Ergod. Th. & Dynam. Sys. (First published online 2010), page 1 of 24*
doi:10.1017/S0143385710000507
*Provisional—final page numbers to be inserted when paper edition is published

Tower systems for linearly repetitive Delone sets

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(*Received* 23 *March* 2010 *and accepted in revised form* 5 *July* 2010)

Abstract. In this paper we study linearly repetitive Delone sets and prove, following the work of Bellissard, Benedetti and Gambaudo, that the hull of a linearly repetitive Delone set admits a properly nested sequence of box decompositions (tower system) with strictly positive and uniformly bounded (in size and norm) transition matrices. This generalizes a result of Durand for linearly recurrent symbolic systems. Furthermore, we apply this result to give a new proof of a classic estimation of Lagarias and Pleasants on the rate of convergence of patch frequencies.

1. *Introduction*

Delone sets arise naturally as mathematical models for the description of solids. In this modelization, the solid is supposed to be infinitely extended and its atoms are represented by points. These atoms interact through a potential (for example a Lennard-Jones potential). For a given specific energy, Delone sets are good candidates to describe the ground state configuration: uniform discreteness corresponds to the existence of a minimum distance between atoms due to the repulsion forces between nuclei, and relative density corresponds to the fact that empty regions can not be arbitrarily big because of the contraction forces. In perfect crystals, atoms are ordered in a repeating pattern extending in all three dimensions and can be modeled by lattices in $\mathbb{R}^3$. Quasi-crystalline solids are those whose X-ray diffraction images have sharp spots indicating *long-range order* but without having a full-lattice of periods. Typically, they exhibit symmetries that are impossible for a perfect crystal (see e.g. [**SBGC84**]).

From the mathematical and physical point of view, linear repetitivity (introduced by Lagarias and Pleasants in [**LP03**]) has become a key feature (see e.g. [**DL06, Sol98**]), and many known examples of quasi-crystalline solids may be modeled using linearly repetitive Delone sets.

A standard tool in the study of a Delone set X is provided by its hull Ω and the natural $\mathbb{R}^d$-action on it. The hull is defined as an appropriate closure of the family $\{X - v \mid v \in \mathbb{R}^d\}$

and $\mathbb{R}^d$ acts over the hull by translation. From this point of view, the hull can be regarded as a generalization to higher dimensions of the standard orbit-closure construction for aperiodic sequences in symbolic dynamics (see e.g. [**Rob04**]). In this context, linear repetitivity corresponds to linear recurrence (see [**Dur00**]).

A powerful combinatorial tool in the study of linearly recurrent subshifts is provided by Kakutani–Rohlin towers (see e.g. [**CDHM03, HPS92**]). In particular, a fundamental result in this context is the following (see [**CDHM03, Dur00**]).

THEOREM 1.1. *Let (X, T) be an aperiodic linearly recurrent subshift. Then there exist a sequence of Kakutani–Rohlin (KR) partitions and $M > 0$ such that the following hold.*
(i) *The number of KR towers of level n is uniformly bounded by M.*
(ii) *Every KR tower of level n crosses all the KR towers of level $n-1$ and the total number of these crossings is uniformly bounded by M.*

An equivalent statement for Theorem 1.1 is that linearly recurrent subshifts admit nested sequences of KR partitions with transition matrices that are strictly positive and uniformly bounded in norm. As a corollary, for instance, it is easy to deduce that linearly recurrent subshifts are uniquely ergodic (see e.g. [**CDHM03, Dur00**]). Other applications include the study of spectral properties of these systems (see [**BDM05, BDM10, CDHM03**]).

In the context of Delone sets, tower systems were introduced in [**BBG06**], and provide a proper generalization for sequences of KR towers (see §3.1 for the definition). A natural question we answer in this paper is whether an analog of Theorem 1.1 for linearly repetitive Delone systems and tower systems holds. To answer this question, we refine the construction in [**BBG06**] to linearly repetitive systems to obtain several estimations on parameters that control the growth of the towers and, therefore, also the norms of the associated transition matrices. Our main result is the following.

MAIN RESULT. (See Theorem 3.6) *Let Ω be the hull of a linearly repetitive Delone set X. Then there exist a tower system and $M > 0$ such that the following hold.*
(i) *The number of boxes at each level is uniformly bounded by M.*
(ii) *Every box of level n crosses every box of level $n-1$ and the total number of crossings is uniformly bounded by M.*

For similar constructions, we refer to [**Bes08b, CGM07, For00, GMPS10, LS05, Pri97, PS01**].

In [**BG03, GM06**], it is proved (using standard arguments) that Delone systems satisfying the assertions of the main result are uniquely ergodic. Hence, as a corollary of this result and our main result, we obtain an alternative proof of the unique ergodicity of the hull of an aperiodic linearly repetitive Delone set X. This is equivalent to say (see e.g. [**LMS02**]) that X has uniform patch frequencies, i.e., for every patch $\mathbf{p}$ in X, if $n_{\mathbf{p}}(U)$ is the number of patches of X that are equivalent to $\mathbf{p}$ and whose center is included in a d-cube U of side N, then $n_{\mathbf{p}}(U)$ divided by the volume of U converges to a limit, called the *frequency* of $\mathbf{p}$ and denoted by $\mathrm{freq}(\mathbf{p})$, when N goes to infinity. The existence of uniform patch frequencies is well-known for linearly repetitive Delone sets, and it is a consequence of the following stronger result of Lagarias and Pleasants [**LP03**].

THEOREM 1.2. (Lagarias–Pleasants theorem) *Let X be a linearly repetitive Delone set. Then, X has uniform patch frequencies. Moreover, there exists $\delta > 0$ such that, for every patch $\mathbf{p}$ of X,*

$$\left| \frac{n_{\mathbf{p}}(D_N)}{\mathrm{vol}(D_N)} - \mathrm{freq}(\mathbf{p}) \right| = O(N^{-\delta}),$$

where D_N is either a d-cube with side N or a ball of radius N.

In this paper, we give an alternative proof of Lagarias and Pleasants' theorem as an application of our main result. A key step in the proof is the introduction of a Markov chain associated with the tower system, whose mixing rate is related to the constant δ. We remark that in the original proof of Lagarias and Pleasants, the constant δ depends on the geometry of D_N. Our new approach suggests that the δ should be defined purely in terms of X and the tower system, and so we expect that the proof can be extended to provide estimations for more general additive ergodic theorems in [**Bes08a, DL06, LS05**]. We also remark that the proof can be applied to self-similar systems, with better bounds, and this will be the subject of a forthcoming paper.

We finish the introduction by giving the organization of the paper. In §2 we review the theory on Delone sets and the dynamical system approach that will be needed in the following. In §3, we review the definition of tower system and prove the main result, by supposing the existence of a tower system satisfying some extra conditions (see Theorem 3.4). In §4 we prove the existence of this tower system. Then, the construction of the Markov chain and the proof of the Lagarias–Pleasant theorem are given, respectively, in §§5 and 6.

2. *Background*

In this section we fix some notation and review some basic definitions about Delone sets and the associated hulls. For details we refer to [**LP03, Rob04**]. For some of the notation, we also follow [**LS05**].

We work with the d-dimensional Euclidean space, denoted by $\mathbb{R}^d$. The set of non-negative integer numbers will be denoted by $\mathbb{N}$, and the set of positive integer numbers by $\mathbb{N}^*$. If $\mathcal{P}$ is a subset of $\mathbb{R}^d$ and v is in $\mathbb{R}^d$, then the set $\{P - v \mid P \in \mathcal{P}\}$ will be denoted $\mathcal{P} - v$. A pair (Λ, Q), where Q is a bounded subset of $\mathbb{R}^d$ and $\Lambda \subseteq Q$ is finite is called a *pattern*. If $Q = \overline{B}_S(x_0)$ is the closed ball of radius $S > 0$ around $x_0 \in \Lambda$, then (Λ, Q) is called a *S-pattern*, *centered* at x_0. The set Q is called the *support* of the pattern. For $t \in \mathbb{R}^d$ and (Λ, Q), we set $(\Lambda, Q) - t = (\Lambda - t, Q - t)$. Two patterns (Λ_1, Q_1) and (Λ_2, Q_2) are *equivalent* if there exists $t \in \mathbb{R}^d$ such that $(\Lambda_1, Q_1) - t = (\Lambda_2, Q_2)$. We refer to the equivalence class of a pattern (Λ, Q) as a pattern-class.

2.1. *Delone sets.*

Let X be a subset of $\mathbb{R}^d$. We say that X is r-discrete if every closed ball of radius r in $\mathbb{R}^d$ intersects X in at most one point. We say that X is R-dense if every closed ball of radius R in $\mathbb{R}^d$ intersects X in at least one point. The set X is a *Delone set* if it is r-discrete and R-dense for some $r, R > 0$.

Let X be a Delone set. A *patch* in X is a pattern of the form $X \wedge Q := (X \cap Q, Q)$. The *$S$-patch* of X *centered* at $x \in X$ is defined as $X \wedge \overline{B}_S(x)$. Two patches in X are *equivalent* if they are equivalent as patterns. The *set of pattern-classes* of X is the set containing the pattern-classes of all patches in X.

For a S-pattern $\mathbf{p} = (\Lambda, \overline{B}_S(x_0))$, an *occurrence* of $\mathbf{p}$ in X is a point $y \in \mathbb{R}^d$ such that $X \wedge \overline{B}_S(y)$ is equivalent to $\mathbf{p}$. A Delone set X is *repetitive* if for every $S > 0$ there is $M > 0$ such that every ball of radius M contains an occurrence of every S-patch of X. The smallest such M is denoted by $M_X(S)$. If there exists $L > 1$ such that $M_X(S) \leq LS$ for all $S > 0$, then X is called *linearly repetitive*. The set X has *finite local complexity* if the number of equivalence classes of S-patches is finite. Clearly, every repetitive Delone set has finite local complexity but not the converse. A Delone set X is called *aperiodic* if $X - v \neq X$ for all $v \neq 0$.

The collection of all Delone sets with finite local complexity is denoted by $\mathcal{D}$. Given two Delone sets X and Y in $\mathcal{D}$, their distance is defined as the smallest $0 < \varepsilon < \sqrt{2}/2$ for which there exist $u, u' \in B_\varepsilon(0)$ such that

$$(X - u) \cap \overline{B}_{1/\varepsilon}(0) = (Y - u') \cap \overline{B}_{1/\varepsilon}(0);$$

if such ε does not exist, then the distance between X and Y is defined to be $\sqrt{2}/2$. It is standard that this gives a distance (see e.g. [**LMS02**]) under which two Delone sets are close whenever they coincide in a big ball around 0 up to a small translation. We refer to the topology induced by this metric as the *tiling* topology (for a discussion of different topologies for Delone sets, see [**Moo97**]).

