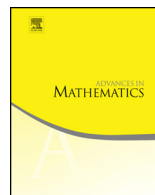




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Robust sensitive dependence of geometric Gibbs states for analytic families of quadratic maps

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ABSTRACT

For quadratic-like maps, we show a phenomenon of sensitive dependence of geometric Gibbs states: There are analytic families of quadratic-like maps for which an arbitrarily small perturbation of the parameter can have a definite effect on the low-temperature geometric Gibbs states. Furthermore, this phenomenon is robust: There is an open set of analytic 2-parameter families of quadratic-like maps that exhibit sensitive dependence of geometric Gibbs states. We introduce a geometric version of the Peierls condition for contour models ensuring that the low-temperature geometric Gibbs states are concentrated near the critical orbit.

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1. Introduction

A central problem in statistical mechanics and the thermodynamic formalism, is the study of phase transitions. Here we focus on “zero-temperature” phase transitions.¹ In good situations, as in the case of contour models satisfying the Peierls condition, Gibbs states converge to a ground state as the temperature drops to zero, see for example [40, II, §2] or the summary in [41, §B.4], and [6,10,12,24] and references therein for other convergence results. There are several examples of non-convergence, see [4,11,42], and the companion paper [15].

Here we focus on the thermodynamic formalism of analytic maps and geometric potentials.² The geometric potential arises naturally in several important problems, like in the construction of physical measures, as in the pioneering work of Sinai [39], Ruelle [38], and Bowen [5]. The pressure of the geometric potential, as a function of the inverse temperature, is also connected to several multifractal spectra, and large deviations rate functions.

The simplest case of interest is that of circle expanding maps. A folklore result asserts that generically there is a unique ground state for the geometric potential, and that geometric Gibbs states converge to this ground state as the temperature drops to zero [16]. On the other hand, some of the non-convergence examples mentioned above can be adapted to the case of smooth circle expanding maps, as shown in [16].³ However, these examples, as well as those in [4,11,15,42], are given by constructions that require infinitely many non-trivial conditions. They are therefore of infinite co-dimension, and a generic finite-dimensional family of circle expanding maps cannot contain such a map.

Our goal is to show that in the next simplest case, of (real and complex) quadratic-like maps on a single variable, the occurrence of zero-temperature phase transitions is a robust phenomenon for 2-parameter families. In fact, our main result implies that there is an open set of 2-parameter families of quadratic-like maps that exhibit a phenomenon of *sensitive dependence of Gibbs states*, which is similar in spirit to the sensitive dependence on initial conditions that is characteristic of chaotic dynamical systems. More precisely, for a 2-parameter family of quadratic-like maps in this open set, an arbitrarily small perturbation of the parameters can have a drastic effect on the low-temperature geometric Gibbs states. In the companion paper [17] we show a similar phenomenon at positive temperature, and in [15] we study it for classical lattice systems. The situation is however significantly simpler in [15], since the potential is independent of the system and there are no differentiability issues.

One of the main technical difficulties to study geometric Gibbs states of quadratic-like maps is the presence of the critical point, which is a serious obstruction to uniform

¹ Also known as the “chaotic dependence of Gibbs states” as the temperature parameter drops to zero.

² Note that in the setting considered in [4,6,10,11,15,12,21,42,24] the map is fixed, and the potential is allowed to vary independently of the map. In contrast, the geometric potential considered here is entirely determined by the map.

³ For real analytic maps it is an open problem to show that for every real analytic circle expanding map the geometric Gibbs states converge to a ground state as the temperature drops to zero, see [16].

hyperbolicity. This leads to some complications, like the fact that there is no obvious characterization of ground states.⁴ In particular, the ergodic optimization approach to study low-temperature Gibbs states, described for example in [3,12,21], breaks down for quadratic-like maps.

The main tool introduced in this paper is the “Geometric Peierls condition”. It ensures that the geometric Gibbs states concentrate on the critical orbit as the temperature drops to zero, under certain circumstances. By considering a critical orbit that accumulates on 2 different periodic orbits with the same Lyapunov exponent,⁵ this creates the non-convergence of geometric Gibbs states. This is why we use 2-parameter families: One parameter is needed to ensure that these 2 periodic orbits have the same Lyapunov exponent and the other parameter is needed to control the combinatorics of the critical orbit. A somewhat similar idea was used by Hofbauer and Keller to produce an example of a quadratic map without a physical measure [22], see also [23]. However, the mechanisms are different: Hofbauer and Keller used long parabolic cascades to control almost every point with respect to the Lebesgue measure; we use a fine control of derivatives of orbits far from the critical orbit to control the mass of the geometric Gibbs states at low temperatures.

To state our results more precisely, we recall the concept of quadratic-like maps of Douady and Hubbard [20]. Given simply-connected subsets U and V of \mathbb{C} such that the closure of U is compact and contained in V , a holomorphic map $f: U \rightarrow V$ is a *quadratic-like map* if it is proper of degree 2. Such a map has a unique point at which the derivative Df vanishes; it is the *critical point* of f . The *filled Julia set* of a quadratic-like map $f: U \rightarrow V$ is

$$K(f) := \{z \in U \mid \text{for every integer } n \geq 1, f^n(z) \in U\}.$$

The *Julia set* $J(f)$ of f is the boundary of $K(f)$, and it coincides with the closure of the repelling periodic points of f .

Given a quadratic-like map f , denote by \mathcal{M}_f the space of all probability measures on $J(f)$ that are invariant by f . For μ in \mathcal{M}_f denote by $h_\mu(f)$ the measure-theoretic entropy of μ , and for each β in \mathbb{R} put

$$P_f(\beta) := \sup \left\{ h_\mu(f) - \beta \int \log |Df| \, d\mu \mid \mu \in \mathcal{M}_f \right\}.$$

It is the *pressure of $f|_{J(f)}$ for the potential $-\beta \log |Df|$* . A measure μ realizing the supremum above is an *equilibrium state of $f|_{J(f)}$ for the potential $-t \log |Df|$* , or a *geometric Gibbs state*.

⁴ In fact, there are quadratic maps without a Lyapunov minimizing measure, see for example [7, Example 5.4], [8, Corollary 2], and [15, Main Theorem].

⁵ In contrast with the usual Peierls condition for contour models, where the ground state is assumed to be supported on a periodic configuration, the Geometric Peierls Condition introduced here is compatible with a non-periodic critical orbit, see Remark 4.2.

A quadratic-like map $f: U \rightarrow V$ is *real* if U and V are invariant under complex conjugation, and if f commutes with complex conjugation. The critical point of such a map is real. A real quadratic-like map with critical point c is *essentially topologically exact*, if $f^2(c)$ is defined and is different from $f(c)$, if f maps the interval $I(f)$ bounded by $f(c)$ and $f^2(c)$ to itself, and if $f|_{I(f)}$ is topologically exact. For such a map f we consider both, the interval map $f|_{I(f)}$, and the complex map f acting on its Julia set $J(f)$.

Let f be a real quadratic-like map that is essentially topologically exact. Denote by $\mathcal{M}_f^{\mathbb{R}}$ the space of all probability measures on $I(f)$ that are invariant by f . For μ in $\mathcal{M}_f^{\mathbb{R}}$ we denote by $h_\mu(f)$ the measure-theoretic entropy of μ , and for each β in \mathbb{R} we put

$$P_f^{\mathbb{R}}(\beta) := \sup \left\{ h_\mu(f) - \beta \int \log |Df| \, d\mu \mid \mu \in \mathcal{M}_f^{\mathbb{R}} \right\}.$$

It is the *pressure of $f|_{I(f)}$ for the potential $-\beta \log |Df|$* . A measure μ realizing the supremum above is an *equilibrium state of $f|_{I(f)}$ for the potential $-t \log |Df|$* , or a *geometric Gibbs state*.

For f as above, we refer to the family of functions $(-\beta \log |Df|)_{\beta > 0}$ as the *geometric potentials* of f . We follow the usual terminology in statistical mechanics, where the parameter β is interpreted as the inverse of the “temperature”.

Definition 1.1 (*Sensitive dependence of Gibbs states*). Let Λ be a topological space, and $(f_\lambda)_{\lambda \in \Lambda}$ a continuous family quadratic-like maps. For λ_0 in Λ the family $(f_\lambda)_{\lambda \in \Lambda}$ has *sensitive dependence of low-temperature geometric Gibbs states at λ_0* , if for every sequence $(\beta_\ell)_{\ell \in \mathbb{N}}$ satisfying $\beta_\ell \rightarrow +\infty$ as $\ell \rightarrow +\infty$, there is a parameter λ in Λ arbitrarily close to λ_0 such that:

1. For each β in $(0, +\infty)$, there is a unique equilibrium state $\rho_t(\lambda)$ of $f|_{J(f_\lambda)}$ for the potential $-\beta \log |Df_\lambda|$.
2. The sequence of equilibrium states $(\rho_{\beta_\ell}(\lambda))_{\ell \in \mathbb{N}}$ does not converge in the weak* topology.

When this holds, we also say that $(f_\lambda)_{\lambda \in \Lambda}$ has *sensitive dependence of low-temperature geometric Gibbs states*.

If in addition for every λ in Λ the quadratic-like map f_λ is real, then for λ_0 in Λ the family $(f_\lambda|_{I(f_\lambda)})_{\lambda \in \Lambda}$ has *sensitive dependence of low-temperature geometric Gibbs states at λ_0* if the properties above are satisfied with $f|_{J(f_\lambda)}$ replaced by $f|_{I(f_\lambda)}$. In this case, we also say that $(f_\lambda|_{I(f_\lambda)})_{\lambda \in \Lambda}$ has *sensitive dependence of low-temperature geometric Gibbs states*.

Our main result is stated as the Main Theorem in §3.2. The following is a simple consequence of this result, which is easier to state.

Sensitive Dependence of Geometric Gibbs States. *There is an open subset Λ_0 of \mathbb{C} intersecting \mathbb{R} , a holomorphic family of quadratic-like maps $(\hat{f}_\lambda)_{\lambda \in \Lambda_0}$, and a Cantor set Λ contained in $\Lambda_0 \cap \mathbb{R}$, such that the following properties hold. For every λ in Λ the map \hat{f}_λ is real, and each of the families $(\hat{f}_\lambda|_{I(f_\lambda)})_{\lambda \in \Lambda}$ and $(\hat{f}_\lambda)_{\lambda \in \Lambda}$ has sensitive dependence of low-temperature geometric Gibbs states at every parameter in Λ .*

In fact, we prove that the conclusions of the Sensitive Dependence of Geometric Gibbs States hold for an open set of holomorphic 2-parameter families of quadratic-like maps, see Remark 3.4. Thus, for quadratic-like maps, the sensitive dependence of geometric Gibbs states is a robust phenomenon for 2-parameter families.

Note that the Sensitive Dependence of Geometric Gibbs States does not say anything about the behavior of the low-temperature geometric Gibbs states of \hat{f}_{λ_0} . We show that the parameter λ_0 can be chosen so that the geometric Gibbs states of \hat{f}_{λ_0} converge as the temperature drops to zero, and that λ_0 can be chosen so that they do not converge, see Remark 3.5. In the latter case we show that the set of accumulation measures of the geometric Gibbs states of \hat{f}_{λ_0} is a segment joining certain periodic measures, see Remark 3.6. In the former case we show that the convergence of the geometric Gibbs states is super-exponential, and that the large deviation principle for Gibbs states studied in [3] holds with a degenerated rate function, see Remark 3.7. Our estimates also show that for every λ in Λ the geometric pressure of \hat{f}_λ is super-exponentially close to its asymptote, see Remark 3.8.

For each λ in Λ the map \hat{f}_λ has a non-recurrent critical point, so it is non-uniformly hyperbolic in a strong sense. In fact, the maps in the family $(\hat{f}_\lambda)_{\lambda \in \Lambda}$ satisfy various non-uniform hyperbolicity conditions with uniform constants. For example, the critical orbit is non-recurrent in a uniform way: There is a neighborhood of $z = 0$ that for each λ in Λ is disjoint of the forward orbit of the critical value of \hat{f}_λ . Furthermore, all the maps in the family $(\hat{f}_\lambda)_{\lambda \in \Lambda}$ satisfy the Collet-Eckmann condition with uniform constants: There are constants C and η satisfying $C > 0$ and $\eta > 1$, such that for every λ in Λ and every n in \mathbb{N} , we have $|D\hat{f}_\lambda^n(\hat{f}_\lambda(0))| \geq C\eta^n$. Moreover, all maps in $(\hat{f}_\lambda)_{\lambda \in \Lambda}$ have uniform “goodness constants” in the sense of [2, Definition 2.2], cf. Proposition 4.3. This supports the idea that the lack of expansion is not responsible for the sensitive dependence of geometric Gibbs states.

The Sensitive Dependence of Geometric Gibbs States provides the first examples of an analytic map having a “zero-temperature” phase transition. In the case of a quadratic-like map f , this completes the classification of phase-transitions for β in $(0, +\infty)$.⁶ Restricting to transitive maps in the real case, there are only 3 types of phase transitions:

High-temperature: A phase transition at the first zero of the geometric pressure function. Such a phase transition appears if and only if f is not uniformly hyperbolic, and if it does not satisfy the Collet-Eckmann condition, see [29, Theorem A]

⁶ Compare with the discussion in the introduction of [13].

or [36, Corollary E] for the real case, and [33, Main Theorem] for the complex case;

Low-temperature: A phase transition occurring after the first zero the geometric pressure function. In this case f cannot be uniformly hyperbolic, and it must satisfy the Collet-Eckmann condition, see [13,14,17];

Zero-temperature: The geometric pressure function is real analytic on $(0, +\infty)$, and for every β in this set there is a unique geometric Gibbs state for the potential $-\beta \log |Df|$, but these measures do not converge as $\beta \rightarrow +\infty$. A map exhibiting such a phase transition must be uniformly hyperbolic, or satisfy the Collet-Eckmann condition.

See [31,32] and the survey article [35] for general results on the thermodynamic formalism of one-dimensional maps.

Roughly speaking, the mechanism responsible for high-temperature phase transitions is the lack of (non-uniform) expansion. However, the lack of (non-uniform) expansion is not responsible for zero-temperature phase transitions. The irregular behavior of the critical orbit seems to be responsible for low and zero-temperature phase transitions. As mentioned above, in [16] we give an example of a smooth circle expanding map having a zero-temperature phase transition. However, it is an open problem if there is a uniformly hyperbolic quadratic-like map having a zero-temperature phase transition.

1.1. Notes and references

The family of quadratic-like maps $(\hat{f}_\lambda)_{\lambda \in \Lambda_0}$ in the theorem is given explicitly in §3.3.

It follows from the proof of the Sensitive Dependence of Geometric Gibbs states that there is a definite oscillation of the geometric Gibbs states. More precisely, there is a continuous function $\varphi: \mathbb{C} \rightarrow [0, 1]$ that only depends on λ_0 , such that for every $(\beta_\ell)_{\ell \in \mathbb{N}}$ and λ as in the definition of sensitive dependence of geometric Gibbs states we have

$$\limsup_{\ell \rightarrow +\infty} \int \varphi \, d\rho_{\beta_\ell}(\lambda) = 1, \text{ and } \liminf_{\ell \rightarrow +\infty} \int \varphi \, d\rho_{\beta_\ell}(\lambda) = 0.$$

In fact, at certain temperatures the geometric Gibbs state is super-exponentially close to a certain periodic measure, and at other temperatures they are close to a different periodic measure, see the Main Theorem in §3.2.

1.2. Organization

After some preliminaries about the quadratic family in §2, we state the Main Theorem in §3, and prove the Sensitive Dependence of Geometric Gibbs states assuming this result (§3.4). The Main Theorem is stated for “uniform families” of quadratic-like maps, which are defined in §3.1. This notion is inspired from the work of Douady and Hubbard [20],

and it is satisfied for a large class of holomorphic families of quadratic-like maps, see Remark 3.2.

The rest of the paper is devoted to the proof of the Main Theorem. In §4 we introduce the Geometric Peierls Condition (Definition 4.1), which roughly speaking requires the derivatives along the orbit of the critical value to outweigh the derivatives of orbits that stay far from the critical point. We also give a criterion for this condition (Proposition 4.3) whose proof occupies §§4.1, 4.2. In §4.3 we make various estimates for uniform families of maps, most of which are deduced from analogous estimates for quadratic maps in [13]. In §5 we implement an inducing scheme (§5.1), analogous to that in [13] for quadratic maps. For a map satisfying the Geometric Peierls Condition, we also show how to control the pressure of the induced map in terms of the derivatives of the map along the orbit of the critical value (Proposition I in §5.2).

The proof of the Main Theorem is given in §6. After introducing a family of itineraries and other combinatorial objects in §6.1, in §6.2 we estimate the postcritical series in terms of certain 2 variables series that only depends on the combinatorics of the postcritical orbit (Lemma 6.1). The main estimates needed in the proof of the Main Theorem can be stated only in terms of these 2 variables series, and are relegated to Appendix A. These are given in an abstract setting that is independent of the rest of the paper. The proof of the Main Theorem is completed in §6.3.

1.3. Acknowledgments

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2. Preliminaries

We use \mathbb{N} to denote the set of integers that are greater than or equal to 1, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For a Borel measure ρ on \mathbb{C} , denote by $\text{supp}(\rho)$ its support.

For an annulus A contained in \mathbb{C} , we use $\text{mod}(A)$ to denote the conformal modulus of A .

2.1. Koebe principle

We use the following version of Koebe distortion theorem that can be found, for example, in [26]. Given an open subset G of \mathbb{C} and a map $f: G \rightarrow \mathbb{C}$ that is a biholomorphism onto its image, the *distortion of f* on a subset C of G is

$$\sup_{x,y \in C} \frac{|Df(x)|}{|Df(y)|}.$$

Koebe Distortion Theorem. For each $A > 0$ there is a constant $\Delta > 1$ such that for each topological disk \widehat{W} contained in \mathbb{C} and each compact set K contained in \widehat{W} and such that $\widehat{W} \setminus K$ is an annulus of modulus at least A , the following property holds: For each open topological disk U contained in \mathbb{C} and every biholomorphic map $f: U \rightarrow \widehat{W}$, for every x, y and z in $f^{-1}(K)$ we have

$$\Delta^{-1}|Df(z)| \leq \frac{|f(x) - f(y)|}{|x - y|} \leq \Delta|Df(z)|.$$

Moreover, the distortion of f on $f^{-1}(K)$ is bounded by Δ .

2.2. Quadratic polynomials, Green's functions, and Böttcher coordinates

In this subsection and the next we recall some basic facts about the dynamics of complex quadratic polynomials, see for instance [9] or [28] for references.

For c in \mathbb{C} we denote by f_c the complex quadratic polynomial

$$f_c(z) := z^2 + c,$$

and by K_c the *filled Julia set* of f_c ; that is, the set of all points z in \mathbb{C} whose forward orbit under f_c is bounded in \mathbb{C} . The set K_c is compact and its complement is the connected set consisting of all points whose orbit converges to infinity in the Riemann sphere. Furthermore, we have $f_c^{-1}(K_c) = K_c$ and $f_c(K_c) = K_c$. The boundary J_c of K_c is the *Julia set* of f_c .

For a parameter c in \mathbb{C} , the *Green's function* of K_c is the function $G_c: \mathbb{C} \rightarrow [0, +\infty)$ that is identically 0 on K_c , and that for z outside K_c is given by the limit,

$$G_c(z) := \lim_{n \rightarrow +\infty} \frac{1}{2^n} \log |f_c^n(z)| > 0. \quad (2.1)$$

The function G_c is continuous, subharmonic, satisfies $G_c \circ f_c = 2G_c$ on \mathbb{C} , and it is harmonic and strictly positive outside K_c . On the other hand, the critical values of G_c are bounded from above by $G_c(0)$, and the open set

$$U_c := \{z \in \mathbb{C} \mid G_c(z) > G_c(0)\}$$

is homeomorphic to a punctured disk. Notice that $G_c(c) = 2G_c(0)$, thus U_c contains c if 0 is not in K_c .

By Böttcher's Theorem there is a unique conformal representation

$$\varphi_c: U_c \rightarrow \{z \in \mathbb{C} \mid |z| > \exp(G_c(0))\},$$

and this map conjugates f_c to $z \mapsto z^2$. It is called the *Böttcher coordinate* of f_c and satisfies $G_c = \log |\varphi_c|$.

2.3. External rays and equipotentials

Let c be in \mathbb{C} . For $v > 0$ the *equipotential v of f_c* is by definition $G_c^{-1}(v)$. A *Green's line of G_c* is a smooth curve on the complement of K_c in \mathbb{C} that is orthogonal to the equipotentials of G_c and that is maximal with this property. Given t in \mathbb{R}/\mathbb{Z} , the *external ray of angle t of f_c* , denoted by $R_c(t)$, is the Green's line of G_c containing

$$\{\varphi_c^{-1}(r \exp(2\pi it)) \mid \exp(G_c(0)) < r < +\infty\}.$$

By the identity $G_c \circ f_c = 2G_c$, for each $v > 0$ and each t in \mathbb{R}/\mathbb{Z} the map f_c maps the equipotential v to the equipotential $2v$ and maps $R_c(t)$ to $R_c(2t)$. For t in \mathbb{R}/\mathbb{Z} the external ray $R_c(t)$ *lands at a point z* , if $G_c: R_c(t) \rightarrow (0, +\infty)$ is a bijection and if $G_c|_{R_c(t)}^{-1}(v)$ converges to z as v converges to 0 in $(0, +\infty)$. By the continuity of G_c , every landing point is in $J_c = \partial K_c$.

The *Mandelbrot set \mathcal{M}* is the subset of \mathbb{C} of those parameters c for which K_c is connected. The function

$$\begin{aligned} \Phi: \mathbb{C} \setminus \mathcal{M} &\rightarrow \mathbb{C} \setminus \text{cl}(\mathbb{D}) \\ c &\mapsto \Phi(c) := \varphi_c(c) \end{aligned}$$

is a conformal representation, see [19, VIII, Théorème 1]. For $v > 0$ the *equipotential v of \mathcal{M}* is by definition

$$\mathcal{E}(v) := \Phi^{-1}(\{z \in \mathbb{C} \mid |z| = v\}).$$

On the other hand, for t in \mathbb{R}/\mathbb{Z} the set

$$\mathcal{R}(t) := \Phi^{-1}(\{r \exp(2\pi it) \mid r > 1\})$$

is called the *external ray of angle t of \mathcal{M}* . We say that $\mathcal{R}(t)$ *lands at a point z in \mathbb{C}* , if $\Phi^{-1}(r \exp(2\pi it))$ converges to z as $r \searrow 1$. When this happens z belongs to $\partial \mathcal{M}$.

2.4. The wake $1/2$

In this subsection we recall a few facts that can be found for example in [19] or [27].

The external rays $\mathcal{R}(1/3)$ and $\mathcal{R}(2/3)$ of \mathcal{M} land at the parameter $c = -3/4$, and these are the only external rays of \mathcal{M} that land at this point, see for example [27, Theorem 1.2]. In particular, the complement in \mathbb{C} of the set

$$\mathcal{R}(1/3) \cup \mathcal{R}(2/3) \cup \{-3/4\}$$

has 2 connected components; we denote by \mathcal{W} the connected component containing the point $c = -2$ of \mathcal{M} .

For each parameter c in \mathcal{W} the map f_c has 2 distinct fixed points; one of them is the landing point of the external ray $R_c(0)$ and it is denoted by $\beta(c)$; the other one is denoted by $\alpha(c)$. The only external ray landing at $\beta(c)$ is $R_c(0)$, and the only external ray landing at $-\beta(c)$ is $R_c(1/2)$.

Moreover, for every parameter c in \mathcal{W} the only external rays of f_c landing at $\alpha(c)$ are $R_c(1/3)$ and $R_c(2/3)$, see for example [27, Theorem 1.2]. The complement of $R_c(1/3) \cup R_c(2/3) \cup \{\alpha(c)\}$ in \mathbb{C} has 2 connected components; one containing $-\beta(c)$ and $z = c$, and the other one containing $\beta(c)$ and $z = 0$. On the other hand, the point $\alpha(c)$ has 2 preimages by f_c : Itself and $\tilde{\alpha}(c) := -\alpha(c)$. The only external rays landing at $\tilde{\alpha}(c)$ are $R_c(1/6)$ and $R_c(5/6)$.

2.5. Yoccoz puzzles and para-puzzle

In this subsection we recall the definitions of Yoccoz puzzles and para-puzzle. We follow [37].

Definition 2.1 (*Yoccoz puzzles*). Fix c in \mathcal{W} and consider the open region $X_c := \{z \in \mathbb{C} \mid G_c(z) < 1\}$. The *Yoccoz puzzle* of f_c is given by the following sequence of graphs $(I_{c,n})_{n=0}^{+\infty}$ defined for $n = 0$ by:

$$I_{c,0} := \partial X_c \cup (X_c \cap \text{cl}(R_c(1/3)) \cap \text{cl}(R_c(2/3))),$$

and for $n \geq 1$ by $I_{c,n} := f_c^{-n}(I_{c,0})$. The *puzzle pieces of depth n* are the connected components of $f_c^{-n}(X_c) \setminus I_{c,n}$. The puzzle piece of depth n containing a point z is denoted by $P_{c,n}(z)$.

Note that for a real parameter c , every puzzle piece intersecting the real line is invariant under complex conjugation. Since puzzle pieces are simply-connected, it follows that the intersection of such a puzzle piece with \mathbb{R} is an interval.

Definition 2.2 (*Yoccoz para-puzzle⁷*). Given an integer $n \geq 0$, put

$$J_n := \{t \in [1/3, 2/3] \mid 2^n t \pmod{1} \in \{1/3, 2/3\}\},$$

let \mathcal{X}_n be the intersection of \mathcal{W} with the open region in the parameter plane bounded by the equipotential $\mathcal{E}(2^{-n})$ of \mathcal{M} , and put

$$\mathcal{I}_n := \partial \mathcal{X}_n \cup \left(\mathcal{X}_n \cap \bigcup_{t \in J_n} \text{cl}(\mathcal{R}(t)) \right).$$

⁷ In contrast to [37], we only consider the para-puzzle in the wake \mathcal{W} .

Then the *Yoccoz para-puzzle of \mathcal{W}* is the sequence of graphs $(\mathcal{I}_n)_{n=0}^{+\infty}$. The *para-puzzle pieces of depth n* are the connected components of $\mathcal{X}_n \setminus \mathcal{I}_n$. The para-puzzle piece of depth n containing a parameter c is denoted by $\mathcal{P}_n(c)$.

Observe that there is only 1 para-puzzle piece of depth 0, and only 1 para-puzzle piece of depth 1; they are bounded by the same external rays but different equipotentials. Both of them contain $c = -2$.

Fix a parameter c in $\mathcal{P}_0(-2)$. There are precisely 2 puzzle pieces of depth 0: $P_{c,0}(\beta(c))$ and $P_{c,0}(-\beta(c))$. Each of them is bounded by the equipotential 1 and by the closures of the external rays landing at $\alpha(c)$. Furthermore, the critical value c of f_c is contained in $P_{c,0}(-\beta(c))$ and the critical point in $P_{c,0}(\beta(c))$. It follows that the set $f_c^{-1}(P_{c,0}(\beta(c)))$ is the disjoint union of $P_{c,1}(-\beta(c))$ and $P_{c,1}(\beta(c))$, so f_c maps each of the sets $P_{c,1}(-\beta(c))$ and $P_{c,1}(\beta(c))$ biholomorphically to $P_{c,0}(\beta(c))$. Moreover, there are precisely 3 puzzle pieces of depth 1:

$$P_{c,1}(-\beta(c)), P_{c,1}(0) \quad \text{and} \quad P_{c,1}(\beta(c));$$

$P_{c,1}(-\beta(c))$ is bounded by the equipotential $1/2$ and by the closures of the external rays that land at $\alpha(c)$; $P_{c,1}(\beta(c))$ is bounded by the equipotential $1/2$ and by the closures of the external rays that land at $\tilde{\alpha}(c)$; and $P_{c,1}(0)$ is bounded by the equipotential $1/2$ and by the closures of the external rays that land at $\alpha(c)$ and at $\tilde{\alpha}(c)$. In particular, the closure of $P_{c,1}(\beta(c))$ is contained in $P_{c,0}(\beta(c))$. It follows from this that for each integer $n \geq 1$ the map f_c^n maps $P_{c,n}(-\beta(c))$ biholomorphically to $P_{c,0}(\beta(c))$.

