_j^\gamma$$

and note that this sum is finite by our hypothesis  $\gamma > \alpha$  and (2.4) in Lemma 2.2. It follows that for all  $k$  in  $\mathbb{N}_0$  and  $y$  in  $[0, x_k]$ , we have

$$(3.7) \quad S_{k+1}(\beta\omega_\gamma)(y) \geq S_{k+1}(\beta\omega_\gamma)(x_k) \geq -\zeta.$$

Recall from §2.2 that  $\mathcal{P}$  is the partition  $\{[0, x_1], (x_1, 1]\}$  of  $[0, 1]$  and that for each  $n$  in  $\mathbb{N}$  we denote its  $n$ -th refinement  $\bigvee_{k=0}^{n-1} \mathcal{P}$  by  $\mathcal{P}_n$ . Let

$$(3.8) \quad \iota_n: \mathcal{P}_n \rightarrow \{0, 1\}^{\{0, \dots, n-1\}}$$

be the itinerary map defined in such a way that for all  $Q \in \mathcal{P}_n$  and  $j \in \{0, \dots, n-1\}$  we have  $\iota_n(Q)_j = 1$  if and only if  $f^j(Q) \subseteq (x_1, 1]$ . Note that  $\iota_n$  is a bijection. For each  $a_0 \cdots a_{n-1}$  in  $\{0, 1\}^{\{0, \dots, n-1\}}$ , put  $|a_0 \cdots a_{n-1}| := \sum_{j=0}^{n-1} a_j$ . It follows that for every  $Q$  in  $\mathcal{P}_n$ , we have

$$(3.9) \quad \sup_Q S_n(\beta\omega_\gamma) \geq -(|\iota_n(Q)| + 1)\zeta.$$

Hence

$$(3.10) \quad \begin{aligned} \sum_{Q \in \mathcal{P}_n} \sup_Q \exp(S_n(\beta\omega_\gamma)) &\geq \exp(-\zeta) \sum_{\underline{a} \in \{0, 1\}^{\{0, \dots, n-1\}}} \exp(-|\underline{a}|\zeta) \\ &= \exp(-\zeta)(1 + \exp(-\zeta))^n. \end{aligned}$$

Taking logarithms, dividing by  $n$ , and then taking the limit as  $n \rightarrow +\infty$ , we obtain by (2.6) in Lemma 2.3

$$(3.11) \quad P(\beta\omega_\gamma) \geq \log(1 + \exp(-\zeta)) > 0.$$

Together with Theorem 1, this implies that  $\beta\varphi$  is in  $\mathcal{I}(\gamma)$ .

To prove item 2, for every  $\beta$  in  $(0, +\infty)$  and each  $n$  in  $\mathbb{N}$  put

$$(3.12) \quad \xi_n(\beta) := \exp\left(-\beta \sum_{j=1}^n x_j^\gamma\right).$$

In view of (2.4) in Lemma 2.2, our hypothesis  $\gamma \leq \alpha$  implies that there is  $\beta_0$  in  $(0, +\infty)$  such that for every  $\beta$  in  $[\beta_0, +\infty)$  we have

$$(3.13) \quad \sum_{j=1}^{+\infty} \xi_j(\beta) < 1.$$

Let  $\beta$  in  $[\beta_0, +\infty)$  be given. In view of (2.7) in Lemma 2.3, to prove  $P(\beta\omega_\gamma) = 0$  it is sufficient to show that the sum

$$(3.14) \quad \sum_{n=1}^{+\infty} \sum_{x \in f^{-n}(1)} \exp(\beta S_n(\omega_\gamma)(x))$$

is finite. Note first that for all  $j$  in  $\mathbb{N}$  and  $x$  in  $J_j$ , we have

$$(3.15) \quad \exp(\beta S_j(\omega_\gamma)(x)) \leq \exp\left(-\beta \sum_{k=1}^j x_{k+1}^\gamma\right) = \exp(\beta x_1^\gamma) \xi_{j+1}(\beta).$$

It follows that (3.14) is bounded from above by

$$(3.16) \quad \left(1 + \exp(\beta x_1^\gamma) \sum_{j=1}^{+\infty} \xi_{j+1}(\beta)\right) \left(\sum_{n=1}^{+\infty} \sum_{\substack{x \in f^{-n}(1) \\ x \in J_0}} \exp(\beta S_n(\omega_\gamma)(x))\right).$$

The first factor above is finite by (3.13). To prove that the second factor above is finite, note that all  $j$  in  $\mathbb{N}$  and  $y$  in  $I_j$ , the point  $f(y)$  is in  $J_{j-1}$ , so by (3.15) we have

$$(3.17) \quad \exp(\beta S_j(\omega_\gamma)(y)) \leq \exp(-\beta y^\gamma) \exp(\beta x_1^\gamma) \xi_j(\beta) \leq \xi_j(\beta).$$

Let  $x$  be a point of  $f^{-n}(1)$  in  $J_0$  and let  $\ell$  in  $\mathbb{N}$  be the integer such that

$$(3.18) \quad m(x) + \cdots + m(F^{\ell-1}(x)) = n.$$

Applying (3.17) repeatedly, we obtain

$$(3.19) \quad \exp(\beta S_n(\omega_\gamma)(x)) \leq \prod_{k=0}^{\ell-1} \xi_{m(F^k(x))}(\beta).$$

Together with (3.13), this implies

$$(3.20) \quad \sum_{n=1}^{+\infty} \sum_{\substack{x \in f^{-n}(1) \\ x \in J_0}} \exp(\beta S_n(\omega_\gamma)(x)) \leq \sum_{\ell=1}^{+\infty} \left(\sum_{j=1}^{+\infty} \xi_j(\beta_0)\right)^\ell < +\infty.$$

This proves that the second factor in (3.16) is finite and completes the proof of  $P(\beta\omega_\gamma) = 0$ . It follows that  $\beta\omega_\gamma$  is outside  $\mathcal{I}(\gamma)$  by Theorem 1. The proof of the lemma is thus complete.  $\square$

**Lemma 3.3.** *For every  $\gamma$  in  $(0, 1]$ , we have*

$$(3.21) \quad C^\gamma(\mathbb{R}) \setminus \mathcal{I}(\gamma) = \{\varphi \in C^\gamma(\mathbb{R}): \delta_0 \text{ is a GIBBS state of } f \text{ for the potential } \varphi\}.$$

*Furthermore, this last set is convex and its interior is equal to  $S(\gamma)$ .*

*Proof.* Putting  $\widehat{\mathcal{S}}(\gamma) := C^\gamma(\mathbb{R}) \setminus \mathcal{I}(\gamma)$ , by Theorem 1 we have

$$(3.22) \quad \begin{aligned} \widehat{\mathcal{S}}(\gamma) &= \{\varphi \in C^\gamma(\mathbb{R}) : P(\varphi) = \varphi(0)\} \\ &= \{\varphi \in C^\gamma(\mathbb{R}) : \delta_0 \text{ is a GIBBS state of } f \text{ for the potential } \varphi\}. \end{aligned}$$

In particular,  $\mathcal{S}(\gamma) \subseteq \text{int}(\widehat{\mathcal{S}}(\gamma))$ . It also follows that  $\widehat{\mathcal{S}}(\gamma)$  is convex, because  $P$  is convex and  $\varphi \mapsto \varphi(0)$  is linear.

To complete the proof of the last assertion, let  $\varphi$  be in  $\text{int}(\widehat{\mathcal{S}}(\gamma))$  and let  $\beta_*$  in  $(0, 1)$  be sufficiently close to 1 so that  $\beta_*\varphi$  is in  $\widehat{\mathcal{S}}(\gamma)$ . Then,  $P(\beta_*\varphi) = \beta_*\varphi(0)$  and by Corollary 4 in §2.2 with  $\beta = 1$  we have that  $\delta_0$  is the unique GIBBS state of  $f$  for the potential  $\varphi$ . Since this holds for every potential in  $\text{int}(\widehat{\mathcal{S}}(\gamma))$  and this set is open, we conclude  $\text{int}(\widehat{\mathcal{S}}(\gamma)) \subseteq \mathcal{S}(\gamma)$ . The proof of the lemma is thus complete.  $\square$

**Lemma 3.4.** *Let  $\gamma$  in  $(0, \min\{\alpha, 1\}]$  and  $\psi_0$  in  $C^\gamma(\mathbb{R}) \setminus \mathcal{I}(\gamma)$  be given. Then, for every  $\tau$  in  $(0, +\infty)$  the potential  $\psi_0 + \tau\omega_\gamma$  is in  $\mathcal{S}(\gamma)$ .*

*Proof.* Let  $\Delta$  in  $C^\gamma(\mathbb{R})$  be such that  $|\Delta|_\gamma \leq \tau$ . By (3.5) with  $\psi = -\Delta$ , we have

$$(3.23) \quad \psi_0 + \tau\omega_\gamma + \Delta \leq \psi_0 + \Delta(0).$$

Together with our hypothesis that  $\psi_0$  is outside  $\mathcal{I}(\gamma)$  and Theorem 1, this implies

$$(3.24) \quad \begin{aligned} P(\psi_0 + \tau\omega_\gamma + \Delta) &\leq P(\psi_0 + \Delta(0)) = P(\psi_0) + \Delta(0) = (\psi_0 + \Delta)(0) \\ &= h_{\delta_0} + \int \psi_0 + \Delta \, d\delta_0 \leq P(\psi_0 + \tau\omega_\gamma + \Delta). \end{aligned}$$

Hence  $\delta_0$  is a GIBBS state for the potential  $\psi_0 + \tau\omega_\gamma + \Delta$ . In view of Lemma 3.3, we conclude that  $\psi_0 + \tau\omega_\gamma$  is in  $\mathcal{S}(\gamma)$ .  $\square$

*Proof of Proposition 3.1.* To prove that  $\mathcal{I}(\gamma)$  contains  $\mathbf{0}$ , let  $\nu_0$  in  $\mathcal{M}$  be such that  $h_{\nu_0} > 0$  (Lemma 2.4(1)) and note that

$$(3.25) \quad \mathbf{0}(0) = 0 < h_{\nu_0} + \int \mathbf{0} \, d\nu_0 \leq P(\mathbf{0}).$$

Together with Theorem 1, this implies that  $\mathbf{0}$  is in  $\mathcal{I}(\gamma)$ .

To prove (3.1), suppose  $\alpha < 1$  and  $\gamma > \alpha$ . Let  $\varphi$  in  $C^\gamma(\mathbb{R})$  be given and observe that by (3.5) with  $\psi = \varphi$  and Proposition 3.2(1) with  $\beta = |\varphi|_\gamma$ , we have

$$(3.26) \quad \varphi - \varphi(0) \geq |\varphi|_\gamma \omega_\gamma \text{ and } P(\varphi) - \varphi(0) = P(\varphi - \varphi(0)) \geq P(|\varphi|_\gamma \omega_\gamma) > 0.$$

Together with Theorem 1, this implies that  $\varphi$  is in  $\mathcal{I}(\gamma)$ .

Henceforth, suppose  $\gamma \leq \alpha$ . To prove item 1, recall that  $\mathcal{I}(\gamma)$  contains  $\mathbf{0}$  and let  $\varphi$  in  $\mathcal{I}(\gamma)$  be given. By Theorem 1, for every  $\beta$  in  $(0, 1)$  we have

$$(3.27) \quad P(\beta\varphi) = \sup_{\nu \in \mathcal{M}} \left\{ h_\nu + \beta \int \varphi \, d\nu \right\} \geq \beta P(\varphi) > \beta\varphi(0).$$

Using Theorem 1 again, we conclude that  $\beta\varphi$  is in  $\mathcal{I}(\gamma)$ .

To prove item 2, note that by Proposition 3.2(2) there is  $\beta_*$  in  $(0, +\infty)$  such that  $\beta_*\omega_\gamma$  is outside  $\mathcal{I}(\gamma)$ . Combined with Lemma 3.4 with  $\psi_0 = \beta_*\omega_\gamma$ , this implies that  $\mathcal{S}(\gamma)$  is unbounded and therefore nonempty. That  $\mathcal{S}(\gamma)$  is convex and the last assertion of item 2, both follow from Lemma 3.3.