Given a Delone set X in $\mathcal{D}$, the hull of X, denoted by Ω, is defined as the closure with respect to the tiling topology of the family $\{X - v \mid v \in \mathbb{R}^d\}$. It is well known that Ω is compact, and the set of pattern-classes of X contains the pattern-classes of all Delone sets in Ω. It is important to observe that this implies that if X is r-dense and R-discrete for some fixed r and $R > 0$, then all the Delone sets in Ω are also r-dense and R-discrete (for the same choices of r and R).

The *translation action* Γ over Ω is the $\mathbb{R}^d$-action defined by

$$\Gamma_v Y = Y - v$$

for all $v \in \mathbb{R}^d$ and $Y \in \Omega$. The pair (Ω, Γ) forms a dynamical system and we refer to it as a *Delone system*. Recall that (Ω, Γ) is said to be *minimal* if every orbit is dense. It is well known that Ω is minimal if and only if X is repetitive, and in this case, Ω is the set containing all the Delone sets that have the same pattern-classes as X. Moreover, if X is repetitive and aperiodic, then all Delone sets in Ω are also aperiodic.

2.2. *Local transversals and return vectors.* Let (Ω, Γ) be an aperiodic minimal Delone system. The *canonical transversal* of Ω is the set composed of all Delone sets in Ω that contain 0. This terminology is motivated by the fact that if Y is in Ω^0, then every small translation of Y will not be in Ω^0. A *cylinder* in Ω is a set of the form

$$C_{Y,S} := \{Z \in \Omega \mid Z \wedge \overline{B}_S(0) = Y \wedge \overline{B}_S(0)\},$$

where $Y \in \Omega$ and $S > 0$ are such that $Y \cap \overline{B}_S(0) \neq \emptyset$. The following proposition is well known (see e.g. [**KP00**]).

PROPOSITION 2.1. *Every cylinder in Ω is a Cantor set. Moreover, a basis for the topology of Ω is given by sets of the form*

$$\{Z - v \mid Z \in C_{Y,S},\ v \in B_\varepsilon(0)\}.$$

In particular, the canonical transversal Ω^0 is a Cantor set.

A *local transversal* in Ω is a clopen (both closed and open) subset of any cylinder in Ω. By Proposition 2.1, a local transversal C is a Cantor set. This implies that

$$\operatorname{rec}(C) := \inf\{S > 0 \mid C_{Y,S} \subseteq C\ \forall Y \in C\}$$

is finite, and the collection

$$\{C_{Y,S} \mid Y \in C,\ S > \operatorname{rec}(C)\}$$

forms a basis for its topology. Indeed, since C is a Cantor set, it is easy to find a finite set $\{Y_1, \ldots, Y_m\}$ in C such that

$$C = \bigcup_{i=1}^{m} C_{Y_i,\operatorname{rec}(C)}.$$

The motivation to define $\operatorname{rec}(C)$ is the following: suppose that we are given a Delone set $Y \in \Omega$ and we want to check if Y belongs to C. Then it suffices to look whether the patch $Y \wedge \overline{B}_{\operatorname{rec}(Y)}(0)$ is equivalent to $Y_i \wedge \overline{B}_{\operatorname{rec}(Y)}(0)$ for some Y_i. Of course, if $C = C_{Y,S}$, then its recognition radius is smaller than S.

Given a local transversal C and $D \subseteq \mathbb{R}^d$, the following notation will be used throughout the paper:

$$C[D] = \{Y - x \mid Y \in C,\ x \in D\}.$$

A very successful way of studying the hull is provided by the set of return vectors to a local transversal. Given a local transversal C and a Delone set $Y \in \Omega$, we define

$$\mathcal{R}_C(Y) = \{x \in \mathbb{R}^d \mid Y - x \in C\}.$$

When Y belongs to C, we refer to $\mathcal{R}_C(Y)$ as the *set of return vectors* of Y to C. The following lemma is standard (see e.g. [**Cor**]).

LEMMA 2.2. *Let C be a local transversal. Then for each $Y \in C$, the set of return vectors $\mathcal{R}_C(Y)$ is a repetitive Delone set. Moreover, the quantities*

$$r(C) = \tfrac{1}{2} \inf\{\|x - y\| \mid x, y \in \mathcal{R}_C(Y),\ x \neq y\} \tag{2.1}$$

and

$$R(C) = \inf\{R > 0 \mid \mathcal{R}_C(Y) \cap \overline{B}_R(y) \neq \emptyset\ \forall y \in \mathbb{R}^d\} \tag{2.2}$$

do not depend on the choice of Y in C.

Remark 2.3. If X is a repetitive Delone set and Ω is its hull, then it follows directly that $R(C_{Y,S}) \leq M_X(S)$ for every $Y \in \Omega^0$ and $S > 0$. Hence, in the linearly repetitive case we have $R(C_{Y,S}) \leq LS$, where $L > 1$ is the constant of linear repetitivity. Moreover, an estimation of $r(C)$ in terms of L (known as a repulsion property) also exists (see [**Len04**]). For reference, these estimations are given below.

LEMMA 2.4. *Let X be a linearly repetitive Delone set with constant $L > 1$. Then, for every cylinder $C_{Y,S}$ with $Y \in \Omega^0$ and $S > 0$ we have*

$$\frac{S}{2(L+1)} \leq r(C_{Y,S}) < R(C_{Y,S}) \leq LS. \tag{2.3}$$

2.3. *Solenoids, boxes and transverse measures.* In this section, we recall some definitions and results of [**BBG06, BG03**] that will be used throughout the paper. Let (Ω, Γ) be an aperiodic minimal Delone system. The hull Ω is locally homeomorphic to the product of a Cantor set and $\mathbb{R}^d$ (see [**AP98, SW03**]). Moreover, there exists an open cover $\{U_i\}_{i=1}^n$ of Ω such that for each $i \in \{1, \ldots, n\}$, there are $Y_i \in \Omega$, $S_i > 0$ and open sets $D_i \subseteq \mathbb{R}^d$ such that $U_i = C_{Y_i,S_i}[D_i]$ and the map $h_i : D_i \times C_{Y_i,S_i} \to U_i$ defined by $h_i(t, Z) = Z - t$ is a homeomorphism. Furthermore, there are vectors $v_{i,j} \in \mathbb{R}^d$ (depending *only* on i and j) such that the transition maps $h_i^{-1} \circ h_j$ satisfy

$$h_i^{-1} \circ h_j(t, Z) = (t - v_{i,j}, Z - v_{i,j}) \tag{2.4}$$

at all points (t, Z) where the composition is defined. Following [**BG03**], we call such a cover an $\mathbb{R}^d$*-solenoid's atlas*. It induces, among others structures, a laminated structure as follows. First, *slices* are defined as sets of the form $h_i(D_i \times \{Z\})$. Equation (2.4) implies that slices are mapped onto slices. Thus, the *leaves* of Ω are defined as the smallest connected subsets that contain all the slices they intersect. It is not difficult to check, using (2.4), that the leaves coincide with the orbits of Ω.

A *box* in Ω is a set of the form $B := C[D]$ where C is a local transversal in Ω, and $D \subseteq \mathbb{R}^d$ is an open set such that the map from $D \times C$ to B given by $(x, Y) \mapsto Y - x$ is a homeomorphism. This is true, for instance, if $D \subseteq B_{r(C)}(0)$ (cf. (2.1)).

A Borel measure μ on Ω is *translation invariant* if $\mu(B - v) = \mu(B)$ for every Borel set B and $v \in \mathbb{R}^d$. Let C be a local transversal and $0 < r < r(C)$. Each translation-invariant measure μ induces a measure ν on C (see [**Ghy99**] for the general construction and e.g. [**CFS82**] for the analog construction for flows): given a Borel subset V of C, its *transverse measure* is defined by

$$\nu(V) = \frac{\mu(V[B_r(0)])}{\mathrm{vol}(B_r(0))}.$$

This gives a measure on each C, which does not depend on r. The collection of all measures defined in this way is called the *transverse invariant measure* induced by μ. It is invariant in the sense that if V is a Borel subset of C and $x \in \mathbb{R}^d$ is such that $V - x$ is a Borel subset of another local transversal C', then $\nu(V - x) = \nu(V)$. Conversely, the measure μ of any box B written as $C[D]$ may be computed by the equation

$$\mu(C[D]) = \mathrm{vol}(D) \times \nu(C).$$

3. *Tower systems*

Let Ω be the hull of an aperiodic repetitive Delone set X. In this section we review the concepts of box decompositions and tower systems introduced in [**BBG06, BG03**] and prove our main result.

3.1. *Box decompositions and derived tilings.* A *box decomposition* is a finite and pairwise-disjoint collection of boxes $\mathcal{B} = \{B_1, \ldots, B_t\}$ in Ω such that the closures of the boxes in $\mathcal{B}$ cover the hull. For simplicity, we always write $B_i = C_i[D_i]$, where C_i and D_i are fixed and C_i is contained in B_i. In particular, the set D_i contains 0. We refer to C_i as the *base* of B_i. In this way, we call the union of all C_i the *base* of $\mathcal{B}$. The reasoning for fixing a local transversal in each B_i comes from the fact that box decompositions can be constructed from the set $\mathcal{R}_C(Y)$ of return vectors to a given local transversal C (see details in §4).

An alternative way of understanding a box decomposition is given by a family of tilings, known as *derived tilings*, which are constructed by intersecting the box decomposition with the orbit of each Delone set in the hull. First, we recall basic definitions about tilings. A *tile* T in $\mathbb{R}^d$ is a compact set that is the closure of its interior (not necessarily connected). A *tiling* $\mathcal{T}$ of $\mathbb{R}^d$ is a countable collection of tiles that cover $\mathbb{R}^d$ and have pairwise disjoint interiors. Tiles can be *decorated*: they may have a color or be punctured at an interior point (or both). Formally, this means that decorated tiles are tuples (T, i, x), where T is a tile, i lies in a finite set of *colors*, and x belongs to the interior of T. Two tiles have the same type if they differ by a translation. If the tiles are punctured, then the translation must also send one puncture to the other, and when they are colored, they must have the same color.