2.6. The uniformly expanding Cantor set

For a parameter c in $\mathcal{P}_3(-2)$, the maximal invariant set Λ_c of f_c^3 in $P_{c,1}(0)$ plays an important rôle in the proof of the Main Theorem.

Fix c in $\mathcal{P}_3(-2)$. There are precisely 2 connected components of $f_c^{-3}(P_{c,1}(0))$ contained in $P_{c,1}(0)$ that we denote by Y_c and \tilde{Y}_c . The closures of these sets are disjoint and contained in $P_{c,1}(0)$. The sets Y_c and \tilde{Y}_c are distinguished by the fact that Y_c contains in its boundary the common landing point of the external rays $R_c(7/24)$ and $R_c(17/24)$, denoted $\gamma(c)$, and that \tilde{Y}_c contains in its boundary the common landing point of the external rays $R_c(5/24)$ and $R_c(19/24)$. The map f_c^3 maps each of the sets Y_c and \tilde{Y}_c biholomorphically to $P_{c,1}(0)$. Thus, if we put

$$\begin{aligned} g_c : Y_c \cup \tilde{Y}_c &\rightarrow P_{c,1}(0) \\ z &\mapsto g_c(z) := f_c^3(z), \end{aligned}$$

then

$$\Lambda_c = \bigcap_{n \in \mathbb{N}} g_c^{-n}(\text{cl}(P_{c,1}(0))).$$

2.7. Parameters

In this subsection we recall the definition of a certain parameter sets in [13, Proposition 3.1] that are important in what follows. To avoid confusions with the notation introduced in §2.4, in this section, and in the rest of the paper, we use the parameter t to denote the inverse temperature.

Given an integer $n \geq 3$, let \mathcal{K}_n be the set of all those real parameters c in $(-\infty, 0)$ such that

$$f_c(c) > f_c^2(c) > \cdots > f_c^{n-1}(c) > 0 \quad \text{and} \quad f_c^n(c) \in \Lambda_c.$$

Note that for a parameter c in \mathcal{K}_n , the first return time of 0 to $P_{c,1}(0)$ is equal to $n+1$. On the other hand, the critical point of f_c cannot be asymptotic to a non-repelling periodic point. This implies that all the periodic points of f_c in \mathbb{C} are hyperbolic repelling and therefore that $K_c = J_c$, see [28]. On the other hand, we have $f_c(c) > c$ and the interval $I_c = [c, f_c(c)]$ is invariant by f_c . This implies that I_c is contained in J_c and hence that for every real number t we have $P_c^{\mathbb{R}}(t) \leq P_c(t)$. Note also that $f_c|_{I_c}$ is not renormalizable, so f_c is topologically exact on I_c , see for example [18, Theorem III.4.1].

Since for c in \mathcal{K}_n the critical point of f_c is not periodic, for every integer $k \geq 0$ we have $f_c^{n+3k}(c) \neq 0$. The *itinerary* of f_c , is the sequence $\iota(c)$ in $\{0, 1\}^{\mathbb{N}_0}$ defined for each k in \mathbb{N}_0 by

$$\iota(c)_k := \begin{cases} 0 & \text{if } f_c^{n+3k}(c) \in Y_c; \\ 1 & \text{if } f_c^{n+3k}(c) \in \tilde{Y}_c. \end{cases}$$

Proposition 2.3. *For each integer $n \geq 3$, the set \mathcal{K}_n is a compact subset of*

$$\mathcal{P}_n(-2) \cap (-2, -3/4),$$

and the function $\iota: \mathcal{K}_n \rightarrow \{0, 1\}^{\mathbb{N}_0}$ is a homeomorphism. Finally, for each $\delta > 0$ there is $n_0 \geq 3$ such that for each integer $n \geq n_0$ the set \mathcal{K}_n is contained in the interval $(-2, -2 + \delta)$.

Proof. Except for the assertion that ι is a homeomorphism, this is [13, Proposition 3.1]. In this last result it is shown that ι is a bijection, so it only remains to observe that, since for each c in $\mathcal{P}_3(-2)$ the map f_c is uniformly expanding on Λ_c [13, §3.3], the map ι is continuous, and therefore a homeomorphism. \square

Remark 2.4. The proposition implies that for every integer n satisfying $n \geq 3$ and every $\underline{\iota}$ in $\{0, 1\}^{\mathbb{N}_0}$, there is a unique c in \mathcal{K}_n for which the itinerary $\iota(c)$ of f_c is equal to $\underline{\iota}$. The real parameter c is uniquely characterized by the following properties:

- For every j in $\{1, \dots, n-1\}$, we have $f_c^j(c) > 0$;

- For every k in \mathbb{N}_0 and r in $\{0, 1, 2\}$, we have

$$f_c^{n+3k+r}(c) \begin{cases} > 0 & \text{if } L_k = 1 \text{ and } r = 0, \text{ or if } r = 2; \\ < 0 & \text{if } L_k = 0 \text{ and } r = 0, \text{ or if } r = 1. \end{cases}$$

3. Main Theorem

In this section we state the Main Theorem, and prove the Sensitive Dependence of Geometric Gibbs States assuming this result.

The Main Theorem is stated in §3.2 for “uniform families” of quadratic-like maps, which are defined in §3.1. By the work of Douady and Hubbard [20], there is a large class of holomorphic families of quadratic-like maps that are uniform, see Remark 3.2. We use this to exhibit in §3.3 a concrete (real) 1-parameter family of quadratic-like maps satisfying the hypotheses of the Main Theorem. This family is used in §3.4 to prove the Sensitive Dependence of Geometric Gibbs States assuming the Main Theorem.

3.1. Uniform families of quadratic-like maps

A quadratic-like map $f: U \rightarrow V$ is *normalized*, if its unique critical point is 0, and if $D^2f(0) = 2$. For such a map f there is a holomorphic function $R_f: U \rightarrow \mathbb{C}$ such that for w in U we have

$$f(w) = f(0) + w^2 + w^3 R_f(w).$$

Note that f is uniquely determined by its critical value $f(0)$, and the function R_f .

In what follows, we endow a given family of normalized quadratic-like maps with a topology that will be used in the statement of the Main Theorem. Let \mathcal{U} be the set of all topological disks in \mathbb{C} containing 0 that are not biholomorphic to \mathbb{C} . Endow \mathcal{U} with the *Carathéodory topology*, which is the topology generated by the following families of subsets of \mathcal{U} :

$$\{\{U \in \mathcal{U} \mid K \subseteq U\} \mid K \subseteq \mathbb{C} \text{ compact}\}$$

and

$$\{\{U \in \mathcal{U} \mid N \not\subseteq U\} \mid N \subseteq \mathbb{C} \text{ open and connected with } 0 \in N\}.$$

Let \mathcal{S} be the set of all holomorphic univalent maps f defined on the open unit disk \mathbb{D} , such that $f(0) = 0$ and $Df(0) > 0$. Equip \mathcal{S} with the topology of locally uniform convergence. The map $R: \mathcal{S} \rightarrow \mathcal{U}$ sending f in \mathcal{S} to $f(\mathbb{D})$ in \mathcal{U} is a bijection by the Riemann mapping theorem, and a homeomorphism by [30, §4]. Let \mathcal{H} be the set of all holomorphic functions defined on \mathbb{D} , equipped with the topology of locally uniform convergence.

Using the homeomorphism R we can inject each normalized family of quadratic-like maps \mathcal{F} into $\mathcal{U} \times \mathcal{H}$, sending $f: U \rightarrow \mathbb{C}$ to $(U, f \circ R^{-1}(U))$. Using this injection, we endow \mathcal{F} with the pull-back of the product topology on $\mathcal{U} \times \mathcal{H}$. We call this topology the *Carathéodory topology* or the *topology locally uniform convergence on \mathcal{F}* . Thus, a sequence $f_n: U_n \rightarrow \mathbb{C}$ in \mathcal{F} converges to $f: U \rightarrow \mathbb{C}$ in \mathcal{F} if and only if $(U_n, f_n \circ R^{-1}(U_n))$ converges to $(U, f \circ R^{-1}(U))$ in $\mathcal{U} \times \mathcal{H}$. Observe that this is equivalent to U_n converging to U in the Carathéodory topology on \mathcal{U} , and that for every compact set K in U the sequence f_n converges uniformly to f on K . The latter is the usual definition of the Carathéodory convergence of maps, see [26, §5.1].

By the straightening theorem of Douady and Hubbard [20], for every quadratic-like map $f: U \rightarrow V$ there is c in \mathbb{C} and a quasi-conformal homeomorphism $h: \mathbb{C} \rightarrow \mathbb{C}$ that conjugates the quadratic polynomial f_c to f on a neighborhood of J_c . In the case f is real, c is real, and h can be chosen so that it commutes with the complex conjugation. In all the cases, the quasi-conformal homeomorphism h can be chosen to be holomorphic on a neighborhood of infinity, and tangent to the identity there.

Put

$$\mathcal{X} := \{c \in \mathbb{C} \mid G_c(c) \leq 1\} \quad \text{and} \quad \widehat{\mathcal{X}} := \{c \in \mathbb{C} \mid G_c(c) \leq 2\},$$

and for c in \mathbb{C} , put

$$X_c := \{z \in \mathbb{C} \mid G_c(z) \leq 1\} \quad \text{and} \quad \widehat{X}_c := \{z \in \mathbb{C} \mid G_c(z) \leq 2\}.$$

Note that X_c is contained in the interior of \widehat{X}_c , and that

$$\mathcal{X} = \{c \in \mathbb{C} \mid c \in X_c\} \quad \text{and} \quad \widehat{\mathcal{X}} = \{c \in \mathbb{C} \mid c \in \widehat{X}_c\}.$$

Definition 3.1 (*Uniform family of quadratic-like maps*). A family \mathcal{F} of normalized quadratic-like maps is *uniform*, if there are constants $K \geq 1$ and $R > 0$, such that for each f in \mathcal{F} there are $c(f)$ in \mathcal{X} and a K -quasi-conformal homeomorphism h_f of \mathbb{C} satisfying the following properties.

1. The homeomorphism h_f conjugates $f_{c(f)}$ on $\widehat{X}_{c(f)}$ to f on $h_f(\widehat{X}_{c(f)})$. Furthermore, if f is real, then h_f commutes with the complex conjugation.
2. The set $\widehat{X}_{c(f)}$ is contained in $B(0, R)$, and the homeomorphism h_f is holomorphic on $\mathbb{C} \setminus \text{cl}(B(0, R))$, and it is tangent to the identity at infinity.

Note that property 1 implies that $h_f(0) = 0$.

Remark 3.2. Although it is not needed in this paper, we remark that a family \mathcal{F} of normalized quadratic-like maps with connected Julia sets is uniform if and only if the following property holds: There is a constant $m > 0$ such that for each $f: U \rightarrow V$ in \mathcal{F} there is an essential annulus in $V \setminus U$ whose conformal modulus is at least m .

Let \mathcal{F} be a uniform family of quadratic-like maps. For each f in \mathcal{F} put

$$X_f := h_f(X_{c(f)}) \quad \text{and} \quad \widehat{X}_f := h_f(\widehat{X}_{c(f)}).$$

By the definition of uniform family, the puzzle pieces of $f_{c(f)}$ can be push-forward to X_f by h_f . We call to these sets the *puzzle pieces of f* . We say that a puzzle piece of f has *depth n* if it is the push-forward of a puzzle piece of $c(f)$ with depth n . The puzzle piece of depth n of f containing w is denoted $P_{f,n}(w)$. Thus, we have

$$P_{f,n}(w) := h_f(P_{c(f),n}(h_f^{-1}(w))).$$

Set

$$\beta(f) := h_f(\beta(c(f))) \quad \text{and} \quad \widetilde{\beta}(f) := h_f(-\beta(c(f))).$$

For every integer $n \geq 0$, put

$$\mathcal{P}_n(\mathcal{F}) := \{f \in \mathcal{F} \mid c(f) \in \mathcal{P}_n(-2)\},$$

and for $n \geq 3$, put

$$\mathcal{K}_n(\mathcal{F}) := \{f \in \mathcal{F} \mid c(f) \in \mathcal{K}_n\}.$$

Moreover, for f in $\mathcal{P}_3(\mathcal{F})$ put

$$Y_f := h_f(Y_{c(f)}), \text{ and } \widetilde{Y}_f := h_f(\widetilde{Y}_{c(f)}),$$

and let $g_f: h_f(Y_{c(f)} \cup \widetilde{Y}_{c(f)}) \rightarrow P_{f,1}(0)$ be defined by $g_f := h_f \circ g_{c(f)} \circ h_f^{-1}$. Denote by $p(f)$ and $p^+(f)$ the unique fixed point of g_f in Y_f and \widetilde{Y}_f , respectively, and denote by $p^-(f)$ the unique fixed point of g_f^2 in \widetilde{Y}_f that is different from $p^+(f)$; it is a periodic point of g_f of minimal period 2. Furthermore, denote by

$$\mathcal{O}^+(f) := \{f^j(p^+(f)) \mid j \in \{0, 1, 2\}\} \quad \text{and} \quad \mathcal{O}^-(f) := \{f^j(p^-(f)) \mid j \in \{0, 1, \dots, 5\}\}$$

the orbits of $p^+(f)$ and $p^-(f)$ under f , respectively.

For every f in \mathcal{F} such that $c(f)$ is real and belongs to $[-2, 0)$, denote by $I(f)$ the image under h_f of the interval $[c(f), f_{c(f)}(c(f))]$. In the case where f is real, the set $I(f)$ is a subinterval of \mathbb{R} . In all of the cases, $I(f)$ is a closed topological arc satisfying $f(I(f)) = I(f)$. A quadratic-like map f in \mathcal{F} is *essentially topologically exact* if $c(f)$ is in $[-2, 0)$, and if $f|_{I(f)}$ is topologically exact. For such a map f we consider both, the map $f|_{I(f)}$, and the complex map f acting on its Julia set $J(f)$. We also define $\mathcal{M}_f^{\mathbb{R}}$, $P_f^{\mathbb{R}}$, and *equilibrium states* or *Geometric Gibbs states* of $f|_{I(f)}$ as in the introduction. In the case f is real, the definitions above coincide with those in the introduction.

Let n be an integer satisfying $n \geq 5$, and let f be in $\mathcal{K}_n(\mathcal{F})$. The *itinerary* of f , is the sequence $\iota(f)$ defined by $\iota(f) := \iota(c(f))$, see §2.7. So, for every k in \mathbb{N}_0 we have

$$\iota(f)_k = \begin{cases} 0 & \text{if } f^{n+3k}(f(0)) \in Y_f; \\ 1 & \text{if } f^{n+3k}(f(0)) \in \tilde{Y}_f. \end{cases}$$

Here is an alternative description of the combinatorics of f in terms of its itinerary $\iota(f)$, in the spirit of kneading theory. Note first that the parameter $c(f)$ is real and belongs to $[-2, 0)$, and that the critical point 0 of f and its orbit are contained in $I(f)$. Removing 0 from the closed arc $I(f)$, we obtain 2 disjoint semi-open arcs

$$I(f)^+ := h_f((0, f_{c(f)}(c(f)))) \text{ and } I(f)^- := h_f([c(f), 0)).$$

In view of Remark 2.4, we have the following properties:

- For every j in $\{1, \dots, n-1\}$, the point $f^j(f(0))$ is in $I(f)^+$;
- For every k in \mathbb{N}_0 and r in $\{0, 1, 2\}$, we have

$$f^{n+3k+r}(f(0)) \begin{cases} \in I(f)^+ & \text{if } \iota(f)_k = 1 \text{ and } r = 0, \text{ or if } r = 2; \\ \in I(f)^- & \text{if } \iota(f)_k = 0 \text{ and } r = 0, \text{ or if } r = 1. \end{cases}$$

3.2. Main Theorem

For every normalized quadratic-like map f , and every periodic point p of f with period m in \mathbb{N} , put

$$\chi_f(p) := \frac{1}{m} \log |Df^m(p)|.$$

In this subsection we state the Main Theorem, which is based on the following concept.

Definition 3.3 (*Admissible family of quadratic-like maps*). A uniform family of quadratic-like maps \mathcal{F} is *admissible*, if for every sufficiently large integer n the following properties hold.

1. If we endow \mathcal{F} with the topology of locally uniform convergence, then there is a continuous function $s_n: \mathcal{K}_n \rightarrow \mathcal{K}_n(\mathcal{F})$ such that $c \circ s_n$ is the identity.
2. For every f in $s_n(\mathcal{K}_n)$, we have

$$\chi_f(p(f)) > \chi_f(p^+(f)) \text{ and } \chi_f(p^+(f)) = \chi_f(p^-(f)). \quad (3.1)$$

Before stating the Main Theorem, we give a rough description of the combinatorics of the maps appearing on it. The Main Theorem asserts that for every admissible uniform

family of quadratic-like maps, there is a continuous subfamily $(f_{\underline{s}})_{\underline{s} \in \{+, -\}^{\mathbb{N}}}$ satisfying some properties regarding the limit behavior of the equilibrium states as the temperature drops to zero. The main feature of this subfamily is that for every map $f_{\underline{s}}$ on it, the orbit of the critical point remains most of the time close to the orbits of the periodic points $p^+(f_{\underline{s}})$ and $p^-(f_{\underline{s}})$. The sequence \underline{s} in $\{+, -\}^{\mathbb{N}}$ indicates how the orbit of the critical point alternates between these periodic orbits, or remains close to them. For example, a large string of repeated $+$'s in \underline{s} indicates that the orbit of the critical point is very close to $p^+(f_{\underline{s}})$ for some period of time. With a careful choice of the family of itineraries $(\iota(f_{\underline{s}}))_{\underline{s} \in \{+, -\}^{\mathbb{N}}}$, we show that for each m in \mathbb{N} there is a certain range of inverse temperatures for which the corresponding equilibrium states of $f_{\underline{s}}$ are either concentrated near the orbit of $p^+(f_{\underline{s}})$, or that of $p^-(f_{\underline{s}})$, depending on whether $\underline{s}_m = +$ or $\underline{s}_m = -$. Another interesting feature of this construction, is that the limit behavior of the equilibrium states of $f_{\underline{s}}$ depends on the tail of the sequence \underline{s} . So, we can have maps of the subfamily that are close among them but with significantly different behavior of its equilibrium states, which is the basic idea behind the sensitive dependence notion stated in the introduction.

Endow the set $\{+, -\}$ with the discrete topology, and $\{+, -\}^{\mathbb{N}}$ with the corresponding product topology.

Main Theorem. *For every $R > 0$ there is a constant $K_0 > 1$ such that if \mathcal{F} is an admissible uniform family of quadratic-like maps with constants K_0 and R , then for every sufficiently large integer n there is a continuous subfamily $(f_{\underline{s}})_{\underline{s} \in \{+, -\}^{\mathbb{N}}}$ of $s_n(K_n)$ such that the following properties hold.*

1. *For each \underline{s} in $\{+, -\}^{\mathbb{N}}$ the map $f_{\underline{s}}$ is essentially topologically exact. Moreover, for each t in $(0, +\infty)$ there is a unique equilibrium state $\rho_t^{\mathbb{R}}(\underline{s})$ (resp. $\rho_t(\underline{s})$) of $f_{\underline{s}}|_{I(f_{\underline{s}})}$ (resp. $f_{\underline{s}}|_{J(f_{\underline{s}})})$ for the potential $-t \log |Df_{\underline{s}}|$.*
2. *There are constants $C_0 > 0$ and $v_0 > 0$, and a continuous function*

$$A: \{+, -\}^{\mathbb{N}} \rightarrow (0, +\infty),$$

such that for every sequence $\underline{s} = (s(m))_{m \in \mathbb{N}}$ in $\{+, -\}^{\mathbb{N}}$, the following properties hold. Let m and \widehat{m} be integers such that

$$\widehat{m} \geq m \geq 1 \quad \text{and} \quad s(m) = \dots = s(\widehat{m}),$$

and let t be in $[A(\underline{s})m, A(\underline{s})\widehat{m}]$. Then the equilibrium state $\rho_t^{\mathbb{R}}(\underline{s})$ (resp. $\rho_t(\underline{s})$) of $f_{\underline{s}}|_{I(f_{\underline{s}})}$ (resp. $f_{\underline{s}}|_{J(f_{\underline{s}})})$ is super-exponentially close to the orbit $\mathcal{O}^{s(m)}(f_{\underline{s}})$ of $p^{s(m)}(f_{\underline{s}})$, in that

$$\rho_t^{\mathbb{R}}(\underline{s}) \left(B \left(\mathcal{O}^{s(m)}(f_{\underline{s}}), \exp(-v_0 t^2) \right) \right) \geq 1 - C_0 \exp(-v_0 t^2)$$

$$\left(\text{resp. } \rho_t(\underline{\varsigma}) \left(B \left(\mathcal{O}^{\varsigma(m)}(f_{\underline{\varsigma}}), \exp(-v_0 t^2) \right) \right) \geq 1 - C_0 \exp(-v_0 t^2) \right).$$

Note that for each $\underline{\varsigma}$ in $\{+, -\}^{\mathbb{N}}$ the map $f_{\underline{\varsigma}}$ has a non-recurrent critical point, so it is non-uniformly hyperbolic in a strong sense.

Remark 3.4 (Robustness). It follows from the theory of quadratic-like maps of Douady and Hubbard [20] that condition 1 in Definition 3.3 is satisfied for every holomorphic 1-parameter family of quadratic-like maps $(\hat{f}_{\lambda})_{\lambda \in \Lambda_0}$ intersecting the combinatorial class of the quadratic map f_{-2} transversally. That is, if there is a parameter λ_0 in Λ_0 such that

$$\hat{f}_{\lambda_0}^2(0) = \beta(\hat{f}_{\lambda_0}), \text{ and } \frac{\partial}{\partial \lambda} \left(\hat{f}_{\lambda}^2(0) - \beta(\hat{f}_{\lambda}) \right) |_{\lambda=\lambda_0} \neq 0.$$

So, condition 1 of Definition 3.3 is satisfied for an open set of holomorphic 1-parameter families of quadratic-like maps. If in addition $\chi_{\hat{f}_{\lambda_0}}(p(\hat{f}_{\lambda_0})) > \chi_{\hat{f}_{\lambda_0}}(p^+(\hat{f}_{\lambda_0}))$, then the inequality in (3.1) is also satisfied for an open set of holomorphic 1-parameter families of quadratic-like maps.

On the other hand, the equality in (3.1) imposes a restriction, but there is an open set of holomorphic 2-parameter families of quadratic-like maps that have a holomorphic 1-parameter subfamily satisfying this condition. Thus, the conclusions of the Main Theorem hold for an open set of holomorphic 2-parameter families of quadratic-like maps.

Remark 3.5 (Sensitivity is compatible with convergence and non-convergence). In the proof of the Main Theorem we show that for any choice of $\underline{\varsigma}_0$ in $\{+, -\}^{\mathbb{N}}$, a uniform family \mathcal{F} as in the Main Theorem has sensitive dependence of low-temperature geometric Gibbs states at $f_{\underline{\varsigma}_0}$. If we choose $\underline{\varsigma}_0$ that is not eventually constant, then the Main Theorem implies that the geometric Gibbs states of $f_{\underline{\varsigma}_0}$ do not converge as the temperature drops to zero. On the other hand, if $\underline{\varsigma}_0$ is eventually constant, then the geometric Gibbs states converge. This shows that in the Sensitive Dependence of Geometric Gibbs states the parameter λ_0 can be chosen so that the geometric Gibbs states of \hat{f}_{λ_0} converge as the temperature drops to zero, and that it can also be chosen so that they do not converge.

Remark 3.6 (Accumulation measures). Our estimates show that for every $\underline{\varsigma}$ in $\{+, -\}^{\mathbb{N}}$, and every t in $(0, +\infty)$ we have

$$\begin{aligned} \rho_t^{\mathbb{R}}(\underline{\varsigma}) \left(B \left(\mathcal{O}^+(f_{\underline{\varsigma}}) \cup \mathcal{O}^-(f_{\underline{\varsigma}}), \exp(-v_0 t^2) \right) \right) &\geq 1 - C_0 \exp(-v_0 t^2) \\ (\text{resp. } \rho_t(\underline{\varsigma}) \left(B \left(\mathcal{O}^+(f_{\underline{\varsigma}}) \cup \mathcal{O}^-(f_{\underline{\varsigma}}), \exp(-v_0 t^2) \right) \right) &\geq 1 - C_0 \exp(-v_0 t^2), \end{aligned}$$

see Remark 6.2. In particular, every accumulation measure of $(\rho_t^{\mathbb{R}}(\underline{\varsigma}))_{t>0}$ and of $(\rho_t(\underline{\varsigma}))_{t>0}$ is supported on $\mathcal{O}^+(f_{\underline{\varsigma}}) \cup \mathcal{O}^-(f_{\underline{\varsigma}})$. Combined with the Main Theorem this implies that, if the sequence $\underline{\varsigma}$ is not eventually constant, then the set of accumulation measures

of $(\rho_t^{\mathbb{R}}(\underline{\zeta}))_{t>0}$ and that of $(\rho_t(\underline{\zeta}))_{t>0}$, are both equal to the segment joining the invariant probability measure supported on $\mathcal{O}^+(f_{\underline{\zeta}})$ to the invariant probability measure supported on $\mathcal{O}^-(f_{\underline{\zeta}})$.

Remark 3.7 (*Speed of convergence to ground states*). If the sequence $\underline{\zeta}$ is eventually constant, then the Main Theorem implies that, as the temperature drops to zero, the geometric Gibbs states of $f_{\underline{\zeta}}$ converge super-exponentially to the periodic measure supported either on $\mathcal{O}^+(f_{\underline{\zeta}})$, or $\mathcal{O}^-(f_{\underline{\zeta}})$. In other situations the convergence is only exponential, as in the case of the shift map and a locally constant potential [6]. For the shift map and a potential admitting a unique ground state, the exponential convergence can be derived from the large deviation principle in [3, §3.1.3], using the fact that the rate function is finite on a dense set. The Main Theorem shows that this large deviation principle holds, and that the corresponding rate function is everywhere equal to $+\infty$, except on $\mathcal{O}^+(f_{\underline{\zeta}})$ or on $\mathcal{O}^-(f_{\underline{\zeta}})$ (depending on the choice of $\underline{\zeta}$) where it vanishes.

Remark 3.8 (*Pressure at low temperatures*). Our estimates show that there is a constant γ in $(0, 1)$ such that for every $\underline{\zeta}$ in $\{+, -\}^{\mathbb{N}}$, and every sufficiently large $t > 0$ we have

$$P_{f_{\underline{\zeta}}}^{\mathbb{R}}(t) \sim P_{f_{\underline{\zeta}}}(t) \sim -t \frac{\chi_{\text{crit}}(f_{\underline{\zeta}})}{2} + \frac{\log 2}{3} \gamma \left(\frac{4}{A(\underline{\zeta})} t \right)^3,$$

see (6.19) and (6.21) for precisions.

3.3. A concrete admissible family

In this subsection we exhibit a concrete (real) 1-parameter family of quadratic-like maps satisfying the hypotheses of the Main Theorem. We use this family to prove the Sensitive Dependence of Geometric Gibbs States, in §3.4 below.

