

PHASE DIAGRAM FOR INTERMITTENT MAPS

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ABSTRACT. We explore the phase diagram for potentials in the space of HÖLDER continuous functions of a given exponent and for the dynamical system generated by a POMEAU–MANNEVILLE, or intermittent, map. There is always a phase where the unique GIBBS state exhibits intermittent behavior. It is the only phase for a specific range of values of the HÖLDER exponent. For the remaining values of the HÖLDER exponent, a second phase with stationary behavior emerges. In this case, a co-dimension 1 submanifold separates the intermittent and stationary phases. It coincides with the set of potentials at which the pressure function fails to be real-analytic. We also describe the relationship between the phase transition locus, (persistent) phase transitions in temperature, and ground states.

1. INTRODUCTION

Inspired by phase diagrams in statistical mechanics and bifurcation diagrams in dynamical systems, we propose studying regions of potentials where the corresponding GIBBS states exhibit qualitatively distinct behaviors for a fixed dynamical system. This paper focuses on one of the simplest cases of interest: The phase diagram for potentials in the space of HÖLDER continuous functions of a given exponent for the dynamical system generated by a POMEAU–MANNEVILLE, or intermittent, map. The simplicity of this setting allows for a detailed analysis of the associated phase diagram. There is always a phase where the unique GIBBS state exhibits intermittent behavior in the sense of POMEAU and MANNEVILLE [PM80]. It is the only phase for a specific range of values of the HÖLDER exponent. For the remaining values of the HÖLDER exponent, a second phase of stationary behavior emerges. In this case, a co-dimension 1 (topological) submanifold separates the intermittent and stationary phases. It coincides with the set of potentials at which the pressure function fails to be real-analytic. We also describe the relationship between the phase transition locus, (persistent) phase transitions in temperature, and ground states.

To state our results more precisely, we introduce some notation and terminology. For concreteness, we fix throughout the rest of the paper α in $(0, +\infty)$ and the POMEAU–MANNEVILLE or *intermittent* map $f: [0, 1] \rightarrow [0, 1]$, defined by

$$(1.1) \quad f(x) := \begin{cases} x(1 + x^\alpha) & \text{if } x(1 + x^\alpha) \leq 1; \\ x(1 + x^\alpha) - 1 & \text{otherwise.} \end{cases}$$

It is a prototypical example of an interval map with a neutral periodic point. It has been extensively studied in smooth ergodic theory, see, for example, [BC23, CV13, CCRV09, Hu04, GIR22, GIR24, Gou04, Klo19, Klo20, LRL14b, LSV99, Lop93, MT12, Pia80, PM80, PS92, PW99, PS92, Tha80, Sar01, Sar02, You99]. Our arguments apply with minor changes to variants of this map, like those studied in [You99, LSV99]. For expository purposes, the map f given

by (1.1) has the advantage that its geometric potential $-\log Df$ is HÖLDER continuous and therefore belongs to (some of) the BANACH spaces of potentials considered here. However, we do not use this fact in our arguments. It is also possible to extend our results to maps with several neutral periodic points, as those considered by THALER [Tha80] and by LIVERANI, SAUSSOL, and VAIENTI [LSV99, §5].

Denote by $C(\mathbb{R})$ the space of continuous real-valued functions defined on $[0, 1]$ endowed with the uniform norm $\|\cdot\|$. For each γ in $(0, 1]$, denote by $C^\gamma(\mathbb{R})$ the space of those functions in $C(\mathbb{R})$ that are HÖLDER continuous of exponent γ . For each φ in $C^\gamma(\mathbb{R})$, put

$$(1.2) \quad |\varphi|_\gamma := \sup \left\{ \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\gamma} : x, y \in [0, 1] \text{ distinct} \right\} \quad \text{and} \quad \|\varphi\|_\gamma := \|\varphi\| + |\varphi|_\gamma.$$

Denote by \mathcal{M} the space of BOREL probability measures on $[0, 1]$ that are invariant by f . For each ν in \mathcal{M} , denote by h_ν the measure-theoretic entropy of ν . For each φ in $C(\mathbb{R})$, the pressure $P(\varphi)$ of f for the potential φ is defined by

$$(1.3) \quad P(\varphi) := \sup \left\{ h_\nu + \int \varphi \, d\nu : \nu \in \mathcal{M} \right\}.$$

A measure at which this supremum is attained is a GIBBS or *equilibrium state* of f for the potential φ . There is at least 1 such measure (Lemma 2.4(1)). The pressure function $P: C(\mathbb{R}) \rightarrow \mathbb{R}$ so defined is convex and LIPSCHITZ continuous.