To construct a derived tiling, the idea is to read the intersection of the boxes in the box decomposition with the orbit of a fixed Delone set in the hull. In the following discussion, it will be convenient to make the following construction. Let $\{C_i\}_{i=1}^t$ be a collection of local transversals and $\{D_i\}_{i=1}^t$ be a collection of bounded open subsets of $\mathbb{R}^d$ containing 0. Define $\mathcal{B} = \{C_i[D_i]\}_{i=1}^t$ and observe that the sets in $\mathcal{B}$ are not necessarily boxes of Ω. For each $Y \in \Omega$, define the (decorated) *derived collection* of $\mathcal{B}$ at Y by

$$\mathcal{T}_{\mathcal{B}}(Y) := \{(\overline{D_i} + v, i, v) \mid i \in \{1, \dots, t\}, v \in \mathcal{R}_{C_i}(Y)\}.$$

The following lemma gives the relation between box decomposition and tilings.

LEMMA 3.1. *Let $\mathcal{B} = \{C_i[D_i]\}_{i=1}^t$, where the C_i are local transversals and the D_i are open bounded subsets of $\mathbb{R}^d$ that contain* 0. *Then $\mathcal{B}$ is a box decomposition if and only if $\mathcal{T}_{\mathcal{B}}(Y)$ is a tiling of $\mathbb{R}^d$ for every $Y \in \Omega$. In this case, we call $\mathcal{T}_{\mathcal{B}}(Y)$ the* derived tiling *of $\mathcal{B}$ at Y.*

Proof. It is easy to see that if $\mathcal{B}$ is a box decomposition, then $\mathcal{T}_{\mathcal{B}}(Y)$ is a tiling for every $Y \in \Omega$. We now show the converse. For convenience, set $C = \bigcup_i C_i$. Fix $Y \in \Omega$ and suppose there are $i, j \in \{1, \dots, t\}$, $Y_1 \in C_i$, $Y_2 \in C_j$, $x_1 \in D_i$ and $x_2 \in D_j$ such that $Y = Y_1 - x_1 = Y_2 - x_2$. This implies that the tiles $\overline{D_i} - x_1$ and $\overline{D_j} - x_2$ of $\mathcal{T}_{\mathcal{B}}(Y)$ meet at an interior point. Since $\mathcal{T}_{\mathcal{B}}(Y)$ is a tiling, these tiles must coincide, and hence $i = j$ and $x_1 = x_2$. We conclude that the maps $h_i : C_i \times D_i \to C_i[D_i]$ given by $(Y, t) \mapsto Y - t$ are one-to-one, and moreover $\mathcal{B}$ is pairwise disjoint.

It rests to prove that the maps h_i are homeomorphisms, and that the closures of the sets in $\mathcal{B}$ cover Ω. Fix $i \in \{1, \dots, t\}$. The map h_i is the restriction of the translation action to $C_i \times D_i$, and therefore it is continuous. The continuity of the inverse of h_i follows from an standard argument involving the compactness of C_i and the boundedness of D_i. Finally, given any Delone set Y in Ω, there is a tile $(\overline{D_i} + x, i, x)$ in $\mathcal{T}_{\mathcal{B}}(Y)$ that contains the origin, which clearly means that Y belongs to the closure of $C_i[D_i]$. □

3.2. *Properly nested box decompositions.* A box decomposition $\mathcal{B}' = \{C'_i[D'_i]\}_{i=1}^{t'}$ is *zoomed out* of another box decomposition $\mathcal{B} = \{C_j[D_j]\}_{j=1}^{t}$ if the following properties are satisfied.

(Z.1) If $Y \in C'_i$ is such that $Y - x \in C_j - y$ for some $x \in \overline{D'_i}$ and $y \in \overline{D_j}$, then $C'_i - x \subseteq C_j - y$.

(Z.2) If $x \in \partial D'_i$, then there exist j and $y \in \partial D_j$ such that $C'_i - x \subseteq C_j - y$.

(Z.3) For every box B' in $\mathcal{B}'$, there is a box B in $\mathcal{B}$ such that $B \cap B' \neq \emptyset$ and $\partial B \cap \partial B' = \emptyset$.

For each $i \in \{1, \ldots, t'\}$ and $j \in \{1, \ldots, t\}$ define

$$O_{i,j} = \{x \in D'_i \mid C'_i - x \subseteq C_j\}. \tag{3.1}$$

(Z.4) For each $i \in \{1, \ldots, t'\}$ and $j \in \{1, \ldots, t\}$,

$$\overline{D'_i} = \bigcup_{j=1}^{t} \bigcup_{x \in O_{i,j}} \overline{D_j} + x,$$

where all the sets in the right-hand side of the equation have pairwise disjoint interiors.

Observe that in the case that D_j is connected, then properties (Z.1) and (Z.2) imply (Z.4).

Since we are considering the C'_i and C_j as the bases of the boxes, we ask the following additional property to be satisfied.

(Z.5) The base of $\mathcal{B}'$ is included in the base of $\mathcal{B}$, that is, $\bigcup_i C'_i \subseteq \bigcup_j C_j$.

By (Z.4), we have that the tiling $\mathcal{T}_{\mathcal{B}'}(Y)$ is a super-tiling of $\mathcal{T}_{\mathcal{B}}(Y)$ in the sense that each tile T in $\mathcal{T}_{\mathcal{B}'}(Y)$ can be decomposed into a finite set of tiles of $\mathcal{T}_{\mathcal{B}}(Y)$. By (Z.3), one of these tiles is included in the interior of T.

LEMMA 3.2. *For every $j \in \{1, \ldots, t\}$ we have*

$$C_j = \bigcup_{i=1}^{t'} \bigcup_{x \in O_{i,j}} C'_i - x.$$

Proof. By the definition of $O_{i,j}$ and (Z.1), it suffices to show that every $Y \in C_j$ belongs to the interior of some box $C'_i[D'_i]$. Suppose not, then $Y \in C'_i - x$ with $x \in \partial D'_i$ for some i since since $\mathcal{B}'$ is a box decomposition. Moreover, by (Z.2) we deduce that Y must be in the boundary of some box $B_{j'}$ in $\mathcal{B}$, which gives a contradiction. □

3.3. *Tower systems.* A *tower system* is a sequence of box decompositions $\mathfrak{T} = (\mathcal{B}_n)_{n\in\mathbb{N}}$ such that $\mathcal{B}_{n+1}$ is zoomed out of $\mathcal{B}_n$ for all $n \in \mathbb{N}$. The following theorem was proved in [**BBG06**].

THEOREM 3.3. *Every aperiodic minimal Delone system possesses a tower system.*

Consider a decreasing sequence $\mathfrak{C} = (C_n)_{n\in\mathbb{N}}$ of local transversals with diameter going to 0, and a tower system $\mathfrak{T}$. We will suppose that $\mathfrak{T}$ is *adapted* to $\mathfrak{C}$, i.e., that for all $n \in \mathbb{N}$ we have $\mathcal{B}_n = \{C_{n,i}[D_{n,i}]\}_{i=1}^{t_n}$ such that $C_n = \bigcup_i C_{n,i}$ and t_n is a positive integer. For each $n \in \mathbb{N}^*$ we define, as in (3.1),

$$O_{i,j}^{(n)} = \{x \in D_{n,i} \mid C_{n,i} - x \subseteq C_{n-1,j}\} \tag{3.2}$$

and

$$m_{i,j}^{(n)} = \sharp O_{i,j}^{(n)}$$

for every $i \in \{1, \ldots, t_n\}$ and $j \in \{1, \ldots, t_{n-1}\}$. The *transition matrix* (associated with $\mathfrak{T}$) of level n is defined as the matrix $M_n = (m_{i,j}^{(n)})_{i,j}$, i.e., M_n has size $t_n \times t_{n-1}$.

Suppose that μ is a translation-invariant probability measure and ν is the induced transverse measure (cf. §2.3). From (Z.4), Lemma 3.2 and the definition of transverse invariant measures, we get

$$\mathrm{vol}(D_{n,i}) = \sum_{j=1}^{t_{n-1}} m_{i,j}^{(n)} \,\mathrm{vol}(D_{n-1,j}) \tag{3.3}$$

and

$$\nu(C_{n-1,j}) = \sum_{i=1}^{t_n} \nu(C_{n,i}) m_{i,j}^{(n)}. \tag{3.4}$$

Fix $n \in \mathbb{N}$. From the relation $\mu(C_{n,i}[D_{n,i}]) = \mathrm{vol}(D_{n,i})\nu(C_{n,i})$ and the fact that $\mathcal{B}_n$ is a box decomposition, it follows that

$$\sum_{j=1}^{t_n} \mathrm{vol}(D_{n,j})\nu(C_{n,j}) = 1. \tag{3.5}$$

Given a box decomposition $\mathcal{B} = \{C_i[D_i]\}_{i=1}^{t}$, define its external and internal radii by

$$R_{\mathrm{ext}}(\mathcal{B}) = \max_{i\in\{1,\ldots,t\}} \inf\{R > 0 : B_R(0) \supseteq D_i\},$$
$$r_{\mathrm{int}}(\mathcal{B}) = \min_{i\in\{1,\ldots,t\}} \sup\{r > 0 : B_r(0) \subseteq D_i\},$$

respectively. Define also $\mathrm{rec}(\mathcal{B}) = \max_{i\in\{1,\ldots,t\}} \mathrm{rec}(C_i)$.

THEOREM 3.4. *Let X be an aperiodic linearly repetitive Delone set with constant $L > 1$ and $0 \in X$. Given $K \geq 6L(L+1)^2$ and $s_0 > 0$, set $s_n = K^n s_0$ for all $n \in \mathbb{N}$ and let $C_n := C_{X,s_n}$ for all $n \in \mathbb{N}$. Then, there exists a tower system $\mathfrak{T}$ of Ω adapted to $(C_n)_{n\in\mathbb{N}}$ that satisfies the following additional properties.*

(i) *For every $n \geq 0$, $C_{n+1} \subseteq C_{n,1}$.*

(ii) *There exist constants*

$$K_1 := \frac{1}{2(L+1)} - \frac{L}{K-1} \quad \text{and} \quad K_2 := \frac{LK}{K-1},$$

which satisfy $0 < K_1 < 1 < K_2$, such that for every $n \in \mathbb{N}$ we have

$$K_1 s_n \leq r_{\mathrm{int}}(\mathcal{B}_n) < R_{\mathrm{ext}}(\mathcal{B}_n) \leq K_2 s_n. \tag{3.6}$$

(iii) *For every $n \in \mathbb{N}$,*

$$\mathrm{rec}(\mathcal{B}_n) \leq (2L+1)s_n. \tag{3.7}$$

The proof is deferred until §4.

3.4. *Tower systems with uniformly bounded transition matrices.* The following lemma allows us to estimate the coefficients of the transition matrices.