To prove item 3, first note that  $S(\gamma)$  is open by definition,  $\mathcal{I}(\gamma)$  is open by Theorem 1, and  $\mathcal{I}(\gamma)$  and  $S(\gamma)$  are disjoint by definition. On the other hand,  $S(\gamma)$  is equal to the interior of  $C^\gamma(\mathbb{R}) \setminus \mathcal{I}(\gamma)$  by Lemma 3.3. This implies that  $S(\gamma)$  is a regular open set, that

$$(3.28) \quad \partial S(\gamma) = C^\gamma(\mathbb{R}) \setminus (\mathcal{I}(\gamma) \cup S(\gamma)),$$

and that  $\mathcal{I}(\gamma) \cup S(\gamma)$  is dense in  $C^\gamma(\mathbb{R})$ . To prove the remaining assertions, we use that the set  $C^\gamma(\mathbb{R}) \setminus \mathcal{I}(\gamma)$  is convex Lemma 3.3 and that its interior  $S(\gamma)$  is nonempty. It follows that  $C^\gamma(\mathbb{R}) \setminus \mathcal{I}(\gamma)$  is equal to the closure of its interior and, as a consequence, the open set  $\mathcal{I}(\gamma)$  is regular and  $\partial \mathcal{I}(\gamma) = \partial S(\gamma)$ . This completes the proof of item 3 and of the proposition.  $\square$

**3.2. The phase transition locus.** In this section we prove Theorem A. The proof is given at the end of this section, after a couple of lemmas.

**Lemma 3.5.** *Let  $\gamma$  in  $(0, \min\{\alpha, 1\}]$  and  $\varphi_0$  in  $C^\gamma(\mathbb{R})$  be given. Then, there is  $\tau_0$  in  $\mathbb{R}$  such that for every  $\tau$  in  $\mathbb{R}$  we have*

$$(3.29) \quad \varphi_0 + \tau \omega_\gamma \in \begin{cases} \mathcal{I}(\gamma) & \text{if } \tau < \tau_0; \\ \mathcal{PT}(\gamma) & \text{if } \tau = \tau_0; \\ S(\gamma) & \text{if } \tau > \tau_0. \end{cases}$$

*Proof.* For each  $\tau$  in  $\mathbb{R}$  put  $\psi_\tau := \varphi_0 + \tau \omega_\gamma$  and let  $g: \mathbb{R} \rightarrow [0, +\infty)$  be defined by

$$(3.30) \quad g(\tau) := P(\psi_\tau) - \psi_\tau(0).$$

Using  $\omega_\gamma \leq 0$  and  $\omega_\gamma(0) = 0$ , for all  $\tau$  and  $\tau'$  in  $\mathbb{R}$  satisfying  $\tau \leq \tau'$  we obtain

$$(3.31) \quad \psi_\tau \geq \psi_{\tau'} \text{ and } g(\tau) = P(\psi_\tau) - \psi_\tau(0) \geq P(\psi_{\tau'}) - \psi_{\tau'}(0) = g(\tau').$$

In particular,  $g$  is nonincreasing. On the other hand, using that  $\delta_1$  is in  $\mathcal{M}$  we obtain that for every sufficiently negative  $\tau$  in  $\mathbb{R}$  we have

$$(3.32) \quad g(\tau) \geq \int \psi_\tau \, d\delta_1 - \psi_\tau(0) = \varphi_0(1) - \tau - \varphi_0(0) > 0.$$

Our next step is to show that  $g$  attains the value 0. To do this, let  $\beta_0$  in  $(0, +\infty)$  be such that  $P(\beta_0 \omega_\gamma) = 0$  (Proposition 3.2(2)) and put  $\tau_\dagger := \beta_0 + |\varphi_0|_\gamma$ . By (3.5) with  $\psi = -\varphi_0$ , we have

$$(3.33) \quad \psi_{\tau_\dagger} = \varphi_0 + (\beta_0 + |\varphi_0|_\gamma) \omega_\gamma \leq \beta_0 \omega_\gamma + \varphi_0(0)$$

and therefore

$$(3.34) \quad P(\psi_{\tau_\dagger}) \leq P(\beta_0 \omega_\gamma + \varphi_0(0)) = P(\beta_0 \omega_\gamma) + \varphi_0(0) = \varphi_0(0) = h_{\delta_0} + \int \psi_{\tau_\dagger} \, d\delta_0 \leq P(\psi_{\tau_\dagger}).$$

This implies  $g(\tau_\dagger) = 0$ , as wanted. It follows that there is  $\tau_0$  in  $\mathbb{R}$  such that  $g$  is strictly positive on  $(-\infty, \tau_0)$  and vanishes identically on  $[\tau_0, +\infty)$ .

In view of Theorem 1 and the definition of  $\mathcal{PT}(\gamma)$ , we conclude that for every  $\tau$  in  $(-\infty, \tau_0)$  the potential  $\psi_\tau$  is in  $\mathcal{I}(\gamma)$  and that for  $\tau = \tau_0$  the potential  $\psi_{\tau_0}$  is in  $\mathcal{PT}(\gamma)$ . Using Lemma 3.4 with  $\psi_0 = \psi_{\tau_0}$ , we obtain that for every  $\tau$  in  $(\tau_0, +\infty)$  the potential  $\psi_\tau$  is in  $S(\gamma)$ .  $\square$

**Lemma 3.6.** *Let  $\gamma$  be in  $(0, \min\{\alpha, 1\}]$ , let  $I$  be an open interval of  $\mathbb{R}$ , and let  $(\varphi_\tau)_{\tau \in I}$  a real-analytic family of potentials in  $C^\gamma(\mathbb{R})$ . Suppose that there is a parameter  $\tau_0$  in  $I$ , such that for every  $\tau$  in  $I$  the potential  $\varphi_\tau$  is in  $\mathcal{I}(\gamma)$  if  $\tau < \tau_0$  and it is outside  $\mathcal{I}(\gamma)$  if  $\tau \geq \tau_0$ . Then,  $\tau \mapsto P(\varphi_\tau)$  fails to be real-analytic at  $\tau_0$ .*

*Proof.* Let  $p: I \rightarrow [0, +\infty)$  be defined by

$$(3.35) \quad p(\tau) := P(\varphi_\tau) - \varphi_\tau(0).$$

In view of Theorem 1, our hypotheses imply that  $p$  is strictly positive on  $(-\infty, \tau_0)$  and that it vanishes identically on  $[\tau_0, +\infty)$ . In particular,  $p$  fails to be real-analytic at  $\tau_0$ . Since the function  $\varphi \mapsto \varphi(0)$  is real-analytic on  $C^\gamma(\mathbb{R})$ , it follows that  $\tau \mapsto P(\varphi_\tau)$  also fails to be real-analytic at  $\tau_0$ .  $\square$

*Proof of Theorem A.* The pressure function  $P$  is real-analytic on  $\mathcal{I}(\gamma)$  (Theorem 1) and it coincides with the real-analytic function  $\varphi \mapsto \varphi(0)$  on  $\mathcal{S}(\gamma)$ . It is thus sufficient to show that for every  $\varphi_0$  in  $\mathcal{PT}(\gamma)$  the pressure function  $P$  fails to be real analytic at  $\varphi_0$ . To do this, consider the real-analytic family  $(\varphi_0 + \tau\omega_\gamma)_{\tau \in \mathbb{R}}$  of potentials in  $C^\gamma(\mathbb{R})$ . By Lemma 3.5 there is  $\tau_0$  in  $\mathbb{R}$  such that for every  $\tau$  in  $\mathbb{R}$  the potential  $\varphi_\tau$  is in  $\mathcal{I}(\gamma)$  if  $\tau < \tau_0$  and it is outside  $\mathcal{I}(\gamma)$  if  $\tau \geq \tau_0$ . Then the function  $\tau \mapsto P(\varphi_\tau)$  fails to be real-analytic at  $\tau_0$  by Lemma 3.6 and therefore the pressure function  $P$  fails to be real-analytic at  $\varphi_0$ .  $\square$

**3.3. Receding from zero temperature.** In this section, we relate the geometry at infinity of the stationary phase and the phase transition locus to the set potentials for which  $\delta_0$  is a ground state of  $f$ . This is summarized in Proposition 3.7 below.

Let  $V$  be a real vector space. A subset  $C$  of  $V$  is a *cone*, if for all  $v$  in  $C$  and  $\beta$  in  $(0, +\infty)$  the vector  $\beta v$  is in  $C$ . In this case,  $C$  is *convex* if for all  $v$  and  $v'$  in  $C$  the vector  $v + v'$  is in  $C$  and it is *salient* if  $C \cap -C$  is reduced to the zero vector of  $V$ . On the other hand, the *recession cone* of a subset  $S$  of  $V$ , is the convex cone  $\text{recc}(S)$  defined by

$$(3.36) \quad \text{recc}(S) := \{v \in V: \text{ for all } s \text{ in } S \text{ and } t \text{ in } (0, +\infty), s + tv \in S\}.$$

Moreover, the *lineality space* of  $S$  is the vector subspace  $\text{recc}(S) \cap -\text{recc}(S)$  of  $V$ .

Recall from (1.8) that for each  $\gamma$  in  $(0, 1]$  we denote by  $\mathcal{G}(\gamma)$  the set of potentials in  $C^\gamma(\mathbb{R})$  for which  $\delta_0$  is a ground state of  $f$ . We also consider the closed vector subspace  $\mathcal{G}(\gamma)$  of  $C^\gamma(\mathbb{R})$ , defined by

$$(3.37) \quad \mathcal{G}(\gamma) := \left\{ \varphi \in C^\gamma(\mathbb{R}): \text{ for every } v \text{ in } \mathcal{M}, \int \varphi \, d\nu = \varphi(0) \right\}.$$

**Proposition 3.7.** *For every  $\gamma$  in  $(0, \min\{\alpha, 1\}]$ , the following properties hold.*

1. We have

$$(3.38) \quad \text{recc}(S(\gamma)) = \text{recc}(S(\gamma) \cup \mathcal{PT}(\gamma)) = \mathcal{G}(\gamma).$$

2. The lineality spaces of  $S(\gamma)$  and  $S(\gamma) \cup \mathcal{PT}(\gamma)$  are both equal to  $\mathcal{G}(\gamma)$ . Moreover, this vector subspace of  $C^\gamma(\mathbb{R})$  is of infinite co-dimension.

*Proof.* To prove item 1, we begin showing

$$(3.39) \quad \mathcal{G}(\gamma) \subseteq \text{recc}(S(\gamma)).$$

Let  $\varphi$  in  $\mathcal{G}(\gamma)$  and  $\varphi_0$  in  $S(\gamma)$  be given and let  $\nu_0$  be a GIBBS state of  $f$  for the potential  $\varphi_0 + \varphi$ . Then we have

$$(3.40) \quad P(\varphi_0 + \varphi) = h_{\nu_0} + \int \varphi_0 + \varphi \, d\nu_0 \leq P(\varphi_0) + \int \varphi \, d\nu_0 \leq P(\varphi_0) + \varphi(0) = (\varphi_0 + \varphi)(0)$$

and therefore  $\varphi_0 + \varphi$  is outside  $\mathcal{I}(\gamma)$  by Theorem 1. Since this holds for every  $\varphi_0$  in  $\mathcal{S}(\gamma)$ , we conclude

$$(3.41) \quad \mathcal{S}(\gamma) + \varphi \subseteq C^\gamma(\mathbb{R}) \setminus \mathcal{I}(\gamma).$$

Using that  $\mathcal{S}(\gamma)$  is open and Proposition 3.1(3), we obtain

$$(3.42) \quad \mathcal{S}(\gamma) + \varphi \subseteq \text{int}(C^\gamma(\mathbb{R}) \setminus \mathcal{I}(\gamma)) = \mathcal{S}(\gamma).$$

Since  $\mathcal{S}(\gamma)$  is convex by Proposition 3.1(2), this yields that  $\varphi$  is in  $\text{recc}(\mathcal{S}(\gamma))$  and proves (3.39).