A measure ν in \mathcal{M} is *exponentially mixing* for f , if there are C in $(0, +\infty)$ and ρ in $(0, 1)$ such that for every measurable bounded function $\psi: [0, 1] \rightarrow \mathbb{R}$, every LIPSCHITZ continuous function $\tilde{\psi}: [0, 1] \rightarrow \mathbb{R}$, and every n in \mathbb{N} , we have

$$(1.4) \quad \left| \int \psi \circ f^n \cdot \tilde{\psi} \, d\nu - \int \psi \, d\nu \int \tilde{\psi} \, d\nu \right| \leq C \|\psi\| \cdot |\tilde{\psi}|_1 \rho^n.$$

1.1. Intermittent and stationary phases. Denote by $\mathbf{0}$ the function defined on $[0, 1]$ that vanishes identically. Together with its unique GIBBS state for the potential $\mathbf{0}$, the map f is measurably isomorphic to a BERNOULLI process (Lemma 2.4(2)). Due to the neutral fixed point at $\mathbf{0}$, typical orbits exhibit intermittent behavior in the sense of POMEAU and MANNEVILLE [PM80]: They alternate irregularly between time intervals spent near $\mathbf{0}$ and random-looking excursions away from it. This intermittent behavior persists for HÖLDER continuous potentials near $\mathbf{0}$. More precisely, for such a potential φ , there is a GIBBS state of f that is exponentially mixing and has strictly positive measure-theoretic entropy. See [IRRL25, Corollary A.4] and [Klo19, Theorems 1.2 and 3.3] for KLOECKNER's effective estimate of the size of this neighborhood of $\mathbf{0}$ for a map similar to f . These strong stochastic properties of the GIBBS state guarantee that typical orbits with respect to this measure also exhibit intermittent behavior. This motivates the following definition.

Definition 1.1. Let γ in $(0, 1]$ be given. The *intermittent phase* $\mathcal{I}(\gamma)$ of f in $C^\gamma(\mathbb{R})$, is the set of potentials in $C^\gamma(\mathbb{R})$ for which there is an exponentially mixing GIBBS state of f of strictly positive measure-theoretic entropy.

For every γ in $(0, 1]$ the intermittent phase $\mathcal{I}(\gamma)$ is nonempty, because it contains (a neighborhood of) $\mathbf{0}$. Moreover, the set $\mathcal{I}(\gamma)$ is open, the pressure function P is real-analytic on $\mathcal{I}(\gamma)$, and for every φ in $\mathcal{I}(\gamma)$ there is a unique GIBBS state of f for the potential φ [IRRL25, Theorem A.2]. See §2.1 for the notion of real-analyticity used in this paper.

In the case where $\alpha < 1$ and $\gamma > \alpha$, the intermittent phase $\mathcal{I}(\gamma)$ is all of $C^\gamma(\mathbb{R})$ and the pressure function is real-analytic on all of $C^\gamma(\mathbb{R})$, see [LRL14a, LRL14b] and, in the case where $\varphi(0) = \varphi(1)$, see also [GIR24, Proposition 11], KLOECKNER's proof based on ideas from transport theory [Klo20, Theorem A], or the Key Lemma in §2.2.

In the remaining case where $\gamma \leq \alpha$, the space $C^\gamma(\mathbb{R})$ contains the geometric potential $-\log Df$. It is the quintessential example of a potential for which every GIBBS state of f different from δ_0^* is at most subexponentially mixing [Hu04, Gou04, Sar02]. That is, $-\log Df$ is outside the intermittent phase $\mathcal{I}(\gamma)$ and therefore $\mathcal{I}(\gamma)$ is strictly smaller than $C^\gamma(\mathbb{R})$. There is in fact an open set of potentials in $C^\gamma(\mathbb{R})$ outside $\mathcal{I}(\gamma)$. For a concrete example, consider the function $\omega_\gamma: [0, 1] \rightarrow \mathbb{R}$ given by $\omega_\gamma(x) := -x^\gamma$. For every sufficiently large constant β and every potential φ in a neighborhood of $\beta\omega_\gamma$ in $C^\gamma(\mathbb{R})$, the measure δ_0 is the unique GIBBS state of f for the potential φ (Proposition 3.2(2)). Such a potential is thus outside $\mathcal{I}(\gamma)$ and for it the system has a stationary behavior localized at the neutral fixed point 0. This motivates the following definition.

Definition 1.2. Let γ in $(0, 1]$ be given. The *stationary phase* $\mathcal{S}(\gamma)$ of f in $C^\gamma(\mathbb{R})$ is the subset of $C^\gamma(\mathbb{R})$ defined by

$$(1.5) \quad \mathcal{S}(\gamma) := \text{int}(\{\varphi \in C^\gamma(\mathbb{R}) : \delta_0 \text{ is the unique GIBBS state of } f \text{ for the potential } \varphi\}).$$

By definition, the stationary phase $\mathcal{S}(\gamma)$ is open and disjoint from $\mathcal{I}(\gamma)$. In the case where $\gamma \leq \alpha$, the stationary phase $\mathcal{S}(\gamma)$ is nonempty and in fact unbounded because for some β_0 in $(0, +\infty)$ it contains $(\beta\omega_\gamma)_{\beta \in (\beta_0, +\infty)}$. We also show that in this case, $\mathcal{I}(\gamma)$ is star-convex at 0 and that $\mathcal{S}(\gamma)$ is convex (Proposition 3.1(1, 2)). In particular, $\mathcal{I}(\gamma)$ and $\mathcal{S}(\gamma)$ are both connected.