LEMMA 3.5. *Let X be an aperiodic repetitive Delone set and $\mathfrak{C} = (C_n)_{n\in\mathbb{N}}$ be a decreasing sequence of clopen subsets of Ω^0 with $\mathrm{diam}(C_n) \to 0$ as $n \to +\infty$. Suppose that $\mathfrak{T}$ is a tower system adapted to $\mathfrak{C}$. Then the following assertions hold.*

(i) *If $M_X(\mathrm{rec}(\mathcal{B}_n)) \leq r_{\mathrm{int}}(\mathcal{B}_{n+1})$ for every $n \in \mathbb{N}$, then the coefficients of the transition matrices M_n are strictly positive.*

(ii) *If $A := \sup_{n\in\mathbb{N}}(R_{\mathrm{ext}}(\mathcal{B}_{n+1})/r_{\mathrm{int}}(\mathcal{B}_n))$ is finite, then $\|M_n\|_\infty \leq A^d$ for all $n \in \mathbb{N}^*$, and, in particular, if the conclusion in (i) also holds then the set $\{M_n\}_{n\in\mathbb{N}}$ is finite. Here $\|M_n\|_\infty$ denotes the maximum absolute row sum of M_n.*

Proof. Fix $n \in \mathbb{N}$, $i \in \{1, \ldots, t_{n+1}\}$ and $Y \in C_{n+1,i}$. First, we prove (i). By hypothesis, all the $\mathrm{rec}(\mathcal{B}_n)$-patches of Y occur in $D_{n+1,i}$. Since every $C_{n,j}$ is determined by a finite number of $\mathrm{rec}(C_{n,i})$-patches and these patches occur in $D_{n+1,i}$, it follows that there are vectors v in $D_{n+1,i}$ such that $Y - v$ belongs to $C_{n,j}$. Hence $m_{i,j}^{(n)} > 0$. We now prove (ii). Fix $n \in \mathbb{N}^*$. Since $D_{n,i}$ is included in a ball of radius $R_{\mathrm{ext}}(\mathcal{B}_n)$ and each $D_{n-1,j}$ contains a ball of radius $r_{\mathrm{int}}(\mathcal{B}_{n-1})$, we deduce from (3.3) that

$$\sum_{j=1}^{t_{n-1}} m_{i,j}^{(n)} \leq \left(\frac{R_{\mathrm{ext}}(\mathcal{B}_n)}{r_{\mathrm{int}}(\mathcal{B}_{n-1})}\right)^d. \tag{3.8}$$

Taking the maximum on i in (3.8) yields $\|M_n\|_\infty \leq A^d$. Assume that the conclusion in (i) also holds. The finiteness of $\{M_n\}_{n\in\mathbb{N}^*}$ now follows from this assumption and the last inequality. □

Finally, we state and prove our main result.

THEOREM 3.6. *Let X be an aperiodic linearly repetitive Delone set. Then, the tower system of Ω obtained in Theorem 3.4 satisfies the following.*

(i) *For every $n \in \mathbb{N}^*$, the matrix M_n has strictly positive coefficients.*

(ii) *The matrices $\{M_n\}_{n\in\mathbb{N}^*}$ are uniformly bounded in size and norm.*

Proof. Take the notations of Theorem 3.4 for $\mathfrak{C}$ and $\mathfrak{T}$. It suffices to prove that $\mathfrak{T}$ satisfies the conditions (i) and (ii) of Lemma 3.5. Indeed, by the definition of linearly repetitivity we have $M_X(\mathrm{rec}(\mathcal{B}_n)) \leq L\,\mathrm{rec}(\mathcal{B}_n)$ for all $n \in \mathbb{N}$. Combining this with (3.7), the left-hand inequality of (3.6) and the definition of s_n we get

$$M_X(\mathrm{rec}(\mathcal{B}_n)) \leq \frac{L(2L+1)}{KK_1} r_{\mathrm{int}}(\mathcal{B}_{n+1}).$$

Since $K \geq 6L(L+1)^2$, it follows that $L(2L+1) \leq K_1 K$ and the condition (i) in Lemma 3.5 is satisfied. To check that the condition in (ii) is also satisfied, use (3.6) twice (with n and $n+1$) and then replace $s_{n+1} = Ks_n$ in the result to obtain

$$\frac{R_{\mathrm{ext}}(\mathcal{B}_{n+1})}{r_{\mathrm{int}}(\mathcal{B}_n)} \leq K\frac{K_2}{K_1},$$

from which it follows that $R_{\mathrm{ext}}(\mathcal{B}_{n+1})/r_{\mathrm{int}}(\mathcal{B}_n)$ is uniformly bounded in $n \in \mathbb{N}$. □

COROLLARY 3.7. *Let X be an aperiodic linearly repetitive Delone set and Ω its hull. Then the system (Ω, Γ) is uniquely ergodic.*

The proof is standard and can be found e.g. in **[BBG06, BG03]**. It is also important (for the remainder of this paper) to remark that this produces an independent proof to the original one by Lagarias and Pleasants in [**LP03**].

4. *The Bellissard–Benedetti–Gambaudo construction in the linearly repetitive case*

In this section we give the proof of Theorem 3.4 by adapting the construction of [**BBG06**]. First, we need to recall some basic facts about Voronoi tilings. Given a Delone set Y and a point $y \in Y$, the *Voronoi cell* $\mathcal{V}_y(Y)$ of y in Y is defined by

$$\mathcal{V}_y(Y) = \{z \in \mathbb{R}^d \mid \forall y' \in Y, \|z - y\| \leq \|z - y'\|\}.$$

It is standard that Voronoi cells are closed convex polyhedra in $\mathbb{R}^d$, and they form the so-called *Voronoi tiling*, which we denote by $\mathcal{T}_Y$. The next lemma gathers well-known facts about Voronoi cells that will be needed in the proof (see [**Sen95**, Proposition 5.2, Corollary 5.2]).

LEMMA 4.1. *Let Y be a Delone set that is R-dense for $R > 0$ and r-discrete for $r > 0$. Then, for all $y \in Y$, the Voronoi cell $\mathcal{V}_y(Y)$ is included in $\overline{B}_R(y)$ and contains the ball $\overline{B}_r(y)$. Moreover, the cell $\mathcal{V}_y(Y)$ is determined by the $2R$-patch of Y centered at y, i.e., if $Y \wedge \overline{B}_{2R}(y)$ is equivalent to $Y \wedge \overline{B}_{2R}(y')$, then*

$$\mathcal{V}_y(Y) - y = \mathcal{V}_{y'}(Y) - y'.$$

The fact that Voronoi cells are locally determined allows us to use them to construct box decompositions with any given clopen subset C as base, as we show in the following result.

PROPOSITION 4.2. *Let C be a clopen subset of Ω^0 and take $k \geq 2R(C) + \mathrm{rec}(C)$. Then, there is a box decomposition $\mathcal{B}(C, k)$ of Ω with C as its base and $\mathrm{rec}(\mathcal{B}(C, k)) \leq k$, $r_{\mathrm{int}}(\mathcal{B}(C, k)) = r(C)$ and $R_{\mathrm{ext}}(\mathcal{B}(C, k)) = R(C)$.*

Proof. Consider

$$\mathcal{A}_{k,C} = \{Y \wedge \overline{B}_k(0) \mid Y \in C\}.$$

By minimality, each patch in $\mathcal{A}_{k,C}$ occurs in X and since X has only finitely many k-patches up to translation, it follows that $\mathcal{A}_{k,C}$ is finite, say $\mathcal{A}_{k,C} = \{\mathbf{p}_1, \ldots, \mathbf{p}_t\}$, where $t \in \mathbb{N}$ and the $\mathbf{p}_i$ are all different. For each $i \in \{1, \ldots, t\}$, we define

$$C_i = \{Y \in \Omega \mid Y \wedge \overline{B}_k(0) = \mathbf{p}_i\}$$

and

$$B_i = \{Y - y \mid Y \in C_i, \, y \in \mathrm{int}(\mathcal{V}_0(\mathcal{R}_C(Y)))\}.$$

Let us show that the B_i are boxes. On the one hand, the inequality $k \geq \mathrm{rec}(C)$ implies that all the C_i are included in C. Hence, they partition C and it follows that each C_i is a clopen subset of C. On the other hand, $\mathcal{R}_C(Y)$ is $R(C)$-dense for every $Y \in C$. Hence, Lemma 4.1 implies that the Voronoi cell $\mathcal{V}_0(\mathcal{R}_C(Y))$ is determined by the patch $\mathbf{p}(Y) := \mathcal{R}_C(Y) \wedge \overline{B}_{2R(C)}(0)$. Since $\mathbf{p}(Y)$ is determined by the patch $Y \wedge \overline{B}_{2R(C)+\mathrm{rec}(C)}(0)$ and $k \geq 2R(C) + \mathrm{rec}(C)$, it follows that $\mathcal{V}_0(\mathcal{R}_C(Y))$ is the same for all $Y \in C_i$ and therefore $B_i = C_i[D_i]$, where $D_i := \mathrm{int}(\mathcal{V}_0(\mathcal{R}_C(Y)))$. To conclude we define

$$\mathcal{B}(C, k) = \{B_1, \ldots, B_t\}$$

and observe that the undecorated version of $\mathcal{T}_{\mathcal{B}(C,k)}(Y)$ is the Voronoi tiling of $\mathcal{R}_C(Y)$, which implies by Lemma 3.1 that $\mathcal{B}(C, k)$ is a box decomposition. The clopen C can be chosen as base of $\mathcal{B}(C, k)$ by construction and the last three relations in the statement of the proposition are direct consequences of Lemma 4.1. □

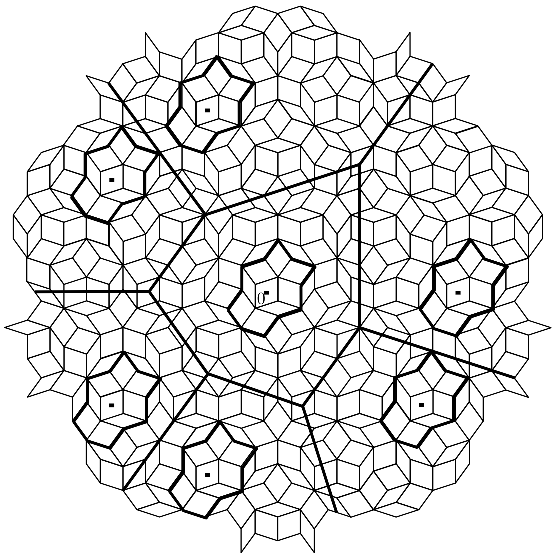

FIGURE 1. Voronoi tiling obtained from the Delone set of centers of translated copies of a patch.