The inclusion  $\text{recc}(\mathcal{S}(\gamma)) \subseteq \text{recc}(\mathcal{S}(\gamma) \cup \mathcal{PT}(\gamma))$  being trivial, to complete the proof of item 1 it is sufficient to show

$$(3.43) \quad \text{recc}(\mathcal{S}(\gamma) \cup \mathcal{PT}(\gamma)) \subseteq \mathcal{G}(\gamma).$$

To do this, let  $\varphi$  in  $\text{recc}(\mathcal{S}(\gamma) \cup \mathcal{PT}(\gamma))$  be given, let  $\nu_\dagger$  in  $\mathcal{M}$  be such that

$$(3.44) \quad \int \varphi \, d\nu_\dagger = \sup_{\nu \in \mathcal{M}} \int \varphi \, d\nu,$$

and choose an arbitrary  $\varphi_S$  in  $\mathcal{S}(\gamma)$ . Then, for every  $t$  in  $(0, +\infty)$  the potential  $\varphi_S + t\varphi$  is in  $\mathcal{S}(\gamma) \cup \mathcal{PT}(\gamma)$ . Together with Proposition 3.1(3) and Theorem 1, this implies

$$(3.45) \quad (\varphi_S + t\varphi)(0) = P(\varphi_S + t\varphi) \geq \int \varphi_S + t\varphi \, d\nu_\dagger = \int \varphi_S \, d\nu_\dagger + t \sup_{\nu \in \mathcal{M}} \int \varphi \, d\nu.$$

Dividing by  $t$  and rearranging, we obtain

$$(3.46) \quad \varphi(0) \geq \sup_{\nu \in \mathcal{M}} \int \varphi \, d\nu - \frac{1}{t} \varphi_S(0) + \frac{1}{t} \int \varphi_S \, d\nu_\dagger \geq \sup_{\nu \in \mathcal{M}} \int \varphi \, d\nu - \frac{2}{t} \|\varphi_S\|.$$

Taking the limit as  $t \rightarrow +\infty$ , we conclude that  $\varphi$  is in  $\mathcal{G}(\gamma)$ . This completes the proof of (3.43) and therefore of (3.38) and item 1.

To prove item 2, note first that for every  $\varphi$  in  $\mathcal{G}(\gamma) \cap -\mathcal{G}(\gamma)$  we have

$$(3.47) \quad \sup_{\nu \in \mathcal{M}} \int \varphi \, d\nu = \varphi(0) = \inf_{\nu \in \mathcal{M}} \int \varphi \, d\nu,$$

so for every  $\nu$  in  $\mathcal{M}$  we have  $\int \varphi \, d\nu = \varphi(0)$  and therefore  $\varphi$  is in  $\mathcal{G}(\gamma)$ . Conversely, every  $\varphi$  in  $\mathcal{G}(\gamma)$  satisfies (3.47) and is therefore in  $\mathcal{G}(\gamma) \cap -\mathcal{G}(\gamma)$ .

To prove the last statement of item 2, denote by  $\mathcal{M}'$  the infinite subset of  $\mathcal{M}$  of those measures supported on a single periodic orbit of  $f$ . Moreover, for each  $\nu$  in  $\mathcal{M}'$  let  $\ell_\nu$  be the linear functional of  $C^\gamma(\mathbb{R})$  defined by

$$(3.48) \quad \ell_\nu(\varphi) := \varphi(0) - \int \varphi \, d\nu.$$

Then

$$(3.49) \quad \mathcal{G}(\gamma) \subseteq \bigcap_{\nu \in \mathcal{M}'} \text{Ker}(\ell_\nu).$$

On the other hand, the set of linear functionals  $\{\ell_\nu: \nu \in \mathcal{M}'\}$  is linearly independent, because the supports of the measures in  $\mathcal{M}'$  are pairwise disjoint. This implies that the right side of (3.49) has infinite co-dimension in  $C^\gamma(\mathbb{R})$  and therefore that the same holds for  $\mathcal{G}(\gamma)$ . This completes the proof of item 2 and of the proposition.  $\square$

**3.4. Structure of the phase transition locus.** This section establishes fundamental properties of the phase transition locus. These are summarized in the proposition below, from which we derive Theorem B at the end of this section.

Note that for every  $\gamma$  in  $(0, \min\{\alpha, 1\}]$ , the GIBBS states of  $f$  for 2 potentials that differ by an element of  $\mathcal{C}(\gamma)$ , coincide. Thus, each of the sets  $\mathcal{I}(\gamma)$ ,  $\mathcal{S}(\gamma)$ , and  $\mathcal{PT}(\gamma)$  is invariant under translations by elements of  $\mathcal{C}(\gamma)$ . In particular,  $\mathcal{PT}(\gamma)$  contains a translate of  $\mathcal{C}(\gamma)$ .

**Proposition 3.8.** *For every  $\gamma$  in  $(0, \min\{\alpha, 1\}]$ , the following properties hold.*

1. *The phase transition locus  $\mathcal{PT}(\gamma)$  is linear homeomorphic to the graph of a real-valued convex function. More precisely, there is a co-dimension 1 vector subspace  $\mathcal{K}$  of  $C^\gamma(\mathbb{R})$ , a convex function  $\tau: \mathcal{K} \rightarrow (0, +\infty)$ , and a linear homeomorphism from  $C^\gamma(\mathbb{R})$  onto  $\mathcal{K} \times \mathbb{R}$  that is the identity on  $\mathcal{K}$  and that maps  $\mathcal{PT}(\gamma)$  onto the graph of  $\tau$ .*
2. *Every affine space contained in  $\mathcal{PT}(\gamma) \cup \mathcal{S}(\gamma)$  is a translate of a vector subspace of  $\mathcal{C}(\gamma)$  and it is of infinite co-dimension in  $C^\gamma(\mathbb{R})$ .*

As mentioned above, for every  $\gamma$  in  $(0, \min\{\alpha, 1\}]$  each of the sets  $\mathcal{I}(\gamma)$ ,  $\mathcal{S}(\gamma)$ , and  $\mathcal{PT}(\gamma)$  is invariant under translations by elements of  $\mathcal{C}(\gamma)$ . That is,

$$(3.50) \quad \mathcal{I}(\gamma) + \mathcal{C}(\gamma) = \mathcal{I}(\gamma), \quad \mathcal{S}(\gamma) + \mathcal{C}(\gamma) = \mathcal{S}(\gamma), \quad \text{and } \mathcal{PT}(\gamma) + \mathcal{C}(\gamma) = \mathcal{PT}(\gamma).$$

Thus, the phase diagram for the intermittent map  $f$  descends naturally to the quotient space  $C^\gamma(\mathbb{R})/\mathcal{C}(\gamma)$ . Proposition 3.8(2) implies that the (convex and unbounded) set

$$(3.51) \quad (\mathcal{PT}(\gamma) \cup \mathcal{S}(\gamma))/\mathcal{C}(\gamma)$$

contains no nontrivial affine subspace of  $C^\gamma(\mathbb{R})/\mathcal{C}(\gamma)$ . Equivalently, the recession cone of this set is salient.

*Proof of Proposition 3.8.* Fix  $\nu$  in  $\mathcal{M} \setminus \{\delta_0\}$  and let  $\ell: C^\gamma(\mathbb{R}) \rightarrow \mathbb{R}$  be the continuous linear functional defined by

$$(3.52) \quad \ell(\psi) := \psi(0) - \int \psi \, d\nu.$$

Our hypothesis  $\nu \neq \delta_0$  implies

$$(3.53) \quad \ell(\omega_\gamma) = \int x^\gamma \, d\nu(x) > 0.$$

We can thus define the linear projection  $P: C^\gamma(\mathbb{R}) \rightarrow \text{Ker}(\ell)$  by

$$(3.54) \quad P(\varphi) := \varphi - \frac{\ell(\varphi)}{\ell(\omega_\gamma)} \omega_\gamma$$

and a linear map  $C^\gamma(\mathbb{R}) \rightarrow \text{Ker}(\ell) \times \mathbb{R}$  by

$$(3.55) \quad \varphi \mapsto \left( P(\varphi), \frac{\ell(\varphi)}{\ell(\omega_\gamma)} \right).$$

This last map is a homeomorphism with inverse  $(\psi, t) \mapsto \psi + t\omega_\gamma$ .

For each  $\varphi$  in  $\text{Ker}(\ell)$ , denote by  $\tau(\varphi)$  the real number  $\tau_0$  given by Lemma 3.5 with  $\varphi_0 = \varphi$ . This lemma implies that the homeomorphism (3.55) maps  $\mathcal{PT}(\gamma) \cup \mathcal{S}(\gamma)$  onto the epigraph of the function  $\tau: \text{Ker}(\ell) \rightarrow \mathbb{R}$  so defined. Since  $\mathcal{PT}(\gamma) \cup \mathcal{S}(\gamma)$  is convex by Proposition 3.1(2, 3), it follows that the function  $\tau$  is convex and therefore continuous.

By construction, the homeomorphism (3.55) maps  $\mathcal{PT}(\gamma)$  onto the graph of  $\tau$ . Hence, to complete the proof of the desired assertion with  $\mathcal{K} = \text{Ker}(\ell)$ , it only remains to show that the function  $\tau$  takes values in  $(0, +\infty)$ . To do this, let  $\varphi$  in  $\text{Ker}(\ell)$  be given and recall that  $\varphi + \tau(\varphi)\omega_\gamma$  is in  $\mathcal{PT}(\gamma)$  by the definition of  $\tau(\varphi)$ . Using Theorem 1, the hypothesis that  $\varphi$  is in  $\text{Ker}(\ell)$ , and the Key Lemma in §2.2, we obtain

$$(3.56) \quad P(\varphi + \tau(\varphi)\omega_\gamma) = (\varphi + \tau(\varphi)\omega_\gamma)(0) = \varphi(0) = \int \varphi \, d\nu < P(\varphi).$$

Together with  $\omega_\gamma \leq 0$ , this implies  $\tau(\varphi) > 0$ .

To prove item 2, let  $\mathcal{A}$  be an affine subspace of  $C^\gamma(\mathbb{R})$  contained in  $\mathcal{PT}(\gamma) \cup \mathcal{S}(\gamma)$ . The vector subspace of  $C^\gamma(\mathbb{R})$  parallel to  $\mathcal{A}$  is contained in the linearity space of  $\mathcal{G}(\gamma)$ , which is equal to  $\mathcal{C}(\gamma)$  and has infinite co-dimension in  $C^\gamma(\mathbb{R})$  by Proposition 3.7(2). It follows that  $\mathcal{A}$  also has infinite co-dimension in  $C^\gamma(\mathbb{R})$ .  $\square$

*Proof of Theorem B.* Let  $\mathcal{K}$  and  $\tau$  be given by Proposition 3.8(1) and denote by

$$(3.57) \quad L: C^\gamma(\mathbb{R}) \rightarrow \mathcal{K} \times \mathbb{R}$$

the linear homeomorphism given by this result. The map  $H: \mathcal{K} \times \mathbb{R} \rightarrow \mathcal{K} \times \mathbb{R}$  defined by

$$(3.58) \quad H(\varphi_0, h) := (\varphi_0, h - \tau(\varphi_0))$$

is a homeomorphism mapping the graph of  $\tau$  onto  $\mathcal{K} \times \{0\}$ . It follows from Proposition 3.8(1) that  $L^{-1} \circ H \circ L$  is a homeomorphism from  $C^\gamma(\mathbb{R})$  onto itself mapping  $\mathcal{PT}(\gamma)$  onto  $\mathcal{K}$ . This implies that  $\mathcal{PT}(\gamma)$  is a submanifold of co-dimension 1 of  $C^\gamma(\mathbb{R})$ .