Recall that for every c in \mathbb{C} , we denote by f_c the complex quadratic polynomial given by $f_c(z) = z^2 + c$. For each parameter λ in $\mathcal{P}_3(-2)$, put $p^-(\lambda) := p^-(f_{\lambda})$ and define the polynomial

$$P_{\lambda}(w) := (w^2 - \beta(\lambda)^2) \prod_{i=0}^2 [(w - f_{\lambda}^i(p(\lambda))) (w - f_{\lambda}^i(p^+(\lambda)))]^2 \\ \cdot (w - p^-(\lambda)) \prod_{j=1}^5 (w - f_{\lambda}^j(p^-(\lambda)))^2.$$

Noting that $DP_{\lambda}(p^-(\lambda)) \neq 0$, define

$$\omega(\lambda) := \frac{2}{p^-(\lambda)^2 DP_{\lambda}(p^-(\lambda))} \left(\frac{(Df_{\lambda}^3(p^+(\lambda)))^2}{Df_{\lambda}^6(p^-(\lambda))} - 1 \right),$$

and the polynomial

$$\widehat{f}_\lambda(w) := \lambda + w^2 + w^3\omega(\lambda)P_\lambda(w).$$

Note that each of the coefficients of \widehat{f}_λ depends holomorphically on λ in $\mathcal{P}_3(-2)$, and that \widehat{f}_λ is real when λ is real. Moreover, we have $\omega(-2) = 0$, so \widehat{f}_{-2} coincides with the quadratic polynomial f_{-2} .

By definition, for each λ in $\mathcal{P}_3(-2)$ the polynomial \widehat{f}_λ coincides with f_λ on $\{\beta(\lambda), -\beta(\lambda)\}$, and on the orbits of $p(\lambda)$, $p^+(\lambda)$, and $p^-(\lambda)$. Moreover, the derivative of \widehat{f}_λ coincides with that of f_λ at every point in the orbit of $p(\lambda)$ and $p^+(\lambda)$, so

$$\chi_{\widehat{f}_\lambda}(p(\lambda)) = \chi_{f_\lambda}(p(\lambda)) \quad \text{and} \quad \chi_{\widehat{f}_\lambda}(p^+(\lambda)) = \chi_{f_\lambda}(p^+(\lambda)). \quad (3.2)$$

On the other hand,

$$D\widehat{f}_\lambda(p^-(\lambda)) = 2p^-(\lambda) \frac{(Df_\lambda^3(p^+(\lambda)))^2}{Df_\lambda^6(p^-(\lambda))},$$

and for each j in $\{1, \dots, 5\}$ the derivative of \widehat{f}_λ coincides with that of f_λ at $f_\lambda^j(p^-(\lambda))$. Thus $D\widehat{f}_\lambda^6(p^-(\lambda)) = (Df_\lambda^3(p^+(\lambda)))^2$ and

$$\chi_{\widehat{f}_\lambda}(p^-(\lambda)) = \chi_{f_\lambda}(p^+(\lambda)) = \chi_{\widehat{f}_\lambda}(p^+(\lambda)). \quad (3.3)$$

Lemma 3.9. *Let $K_0 > 1$ be given. For each λ in $\mathcal{P}_3(-2)$, put $U_\lambda := \widehat{f}_\lambda^{-1}(B(0, 80))$. Then there are $r_\# > 0$ and a map $\chi : B(-2, r_\#) \rightarrow \mathcal{X}$ such that for every λ in $B(-2, r_\#)$ the map $\widehat{f}_\lambda : U_\lambda \rightarrow B(0, 80)$ is a normalized quadratic-like map, and the family*

$$\mathcal{F}_0 := \{\widehat{f}_\lambda : U_\lambda \rightarrow B(0, 80) \mid \lambda \in B(-2, r_\#)\}$$

is uniform with constants K_0 and $R = 80$, and with $c(\widehat{f}_\lambda) = \chi(\lambda)$. Moreover, there is $\delta > 0$ such that χ maps $[-2, -2 + r_\#)$ homeomorphically onto $[-2, -2 + \delta)$, and there is $n_\# \geq 1$ such that for every integer $n \geq n_\#$ there is a continuous map $\sigma_n : \mathcal{K}_n \rightarrow [-2, -2 + r_\#)$ such that $c \mapsto \chi(\sigma_n(c))$ is the identity on \mathcal{K}_n . In particular,

$$\{\widehat{f}_\lambda \in \mathcal{F}_0 \mid \lambda \in \sigma_n(\mathcal{K}_n)\} \subseteq \mathcal{K}_n(\mathcal{F}_0),$$

the map $s_n : \mathcal{K}_n \rightarrow \mathcal{K}_n(\mathcal{F}_0)$ given by $s_n(c) := \widehat{f}_{\sigma_n(c)}$ is continuous, and the family \mathcal{F}_0 is admissible.

Proof. Since $\omega(-2) = 0$, and ω and P_λ are holomorphic in λ , we can choose $r_1 > 0$ such that $B(-2, r_1)$ is contained in $\mathcal{P}_3(-2)$, and such that for every λ in $B(-2, r_1)$ the closure of the open set U_λ is contained in $B(0, 80)$ and $\widehat{f}_\lambda : U_\lambda \rightarrow B(0, 80)$ is a quadratic-like map. For each r in $(0, r_1]$, consider the family of quadratic-like maps

$$\mathcal{F}(r) := (\widehat{f}_\lambda : U_\lambda \rightarrow B(0, 80))_{\lambda \in B(0, r)}.$$

Noting that for λ close to -2 the set ∂U_λ is an analytic Jordan curve that is close to ∂U_{-2} in the C^1 topology, it follows that there is r_2 in $(0, r_1)$ such that the map $B(0, r_2) \rightarrow \mathcal{U}$ given by $\lambda \mapsto U_\lambda$ is continuous with respect to the Carathéodory topology on \mathcal{U} , and that the family of quadratic-like maps $\mathcal{F}(r_2)$ is analytic in the sense of [20, §II, 1]. Moreover, the considerations in [20, §II] imply that for every r_3 sufficiently small in $(0, r_2)$ the family $\mathcal{F}(r_3)$ satisfies the following. There are a map $\chi : B(0, r_3) \rightarrow \mathcal{X}$, and a constant K_0 such that the family $\mathcal{F}(r_3)$ is uniform with constants K_0 and $R = 80$, and with the function $c : \mathcal{F}_0 \rightarrow \mathcal{X}$ given by $c(f) := \chi(f(0))$. Note that for every λ in $B(0, r_3)$, we have

$$\widehat{f}_\lambda(0) = \lambda \text{ and } c(\widehat{f}_\lambda) = \chi(\lambda).$$

Moreover, for every real parameter λ in $B(0, r_3)$ the conjugacy $h_{\widehat{f}_\lambda}$ commutes with the complex conjugation, and therefore $c(\widehat{f}_\lambda)$ is real. If λ in $B(-2, r_3)$ is real and satisfies $\lambda > -2$, then we have $\widehat{f}_\lambda(0) = \lambda > -\beta(\lambda)$. Together with

$$\widehat{f}_\lambda(-\beta(\lambda)) = \widehat{f}_\lambda(\beta(\lambda)) = \beta(\lambda),$$

this implies that $c(\widehat{f}_\lambda) > -2$. By [13, Lemma A.1] there is $\varrho > 0$, such that for every λ in $(-2, -2 + \varrho)$ we have by (3.2)

$$\chi_{\widehat{f}_\lambda}(p(\lambda)) = \chi_{f_\lambda}(p(\lambda)) > \chi_{f_\lambda}(p^+(\lambda)) = \chi_{\widehat{f}_\lambda}(p^+(\lambda)).$$

Thus, reducing r_3 if necessary, the inequality in (6.1) is satisfied.

Note that [20, Proposition 17 and Theorem 4] implies that there is $r_\#$ in $(0, r_3)$ such that χ is locally injective at each point of $B(-2, r_\#) \setminus \{-2\}$. Thus, there is $\delta > 0$ such that χ maps $[-2, -2 + r_\#)$ homeomorphically onto $[-2, -2 + \delta)$. Since by Proposition 2.3 there is $n_\# \geq n_1$ such that for every integer $n \geq n_\#$ the set \mathcal{K}_n is contained in $(-2, -2 + \delta)$, we conclude that for every integer $n \geq n_\#$ there are continuous map $\sigma_n : \mathcal{K}_n \rightarrow [-2, -2 + r_\#)$ such that $c \mapsto \chi(\sigma_n(c))$ is the identity on \mathcal{K}_n . In particular,

$$\{\widehat{f}_\lambda \in \mathcal{F}_0 \mid \lambda \in \sigma_n(\mathcal{K}_n)\} \subseteq \mathcal{K}_n(\mathcal{F}_0).$$

To finish the proof, note that the map $[-2, -2 + r_\#) \rightarrow \mathcal{F}_0$ given by $\lambda \mapsto \widehat{f}_\lambda$ is continuous and thus, for every integer $n \geq n_\#$ the map $s_n : \mathcal{K}_n \rightarrow \mathcal{K}_n(\mathcal{F}_0)$ given by $s_n(c) := \widehat{f}_{\sigma_n(c)}$ is continuous, and $c \circ s_n$ is the identity on \mathcal{K}_n . This completes the proof that \mathcal{F}_0 is an admissible family, and of the lemma. \square

3.4. Proof of the Sensitive Dependence of Geometric Gibbs States assuming the Main Theorem

Let K_0 be the constant given by the Main Theorem with $R = 80$, and let $r_\# > 0$ and the family of quadratic-like maps \mathcal{F}_0 be given by Lemma 3.9 for this choice

of K_0 . Putting $\Lambda_0 := B(-2, r_\#)$, we have $\mathcal{F}_0 = (\widehat{f_\lambda})_{\lambda \in \Lambda_0}$. On the other hand, \mathcal{F}_0 is uniform with constants K_0 and 80, and admissible by Lemma 3.9. Fix a sufficiently large integer n for which the conclusions of the Main Theorem are satisfied with $\mathcal{F} = \mathcal{F}_0$, and let $(f_\varsigma)_{\varsigma \in \{+, -\}^\mathbb{N}}$ and A be as in the statement of the Main Theorem. Given ς in $\{+, -\}^\mathbb{N}$, denote by $\lambda(\varsigma)$ the unique parameter in Λ_0 such that $\widehat{f_{\lambda(\varsigma)}} = f_\varsigma$. By Lemma 3.9, the parameter $\lambda(\varsigma)$ is real. Then we prove the Sensitive Dependence of Geometric Gibbs States with $\Lambda = \{\lambda(\varsigma) \mid \varsigma \in \{+, -\}^\mathbb{N}\}$.

Put

$$A_{\sup} := \sup_{\varsigma \in \{+, -\}^\mathbb{N}} A(\varsigma) \quad \text{and} \quad A_{\inf} := \inf_{\varsigma \in \{+, -\}^\mathbb{N}} A(\varsigma).$$

Let $(\beta_\ell)_{\ell \in \mathbb{N}}$ be a sequence of inverse temperatures such that $\beta_\ell \rightarrow +\infty$ as $\ell \rightarrow +\infty$. Replacing $(\beta_\ell)_{\ell \in \mathbb{N}}$ by a subsequence if necessary, assume that $\beta_1 \geq A_{\sup}$, and that for every ℓ in \mathbb{N} we have

$$\beta_{\ell+1} \geq A_{\sup} \left(\frac{\beta_\ell}{A_{\inf}} + 2 \right). \quad (3.4)$$

For each ℓ in \mathbb{N} put $m(\ell) := \lfloor \beta_\ell / A_{\sup} \rfloor$, and note that $m(1) \geq 1$ and that

$$m(\ell+1) \geq \frac{\beta_{\ell+1}}{A_{\sup}} - 1 \geq \frac{\beta_\ell}{A_{\inf}} + 1 \geq m(\ell) + 1.$$

Fix a sequence $\varsigma_0 = (\varsigma_0(m))_{m \in \mathbb{N}}$ in $\{+, -\}^\mathbb{N}$, let λ_0 in Λ_0 be such that $\widehat{f_{\lambda_0}} = f_{\varsigma_0}$, and let $\varepsilon > 0$ be given. Then there is $\ell_0 \geq 1$ such that for every $\varsigma := (\varsigma(m))_{m \in \mathbb{N}}$ in $\{+, -\}^\mathbb{N}$ such that for every m in $[0, m(\ell_0) - 1]$ we have $\varsigma(m) = \varsigma_0(m)$, the parameter λ in Λ_0 such that $\widehat{f_\lambda} = f_\varsigma$, satisfies $|\lambda - \lambda_0| < \varepsilon$. Let ς be the unique such sequence, such that in addition for every even (resp. odd) integer $\ell \geq \ell_0 + 1$, and every m in $[m(\ell), m(\ell) + 1]$, we have $\varsigma(m) = +$ (resp. $\varsigma(m) = -$).

For every integer $\ell \geq \ell_0 + 1$, we have

$$\beta_\ell \geq A_{\sup} m(\ell) \geq A(\varsigma) m(\ell),$$

and by (3.4)

$$A(\varsigma)(m(\ell+1) - 1) \geq A_{\inf} \left(\frac{\beta_{\ell+1}}{A_{\sup}} - 2 \right) \geq \beta_\ell.$$

This proves that β_ℓ is in $[A(\varsigma)m(\ell), A(\varsigma)(m(\ell+1) - 1)]$. Since by definition of ς for every $\ell \geq \ell_0$ we have

$$\varsigma(m(\ell)) = \dots = \varsigma(m(\ell+1) - 1),$$

and since $\varsigma(m(\ell))$ alternates between $+$ and $-$ according to whether ℓ is even or odd, the Main Theorem implies the desired assertion for the map $\widehat{f_\lambda} = f_\varsigma$.

4. The Geometric Peierls condition, and uniform estimates

In this section we introduce the Geometric Peierls condition, and give a criterion for maps in a uniform family to satisfy this condition with uniform constants. We also make other uniform estimates that are used in the rest of the paper, which are mostly deduced from analogous estimates for quadratic maps in [13].

To state the Geometric Peierls condition, we introduce some notation. For every normalized quadratic-like map f , put

$$\chi_{\text{crit}}(f) := \liminf_{n \rightarrow +\infty} \frac{1}{n} \log |Df^n(f(0))|.$$

Let \mathcal{F} be a uniform family of quadratic-like maps, n an integer satisfying $n \geq 5$, and f a map in $\mathcal{K}_n(\mathcal{F})$. Put

$$V_f := P_{f,n+1}(0) = f^{-1}(P_{f,n}(\tilde{\beta}(f))),$$

and

$$D'_f := \{w \in \mathbb{C} \setminus V_f \mid f^m(w) \in V_f \text{ for some } m \in \mathbb{N}\}.$$

For w in D'_f denote by $m_f(w)$ the least m in \mathbb{N} such that $f^m(w) \in V_f$, and call it the *first landing time of w to V_f* . The *first landing map to V_f* is the map $L_f: D'_f \rightarrow V_f$ defined by $L_f(w) := f^{m_f(w)}(w)$.

Definition 4.1 (*Geometric Peierls Condition*). Let \mathcal{F} be a uniform family of quadratic-like maps, let n be an integer satisfying $n \geq 5$ an integer, and f a map in $\mathcal{K}_n(\mathcal{F})$. Given $\kappa > 0$ and $v > 0$, a quadratic-like map f in \mathcal{F} satisfies the *Geometric Peierls Condition with constants κ and v* , if for every z in $L_f^{-1}(V_f)$ we have

$$|DL_f(z)| \geq \kappa \exp((\chi_{\text{crit}}(f)/2 + v)m_f(z)). \quad (4.1)$$

Remark 4.2. The analogy between (4.1) and the usual Peierls conditions for contour models is as follows. We use the terminology in [40, II]. As usual, the one-point interaction energy corresponds to the geometric potential $-\log |Df|$. The (orbit of the) critical point $z = 0$ of f plays the rôle of the unique ground state. In contrast to the usual Peierls condition for contour models where the ground state is assumed to be supported on a periodic configuration, it is crucial for the Main Theorem to allow the orbit of 0 to be nonperiodic. However, we do require later that the Lyapunov exponent $\lim_{n \rightarrow +\infty} \frac{1}{n} \log |Df^n(f(0))|$ exists, so that the “lim inf” that defines $\chi_{\text{crit}}(f)$ is actually a limit. Consider an initial condition w near the critical point 0 of f . Following the definition of the boundary of a configuration [40, II, Definition 2.2], we see that for the “boundary” of w with respect to 0 to be finite, it is enough to assume that for some

integer $\tau \geq 1$ we have $f^\tau(w) = f^\tau(0)$. The orbit of w shadows that of 0 up to a certain time ℓ , so that the derivatives of $f^{\ell-1}$ at $f(w)$ and at $f(0)$ are comparable. After time ℓ , the orbit of w can be significantly different from that of 0 . To simplify, assume that the point z defined by $z := f^\ell(w)$ satisfies $L_f(z) = 0$, so we have

$$f^{\ell+m_f(z)}(w) = f^{m_f(z)}(z) = L_f(z) = 0,$$

and therefore the boundary of w with respect to 0 is bounded from above by $m_f(z)$. Up to a uniform distortion constant, the Hamiltonian at z relative to w is equal to⁸

$$-\log |Df^{\ell+m_f(z)}(w)| - \left(-\frac{\ell + m_f(z)}{2} \chi_{\text{crit}}(f) \right) \sim -\log |DL_f(z)| + \frac{m_f(z)}{2} \chi_{\text{crit}}(f).$$

Thus, condition (4.1) becomes Peierls condition as in [40, II, Definition 2.3].⁹

The following is a criterion for the Geometric Peierls Condition. For future reference, it is stated in a slightly stronger form than what is needed for this paper.

Proposition 4.3. *For every $v > 0$ satisfying $v < \frac{1}{2} \log 2$ and every $R > 0$, there are constants $K_1 > 1$, $n_1 \geq 6$, and $\kappa_1 > 0$, such that the following property holds. If the family of quadratic-like maps \mathcal{F} is uniform with constants K_1 and R , then for every integer $n \geq n_1$, every element f of $\mathcal{K}_n(\mathcal{F})$ satisfies the Geometric Peierls Condition with constants κ_1 and v . Furthermore, we have*

$$\chi_f(\beta(f)) > \chi_{\text{crit}}(f) + 2v, \chi_{\text{crit}}(f) > 2v, \text{ and } \chi_f(p(f)) < \chi_f(p^+(f)) + \frac{v}{4}.$$

Roughly speaking, to prove the geometric Peierls condition we show that $\chi_{\text{crit}}(f) \sim \log 2$ (Lemma 4.7), and that for every z in $L_f^{-1}(V_f)$ we have $L_f(z) \sim 2^{m_f(z)}$. Assuming that the dilatation constant of the uniform family is sufficiently close to 1, we derive these estimates from the analogous estimates for the quadratic map $f_{c(f)}$ established in [13]. This is done with the help of the uniform geometric estimates established in §4.1.

After some uniform geometric estimates in §4.1, the proof of Proposition 4.3 is given in §4.2. In §4.3 we make various uniform estimates.

4.1. Uniform geometric estimates

In this subsection we use Mori's theorem on the modulus of continuity of normalized quasi-conformal maps, to obtain some preliminary estimates for quadratic-like maps in a given uniform family.

⁸ Here we replaced $\log |Df^{\ell+m_f(z)}(0)| = -\infty$ by $(\ell + m_f(z))\chi_{\text{crit}}(f)/2$, which is what appears naturally in several estimates, see for example Lemmas 4.10 and 5.3.

⁹ To follow the analogy, we should require the constant κ to be larger than the implicit distortion constant in the computation above. Later on we compensate a possible small value of κ by assuming that the map f is in $\mathcal{K}_n(\mathcal{F})$ for a sufficiently large integer n .

Lemma 4.4. *Given $K > 1$ and $R > 0$ there is a constant $C_1 > 1$ such that for every uniform family \mathcal{F} of quadratic-like maps with constants K and R , the following property holds for every f in \mathcal{F} and w in \widehat{X}_f :*

$$C_1^{-1}|w| \leq |f(w) - f(0)|^{1/2} \leq C_1|w|. \quad (4.2)$$

Moreover, if in addition w is in X_f , then

$$C_1^{-1}|w| \leq |Df(w)| \leq C_1|w|, \quad (4.3)$$

$$C_1^{-1}|Df_{c(f)}(h_f^{-1}(w))|^K \leq |Df(w)| \leq C_1|Df_{c(f)}(h_f^{-1}(w))|^{\frac{1}{K}}. \quad (4.4)$$

The proof of this lemma is after the following one.

Lemma 4.5. *For each $R > 0$ there is a constant $C_2 > 1$ such that the following property holds. Let $K \geq 1$ be given, and let h be a K -quasi-conformal homeomorphism of \mathbb{C} that is holomorphic outside $\text{cl}(B(0, R))$ and that is tangent to the identity at infinity. Then for every z and z' in $B(0, 2R)$ we have*

$$C_2^{-K}|z - z'|^K \leq |h(z) - h(z')| \leq C_2|z - z'|^{\frac{1}{K}}.$$

Proof. Replacing h by $h - h(0)$ if necessary, assume $h(0) = 0$. Put $D := h(B(0, 2R))$, let $\varphi: D \rightarrow B(0, 2R)$ be a bi-holomorphic map fixing $z = 0$, and note that $\varphi \circ h|_{B(0, 2R)}$ is a K -quasi-conformal homeomorphism of $B(0, 2R)$ fixing $z = 0$. Thus, Mori's theorem implies that for every z and z' in $B(0, 2R)$, we have

$$(16^K 2R^{K-1})^{-1}|z - z'|^K \leq |\varphi \circ h(z) - \varphi \circ h(z')| \leq 16(2R)^{1-\frac{1}{K}}|z - z'|^{1/K}, \quad (4.5)$$

see for example [1, p. 47]. It remains to estimate the distortion of φ on D . Note first that the holomorphic function $g: B(0, R^{-1}) \setminus \{0\} \rightarrow \mathbb{C}$ defined by $g(\zeta) := h(\zeta^{-1})^{-1}$ extends holomorphically to $\zeta = 0$, and that the extension, also denoted by g , satisfies $g(0) = 0$ and $Dg(0) = 1$. By Koebe's $\frac{1}{4}$ -theorem and the version of the Koebe Distortion theorem in [9, Theorem 1.6], we have

$$B(0, (8R)^{-1}) \subset g(B(0, (2R)^{-1})) \subset B(0, 2R^{-1}),$$

and therefore,

$$B(0, R/2) \subset D \subset B(0, 8R).$$

By Schwarz' Lemma and Koebe's $\frac{1}{4}$ -theorem we have $\frac{1}{4} \leq |D\varphi(0)| \leq 4$. Next we show that φ has a univalent extension to $\widehat{D} := h(B(0, 4R))$. Note first that φ extends continuously to $\text{cl}(D)$, since ∂D is a Jordan curve; we denote this extension also by φ . Consider the holomorphic involution ι of $A := h(B(0, 4R) \setminus \text{cl}(B(0, R)))$ defined

by $\iota(w) := h(4R^2/h^{-1}(w))$. Let $\widehat{\varphi}: \widehat{D} \rightarrow \mathbb{C}$ be the function that coincides with φ on $\text{cl}(D)$, and that for z in $\widehat{D} \setminus \text{cl}(D)$ is given by $\widehat{\varphi}(z) := 4R^2/\varphi(\iota(z))$. Then $\widehat{\varphi}$ is homeomorphism from \widehat{D} to $B(0, 4R)$, and by Schwarz reflection principle it is holomorphic. By the Koebe Distortion Theorem, there is a universal constant $\Delta > 1$ independent of h such that for every distinct z and z' in $B(0, 2R)$ we have

$$(4\Delta)^{-1} \leq \Delta^{-1} |D\varphi(0)| \leq \frac{|\varphi \circ h(z) - \varphi \circ h(z')|}{|h(z) - h(z')|} \leq \Delta |D\varphi(0)| \leq 4\Delta.$$

Together with (4.5), this proves the desired chain of inequalities with $C_2 = 16 \cdot 2R \cdot 4\Delta$. \square

Proof of Lemma 4.4. Let $C_2 > 0$ be the constant given by Lemma 4.5. Then

$$\widehat{D} := \sup_{f \in \mathcal{F}} \text{diam}(f(\widehat{X}_f)) \leq C_2 \left(\sup_{c \in \widehat{\mathcal{X}}} \text{diam}(f_c(\widehat{X}_c)) \right)^{\frac{1}{K}} < +\infty.$$

Observe that, since $\widehat{X}_f \subset f(\widehat{X}_f)$ and since \widehat{X}_f contains 0, for every w in \widehat{X}_f we have $|w| \leq \widehat{D}$.

On the other hand, since for c in $\widehat{\mathcal{X}}$ the sets

$$\partial X_c := \{z \in \mathbb{C} \mid G_c(z) = 1\} \quad \text{and} \quad \partial \widehat{X}_c = \{z \in \mathbb{C} \mid G_c(z) = 2\}$$

are disjoint and depend continuously with c , we have

$$r := \inf_{c \in \widehat{\mathcal{X}}} \text{dist}(\partial X_c, \partial \widehat{X}_c) > 0.$$

In particular, for every c in $\widehat{\mathcal{X}}$ the set \widehat{X}_c contains $B(0, r)$. Combined with Lemma 4.5 this implies that, if we put $\widehat{r} := (r/C_2)^K$, then for every f in \mathcal{F} and w in X_f the set \widehat{X}_f contains $B(w, \widehat{r})$.