1.2. Phase transition locus. Fix γ in $(0, \min\{\alpha, 1\}]$, so that the stationary phase $\mathcal{S}(\gamma)$ is nonempty. We show that the union of intermittent phase $\mathcal{I}(\gamma)$ and the stationary phase $\mathcal{S}(\gamma)$ is dense in $C^\gamma(\mathbb{R})$ and that the boundaries of these sets coincide (Proposition 3.1(3)). So, the following definition arises naturally.

Definition 1.3. Let γ in $(0, \min\{\alpha, 1\}]$ be given. The *phase transition locus* $\mathcal{PT}(\gamma)$ of f in $C^\gamma(\mathbb{R})$, is the common boundary of the intermittent $\mathcal{I}(\gamma)$ and stationary $\mathcal{S}(\gamma)$ phases.

The phase transition locus $\mathcal{PT}(\gamma)$ is thus closed, has empty interior, and satisfies

$$(1.6) \quad \mathcal{PT}(\gamma) = C^\gamma(\mathbb{R}) \setminus (\mathcal{I}(\gamma) \cup \mathcal{S}(\gamma)).$$

Our first main result characterizes the phase transition locus $\mathcal{PT}(\gamma)$ as the singular set of the pressure function P .

Theorem A. For every γ in $(0, 1]$, the phase transition locus $\mathcal{PT}(\gamma)$ coincides with the set of potentials at which the pressure function P fails to be real-analytic.

This result has natural applications to 1-parameter families of potentials. Let I be an open interval of \mathbb{R} and let $(\varphi_\tau)_{\tau \in I}$ be a real-analytic family of potentials in $C^\gamma(\mathbb{R})$. Theorem A implies that for every parameter τ_0 at which the function $\tau \mapsto P(\varphi_\tau)$ fails to be real-analytic, the potential φ_{τ_0} must be in $\mathcal{PT}(\gamma)$. In other words, every phase transition parameter for the pressure function occurs at a potential in the phase transition locus. In the opposite direction, we show that if τ_0 is a parameter such that φ_τ is in $\mathcal{I}(\gamma)$ if $\tau < \tau_0$ and outside $\mathcal{I}(\gamma)$

*Strictly speaking, δ_0 is exponentially mixing for trivial reasons.

if $\tau \geq \tau_0$, then the function $\tau \mapsto P(\varphi_\tau)$ fails to be real-analytic at τ_0 (Lemma 3.6). Together with the fact that the intermittent phase $\mathcal{I}(\gamma)$ is star-convex at $\mathbf{0}$ (Proposition 3.1(1)) and equal to $\{\varphi \in C^\gamma(\mathbb{R}) : P(\varphi) > \varphi(0)\}$ [IRRL25, Theorem A.2], this implies the following corollary as an immediate consequence.

Corollary 1 (Phase transitions in temperature). *For every HÖLDER continuous potential φ , precisely one of the following properties holds.*

1. *The family $(\beta\varphi)_{\beta \in (0, +\infty)}$ is contained in $\mathcal{I}(\gamma)$ and the function $\beta \mapsto P(\beta\varphi)$ is real-analytic and strictly larger than $\beta \mapsto \beta\varphi(0)$ on $(0, +\infty)$.*
2. *There is β_* in $(0, +\infty)$ such that for every β in $(0, +\infty)$, the potential $\beta\varphi$ is in $\mathcal{I}(\gamma)$ if β is in $(0, \beta_*)$ and outside $\mathcal{I}(\gamma)$ if β is in $[\beta_*, +\infty)$. Thus, $\beta_*\varphi$ is in $\mathcal{PT}(\gamma)$ and the function $\beta \mapsto P(\beta\varphi)$ is real-analytic and strictly larger than $\beta \mapsto \beta\varphi(0)$ on $(0, \beta_*)$, it coincides with $\beta \mapsto \beta\varphi(0)$ on $[\beta_*, +\infty)$, and it only fails to be real-analytic on $(0, +\infty)$ at β_* .*

In the case where the latter property in Corollary 1 holds, the potential φ undergoes a phase transition in temperature and β_* is the phase transition parameter of φ . This terminology comes from statistical mechanics, where β is interpreted as the inverse temperature. Thus, a HÖLDER continuous potential φ undergoes a phase transition in temperature if and only if the function $\beta \mapsto P(\beta\varphi)$ fails to be real-analytic on $(0, +\infty)$. The corollary above implies that these equivalent conditions hold if and only if at low temperatures δ_0 is a GIBBS state of f for φ .

The geometric potential $-\log Df$ is a well-known example of a potential having a phase transition in temperature. See, for example, Proposition 4.5, [PS92], [CR25, Proposition 4.2], as well as [BC23, Theorem B] for an analogous result for local diffeomorphisms of the circle. For potentials that have a precise asymptotic behavior near 0 and are sufficiently regular, phase transitions in temperature were studied by SARIG [Sar01, Proposition 1] and in the companion paper [CR25].