Finally, note that if  $\mathcal{PT}(\gamma)$  were an affine subspace of  $C^\gamma(\mathbb{R})$ , then it would be of co-dimension 1. This would contradict Proposition 3.8(2), so  $\mathcal{PT}(\gamma)$  cannot be an affine subspace of  $C^\gamma(\mathbb{R})$ . The proof of the theorem is thus complete.  $\square$

#### 4. ON THE PERSISTENCE OF PHASE TRANSITIONS IN TEMPERATURE

This section investigates the persistence of phase transitions in temperature. In §4.1, we relate, for each  $\gamma$  in  $(0, \min\{\alpha, 1\}]$ , the stationary phase  $\mathcal{S}(\gamma)$  to the interior of the cone  $\mathcal{G}(\gamma)$  (Proposition 4.1) and prove Theorem C. In §4.2, we state and prove our results in the case where  $\gamma < \alpha$  (Proposition 4.4). In §4.3, we give several characterizations of the potentials that undergo a phase transition in temperature in the remaining case, where  $\alpha \leq 1$  and  $\gamma = \alpha$  (Theorem D' in §4.3). This characterization is the main ingredient in the proof of Theorem D, which is given in §4.4.

**4.1. From zero to low temperatures.** In this section, we prove Theorem C. The main new ingredient is the following proposition.

**Proposition 4.1.** *For every  $\gamma$  in  $(0, \min\{\alpha, 1\}]$ , we have*

$$(4.1) \quad \text{int}(\mathcal{G}(\gamma)) = \bigcup_{\beta \in (0, +\infty)} \beta \mathcal{S}(\gamma).$$

*In particular,*

$$(4.2) \quad \mathcal{S}(\gamma) \subseteq \text{int}(\mathcal{G}(\gamma)) \text{ and } \mathcal{PT}(\gamma) \subseteq \text{cl}(\mathcal{S}(\gamma)) \subseteq \mathcal{G}(\gamma).$$

*Proof.* For all  $\beta$  in  $(0, +\infty)$  and  $\varphi$  in  $\beta S(\gamma)$ , we have

$$(4.3) \quad \beta^{-1}\varphi(0) = P(\beta^{-1}\varphi) \geq \sup_{v \in \mathcal{M}} \int \beta^{-1}\varphi \, d\nu \geq \beta^{-1}\varphi(0).$$

This implies that  $\varphi$  is in  $\mathcal{G}(\gamma)$  and  $\beta S(\gamma) \subseteq \mathcal{G}(\gamma)$ . Since  $S(\gamma)$  is open and this holds for every  $\beta$  in  $(0, +\infty)$ , we obtain

$$(4.4) \quad \bigcup_{\beta \in (0, +\infty)} \beta S(\gamma) \subseteq \text{int}(\mathcal{G}(\gamma)).$$

To prove the reverse inclusion, let  $\varphi$  in  $\text{int}(\mathcal{G}(\gamma))$  be given and choose an arbitrary  $\varphi_*$  in  $S(\gamma)$ . If  $\varphi$  and  $\varphi_*$  are linearly dependent, then there is  $\beta$  in  $(0, +\infty)$  such that  $\varphi$  is in  $\beta S(\gamma)$ . Suppose  $\varphi$  and  $\varphi_*$  are linearly independent, denote by  $\mathcal{P}$  the plane in  $C^\gamma(\mathbb{R})$  generated by  $\varphi$  and  $\varphi_*$ , and put

$$(4.5) \quad \mathcal{G} := \mathcal{G}(\gamma) \cap \mathcal{P}.$$

The set  $\mathcal{G}$  is a closed convex cone and therefore there are linear functionals  $\ell$  and  $\ell'$  of  $\mathcal{P}$ , so that

$$(4.6) \quad \mathcal{G} = \{\psi \in \mathcal{P}: \ell(\psi) \geq 0, \ell'(\psi) \geq 0\}.$$

Since  $\varphi$  is in the interior of  $\mathcal{G}(\gamma)$ , we have  $\ell(\varphi) > 0$  and  $\ell'(\varphi) > 0$ . Let  $t$  in  $(0, +\infty)$  be sufficiently large so that

$$(4.7) \quad t\ell(\varphi) \geq \ell(\varphi_*) \text{ and } t\ell'(\varphi) \geq \ell'(\varphi_*),$$

and put  $\hat{\varphi} := t\varphi - \varphi_*$ . Then  $\ell(\hat{\varphi}) \geq 0$ ,  $\ell'(\hat{\varphi}) \geq 0$ , and  $\hat{\varphi}$  is in  $\mathcal{G}$  by (4.6). In particular,  $\hat{\varphi}$  is in  $\mathcal{G}(\gamma)$  and therefore in  $\text{recc}(S(\gamma))$  by (3.38) in Proposition 3.7(1). Since  $\varphi_*$  is in  $S(\gamma)$  and  $t\varphi = \varphi_* + \hat{\varphi}$ , we conclude that  $t\varphi$  is in  $S(\gamma)$ . This proves that  $\varphi$  is in  $t^{-1}S(\gamma)$  and completes the proof of (4.1).

The first inclusion in (4.2) is a direct consequence of (4.1), the second is a direct consequence of the definition of  $\mathcal{PT}(\gamma)$ , and the last is a direct consequence of the first inclusion. The proof of the proposition is thus complete.  $\square$

The proof of Theorem C also relies on the following lemma.

**Lemma 4.2.** *Let  $\gamma$  in  $(0, \min\{\alpha, 1\}]$  and  $\varphi$  in  $\text{int}(\mathcal{G}(\gamma))$  be given. Then  $\varphi$  undergoes a phase transition in temperature and, if we denote by  $\beta_*$  the phase transition parameter of  $\varphi$ , then for every  $\beta$  in  $(\beta_*, +\infty)$  the potential  $\beta\varphi$  is in  $S(\gamma)$ .*

*Proof.* By Proposition 4.1 there is  $\beta_0$  in  $(0, +\infty)$  such that  $\beta_0 S(\gamma)$  contains  $\varphi$ . It follows that  $\varphi$  undergoes a phase transition in temperature by Corollary 1 in §1.2. Denote by  $\beta_*$  the phase transition parameter of  $\varphi$  and note that  $\beta_*\varphi$  is in  $\partial S(\gamma)$  and that  $\beta_* < \beta_0^{-1}$  by Corollary 1 in §1.2. It follows that  $\beta_*\varphi$  is in  $\partial S(\gamma)$  by Proposition 3.1(3). On the other hand,  $\varphi$  is in  $\text{recc}(S(\gamma))$  by Proposition 3.7(1) and therefore for every  $\beta$  in  $[\beta_0^{-1}, +\infty)$  the potential  $\beta\varphi$  is in  $S(\gamma)$ .

To complete the proof of the lemma, let  $\beta$  in  $(\beta_*, \beta_0^{-1})$  be given and put

$$(4.8) \quad \eta_* := \frac{\beta_0^{-1} - \beta}{\beta_0^{-1} - \beta_*}, \quad \eta_0 := \frac{\beta - \beta_*}{\beta_0^{-1} - \beta_*}, \quad \text{and} \quad \mathcal{U} := \eta_*(\beta_*\varphi) + \eta_0\beta_0^{-1}S(\gamma).$$

Note that  $\eta_*$  and  $\eta_0$  are both in  $(0, 1)$  and that we have

$$(4.9) \quad \eta_* + \eta_0 = 1 \text{ and } \eta_*\beta_* + \eta_0\beta_0^{-1} = \beta.$$

In particular,  $\mathcal{U}$  contains  $\beta\varphi$ . On the other hand, since  $\beta_*\varphi$  is in  $\partial\mathcal{S}(\gamma)$ ,  $\beta_0^{-1}\varphi$  is in  $\mathcal{S}(\gamma)$ , and  $\text{cl}(\mathcal{S}(\gamma))$  is convex by Proposition 3.1(2), we have  $\mathcal{U} \subseteq \text{cl}(\mathcal{S}(\gamma))$ . But  $\mathcal{U}$  is open, so  $\mathcal{U} \subseteq \mathcal{S}(\gamma)$  and therefore  $\beta\varphi$  is in  $\mathcal{S}(\gamma)$ . The proof of the lemma is thus complete.  $\square$

*Proof of Theorem C.* The last assertion is a direct consequence of Corollary 1 in §1.2 and Corollary 4 in §2.2.

To prove the implication  $1 \Rightarrow 2$ , suppose that  $\varphi$  undergoes a persistent phase transition in temperature in  $C^\gamma(\mathbb{R})$  and let  $\mathcal{U}$  be an open neighborhood of  $\varphi$  in  $C^\gamma(\mathbb{R})$  of potentials undergoing a phase transition in temperature. By Corollary 1 in §1.2, we have  $\mathcal{U} \subseteq \mathcal{G}(\gamma)$ . But  $\mathcal{U}$  is open, so  $\mathcal{U} \subseteq \text{int}(\mathcal{G}(\gamma))$  and therefore  $\varphi$  is in  $\text{int}(\mathcal{G}(\gamma))$ .

To prove the implication  $2 \Rightarrow 1$ , suppose that  $\varphi$  is in  $\text{int}(\mathcal{G}(\gamma))$  and let  $\eta$  in  $(0, 1)$  be given. By Lemma 4.2 the potential  $\varphi$  undergoes a phase transition in temperature. Denote by  $\beta_*$  the phase transition parameter of  $\varphi$  and put  $\varphi_* := \beta_*\varphi$ . Then  $\eta\varphi_*$  is in  $\mathcal{I}(\gamma)$  by Corollary 1 in §1.2 and  $\eta^{-1}\varphi_*$  is in  $\mathcal{S}(\gamma)$  by Lemma 4.2. Let  $\mathcal{U}_*$  be a sufficiently small neighborhood of  $\varphi_*$ , such that the open neighborhood  $\beta_*^{-1}\mathcal{U}_*$  of  $\varphi$  is contained in  $\text{int}(\mathcal{G}(\gamma))$  and we have

$$(4.10) \quad \eta\mathcal{U}_* \subseteq \mathcal{I}(\gamma) \text{ and } \eta^{-1}\mathcal{U}_* \subseteq \mathcal{S}(\gamma).$$

Thus, every  $\tilde{\varphi}$  in  $\beta_*^{-1}\mathcal{U}_*$  undergoes a phase transition in temperature by Lemma 4.2 and satisfies

$$(4.11) \quad (\eta\beta_*)\tilde{\varphi} \in \mathcal{I}(\gamma) \text{ and } (\eta^{-1}\beta_*)\tilde{\varphi} \in \mathcal{S}(\gamma).$$

Together with Corollary 1 in §1.2, this implies that the phase transition parameter  $\tilde{\beta}$  of  $\tilde{\varphi}$  satisfies

$$(4.12) \quad \eta\beta_* < \tilde{\beta} < \eta^{-1}\beta_*.$$

This proves that  $\varphi$  undergoes a persistent phase transition in temperature in  $C^\gamma(\mathbb{R})$  and completes the proof of the theorem.  $\square$

**4.2. Nonpersistent phase transitions in temperature when  $\gamma < \alpha$ .** In this section, we state and prove our results for  $\gamma$  in  $(0, 1]$  satisfying  $\gamma < \alpha$  (Proposition 4.4). We begin with the following result, which is also used in the proof of Theorem D in §4.4.

**Proposition 4.3.** *For every  $\gamma$  in  $(0, \min\{\alpha, 1\}]$ , the function  $\beta_{\text{crit},\gamma}$  is lower semicontinuous and satisfies*

$$(4.13) \quad \beta_{\text{crit},\gamma}^{-1}(1) \subseteq \mathcal{PT}(\gamma) \text{ and } \beta_{\text{crit},\gamma}^{-1}(1) \cap \text{int}(\mathcal{G}(\gamma)) = \mathcal{PT}(\gamma) \cap \text{int}(\mathcal{G}(\gamma)).$$

*Proof.* To prove that  $\beta_{\text{crit},\gamma}$  is lower semicontinuous, let  $\varphi$  in  $C^\gamma(\mathbb{R})$  and  $\beta$  in  $(0, \beta_{\text{crit},\gamma}(\varphi))$  be given. By Corollary 1 in §1.2 the potential  $\beta\varphi$  is in  $\mathcal{I}(\gamma)$ . Let  $\mathcal{V}$  be a neighborhood of  $\varphi$  in  $C^\gamma(\mathbb{R})$  such that  $\beta\mathcal{V} \subseteq \mathcal{I}(\gamma)$  and let  $\tilde{\varphi}$  be in  $\mathcal{V}$ . If  $\tilde{\varphi}$  undergoes a phase transition in temperature, then we have  $\beta_{\text{crit},\gamma}(\tilde{\varphi}) \geq \beta$  by Corollary 1 in §1.2. Otherwise,  $\beta_{\text{crit},\gamma}(\varphi) = +\infty$ . In all of the cases,  $\beta_{\text{crit},\gamma}(\tilde{\varphi}) \geq \beta$ .