To prove (4.2), note that for every f in \mathcal{F} and w in \widehat{X}_f we have

$$|w^2 + w^3 R_f(w)| = |f(w) - f(0)| \leq \text{diam}(f(\widehat{X}_f)) \leq \widehat{D}, \quad (4.6)$$

and therefore $|w^3 R_f(w)| \leq \widehat{D} + \widehat{D}^2$. So, if we put $\widetilde{R} := (\widehat{D} + \widehat{D}^2)/\widehat{r}^3$, then the maximum principle implies $|R_f| \leq \widetilde{R}$ on $B(0, \widehat{r})$. Letting $\widehat{r}_0 := \min \left\{ \widehat{r}, 1/(2\widetilde{R}) \right\}$, for every w in $B(0, \widehat{r}_0)$ we have $|w R_f(w)| \leq 1/2$, and therefore

$$\frac{1}{2} \leq \left| \frac{f(w) - f(0)}{w^2} \right| \leq \frac{3}{2}.$$

Let w in $\widehat{X}_f \setminus B(0, \widehat{r}_0)$ be given and put $z := h_f^{-1}(w)$. Applying Lemma 4.5 with $K = K$ twice and using $h_f(0) = 0$ we obtain

$$\begin{aligned}
 |f(w) - f(0)| &= |h_f(f_{c(f)}(z)) - h_f(f_{c(f)}(0))| \geq C_2^{-K} |f_{c(f)}(z) - f_{c(f)}(0)|^K \\
 &= C_2^{-K} |z|^{2K} \geq C_2^{-K-2K^2} |w|^{2K^2} \geq C_2^{-K-2K^2} \widehat{r}_0^{2K^2}.
 \end{aligned}$$

Together with (4.6) these estimates imply (4.2) with

$$C_1 = \max \left\{ 2, C_2^{K+2K^2} \widehat{D}^2 \widehat{r}_0^{-2K^2}, \widehat{D} \widehat{r}_0^{-2} \right\}^{\frac{1}{2}}.$$

To prove (4.3) and (4.4), note first that by Schwarz' Lemma for every w in $B(0, \widehat{r}/2)$ we have $|DR_f(w)| \leq 2\widetilde{R}/\widehat{r}$. Then, putting $\widehat{r}_1 := \min \left\{ \widehat{r}, 1/(10\widetilde{R}) \right\}$, for every w in $B(0, \widehat{r}_1)$ we have $|3wR_f(w) + w^2DR_f(w)| \leq 1/2$, so

$$\frac{3}{2} \leq \left| \frac{Df(w)}{w} \right| \leq \frac{5}{2}.$$

Let w in $X_c \setminus B(0, \widehat{r}_1)$ be given. Using that $B(w, \widehat{r})$ is contained in \widehat{X}_f and the definition of \widehat{D} , Schwarz' lemma implies $|Df(w)|/|w| \leq (\widehat{D}/\widehat{r})/\widehat{r}_1$. To estimate $|Df(w)|/|w|$ from below, put

$$z := h_f^{-1}(w), r_1 := \min\{C_2^{-K}\widehat{r}_1^K, r\}, \quad \text{and} \quad B(z) := B(z, r_1).$$

By Lemma 4.5 we have $r_1 \leq |z|$, so $f_{c(f)}$ is injective on $B(z)$. By Koebe's $\frac{1}{4}$ -theorem, the set $f_{c(f)}(B(z))$ contains

$$B(f_{c(f)}(z), |Df_{c(f)}(z)|r_1/4),$$

and therefore $B(f_{c(f)}(z), r_1^2/2)$. By Lemma 4.5 we have

$$h_f(B(z)) \subset B\left(w, C_2 r_1^{\frac{1}{K}}\right)$$

and

$$\widehat{B}(w) := B(f(w), C_2^{-K}(r_1^2/2)^K) \subset h_f(f_{c(f)}(B(z))).$$

Thus, if we put $\varepsilon := C_2^{-1-K} 2^{-K} r_1^{2K - \frac{1}{K}}$, then Schwarz' lemma applied to $f^{-1}|_{\widehat{B}(w)}$ implies

$$|Df(w)| \geq \varepsilon \quad \text{and} \quad \frac{|Df(w)|}{|w|} \geq \frac{\varepsilon}{\widehat{D}}.$$

This completes the proof of (4.3) with C_1 equal to $\widetilde{C} := \max\{3, \widehat{D}/(\widehat{r}\widehat{r}_1), \widehat{D}/\varepsilon\}$. Combined with Lemma 4.5 this last estimate implies,

$$\begin{aligned} \frac{\tilde{C}^{-1}}{(2C_2)^K} |Df_{c(f)}(z)|^K &= \frac{\tilde{C}^{-1}}{C_2^K} |z|^K \leq \tilde{C}^{-1} |h_f(z)| = \tilde{C}^{-1} |w| \\ &\leq |Df(w)| \leq \tilde{C} |w| = \tilde{C} |h_f(z)| \leq \tilde{C} C_2 |z|^{\frac{1}{K}} = \frac{\tilde{C} C_2}{2^{1/K}} |Df_{c(f)}(z)|^{\frac{1}{K}}. \end{aligned} \quad (4.7)$$

This proves (4.4) with $C_1 = \tilde{C}(2C_2)^K$, and completes the proof of the lemma. \square

4.2. Proving the Geometric Peierls Condition

In this subsection we prove Proposition 4.3. The proof is given after the following lemmas.

Lemma 4.6. *Given $K > 1$ and $R > 0$ there is a constant $C_3 > 1$ such that for every uniform family of quadratic-like maps \mathcal{F} with constants K and R , every integer $n \geq 6$, and every f in $\mathcal{K}_n(\mathcal{F})$, the following property holds. Let w be a point in X_f and $m \geq 1$ an integer such that f^m maps a neighborhood of w biholomorphically onto $P_{f,1}(0)$. Then*

$$C_3^{-1} |Df_{c(f)}^m(h_f^{-1}(w))|^{\frac{1}{K}} \leq |Df^m(w)| \leq C_3 |Df_{c(f)}^m(h_f^{-1}(w))|^K.$$

Proof. Let C_2 be the constant given by Lemma 4.5. From the proof of [13, Lemma 5.4], we have

$$\Xi := \inf_{c \in \mathcal{P}_4(-2)} \text{diam}(P_{c,1}(0)) > 0.$$

We prove the lemma with $C_3 = C_2^{1+K} \Xi^{\frac{1}{K}-K}$. Let w be as in the statement of the lemma. By hypothesis, there is a neighborhood W of w that is mapped biholomorphically onto $P_{f,1}(0)$ by f^m . Take arbitrary points u and v in W , and put

$$z := h_f^{-1}(w), x := h_f^{-1}(u), \text{ and } y := h_f^{-1}(v).$$

By Lemma 4.5, we have

$$\begin{aligned} \frac{|f^m(u) - f^m(v)|}{|u - v|} &= \frac{|f^m(h_f(x)) - f^m(h_f(y))|}{|h_f(x) - h_f(y)|} \\ &\leq C_2^{1+K} \frac{|f_{c(f)}^m(x) - f_{c(f)}^m(y)|^{\frac{1}{K}}}{|x - y|^K} \\ &= C_2^{1+K} |f_{c(f)}^m(x) - f_{c(f)}^m(y)|^{\frac{1}{K}-K} \frac{|f_{c(f)}^m(x) - f_{c(f)}^m(y)|^K}{|x - y|^K} \\ &\leq C_2^{1+K} \text{diam}(P_{c(f),1}(0))^{\frac{1}{K}-K} \frac{|f_{c(f)}^m(x) - f_{c(f)}^m(y)|^K}{|x - y|^K}. \end{aligned}$$

Since u and v are arbitrary points of W , we conclude that

$$|Df^m(w)| \leq C_2^{1+K} \operatorname{diam}(P_{c(f),1}(0))^{\frac{1}{K}-K} |Df_{c(f)}^m(z)|^K \leq C_2^{1+K} \Xi^{\frac{1}{K}-K} |Df_{c(f)}^m(z)|^K.$$

The proof of the other inequality follows similar arguments. \square

Lemma 4.7. *For every $K > 1$, $R > 0$, and $\varepsilon > 0$ there is an integer $n_2 \geq 5$ such that for every uniform family of quadratic-like maps \mathcal{F} with constants K and R , the following property holds. For every integer $n \geq n_2$, and every f in $\mathcal{K}_n(\mathcal{F})$, we have*

$$K^{-1}(1-\varepsilon)\log 2 \leq \chi_{\text{crit}}(f) \leq K(1+\varepsilon)\log 2, \quad (4.8)$$

and for every periodic point p of f in $h_f(\Lambda_{c(f)})$, we have

$$K^{-1}(1-\varepsilon)\log 2 \leq \chi_f(p) \leq K(1+\varepsilon)\log 2. \quad (4.9)$$

Proof. Let C_3 be the constant given by Lemma 4.6.

Combining [13, Lemma 4.2] and [13, Lemma 5.3] with $m_1 = 4$, we conclude that there are constants $\widehat{C}_0 > 0$ and $n_0 \geq 3$ such that for each integer $n \geq n_0$ and each parameter c in \mathcal{K}_n , we have for every z in Λ_c and every integer $m \geq 1$,

$$\widehat{C}_0^{-1} 2^{(1-\varepsilon)m} \leq |Df_c^m(z)| \leq \widehat{C}_0 2^{(1+\varepsilon)m}.$$

Note that $f_c^n(c)$ is in Λ_c , so we can take $z = f_c^n(c)$ above. Noting that for every f in $\mathcal{K}_n(\mathcal{F})$ and every integer $k \geq 1$ the map f^{3k} maps a neighborhood of $h_f(f_c^{n+1}(0)) = f^{n+1}(0)$ biholomorphically onto $P_{f,1}(0)$ (cf., [13, Lemma 5.1]), by Lemma 4.6 for every integer $m \geq 1$ we have

$$C_3^{-1} \widehat{C}_0^{\frac{1}{K}} 2^{K(1+\varepsilon)m} \leq |Df^m(f^{n+1}(0))| \leq C_3 \widehat{C}_0^K 2^{K(1+\varepsilon)m},$$

and

$$C_3^{-1} \widehat{C}_0^{\frac{1}{K}} 2^{K(1+\varepsilon)m} \leq |Df^m(p)| \leq C_3 \widehat{C}_0^K 2^{K(1+\varepsilon)m}.$$

Taking logarithms, dividing by m , and letting $m \rightarrow +\infty$, we conclude the proof of the lemma. \square

Proof of Proposition 4.3. Let $\varepsilon > 0$ be sufficiently small so that

$$\varepsilon < \frac{2}{3} \left(\frac{1}{2} - \frac{v}{\log 2} \right) \text{ and } \varepsilon < \frac{v}{8 \log 2},$$

and let $K_1 > 1$ be sufficiently close to 1 so that

$$v' := \left(\frac{1-\varepsilon}{K_1} - \frac{K_1(1+\varepsilon)}{2} \right) \log 2 > v > 4 \left(K_1(1+\varepsilon) - \frac{1-\varepsilon}{K_1} \right) \log 2.$$

Let n_2 be given by Lemma 4.7 for this value of ε . In view of Proposition 2.3 and of the formula $Df_{-2}(\beta(-2)) = 4$, we can take n_2 larger if necessary so that for every integer $n \geq n_2$ and every parameter c in \mathcal{K}_n we have $\chi_f(\beta(f)) \geq (1-\varepsilon) \log 4$. Assume \mathcal{F} is uniform with constants R and K_1 , and let C_3 be the constant given by Lemma 4.6 with $K = K_1$. Note that C_3 depends on R and v only.

By Lemma 4.7, for every integer $n \geq n_2$ and every f in $\mathcal{K}_n(\mathcal{F})$ we have (4.8) with K replaced by K_1 . On the other hand, by [13, Proposition B] there are $\widehat{\kappa}_1 > 0$ and $\widehat{n}_1 \geq 4$, such that for every integer $n \geq \widehat{n}_1$, every parameter c in \mathcal{K}_n , and every z in $L_c^{-1}(V_c)$, we have

$$|DL_c(z)| \geq \widehat{\kappa}_1 2^{(1-\varepsilon)m_c(z)}.$$

Noting that for each f in $\mathcal{K}_n(\mathcal{F})$ and each w in $L_f^{-1}(V_f)$ the map $f^{m_f(z)}$ maps a neighborhood of w biholomorphically onto $P_{f,1}(0)$, by Lemma 4.6 we have

$$|DL_f(w)| \geq C_3 \widehat{\kappa}_1^{K_1} 2^{\frac{1-\varepsilon}{K_1} m_f(z)}.$$

Noting that by definition of v' we have

$$2^{\frac{1-\varepsilon}{K_1}} = 2^{\frac{K_1(1+\varepsilon)}{2}} \exp(v') \geq \exp(\chi_{\text{crit}}(f)/2 + v'),$$

inequality (4.8) implies the first assertion of the proposition with

$$n_1 = \min\{n_2, \widehat{n}_1\} \text{ and } \kappa_1 = C_3 \widehat{\kappa}_1^{K_1}.$$

It remains to prove the inequalities. The second inequality follows from (4.8), and the definition of v' . To prove the third inequality, note that by (4.8) and the definition of v' , we have

$$\chi_f(p(f)) - \chi_f(p^+(f)) \leq \left(K_1(1+\varepsilon) - \frac{1}{K_1}(1-\varepsilon) \right) \log 2 < \frac{v}{4}.$$

To prove the first inequality, note that by Lemma 4.6 and our choice of n_2 , we have

$$\chi_f(\beta(f)) \geq \frac{1}{K_1} \chi_{f_{c(f)}}(\beta(c(f))) \geq \frac{1-\varepsilon}{K_1} \log 4.$$

Combined with (4.8) and the definition of v' , this implies

$$\chi_f(\beta(f)) - \chi_{\text{crit}}(f) \geq \left(\frac{1-\varepsilon}{K_1} - \frac{K_1(1+\varepsilon)}{2} \right) \log 4 = 2v' > 2v.$$

This we conclude the proof of the proposition. \square

4.3. Uniform estimates

In this subsection we prove various uniform estimates. Throughout this subsection we fix a uniform family of quadratic-like maps \mathcal{F} , with constants K and R .

Lemma 4.8. *There is a constant $\Delta_1 > 1$ that only depends on K and R , such that for each f in $\mathcal{P}_2(\mathcal{F})$ the following properties hold for each integer $k \geq 2$: For each point y in $P_{f,k}(\tilde{\beta}(f))$ or in $P_{f,k}(\beta(f))$ we have*

$$\Delta_1^{-1} |Df(\beta(f))|^k \leq |Df^k(y)| \leq \Delta_1 |Df(\beta(f))|^k.$$

Proof. The proof follows the same lines that [13, Lemma 3.6], and we only need to check that some constants are finite and others are positive. Let C_1 be the constant given by Lemma 4.4, and let C_2 be that given by Lemma 4.5. By the proof of [13, Lemma 3.6] we have

$$\begin{aligned} \Xi_1 &:= \sup_{c \in \mathcal{P}_0(-2)} \sup_{z \in P_{c,1}(\beta(c))} |Df_c(z)| < +\infty, \\ \Xi_2 &:= \inf_{c \in \mathcal{P}_2(-2)} \inf_{z \in P_{c,1}(\beta(c))} |Df_c(z)| > 0, \end{aligned}$$

and

$$\Xi_3 := \inf_{c \in \mathcal{P}_2(-2)} \text{mod}(P_{c,0}(\beta(c)) \setminus \text{cl}(P_{c,1}(\beta(c)))) > 0.$$

By Lemmas 4.4 and 4.5, we have

$$\begin{aligned} \hat{\Xi}_1 &:= \sup_{f \in \mathcal{P}_0(\mathcal{F})} \sup_{w \in P_{f,1}(\beta(f))} |Df(w)| \leq C_1 \Xi_1^{\frac{1}{K}} < +\infty, \\ \hat{\Xi}_2 &:= \inf_{f \in \mathcal{P}_2(\mathcal{F})} \inf_{w \in P_{f,1}(\beta(f))} |Df(w)| \geq C_1^{-1} \Xi_2^K > 0, \end{aligned}$$

and since for every f in \mathcal{F} the conjugacy h_f is K -quasi-conformal

$$\hat{\Xi}_3 := \inf_{f \in \mathcal{P}_2(\mathcal{F})} \text{mod}(P_{f,0}(\beta(f)) \setminus \text{cl}(P_{f,1}(\beta(f)))) \geq \frac{1}{K} \Xi_3 > 0.$$

Let $\Delta > 1$ be the constant given by Koebe Distortion Theorem with $A = \hat{\Xi}_3$. The desired inequalities follow from the fact that f^{k-1} maps each of the sets $P_{f,k}(\beta(f))$ and $P_{f,k}(\tilde{\beta}(f))$ biholomorphically to $P_{f,1}(\beta(f))$ with $\Delta_1 = \Delta \hat{\Xi}_1 \hat{\Xi}_2^{-1}$. \square

For a parameter c in $\mathcal{P}_2(-2)$ the external rays $R_c(7/24)$ and $R_c(17/24)$ land at the point $\gamma(c)$ in $P_{c,1}(0)$, see [13, Section 3.3]. Let \hat{U}_c be the open disk containing $-\beta(c)$ that is bounded by the equipotential 2 and by

$$R_c(7/24) \cup \{\gamma(c)\} \cup R_c(17/24).$$

Put $\widehat{W}_c := f_c^{-1}(\widehat{U}_c)$, and for every $n \geq 3$ and every f in $\mathcal{K}_n(\mathcal{F})$ put $\widehat{W}_f := h_f(\widehat{W}_{c(f)})$.

Lemma 4.9 (Uniform distortion bound). *There is a constant $\Delta_2 > 1$ that only depends on K and R , such that for each integer $n \geq 4$, and each f in $\mathcal{K}_n(\mathcal{F})$ the following properties hold: For each integer $m \geq 1$ and each connected component W of $f^{-m}(P_{f,1}(0))$ on which f^m is univalent, f^m maps a neighborhood of W biholomorphically to \widehat{W}_f and the distortion of this map on W is bounded by Δ_2 .*

Proof. We follow the proof of [13, Lemma 4.3]. From that proof we have that for each parameter c in $\mathcal{P}_4(-2)$ the set \widehat{W}_c contains the closure of $P_{c,1}(0)$ and

$$\widetilde{A} := \inf_{c \in \mathcal{P}_4(-2)} \text{mod}(\widehat{W}_c \setminus \text{cl}(P_{c,1}(0))) > 0.$$

Since for every f in \mathcal{F} the conjugacy h_f is K -quasi-conformal, we have

$$\widehat{A} := \inf_{f \in \mathcal{P}_4(\mathcal{F})} \text{mod}(\widehat{W}_f \setminus \text{cl}(P_{f,1}(0))) \geq \frac{\widetilde{A}}{K} > 0.$$

By [13, Lemma 4.2], f_c^m maps a neighborhood of $h_f^{-1}(W)$ biholomorphically to $\widehat{W}_{c(f)}$. By conjugacy, f^m maps a neighborhood of W biholomorphically to \widehat{W}_f . The conclusion follows from Koebe Distortion Theorem with $A = \widehat{A}$. \square

Lemma 4.10. *There is a constant $C_4 > 1$ that only depends on K and R , such that for each integer $n \geq 4$ and each f in $\mathcal{K}_n(\mathcal{F})$, the following properties hold for each integer $q \geq 1$: For each open set W that is mapped biholomorphically to $P_{f,1}(0)$ by f^q , and each x in W , we have*

$$|Df(x)| \geq C_4^{-1} |Df^{q-1}(f(x))|^{-\frac{1}{2}}.$$

Proof. We follow the proof of [13, Lemma 5.4]. Let C_1 be the constant given by Lemma 4.4, and let C_2 be that given by Lemma 4.5. Let $\Delta_1 > 1$ and $\Delta_2 > 1$ be the constants given by Lemmas 4.8 and 4.9, respectively. From the proof of [13, Lemma 5.4], we have

$$\Xi_1 := \inf_{c \in \mathcal{P}_4(-2)} \text{diam}(P_{c,1}(0)) > 0 \quad \text{and} \quad \Xi_2 := \sup_{c \in \mathcal{P}_4(-2)} |Df_c(\beta(c))| < +\infty,$$

and that for each c in $\mathcal{P}_3(-2)$ the closure of $P_{c,1}(0)$ is contained in \widehat{W}_c and

$$\Xi_3 := \inf_{c \in \mathcal{P}_4(-2)} \text{mod}(\widehat{W}_c \setminus \text{cl}(P_{c,1}(0))) > 0.$$

Since for every f in \mathcal{F} the conjugacy h_f is K -quasi-conformal, we have

$$\widehat{\Xi}_3 := \inf_{f \in \mathcal{P}_4(\mathcal{F})} \text{mod}(\widehat{W}_f \setminus \text{cl}(P_{f,1}(0))) \geq \frac{\Xi_3}{K} > 0,$$

and by Lemma 4.5 and inequality (4.4), we have

$$\widehat{\Xi}_1 := \inf_{f \in \mathcal{P}_4(\mathcal{F})} \text{diam}(P_{f,1}(0)) \geq C_2^{-K} \Xi_1^K > 0$$

and

$$\widehat{\Xi}_2 := \sup_{f \in \mathcal{P}_4(\mathcal{F})} |Df(\beta(f))| \leq C_1 \Xi_2^{\frac{1}{K}} < +\infty.$$

Let $n \geq 4$ be a integer and f in $\mathcal{K}_n(\mathcal{F})$. Note that f^q maps a neighborhood \widetilde{W} of W biholomorphically to \widetilde{W}_f (Lemma 4.9). So, if we put $\widetilde{W}' := f(\widetilde{W})$, then $f(0)$ is not in \widetilde{W}' and f^{q-1} maps \widetilde{W}' biholomorphically to \widetilde{W}_f ; in particular we have

$$\text{mod}(\widetilde{W}' \setminus \text{cl}(f(W))) = \text{mod}(\widehat{W}_f \setminus \text{cl}(P_{f,1}(0))) \geq \widehat{\Xi}_3.$$

Thus there is a constant $A_1 > 0$ independent of n , f and q such that for every x in W , we have

$$|f(x) - f(0)| \geq \text{dist}(f(W), f(0)) \geq \text{dist}(f(W), \partial \widetilde{W}') \geq A_1 \text{diam}(f(W))$$

(cf., [25, Teichmüller's module theorem, II, §1.3]). Thus, if we put

$$A_2 := C_1^{-2} (A_1 \Delta_2^{-1} \widehat{\Xi}_1)^{1/2},$$

then by Lemmas 4.4 and 4.9 with $m = q - 1$ and with W replaced by $f(W)$, we have

$$|Df(x)| \geq C_1^{-2} A_1^{1/2} \text{diam}(f(W))^{1/2} \geq A_2 |Df^{q-1}(f(x))|^{-1/2}.$$

This proves the lemma with constant $C_4 = A_2^{-1}$. \square

Lemma 4.11. *There are constants $C_5 > 0$ and $v_1 > 0$ that only depend on K and R , such that for every f in $\mathcal{P}_5(\mathcal{F})$, every ℓ in \mathbb{N} , and every connected component W of $g_f^{-\ell}(P_{f,1}(0))$, we have*

$$\max\{\text{diam}(W), \text{diam}(f(W)), \text{diam}(f^2(W))\} \leq C_5 \exp(-v_1 \ell).$$

Proof. Let $K \geq 1$ and $R > 0$ be the constants of the family \mathcal{F} . For this R , let C_2 be the constant of Lemma 4.5. By [14, Lemma 2.4] there are constants $C'_0 > 0$ and $v'_0 > 0$

such that for every c in $\mathcal{P}_5(-2)$, every ℓ in \mathbb{N} , and every connected component W' of $g_c^{-\ell}(P_{c,1}(0))$, we have

$$\text{diam}(W') \leq C'_0 \exp(-v'_0 \ell).$$

Put

$$\Xi_0 := \inf_{c \in \mathcal{P}_5(-2)} \text{dist}(Y_c \cup \tilde{Y}_c, \partial P_{c,1}(0)) > 0,$$

$$\Xi_1 := \sup_{c \in \mathcal{P}_0(-2)} \text{diam}(\{z \in \mathbb{C} \mid G_c(z) \leq 2\}) < +\infty,$$

and $\widehat{C}_0 := \max\{C'_0, \Xi_0^{-1} \Xi_1 C'_0\}$. Fix c in $\mathcal{P}_5(-2)$, ℓ in \mathbb{N} , and a connected component W' of $g_c^{-\ell}(P_{c,1}(0))$. For every w in $Y_c \cup \tilde{Y}_c$ define the holomorphic maps

$$z \mapsto \frac{f_c(z) - f_c(w)}{z - w} \text{ and } z \mapsto \frac{f_c^2(z) - f_c^2(w)}{z - w}$$

on X_c . Notice that for z in $\partial P_{c,1}(0)$ both maps are bounded from above by $\Xi_0^{-1} \Xi_1$. By the maximum principle for every z and w in $Y_c \cup \tilde{Y}_c$ we have

$$\max\{|f_c(z) - f_c(w)|, |f_c^2(z) - f_c^2(w)|\} \leq \Xi_0^{-1} \Xi_1 |z - w|.$$

In particular,

$$\max\{\text{diam}(W'), \text{diam}(f_c(W')), \text{diam}(f_c^2(W'))\} \leq \widehat{C}_0 \exp(-v'_0 \ell).$$

From Lemma 4.5 by putting $C_5 := C_2 \widehat{C}_0^{\frac{1}{K}}$ and $v_1 := v'_0/K$, we conclude the proof of the lemma. \square

5. Estimating the geometric pressure function

The aim of this section is to prove Proposition I, stated at the beginning of §5.2. For a uniform family, this proposition allows us to control the geometric pressure function with the itinerary of the critical point. We achieve this aim using an inducing scheme and by adapting a similar result for quadratic maps [13, Proposition D]. The general scheme of this adaptation is the following. The arguments in the original proof can be grouped into 3 types. Purely combinatorial arguments depending only on the combinatorics of the Yoccoz puzzle. Geometric estimates of the sizes of the puzzle pieces. An estimate of the derivative of the first landing map to a neighborhood of the critical point and an estimate of the Lyapunov exponent of the critical value. In the adaptation, the combinatorial arguments follow directly from the conjugacy, and the geometric estimates follow from the Hölder continuity of the conjugacy (cf., Lemma 4.5). The third type of arguments

uses the Geometric Peierls condition in a crucial way. This condition is introduced in Definition 4.1, and it is used in Propositions I, II and 5.6, and in Lemmas 5.4 and 5.5.

In §5.1 we introduce an inducing scheme, and we prove a result on the existence of conformal measures and equilibrium states (Proposition 5.2) that is analogous to general results in [31]. In §5.2 we state Proposition I, and prove a Bowen type formula, and other general properties of the geometric pressure function. Finally, the proof of Proposition I is given in §5.3.

Throughout this section we fix a uniform family of quadratic-like maps \mathcal{F} , with constants K and R .

5.1. Inducing scheme

In this subsection we introduce the inducing scheme to estimate the geometric pressure function for maps in $\mathcal{K}_n(\mathcal{F})$.

Let n be an integer satisfying $n \geq 5$ and let f be in $\mathcal{K}_n(\mathcal{F})$. Recall that $V_f = P_{f,n+1}(0)$, and put

$$D_f := \{z \in V_f \mid f^m(z) \in V_f \text{ for some } m \geq 1\}.$$

For w in D_f put $m_f(w) := \min\{m \in \mathbb{N} \mid f^m(w) \in V_f\}$, and call it the *first return time* of w to V_f . The *first return map* to V_f is defined by

$$\begin{aligned} F_f: D_f &\rightarrow V_f \\ w &\mapsto F_f(w) := f^{m_f(w)}(w). \end{aligned}$$

It is easy to see that D_f is a disjoint union of puzzle pieces; so each connected component of D_f is a puzzle piece. Note furthermore that in each of these puzzle pieces W , the return time function m_f is constant; denote the common value of m_f on W by $m_f(W)$.

Throughout the rest of this subsection we put $\widehat{V}_f := P_{f,4}(0)$. The proof of the following lemma is the same as for [13, Lemma 6.1]. The reason is that the combinatorics and Koebe space are preserved by the conjugacy.

Lemma 5.1 (*Uniform distortion bound*). *There is a constant $\Delta_3 > 1$ that only depends on K and R , such that for each integer $n \geq 5$, and each f in $\mathcal{K}_n(\mathcal{F})$ the following property holds: For every connected component W of D_f the map $F_f|_W$ is univalent and its distortion is bounded by Δ_3 . Furthermore, the inverse of $F_f|_W$ admits a univalent extension to \widehat{V}_f taking images in V_f . In particular, F_f is uniformly expanding with respect to the hyperbolic metric on \widehat{V}_f .*

Denote by \mathfrak{D}_f the collection of connected components of D_f and if $c(f)$ is real denote by $\mathfrak{D}_f^{\mathbb{R}}$ the sub-collection of \mathfrak{D}_f of those sets intersecting $I(f)$. For each W in \mathfrak{D}_f denote by $\phi_W: \widehat{V}_f \rightarrow V_f$ the extension of $f|_W^{-1}$ given by Lemma 5.1. Given an integer $\ell \geq 1$

we denote by $E_{f,\ell}$ (resp. $E_{f,\ell}^{\mathbb{R}}$) the set of all words of length ℓ in the alphabet \mathfrak{D}_f (resp. $\mathfrak{D}_f^{\mathbb{R}}$). Again by Lemma 5.1, for each integer $\ell \geq 1$ and each word $W_1 \cdots W_\ell$ in $E_{f,\ell}$ the composition

$$\phi_{W_1 \cdots W_\ell} = \phi_{W_1} \circ \cdots \circ \phi_{W_\ell}$$

is defined on \widehat{V}_f . We also put

$$m_f(W_1 \cdots W_\ell) = m_f(W_1) + \cdots + m_f(W_\ell).$$

For t, p in \mathbb{R} and an integer $\ell \geq 1$ put

$$Z_\ell(t, p) := \sum_{\underline{W} \in E_{f,\ell}} \exp(-m_f(\underline{W})p) (\sup\{|D\phi_{\underline{W}}(z)| \mid z \in V_f\})^t$$

and

$$Z_\ell^{\mathbb{R}}(t, p) := \sum_{\underline{W} \in E_{f,\ell}^{\mathbb{R}}} \exp(-m_f(\underline{W})p) (\sup\{|D\phi_{\underline{W}}(z)| \mid z \in V_f\})^t.$$

For a fixed t and p in \mathbb{R} the sequence

$$\left(\frac{1}{\ell} \log Z_\ell(t, p) \right)_{\ell=1}^{+\infty} \left(\text{resp. } \left(\frac{1}{\ell} \log Z_\ell^{\mathbb{R}}(t, p) \right)_{\ell=1}^{+\infty} \right)$$

converges to the pressure function of F_f (resp. $F_f|_{D_f \cap I(f)}$) for the potential $-t \log |DF_f| - pm_f$; we denote it by $\mathscr{P}_f(t, p)$ (resp. $\mathscr{P}_f^{\mathbb{R}}(t, p)$). On the set where it is finite, the function \mathscr{P}_f (resp. $\mathscr{P}_f^{\mathbb{R}}$) so defined is continuous and strictly decreasing in each of its variables.