The next step in the proof consists in, given a box decomposition $\mathcal{B}$ based at C, constructing a box decomposition zoomed out of $\mathcal{B}$. Before stating this precisely, we sketch the idea of the proof. Consider $\mathcal{T}$ the derived tiling of $\mathcal{B}$ at some Delone set Y of Ω. Let C' be a clopen subset of C and consider the Voronoi tiling $\mathcal{V}$ of $\mathcal{R}_{C'}(Y)$. Without considering colors, the tiling $\mathcal{V}$ is the derived tiling of the box decomposition $\mathcal{B}(C', k)$ given by Proposition 4.2, where k is chosen large enough. In general, $\mathcal{B}(C', k)$ is not zoomed out of $\mathcal{B}$ because (see e.g. Figure 1), the tiles of $\mathcal{V}$ cannot be decomposed into tiles of $\mathcal{T}$.

Next, we modify the tiles of $\mathcal{V}$ around their boundaries (without modifying their punctures so we do not modify the bases of the boxes) in such a way that, after the modification, the tiles can be decomposed into tiles of $\mathcal{T}$ (see Figure 2). The idea is to replace each tile V in $\mathcal{V}$ with the union of tiles having their $\mathcal{T}$ with punctures in V. In the case that there is a tile of $\mathcal{T}$ with its puncture in the boundary of a tile of $\mathcal{V}$, then the new collection will not form a tiling (since this tile would belong to two different tiles of $\mathcal{V}$, which will, consequently, overlap). To solve this problem, a choice has to be made so each point of $\mathbb{R}^d$ belongs to a unique tile of $\mathcal{V}$. One way of achieving this is the following (see [**GMPS10**]). Given any subset D of $\mathbb{R}^d$, we define

$$D^* = \{p \in \mathbb{R}^d \mid p + (\varepsilon, \varepsilon^2, \dots, \varepsilon^d) \in D \text{ for all sufficiently small } \varepsilon > 0\}. \tag{4.1}$$

We have that $(D + v)^* = D^* + v$ for every $v \in \mathbb{R}^d$, and if $\mathcal{S}$ is any tiling of $\mathbb{R}^d$, then $\{S^* \mid S \in \mathcal{S}\}$ is a partition. Thus, we replace each tile V of $\mathcal{V}$ with the tiles of $\mathcal{T}$ whose puncture belong to V^*.

LEMMA 4.3. *Let $\mathcal{B}$ be a box decomposition of Ω based at a clopen subset C of Ω^0 and C' be a clopen subset of C satisfying $r(C') \geq 2R_{\text{ext}}(\mathcal{B})$. Then for each*

$$k' \geq \max\{2R(C') + \operatorname{rec}(C'),\ R(C') + 2R_{\text{ext}}(\mathcal{B}) + \operatorname{rec}(\mathcal{B})\}$$

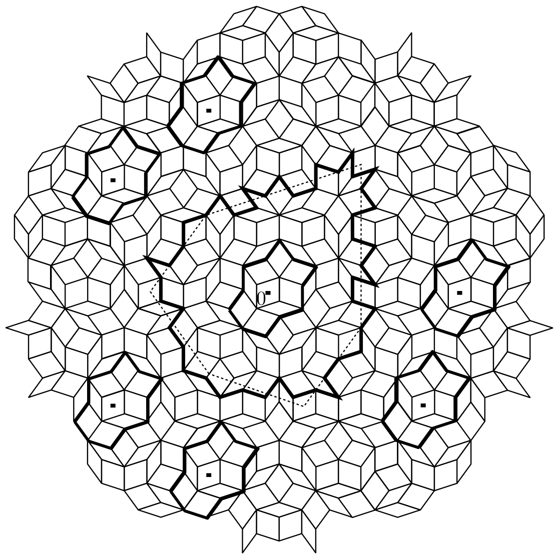

FIGURE 2. Patch of the Penrose tiling constructed from the tiles intersecting the Voronoi tile.

there is a box decomposition $\mathcal{B}'$ zoomed out of $\mathcal{B}$ and based at C' satisfying

$$\operatorname{rec}(\mathcal{B}') \leq k', \tag{4.2}$$

$$r_{\text{int}}(\mathcal{B}') \geq r(C') - R_{\text{ext}}(\mathcal{B}), \tag{4.3}$$

$$R_{\text{ext}}(\mathcal{B}') \leq R(C') + R_{\text{ext}}(\mathcal{B}). \tag{4.4}$$

Proof. Let $\mathcal{B} = \{C_j[D_j]\}_{j=1}^{t}$ and $\mathcal{B}(C', k') = \{C_i'[E_i]\}_{i=1}^{t'}$. The proof consists in modifying the E_i in $\mathcal{B}(C', k')$ to obtain a box decomposition $\mathcal{B}'$ that is zoomed out of $\mathcal{B}$. We proceed by steps and work with derived tilings.

Step 1. For each $Y \in \Omega$, we 'deform' each tile of $\mathcal{T}_{\mathcal{B}(C',k')}(Y)$ into a tile that is the support of a patch of $\mathcal{T}_{\mathcal{B}}(Y)$. More precisely, for each tile $(\overline{E_i} + v, i, v)$ with $i \in \{1, \ldots, t'\}$ and $v \in \mathcal{R}_{C_i'}(Y)$ we define $D'_{v,Y}$ as

$$D'_{v,Y} = \bigcup_{j=1}^{t} \bigcup_{w \in \mathcal{R}_{C_j}(Y) \cap (E_i^* + v)} \overline{D_j} + w,$$

where $E_i^* + v$ is defined by (4.1). It is easy to see that $D'_{v,Y}$ is the support of the tiles of $\mathcal{T}_{\mathcal{B}}(Y)$ whose punctures are in $E_i^* + v$. Since $\{E_i^* + v \mid i \in \{1, \ldots, t\}, v \in \mathcal{R}_{C_i'}(Y)\}$ is a partition of $\mathbb{R}^d$, it follows that $\mathcal{T}'(Y) = \{(D'_{v,Y}, i, v) \mid i \in \{1, \ldots, t\}, v \in \mathcal{R}_{C_i'}(Y)\}$ is a decorated tiling (the tiles are not necessarily connected), which we view as a deformation of $\mathcal{T}_{\mathcal{B}(C',k')}(Y)$.

Step 2. Fix $i \in \{1, \ldots, t'\}$ and denote $S = R(C') + \operatorname{rec}(\mathcal{B})$. We show that $D'_{w,Z} = D'_{v,Y} - v + w$ for every $Y, Z \in \Omega$, $v \in \mathcal{R}_{C_i'}(Y)$ and $w \in \mathcal{R}_{C_i'}(Z)$. Indeed, since E_i^* included in $\overline{E_i}$, from Lemma 4.1 we get that $E_i^* \subseteq \overline{B}_{R(C')}(0)$. It follows that the set

$$\mathcal{R}_{C_j}(Y) \cap (E_i + v)^*$$

is determined by $Y \wedge \overline{B}_S(v)$. From the proof of Proposition 4.2, there exists $Y_i \in \Omega$ such that $C_i' = C_{Y_i,k'}$. Hence, $Y \wedge \overline{B}_{k'}(v)$ is equivalent to $Z \wedge \overline{B}_{k'}(w)$. Since $k' \geq S$, it follows that $D'_{w,Z} = D'_{v,Y} - v + w$.

For each $i \in \{1, \ldots, t'\}$, we set $D_i' = D'_{v,Y} - v$ and $\mathcal{B}' = \{C_i'[D_i']\}_{i=1}^{t'}$, where $Y \in \Omega$ and $v \in \mathcal{R}_{C_i'}(Y)$ are arbitrary. It is clear that $\mathcal{T}_{\mathcal{B}'}(Y) = \mathcal{T}'(Y)$, which by Lemma 3.1 implies that $\mathcal{B}'$ is a box decomposition.

Step 3. We check (4.2)–(4.4). Indeed, (4.2) follows directly from Proposition 4.2. Also from Proposition 4.2, we have that $B_{r(C')}(0) \subseteq E_i \subseteq B_{R(C')}(0)$ for every $i \in \{1, \ldots, t'\}$. Since $D_j \subseteq B_{R_{\text{ext}}(\mathcal{B})}(0)$ for all $j \in \{1, \ldots, t\}$, from Step 2 we deduce that each D_i' contains the ball of radius $r(C') - R_{\text{ext}}(\mathcal{B})$ around 0 and is contained in the ball of radius $R(C') + R_{\text{ext}}(\mathcal{B})$ around 0, which yields (4.3) and (4.4).

Step 4. We check (Z.1). Indeed, suppose that $Y \in C_i' - y$ belongs to $C_j - x$ for some $j \in \{1, \ldots, t\}$, $y \in \overline{D_i'}$ and $x \in \overline{D_j}$. Let $Z \in C_i' - y$. We need to show that $Z \in C_j - x$, which is equivalent to $Z \wedge \overline{B}_{\text{rec}(\mathcal{B})}(-x) = Y \wedge \overline{B}_{\text{rec}(\mathcal{B})}(-x)$. From (4.4) we get $\|y\| \leq R(C') + R_{\text{ext}}(\mathcal{B})$. Hence we have $k' - \|x - y\| \geq \text{rec}(\mathcal{B})$. Since by hypothesis $Y \wedge \overline{B}_{k'}(-y) = Z \wedge \overline{B}_{k'}(-y)$, it follows that $\overline{B}_{\text{rec}(\mathcal{B})}(-x) \subseteq \overline{B}_{k'}(-y)$ and the proof of (Z.1) is done.

We check (Z.3). Let $Y \in C_i'$. From (4.3) and $r(C') \geq 2R_{\text{ext}}(\mathcal{B})$, we get $r_{\text{int}}(\mathcal{B}') > R_{\text{ext}}(\mathcal{B})$. Since $C' \subseteq C$, there is $j \in \{1, \ldots, t\}$ such that $Y \in C_j$, which means that the tile of $\mathcal{T}'(Y)$ containing the origin contain the tile of $\mathcal{T}_{\mathcal{B}}(Y)$ containing the origin in its interior. Since Y was arbitrary, this implies (Z.3).

Finally, (Z.2), (Z.4) and (Z.5) are direct consequences of the construction. □

Now we are ready to give the proof of Theorem 3.4.

Proof of Theorem 3.4. For each $n \in \mathbb{N}$ we define $k_n = 2R(C_n) + s_n$. We construct a tower system by induction on n and use Lemma 4.3 as a key part in the induction step. The estimates given by Lemma 4.3 are then used to prove properties (i)–(iii).