The first inclusion in (4.13) is a direct consequence of Corollary 1 in §1.2, the definition of  $\beta_{\text{crit},\gamma}$ , and the equality  $\partial\mathcal{I}(\gamma) = \mathcal{PT}(\gamma)$ . The second one is given by (4.2) in Proposition 4.1.

To prove the equality in (4.13), let  $\varphi$  in  $\text{int}(\mathcal{G}(\gamma))$  be given. If  $\beta_{\text{crit},\gamma}(\varphi) = 1$ , then  $\varphi$  is in  $\mathcal{PT}(\gamma)$  by the first inclusion in (4.13). On the other hand,  $\varphi$  is in  $\mathcal{I}(\gamma)$  if  $\beta_{\text{crit},\gamma}(\varphi) > 1$  by

Corollary 1 in §1.2 and it is in  $S(\gamma)$  if  $\beta_{\text{crit},\gamma}(\varphi) < 1$  by Lemma 4.2. In both of these cases  $\varphi$  is outside  $\mathcal{PT}(\gamma)$ .  $\square$

The following proposition gathers our results for  $\gamma < \alpha$ . Recall that the geometric potential  $-\log Df$  undergoes a phase transition in temperature, and  $\beta_{\text{crit},\gamma}(-\log Df) = 1$ , see Proposition 4.5 below.

**Proposition 4.4.** *For every  $\gamma$  in  $(0, 1]$  satisfying  $\gamma < \alpha$ , the following properties hold.*

1. *The phase transition in temperature of  $-\log Df$  is nonpersistent in  $C^\gamma(\mathbb{R})$  and  $\beta_{\text{crit},\gamma}$  is discontinuous at  $-\log Df$ .*
2. *The potential  $-\log Df$  is in  $\partial\mathcal{G}(\gamma)$  and for each  $\beta$  in  $(0, +\infty)$  the potential  $-\beta \log Df$  is in  $\mathcal{PT}(\gamma)$  if and only if  $\beta$  is in  $[1, +\infty)$ .*
3. *The phase transition locus  $\mathcal{PT}(\gamma)$  fails to be a co-dimension 1 real-analytic subset of  $C^\gamma(\mathbb{R})$  at  $-\log Df$ .*

In particular, the set  $\mathcal{PT}(\gamma)$  is not contained in  $\text{int}(\mathcal{G}(\gamma))$ , the function  $\beta_{\text{crit},\gamma}$  is discontinuous, and  $\beta_{\text{crit},\gamma}^{-1}(1)$  is strictly contained in  $\mathcal{PT}(\gamma)$ .

To prove this proposition, we rely on the following well-known properties of the geometric potential. We provide a proof for completeness, see also [PS92] and [CR25, Proposition 4.2], as well as [CT13, Theorem 4.3] in the case where  $\alpha$  is in  $(0, 1)$ .

**Proposition 4.5.** *For every  $\beta$  in  $(0, +\infty)$ , we have  $P(-\beta \log Df) \geq 0$  with equality if and only if  $\beta$  is in  $[1, +\infty)$ . In particular,  $-\log Df$  undergoes a phase transition in temperature and*

$$(4.14) \quad \beta_{\text{crit},\gamma}(-\log Df) = 1$$

The proof of Proposition 4.5 relies on the the following lemma.

**Lemma 4.6** ([CR25, Lemma 3.2]). *There are  $\varepsilon_1$  and  $C_1$  in  $(0, +\infty)$ , such that the following properties hold. For every  $n$  in  $\mathbb{N}$  and every connected component  $J$  of  $f^{-n}(J_0)$ , we have*

$$(4.15) \quad |J| \leq \frac{|J_0|}{(1 + \varepsilon_1 n)^{\frac{1}{\alpha} + 1}}$$

and for all  $x$  and  $y$  in  $J$  we have

$$(4.16) \quad Df^n(x) \geq (1 + \varepsilon_1 n)^{\frac{1}{\alpha} + 1} \text{ and } C_1^{-1} \leq \frac{Df^n(x)}{Df^n(y)} \leq C_1.$$

*Proof of Proposition 4.5.* First note that for every  $\beta$  in  $\mathbb{R}$ , we have

$$(4.17) \quad P(-\beta \log Df) \geq h_{\delta_0} - \beta \int \log Df \, d\delta_0 = 0.$$

On the other hand, if  $C_1$  is the constant from Lemma 4.6, then for every  $n$  in  $\mathbb{N}$  we have

$$(4.18) \quad 1 \geq \sum_{\substack{J \text{ connected} \\ \text{component of } f^{-n}(J_0)}} |J| \geq C_1^{-1} |J_0| \sum_{x \in f^{-n}(1)} Df^n(x)^{-1}$$

and therefore

$$(4.19) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in f^{-n}(1)} Df^n(x)^{-1} \leq 0.$$

21

By (2.7) in Lemma 2.3 with  $\varphi = -\log Df$  and  $x = 1$ , the left side of the previous inequality is equal to  $P(-\log Df)$ . Thus, for every  $\beta$  in  $[1, +\infty)$  we have

$$(4.20) \quad P(-\beta \log Df) \leq P(-\log Df) \leq 0$$

and therefore  $P(-\beta \log Df) = 0$ . To complete the proof of item 1, it remains to prove that for every  $\beta$  in  $(0, 1)$  we have  $P(-\beta \log Df) > 0$ . For each  $n$  in  $\mathbb{N}$ , denote by  $K_n$  the maximal invariant set of  $f$  on  $[x_n, 1]$ . The map  $f$  is uniformly expanding and topologically exact on  $K_n$ . So, by Bowen's formula, for every  $\beta$  in  $(0, \text{HD}(K_n))$  we have

$$(4.21) \quad P(-\beta \log Df) \geq P(f|_{K_n}, -\beta \log Df) > 0,$$

see, for example, [PU10, Theorem 9.1.6 and Corollary 9.1.7]. Thus, to complete the proof of item 1 it sufficient to show

$$(4.22) \quad \sup_{n \in \mathbb{N}} \text{HD}(K_n) = 1.$$

This is given, for example, by [URM22, Theorem 19.6.4] applied to the "CGDMS" generated by the first return map of  $f$  to  $J_0$ .  $\square$

The proof of Proposition 4.3 also relies on the following lemma. Recall that for  $\gamma$  in  $(0, +\infty)$  we denote by  $\omega_\gamma$  the function in  $C^\gamma(\mathbb{R})$  given by  $\omega_\gamma(x) = -x^\gamma$ .

**Lemma 4.7.** *For every  $\gamma$  in  $(0, 1]$  satisfying  $\gamma < \alpha$  and every  $\tau$  in  $(-\infty, 0)$ , the potential  $\tau\omega_\gamma - \log Df$  is outside  $\mathcal{G}(\gamma)$  and satisfies*

$$(4.23) \quad \beta_{\text{crit}, \gamma}(\tau\omega_\gamma - \log Df) = +\infty.$$

*Proof.* Fix  $\tau$  in  $(-\infty, 0)$ , put

$$(4.24) \quad \varphi_0 := \tau\omega_\gamma - \log Df,$$

and let  $\varepsilon$  in  $(0, -\tau)$  be given. By our hypothesis  $\gamma < \alpha$ , there is  $N$  in  $\mathbb{N}$  such that for every  $x$  in  $[0, x_N]$  we have  $\varphi_0(x) \geq \varepsilon x^\gamma$ . On the other hand, by (2.4) in Lemma 2.2 there is  $C_\bullet$  in  $(0, +\infty)$  such that for every  $i$  in  $\mathbb{N}_0$  we have

$$(4.25) \quad x_i^\gamma \geq \frac{C_\bullet}{i^\alpha}.$$

Thus, if for a given  $n$  in  $\mathbb{N}$  satisfying  $n \geq N + 1$  we denote by  $p_n$  the unique periodic point of  $f$  of period  $n$  in  $J_{n-1}$ , then we have

$$(4.26) \quad S_{n-N}(\varphi_0)(p_n) \geq \varepsilon \sum_{j=0}^{n-N-1} f^j(p_n)^\gamma \geq \varepsilon \sum_{i=N+1}^n x_i^\gamma \geq \varepsilon C_\bullet \sum_{i=N+1}^n \frac{1}{i^\alpha}.$$

Hence,

$$(4.27) \quad S_n(\varphi_0)(p_n) \geq \varepsilon C_\bullet \sum_{i=N+1}^n \frac{1}{i^\alpha} + N \inf_{[0, 1]} \varphi_0 \text{ and } \lim_{n \rightarrow +\infty} S_n(\varphi_0)(p_n) = +\infty.$$

This proves that for every sufficiently large  $n$ , we have

$$(4.28) \quad \frac{1}{n} S_n(\varphi_0)(p_n) > 0 = \varphi_0(0)$$

and therefore  $\delta_0$  is not a ground state of  $f$  for the potential  $\varphi_0$ . That is,  $\varphi_0$  is outside  $\mathcal{G}(\gamma)$ . Together with Corollary 2 in §1.3, this yields (4.23).  $\square$

*Proof of Proposition 4.4.* Item 1 is a direct consequence of the equality  $\beta_{\text{crit},\gamma}(-\log Df) = 1$  given by Proposition 4.5 and of (4.23) in Lemma 4.7.

To prove item 2, note that the equality  $\beta_{\text{crit},\gamma}(-\log Df) = 1$  and Corollary 1 in §1.2 imply that  $-\log Df$  is in  $\mathcal{G}(\gamma)$ . Together with Lemma 4.7, this implies that  $-\log Df$  is in  $\partial\mathcal{G}(\gamma)$ . To prove the last assertion, note that by the equality  $\beta_{\text{crit},\gamma}(-\log Df) = 1$  and Corollary 1 in §1.2, for every  $\beta$  in  $(0, 1)$  the potential  $-\beta \log Df$  is in  $\mathcal{I}(\gamma)$  and for every  $\beta$  in  $[1, +\infty)$  the potential  $-\beta \log Df$  is outside  $\mathcal{I}(\gamma)$ . On the other hand, Corollaries 1 in §1.2 and (4.23) in Lemma 4.7 imply that for all  $\beta$  in  $(0, +\infty)$  and  $\tau$  in  $(-\infty, 0)$  we have

$$(4.29) \quad \beta_{\text{crit},\gamma}(\tau\omega_\gamma - \beta \log Df) = \beta^{-1}\beta_{\text{crit},\gamma}\left(\frac{\tau}{\beta}\omega_\gamma - \log Df\right) = +\infty$$

and therefore the potential  $\tau\omega_\gamma - \beta \log Df$  is in  $\mathcal{I}(\gamma)$  by Corollary 1 in §1.2. Fixing  $\beta$  in  $[1, +\infty)$  and letting  $\tau \rightarrow 0$ , we conclude that  $-\beta \log Df$  is in  $\partial\mathcal{I}(\gamma)$ . This last set is equal to  $\mathcal{PT}(\gamma)$ , so  $-\beta \log Df$  is in  $\mathcal{PT}(\gamma)$ .

To prove item 3, recall that  $\mathcal{PT}(\gamma)$  is closed by definition and note the function  $\mathbb{R} \mapsto C^\gamma(\mathbb{R})$  given by  $\beta \mapsto -\beta \log Df$  is real-analytic. If  $\mathcal{PT}(\gamma)$  were a co-dimension 1 real-analytic subset of  $C^\gamma(\mathbb{R})$ , then combining item 2 with Lemma 2.1 would lead to a contradiction. This completes the proof of item 3.