Given $t > 0$ and p in \mathbb{R} , a finite measure $\tilde{\mu}$ on \mathbb{C} that is supported on the maximal invariant set of $F|_{D_f \cap \mathbb{R}}$ (resp. F) is (t, p) -conformal for F_f , if for every W in $\mathfrak{D}_f^{\mathbb{R}}$ (resp. \mathfrak{D}_f), and every Borel subset U of $W \cap \mathbb{R}$ (resp. W), we have

$$\tilde{\mu}(F_f(U)) = \exp(pm_f(W)) \int_U |DF_f|^t d\tilde{\mu}.$$

Note that in this case we have

$$\exp(-pm_f(W)) \inf_{z \in W} |DF_f(z)|^{-t} \leq \tilde{\mu}(W) \leq \exp(-pm_f(W)) \sup_{z \in W} |DF_f(z)|^{-t}. \quad (5.1)$$

Proposition 5.2. *Let $n \geq 5$ be an integer, f in $\mathcal{K}_n(\mathcal{F})$, and $t > 0$ such that*

$$\mathscr{P}_f^{\mathbb{R}}(t, P_f^{\mathbb{R}}(t)) = 0 \text{ (resp. } \mathscr{P}_f(t, P_f(t)) = 0 \text{)}. \quad (5.2)$$

Then there is a $(t, P_f^{\mathbb{R}}(t))$ -conformal (resp. $(t, P_f(t))$ -conformal) probability measure $\tilde{\mu}$ for F_f , and there is a probability measure $\tilde{\rho}$ that is invariant by F_f , absolutely continuous with respect to $\tilde{\mu}$, and whose density satisfies

$$\Delta_3^{-t} \leq \frac{d\tilde{\rho}}{d\tilde{\mu}} \leq \Delta_3^t. \quad (5.3)$$

If in addition

$$\sum_{W \in \mathfrak{D}_f^{\mathbb{R}}} m_f(W) \cdot \exp(-m_f(W)P_f^{\mathbb{R}}(t)) \sup_{w \in W \cap \mathbb{R}} |DF_f(w)|^{-t} \left(\text{resp. } \sum_{W \in \mathfrak{D}_f} m_f(W) \cdot \exp(-m_f(W)P_f(t)) \sup_{w \in W} |DF_f(w)|^{-t} \right) \quad (5.4)$$

is finite, then the measure

$$\hat{\rho} := \sum_{W \in \mathfrak{D}_f^{\mathbb{R}}} \sum_{j=0}^{m_f(W)-1} (f^j)_* (\tilde{\rho}|_{W \cap \mathbb{R}}) \left(\text{resp. } \sum_{W \in \mathfrak{D}_f} \sum_{j=0}^{m_f(W)-1} (f^j)_* (\tilde{\rho}|_W) \right)$$

is finite and the probability measure proportional to $\hat{\rho}$ is the unique equilibrium state of $f|_{I(f)}$ (resp. $f|_{J(f)}$) for the potential $-\log |Df|$.

Proof. The proof is standard, refer to [31, §4] for precisions. The existence of the conformal measure follows from the same arguments given in [31, Theorem A in §4 and Proposition 4.3]. To construct an absolutely continuous invariant measure, let $\ell \geq 1$ be an integer, and let \underline{W} be a word in $E_{f,\ell}$. Then by Lemma 5.1 and (5.1), for every integer $\ell' \geq 1$ we have in the complex case

$$\begin{aligned} \tilde{\mu}(F_f^{-\ell'}(\phi_{\underline{W}}(V_f))) &= \sum_{\underline{W}' \in E_{f,\ell'}} \tilde{\mu}(\phi_{\underline{W}'} \circ \phi_{\underline{W}}(V_f)) \\ &\geq \tilde{\mu}(\phi_{\underline{W}}(V_f)) \sum_{\underline{W}' \in E_{f,\ell'}} \exp(-m_f(\underline{W}')P_f(t)) \inf_{z \in \phi_{\underline{W}'}(V_f)} |D\phi_{\underline{W}'}(z)|^t \\ &\geq \Delta_3^{-t} \tilde{\mu}(\phi_{\underline{W}}(V_f)) \sum_{\underline{W}' \in E_{f,\ell'}} \tilde{\mu}(\phi_{\underline{W}'}(V_f)) \\ &= \Delta_3^{-t} \tilde{\mu}(\phi_{\underline{W}}(V_f)). \end{aligned}$$

A similar argument shows that $\tilde{\mu}(F_f^{-\ell'}(\phi_{\underline{W}}(V_f))) \leq \Delta_3^t \tilde{\mu}(\phi_{\underline{W}}(V_f))$. Analogous inequalities also hold in the real case. Since these inequalities hold for every $\ell \geq 1$, and every \underline{W} in $E_{f,\ell}$, it follows that any weak* accumulation measure of $\left(\frac{1}{k} \sum_{\ell=0}^{k-1} (F_f)_{*}^{\ell} \tilde{\mu} \right)_{k=1}^{+\infty}$ is an invariant probability measure satisfying the desired properties.

To prove the last statement, note that by (5.1) and (5.3), our hypothesis (5.4) implies that

$$\sum_{W \in \mathfrak{D}_f^{\mathbb{R}}} m_f(W) \tilde{\rho}(W) \left(\text{resp. } \sum_{W \in \mathfrak{D}_f} m_f(W) \tilde{\rho}_t(W) \right)$$

is finite, so the measure $\hat{\rho}$ is finite. The last statement of the proposition follows as in the proof of [13, Proposition A]. \square

5.2. Estimating the 2 variables pressure function

The following is our main tool to estimate the 2 variables pressure function, in order to verify the hypotheses of Proposition 5.2.

Proposition I. *Let $\kappa > 0$ and $v > 0$ be given. Then there are $n_3 \geq 5$ and $C_6 > 1$ that only depend on K, R, κ , and v , such that for every integer $n \geq n_3$, and every f in $\mathcal{K}_n(\mathcal{F})$ satisfying the Geometric Peierls Condition with constants κ and v , the following properties hold for each $t \geq 2 \log 2/v$.*

1. For p in $[-t\chi_{\text{crit}}(f)/2, 0)$ satisfying

$$\sum_{k=0}^{+\infty} \exp(-(n+3k)p) |Df^{n+3k}(f(0))|^{-t/2} \geq C_6^t,$$

we have $\mathcal{P}_f^{\mathbb{R}}(t, p) > 0$ and $P_f^{\mathbb{R}}(t) > p$. If in addition the sum above is finite, then $\mathcal{P}_f(t, p)$ is finite.

2. For $p \geq -t\chi_{\text{crit}}(f)/2$ satisfying

$$\sum_{k=0}^{+\infty} \exp(-(n+3k)p) |Df^{n+3k}(f(0))|^{-t/2} \leq C_6^{-t},$$

we have $\mathcal{P}_f(t, p) < 0$ and $P_f(t) \leq p$.

3. For $p \geq -t\chi_{\text{crit}}(f)/2$ satisfying

$$\sum_{k=0}^{+\infty} k \cdot \exp(-(n+3k)p) |Df^{n+3k}(f(0))|^{-t/2} < +\infty,$$

we have

$$\sum_{W \in \mathfrak{D}_f} m_f(W) \cdot \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t} < +\infty.$$

The proof of this proposition is given in §5.3. In the rest of this subsection we prove 2 results that are used in the proof of the proposition above. The first is a Bowen type formula relating $P_f^{\mathbb{R}}$ (resp. P_f) to the 2 variables pressure function of F_f (Proposition II). The second is a lower bound for the pressure function (Proposition III).

Proposition II (Bowen type formula). *For every $\kappa > 0$, every $v > 0$, every integer $n \geq 5$, and every f in $\mathcal{K}_n(\mathcal{F})$ satisfying the Geometric Peierls Condition with constants κ and v , we have for each $t \geq 2 \log 2/v$,*

$$P_f^{\mathbb{R}}(t) = \inf \left\{ p \mid \mathcal{P}_f^{\mathbb{R}}(t, p) \leq 0 \right\} \text{ (resp. } P_f(t) = \inf \{ p \mid \mathcal{P}_f(t, p) \leq 0 \} \text{)}.$$

The proof of this proposition is at the end of the subsection. It uses several results is also used in the next subsection. The proof that the geometric pressure function is smaller or equal than the infimum is simple and depends basically on Lemma 5.1. The other inequality is much more involved. It requires the Geometric Peierls condition, and a lower bound on the pressure function that we proceed to state and prove.

Proposition III (Critical line). *For every integer $n \geq 5$, and every f in $\mathcal{K}_n(\mathcal{F})$, we have*

$$\chi_{\inf}^{\mathbb{R}} := \inf \left\{ \int \log |Df| \, d\mu \mid \mu \in \mathcal{M}_f^{\mathbb{R}} \right\} \leq \chi_{\text{crit}}(f)/2.$$

In particular, for each $t > 0$ we have

$$P_f(t) \geq P_f^{\mathbb{R}}(t) \geq -t\chi_{\text{crit}}(f)/2.$$

The proof of this proposition is given after the following lemma.

Lemma 5.3. *There is a constant $C_7 > 0$ that only depends on K and R , such that for each integer $n \geq 5$ and each f in $\mathcal{K}_n(\mathcal{F})$, the following property holds: For every integer $k \geq 0$ there is a connected component \widehat{W} of D_f contained in $P_{f, n+3k+2}(0)$, such that $h_f^{-1}(\widehat{W})$ intersects \mathbb{R} , and such that $m_f(\widehat{W}) = n + 3k + 3$ and*

$$\sup_{z \in \widehat{W}} |DF_f(z)| \leq C_7 |Df^{n+3k}(f(0))|^{1/2}.$$

Proof. We follow the proof of [13, Lemma 6.3]. Let C_1 be the constant given by Lemma 4.4 and let C_2 be the constant given by Lemma 4.5. Let $\Delta_2 > 1$ and $\Delta_3 > 1$ be the constants given by Lemmas 4.9 and 5.1, respectively. From the proof of [13, Lemma 6.3], we have

$$\Xi_0 := \sup_{c \in \mathcal{P}_4(-2)} \text{diam}(P_{c,1}(0)) < +\infty$$

and

$$\Xi_1 := \sup_{c \in \mathcal{P}_4(-2)} \sup_{z \in P_{c,1}(0)} |Df_c^2(z)| < +\infty.$$

By Lemma 4.5 and inequality (4.4), we have

$$\widehat{\Xi}_0 := \sup_{f \in \mathcal{P}_4(\mathcal{F})} \text{diam}(P_{f,1}(0)) \leq C_2 \Xi_0^{\frac{1}{K}} < +\infty,$$

and

$$\widehat{\Xi}_1 := \sup_{f \in \mathcal{P}_4(\mathcal{F})} \sup_{z \in P_{f,1}(0)} |Df^2(z)| \leq C_1^2 \Xi_1^{\frac{2}{K}} < +\infty.$$

Fix an integer $n \geq 5$, a f in $\mathcal{K}_n(\mathcal{F})$, and an integer $k \geq 0$. By the proof of [13, Lemma 6.3] there is a connected component W of $D_{c(f)}$ contained in $P_{c(f),n+3k+2}(0)$, that intersects \mathbb{R} , and such that $m_{c(f)}(W) = n + 3k + 3$. The set $\widehat{W} = h_f(W)$ verifies the desired properties of the lemma. To finish the proof it remains to prove the inequality.

Let $z_{\widehat{W}}$ be the unique point in \widehat{W} such that $f^{n+3k+3}(z_{\widehat{W}}) = 0$. Then $f^{n+3k+1}(z_{\widehat{W}})$ belongs to $P_{f,1}(0)$, so by definition of $\widehat{\Xi}_0$ we have

$$|f^{n+3k+1}(z_{\widehat{W}}) - f^{n+3k}(f(0))| \leq \text{diam}(P_{f,1}(0)) \leq \widehat{\Xi}_0. \quad (5.5)$$

Since f_c^n maps $P_{c,n+1}(c)$ biholomorphically to $P_{c,1}(0)$ and $f_c^n(c) \in \Lambda_c$, it follows that f_c^{n+3k} maps $P_{c,n+3k+1}(c)$ biholomorphically to $P_{c,1}(0)$, and the same holds for f^{n+3k} and the sets $P_{f,n+3k+1}(f(0))$ and $P_{f,1}(0)$; so the distortion of f^{n+3k} on $P_{f,n+3k+1}(f(0))$ is bounded by Δ_2 (Lemma 4.9) and for each point y in $P_{f,n+3k+1}(f(0))$ we have

$$\Delta_2^{-1} |Df^{n+3k}(f(0))| \leq |Df^{n+3k}(y)| \leq \Delta_2 |Df^{n+3k}(f(0))|. \quad (5.6)$$

Together with (5.5) this implies that,

$$|f(z_{\widehat{W}}) - f(0)| \leq \Delta_2 \widehat{\Xi}_0 |Df^{n+3k}(f(0))|^{-1}$$

and by Lemma 4.4,

$$|Df(z_{\widehat{W}})| \leq C_1^2 \Delta_2^{1/2} \widehat{\Xi}_0^{1/2} |Df^{n+3k}(f(0))|^{-1/2}.$$

Combined with (5.6) with $y = f(z_{\widehat{W}})$, this implies

$$|Df^{n+3k+1}(z_{\widehat{W}})| \leq C_1^2 \Delta_2^{3/2} \widehat{\Xi}_0^{1/2} |Df^{n+3k}(f(0))|^{1/2}.$$

Putting $C_7 := C_1^2 \Delta_3 \widehat{\Xi}_1 \Delta_2^{3/2} \widehat{\Xi}_0^{1/2}$, we get by Lemma 5.1

$$\begin{aligned}
\sup_{z \in \widehat{W}} |DF_f(z)| &\leq \Delta_3 |Df^{n+3k+3}(z_{\widehat{W}})| \\
&\leq \Delta_3 \widehat{\Xi}_1 |Df^{n+3k+1}(z_{\widehat{W}})| \\
&\leq C_7 |Df^{n+3k}(f(0))|^{1/2}. \quad \square
\end{aligned}$$

Proof of Proposition III. The proof follows from Lemma 5.3 by constructing a sequence of measures supported in periodic points whose Lyapunov exponents converge to $\chi_{\text{crit}}(f)$. For details see the proof of [13, Proposition 6.2]. \square

Lemma 5.4. *Let $\kappa > 0$ and $v > 0$ be given. Then there is $C_8 > 1$ that only depends on K , R , κ , and v , such that for every integer $n \geq 5$, and every f in $\mathcal{K}_n(\mathcal{F})$ satisfying the Geometric Peierls Condition with constants κ and v , the following property holds: For every*

$$t \geq 2 \log 2/v, p \geq -t(\chi_{\text{crit}}(f) + \frac{2v}{3})/2,$$

and y in V_f we have

$$\widetilde{L}_{t,p}(y) := 1 + \sum_{z \in L_f^{-1}(y)} (m_f(z) + 1) \exp(-m_f(z)p) |DL_f(z)|^{-t} \leq C_8^t.$$

Proof. Fix $\kappa > 0$ and $v > 0$, and put $t_0 := 2 \log 2/v$. We prove the lemma with

$$C_8 := \max\{1, \kappa^{-1}\} (1 - \exp(\log 2 - (2/3)vt_0))^{-2/t_0}.$$

Fix $n \geq 5$ and f in $\mathcal{K}_n(\mathcal{F})$ satisfying the Geometric Peierls Condition with constants κ and v .

Fix $t \geq t_0$, $p \geq -t(\chi_{\text{crit}}(f)/2 + v/2)$, and y in V_f . Since f satisfies the Geometric Peierls Condition, for every $z \in L_f^{-1}(y)$ we have

$$\exp(-m_f(z)p) |DL_f(z)|^{-t} \leq \kappa^{-t} \exp(-(2/3)v \cdot m_f(z)t).$$

On the other hand, for each integer $m \geq 1$ the set $\{z \in L_f^{-1}(y) \mid m_f(z) = m\}$ is contained in $f^{-m}(y)$ and therefore it contains at most 2^m points. So, we have

$$\widetilde{L}_{t,p}(y) \leq \max\{1, \kappa^{-1}\}^t \sum_{m=0}^{+\infty} (m+1) \exp(m(\log 2 - (2/3)vt)) \leq C_8^t. \quad \square$$

Lemma 5.5. *Given an integer $n \geq 5$ and f in $\mathcal{K}_n(\mathcal{F})$, the following property holds for every t in $(0, +\infty)$ and every real number p : If $\mathcal{P}_f^{\mathbb{R}}(t, p) > 0$ (resp. $\mathcal{P}_f(t, p) > 0$), then the series*

$$\sum_{j=1}^{+\infty} \exp(-jp) \sum_{y \in f|_{I(f)}^{-j}(0)} |Df^j(y)|^{-t} \left(\text{resp.} \sum_{j=1}^{+\infty} \exp(-jp) \sum_{y \in f^{-j}(0)} |Df^j(y)|^{-t} \right) \quad (5.7)$$

diverges. On the other hand, if for some $\kappa > 0$ and $v > 0$ the map f satisfies the Geometric Peierls Condition with constants κ and v , then for every

$$t \geq 2 \log 2/v \text{ and } p \geq P_f^{\mathbb{R}}(t) - t \frac{v}{3} \text{ (resp. } p \geq P_f(t) - t \frac{v}{3} \text{)}$$

satisfying $\mathcal{P}_f^{\mathbb{R}}(t, p) < 0$ (resp. $\mathcal{P}_f(t, p) < 0$), the series above converges.

Proof. The proof of the first part of the lemma depends on Lemma 5.1 and it follows the same lines that the first part of [13, Lemma 6.5].

We prove the last assertion concerning $f|_{J_f}$; the arguments apply without change to $f|_{I(f)}$. Fix some positive constants κ and v and let $C_8 > 1$ be given by Lemma 5.4 for the constants κ and v . Let f be a map in \mathcal{F} satisfying the Geometric Peierls Condition with constants κ and v , and let

$$t \geq 2 \log 2/v \text{ and } p \geq P_f(t) - t \frac{v}{3}$$

be such that $\mathcal{P}_f(t, p) < 0$. By Proposition III we have

$$p \geq -t(\chi_{\text{crit}}(f) + \frac{2}{3}v)/2,$$

so t and p satisfy the hypotheses of Lemma 5.4. Given an integer $m \geq 1$ and a point z in $f^{-m}(0)$ denote by $\ell(z)$ the number of those j in $\{0, \dots, m-1\}$ such that $f^j(z)$ is in V_f . In the case where z is not in V_f , this point is in the domain of L_f and we have $\ell(z) = 0$ if and only if $L_f(z) = 0$. Moreover, if z is not in V_f and $\ell(z) \geq 1$, then $L_f(z)$ is in the domain of $F_f^{\ell(z)}$ and $F_f^{\ell(z)}(L_f(z)) = 0$. So, if z is not in V_f we have in all the cases,

$$|Df_f^m(z)| = |DF_f^{\ell(z)}(L_f(z))| \cdot |DL_f(z)|.$$

Then Lemma 5.4 and our hypothesis $\mathcal{P}_f(t, p) < 0$ imply that the series (5.7) is bounded from above by

$$\begin{aligned} \tilde{L}_{t,p}(0) + \sum_{\ell=1}^{+\infty} \sum_{y \in F_f^{-\ell}(0)} \tilde{L}_{t,p}(y) \exp(-(m_f(F_f^{\ell-1}(y)) + \dots + m_f(y))p) |DF_f^{\ell}(y)|^{-t} \\ \leq C_8^t \left(1 + \sum_{\ell=0}^{+\infty} Z_{f,\ell}(t, p) \right) < +\infty. \quad \square \end{aligned}$$

Proof of Proposition II. We follow the proof of [13, Proposition C].

We prove the assertion for $f|_{J(f)}$; the arguments apply without change to $f|_{I(f)}$. Let $\Delta_3 > 1$ be given by Lemma 5.1. Let $n \geq 5$ be an integer and let f be in $\mathcal{K}_n(\mathcal{F})$. We use that fact that for each $t > 0$ we have

$$P_f(t) = \limsup_{m \rightarrow +\infty} \frac{1}{m} \log \sum_{y \in f^{-m}(0)} |Df^m(y)|^{-t}, \quad (5.8)$$

see for example [34].

Fix $t \geq 2 \log 2/v$. We use the fact that the function $p \mapsto \mathcal{P}_f(t, p)$ is strictly decreasing where it is finite, see §5.1. In particular, for each p satisfying $p < p_0 := \inf\{p \mid \mathcal{P}_f(t, p) \leq 0\}$ we have $\mathcal{P}_f(t, p) > 0$. Lemma 5.5 implies that for such p the series (5.7) diverges and by (5.8) we have $P_f(t) \geq p$. It follows that, $P_f(t) \geq p_0$. To prove the reverse inequality, suppose by contradiction $p_0 < P_f(t)$ and let p be in the interval $(p_0, P_f(t))$ satisfying $p \geq P_f(t) - t\frac{v}{3}$. Then $\mathcal{P}_f(t, p) < 0$ and by Lemma 5.5 the series (5.7) converges. Then (5.8) implies $P_f(t) \leq p$ and we obtain a contradiction that completes the proof of the proposition. \square

5.3. Proof of Proposition I

The final step in the proof of Proposition I is given after the following proposition, which estimates the partition function of the induced map in terms of the derivative of the iterates of the map at its critical value.

Let $n \geq 4$ be an integer and f in $\mathcal{K}_n(\mathcal{F})$. Since the critical point $z = 0$ does not belong to D_f (cf., [13, Lemma 4.2]), for each integer $\ell \geq 1$, each connected component of D_f intersecting $P_{f,\ell}(0)$ is contained in $P_{f,\ell}(0)$. Define the *level* of a connected component W of D_f as the largest integer $k \geq 0$ such that W is contained in $P_{f,n+3k+2}(0)$. Given an integer $k \geq 0$ denote by $\mathfrak{D}_{f,k}$ the collection of all connected components of D_f of level k ; we have $\mathfrak{D}_f = \bigcup_{k=0}^{+\infty} \mathfrak{D}_{f,k}$, and for every W in $\mathfrak{D}_{f,k}$ we have $m_f(W) \geq n + 3k + 1$.

Proposition 5.6. *Let $\kappa > 0$ and $v > 0$ be given. Then there are $n_4 \geq 5$ and $C_9 > 1$ that only depend on K , R , κ , and v , such that for every integer $n \geq n_4$, and every f in $\mathcal{K}_n(\mathcal{F})$ satisfying the Geometric Peierls Condition with constants κ and v , the following properties hold for each $t \geq 2 \log 2/v$ and each integer $k \geq 0$:*

1. For each p in $(-\infty, 0)$, we have

$$\begin{aligned} \sum_{W \in \mathfrak{D}_{f,k} \cap \mathfrak{D}_f^{\mathbb{R}}} \exp(-m_f(W)p) \inf_{z \in W} |DF_f(z)|^{-t} \\ > C_9^{-t} \exp(-(n+3k)p) |Df^{n+3k}(f(0))|^{-t/2}. \end{aligned}$$

2. For each $p \geq -t\chi_{\text{crit}}(f)/2 - tv/3$, we have

$$\begin{aligned}
& \sum_{W \in \mathfrak{D}_{f,k}} \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t} \\
& \leq \sum_{W \in \mathfrak{D}_{f,k}} (m_f(W) - (n + 3k)) \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t} \\
& < C_9^t \exp(-(n + 3k)p) |Df^{n+3k}(f(0))|^{-t/2}.
\end{aligned}$$

The proof of this proposition is given after the following lemma.

Lemma 5.7. *There is $C_{10} > 1$ that only depends on K and R , such that for each integer $n \geq 5$, each f in $\mathcal{K}_n(\mathcal{F})$, each integer $k \geq 0$, and each pair of real numbers $t > 0$ and p , we have*

$$\begin{aligned}
& \sum_{W \in \mathfrak{D}_{f,k}} \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t} \\
& \leq 2C_{10}^t \exp(-(n + 3k + 1)p) |Df^{n+3k}(f(0))|^{-t/2} \\
& \quad \cdot \left(1 + \sum_{w \in L_f^{-1}(0) \text{ in } P_{f,1}(0)} \exp(-m_f(w)p) |DL_f(w)|^{-t} \right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \sum_{W \in \mathfrak{D}_{f,k}} (m_f(W) - (n + 3k)) \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t} \\
& \leq 2C_{10}^t \exp(-(n + 3k + 1)p) |Df^{n+3k}(f(0))|^{-t/2} \\
& \quad \cdot \left(1 + \sum_{w \in L_f^{-1}(0) \text{ in } P_{f,1}(0)} (m_f(w) + 1) \exp(-m_f(w)p) |DL_f(w)|^{-t} \right).
\end{aligned}$$

Proof. The proof follows the same lines that the proof of [13, Lemma 7.1], and it depends on Lemmas 4.9, 4.10, and 5.1, as well as on [13, Lemma 5.1]. The second inequality does not appear in [13, Lemma 7.1]. It follows from the first displayed equation in the proof of [13, Lemma 7.1], and from [13, (7.1)]. See the proof of [13, Lemma 7.1] for further details. \square

Proof of Proposition 5.6. The proof depends on Lemmas 5.3, 5.4, and 5.7, and on Proposition II, and it follows the same lines that the proof of [13, Lemma 7.2]. There are some differences in item 2 since the condition on p is slightly different and we add a new inequality. We include the proof of item 2 here.

Let $C_8 > 0$ and $C_{10} > 0$ be given by Lemmas 5.4 and 5.7, respectively. Let n_2 be the integer given by Lemma 4.7 with $\varepsilon = \frac{1}{10}$. Put $t_0 := 2 \log 2/v$. We prove the lemma for $n_4 = n_2$. Fix an integer $n \geq n_2$, a map f in $\mathcal{K}_n(\mathcal{F})$, $t \geq t_0$, and an integer $k \geq 0$.

To prove item 2, note that the first inequality follows from the fact that for every W in $\mathfrak{D}_{f,k}$ we have $m_f(W) \geq n + 3k + 1$. To prove the second inequality, let $p \geq -t\chi_{\text{crit}}(f)/2 - tv/3$ be given. By Lemma 4.7, we have

$$\chi_{\text{crit}}(f) \leq 1.1K \log 2.$$

Thus $-p \leq t(0.55K + \frac{v}{3 \log 2}) \log 2 < t(0.55K + 1/t_0) \log 2$ and therefore $2 \exp(-p) < 2^{t(0.55K+2/t_0)}$. Combined with Lemmas 5.4 and 5.7, we obtain item 2 of the proposition with $C_9 = 2^{0.55K+2/t_0} C_{10} C_8$. \square

Proof of Proposition I. We follow the proof of [13, Proposition D]. Let n_4 and C_9 be given by Proposition 5.6, and put $t_0 := 2 \log 2/v$. To prove the proposition, fix an integer n satisfying $n \geq n_4$, a map f in $\mathcal{K}_n(\mathcal{F})$, and a real number t satisfying $t \geq t_0$.