For the basis step, we set $\mathcal{B}_0 = \mathcal{B}(C_0, k_0)$. Since $s_0 \geq \text{rec}(C_0)$ by definition of C_0, $k_0 \geq 2R(C_0) + \text{rec}(C_0)$. Thus Proposition 4.2 ensures that $\mathcal{B}(C_0, k_0)$ is a box decomposition with $\text{rec}(B_0) \leq k_0$ and $R_{\text{ext}}(\mathcal{B}_0) = R(C_0)$. By permuting the indices if necessary, we obtain that $X \in C_{0,1}$.

For the inductive step, we fix $n \in \mathbb{N}^*$ and suppose that $\mathcal{B}_{n-1}$ is a box decomposition that satisfies

$$\text{rec}(\mathcal{B}_{n-1}) \leq k_{n-1}, \tag{4.5}$$

$$R_{\text{ext}}(\mathcal{B}_{n-1}) \leq \frac{L}{K-1} s_n. \tag{4.6}$$

We need to show that $\mathcal{B}_{n-1}$ and k_n satisfy the hypotheses of Lemma 4.3:

$$k_n \geq 2R(C_n) + \text{rec}(C_n), \tag{4.7}$$

$$k_n \geq R(C_n) + 2R_{\text{ext}}(\mathcal{B}_{n-1}) + \text{rec}(\mathcal{B}_{n-1}), \tag{4.8}$$

$$r(C_n) \geq 2R_{\text{ext}}(\mathcal{B}_{n-1}). \tag{4.9}$$

The inequality (4.7) is clear since $s_n \geq \text{rec}(C_n)$ by the definition of C_n. To check (4.8), recall that $R(C_{n-1}) \leq L s_{n-1}$ by linearly repetitivity. Replacing this inequality and

$s_n = Ks_{n-1}$ in the definition of k_n yields

$$k_{n-1} \leq \frac{2L+1}{K} s_n. \tag{4.10}$$

From (4.10), (4.5) and (4.6) it is easy to deduce

$$2R_{\text{ext}}(\mathcal{B}_{n-1}) + \text{rec}(\mathcal{B}_{n-1}) \leq \left(\frac{2L}{K-1} + \frac{2L+1}{K}\right) s_n. \tag{4.11}$$

An easy computation shows that the right-hand side of (4.11) is smaller or equal than s_n and thus (4.8) follows from the definition of k_n. Finally, (4.9) follows from an easy computation involving the left-hand side of (2.3) and (4.6).

Applying Lemma 4.3 we get a box decomposition $\mathcal{B}_n = \{C_{n,i}[D_{n,i}]\}_{i=1}^{t_n}$ zoomed out of $\mathcal{B}_{n-1}$ that satisfies

$$\text{rec}(\mathcal{B}_n) \leq k_n, \tag{4.12}$$

$$r(C_n) - R_{\text{ext}}(\mathcal{B}_{n-1}) \leq r_{\text{int}}(\mathcal{B}_n), \tag{4.13}$$

$$R_{\text{ext}}(\mathcal{B}_n) \leq R(C_n) + R_{\text{ext}}(\mathcal{B}_{n-1}). \tag{4.14}$$

To finish the inductive step, it remains to show that

$$R_{\text{ext}}(\mathcal{B}_n) \leq \frac{L}{K-1} s_{n+1}. \tag{4.15}$$

This is proved easily by applying (2.3) and (4.6) to (4.14). Hence, applying the induction above we obtain a sequence of box decompositions $(\mathcal{B}_n)_{n\in\mathbb{N}}$ such that $\mathcal{B}_n$ is zoomed out of $\mathcal{B}_{n-1}$ and satisfies (4.12), (4.13) and (4.15).

Finally we check properties (i)–(iii). After permuting indices we have that $X \in C_{n,1}$ for every $n \in \mathbb{N}$. Since $s_{n+1} > k_n \geq \text{rec}(\mathcal{B}_n)$, it follows that $C_{n+1} = C_{X,s_{n+1}} \subseteq C_{n,1}$ and (i) holds.

Applying (2.3) and (4.6) to (4.13) we obtain

$$r_{\text{int}}(\mathcal{B}_n) \geq s_n \left(\frac{1}{2(L+1)} - \frac{L}{K-1}\right) \quad \text{for all } n \in \mathbb{N}. \tag{4.16}$$

Hence, property (ii) follows from (4.16) and (4.15).

Finally, property (iii) follows directly from (4.10) and (4.12). □

5. *Markov chain induced by a tower system*

Suppose that Ω is the hull of an aperiodic linearly repetitive Delone set with constant $L > 1$. Take K, $\mathfrak{C}$ and $\mathfrak{T} = \{\mathcal{B}_n\}_{n\in\mathbb{N}}$ as in Theorem 3.4. Let μ be the unique translation-invariant probability measure on Ω and denote by ν the induced transverse measure (cf. §2.3).

The tower system $\mathfrak{T}$ induces a random process $\beta = (\beta_n)_{n\in\mathbb{N}}$ on (Ω, μ) as follows. For every $Y \in \Omega$ and $n \in \mathbb{N}$, define $\beta_n(Y)$ by

$$\beta_n(Y) = i \text{ if and only if } Y \text{ belongs to } B_{n,i},$$

where $\mathcal{B}_n = \{B_{n,i}\}_{i=1}^{t_n}$. Observe that β_n is not defined at the boundaries of the boxes in $\mathcal{B}_n$. Since the boundaries have measure zero, it follows that β is well defined in a full-measure set. We also set

$$c_{\mathfrak{T}} = 1 - \left(\sup_{n \in \mathbb{N}} \|M_n\|_1^{-1} \|M_{n+1}\|_1^{-1} \right),$$

where M_n are the transition matrices of $\mathfrak{T}$ and $\|M_n\|_1$ is the maximum absolute column sum of M_n, which are bounded in norm by Theorem 3.6. The following proposition summarizes the properties of β.

PROPOSITION 5.1. *The process β is a non-stationary Markov chain with its transition probabilities given by*

$$\mu(\beta_n = i \mid \beta_{n-1} = j) = \frac{\nu(C_{n,i})}{\nu(C_{n-1,j})} m_{i,j}^{(n)}. \tag{5.1}$$

Moreover, for every $n, m \in \mathbb{N}$ with $n > m$ we have

$$\max_{\substack{1 \le j \le t_m \\ 1 \le i \le t_n}} |\mu(\beta_n = i \mid \beta_m = j) - \mu(\beta_n = i)| \le c_{\mathfrak{T}}^{n-m}. \tag{5.2}$$

The proof will be divided into several lemmas. First, we introduce some notation. For each $n \in \mathbb{N}^*$, we define the matrix $Q_n = (q_{j,i}^{(n)})$ with $j \in \{1, \ldots, t_{n-1}\}$ and $i \in \{1, \ldots, t_n\}$ by

$$q_{j,i}^{(n)} = \frac{\nu(C_{n,i})}{\nu(C_{n-1,j})} m_{i,j}^{(n)}. \tag{5.3}$$

For $n > m$, we define the products $Q(n, m) := Q_{m+1} \cdot \cdots \cdot Q_n$ and $P(n, m) := M_n \cdot \cdots \cdot M_{m+1}$, and denote their coefficients by $q_{j,i}^{(n,m)}$ and $p_{i,j}^{(n,m)}$, respectively. It is not difficult to check, using induction and (5.3), that

$$q_{j,i}^{(n,m)} = \frac{\nu(C_{n,i})}{\nu(C_{m,j})} p_{i,j}^{(n,m)} \tag{5.4}$$

for all $n, m \in \mathbb{N}$ with $n > m$, $i \in \{1, \ldots, t_n\}$ and $j \in \{1, \ldots, t_m\}$.

LEMMA 5.2. *Fix $n, m \in \mathbb{N}$ with $m < n$. Then for every sequence $(i_k)_{k=m}^n$ with $i_k \in \{1, \ldots, t_k\}$ we have*

$$\mu(B_{n,i_n} \cap B_{n-1,i_{n-1}} \cdots \cap B_{m,i_m}) = \left(\prod_{k=m}^{n-1} m_{i_{k+1},i_k}^{(k+1)} \right) \operatorname{vol}(D_{m,i_m}) \nu(C_{n,i_n}). \tag{5.5}$$

Proof. From (Z.4) and (3.2) it is easy to deduce that

$$B_{m+1,i_{m+1}} \cap B_{m,i_m} = \bigcup_{x \in O_{i_{m+1},i_m}^{(m+1)}} C_{m+1,i_{m+1}}[D_{m,i_m} + x].$$

Using induction, it is possible to obtain

$$\begin{aligned} &B_{n,i_n} \cap \cdots \cap B_{m+1,i_{m+1}} \cap B_{m,i_m} \\ &= \bigcup_{x_n \in O_{i_n,i_{n-1}}^{(n)}} \cdots \bigcup_{x_{m+1} \in O_{i_{m+1},i_m}^{(m+1)}} C_{n,i_n}\left[D_{m,i_m} + \sum_{k=m+1}^{n} x_k \right]. \end{aligned}$$

Since the boxes in the right-hand side of the last equation have disjoint interiors, using the invariance of μ we obtain (5.5). □

For each $n \in \mathbb{N}^*$, define $c(Q_n) := 1 - \min_{i,j} q_{j,i}^{(n)}$.