Our next result describes fundamental properties of the phase transition locus.

Theorem B. *For every γ in $(0, \min\{\alpha, 1\}]$, the phase transition locus $\mathcal{PT}(\gamma)$ is a topological submanifold of $C^\gamma(\mathbb{R})$ of co-dimension 1. Furthermore, this submanifold is homeomorphic to a vector subspace of $C^\gamma(\mathbb{R})$ of co-dimension 1, but it is not an affine subspace of $C^\gamma(\mathbb{R})$.*

In fact, we show that the phase transition locus is linear homeomorphic to the graph of a real-valued convex function and that every affine subspace contained in the phase transition locus is of infinite co-dimension (Proposition 3.8).

1.3. On the persistence of phase transitions in temperature. Our next result is reminiscent of the PEIERLS condition for contour models, which ensures that ground states persist as GIBBS states at low temperatures [Pei36, Sin82]. In various situations of interest, like the ISING model on the square lattice [Ons44], this leads to the existence of phase transitions [Geo11, Chapter 6]. For intermittent maps and a HÖLDER continuous potential φ , we show that the condition of δ_0 being robustly a ground state implies that δ_0 is also a GIBBS state at low temperatures. As a consequence, such a potential φ undergoes a phase transition in temperature (Theorem C).

To state this result more precisely, we introduce some terminology. A *ground state of f for a continuous potential φ* , is a measure ν in \mathcal{M} satisfying

$$(1.7) \quad \int \varphi \, d\nu = \sup_{\nu' \in \mathcal{M}} \int \varphi \, d\nu'.$$

Recall from §1.2 that for a HÖLDER continuous potential φ having a phase transition in temperature, the phase transition parameter of φ is the unique parameter in $(0, +\infty)$ at which the function $\beta \mapsto P(\beta\varphi)$ fails to be real-analytic.

Definition 1.4. Let γ in $(0, 1]$ be given. A potential φ in $C^\gamma(\mathbb{R})$ undergoes a *persistent phase transition in temperature in $C^\gamma(\mathbb{R})$* , if it undergoes a phase transition in temperature and if every potential close to φ in $C^\gamma(\mathbb{R})$ undergoes a phase transition in temperature whose phase transition parameter is close to that of φ .

By definition, for a potential φ undergoing a persistent phase transition in temperature in $C^\gamma(\mathbb{R})$, the phase transition parameter is defined on a neighborhood of φ in $C^\gamma(\mathbb{R})$ and it is continuous at φ .

Theorem C (From zero to low temperatures). *For all γ in $(0, 1]$ and φ in $C^\gamma(\mathbb{R})$, the following properties are equivalent.*

1. *The potential φ undergoes a persistent phase transition in temperature in $C^\gamma(\mathbb{R})$.*
2. *The measure δ_0 is a ground state of f for every potential in $C^\gamma(\mathbb{R})$ close to φ .*

When these equivalent conditions hold, δ_0 is the unique ground state of f for the potential φ .

The following corollary is a direct consequence of Theorem C. For each γ in $(0, 1]$, put

$$(1.8) \quad \mathcal{G}(\gamma) := \{\varphi \in C^\gamma(\mathbb{R}) : \delta_0 \text{ is a ground state of } f \text{ for the potential } \varphi\}$$

and note that $\mathcal{G}(\gamma)$ is a closed convex cone. Moreover, let $\beta_{\text{crit}, \gamma} : C^\gamma(\mathbb{R}) \rightarrow (0, +\infty)$ be the function defined for φ in $C^\gamma(\mathbb{R})$ as follows. If φ undergoes a phase transition in temperature, then $\beta_{\text{crit}, \gamma}(\varphi)$ is its phase transition parameter. Otherwise, $\beta_{\text{crit}, \gamma}(\varphi) := +\infty$. Note that by Corollary 1 in §1.2 every potential in $C^\gamma(\mathbb{R})$ undergoing a phase transition in temperature is in $\mathcal{G}(\gamma)$ and we have

$$(1.9) \quad \beta_{\text{crit}, \gamma}^{-1}((1, +\infty]) = \mathcal{G}(\gamma) \text{ and } \beta_{\text{crit}, \gamma}^{-1}(1) \subseteq \mathcal{PT}(\gamma).$$

Corollary 2. *For every γ in $(0, 1]$, the function $\beta_{\text{crit}, \gamma}$ is constant equal to $+\infty$ outside $\mathcal{G}(\gamma)$ and it is finite and continuous on $\text{int}(\mathcal{G}(\gamma))$.*

To introduce our next result, recall that the geometric potential $-\log Df$ undergoes a phase transition in temperature and that its phase transition parameter $\beta_{\text{crit}, \gamma}(-\log Df)$ is equal to 1 (Proposition 4.5). This function is HÖLDER continuous of exponent $\min\{\alpha, 1\}$, so for every γ in $(0, \min\{\alpha, 1\}]$, it belongs to $C^\gamma(\mathbb{R})$. In the case where $\gamma < \alpha$, we show that the phase transition in temperature of $-\log Df$ is nonpersistent in $C^\gamma(\mathbb{R})$, and thus $\beta_{\text{crit}, \gamma}$ is discontinuous at this potential. Moreover, we show that the phase transition locus fails to be a co-dimension 1 real-analytic subset of $C^\gamma(\mathbb{R})$ at $-\log Df$. See Proposition 4.4 for a precise statement of these results.