To prove item 1, let p in $[-t\chi_{\text{crit}}(f)/2, 0)$ be such that the sum

$$\sum_{k=0}^{+\infty} \exp(-(n+3k)p) |Df^{n+3k}(f(0))|^{-t/2} \quad (5.9)$$

is greater than or equal to $(2C_9)^t$. Then, we can choose p' in $(p, 0)$ such that the sum above with p replaced by p' is strictly larger than C_9^t . By item 1 of Proposition 5.6, we have $\mathcal{P}_f^{\mathbb{R}}(t, p) \geq \mathcal{P}_f^{\mathbb{R}}(t, p') > 0$ and by Proposition II we have $P_f^{\mathbb{R}}(t) \geq p' > p$. This proves the first assertion of item 1 with $C_6 = 2C_9$. To prove the second assertion of item 1, suppose that is finite. Then, by item 2 of Proposition 5.6 the sum

$$\sum_{W \in \mathfrak{D}_f} \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t}$$

is finite, so $\mathcal{P}_f^{\mathbb{R}}(t, p)$ is also finite. This completes the proof of item 1.

To prove item 2, let $p \geq -t\chi_{\text{crit}}(f)/2$ be given. By item 2 of Proposition 5.6, if (5.9) is less than or equal to C_9^{-t} , then $\mathcal{P}_f(t, p) < 0$ and by Proposition II we have $P_f(t) \leq p$. This proves item 2 of the proposition with $C_6 = C_9$.

To prove item 3, let $p \geq -t\chi_{\text{crit}}(f)/2$ be given and put $p' := p - t\frac{v}{3}$. By item 2 of Proposition 5.6 with $k = 0$, the sum

$$\sum_{W \in \mathfrak{D}_{f,0}} \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t}$$

is finite. Let $A > 0$ be a constant such that for every pair of integers $k \geq 1$ and $m \geq 3k+1$, we have

$$m \leq Ak \exp(t_0 v(m - 3k)/3).$$

Applying item 2 of Proposition 5.6 with p replaced by p' , we obtain that for each integer $k \geq 1$ we have

$$\begin{aligned}
& \sum_{W \in \mathfrak{D}_{f,k}} m_f(W) \cdot \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t} \\
& \leq \sum_{W \in \mathfrak{D}_{f,k}} Ak \exp(tv(m_f(W) - 3k)/3) \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t} \\
& = Ak \exp(tvk) \sum_{W \in \mathfrak{D}_{f,k}} \exp(-m_f(W)p') \sup_{z \in W} |DF_f(z)|^{-t} \\
& \leq (AC_9^t \exp(-tvn/3)) k \cdot \exp(-(n + 3k)p) |Df^{n+3k}(f(0))|^{-t/2}.
\end{aligned}$$

Summing over $k \geq 0$ we obtain the desired assertion. \square

6. Sensitive dependence of geometric Gibbs states

In this section we prove the Main Theorem. In §6.1 we define the family of itineraries of the maps used in the Main Theorem, as well as other combinatorial objects used in the proof. In §6.2 we estimate the postcritical series in terms of certain 2 variables series that only depends on the combinatorics of the postcritical orbit (Lemma 6.1). The main estimates needed in the proof of the Main Theorem can be stated only in terms of these 2 variables series, and are relegated to Appendix A. The proof of the Main Theorem is given in §6.3, and it is divided in 3 parts. In the first part (§6.3.1), we introduce the family of maps $(f_{\underline{s}})_{\underline{s} \in \{+, -\}^{\mathbb{N}}}$, which is mostly defined through the combinatorics of the postcritical orbit. In the second part (§6.3.2), we estimate the geometric pressure, and prove the existence and uniqueness of geometric Gibbs states, as well as the existence of conformal measures. The third, and most difficult part of the proof is given in §6.3.3, where we show that on certain intervals of inverse temperatures the geometric Gibbs states are concentrated near the orbit of p^+ , or the orbit of p^- .

6.1. The family of itineraries

In this section we define several combinatorial objects used in the proof of the Main Theorem in §§6.2, 6.3. One of the most important ones, is a family of itineraries $(\iota(\underline{s}))_{\underline{s} \in \{+, -\}^{\mathbb{N}}}$ in $\{0, 1\}^{\mathbb{N}_0}$. Its definition depends on a choice of nonnegative integers q and Ξ , satisfying

$$q \geq 50(\Xi + 1) \text{ and } q + \Xi \equiv 0 \pmod{2}. \quad (6.1)$$

These integers are chosen in §6.3.1. They depend on the uniform family \mathcal{F} and of a choice of a sufficiently large integer n , as in the Main Theorem.

Endow each of the sets $\{+, -\}$, $\{0, 1\}$, and $\{0, 1^+, 1^-\}$ with the discrete topology, and each of the sets $\{+, -\}^{\mathbb{N}}$, $\{0, 1\}^{\mathbb{N}_0}$, and $\{0, 1^+, 1^-\}^{\mathbb{N}_0}$ with the corresponding product topology. Roughly speaking, for a map f in $\mathcal{K}_n(\mathcal{F})$ a large block formed by repeated 0's (resp. 1's, 10's) in the itinerary $\iota(f)$ of f indicates that a large portion of the critical orbit

is close to the orbit of $p(f)$ (resp. $p^+(f)$, $p^-(f)$). To make this encoding more transparent, we use the auxiliary symbolic space $\{0, 1^+, 1^-\}^{\mathbb{N}_0}$ in which the symbol 0 (resp. 1^+ , 1^-) represents the orbit of $p(f)$ (resp. $p^+(f)$, $p^-(f)$). Thus, to define the family of itineraries $(\iota(\underline{\varsigma}))_{\underline{\varsigma} \in \{+, -\}^{\mathbb{N}}}$ in $\{0, 1\}^{\mathbb{N}_0}$, we first define a family of sequences $(\hat{x}(\underline{\varsigma}))_{\underline{\varsigma} \in \{+, -\}^{\mathbb{N}}}$ in $\{0, 1^+, 1^-\}^{\mathbb{N}_0}$. Denote by $\hat{\Sigma}$ the subspace of $\{0, 1^+, 1^-\}^{\mathbb{N}_0}$, given by

$$\hat{\Sigma} := \left\{ (\hat{x}_j)_{j \in \mathbb{N}_0} \in \{0, 1^+, 1^-\}^{\mathbb{N}_0} \mid \hat{x}_j = 1^+ \Rightarrow \hat{x}_{j+1} \neq 1^-, \hat{x}_j = 1^- \Rightarrow \hat{x}_{j+1} \neq 1^+ \right\}. \quad (6.2)$$

For each s in \mathbb{N}_0 , define the integers

$$a_s := 2^{qs^3} \text{ and } b_s := 2^{qs^3} + q(2s + 1) + \Xi.$$

Note that $b_s < a_{s+1}$ and that in the case where $s \geq 1$ both a_s and b_s are even. Define the intervals I_s and J_s of \mathbb{R} , by

$$I_s := [a_s, b_s) \text{ and } J_s := [b_s, a_{s+1}).$$

As s runs through \mathbb{N}_0 , the intervals I_s and J_s form a partition of $[1, +\infty)$. Define functions

$$N: \mathbb{N}_0 \rightarrow \mathbb{N}_0 \text{ and } B: \mathbb{N}_0 \rightarrow \mathbb{N}_0$$

by $N(0) := 0$, $B(0) := 0$, for k in \mathbb{N} by

$$N(k) := \# \left\{ j \in \{0, \dots, k-1\} \mid j+1 \in \bigcup_{s \in \mathbb{N}_0} I_s \right\},$$

and for s in \mathbb{N}_0 by

$$B^{-1}(2s+1) = I_s \text{ and } B^{-1}(2(s+1)) = J_s. \quad (6.3)$$

Note that for every k in \mathbb{N} we have $B(k) \leq 2N(k) + 1$, and that

$$\lim_{k \rightarrow +\infty} \frac{N(k)}{k} = 0. \quad (6.4)$$

For each $\underline{\varsigma}$ in $\{+, -\}^{\mathbb{N}}$, let $\hat{x}(\underline{\varsigma})$ be the sequence in $\{0, 1^+, 1^-\}^{\mathbb{N}_0}$ defined for j in \mathbb{N}_0 by

$$\hat{x}(\underline{\varsigma})_j := \begin{cases} 0 & \text{if for some } s \text{ in } \mathbb{N}_0 \text{ we have } j+1 \in I_s; \\ 1^+ & \text{if } j+1 \in J_0; \\ 1^{\varsigma(m)} & \text{for } j+1 \in J_{4m-3} \cup J_{4m-2} \cup J_{4m-1} \cup J_{4m}. \end{cases} \quad (6.5)$$

Note that the first q entries of this sequence are equal to 0, that $\widehat{x}(\underline{\varsigma})$ is in $\widehat{\Sigma}$, and that the map $\{+, -\}^{\mathbb{N}} \rightarrow \widehat{\Sigma}$ given by $\underline{\varsigma} \mapsto \widehat{x}(\underline{\varsigma})$ is continuous. Moreover, the length of each maximal block of 1^- 's in $\widehat{x}(\underline{\varsigma})$ is even, and for every k in \mathbb{N} we have

$$N(k) = \#\{j \in \{0, \dots, k-1\} \mid \widehat{x}_j = 0\}, \quad (6.6)$$

and, if in addition $k \geq 2$, then we have

$$B(k) = 1 + \#\{j \in \{0, \dots, k-2\} \mid \widehat{x}_j \neq \widehat{x}_{j+1}\}. \quad (6.7)$$

Thus, for every k in \mathbb{N} the number $B(k)$ is equal to the number of maximal blocks of 0's, 1^+ 's, and 1^- 's in the sequence $(\widehat{x}_j)_{j=0}^{k-1}$.

For $\underline{\varsigma}$ in $\{+, -\}^{\mathbb{N}}$, define the itinerary $\iota(\underline{\varsigma})$ in $\{0, 1\}^{\mathbb{N}_0}$ for each j in \mathbb{N}_0 by

$$\iota(\underline{\varsigma})_j = \begin{cases} 0 & \text{if } \widehat{x}(\underline{\varsigma})_j = 0; \\ 1 & \text{if } \widehat{x}(\underline{\varsigma})_j = 1^+; \\ 0 & \text{if } \widehat{x}(\underline{\varsigma})_j = 1^- \text{ and } j \text{ is even;} \\ 1 & \text{if } \widehat{x}(\underline{\varsigma})_j = 1^- \text{ and } j \text{ is odd.} \end{cases} \quad (6.8)$$

Note that the first q entries of $\iota(\underline{\varsigma})$ are equal to 0, and that the map $\{+, -\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}_0}$ given by $\underline{\varsigma} \mapsto \iota(\underline{\varsigma})$ is continuous.

6.2. The 2 variables series

Given a real number ξ , define for each k in \mathbb{N}_0 and each (τ, λ) in $[0, +\infty) \times [0, +\infty)$ the number

$$\pi_k^{\pm}(\tau, \lambda) := 2^{-\lambda k - \tau N(k) \pm \xi \tau B(k)}.$$

In this subsection we fix a uniform family of quadratic-like maps \mathcal{F} , let Δ_2 be the constant given by Lemma 4.9, and recall that $\Delta_2 > 1$. For each integer n satisfying $n \geq 5$, and each f in $\mathcal{K}_n(\mathcal{F})$, put

$$\theta(f) := \left| \frac{Dg_f(p(f))}{Dg_f(p^+(f))} \right|^{1/2}. \quad (6.9)$$

Note that the condition $\theta(f) > 1$ is equivalent to $\chi_f(p(f)) > \chi_f(p^+(f))$. When this holds, define

$$\xi(f) := \frac{\log \Delta_2}{2 \log \theta(f)}.$$

The purpose of this subsection is to prove the following lemma.

Lemma 6.1. Let Δ_1 be the constant given by Lemma 4.8, let q and Ξ be nonnegative integers satisfying (6.1), and let $(\iota(\underline{\varsigma}))_{\underline{\varsigma} \in \{+, -\}^{\mathbb{N}}}$ be the family of itineraries in $\{0, 1\}^{\mathbb{N}_0}$ defined in §6.1 for these choices of q and Ξ . Then, for every $\underline{\varsigma}$ in $\{+, -\}^{\mathbb{N}}$, every integer n satisfying $n \geq 6$, and every f in $\mathcal{K}_n(\mathcal{F})$ such that

$$\iota(f) = \iota(\underline{\varsigma}) \text{ and } \chi(p^-(f)) = \chi(p^+(f)),$$

we have

$$\chi_{\text{crit}}(f) = \frac{1}{3} \log |Dg_f(p^+(f))|. \quad (6.10)$$

If in addition

$$\chi_f(p(f)) > \chi_f(p^+(f)),$$

then the following property holds for every number ξ satisfying $\xi \geq \xi(f)$. For every k in \mathbb{N}_0 , and all real numbers t and δ satisfying $t > 0$ and $\delta \geq 0$, we have

$$\begin{aligned} \Delta_1^{-\frac{t}{2}} \exp(-n\delta) \left(\frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{\frac{t}{2}n} \pi_k^- \left(\frac{\log \theta(f)}{\log 2} t, \frac{3\delta}{\log 2} \right) \\ \leq \exp \left(-(n+3k) \left(-t \frac{\chi_{\text{crit}}(f)}{2} + \delta \right) \right) |Df^{n+3k}(f(0))|^{-\frac{t}{2}} \\ \leq \Delta_1^{\frac{t}{2}} \exp(-n\delta) \left(\frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{\frac{t}{2}n} \pi_k^+ \left(\frac{\log \theta(f)}{\log 2} t, \frac{3\delta}{\log 2} \right). \end{aligned} \quad (6.11)$$

Proof. Put $\widehat{c} := f^{n+1}(0)$. For every k in \mathbb{N} and every j in $\{0, 1, 2\}$, we have by the chain rule

$$\begin{aligned} Df^{3k+j}(f(0)) &= Df^j((f^{3k})(\widehat{c})) \cdot Df^{3k}(\widehat{c}) \cdot Df^n(f(0)) \\ &= Df^j(g_f^k(\widehat{c})) \cdot Dg_f^k(\widehat{c}) \cdot Df^n(f(0)). \end{aligned}$$

Since $|Df^j((g_f^k)(\widehat{c}))|$ is bounded independently of k and j , we have

$$\chi_{\text{crit}}(f) = \liminf_{m \rightarrow +\infty} \frac{1}{m} \log |Df^m(f(0))| = \frac{1}{3} \liminf_{k \rightarrow +\infty} \frac{1}{k} \log |Dg_f^k(\widehat{c})|. \quad (6.12)$$

On the other hand, by Lemma 4.9, the hypothesis $\chi(p^-(f)) = \chi(p^+(f))$, (6.6), (6.7), and the fact that the maximal blocks of 1's in $\widehat{x}(\underline{\varsigma})$ have even length, we have that for each integer k in \mathbb{N} ,

$$\Delta_2^{-B(k)} \leq \frac{|Dg_f^k(\widehat{c})|}{|Dg_f(p^+(f))|^{k-N(k)} |Dg_f(p(f))|^{N(k)}} \leq \Delta_2^{B(k)}.$$

Taking logarithms yields

$$\begin{aligned} -B(k) \log \Delta_2 + N(k) \log \frac{|Dg_f(p(f))|}{|Dg_f(p^+(f))|} &\leq \log |Dg_f^k(\hat{c})| - k \log |Dg_f(p^+(f))| \\ &\leq B(k) \log \Delta_2 + N(k) \log \frac{|Dg_f(p(f))|}{|Dg_f(p^+(f))|}. \end{aligned}$$

Since for each k in \mathbb{N} we have $B(k) \leq 2N(k) + 1$, by (6.4) we conclude that

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \log |Dg_f^k(\hat{c})| = \log |Dg_f(p^+(f))|.$$

Combined with (6.12), this completes the proof of (6.10).

In the case where $k = 0$, the chain of inequalities (6.11) is given by Lemma 4.8. Fix k in \mathbb{N} and a number t satisfying $t > 0$. Using

$$Df^{n+3k}(f(0)) = Dg_f^k(f^{n+1}(0)) \cdot Df^n(f(0))$$

and Lemmas 4.8 and 4.9, the hypothesis $\chi(p^-(f)) = \chi(p^+(f))$, (6.6), (6.7), and the fact that the maximal blocks of 1^- 's in $\hat{x}(\underline{\zeta})$ have even length, we have

$$\begin{aligned} \Delta_1^{-t} \theta(f)^{-2tN(k)} \Delta_2^{-tB(k)} &\leq \frac{|Df^{n+3k}(f(0))|^{-t}}{|Dg_f(p^+(f))|^{-tk} |Df(\beta(f))|^{-tn}} \\ &\leq \Delta_1^t \theta(f)^{-2tN(k)} \Delta_2^{tB(k)}. \end{aligned} \quad (6.13)$$

Since by (6.10) we have

$$\exp((n+3k)t\chi_{\text{crit}}(f)) = \exp(nt\chi_{\text{crit}}(f)) |Dg_f(p^+(f))|^{tk},$$

if we multiply each term in the chain of inequalities (6.13) by

$$\left(\frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{tn},$$

then we get

$$\begin{aligned} \Delta_1^{-t} \left(\frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{tn} \theta(f)^{-2tN(k)} \Delta_2^{-tB(k)} &\leq \exp((n+3k)t\chi_{\text{crit}}(f)) |Df^{n+3k}(f(0))|^{-t} \\ &\leq \Delta_1^t \left(\frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{tn} \theta(f)^{-2tN(k)} \Delta_2^{tB(k)}. \end{aligned}$$

Taking square roots, and then by multiplying by $\exp(-(n+3k)\delta)$ in each of the terms of the chain of inequalities above, we obtain

$$\begin{aligned}
\Delta_1^{-t/2} \exp(-n\delta) \left(\frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{\frac{t}{2}n} \exp(-3k\delta)\theta(f)^{-tN(k)} \Delta_2^{-tB(k)/2} \\
\leq \exp \left(-(n+3k) \left(-t \frac{\chi_{\text{crit}}(f)}{2} + \delta \right) \right) |Df^{n+3k}(f(0))|^{-\frac{t}{2}} \\
\leq \Delta_1^{t/2} \exp(-n\delta) \left(\frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{\frac{t}{2}n} \exp(-3k\delta)\theta(f)^{-tN(k)} \Delta_2^{tB(k)/2}.
\end{aligned}$$

Together with our hypothesis $\xi \geq \xi(f)$, and our definition of π_k^\pm , this implies the desired chain of inequalities. \square

6.3. Proof of the Main Theorem

Put $v := \frac{1}{4} \log 2$, let $R > 0$ be given, and let K_1, κ_1 , and n_1 be given by Proposition 4.3. We prove that the Main Theorem holds with $K_0 = K_1$. Let \mathcal{F} be a uniform family of quadratic-like maps with constants K_1 and R that is admissible. By Proposition 4.3, for every $n \geq n_1$, every f in $\mathcal{K}_n(\mathcal{F})$ satisfies the Geometric Peierls Condition with constants κ_1 and v , and we have

$$\chi_f(\beta(f)) > \chi_{\text{crit}}(f) + 2v. \quad (6.14)$$

Taking n_1 larger if necessary, assume that for every $n \geq n_1$ there is a continuous function $s_n: \mathcal{K}_n \rightarrow \mathcal{K}_n(\mathcal{F})$ such that $c \circ s_n$ is the identity, and that (3.1) holds for every f in $s_n(\mathcal{K}_n)$.

Let $\Delta_1, \Delta_2, C_5, v_1$ and Δ_3 be the constants given by Lemmas 4.8, 4.9, 4.11, and 5.1, respectively. Moreover, let n_3 and C_6 the constants given by Proposition I, and $\kappa = \kappa_1$, let n_4 and C_9 be given by Proposition 5.6, and let $n_\& \geq \max\{6, n_3, n_4\}$ be sufficiently large so that

$$\exp(n_\&v) \geq \Delta_1^{\frac{1}{2}} C_6 (2 + \Delta_2). \quad (6.15)$$

6.3.1. The subfamily

In this subsection we define the family $(f_\pm)_{\pm \in \{+, -\}^\mathbb{N}}$, as in the statement of the Main Theorem.

Fix an integer n satisfying $n \geq n_\&$, let $c_\&$ in \mathcal{K}_n be such that the itinerary $\iota(c_\&)$ is the constant sequence equal to 0 (Proposition 2.3), and put $f_\& := s_n(c_\&)$. By (3.1) we have $\theta(f_\&) > 1$, so there is $r_\& > 0$ such that for every c in $B(c_\&, r_\&) \cap \mathcal{K}_n$ the number $\theta(s_n(c))$ is defined, and depends continuously on c . Reducing $r_\&$ if necessary, assume that for all c and c' in $B(c_\&, r_\&) \cap \mathcal{K}_n$ we have $\theta(s_n(c)) \leq \theta(s_n(c'))^2$. By Proposition 2.3 it follows that there is an integer $q_\& \geq 0$ such that the set

$$\{c \in \mathcal{K}_n \mid \text{for every } j \text{ in } \{0, \dots, q_\&\}, \iota(c)_j = 0\}$$

is a compact set contained in $B(c_{\&}, r_{\&})$. On the other hand, by (3.1) for each c in \mathcal{K}_n we can define the number $\xi(s_n(c))$ as in §6.2. Since s_n is continuous, it follows that this number depends continuously on c in \mathcal{K}_n , so

$$\sup_{c \in \mathcal{K}_n \cap B(c_{\&}, r_{\&})} \xi(s_n(c)) < +\infty.$$

Denote the supremum on the left side by ξ .

Put $\Xi := \lceil 2\xi \rceil + 1$ as in §A.2, and let q be an integer satisfying $q \geq q_{\&}$ and (6.1). Moreover, let

$$(\widehat{x}(\underline{\varsigma}))_{\underline{\varsigma} \in \{+, -\}^{\mathbb{N}}} \text{ and } (\iota(\underline{\varsigma}))_{\underline{\varsigma} \in \{+, -\}^{\mathbb{N}}}$$

be given by (6.5) and (6.8), respectively, in §6.1. Given $\underline{\varsigma}$ in $\{+, -\}^{\mathbb{N}}$, let $c(\underline{\varsigma})$ in \mathcal{K}_n be the unique parameter such that $\iota(f_{c(\underline{\varsigma})}) = \iota(\underline{\varsigma})$ (Proposition 2.3), and put $f_{\underline{\varsigma}} := s_n(c(\underline{\varsigma}))$. Note that the function $\underline{\varsigma} \mapsto f_{\underline{\varsigma}}$ so defined is continuous. On the other hand, since for each $\underline{\varsigma}$ in $\{+, -\}^{\mathbb{N}}$ the first q entries of $\iota(\underline{\varsigma})$ are equal to 0 and we have $q \geq q_{\&}$ and $\iota(f_{\underline{\varsigma}}) = \iota(\underline{\varsigma})$, we conclude that the parameter $c(\underline{\varsigma})$ is in $B(\lambda_{\&}, r_{\&})$. So, for all $\underline{\varsigma}$ and $\underline{\varsigma}'$ in $\{+, -\}^{\mathbb{N}}$ we have

$$\theta(f_{\underline{\varsigma}}) \leq \theta(f_{\underline{\varsigma}'})^2. \quad (6.16)$$

6.3.2. Pressure estimates, and the existence of equilibria

The purpose of this subsection is to prove item 1 of the Main Theorem, and at the same time to estimate for each $\underline{\varsigma}$ in $\{+, -\}^{\mathbb{N}}$ the pressure functions of $f_{\underline{\varsigma}}$ at large values of t . That for each $\underline{\varsigma}$ in $\{+, -\}^{\mathbb{N}}$ the interval map $f_{\underline{\varsigma}}|_{I(f_{\underline{\varsigma}})}$ is topologically exact follows from the fact that this map is not renormalizable, see [13, §3] for details. Thus, to prove item 1 of the Main Theorem we only need to prove the assertions about equilibrium states.

Let $N: \mathbb{N}_0 \rightarrow \mathbb{N}_0$ and $B: \mathbb{N}_0 \rightarrow \mathbb{N}_0$ be the functions defined in §6.1 for our choices of Ξ and q in §6.3.1. Let Π^{\pm} be the 2 variables series, and for each s in \mathbb{N}_0 , let I_s^{\pm} and J_s^{\pm} be the series defined in §A.2. They satisfy

$$\Pi^{\pm} = \sum_{k=0}^{+\infty} \pi_k^{\pm}, I_s^{\pm} = \sum_{k \in I_s} \pi_k^{\pm}, \text{ and } J_s^{\pm} = \sum_{k \in J_s} \pi_k^{\pm}.$$

Furthermore, for each nonnegative real number s put $\lambda(s) := |J_s|^{-1}$ as in §A.2.