To prove the last assertion of the proposition, note that for every  $\beta$  in  $(1, +\infty)$  the potential  $-\beta \log Df$  is in  $\mathcal{PT}(\gamma)$  by item 2 and we have

$$(4.30) \quad \beta_{\text{crit},\gamma}(-\beta \log Df) = \beta\beta_{\text{crit},\gamma}(-\log Df) = \beta > 1$$

by Proposition 4.5. Together with Proposition 4.3 this proves that  $\beta_{\text{crit},\gamma}^{-1}(1)$  is strictly contained in  $\mathcal{PT}(\gamma)$ . The remaining assertions are a direct consequence of items 1 and 2 and of Proposition 4.3.  $\square$

*Example 4.8.* Fix  $\beta_0$  in  $(1, +\infty)$  and let  $(\tilde{\varphi}_\tau)_{\tau \in \mathbb{R}}$  be the real-analytic family of potentials in  $C^\gamma(\mathbb{R})$ , defined by

$$(4.31) \quad \tilde{\varphi}_\tau := \tau\omega_\gamma - \beta_0 \log Df.$$

We show that the function  $\tau \mapsto P(\tilde{\varphi}_\tau)$  is real-analytic on  $\mathbb{R} \setminus \{0\}$ , but fails to be real-analytic at 0. Since for  $\tau = 0$  we have

$$(4.32) \quad \beta_{\text{crit},\gamma}(\tilde{\varphi}_0) = \beta_0^{-1}\beta_{\text{crit},\gamma}(-\log Df) = \beta_0^{-1} < 1$$

by Proposition 4.5, this provides an example of a phase transition at a potential where no phase transition in temperature takes place. Note first that for every  $\tau$  in  $(-\infty, 0)$ , we have

$$(4.33) \quad \beta_{\text{crit},\gamma}(\tilde{\varphi}_\tau) = \beta_0^{-1}\beta_{\text{crit},\gamma}\left(\frac{\tau}{\beta_0}\omega_\gamma - \log Df\right) = +\infty$$

by Lemma 4.7 and therefore the potential  $\tilde{\varphi}_\tau$  is in  $\mathcal{I}(\gamma)$  by Corollary 1 in §1.2. On the other hand, for  $\tau = 0$  the potential  $\tilde{\varphi}_0$  is in  $\mathcal{PT}(\gamma)$  by Proposition 4.4(2). It follows that for every  $\tau$  in  $(0, +\infty)$  the potential  $\varphi_\tau$  is in  $\mathcal{S}(\gamma)$  by Lemma 3.5. Combined with Theorem A, this implies that  $\tau \mapsto P(\tilde{\varphi}_\tau)$  is real-analytic on  $\mathbb{R} \setminus \{0\}$ . On the other hand, the function  $\tau \rightarrow P(\tilde{\varphi}_\tau)$  fails to be real-analytic at 0 by Lemma 3.6.

**4.3. Persistence of phase transitions in temperature when  $\gamma = \alpha$ .** In this section, we prove the following characterization of potentials undergoing a phase transition in temperature in the case where  $\alpha \leq 1$  and  $\gamma = \alpha$ . This result is the main ingredient in the proof of Theorem D, which is given in §4.4.

**Theorem D'.** *If  $\alpha$  is in  $(0, 1]$ , then for every  $\varphi$  in  $C^\alpha(\mathbb{R})$  the following properties are equivalent.*

1. *The potential  $\varphi$  is in  $\text{int}(\mathcal{G}(\alpha))$ .*
2. *The potential  $\varphi$  undergoes a phase transition in temperature.*
3. *The measure  $\delta_0$  is the unique ground state of  $f$  for the potential  $\varphi$  and there are  $C$  and  $\delta$  in  $(0, +\infty)$ , such that for every  $n$  in  $\mathbb{N}$  we have*

$$(4.34) \quad \exp(S_n(\varphi)(x_n) - n\varphi(0)) \leq \frac{C}{n^\delta}.$$

4. *There are constants  $\tilde{C}$  and  $\tilde{\delta}$  in  $(0, +\infty)$ , such that the following property holds. For every  $\tilde{\varphi}$  in  $C^\alpha(\mathbb{R})$  close to  $\varphi$  and all  $n$  in  $\mathbb{N}$  and  $x$  in  $(0, 1]$  such that  $f^n(x)$  is in  $J_0$ , we have*

$$(4.35) \quad \exp(S_n(\tilde{\varphi})(x) - n\tilde{\varphi}(0)) \leq \frac{\tilde{C}}{n^{\tilde{\delta}}}.$$

The proof of this proposition is at the end of this section, after a couple of lemmas.

**Lemma 4.9.** *If  $\alpha$  is in  $(0, 1]$ , then there is  $D$  in  $(0, +\infty)$  such that for every  $\varphi$  in  $C^\alpha(\mathbb{R})$  the following property holds. For every  $n$  in  $\mathbb{N}$ , every connected component  $J$  of  $f^{-n}(J_0)$ , and all  $x$  and  $x'$  in  $J$ , we have*

$$(4.36) \quad |S_n(\varphi)(x) - S_n(\varphi)(x')| \leq D|\varphi|_\alpha.$$

*Proof.* Let  $\varepsilon_1$  be the constant from Lemma 4.6 and put

$$(4.37) \quad D := |J_0|^\alpha \sum_{k=1}^{+\infty} \frac{1}{(1 + \varepsilon_1 k)^{1+\alpha}}.$$

Let  $n$ ,  $x$ ,  $x'$ , and  $J$  be as in the statement of the lemma. By Lemma 4.6, for every  $j$  in  $\{0, \dots, n-1\}$  we have

$$(4.38) \quad |f^j(J)| \leq \frac{|J_0|}{(1 + \varepsilon_1(n-j))^{\frac{1}{\alpha}+1}}.$$

Hence

$$(4.39) \quad |S_n(\varphi)(x) - S_n(\varphi)(x')| \leq |\varphi|_\alpha \sum_{j=0}^{n-1} |f^j(x) - f^j(x')|^{\alpha} \leq |\varphi|_\alpha \sum_{j=0}^{n-1} |f^j(J)|^\alpha \leq D|\varphi|_\alpha. \quad \square$$

**Lemma 4.10.** *Suppose  $\alpha$  is in  $(0, 1]$  and let  $D$  be the constant from Lemma 4.9. If  $\varphi$  in  $C^\alpha(\mathbb{R})$  satisfies*

$$(4.40) \quad \sum_{n=1}^{+\infty} \exp(S_n(\varphi)(x_n) - n\varphi(0)) > \exp(D|\varphi|_\alpha),$$

*then  $P(\varphi) > \varphi(0)$ .*

*Proof.* Consider the formal power series  $\Xi$  and  $\Phi$  in the variable  $s$ , defined by

$$(4.41) \quad \Xi(s) := \sum_{n=1}^{+\infty} \sum_{x \in f^{-n}(1)} \exp(S_n(\varphi)(x) - n\varphi(0)) s^n$$

and

$$(4.42) \quad \Phi(s) := \exp(-D|\varphi|_\alpha) \sum_{n=1}^{+\infty} \exp(S_n(\varphi)(x_n) - n\varphi(0)) s^n.$$

By (2.7) in Lemma 2.3, the convergence radius  $R$  of  $\Xi(s)$  satisfies

$$(4.43) \quad R = \exp(\varphi(0) - P(\varphi)).$$

To prove that  $R$  is strictly less than 1, note that  $f|_{J_0}^{-1}$  induces a bijection between the iterated preimages of 1 by  $f$  and the iterated preimages of 1 by  $F$ . Furthermore, for every  $n$  in  $\mathbb{N}$  and  $x$  in  $f^{-n}(1)$  we have the following estimate. Put  $y := f|_{J_0}^{-1}(x)$ , let  $k$  in  $\mathbb{N}$  be such that

$$(4.44) \quad \sum_{j=0}^{k-1} m(F^j(y)) = n + 1,$$

and note that  $F^k(y) = 1$ . Applying Lemma 4.9 repeatedly, we obtain

$$(4.45) \quad S_n(\varphi)(x) \geq S_{n+1}(\varphi)(y) - \sup_{J_0} \varphi \geq \sum_{j=0}^{k-1} (S_{m(F^j(y))}(\varphi)(x_{m(F^j(y))}) - D|\varphi|_\alpha) - \sup_{J_0} \varphi.$$

We conclude that the power series in  $s$

$$(4.46) \quad \exp\left(\varphi(0) - \sup_{J_0} \varphi\right) (\Phi(s) + \Phi(s)^2 + \dots)$$

has coefficients smaller than or equal to the corresponding coefficients of  $\Xi(s)$ . In particular, the convergence radius  $\tilde{R}$  of (4.46) satisfies  $R \leq \tilde{R}$ . On the other hand, our hypothesis (4.40) implies that evaluating  $\Phi(s)$  at  $s = 1$ , we obtain  $\Phi(1) > 1$ . It thus follows that there is  $s_0$  in  $(0, 1)$ , such that  $\Phi(s_0)$  is finite and  $\Phi(s_0) \geq 1$ . Together with (4.43), this implies

$$(4.47) \quad \exp(\varphi(0) - P(\varphi)) = R \leq \tilde{R} \leq s_0 < 1 \text{ and } P(\varphi) > \varphi(0). \quad \square$$

*Proof of Theorem D'.* The implication  $1 \Rightarrow 2$  is a direct consequence of Theorem C. The proofs of the implications  $2 \Rightarrow 3$ ,  $3 \Rightarrow 4$ , and  $4 \Rightarrow 1$  are below. Let  $D$  be the constant from Lemma 4.9 and note that by (2.4) in Lemma 2.2, there is  $C_\bullet$  in  $(0, +\infty)$  such that for every  $i$  in  $\mathbb{N}_0$  we have

$$(4.48) \quad x_i^\alpha \geq \frac{C_\bullet}{i}.$$

Given a continuous potential  $\psi: [0, 1] \rightarrow \mathbb{R}$  and  $n$  in  $\mathbb{N}$ , write

$$(4.49) \quad \zeta_n(\psi) := \exp(S_n(\psi)(x_n) - n\psi(0)).$$

**2  $\Rightarrow$  3.** Suppose  $\varphi$  undergoes a phase transition in temperature. Then there is  $\beta$  in  $(0, +\infty)$  such that  $P(\beta\varphi) = \beta\varphi(0)$  (Corollary 1 in §1.2) and  $\delta_0$  is the unique ground state of  $f$  for the

potential  $\varphi$  (Corollary 4 in §2.2). So, by Lemma 4.10 with  $\varphi$  replaced by  $\beta\varphi$  we have

$$(4.50) \quad \sum_{n=1}^{+\infty} \zeta_n(\varphi)^\beta \leq \exp(D\beta|\varphi|_\alpha).$$

On the other hand, by (3.5) with  $\gamma = \alpha$  and  $\psi = \varphi$  and by (4.48), for all  $k$  and  $\ell$  in  $\mathbb{N}$  we have

$$(4.51) \quad S_\ell(\varphi)(x_{k+\ell}) - \ell\varphi(0) \geq -|\varphi|_\alpha \sum_{j=0}^{\ell-1} x_{k+\ell-j}^a \geq -|\varphi|_\alpha C_* \sum_{i=k+1}^{k+\ell} \frac{1}{i} \geq -|\varphi|_\alpha C_* \log \left( \frac{k+\ell}{k} \right).$$

Assuming in addition  $\ell \leq k$ , we obtain

$$(4.52) \quad \zeta_{k+\ell}(\varphi) \geq \exp(-(C_* \log 2)|\varphi|_\alpha) \zeta_k(\varphi).$$

Combined with (4.50), this yields

$$(4.53) \quad \exp(\beta D|\varphi|_\alpha) \geq \sum_{n=1}^{+\infty} \zeta_n(\varphi)^\beta \geq \sum_{n=k+1}^{2k} \zeta_n(\varphi)^\beta \geq k \exp(-(\beta C_* \log 2)|\varphi|_\alpha) \zeta_k(\varphi)^\beta$$

and therefore

$$(4.54) \quad \zeta_k(\varphi) \leq \frac{\exp((D + C_* \log 2)|\varphi|_\alpha)}{k^{\frac{1}{\beta}}}.$$

This completes the proof of item 3.