In the remaining case where α is in $(0, 1]$ and $\gamma = \alpha$, the following result shows that the situation is different.

Theorem D. *If α is in $(0, 1]$, then every phase transition in temperature is persistent in $C^\alpha(\mathbb{R})$, the function $\beta_{\text{crit}, \alpha}$ is continuous, and we have*

$$(1.10) \quad \beta_{\text{crit}, \alpha}^{-1}(1) = \mathcal{PT}(\alpha) \subseteq \text{int}(\mathcal{G}(\alpha)) \text{ and } \beta_{\text{crit}, \gamma}^{-1}((0, +\infty)) = \text{int}(\mathcal{G}(\alpha)).$$

The following corollary is a direct consequence of Theorems A and D.

Corollary 3. *Suppose α is in $(0, 1]$, let I be an open subinterval of \mathbb{R} containing 0, and let $(\varphi_\tau)_{\tau \in I}$ be a real-analytic family of potentials in $C^\alpha(\mathbb{R})$ having a phase transition at 0. Then, φ_0 undergoes a phase transition in temperature and its phase transition parameter $\beta_{\text{crit}, \alpha}(\varphi_0)$ is equal to 1.*

See Example 4.8 for a concrete example showing that the analogous statement is false for every γ in $(0, 1]$ satisfying $\gamma < \alpha$.

The main ingredient in the proof of Theorem D is a characterization of those potentials in $C^\alpha(\mathbb{R})$ having a phase transition in temperature (Theorem D' in §4.3).

As mentioned above, for every γ in $(0, 1]$ satisfying $\gamma < \alpha$, the phase transition locus $\mathcal{PT}(\gamma)$ fails to be a co-dimension 1 real-analytic subset of $C^\gamma(\mathbb{R})$. In the case where α is in $(0, 1]$, it is unclear to us whether $\mathcal{PT}(\alpha)$ is a co-dimension 1 real-analytic subset of $C^\alpha(\mathbb{R})$.

1.4. Notes and references. The intermittent behavior of a positive entropy GIBBS state reflects the concentration near the neutral fixed point 0 of its density with respect to the conformal measure. See, for example, [LSV99, Tha80].

We show that for each HÖLDER continuous potential φ undergoing a phase transition in temperature, δ_0 is the unique ground state of f for the potential φ (Corollary 1 in §1.2 and Corollary 4 in §2.2). The following is a counterexample for the reverse implication. Choose γ in $(\alpha, +\infty)$ and let $\omega_\gamma: [0, 1] \rightarrow \mathbb{R}$ be the function defined by $\omega_\gamma(x) := -x^\gamma$. It is HÖLDER continuous of exponent $\min\{1, \gamma\}$. Clearly, δ_0 is the unique ground state of f for the potential ω_γ . However, ω_γ has no phase transition in temperature by Proposition 3.2.

In [CR25], we studied phase transitions in temperature for a class of HÖLDER continuous potentials introduced by SARIG in [Sar01]. Using the notion of “robust phase transition in temperature,” we proved results analogous to Theorems C and D [CR25, Theorem 3 and Corollary 1.3]. For γ in $(0, 1]$, this notion can be formulated in the space $C^\gamma(\mathbb{R})$ as follows: A potential φ in $C^\gamma(\mathbb{R})$ exhibits a *robust phase transition in temperature in $C^\gamma(\mathbb{R})$* if every potential sufficiently close to φ in $C^\gamma(\mathbb{R})$ undergoes a phase transition in temperature. Every phase transition in temperature persistent in $C^\gamma(\mathbb{R})$ is also robust in $C^\gamma(\mathbb{R})$. Corollary 2 in §1.3 shows that these notions coincide in $C^\gamma(\mathbb{R})$. However, we have not investigated if these notions coincide in the space of potentials in [CR25, Sar01].

The inducing scheme has been successful in establishing fundamental properties of intermittent maps. However, for γ in $(0, 1]$ satisfying $\gamma < \alpha/(\alpha + 1)$, the inducing scheme approach breaks down for potentials in $C^\gamma(\mathbb{R})$ because the distortion is, in general, unbounded. We have thus avoided the inducing scheme approach entirely.