Let $A: \{+, -\}^{\mathbb{N}} \rightarrow (0, +\infty)$ be the continuous function defined by

$$A(\underline{\varsigma}) := \frac{4 \log 2}{\log \theta(f_{\underline{\varsigma}})},$$

and define

$$A_{\sup} := \sup_{\underline{\varsigma} \in \{+, -\}^{\mathbb{N}}} A(\underline{\varsigma}),$$

and

$$\eta_0 := \sup \left\{ \exp(\chi_{f_{\underline{\varsigma}}}(\beta(f_{\underline{\varsigma}})) - \chi_{\text{crit}}(f_{\underline{\varsigma}})) \mid \underline{\varsigma} \in \{+, -\}^{\mathbb{N}} \right\}.$$

Moreover, let $t_{\&}$ be a sufficiently large number so that

$$t_{\&} \geq \frac{2 \log 2}{v}, t_{\&} \geq \left(\inf_{\underline{\varsigma} \in \{+, -\}^{\mathbb{N}}} \chi_{f_{\underline{\varsigma}}}(p^+(f_{\underline{\varsigma}})) \right)^{-1}, t_{\&} \geq \frac{25}{2} A_{\sup},$$

$$2^{\left(\frac{4}{A_{\sup}}\right)^2 t_{\&}} \geq 2^n \Delta_1^{\frac{1}{2}} C_6 \eta_0^{\frac{n}{2}}, \text{ and } \frac{\log C_5}{t_{\&}^2} \leq v_1 \frac{8}{A_{\sup}^2}. \quad (6.17)$$

For the rest of this subsection we fix $\underline{\varsigma}$ in $\{+, -\}^{\mathbb{N}}$, and put

$$f := f_{\underline{\varsigma}}, p^+ := p^+(f_{\underline{\varsigma}}), p^- := p^-(f_{\underline{\varsigma}}),$$

$$P^{\mathbb{R}} := P_{f_{\underline{\varsigma}}}^{\mathbb{R}}, \mathcal{P}^{\mathbb{R}} := \mathcal{P}_{f_{\underline{\varsigma}}}^{\mathbb{R}}, P := P_{f_{\underline{\varsigma}}}, \text{ and } \mathcal{P} := \mathcal{P}_{f_{\underline{\varsigma}}}.$$

Moreover, fix a real number t satisfying $t \geq t_{\&}$, and put

$$\tau := \frac{4}{A(\underline{\varsigma})} t,$$

$$P^+ := -t \frac{\chi_{\text{crit}}(f)}{2} + \frac{\log 2}{3} \lambda(\tau - 1), \quad \text{and} \quad P^- := -t \frac{\chi_{\text{crit}}(f)}{2} + \frac{\log 2}{3} \lambda(\tau).$$

Note that

$$\tau = \frac{\log \theta(f)}{\log 2} t,$$

that by (6.10) we have $\chi_{\text{crit}}(f) = \chi_f(p^+)$, and that by (6.17) we have

$$\tau \geq 50 \text{ and } P^- < P^+ < 0.$$

Moreover, by (6.16) we have

$$2^{\tau \xi} = \theta(f)^{t \xi} \leq \theta(f)^{2t \xi(f)} = \Delta_2^t.$$

Combined with (3.1), Lemma 6.1 with $\delta = \frac{\log 2}{3} \lambda(\tau - 1)$, (6.14), (6.15), and item 1 of Lemma A.2, this implies

$$\begin{aligned}
& \sum_{k=0}^{+\infty} \exp(-(n+3k)P^+) |Df^{n+3k}(f(0))|^{-\frac{t}{2}} \\
& \leq \Delta_1^{\frac{t}{2}} \left(\frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{\frac{t}{2}n} \Pi^+(\tau, \lambda(\tau-1)). \\
& \leq \left(\Delta_1^{\frac{1}{2}} \exp(-nv) \right)^t (2 + 2^{\tau\xi}) \\
& \leq \left(\Delta_1^{\frac{1}{2}} \exp(-nv) (2 + \Delta_2) \right)^t \\
& \leq C_6^{-t}.
\end{aligned} \tag{6.18}$$

Together with item 2 of Proposition I, this implies

$$P^{\mathbb{R}}(t) \leq P(t) \leq P^+ \quad \text{and} \quad \mathcal{P}^{\mathbb{R}}(t, P^+) \leq \mathcal{P}(t, P^+) < 0. \tag{6.19}$$

On the other hand, by (3.1), Lemma 6.1 with $\delta = \frac{\log 2}{3} \lambda(\tau) \leq \log 2$, the definition of η_0 , (6.17), and item 2 of Lemma A.2, we have

$$\begin{aligned}
& \sum_{k=0}^{+\infty} \exp(-(n+3k)P^-) |Df^{n+3k}(f(0))|^{-\frac{t}{2}} \\
& \geq \Delta_1^{-\frac{t}{2}} \exp\left(-n \frac{\log 2}{3} \lambda(\tau)\right) \left(\frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{\frac{t}{2}n} \Pi^-(\tau, \lambda(\tau)). \\
& \geq 2^{-n} \left(\Delta_1^{\frac{1}{2}} \eta_0^{\frac{n}{2}} \right)^{-t} 2^{\tau^2} \\
& \geq \left(2^n \Delta_1^{\frac{1}{2}} \eta_0^{\frac{n}{2}} 2^{-\left(\frac{4}{A(\underline{s})}\right)^2 t_{\&}} \right)^{-t} \\
& \geq C_6^t.
\end{aligned} \tag{6.20}$$

Then item 1 of Proposition I implies

$$P(t) \geq P^{\mathbb{R}}(t) > P^- \quad \text{and} \quad \mathcal{P}(t, P^-) \geq \mathcal{P}^{\mathbb{R}}(t, P^-) > 0. \tag{6.21}$$

We proceed to prove the existence and uniqueness of equilibrium states. Combining (6.19) and (6.21), we have that the number $\chi_{\text{inf}}^{\mathbb{R}}(f)$ defined in the statement of Proposition III satisfies

$$\chi_{\text{inf}}^{\mathbb{R}}(f) = - \lim_{t \rightarrow +\infty} \frac{P^{\mathbb{R}}(t)}{t} = \frac{\chi_{\text{crit}}(f)}{2}.$$

Similarly,

$$\chi_{\text{inf}}(f) := \inf \left\{ \int \log |Df| \, d\mu \mid \mu \in \mathcal{M}_f \right\} = \frac{\chi_{\text{crit}}(f)}{2}.$$

Using (6.21) again, we conclude that for every $t > 0$ we have

$$P^{\mathbb{R}}(t) > -t\chi_{\inf}^{\mathbb{R}}(f) \quad \text{and} \quad P(t) > -t\chi_{\inf}(f).$$

The existence and uniqueness of equilibrium states follows from [32, Theorem A] in the real case. In the complex case it is proved in [31, Main Theorem] for rational maps, and the proof applies without changes to quadratic-like maps. This completes the proof of item 1 of the Main Theorem.

6.3.3. Temperature dependence

In this subsection we complete the proof of the Main Theorem by showing item 2. We give the proof in the complex setting; except for the obvious notational changes, it applies to the real case without modifications. We adopt the notation introduced in the previous subsections.

Fix a number t satisfying $t \geq t_{\&}$, and let m_0 be the integer in \mathbb{N} such that t is in $(A(\zeta)(m_0 - 1), A(\zeta)m_0]$. Note that $\tau \geq 50$, and that the integer $\tau_0 := \lceil \tau \rceil$ satisfies $4m_0 - 3 \leq \tau_0 \leq 4m_0$. On the other hand, by (6.19) and (6.21) there is $s^{\mathbb{C}}$ in $[\tau - 1, \tau]$ such that

$$P(t) = -t \frac{\chi_{\text{crit}}(f)}{2} + \frac{\log 2}{3} \lambda(s^{\mathbb{C}}). \quad (6.22)$$

Put $s_0 := \lceil s^{\mathbb{C}} \rceil$ and note that s_0 is either equal to $\tau_0 - 1$ or τ_0 .

We first prove that the hypotheses of Proposition 5.2 are satisfied for this value of t . By (3.1), Lemma 6.1 with $\delta = \frac{\log 2}{3} \lambda(\tau)$, and item 1 of Lemma A.3, we have

$$\sum_{k=0}^{+\infty} k \cdot \exp(-(n+3k)P^-) |Df^{n+3k}(f(0))|^{-\frac{t}{2}} < +\infty. \quad (6.23)$$

In particular, this implies that the sum in (6.20) is finite, so by item 1 of Proposition I we have $\mathcal{P}(t, P^-) < +\infty$. This implies that $\mathcal{P}(t, \cdot)$ is continuous and strictly decreasing on $[P^-, +\infty)$, so by (6.19), (6.21), and Proposition II we have the second equality in (5.2). Finally, combining (6.23), and item 3 of Proposition I, we obtain that the second sum in (5.4), with $P_f(t)$ replaced by P^- , is finite. In view of (6.21), this implies that the second sum in (5.4) is finite. This completes the proof that the hypotheses of Proposition 5.2 are satisfied.

Let $\tilde{\rho}$, and $\hat{\rho}$ be the measures given by Proposition 5.2. Moreover, put $\mathfrak{D} := \mathfrak{D}_f$, $F := F_f$, and for every k in \mathbb{N}_0 put $\mathfrak{D}_k := \mathfrak{D}_{f,k}$. Recall from §6.1, that $a_0 = 1$ and that for every s in \mathbb{N}_0 we have

$$I_s = [a_s, b_s) \quad \text{and} \quad J_s = [b_s, a_{s+1}).$$

Note that by item (b) of Lemma A.1, and the hypothesis $q \geq 50(\Xi + 1)$, we have $a_{s+1} - b_s = |J_s| \geq (s+1)^2$. For each integer ς in $[\tau_0 - 3, s_0]$ put

$$\tilde{\rho}'_{\varsigma} := \sum_{k=b_{\varsigma}+\varsigma^2}^{a_{\varsigma+1}-1} \sum_{j=n+3b_{\varsigma}-2}^{n+3(k+1-\varsigma^2)} \sum_{W \in \mathfrak{D}_k} (f^j)_* (\tilde{\rho}|_W),$$

and put

$$\tilde{\rho}'' := \sum_{k \in J_{s_0}} \sum_{j=n+3b_{s_0}-2}^{n+3(a_{s_0}+1-s_0^2)} \sum_{W \in \mathfrak{D}_k} (f^j)_* (\tilde{\rho}|_W).$$

In part 1 we estimate the total mass of the measure

$$\hat{\rho}' := \left(\sum_{\varsigma=\tau_0-3}^{s_0} \tilde{\rho}'_{\varsigma} \right) + \tilde{\rho}''$$

from below, and in part 2 we show that the total mass of $\hat{\rho} - \hat{\rho}'$ is small in comparison to that of $\hat{\rho}'$. In part 3 we complete the proof of item 2 of the Main Theorem by showing that $\hat{\rho}'$ is supported on a small neighborhood of the orbit of p^+ or p^- .

The following series defined in §A.3, are used in parts 1 and 2 below: $\tilde{\Pi}^{\pm}$, and for each s in \mathbb{N}_0 , the series $\tilde{\Pi}_s^{\pm}$, \tilde{I}_s^+ , \tilde{J}_s^+ , and \hat{J}_s^{\pm} . They satisfy

$$\begin{aligned} \tilde{\Pi}^+ &= \sum_{k=0}^{+\infty} k \cdot \pi_k^+, \tilde{I}_s^+ = \sum_{k \in I_s} k \cdot \pi_k^+, \tilde{J}_s^+ = \sum_{k \in J_s} k \cdot \pi_k^+, \text{ and} \\ \hat{J}_s^{\pm} &= \sum_{k=b_s+s^2}^{a_{s+1}-1} (k+1-b_s-s^2) \pi_k^{\pm}. \end{aligned}$$

1. To estimate the total mass of $\hat{\rho}'$ from below, put $\Upsilon_1 := \Delta_3 C_9 \Delta_1^{\frac{1}{2}} \eta_0^{\frac{n}{2}} 2^n$, and for each ς in $[\tau_0 - 3, s_0]$, put

$$H_{\varsigma} := \{k \in \mathbb{N}_0 \mid b_{\varsigma} + \varsigma^2 \leq k \leq a_{\varsigma+1} - 1\}.$$

By item 1 of Proposition 5.6, (3.1), Lemma 6.1 with $\delta = \frac{\log 2}{3} \lambda(s^C) \leq \log 2$, the definition of η_0 , (5.1), (5.3), and (6.22), we have

$$\begin{aligned} |\tilde{\rho}'| &\geq \sum_{\varsigma=\tau_0-3}^{s_0} |\tilde{\rho}'_{\varsigma}| \\ &= \sum_{\varsigma=\tau_0-3}^{s_0} \sum_{k \in H_{\varsigma}} 3(k+1-b_{\varsigma}-\varsigma^2) \sum_{W \in \mathfrak{D}_k} \tilde{\rho}(W) \\ &\geq (\Delta_3 C_9)^{-t} \sum_{\varsigma=\tau_0-3}^{s_0} \sum_{k \in H_{\varsigma}} 3(k+1-b_{\varsigma}-\varsigma^2) \exp(-(n+3k)P(t)) |Df^{n+3k}(f(0))|^{-t/2} \end{aligned}$$

$$\begin{aligned}
&\geq \left(\Delta_3 C_9 \Delta_1^{\frac{1}{2}} \left(\frac{|Df(\beta(f))|}{\exp(\chi_{\text{crit}}(f))} \right)^{\frac{n}{2}} \right)^{-t} 2^{-n} \sum_{\varsigma=\tau_0-3}^{s_0} \sum_{k \in H_\varsigma} 3(k+1-b_\varsigma-\varsigma^2) \pi_k^-(\tau, \lambda(s^{\mathbb{C}})) \\
&\geq 3\Upsilon_1^{-t} \sum_{\varsigma=\tau_0-3}^{s_0} \widehat{J}_\varsigma^-(\tau, \lambda(s^{\mathbb{C}})). \tag{6.24}
\end{aligned}$$

2. By (5.1), (5.3), item 2 of Proposition 5.6, and (6.22), we have

$$\begin{aligned}
|\widehat{\rho} - \widehat{\rho}'| &= \sum_{\substack{k \in \mathbb{N}_0 \\ k \notin \bigcup_{\varsigma=\tau_0-3}^{s_0} H_\varsigma}} \sum_{W \in \mathfrak{D}_k} m_f(W) \widetilde{\rho}(W) \\
&\quad + \sum_{\varsigma=\tau_0-3}^{s_0-1} \sum_{k \in H_\varsigma} \sum_{W \in \mathfrak{D}_k} (m_f(W) - 3(k+2-b_\varsigma-\varsigma^2)) \widetilde{\rho}(W) \\
&\quad + \sum_{k \in H_{s_0}} \sum_{W \in \mathfrak{D}_k} (m_f(W) - 3(k+4-b_{s_0}-2s_0^2+|J_{s_0-1}|)) \widetilde{\rho}(W) \\
&\leq (\Delta_3 C_9)^t \left[\sum_{\substack{k \in \mathbb{N}_0 \\ k \notin \bigcup_{\varsigma=\tau_0-3}^{s_0} H_\varsigma}} (n+3k+1) \exp(-(n+3k)P(t)) |Df^{n+3k}(f(0))|^{-\frac{t}{2}} \right. \\
&\quad + \sum_{\varsigma=\tau_0-3}^{s_0-1} \sum_{k \in H_\varsigma} (n+3(b_\varsigma+\varsigma^2)) \exp(-(n+3k)P(t)) |Df^{n+3k}(f(0))|^{-\frac{t}{2}} \\
&\quad \left. + \sum_{k \in H_{s_0}} (n+3(b_{s_0}+2s_0^2-|J_{s_0-1}|)) \exp(-(n+3k)P(t)) |Df^{n+3k}(f(0))|^{-\frac{t}{2}} \right].
\end{aligned}$$

Thus, if we put $\Upsilon_2 := \Delta_3 C_9 \Delta_1^{\frac{1}{2}} \exp(-nv)$, then by (3.1), Lemma 6.1 with $\delta = \frac{\log 2}{3} \lambda(s^{\mathbb{C}})$, and item 2 of Lemma A.3,

$$\begin{aligned}
|\widehat{\rho} - \widehat{\rho}'| &\leq (n+4) \Upsilon_2^t \left[\widetilde{\Pi}^+(\tau, \lambda(s^{\mathbb{C}})) - \sum_{\varsigma=\tau_0-3}^{s_0} \widehat{J}_\varsigma^+(\tau, \lambda(s^{\mathbb{C}})) - (|J_{s_0-1}| - s_0^2) \widehat{J}_{s_0}^+(\tau, \lambda(s^{\mathbb{C}})) \right] \\
&\leq (n+4) \Upsilon_2^t 2^{-q\tau^2} \sum_{\varsigma=\tau_0-3}^{s_0} \widehat{J}_\varsigma^-(\tau, \lambda(s^{\mathbb{C}})).
\end{aligned}$$

Together with (6.24) and the definitions of τ and A_{sup} , the previous chain of inequalities implies

$$|\widehat{\rho} - \widehat{\rho}'| \leq 3(n+4) (\Upsilon_1 \Upsilon_2)^t 2^{-q\tau^2} |\widehat{\rho}'| \leq 3(n+4) (\Upsilon_1 \Upsilon_2)^t 2^{-q\left(\frac{4}{A_{\text{sup}}}\right)^2 t^2} |\widehat{\rho}'|.$$

Thus, if we put

$$v'_0 := \frac{1}{2}q \left(\frac{4}{A_{\sup}} \right)^2 \log 2, \text{ and } C'_0 := 3(n+4) \exp \left(\frac{(\log(\Upsilon_1 \Upsilon_2))^2}{4v'_0} \right),$$

then

$$\frac{|\widehat{\rho} - \widetilde{\rho}'|}{|\widehat{\rho}|} \leq \frac{|\widehat{\rho} - \widetilde{\rho}'|}{|\widetilde{\rho}'|} \leq C'_0 \exp(-v'_0 t^2). \quad (6.25)$$

3. Using the inequality $\tau \geq 50$, and the definitions of τ and A_{\sup} we have for every ς in $[\tau_0 - 3, s_0]$,

$$\varsigma^2 \geq \frac{\tau^2}{2} \geq \frac{8}{A_{\sup}^2} t^2.$$

So, if we put $v''_0 := v_1 \frac{8}{A_{\sup}^2} - \frac{\log C_5}{t_{\<}^2} > 0$, then

$$C_5 \exp(-v_1 \varsigma^2) \leq \exp(-v''_0 t^2). \quad (6.26)$$

For $\varsigma \in \{+, -\}$, denote by \mathcal{O}^ς the forward orbit of p^ς under f . Let ς in $[\tau_0 - 3, s_0]$ be given, and put $m(\varsigma) := \lceil \varsigma/4 \rceil$, so that $4m(\varsigma) - 3 \leq \varsigma \leq 4m(\varsigma)$. For every integer j such that $j+1$ is in J_ς we have $\widehat{x}(\underline{\varsigma})_j = 1^{\varsigma(m(\varsigma))}$, so

$$\iota(\underline{\varsigma})_j = \begin{cases} 1 & \text{if } \varsigma(m(\varsigma)) = +; \\ 0 & \text{if } \varsigma(m(\varsigma)) = - \text{ and } j \text{ is even;} \\ 1 & \text{if } \varsigma(m(\varsigma)) = - \text{ and } j \text{ is odd.} \end{cases}$$

Since b_ς is even, for every ℓ in $[0, a_{\varsigma+1} - 1 - b_\varsigma]$ the points $f^{n+1+3(b_\varsigma+\ell-1)}(0)$ and $f^{3\ell}(p^{\varsigma(m(\varsigma))})$ are both in Y_f or both in \widetilde{Y}_f . It follows that

$$P_{f,3(a_{\varsigma+1}-1-b_\varsigma)+4}(f^{n+1+3(b_\varsigma-1)}(0)) = P_{f,3(a_{\varsigma+1}-1-b_\varsigma)+4}(p^{\varsigma(m(\varsigma))}).$$

Then, for each integer j in $[b_\varsigma - 1, a_{\varsigma+1} - 2]$ we have

$$P_{f,3(a_{\varsigma+1}-2-j)+4}(f^{n+1+3j}(0)) = P_{f,3(a_{\varsigma+1}-2-j)+4}(f^{3j}(p^{\varsigma(m(\varsigma))})), \quad (6.27)$$

which implies that for each integer k in J_ς and for each integer j in $[b_\varsigma - 1, k - 1]$,

$$P_{f,3(k-j)+1}(f^{n+1+3j}(0)) = P_{f,3(k-j)+1}(f^{3j}(p^{\varsigma(m(\varsigma))})).$$

Note that by definition of \mathfrak{D}_k , every element W of \mathfrak{D}_k is contained in $P_{f,n+3k+2}(0)$, so, if in addition we have $k \geq b_\varsigma + \varsigma^2$ and $j \leq k - \varsigma^2$, then by (6.26) and Lemma 4.11 we obtain

$$f^{n+1+3j}(W) \cup f^{(n+1+3j)+1}(W) \cup f^{(n+1+3j)+2}(W) \subset B(\mathcal{O}^{\varsigma(m(\varsigma))}, \exp(-v''_0 t^2)).$$

This proves that $\hat{\rho}'_\varsigma$ is supported on $B(\mathcal{O}^{\varsigma(m(\varsigma))}, \exp(-v''_0 t^2))$.

On the other hand, for each integer k in J_{s_0} , every element W of \mathfrak{D}_k is contained in $P_{f,n+3k+2}(0)$, and hence in $P_{f,n+3(a_{s_0}-1)+2}(0)$. Thus, by (6.27) with $\varsigma = s_0 - 1$, (6.26), and Lemma 4.11, we have that for every integer j in $[b_{s_0-1} - 1, a_{s_0} - s_0^2]$,

$$f^{n+1+3j}(W) \cup f^{(n+1+3j)+1}(W) \cup f^{(n+1+3j)+2}(W) \subset B(\mathcal{O}^{\varsigma(m(s_0-1))}, \exp(-v''_0 t^2)),$$

which proves that $\hat{\rho}''$ is supported on $B(\mathcal{O}^{\varsigma(m(s_0-1))}, \exp(-v''_0 t^2))$.

Assume that there are integers m and \hat{m} as in the statement of the Main Theorem, so that

$$\hat{m} \geq m \geq 1, \varsigma(m) = \cdots = \varsigma(\hat{m}), \text{ and } t \in [A(\varsigma)m, A(\varsigma)\hat{m}]. \quad (6.28)$$

Then $4m \leq \tau_0 \leq 4\hat{m}$, so for every ς in $[\tau_0 - 3, s_0]$ we have $\varsigma(m(\varsigma)) = \varsigma(m)$. It follows from the considerations above that the measure $\hat{\rho}'$ is supported on $B(\mathcal{O}^{\varsigma(m_0)}, \exp(-v''_0 t^2))$. Since the equilibrium state ρ_t of $f|_{J(f)}$ for the potential $-t \log |Df|$ is the probability measure proportional to $\hat{\rho}$, by (6.25) we have

$$\rho_t(\mathbb{C} \setminus B(\mathcal{O}^{\varsigma(m_0)}, \exp(-v''_0 t^2))) \leq \frac{|\hat{\rho} - \hat{\rho}'|}{|\hat{\rho}|} \leq C'_0 \exp(-v'_0 t^2).$$

Under our assumption $t \geq t_\&$, this proves item 2 of the Main Theorem with $v_0 = \min\{v'_0, v''_0\}$ and $C_0 = C'_0$. In the case where t is in $(0, t_\&)$, it suffices to take the same value of v_0 and replace C_0 by a constant bounded from below by $\exp(v_0 t_\&^2)$, if necessary. The proof of the Main Theorem is thus complete.

Remark 6.2. Without assuming the existence of m and \hat{m} satisfying (6.28), the measure $\hat{\rho}'$ is supported on $B(\mathcal{O}^+ \cup \mathcal{O}^-, \exp(-v''_0 t^2))$, and the estimate above gives that for every $t > 0$ we have

$$\rho_t(\mathbb{C} \setminus B(\mathcal{O}^+ \cup \mathcal{O}^-, \exp(-v_0 t^2))) \leq C_0 \exp(-v_0 t^2).$$

Appendix A. Estimating the 2 variables series

In this appendix we make some of the main estimates in the proof of the Main Theorem, in an abstract setting that is independent of the rest of the paper.

After some preliminary estimates in §A.1, the main estimates are given in §A.2, and §A.3.

A.1. Preliminary estimates

Fix a nonnegative integer Ξ , and an integer q satisfying $q \geq 50(\Xi + 1)$. For each s in $[0, +\infty)$, define

$$a_s := 2^{qs^3} \text{ and } b_s := 2^{qs^3} + q(2s+1) + \Xi,$$

$$I_s := [a_s, b_s) \text{ and } J_s := [b_s, a_{s+1}).$$

In the case where s is an integer, these definitions coincide with those in §6.1. For s in $[0, +\infty)$ that is not necessarily an integer, the interval J_s is used in the proof of Lemmas A.2 and A.3 in §A.2. Note that $|I_0| = q + \Xi$, and that for integer values of s the intervals I_s and J_s form a partition of $[1, +\infty)$.

Let $N: \mathbb{N}_0 \rightarrow \mathbb{N}_0$ and $B: \mathbb{N}_0 \rightarrow \mathbb{N}_0$ be as in §6.1. Observe that for every s in \mathbb{N}_0 , we have for every k in J_s

$$N(k) = \sum_{j=0}^s |I_j| = \sum_{j=0}^s (q(2j+1) + \Xi) = q(s+1)^2 + \Xi \cdot (s+1) \quad (\text{A.1})$$

and for every k in I_s

$$N(k) = k - (2^{qs^3} - 1) + qs^2 + \Xi s. \quad (\text{A.2})$$

Lemma A.1. *The following properties hold.*

- (a) *For each real number $s \geq 0$, we have $b_s \leq a_{s+1}/2$.*
- (b) *For each real number $s \geq 0$, we have $a_{s+1}/2 \leq |J_s|$.*
- (c) *For each real number $s \geq 1$, we have $b_s/a_s \leq 5/4$.*

Proof. Item (a) with $s = 0$ follows from our hypothesis $q \geq 50(\Xi + 1)$. For $s > 0$, it follows from this and from the fact that the derivative of the function

$$s \mapsto 2^{q(s+1)^3-1} - (2^{qs^3} + q(2s+1) + \Xi)$$

is strictly positive on $[0, +\infty)$. Item (b) follows easily from item (a). For item (c) notice that by our hypothesis $q \geq 50(\Xi + 1)$ it is enough to prove that for every $s \geq 1$ we have $2q(s+1) \leq (1/4) \cdot 2^{qs^3}$. The case $s = 1$ is given by our hypothesis $q \geq 50(\Xi + 1)$. For $s > 1$, it follows from this and from the fact that the derivative of the function

$$s \mapsto 2^{qs^3} - 8q(s+1)$$

is strictly positive on $[1, +\infty)$. \square

A.2. Estimating the 2 variables series

Let $\xi > 0$ be given, put $\Xi := \lceil 2\xi \rceil + 1$, and let q , N , and B be as in the previous subsection. For s in \mathbb{N}_0 define the following 2 variables series on $[0, +\infty) \times [0, +\infty)$,

$$I_s^\pm(\tau, \lambda) := \sum_{k \in I_s} 2^{-\lambda k - \tau N(k) \pm \tau \xi B(k)} \quad \text{and} \quad J_s^\pm(\tau, \lambda) := \sum_{k \in J_s} 2^{-\lambda k - \tau N(k) \pm \tau \xi B(k)},$$

and put

$$\Pi^\pm(\tau, \lambda) := 1 + \sum_{s=0}^{+\infty} I_s^\pm(\tau, \lambda) + \sum_{s=0}^{+\infty} J_s^\pm(\tau, \lambda).$$

Note that by (6.3) and (A.1), for every j in \mathbb{N}_0 and every $\tau > 0$ we have

$$J_j^\pm(\tau, \lambda) = 2^{-q\tau \cdot (j+1)^2 - (\Xi \mp 2\xi)\tau \cdot (j+1)} \sum_{k \in J_j} 2^{-\lambda k}. \quad (\text{A.3})$$

Moreover, for every real number s in $[0, +\infty)$ define

$$\lambda(s) := \frac{1}{|J_s|}.$$

Lemma A.2. *For every $\tau \geq 2$ the following inequalities hold:*

1. $\Pi^+(\tau, \lambda(\tau-1)) \leq 2 + 2^{\tau\xi}$.
2. $2^{\tau^2} \leq \Pi^-(\tau, \lambda(\tau))$.

Proof. 1. By (6.3), (A.2), our hypothesis $\tau \geq 2$, and the inequality $\Xi - 2\xi \geq 1$, for every $\lambda \geq 0$ we have

$$\begin{aligned} \sum_{s=0}^{+\infty} I_s^+(\tau, \lambda) &\leq \sum_{s=0}^{+\infty} \sum_{m=1}^{|I_s|} 2^{-\tau(qs^2 + \Xi s + m) + \tau\xi \cdot (2s+1)} \\ &\leq 2^{\tau\xi} \sum_{s=0}^{+\infty} 2^{-(\Xi - 2\xi)\tau s} \sum_{m=1}^{|I_s|} 2^{-\tau m} \\ &\leq 2^{\tau\xi} \frac{2^{-\tau}}{1 - 2^{-\tau}} \sum_{s=0}^{+\infty} 2^{-(\Xi - 2\xi)\tau s} \\ &\leq 2^{\tau\xi} \frac{2^{-\tau}}{(1 - 2^{-\tau})^2} \\ &\leq 2^{\tau\xi}. \end{aligned} \quad (\text{A.4})$$

To complete the proof of item 1, note that

$$\sum_{m=1}^{+\infty} 2^{-\lambda(\tau-1)m} = \frac{1}{2^{\lambda(\tau-1)} - 1} \leq \frac{1}{\lambda(\tau-1) \log 2} \leq 2|J_{\tau-1}| \leq 2 \cdot a_\tau. \quad (\text{A.5})$$

Combined with (A.3) and the inequality $\Xi - 2\xi \geq 1$, the previous chain of inequalities implies that for every j in \mathbb{N}_0 we have

$$\begin{aligned} J_j^+(\tau, \lambda(\tau - 1)) &\leq 2 \cdot 2^{q\tau^3 - q\tau \cdot (j+1)^2 - (\Xi - 2\xi)\tau \cdot (j+1)} \\ &\leq 2 \cdot 2^{q\tau^3 - q\tau \cdot (j+1)^2 - \tau \cdot (j+1)}. \end{aligned}$$

We obtain for every integer $j \geq \lfloor \tau \rfloor \geq \tau - 1$

$$J_j^+(\tau, \lambda(\tau - 1)) \leq 2 \cdot 2^{q\tau^3 - q\tau \cdot (j+1)^2 - \tau \cdot (j+1)} \leq 2 \cdot 2^{-\tau \cdot (j+1)}.$$

To estimate $J_j^+(\tau, \lambda(\tau - 1))$ for j in $\{0, \dots, \lfloor \tau \rfloor - 1\}$, note that

$$\sum_{m=1}^{|J_j|} 2^{-\lambda(\tau-1)m} \leq |J_j| \leq a_{j+1}.$$

Combined with (A.3) and the inequality $\Xi - 2\xi \geq 1$, this implies that for every integer j in $\{0, \dots, \lfloor \tau \rfloor - 1\}$ we have

$$J_j^+(\tau, \lambda(\tau - 1)) \leq 2^{q(j+1)^3 - \tau q(j+1)^2 - (\Xi - 2\xi)\tau \cdot (j+1)} \leq 2^{-\tau \cdot (j+1)}. \quad (\text{A.6})$$

Thus,

$$\sum_{j=0}^{+\infty} J_j^+(\tau, \lambda(\tau - 1)) \leq 2 \sum_{j=0}^{+\infty} 2^{-\tau \cdot (j+1)} = 2 \frac{2^{-\tau}}{1 - 2^{-\tau}} \leq 2.$$

Together with (A.4) this implies the desired inequality.