**3  $\Rightarrow$  4.** Suppose item 3 holds, put

$$(4.55) \quad \tilde{\delta} := \frac{\delta}{3}, \quad \tilde{D} := \exp(D(|\varphi|_\alpha + C_*^{-1}\tilde{\delta})),$$

and let  $m_0$  in  $\mathbb{N}$  be sufficiently large so that for every  $m$  in  $\mathbb{N}$  satisfying  $m \geq m_0$  we have

$$(4.56) \quad \exp(|\varphi|_\alpha + C_*^{-1}\tilde{\delta}) \tilde{D} \frac{C \exp(\tilde{\delta})}{(m-1)^{2\tilde{\delta}}} \leq \frac{\tilde{D}^{-1}}{(m+1)^{\tilde{\delta}}}.$$

Furthermore, put

$$(4.57) \quad \iota := \sup \left\{ \int \varphi \, d\nu : \nu \in \mathcal{M}, \text{supp}(\nu) \subseteq [x_{m_0}, 1] \right\} \text{ and } \kappa := \exp(\iota - \varphi(0)).$$

The supremum above is attained, so our hypothesis that  $\delta_0$  is the unique ground state of  $f$  for the potential  $\varphi$  implies

$$(4.58) \quad \iota < \varphi(0) \text{ and } \kappa < 1.$$

Choose  $\tilde{\kappa}$  in  $(\kappa, 1)$  and let  $\tilde{K}$  in  $(0, +\infty)$  be sufficiently large, so that for every  $m$  in  $\mathbb{N}$  we have

$$(4.59) \quad \tilde{\kappa}^m \leq \frac{\tilde{K}}{(m+1)^{\tilde{\delta}}}.$$

Let  $\Delta$  in  $C^\alpha(\mathbb{R})$  be such that

$$(4.60) \quad |\Delta|_\alpha < \min \left\{ \log \frac{\tilde{\kappa}}{\kappa}, C_*^{-1}\tilde{\delta} \right\}$$

and put  $\tilde{\varphi} := \varphi + \Delta$ . Note that for every measure  $\nu$  in  $\mathcal{M}$  satisfying  $\text{supp}(\nu) \subseteq [x_{m_0}, 1]$ , we have

$$(4.61) \quad \exp\left(\int \tilde{\varphi} d\nu - \tilde{\varphi}(0)\right) \leq \kappa \exp\left(\int \Delta d\nu - \Delta(0)\right) \leq \tilde{\kappa}.$$

On the other hand, by the definition of  $\tilde{D}$  in (4.55) we have

$$(4.62) \quad \exp(D|\tilde{\varphi}|_\alpha) \leq \tilde{D}.$$

Before proving (4.35), we establish a couple of preliminary estimates. Let  $x'$  in  $J_0$  and  $k$  in  $\mathbb{N}$  be such that for every  $\ell$  in  $\{0, \dots, k-1\}$  we have  $m(F^\ell(x')) \leq m_0$ . Put  $n' := \sum_{\ell=0}^{k-1} m(F^\ell(x'))$  and note that  $f^{n'}(x') = F^k(x')$ . Then, for every  $i$  in  $\{0, \dots, n'-1\}$  the connected component  $J$  of  $f^{-n'}(J_0)$  containing  $x'$  satisfies

$$(4.63) \quad f^i(J) \subseteq \bigcup_{m=0}^{m_0-1} J_m = (x_{m_0}, 1].$$

In particular, if  $p$  is the unique periodic point of  $f$  of period  $n'$  in  $J$ , then the orbit of  $p$  is contained in  $(x_{m_0}, 1]$ . Thus, by Lemma 4.9, (4.61), and (4.62) we have

$$(4.64) \quad \exp(S_{n'}(\tilde{\varphi})(x') - n'\tilde{\varphi}(0)) \leq \exp(D|\tilde{\varphi}|_\alpha) \exp(S_{n'}(\tilde{\varphi})(p) - n'\tilde{\varphi}(0)) \leq \tilde{D}\tilde{\kappa}^{n'}.$$

To establish the second preliminary estimate, note that by (4.48) for every  $m$  in  $\mathbb{N}$  we have

$$(4.65) \quad S_m(\omega_\alpha)(x_m) = \sum_{j=0}^{m-1} x_{m-j}^\alpha \geq -C_* \sum_{i=1}^m \frac{1}{i} \geq -C_*(1 + \log m).$$

Together with (3.5) with  $\gamma = \alpha$  and  $\psi = \Delta$  and with (4.34) and (4.60), this implies

$$(4.66) \quad \zeta_m(\tilde{\varphi}) = \zeta_m(\varphi)\zeta_m(\Delta) \leq \frac{C}{m^\delta} \exp(-|\Delta|_\alpha S_m(\omega_\alpha)(x_m)) \leq \frac{C \exp(C_*|\Delta|_\alpha)}{m^{\delta-C_*|\Delta|_\alpha}} < \frac{C \exp(\tilde{\delta})}{m^{2\tilde{\delta}}}.$$

We now proceed to the proof of (4.35). Let  $n$  in  $\mathbb{N}$  and  $x$  in  $(0, 1]$  be such that  $f^n(x)$  is in  $J_0$ . Suppose first that  $x$  is in  $J_0$ . Let  $k$  in  $\mathbb{N}$  be such that  $\sum_{\ell=0}^{k-1} m(F^\ell(x)) = n$ , note that  $F^k(x) = f^n(x)$ , and for each  $\ell$  in  $\{0, \dots, k-1\}$  put

$$(4.67) \quad z_\ell := F^\ell(x) \text{ and } m_\ell := m(z_\ell).$$

If for every  $\ell$  in  $\{0, 1, \dots, k-1\}$  we have  $m_\ell \leq m_0$ , then by (4.59) and (4.64) we have

$$(4.68) \quad \exp(S_n(\tilde{\varphi})(x) - n\tilde{\varphi}(0)) \leq \tilde{D}\tilde{\kappa}^n \leq \tilde{D} \frac{\tilde{K}}{(n+1)^{\tilde{\delta}}}.$$

Suppose there is  $\ell$  in  $\{0, 1, \dots, k-1\}$  satisfying  $m(F^\ell(x)) \geq m_0 + 1$ . Let  $(\ell_j)_{j=1}^s$  be the increasing sequence of all such. Then, for each  $j$  in  $\{1, \dots, s\}$  the point  $f(z_{\ell_j})$  is in  $J_{m_{\ell_j}-1}$  and therefore by Lemma 4.9, (4.56) with  $m = m_{\ell_j}$ , (4.60), (4.62), and (4.66) with  $m = m_{\ell_j} - 1$ , we have

$$(4.69) \quad \begin{aligned} \exp(S_{m_{\ell_j}}(\tilde{\varphi})(z_{\ell_j}) - m_{\ell_j}\tilde{\varphi}(0)) &\leq \exp(\tilde{\varphi}(z_{\ell_j}) - \tilde{\varphi}(0)) \tilde{D}\zeta_{m_{\ell_j}-1}(\tilde{\varphi}) \\ &\leq \exp(|\tilde{\varphi}|_\alpha) \tilde{D} \frac{C \exp(\tilde{\delta})}{(m_{\ell_j} - 1)^{2\tilde{\delta}}} \leq \frac{\tilde{D}^{-1}}{(m_{\ell_j} + 1)^{\tilde{\delta}}}. \end{aligned}$$

If  $s = k$ , then for every  $\ell$  in  $\{0, \dots, k-1\}$  we have  $m_\ell \geq m_0 + 1$  and

$$(4.70) \quad \exp(S_n(\tilde{\varphi})(x) - n\tilde{\varphi}(0)) = \prod_{\ell=0}^{k-1} \exp(S_{m_\ell}(\tilde{\varphi})(z_\ell) - m_\ell \tilde{\varphi}(0)) \leq \tilde{D}^{-k} \prod_{\ell=0}^{k-1} \frac{1}{(m_\ell + 1)^{\tilde{\delta}}} \\ \leq \tilde{D}^{-k} \frac{1}{(n+1)^{\tilde{\delta}}}.$$

Suppose  $s < k$ , put  $\ell_0 := -1$  and  $\ell_{s+1} := k$ , and let  $(j(i))_{i=1}^t$  be the increasing sequence of all those  $j$  in  $\{0, \dots, s\}$  such that  $\ell_{j+1} \geq \ell_j + 2$ . For each  $i$  in  $\{1, \dots, t\}$  put

$$(4.71) \quad M_i := \sum_{\ell=\ell_{j(i)}+1}^{\ell_{j(i)+1}-1} m_\ell$$

and note that for every  $\ell$  in  $\{\ell_{j(i)} + 1, \dots, \ell_{j(i)+1} - 1\}$  we have  $m_\ell \leq m_0$ . Furthermore,

$$(4.72) \quad t \leq s+1 \text{ and } \sum_{i=1}^t M_i = n - \sum_{j=1}^s m_{\ell_j}.$$

Thus, by (4.64) for every  $i$  in  $\{1, \dots, t\}$  we have

$$(4.73) \quad \exp(S_{M_i}(\tilde{\varphi})(z_{\ell_{j(i)+1}}) - M_i \tilde{\varphi}(0)) \leq \tilde{D} \tilde{\kappa}^{M_i}.$$

Together with (4.59), (4.69), and (4.72), this implies

$$(4.74) \quad \exp(S_n(\tilde{\varphi})(x) - n\tilde{\varphi}(0)) \\ = \prod_{j=1}^s \exp(S_{m_{\ell_j}}(\tilde{\varphi})(z_{\ell_j}) - m_{\ell_j} \tilde{\varphi}(0)) \cdot \prod_{i=1}^t \exp(S_{M_i}(\tilde{\varphi})(z_{\ell_{j(i)+1}}) - M_i \tilde{\varphi}(0)) \\ \leq \tilde{D}^{t-s} \left( \prod_{j=1}^s \frac{1}{(m_{\ell_j} + 1)^{\tilde{\delta}}} \right) \tilde{\kappa}^{\sum_{i=1}^t M_i} \\ \leq \tilde{D} \tilde{\kappa} \frac{1}{\left( \sum_{j=1}^s m_{\ell_j} + 1 \right)^{\tilde{\delta}}} \cdot \frac{1}{\left( \sum_{i=1}^t M_i + 1 \right)^{\tilde{\delta}}} \\ \leq \tilde{D} \tilde{\kappa} \frac{1}{(n+1)^{\tilde{\delta}}}.$$

Together with (4.68) and (4.70) this completes the proof that in the case where  $x$  is in  $J_0$ , we have

$$(4.75) \quad \exp(S_n(\tilde{\varphi})(x) - n\tilde{\varphi}(0)) \leq \frac{\max\{\tilde{D}\tilde{\kappa}, \tilde{D}^{-1}\}}{(n+1)^{\tilde{\delta}}}.$$

To treat the general case, put

$$(4.76) \quad \hat{C} := \tilde{D} C \exp(\tilde{\delta}) \max\{\tilde{D}\tilde{\kappa}, \tilde{D}^{-1}\},$$

suppose  $x$  is not  $J_0$ , and let  $m$  in  $\mathbb{N}$  be such that  $J_m$  contains  $x$ . Then  $m \leq n$ ,  $f^m(x)$  is in  $J_0$ , and by Lemma 4.9 with  $n$  replaced by  $m$  and with  $x' = x_m$ , (4.62), and (4.66) we have

$$(4.77) \quad \exp(S_m(\tilde{\varphi})(x) - m\tilde{\varphi}(0)) \leq \tilde{D}\zeta_m(\tilde{\varphi}) \leq \frac{\tilde{D}C \exp(\tilde{\delta})}{m^{2\tilde{\delta}}}.$$

In the case where  $n > m$ , by (4.75) with  $n$  replaced by  $n - m$  and  $x$  replaced by  $f^m(x)$  this implies

$$(4.78) \quad \begin{aligned} \exp(S_n(\tilde{\varphi})(x) - n\tilde{\varphi}(0)) &\leq \frac{\tilde{D}C \exp(\tilde{\delta})}{m^{2\tilde{\delta}}} \exp(S_{n-m}(\tilde{\varphi})(f^m(x)) - (n-m)\tilde{\varphi}(0)) \\ &\leq \frac{\hat{C}}{m^{2\tilde{\delta}} \cdot (n-m+1)^{\tilde{\delta}}} \leq \frac{\hat{C}}{n^{\tilde{\delta}}}. \end{aligned}$$

Together with (4.75) and (4.77), this completes the proof of item 4 in all of the cases.