1.5. Organization. After some preliminaries in §2, we explore the phase diagram in §3. In §3.1, we examine the intermittent and stationary phases. In §3.2, we prove that the phase transition locus coincides with the set where the pressure function fails to be real-analytic (Theorem A). In §3.3, we analyze the geometry at infinity of both the stationary phase and the phase transition locus. Finally, in §3.4, we use this analysis to establish fundamental properties of the phase transition locus and to derive Theorem B.

In §4 we study the persistence of phase transitions in temperature. In §4.1, we relate, for each γ in $(0, \min\{\alpha, 1\}]$, the stationary phase $\mathcal{S}(\gamma)$ to the interior of the cone $\mathcal{G}(\gamma)$ and prove Theorem C. In §4.2, we state and prove our results in the case where $\gamma < \alpha$. In §4.3, we characterize the potentials that undergo a phase transition in temperature in the remaining case, where $\alpha \leq 1$ and $\gamma = \alpha$. This characterization is the main ingredient in the proof of Theorem D, which is given in §4.4.

2. PRELIMINARIES

Let X be a topological space. For a subset S of X , denote by $\text{int}(S)$ and $\text{cl}(S)$ the interior and the closure of S , respectively. A subset of X is a *regular open set*, if it is equal to the interior of its closure.

Given a subset K of \mathbb{R} , denote by $\text{HD}(K)$ its HAUSDORFF dimension.

For each ν in \mathcal{M} , denote by $\text{supp}(\nu)$ its support.

Denote by $C(\mathbb{C})$ the space of continuous complex-valued functions defined on $[0, 1]$ and for φ in $C(\mathbb{C})$, put

$$(2.1) \quad \|\varphi\| := \sup_{[0,1]} |\varphi|.$$

Furthermore, denote by $C(\mathbb{R})$ the subspace of $C(\mathbb{C})$ of real-valued functions and by $\mathbf{0}$ the function in $C(\mathbb{R})$ that is constant equal to 0.

For each γ in $(0, 1]$, denote by $C^\gamma(\mathbb{C})$ the space of those functions in $C(\mathbb{C})$ that are HÖLDER continuous of exponent γ . For each φ in $C^\gamma(\mathbb{C})$, put

$$(2.2) \quad |\varphi|_\gamma := \sup \left\{ \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\gamma} : x, y \in [0, 1] \text{ distinct} \right\} \text{ and } \|\varphi\|_\gamma := \|\varphi\| + |\varphi|_\gamma.$$

Furthermore, denote by $C^\gamma(\mathbb{R})$ the subspace of $C^\gamma(\mathbb{C})$ of real-valued functions. The normed spaces $(C^\gamma(\mathbb{R}), \|\cdot\|_\gamma)$ and $(C^\gamma(\mathbb{C}), \|\cdot\|_\gamma)$ are both BANACH.

2.1. Holomorphic and real-analytic functions and subsets. Fix γ in $(0, 1]$.

Given an open subset Δ of \mathbb{C} , a function $H: \Delta \rightarrow C^\gamma(\mathbb{C})$ is *holomorphic* if for every z in Δ there is φ_z in $C^\gamma(\mathbb{C})$ such that

$$(2.3) \quad \lim_{h \rightarrow 0} \left\| \varphi_z - \frac{H(z+h) - H(z)}{h} \right\|_\gamma = 0.$$

On other hand, given an open subset \mathcal{U} of $C^\gamma(\mathbb{C})$, a function $h: \mathcal{U} \rightarrow \mathbb{C}$ is *holomorphic* if for every open subset Δ of \mathbb{C} and every holomorphic function $H: \Delta \rightarrow \mathcal{U}$ the composition $h \circ H$ is holomorphic.

A real-valued function defined on an open subset of $C^\gamma(\mathbb{R})$ is *real-analytic*, if it extends to a complex-valued holomorphic function defined on an open subset of $C^\gamma(\mathbb{C})$. Similarly, a function defined on an open subset of \mathbb{R} and taking values in $C^\gamma(\mathbb{R})$ is *real-analytic*, if it extends to a function defined on an open subset of \mathbb{C} and taking values in $C^\gamma(\mathbb{C})$.

A subset S of $C^\gamma(\mathbb{R})$ is a *co-dimension 1 real-analytic subset of $C^\gamma(\mathbb{R})$* , if for every φ in S there is an open neighborhood \mathcal{U} of φ and a nonconstant real-analytic function $\mathcal{U} \rightarrow \mathbb{R}$ vanishing precisely on $\mathcal{U} \cap S$.

Lemma 2.1. Fix γ in $(0, 1]$ and let S be a co-dimension 1 real-analytic subset of $C^\gamma(\mathbb{R})$ that is closed. Moreover, let I be an open interval of \mathbb{R} and let $A: I \rightarrow C^\gamma(\mathbb{R})$ be a real-analytic function. Then, either $A(I) \subseteq S$ or every point of $A^{-1}(S)$ is isolated in I .