2. Put $\tau_0 := \lceil \tau \rceil$. By item (b) of Lemma A.1 and the definition of $\lambda(\tau)$, we have

$$\lambda(\tau) = |J_\tau|^{-1} \leq \frac{2}{a_{\tau+1}} \leq \frac{2}{a_{\tau_0}}.$$

From this inequality, item (c) of Lemma A.1 and our hypothesis $\tau \geq 2$, we obtain

$$\lambda(\tau)(b_{\tau_0} - 1) \leq 2 \frac{b_{\tau_0}}{a_{\tau_0}} \leq 3. \quad (\text{A.7})$$

On the other hand, note that for every m in $\{1, \dots, \lfloor |J_\tau| \rfloor\}$ we have $\lambda(\tau)m \leq 1$, so by item (b) of Lemma A.1 we have

$$\sum_{m=1}^{\lfloor |J_\tau| \rfloor} 2^{-\lambda(\tau)m} \geq \frac{1}{2}(|J_\tau| - 1) \geq \frac{1}{2^2}|J_\tau| \geq \frac{1}{2^3}2^{q(\tau+1)^3}.$$

Suppose $\tau \geq \tau_0 - 1/3$. In view of (A.3) and (A.7), the previous chain of inequalities implies

$$\begin{aligned}
& \frac{1}{2^6} 2^{q(\tau+1)^3 - q\tau \cdot (\tau_0+1)^2 - (\Xi+2\xi)\tau \cdot (\tau_0+1)} \\
& \leq \frac{1}{2^3} \left(\sum_{m=1}^{|J_{\tau_0}|} 2^{-\lambda(\tau)m} \right) 2^{-q\tau \cdot (\tau_0+1)^2 - (\Xi+2\xi)\tau \cdot (\tau_0+1)} \\
& \leq \left(\sum_{m=1}^{|J_{\tau_0}|} 2^{-\lambda(\tau)m} \right) 2^{-\lambda(\tau)(b_{\tau_0}-1) - q\tau \cdot (\tau_0+1)^2 - (\Xi+2\xi)\tau \cdot (\tau_0+1)} \quad (\text{A.8}) \\
& = J_{\tau_0}^-(\tau, \lambda(\tau)).
\end{aligned}$$

On the other hand, by our assumption $\tau \geq \tau_0 - 1/3$ we have

$$(\tau+1)^3 - \tau(\tau_0+1)^2 \geq (\tau+1)^3 - \tau \left(\tau + \frac{4}{3} \right)^2 = \frac{\tau^2}{3} + \frac{11\tau}{9} + 1 \geq \frac{\tau^2}{4} + \frac{\tau(\tau_0+1)}{12}.$$

Combined with our hypotheses $q \geq 50(\Xi+1) \geq 25(\Xi+2\xi+3)$ and $\tau \geq 2$, and with (A.8), this implies item 2 of the lemma when $\tau \geq \tau_0 - 1/3$.

To complete the proof, suppose $\tau \leq \tau_0 - 1/3$. Similarly as above we have

$$\begin{aligned}
& \frac{1}{2^6} 2^{q\tau_0^3 - q\tau\tau_0^2 - (\Xi+2\xi)\tau\tau_0} \leq \frac{1}{2^3} \left(\sum_{m=1}^{|J_{\tau_0-1}|} 2^{-\lambda(\tau_0-1)m} \right) 2^{-q\tau\tau_0^2 - (\Xi+2\xi)\tau\tau_0} \\
& \leq \frac{1}{2^3} \left(\sum_{m=1}^{|J_{\tau_0-1}|} 2^{-\lambda(\tau)m} \right) 2^{-q\tau\tau_0^2 - (\Xi+2\xi)\tau\tau_0} \quad (\text{A.9}) \\
& \leq \left(\sum_{m=1}^{|J_{\tau_0-1}|} 2^{-\lambda(\tau)m} \right) 2^{-\lambda(\tau)(b_{\tau_0-1}-1) - q\tau\tau_0^2 - (\Xi+2\xi)\tau\tau_0} \\
& = J_{\tau_0-1}^-(\tau, \lambda(\tau)).
\end{aligned}$$

On the other hand, our assumption $\tau \leq \tau_0 - 1/3$ implies

$$\tau_0^3 - \tau\tau_0^2 \geq \frac{\tau_0^2}{3} \geq \frac{\tau^2}{4} + \frac{\tau\tau_0}{12}.$$

Combined with our hypotheses $q \geq 50(\Xi+1) \geq 25(\Xi+2\xi+3)$ and $\tau \geq 2$, and with (A.9), we obtain item 2 of the lemma when $\tau \leq \tau_0 - 1/3$. The proof of the lemma is thus complete. \square

A.3. Estimating the weighted 2 variables series

For each s in \mathbb{N}_0 , $\tau > 0$, and $\lambda \geq 0$ put

$$\tilde{I}_s^+(\tau, \lambda) := \sum_{k \in I_s} k \cdot 2^{-\lambda k - \tau N(k) + \tau \xi B(k)},$$

$$\tilde{J}_s^+(\tau, \lambda) := \sum_{k \in J_s} k \cdot 2^{-\lambda k - \tau N(k) + \tau \xi B(k)},$$

and

$$\tilde{\Pi}^+(\tau, \lambda) := 1 + \sum_{s=0}^{+\infty} \tilde{I}_s^+(\tau, \lambda) + \sum_{s=0}^{+\infty} \tilde{J}_s^+(\tau, \lambda).$$

Noting that by item (b) of Lemma A.1 we have $a_{s+1} - b_s = |J_s| \geq s^2 + 1$, define for each $\tau > 0$ and $\lambda \geq 0$,

$$\hat{J}_s^\pm(\tau, \lambda) := \sum_{k=b_s+s^2}^{a_{s+1}-1} (k+1-b_s-s^2) \cdot 2^{-\lambda k - \tau N(k) \pm \tau \xi B(k)}.$$

Lemma A.3. *For each $\tau \geq 50$, the following properties hold:*

1. $\tilde{\Pi}^+(\tau, \lambda(\tau)) < +\infty$.
2. Let s in $[\tau-1, \tau]$ be given, put $s_0 := \lceil s \rceil$ and $\tau_0 := \lceil \tau \rceil$, and note that s_0 is equal to either $\tau_0 - 1$ or τ_0 . Then

$$\begin{aligned} \Pi^+(\tau, \lambda(s)) &\leq \tilde{\Pi}^+(\tau, \lambda(s)) - \sum_{\varsigma=\tau_0-3}^{s_0} \hat{J}_\varsigma^+(\tau, \lambda(s)) - (|J_{s_0-1}| - s_0^2) J_{s_0}^+(\tau, \lambda(s)) \\ &\leq 2^{-q\tau^2} \sum_{\varsigma=\tau_0-3}^{s_0} \hat{J}_\varsigma^-(\tau, \lambda(s)). \end{aligned}$$

The proof of this lemma is given after the following one.

Sublemma A.4. *Given $\tau \geq 50$ and s in $[\tau-1, \tau]$, put $\tau_0 := \lceil \tau \rceil$ and $s_0 := \lceil s \rceil$. Then the following properties hold.*

1. $\hat{J}_{s_0}^-(\tau, \lambda(s)) \geq 2^{2q(s+1)^3 - q\tau(s_0+1)(s_0+2)}$.
2. $\hat{J}_{s_0}^-(\tau, \lambda(s)) \geq 2^{q\tau^2(\tau-4)}$.
3. For every integer ς in $[\tau_0-3, s_0-1]$ we have

$$(b_\varsigma + \varsigma^2) J_\varsigma^+(\tau, \lambda(s)) \leq \frac{1}{20} 2^{-q\tau^2} \cdot \hat{J}_\varsigma^-(\tau, \lambda(s)).$$

4. $(b_{s_0} - |J_{s_0-1}| + 2s_0^2) J_{s_0}^+(\tau, \lambda(s)) \leq \frac{1}{4} 2^{-q\tau^2} \cdot \hat{J}_{s_0}^-(\tau, \lambda(s)).$

Proof. 1. By item (b) of Lemma A.1 and the definition of $\lambda(s)$, we have

$$\lambda(s) = |J_s|^{-1} \leq \frac{2}{a_{s+1}} \leq \frac{2}{a_{s_0}}.$$

On the other hand,

$$\lambda(s_0)s_0^2 = \frac{s_0^2}{|J_{s_0}|} \leq \frac{s_0^2}{2^{qs_0^3-1}} \leq \frac{1}{qs_0} \leq \frac{1}{100}. \quad (\text{A.10})$$

From these 2 inequalities and item (c) of Lemma A.1, we obtain

$$\lambda(s)(b_{s_0} + s_0^2) \leq \frac{2b_{s_0}}{a_{s_0}} + \frac{1}{100} \leq 3. \quad (\text{A.11})$$

By (6.3), (A.1), and (A.11), we have

$$\begin{aligned} \widehat{J}_{s_0}^-(\tau, \lambda(s)) &= 2^{-\tau(q(s_0+1)^2 + \Xi \cdot (s_0+1)) - 2\tau\xi \cdot (s_0+1)} \sum_{k=b_{s_0}+s_0^2}^{a_{s_0+1}-1} (k+1-b_{s_0}-s_0^2) \cdot 2^{-\lambda(s)k} \\ &\geq \frac{1}{2^3} 2^{-q\tau \cdot (s_0+1)^2 - (\Xi+2\xi)\tau \cdot (s_0+1)} \sum_{m=1}^{|J_{s_0}|-s_0^2} m \cdot 2^{-\lambda(s)m}. \end{aligned} \quad (\text{A.12})$$

Noticing that for every integer $N \geq 1$ we have

$$\sum_{m=1}^N m \cdot 2^{-\lambda(s)m} = \frac{2^{\lambda(s)}}{(2^{\lambda(s)} - 1)^2} \left(1 - (N+1)2^{-\lambda(s)N} + N2^{-\lambda(s)(N+1)} \right),$$

and that the function

$$\eta \mapsto 1 - (N+1)\eta^N + N\eta^{N+1}$$

is decreasing on $[0, 1]$, we have by (A.10) and the inequality $1 - 2^{-\lambda(s_0)} \leq \lambda(s_0) \log 2$

$$\begin{aligned} &\sum_{m=1}^{|J_{s_0}|-s_0^2} m \cdot 2^{-\lambda(s)m} \\ &\geq \frac{2^{\lambda(s)}}{(2^{\lambda(s)} - 1)^2} \cdot \left(1 - (|J_{s_0}| - s_0^2 + 1)2^{-\lambda(s_0)(|J_{s_0}|-s_0^2)} + (|J_{s_0}| - s_0^2)2^{-\lambda(s_0)(|J_{s_0}|-s_0^2+1)} \right) \\ &= \frac{2^{\lambda(s)}}{(2^{\lambda(s)} - 1)^2} \cdot \left(1 - 2^{\lambda(s_0)s_0^2-1} - 2^{\lambda(s_0)s_0^2-1}(|J_{s_0}| - s_0^2) \left(1 - 2^{-\lambda(s_0)} \right) \right) \\ &\geq \frac{2^{\lambda(s)}}{(2^{\lambda(s)} - 1)^2} \left(1 - 2^{\lambda(s_0)s_0^2-1}(1 + \log 2) \right) \\ &\geq \frac{1}{2^4} \frac{1}{(2^{\lambda(s)} - 1)^2}. \end{aligned} \quad (\text{A.13})$$

Note that by $\lambda(s) \leq 1$, we have $2^{\lambda(s)} - 1 \leq \lambda(s)$. Thus, together with item (b) of Lemma A.1, the previous chain of inequalities implies

$$\sum_{m=1}^{|J_{s_0}| - s_0^2} m \cdot 2^{-\lambda(s)m} \geq \frac{1}{2^4} \cdot |J_s|^2 \geq \frac{1}{2^6} \cdot 2^{2q(s+1)^3}.$$

Together with (A.12), the inequality $\Xi \geq 2\xi$, and our hypotheses $q \geq 50(\Xi + 1)$ and $\tau \geq 50$, this implies

$$\begin{aligned} \widehat{J}_{s_0}^-(\tau, \lambda(s)) &\geq \frac{1}{2^9} 2^{2q(s+1)^3 - q\tau \cdot (s_0+1)^2 - (\Xi+2\xi)\tau \cdot (s_0+1)} \\ &\geq 2^{2q(s+1)^3 - q\tau \cdot (s_0+1)(s_0+2)}. \end{aligned}$$

This proves item 1.

2. When $s_0 \leq \tau$ we have by our hypothesis $\tau \geq 50$,

$$2(s+1)^3 - \tau \cdot (s_0+1)(s_0+2) \geq 2\tau^3 - \tau(\tau+1)(\tau+2) \geq \tau^2(\tau-4).$$

On the other hand, in the case where $s_0 \geq \tau$ we have by our hypothesis $\tau \geq 50$,

$$\begin{aligned} 2(s+1)^3 - \tau \cdot (s_0+1)(s_0+2) &\geq 2\tau s_0^2 - \tau \cdot (s_0+1)(s_0+2) \\ &= \tau s_0(s_0-3) - 2\tau \geq \tau^2(\tau-3) - 2\tau \geq \tau^2(\tau-4). \end{aligned}$$

In all the cases, item 2 follows from item 1.

3. Let ς be an integer in $[\tau_0 - 3, s_0]$ and note that by (6.3), (A.3), and the definition of $\widehat{J}_{\varsigma}^-$, we have

$$\frac{J_{\varsigma}^+(\tau, \lambda(s))}{\widehat{J}_{\varsigma}^-(\tau, \lambda(s))} = 2^{4\tau\xi \cdot (\varsigma+1) + \lambda(s)\varsigma^2} \frac{\sum_{m=1}^{|J_{\varsigma}|} 2^{-\lambda(s)m}}{\sum_{m=1}^{|J_{\varsigma}| - \varsigma^2} m 2^{-\lambda(s)m}}. \quad (\text{A.14})$$

Suppose ς is in $[\tau_0 - 3, s_0 - 1]$. Then $\lambda(s)|J_{\varsigma}| \leq 1$, so

$$\begin{aligned} \frac{J_{\varsigma}^+(\tau, \lambda(s))}{\widehat{J}_{\varsigma}^-(\tau, \lambda(s))} &\leq 2 \cdot 2^{4\tau\xi \cdot (\varsigma+1) + \lambda(s)\varsigma^2} \frac{\sum_{m=1}^{|J_{\varsigma}|} 2^{-\lambda(s)m}}{\sum_{m=1}^{|J_{\varsigma}| - \varsigma^2} m} \\ &\leq 2^2 \cdot 2^{4\tau\xi \cdot (\varsigma+1) + \lambda(s)\varsigma^2} \frac{|J_{\varsigma}|}{(|J_{\varsigma}| - \varsigma^2)^2}. \end{aligned}$$

Noting that by items (a) and (b) of Lemma A.1 we have

$$\lambda(s)\varsigma^2 \leq \varsigma^2/|J_{\varsigma}| \leq 1 \quad \text{and} \quad |J_{\varsigma}| \leq 2(|J_{\varsigma}| - \varsigma^2),$$

by item (c) of Lemma A.1, the inequality $\Xi \geq 2\xi$, and our hypotheses $q \geq 50(\Xi + 1)$ and $\tau \geq 50$ we obtain

$$\begin{aligned} (b_\varsigma + \varsigma^2) \frac{J_\varsigma^+(\tau, \lambda(s))}{\widehat{J}_\varsigma^-(\tau, \lambda(s))} &\leq 2^6 a_s 2^{4\tau\xi \cdot (\varsigma+1)} |J_\varsigma|^{-1} \\ &\leq 2^7 \cdot 2^{q\varsigma^3 + 4\tau\xi \cdot (\varsigma+1) - q(\varsigma+1)^3} \\ &\leq \frac{1}{20} 2^{-q\tau^2}. \end{aligned}$$

This proves item 3.

4. By our hypotheses $q \geq 50(\Xi + 1)$ and $\tau \geq 50$ we have

$$b_{s_0} - |J_{s_0-1}| + 2s_0^2 = 2^{q(s_0-1)^3} + 2s_0^2 + 4qs_0 + 2\Xi \leq 2^{q(s_0-1)^3} + qs_0^2 \leq 2 \cdot 2^{q(s_0-1)^3}.$$

Thus, by item (b) of Lemma A.1, (A.13), (A.14), and the inequality $\lambda(s) \leq 1$, we have

$$\begin{aligned} (b_{s_0} - |J_{s_0-1}| + 2s_0^2) \frac{J_{s_0}^+(\tau, \lambda(s))}{\widehat{J}_{s_0}^-(\tau, \lambda(s))} &\leq 2^5 \cdot 2^{q(s_0-1)^3 + 4\tau\xi \cdot (s_0+1) + \lambda(s)s_0^2} (2^{\lambda(s)} - 1) \\ &\leq 2^5 \lambda(s) \cdot 2^{q(s_0-1)^3 + 4\tau\xi \cdot (s_0+1) + \lambda(s)s_0^2} \\ &\leq 2^6 \cdot 2^{-q(s+1)^3 + q(s_0-1)^3 + 4\tau\xi \cdot (s_0+1) + \lambda(s)s_0^2} \end{aligned}$$

Using $\lambda(s)s_0^2 \leq s_0^2|J_s|^{-1} \leq 1$, the inequality $\Xi \geq 2\xi$, and our hypotheses $q \geq 50(\Xi + 1)$ and $\tau \geq 50$, we have

$$\begin{aligned} (b_{s_0} - |J_{s_0-1}| + 2s_0^2) \frac{J_{s_0}^+(\tau, \lambda(s))}{\widehat{J}_{s_0}^-(\tau, \lambda(s))} &\leq 2^{-qs_0^3 + q(s_0-1)^3 + q\tau \cdot (s_0+1)} \\ &\leq 2^{-3qs_0(s_0-1) + q\tau \cdot (s_0+1)} \\ &\leq \frac{1}{4} 2^{-q\tau^2}. \end{aligned}$$

This completes the proof of item 4 and of the lemma. \square

Proof of Lemma A.3. 1. Note that for every $s \geq 0$, we have $\lambda(s) \leq 1$ and

$$\sum_{m=1}^{+\infty} m \cdot 2^{-\lambda(s)m} = \frac{2^{\lambda(s)}}{(2^{\lambda(s)} - 1)^2} \leq \frac{2^{\lambda(s)}}{(\lambda(s) \log 2)^2} \leq 2^3 |J_s|^2 \leq (2^3) 2^{2q(s+1)^3}. \quad (\text{A.15})$$

Together with (6.3), (A.1), (A.2), and the inequality $\Xi - 2\xi \geq 1$, for every j in \mathbb{N}_0 we have

$$\begin{aligned} \tilde{J}_j^+(\tau, \lambda(s)) + \tilde{I}_{j+1}^+(\tau, \lambda(s)) &\leq 2^{-\tau(q(j+1)^2 + \Xi \cdot (j+1)) + \tau \xi \cdot (2j+3)} \sum_{k \in J_j \cup I_{j+1}} k \cdot 2^{-\lambda(s)k} \\ &\leq (2^{\tau \xi + 3}) 2^{2q(s+1)^3 - q\tau \cdot (j+1)^2 - \tau \cdot (j+1)}. \end{aligned} \quad (\text{A.16})$$

Taking $s = \tau$, for every $j \geq 2\tau + 1$ we have

$$\tilde{J}_j^+(\tau, \lambda(\tau)) + \tilde{I}_{j+1}^+(\tau, \lambda(\tau)) \leq (2^{\tau \xi + 3}) 2^{-\tau \cdot (j+1)}.$$

This implies that $\tilde{\Pi}^+(\tau, \lambda(\tau))$ is finite, as wanted.

2. The first inequality follows directly from the definitions. To prove the second inequality, note that by (6.3) and (A.2), and our hypotheses $\tau \geq 50$ and $\xi > 0$, we have

$$1 + \tilde{I}_0^+(\tau, \lambda(s)) \leq 1 + \sum_{k=1}^{+\infty} k \cdot 2^{-\tau k + \tau \xi} = 1 + 2^{\tau \xi} \frac{2^{-\tau}}{(1 - 2^{-\tau})^2} \leq 2^{\tau \xi}. \quad (\text{A.17})$$

On the other hand, by item (c) of Lemma A.1, (6.3), (A.2), and the inequality $\Xi - 2\xi \geq 1$, for every integer $j \geq 1$ we have

$$\begin{aligned} \tilde{I}_j^+(\tau, \lambda(s)) &\leq (2^{\tau \xi}) 2^{-\tau(qj^2 + (\Xi - 2\xi)j)} \sum_{m=1}^{|I_j|} (2^{qj^3} + m) 2^{-\tau m} \\ &\leq (2^{\tau \xi + 1}) 2^{qj^3 - q\tau j^2 - (\Xi - 2\xi)\tau j} \frac{1}{1 - 2^{-\tau}} \\ &\leq (2^{\tau \xi + 2}) 2^{q(j-\tau)j^2 - \tau j}. \end{aligned}$$

Combined with (A.17), the inequality $\Xi \geq 2\xi$, and our hypotheses $q \geq 50(\Xi + 1)$ and $\tau \geq 50$, this implies

$$1 + \sum_{j=0}^{\tau_0+1} \tilde{I}_j^+(\tau, \lambda(s)) \leq (2^{\tau \xi + 2}) \frac{2^{2q(\tau+2)^2}}{1 - 2^{-\tau}} \leq 2^{2q\tau(\tau+6)}.$$

Together with item 2 of Sublemma A.4 and our hypothesis $\tau \geq 50$, this chain of inequalities implies

$$1 + \sum_{j=0}^{\tau_0+1} \tilde{I}_j^+(\tau, \lambda(s)) \leq \frac{1}{2^3} 2^{-q\tau^2} \cdot \hat{J}_{s_0}^-(\tau, \lambda(s)). \quad (\text{A.18})$$

On the other hand, by (6.3), (A.1), and our hypothesis $\tau \geq 50$, for every j in $\{0, \dots, \tau_0 - 4\}$ we have

$$\begin{aligned}
\tilde{J}_j^+(\tau, \lambda(s)) &= 2^{-\tau(q(j+1)^2 + \Xi \cdot (j+1) + 2\tau\xi \cdot (j+1))} \sum_{k \in J_j} k \cdot 2^{-\lambda(s)k} \\
&\leq |J_j| 2^{q(j+1)^3 - q\tau \cdot (j+1)^2 - (\Xi - 2\xi)\tau \cdot (j+1)} \\
&\leq 2^{2q(j+1)^3 - q\tau \cdot (j+1)^2 - (\Xi - 2\xi)\tau \cdot (j+1)} \\
&\leq 2^{q(j+1)^2(2j+2-\tau)} \\
&\leq 2^{q(\tau-2)^2(\tau-4)} \\
&\leq 2^{q\tau^2(\tau-7)}.
\end{aligned}$$

Together with item 2 of Sublemma A.4 and with our hypothesis $\tau \geq 50$, this implies

$$\frac{\sum_{j=0}^{\tau_0-4} \tilde{J}_j^+(\tau, \lambda(s))}{\widehat{J}_{s_0}^-(\tau, \lambda(s))} \leq \tau 2^{-3q\tau^2} \leq \frac{1}{2^2} 2^{-q\tau^2}. \quad (\text{A.19})$$

On the other hand, by (A.16), item 1 of Sublemma A.4, the inequality $\Xi \geq 2\xi$ and our hypothesis $q \geq 50(\Xi + 1)$, for every integer $j \geq s_0 + 1$ we have

$$\begin{aligned}
\frac{\tilde{J}_j^+(\tau, \lambda(s)) + \tilde{I}_{j+1}^+(\tau, \lambda(s))}{\widehat{J}_{s_0}^-(\tau, \lambda(s))} &\leq (2^{\tau\xi+3}) 2^{-q\tau \cdot ((j+1)^2 - (s_0+1)(s_0+2))} \\
&\leq (2^{\tau\xi+3}) 2^{-q\tau \cdot (s_0+2)(j-s_0)} \\
&\leq (2^{\tau\xi+3}) 2^{-q\tau^2(j-s_0) - q\tau} \\
&\leq \frac{1}{2^3} 2^{-q\tau^2(j-s_0)}.
\end{aligned}$$

Summing over $j \geq s_0 + 1$ and using our hypotheses $q \geq 50(\Xi + 1)$ and $\tau \geq 50$, we obtain

$$\frac{\sum_{j=s_0+1}^{+\infty} (\tilde{J}_j^+(\tau, \lambda(s)) + \tilde{I}_{j+1}^+(\tau, \lambda(s)))}{\widehat{J}_{s_0}^-(\tau, \lambda(s))} \leq \frac{1}{2^3} \frac{2^{-q\tau^2}}{1 - 2^{-q\tau^2}} \leq \frac{1}{2^2} 2^{-q\tau^2}.$$

Combined with (A.18) and (A.19), this implies

$$\tilde{\Pi}^+(\tau, \lambda(s)) - \sum_{\varsigma=\tau_0-3}^{s_0} \tilde{J}_{\varsigma}^+(\tau, \lambda(s)) \leq \frac{1}{2} 2^{-q\tau^2} \cdot \widehat{J}_{s_0}^-(\tau, \lambda(s)). \quad (\text{A.20})$$

For each integer ς in $[\tau_0 - 3, s_0]$, we have

$$\begin{aligned}
&\tilde{J}_{\varsigma}^+(\tau, \lambda(s)) - \widehat{J}_{\varsigma}^+(\tau, \lambda(s)) \\
&\leq \sum_{k=b_{\varsigma}}^{b_{\varsigma}+\varsigma^2-1} k \cdot 2^{-\lambda k - \tau N(k) + \tau \xi B(k)} + \sum_{k=b_{\varsigma}+\varsigma^2}^{a_{\varsigma+1}-1} (b_{\varsigma} + \varsigma^2) 2^{-\lambda k - \tau N(k) + \tau \xi B(k)} \\
&\leq (b_{\varsigma} + \varsigma^2) J_{\varsigma}^+(\tau, \lambda(s)).
\end{aligned} \quad (\text{A.21})$$

Together with item 3 of Sublemma A.4, this implies that for ς in $[\tau_0 - 3, s_0 - 1]$ we have

$$\tilde{J}_{\varsigma}^{+}(\tau, \lambda(s)) - \hat{J}_{\varsigma}^{+}(\tau, \lambda(s)) \leq \frac{1}{20} 2^{-q\tau^2} \cdot \hat{J}_{\varsigma}^{-}(\tau, \lambda(s)). \quad (\text{A.22})$$

On the other hand, (A.21) with $\varsigma = s_0$ and item 4 of Sublemma A.4 imply

$$\tilde{J}_{\varsigma}^{+}(\tau, \lambda(s)) - \hat{J}_{\varsigma}^{+}(\tau, \lambda(s)) - (|J_{s_0-1}| - s_0^2) J_{s_0}^{+}(\tau, \lambda(s)) \leq \frac{1}{4} 2^{-q\tau^2} \cdot \hat{J}_{s_0}^{-}(\tau, \lambda(s)).$$

Together with (A.20) and (A.22), this implies the desired inequality and completes the proof of the lemma. \square

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