LEMMA 5.3. *If* $c := \sup_n(c(Q_n)) < 1$*, then for every* $n, m \in \mathbb{N}$ *with* $n > m$ *we have*

$$\max_{i,j,s} |q_{i,s}^{(n,m)} - q_{j,s}^{(n,m)}| \leq c^{n-m}.$$

Proof. The proof follows by applying [**Sen81**, Equations (4.6) and (4.7) pp. 137–138], which remain true in our setting. □

LEMMA 5.4. *For every* $n \in \mathbb{N}$ *we have*

$$\sup_n c(Q_n) \leq c_{\mathfrak{T}}.$$

Proof. Fix $n \in \mathbb{N}^*$. First, we estimate $\nu(C_{n,k})/\nu(C_{n,j})$ for all j and k in $\{1, \dots t_n\}$. From (3.4) we get

$$\frac{\nu(C_{n,k})}{\nu(C_{n,j})} = \sum_{i=1}^{t_{n+1}} \frac{\nu(C_{n+1,i})}{\nu(C_{n,j})} m_{ik}^{(n+1)}. \tag{5.6}$$

Since $m_{i,j}^{(n+1)} \geq 1$ for all $i \in \{1, \dots, t_{n+1}\}$, it follows from (3.4) that $\nu(C_{n,j}) > \nu(C_{n+1,i})$ and hence from (5.6) we get

$$\frac{\nu(C_{n,k})}{\nu(C_{n,j})} \leq \sum_{i=1}^{t_{n+1}} m_{i,k}^{(n+1)} \leq \|M_{n+1}\|_1. \tag{5.7}$$

Next, we estimate $\nu(C_{n-1,l})/\nu(C_{n,j})$ for all $j \in \{1, \dots t_n\}$ and $l \in \{1, \dots t_{n-1}\}$. Plugging in (3.4) and (5.7) we obtain

$$\frac{\nu(C_{n-1,l})}{\nu(C_{n,j})} = \sum_{k=1}^{t_n} \frac{\nu(C_{n,k})}{\nu(C_{n,j})} m_{kl}^{(n)} \leq \|M_n\|_1 \|M_{n+1}\|_1. \tag{5.8}$$

Finally, we estimate $q_{l,j}^{(n)}$. Plugging (5.8) in (5.3) yields

$$q_{l,j}^{(n)} \geq \|M_n\|_1^{-1} \|M_{n+1}\|_1^{-1} m_{j,l}^{(n)} \geq \|M_n\|_1^{-1} \|M_{n+1}\|_1^{-1}, \tag{5.9}$$

where we used that $m_{i,j}^{(n)} \geq 1$, and the conclusion now follows. □

Proof of Proposition 5.1. The fact that β is a Markov chain with transition probabilities given by (5.1) can be proved by a simple computation using Lemma 5.2. To check (5.2), using $\sum_i \mu(B_{n,i}) = 1$ we may write

$$|\mu(\beta_n = i \mid \beta_m = j) - \mu(\beta_n = i)| \leq \max_{l,j} |\mu(\beta_n = i \mid \beta_m = j) - \mu(\beta_n = i \mid \beta_m = l)|$$

for all $n, m \in \mathbb{N}$ with $n > m$. Since the coefficients of M_n are all positive, $\|M_n\|_1 > 1$ and hence $c_{\mathfrak{T}} < 1$ and the conclusion now follows from Lemmas 5.4 and 5.3. □

6. *An alternative proof of a theorem of Lagarias and Pleasants*

Let X be a linearly repetitive Delone set with constant $L > 1$ and $\mathfrak{T}$ be the tower system given by Theorem 3.6. By Corollary 3.7, the hull Ω is uniquely ergodic, which means that X has uniform patch frequencies, i.e., each S-patch has a well-defined frequency freq($\mathbf{p}$). We denote by μ the unique translation-invariant probability measure and by ν its induced transverse measure.

We say that $U \subseteq \mathbb{R}^d$ is a d-cube of side N if $U = [0, N]^d + x$ for some $x \in \mathbb{R}^d$. Given a d-cube $U \in \mathbb{R}^d$ and an S-patch $\mathbf{p}$ of X with $S > 0$, we estimate the *deviation* of $\mathbf{p}$ in U, which is defined as

$$\operatorname{dev}_{\mathbf{p}}(U) = n_{\mathbf{p}}(U) - \operatorname{vol}(U)\operatorname{freq}(\mathbf{p}),$$

and obtain the following theorem.

THEOREM 6.1. *Let X be a linearly repetitive Delone set, Ω be its hull and $\mathfrak{T}$ be the tower system of Theorem 3.6. Define*

$$\delta_{\mathfrak{T}} = -\log_K c_{\mathfrak{T}}, \tag{6.1}$$

where $\log_K$ is the logarithm to base K. Then, for every $S > 0$ and every S-patch $\mathbf{p}$ in X we have

$$|\operatorname{dev}_{\mathbf{p}}(U_N)| = O(N^{d-\delta_{\mathfrak{T}}})$$

for all $N \in \mathbb{N}$, where U_N is a d-cube of side N and the O-constant depends only on Ω and $\mathbf{p}$.

This produces an alternative proof of Theorem 1.2 (cf. [**LP03**, Theorem 6.1]). The main difference between our result and Theorem 1.2 comes from the fact that Theorem 6.1 relates the growth rate of the deviation to the mixing rate of the Markov chain associated with $\mathfrak{T}$ through (6.1). The proof will be given at the end of the section. In the remainder of this section, we denote by $(\mathcal{T}_n = \mathcal{T}_n(X))_{n\in\mathbb{N}}$ the sequence of derived tilings of X associated with $\mathfrak{T}$.

Now we introduce a decomposition argument to estimate the deviation of an S-patch on a d-cube U of side N, but first we need some notation. For each $S > 0$, define $n_0 = n_0(S)$ to be the smallest integer n such that

$$Y, Z \in C_{n,i} - x \quad \text{implies} \quad Y \wedge \overline{B}_S(0) = Z \wedge \overline{B}_S(0)$$

for all $i \in \{1, \dots, t_n\}$ and all $x \in D_{n,i}$. We check that n_0 is finite. Indeed, from the construction of the towers (proof of Theorem 3.4), it follows that if Y and Z are in $C_{n,i} - x$, then they coincide in a ball or radius

$$k_n - R_{\text{ext}}(\mathcal{B}_n) \tag{6.2}$$

around 0, where $k_n = 2R(C_n) + s_n$. It suffices to show that (6.2) goes to infinity as n goes to infinity. Indeed, from (4.14) and (3.6), we get $R_{\text{ext}}(\mathcal{B}_n) \le R(C_n) + s_n L/(K-1)$. From the definition of k_n and since $K - 1 > L$ we conclude that (6.2) goes to infinity, and therefore n_0 is finite.

From the definition of n_0, we check that if $n \ge n_0$, then the number of S-patches of a Delone set Y that are equivalent to a given S-patch $\mathbf{p}$ and have their centers inside $D_{n,i}$ is the same for all $Y \in C_{n,i}$. Therefore, we write this number as $n_{\mathbf{p}}(D_{n,i})$ and let

$$\operatorname{dev}_{\mathbf{p}}(D_{n,i}) := n_{\mathbf{p}}(D_{n,i}) - \operatorname{freq}(\mathbf{p})\operatorname{vol}(D_{n,i}).$$

Next, we define $n_1 = n_1(U)$ to be the biggest integer n such that there is a tile of $\mathcal{T}_n$ included in U. First, we prove the following lemma.

LEMMA 6.2. *For each S-patch $\mathbf{p}$ of X, and for all d-cubes U of side N in $\mathbb{R}^d$,*

$$|\mathrm{dev}_{\mathbf{p}}(U)| = O\bigg(N^{d-1}\bigg(1 + \sum_{n=n_0}^{n_1} s_{n+1}^{1-d} \max_{1\le i\le t_n} |\mathrm{dev}_{\mathbf{p}}(D_{n,i})|\bigg)\bigg).$$

The proof of Lemma 6.2 follows the proof of the following lemma.

LEMMA 6.3. *There exists $M > 0$ such that for every d-cube U of side N and every $n \in \mathbb{N}$ smaller than $n_1 + 1$, the number of tiles of $\mathcal{T}_n$ whose supports intersect U but are not included in U is bounded above by $MN^{d-1}s_n^{1-d}$.*

Proof. Fix a cube U of side $N \in \mathbb{N}$ and $n \in \mathbb{N}$, $n \le n_1 + 1$. Denote by $\mathcal{A}$ the set of tiles of $\mathcal{T}_n$ whose supports intersect U but they are not included in U. From (3.6) it follows that each tile in $\mathcal{A}$ contains a ball of radius $K_1 s_n$ and is included in a ball of radius $K_2 s_n$. Since tiles in $\mathcal{A}$ do not overlap, this implies that

$$K_1^d s_n^d \,\mathrm{vol}(B_1(0))\#\mathcal{A} \le \mathrm{vol}((\partial U)^{+2K_2 s_n}), \tag{6.3}$$

where $(\partial U)^{+2K_2 s_n} = \{x \in \mathbb{R}^d \mid \mathrm{dist}(x, \partial U) \le 2K_2 s_n\}$. It is easy to check that

$$\mathrm{vol}((\partial U)^{+2K_2 s_n}) \le 2dK_2 s_n(2K_2 s_n + N)^{d-1}. \tag{6.4}$$

By definition of n_1, there is a tile T in $\mathcal{T}_{n_1}$ that is included in U. The tile T contains, by (3.6), a ball of radius $K_1 s_{n_1}$ and hence

$$K_1 s_{n_1} \le N. \tag{6.5}$$

It follows that $s_n/N \le K_1^{-1}$ since $(s_n)_{n\in\mathbb{N}}$ is increasing. Hence (6.4) implies

$$\mathrm{vol}((\partial U)^{+2K_2 s_n}) \le 2dK_2\bigg(2\frac{K_2}{K_1} + 1\bigg)^{d-1} N^{d-1} s_n. \tag{6.6}$$

The conclusion now follows from (6.3) and (6.6) with M being defined by

$$M = 2dK_2(K_1^d \,\mathrm{vol}(B_1(0)))^{-1} K\bigg(2\frac{K_2}{K_1} + 1\bigg)^{d-1}. \qquad \square$$

In the following proof, we abuse the notation and identify the tiles of $\mathcal{T}_n$ (which are decorated) with their undecorated versions. In particular, if $T \in (\overline{D_{n,i}}, i, v)$ is a tile of $\mathcal{T}_n$, we write $n_{\mathbf{p}}(T)$ and $\mathrm{vol}(T)$ for $n_{\mathbf{p}}(D_{n,i})$ and $\mathrm{vol}(D_{n,i})$.

Proof of Lemma 6.2. Fix an S-patch $\mathbf{p}$ of X and a d-cube U of side N in $\mathbb{R}^d$. The idea of the proof is to decompose U into smaller pieces that are tiles of $\mathcal{T}_n$ for some $n \in \{n_0, \ldots, n_1\}$. Since the tiles of $\mathcal{T}_n$ are tiled by tiles of $\mathcal{T}_m$ for all $m \le n$, we ask this decomposition to contain tiles as big as possible.