Proof. Put $K := A^{-1}(S)$. Since S is closed by hypothesis, K is closed in I . It is thus sufficient to show that every accumulation point τ_0 of K in I is in the interior of K . Let \mathcal{U} be an open neighborhood of $A(\tau_0)$ in $C^\gamma(\mathbb{R})$ and $a: \mathcal{U} \rightarrow \mathbb{R}$ a real-analytic function vanishing precisely on $\mathcal{U} \cap S$. Furthermore, let I_0 be an open subinterval of I such that $A(I_0) \subseteq \mathcal{U}$. Then, the composition $a \circ A|_{I_0}$ vanishes on each point of $K \cap I_0$ and it extends to a holomorphic function defined on a neighborhood of I_0 in \mathbb{C} . Combined with our hypothesis that τ_0 is not isolated in $K \cap I_0$ and the Identity Principle, it follows that $a \circ A|_{I_0}$ vanishes on all of I_0 . In particular, K contains I_0 and therefore τ_0 is in the interior of K . The proof of the lemma is thus complete. \square

2.2. Dynamics and thermodynamic formalism of intermittent maps. Note that the function $\log Df$ is strictly increasing and HÖLDER continuous of exponent $\min\{1, \alpha\}$.

Denote by x_1 the unique discontinuity of f . Then $f(x_1) = 1$ and the map $f: [0, x_1] \rightarrow [0, 1]$ is a diffeomorphism. Put $x_0 := 1$ and for each integer j satisfying $j \geq 2$, put $x_j := f|_{[0, x_1]}^{-(j-1)}(x_1)$. Moreover, for every j in \mathbb{N}_0 put $J_j := (x_{j+1}, x_j]$ and note that $f(x_{j+1}) = x_j$ and $f(J_{j+1}) = J_j$.

Lemma 2.2 ([CR25, Lemma 3.1]). We have

$$(2.4) \quad \lim_{n \rightarrow +\infty} n \cdot x_n^\alpha = \frac{1}{\alpha}.$$

Moreover, there is ε_0 in $(0, 1)$ such that for all n in \mathbb{N} and x in J_n we have

$$(2.5) \quad (1 + \varepsilon_0 n)^{\frac{1}{\alpha} + 1} \leq Df^n(x) \leq (1 + \varepsilon_0^{-1} n)^{\frac{1}{\alpha} + 1}.$$

Let \mathcal{P} be the partition of $[0, 1]$ defined by $\mathcal{P} := \{[0, x_1], (x_1, 1]\}$ and for every n in \mathbb{N} put $\mathcal{P}_n := \bigvee_{k=0}^{n-1} \mathcal{P}$.

Lemma 2.3. [CR25, Lemma 3.6] For every continuous potential φ , we have

$$(2.6) \quad P(\varphi) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{Q \in \mathcal{P}_n} \sup_Q \exp(S_n(\varphi))$$

and that for every x in $(0, 1]$ we have

$$(2.7) \quad P(\varphi) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{x' \in f^{-n}(x)} \exp(S_n(\varphi)(x')).$$

Lemma 2.4. The following properties hold.

1. The measure-theoretic entropy is upper-semicontinuous and for every continuous potential there is at least 1 GIBBS state.
2. We have

$$(2.8) \quad \sup_{\nu \in \mathcal{M}} h_\nu = \log 2$$

and there is a unique measure ν_{\max} realizing this supremum. Furthermore, (f, ν_{\max}) is measurably isomorphic to a BERNOULLI process. That is, there is a measurable map $[0, 1] \rightarrow \{0, 1\}^{\mathbb{N}_0}$ mapping ν_{\max} to the BERNOULLI measure with weights $\frac{1}{2}$ and $\frac{1}{2}$ and conjugating f to the shift map on a set of full measure.

Proof. Denote by $\sigma: \{0, 1\}^{\mathbb{N}_0} \rightarrow \{0, 1\}^{\mathbb{N}_0}$ the shift map, by \mathcal{M}_σ the space of Borel probability measures on $\{0, 1\}^{\mathbb{N}_0}$ invariant by σ , and, for each μ in \mathcal{M}_σ , denote by $h_\mu(\sigma)$ the measure-theoretic entropy of μ .

By [CR25, Lemma 3.3], there is a continuous surjective map $\pi: \{0, 1\}^{\mathbb{N}_0} \rightarrow [0, 1]$ such that $\pi \circ \sigma = f \circ \pi$. Moreover, for every invariant measure ν for f , there is a unique invariant measure μ for σ such that $\pi_*\mu = \nu$, and the systems $(\{0, 1\}^{\mathbb{N}_0}, \sigma, \mu)$ and $([0, 1], f, \nu)$ are isomorphic in measure. In particular, the map $\pi_*: \mathcal{M}_\sigma \rightarrow \mathcal{M}$ is one-to-one. Since \mathcal{M}_σ is compact and π is continuous, it follows that π_* is a homeomorphism. Together with the fact that the measure-theoretic entropy of σ is upper semicontinuous, see, for example, [Bow08, Proposition 2.19], this implies the same property for f . Since \mathcal{M} is compact, it follows that for every continuous potential there is at least 1 GIBBS state. This proves item 1.