**4  $\Rightarrow$  1.** Suppose there are  $\tilde{C}$  and  $\tilde{\delta}$  in  $(0, +\infty)$  such that (4.35) holds for every  $\tilde{\varphi}$  in  $C^\alpha(\mathbb{R})$  close to  $\varphi$ . Let  $\nu$  be an ergodic measure in  $\mathcal{M}$  different from  $\delta_0$ . Then,  $\nu$  charges  $J_0$  and by the BIRKHOFF ergodic theorem there is  $x_\bullet$  in  $J_0$  such that

$$(4.79) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} S_n(\tilde{\varphi})(x_\bullet) = \int \tilde{\varphi} d\nu$$

and such that there are arbitrarily large  $n$  for which  $f^n(x_\bullet)$  is in  $J_0$ . For every such  $n$ , we have by (4.35) with  $x = x_\bullet$ .

$$(4.80) \quad S_n(\tilde{\varphi})(x_\bullet) - n\tilde{\varphi}(0) \leq -\tilde{\delta} \log n + \log \tilde{C}.$$

Together with (4.79), this implies

$$(4.81) \quad \int \tilde{\varphi} d\nu \leq \tilde{\varphi}(0).$$

That is,  $\delta_0$  is a ground state of  $f$  for the potential  $\tilde{\varphi}$  and therefore  $\tilde{\varphi}$  is in  $\mathcal{G}(\alpha)$ . This proves that  $\mathcal{G}(\alpha)$  contains a neighborhood of  $\varphi$  in  $C^\alpha(\mathbb{R})$  and therefore  $\varphi$  is in  $\text{int}(\mathcal{G}(\alpha))$ .  $\square$

**4.4. Proof of Theorem D.** That every phase transition in temperature is persistent follows from the equivalence  $1 \Leftrightarrow 2$  in Theorem D' in §4.3.

To prove (1.10), let  $\varphi$  in  $\mathcal{PT}(\alpha)$  be given. Then  $P(\varphi) = \varphi(0)$  by Theorem 1 and  $\varphi$  undergoes a phase transition in temperature by Corollary 1 in §1.2. This phase transition in temperature is persistent in  $C^\alpha(\mathbb{R})$ , so Theorem C implies that  $\varphi$  is in  $\text{int}(\mathcal{G}(\alpha))$ . This proves  $\mathcal{PT}(\alpha) \subseteq \text{int}(\mathcal{G}(\alpha))$ . Together with Proposition 4.3, this inclusion implies

$$(4.82) \quad \mathcal{PT}(\alpha) = \mathcal{PT}(\alpha) \cap \text{int}(\mathcal{G}(\alpha)) = \{\varphi \in \text{int}(\mathcal{G}(\alpha)): \beta_{\text{crit},\alpha}(\varphi) = 1\}.$$

Thus, to complete the proof of (1.10), it is sufficient to show that  $\beta_{\text{crit},\alpha} = +\infty$  on  $C^\alpha(\mathbb{R}) \setminus \text{int}(\mathcal{G}(\alpha))$ . To do this, let  $\varphi$  be in  $C^\alpha(\mathbb{R}) \setminus \text{int}(\mathcal{G}(\alpha))$  and note that for every  $\beta$  in  $(0, +\infty)$  the potential  $\beta\varphi$  is outside  $S(\alpha)$  by Proposition 4.1 and it is outside  $\mathcal{PT}(\alpha)$  because  $\mathcal{PT}(\alpha) \subseteq \text{int}(\mathcal{G}(\alpha))$ . Hence  $\beta\varphi$  is in  $\mathcal{I}(\gamma)$ . Since this holds for every  $\beta$  in  $(0, +\infty)$ , by Corollary 1 in §1.2 the potential  $\varphi$  does not undergo a phase transition in temperature and we have  $\beta_{\text{crit},\alpha}(\varphi) = +\infty$  by definition. The proof of (1.10) is thus complete.

It remains to show that  $\beta_{\text{crit},\alpha}$  is continuous. Since  $\beta_{\text{crit},\alpha} = +\infty$  on  $C^\alpha(\mathbb{R}) \setminus \text{int}(\mathcal{G}(\alpha))$ , this follows from Corollary 2 in §1.3 and the lower semicontinuity of  $\beta_{\text{crit},\alpha}$  given by Proposition 4.3.

**Data availability.** No datasets were generated or analyzed during the current study.

**Conflict of interest.** The authors declare that they have no conflict of interest.

## REFERENCES

- [BC23] Thiago Bomfim and Victor Carneiro. Thermodynamical and spectral phase transition for local diffeomorphisms in the circle. *Nonlinearity*, 36(12):6513–6545, 2023.
- [Bow08] Rufus Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, revised edition, 2008. With a preface by David Ruelle, Edited by Jean-René Chazottes.
- [CCRV09] J.-R. Chazottes, P. Collet, F. Redig, and E. Verbitskiy. A concentration inequality for interval maps with an indifferent fixed point. *Ergodic Theory Dynam. Systems*, 29(4):1097–1117, 2009.
- [CR25] Daniel Coronel and Juan Rivera-Letelier. Phase transitions in temperature for intermittent maps. *arXiv e-prints*, page arXiv:2504.01166, April 2025.
- [CT13] Vaughn Climenhaga and Daniel J. Thompson. Equilibrium states beyond specification and the Bowen property. *J. Lond. Math. Soc. (2)*, 87(2):401–427, 2013.
- [CV13] A. Castro and P. Varandas. Equilibrium states for non-uniformly expanding maps: decay of correlations and strong stability. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 30(2):225–249, 2013.
- [Geo11] Hans-Otto Georgii. *Gibbs measures and phase transitions*, volume 9 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 2011.
- [GIR22] Eduardo Garibaldi and Irene Inoquio-Renteria. A Ruelle-Perron-Frobenius theorem for expanding circle maps with an indifferent fixed point. *Lett. Math. Phys.*, 112(4):Paper No. 76, 25, 2022.
- [GIR24] Eduardo Garibaldi and Irene Inoquio-Renteria. Exponential rate of decay of correlations of equilibrium states associated with non-uniformly expanding circle maps. *Nonlinearity*, 37(3):Paper No. 035018, 12, 2024.
- [Gou04] Sébastien Gouëzel. Sharp polynomial estimates for the decay of correlations. *Israel J. Math.*, 139:29–65, 2004.
- [Hu04] Huyi Hu. Decay of correlations for piecewise smooth maps with indifferent fixed points. *Ergodic Theory Dynam. Systems*, 24(2):495–524, 2004.
- [IRRL12] Irene Inoquio-Renteria and Juan Rivera-Letelier. A characterization of hyperbolic potentials of rational maps. *Bull. Braz. Math. Soc. (N.S.)*, 43(1):99–127, 2012.
- [IRRL25] I. Inoquio-Rentería and J. Rivera-Letelier. -. In preparation, 2025.
- [Klo19] Benoît R. Kloeckner. Effective high-temperature estimates for intermittent maps. *Ergodic Theory Dynam. Systems*, 39(8):2159–2175, 2019.
- [Klo20] Benoît R. Kloeckner. An optimal transportation approach to the decay of correlations for non-uniformly expanding maps. *Ergodic Theory Dynam. Systems*, 40(3):714–750, 2020.
- [Li15] Huaibin Li. Equivalent characterizations of hyperbolic Hölder potential for interval maps. *Proc. Amer. Math. Soc.*, 143(5):2129–2141, 2015.
- [Lop93] A. O. Lopes. The zeta function, nondifferentiability of pressure, and the critical exponent of transition. *Adv. Math.*, 101(2):133–165, 1993.
- [LRL14a] Huaibin Li and Juan Rivera-Letelier. Equilibrium states of interval maps for hyperbolic potentials. *Nonlinearity*, 27(8):1779–1804, 2014.
- [LRL14b] Huaibin Li and Juan Rivera-Letelier. Equilibrium states of weakly hyperbolic one-dimensional maps for Hölder potentials. *Comm. Math. Phys.*, 328(1):397–419, 2014.
- [LSV99] Carlangelo Liverani, Benoît Saussol, and Sandro Vaienti. A probabilistic approach to intermittency. *Ergodic Theory Dynam. Systems*, 19(3):671–685, 1999.
- [MT12] Ian Melbourne and Dalia Terhesiu. Operator renewal theory and mixing rates for dynamical systems with infinite measure. *Invent. Math.*, 189(1):61–110, 2012.
- [Ons44] Lars Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev. (2)*, 65:117–149, 1944.
- [Pei36] R. Peierls. On ising's model of ferromagnetism. *Proceedings of the Cambridge Philosophical Society*, 32:477–481, 1936.

- [Pia80] Giulio Pianigiani. First return map and invariant measures. *Israel J. Math.*, 35(1-2):32–48, 1980.
- [PM80] Yves Pomeau and Paul Manneville. Intermittent transition to turbulence in dissipative dynamical systems. *Comm. Math. Phys.*, 74(2):189–197, 1980.
- [PS92] Thomas Prellberg and Joseph Slawny. Maps of intervals with indifferent fixed points: thermodynamic formalism and phase transitions. *J. Statist. Phys.*, 66(1-2):503–514, 1992.
- [PU10] Feliks Przytycki and Mariusz Urbański. *Conformal fractals: ergodic theory methods*, volume 371 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2010.
- [PW99] Mark Pollicott and Howard Weiss. Multifractal analysis of Lyapunov exponent for continued fraction and Manneville-Pomeau transformations and applications to Diophantine approximation. *Comm. Math. Phys.*, 207(1):145–171, 1999.
- [Sar01] Omri M. Sarig. Phase transitions for countable Markov shifts. *Comm. Math. Phys.*, 217(3):555–577, 2001.
- [Sar02] Omri Sarig. Subexponential decay of correlations. *Invent. Math.*, 150(3):629–653, 2002.
- [Sin82] Ya. G. Sinai. *Theory of phase transitions: rigorous results*, volume 108 of *International Series in Natural Philosophy*. Pergamon Press, Oxford-Elmsford, N.Y., 1982. Translated from the Russian by J. Fritz, A. Krámli, P. Major and D. Szász.
- [Tha80] Maximilian Thaler. Estimates of the invariant densities of endomorphisms with indifferent fixed points. *Israel J. Math.*, 37(4):303–314, 1980.
- [URM22] Mariusz Urbański, Mario Roy, and Sara Munday. *Non-invertible dynamical systems. Vol. 2. Finer thermodynamic formalism—distance expanding maps and countable state subshifts of finite type, conformal GDMs, Lasota-Yorke maps and fractal geometry*, volume 69.2 of *De Gruyter Expositions in Mathematics*. De Gruyter, Berlin, [2022] ©2022.
- [You99] Lai-Sang Young. Recurrence times and rates of mixing. *Israel J. Math.*, 110:153–188, 1999.

FACULTAD DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, CAMPUS SAN JOAQUÍN, AVENIDA VICUÑA MACKENNA 4860, SANTIAGO, CHILE.

Email address: acoronel@uc.cl

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER. HYLAN BUILDING, ROCHESTER, NY 14627, U.S.A.

Email address: riveraletelier@gmail.com

URL: <http://rivera-letelier.org/>