More precisely, we define

$$P_{n_1} = \{T \in \mathcal{T}_{n_1} \mid T \subseteq U\}$$

and

$$Q_{n_1} = \bigcup_{T\in P_{n_1}} T.$$

For $n \in \{n_0, \ldots, n_1 - 1\}$, Q_n and P_n are defined recursively as follows

$$P_n = \{T \in \mathcal{T}_n \mid T \subseteq \overline{U \setminus Q_{n+1}}\},$$
$$Q_n = \bigcup_{T \in P_n} T.$$

Now we estimate, for every $n \in \{n_0, \ldots, n_1\}$, the cardinality of P_n. Fix $n \in \{n_0, \ldots, n_1\}$. By definition, each tile in P_n lies inside a tile of $\mathcal{T}_{n+1}$ whose support intersects U but it is not included in U. By Lemma 6.3 there is at most $MN^{d-1}s_{n+1}^{1-d}$ of these tiles for some constant $M > 0$ that does not depend on n. By property (ii) in Theorem 3.6 there is a uniform bound $\alpha > 0$ for the number of tiles in $\mathcal{T}_n$ that form a tile in $\mathcal{T}_{n+1}$. Hence,

$$\#P_n \leq M\alpha^d N^{d-1} s_{n+1}^{1-d} \quad \text{for all } n \in \{n_0, \ldots, n_1\}. \tag{6.7}$$

Let $W = U \setminus \bigcup_{n=n_0}^{n_1} Q_n$. Since the Q_n do not overlap, we have

$$\operatorname{vol}(U) = \operatorname{vol}(W) + \sum_{n=n_0}^{n_1} \sum_{T \in P_n} \operatorname{vol}(T). \tag{6.8}$$

Moreover, since for every $T \in \mathcal{T}_n$, $X \cap \partial T = \emptyset$ we have that $n_{\mathbf{p}}(T) = n_{\mathbf{p}}(\mathring{T})$. Hence,

$$n_{\mathbf{p}}(U) = n_{\mathbf{p}}(W) + \sum_{n=n_0}^{n_1} \sum_{T \in P_n} n_{\mathbf{p}}(T). \tag{6.9}$$

By the definition of derived tiling, every tile in $\mathcal{T}_n$ is a translation of some tile $T_{n,i}$ for some $i \in \{1, \ldots, t_n\}$. Since $n \geq n_0$ we obtain that $n_{\mathbf{p}}(T) = n_{\mathbf{p}}(T_{n,i})$. Hence

$$|n_{\mathbf{p}}(T) - \operatorname{vol}(T) \operatorname{freq}(\mathbf{p})| \leq \max_{i \in \{1,\ldots,t_n\}} |n_{\mathbf{p}}(D_{n,i}) - \operatorname{vol}(D_{n,i}) \operatorname{freq}(\mathbf{p})| \tag{6.10}$$

for every $t \in P_n$. Thus, from (6.9), (6.8) and (6.10) we obtain

$$\begin{aligned} |n_{\mathbf{p}}(U) - \operatorname{freq}(\mathbf{p}) \operatorname{vol}(U)| \leq {} & \sum_{n=n_0}^{n_1} \#P_n \max_{i \in \{1,\ldots,t_n\}} |n_{\mathbf{p}}(D_{n,i}) - \operatorname{vol}(D_{n,i}) \operatorname{freq}(\mathbf{p})| \\ & + |n_{\mathbf{p}}(W) - \operatorname{vol}(W) \operatorname{freq}(\mathbf{p})|. \end{aligned} \tag{6.11}$$

By Lemma 6.3,

$$\operatorname{vol}(W) \leq MN^{d-1} s_{n_0}^{1-d} \max_{i \in \{1,\ldots,t_{n_0}\}} \operatorname{vol}(D_{n_0,i}). \tag{6.12}$$

Then, plugging (6.12) and (6.7) into (6.11) gives the conclusion of the lemma. □

The next lemma allows us to estimate $\operatorname{dev}_{\mathbf{p}}(D_{n,i})$ in terms of the coefficients of the transition matrices of $\mathfrak{T}$ (cf. §5).

LEMMA 6.4. *For all $n \geq n_0$ and $i \in \{1, \ldots, t_n\}$,*

$$\operatorname{dev}_{\mathbf{p}}(D_{n,i}) = \sum_{k=1}^{t_{n_0}} n_{\mathbf{p}}(D_{n_0,k})(p_{ik}^{(n,n_0)} - \operatorname{vol}(D_{n,i})\nu(C_{n_0,k})).$$

Proof. Suppose that $\mathbf{p} = X \wedge B_S(x)$ with $x \in X$. It is well known (see e.g. [**LMS02**]) that $\mathrm{freq}(\mathbf{p}) = \nu(C_{\mathbf{p}})$, where

$$C_{\mathbf{p}} = \{Y \in \Omega \mid Y \wedge B_S(0) = (X - x) \wedge B_S(0)\}.$$

On the one hand, from (3.2), the definition of $p_{ik}^{(n,n_0)}$ (cf. §5) and the additivity of $n_{\mathbf{p}}$ we deduce that

$$n_{\mathbf{p}}(D_{n,i}) = \sum_{k=1}^{t_{n_0}} n_{\mathbf{p}}(D_{n_0,k}) p_{ik}^{(n,n_0)}. \tag{6.13}$$

On the other hand, since there is no occurrence of $\mathbf{p}$ in the border of a box of $\mathcal{B}_{n_0}$, for every $Y \in C_{\mathbf{p}}$, there are $k \in \{1, \ldots, t_{n_0}\}$, $Z \in C_{n_0,k}$ and $z \in D_{n_0,k}$ such that $Z - z \in C_{\mathbf{p}}$. Moreover, the number of z as above such that $Z - z \in C_{\mathbf{p}}$ (with k and Z fixed) is exactly $n_{\mathbf{p}}(D_{n_0,k})$. It follows from the definition of n_0 that there are exactly $n_{\mathbf{p}}(D_{n_0,k})$ copies of $C_{n_0,k}$ inside $C_{\mathbf{p}}$ for all k and hence

$$\nu(C_{\mathbf{p}}) = \sum_{k=1}^{t_{n_0}} n_{\mathbf{p}}(D_{n_0,k}) \nu(C_{n_0,k}). \tag{6.14}$$

The conclusion now follows by (6.14) multiplied by $\mathrm{vol}(D_{n,i})$ from (6.13). □

The last lemma before the proof of Theorem 6.1 estimates the deviation of $p_{i,k}^{(n,n_0)}/\mathrm{vol}(D_{n,i})$ with respect to its limit $\nu(C_{n_0,k})$, in terms of the mixing rate $c_{\mathfrak{T}}$ of the transition matrices.

LEMMA 6.5. *For every $n > m \geq 0$ we have*

$$\max_{\substack{1 \leq j \leq t_m \\ 1 \leq i \leq t_n}} \left| \frac{p_{i,j}^{(n,m)}}{\mathrm{vol}(D_{n,i})} - \nu(C_{m,j}) \right| = O(\nu(C_m) c_{\mathfrak{T}}^{n-m}).$$

Proof. Let $M := \sup_{n \in \mathbb{N}^*} \|M_n\|_\infty \|M_{n+1}\|_\infty > 1$. We prove that for every $n \in \mathbb{N}$ and $1 \leq i \leq t_n$ we have

$$\mathrm{vol}(D_{n,i}) \nu(C_{n,i}) \geq \frac{1}{M}. \tag{6.15}$$

Indeed, by an argument analogous to the one used in the proof of Lemma 5.4, we get

$$\frac{\mathrm{vol}(D_{n+1,k})}{\mathrm{vol}(D_{n,i'})} \leq \|M_{n+1}\|_\infty \|M_n\|_\infty \tag{6.16}$$

for all $n > 0$, $i' \in \{1, \ldots, t_n\}$ and $k \in \{1, \ldots, t_{n+1}\}$. Now, from (3.4) and $m_{i,j}^{(n)} \geq 1$ we deduce that $\nu(C_{n,i}) \geq \sum_{j=1}^{t_{n+1}} \nu(C_{n+1,j})$. Hence, we have

$$\mathrm{vol}(D_{n,i}) \nu(C_{n,i}) \geq \sum_{j=1}^{t_{n+1}} \mathrm{vol}(D_{n+1,j}) \nu(C_{n+1,j}) \frac{\mathrm{vol}(D_{n,i})}{\mathrm{vol}(D_{n+1,j})}.$$

Thus, plugging (6.16) and (3.5) into the last inequality we get (6.15). Finally, the conclusion of the lemma follows from (5.4), (5.2) and (6.15). □

Proof of Theorem 6.1. We do the proof for the case $\delta_{\mathfrak{T}} < 1$ (the other cases giving better estimates). By Lemma 6.2, it suffices to show that

$$\sum_{n=n_0}^{n_1} s_n^{1-d} \max_{1\leq i\leq t_n} |\mathrm{dev}_{\mathbf{p}}(D_{n,i})| = O(N^{1-\delta_{\mathfrak{T}}}). \tag{6.17}$$

Indeed, using Lemmas 6.4 and 6.5 we obtain

$$|\mathrm{dev}_{\mathbf{p}}(D_{n,i})| = \sum_{k=1}^{t_{n_0}} n_{\mathbf{p}}(D_{n_0,k}) O(c_{\mathfrak{T}}^{n-n_0} \operatorname{vol}(D_{n,i})).$$

Recall that from (3.6) we have $\operatorname{vol}(D_{n,i}) = O(s_n^d)$. Hence the left-hand side of (6.17) can be estimated as

$$\begin{aligned}\sum_{n=n_0}^{n_1} s_n^{1-d} \max_{1\leq i\leq t_n} |\mathrm{dev}_{\mathbf{p}}(D_{n,i})| &= \sum_{n=n_0}^{n_1} s_n^{1-d} \sum_{k=1}^{t_{n_0}} n_{\mathbf{p}}(D_{n_0,k}) O(c_{\mathfrak{T}}^{n-n_0} s_n^d)\\ &= \sum_{n=n_0}^{n_1} O(s_n c_{\mathfrak{T}}^{n-n_0}).\end{aligned}$$

Plugging $s_n = K^{n-n_0} s_{n_0}$ into the last equation and using $K^{1-\delta_{\mathfrak{T}}} > 1$ we obtain

$$\sum_{n=n_0}^{n_1} s_n^{1-d} \max_{1\leq i\leq t_n} |\mathrm{dev}_{\mathbf{p}}(D_{n,i})| = O((K^{1-\delta_{\mathfrak{T}}})^{n_1-n_0}).$$

Finally, the conclusion follows from (6.5) in the proof of Lemma 6.3. □

Acknowledgements. The authors would like to thank Jarek Kwapisz for several comments and suggestions on the presentation of this work. They also thank Alejandro Maass and Jean-Marc Gambaudo for enlightening discussions about the tower system constructions. Part of this work was done while J.A.-P. was at the Erwin Scrhödinger Institute in Vienna with the support of a Junior Research Fellowship. J.A.-P. is funded by Fondecyt Postdoctoral grant 3100097 and D.C. is funded by Fondecyt Postdoctoral grant 3100092 and PBCT-Conicyt Research Project ADI-17. Finally, the authors acknowledge support from grant Basal-CMM.

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