To prove item 2, note that

$$(2.9) \quad \sup_{\mu \in \mathcal{M}_f} h_\mu = \sup_{\mu \in \mathcal{M}_\sigma} h_\mu(\sigma) = \log 2$$

and recall that the BERNOULLI measure ρ with weights $\frac{1}{2}$ and $\frac{1}{2}$ on $\{0, 1\}^{\mathbb{N}_0}$ is the unique measure of maximal entropy for σ . It follows that $\pi_*\rho$ is the unique maximal entropy measure for f , and the systems (σ, ρ) and $(f, \pi_*\rho)$ are isomorphic in measure. \square

The following is a version for intermittent maps of a result originated in [IRRL12] in the context of complex rational maps, see also the version for multimodal maps in [L15]. See [CR25, Appendix A] for a simplified proof for intermittent maps.

Key Lemma. *For every HÖLDER continuous potential φ and every ν in \mathcal{M} distinct from δ_0 , we have*

$$(2.10) \quad P(\varphi) > \int \varphi \, d\nu.$$

Corollary 4. *Let φ be a HÖLDER continuous potential such that for some β_* in $(0, +\infty)$ we have $P(\beta_*\varphi) = \beta_*\varphi(0)$. Then, δ_0 is the unique ground state of f for the potential φ and for every β in $(\beta_*, +\infty)$ the measure δ_0 is the unique GIBBS state of f for the potential φ .*

Proof. To prove the first assertion, note that

$$(2.11) \quad \beta\varphi(0) = P(\beta\varphi) \geq \beta \sup_{\nu \in \mathcal{M}} \int \varphi \, d\nu \geq \beta\varphi(0).$$

Thus, δ_0 is a ground state of f for the potential φ . To prove uniqueness, note that for every ground state ν_0 of f for φ we have

$$(2.12) \quad \int \varphi \, d\nu_0 = \sup_{\nu \in \mathcal{M}} \int \varphi \, d\nu = \varphi(0) = P(\varphi) \geq h_{\nu_0} + \int \varphi \, d\nu_0$$

and therefore

$$(2.13) \quad h_{\nu_0} = 0 \text{ and } P(\varphi) = \int \varphi \, d\nu_0.$$

Together with the Key Lemma, this implies $\nu_0 = \delta_0$.

To prove the second assertion, let β in $(\beta_*, +\infty)$ be given and let ν be a GIBBS state of f for the potential $\beta\varphi$ (Lemma 2.4(2)). Then we have

$$(2.14) \quad \beta\varphi(0) = \frac{\beta}{\beta_*} P(\beta_*\varphi) \geq \frac{\beta}{\beta_*} h_{\nu} + \beta \int \varphi \, d\nu = \frac{\beta - \beta_*}{\beta_*} h_{\nu} + P(\beta\varphi) \geq \frac{\beta - \beta_*}{\beta_*} h_{\nu} + \beta\varphi(0)$$

and therefore $h_\nu = 0$ and $P(\beta\varphi) = \beta\varphi(0)$. The latter implies that δ_0 is a GIBBS state of f for the potential φ . Together with the Key Lemma, the former implies $\chi_\nu(f) = 0$ and therefore $\nu = \delta_0$. That is, δ_0 is the unique GIBBS state of f for the potential φ . \square

3. PHASE DIAGRAM

This section explores the phase diagram. In §3.1, we examine the intermittent and stationary phases (Proposition 3.1). In §3.2, we prove that the phase transition locus coincides with the set where the pressure function fails to be real-analytic (Theorem A). In §3.3, we analyze the geometry at infinity of both the stationary phase and the phase transition locus (Proposition 3.7). Building on this analysis, we establish fundamental facts about the structure of the phase transition locus (Proposition 3.8) and derive Theorem B in §3.4.

3.1. Intermittent and stationary phases. The goal of this section is to prove the following.

Proposition 3.1. *Let γ in $(0, 1]$ be given. The intermittent phase $\mathcal{I}(\gamma)$ contains 0 and it is therefore nonempty. In the case where $\alpha < 1$ and $\gamma > \alpha$, we have*

$$(3.1) \quad \mathcal{I}(\gamma) = C^\gamma(\mathbb{R}) \text{ and } S(\gamma) = \emptyset.$$

In the remaining case where $\gamma \leq \alpha$, the stationary phase $S(\gamma)$ is nonempty and the following properties hold.

1. *The intermittent phase $\mathcal{I}(\gamma)$ is star-convex at 0. In particular, $\mathcal{I}(\gamma)$ is connected.*
2. *The stationary phase $S(\gamma)$ is unbounded, convex, and coincides with the interior of*

$$(3.2) \quad \{\varphi \in C^\gamma(\mathbb{R}) : \delta_0 \text{ is a GIBBS state of } f \text{ for the potential } \varphi\}.$$

In particular, $S(\gamma)$ is connected.

3. *Each of the phases $\mathcal{I}(\gamma)$ and $S(\gamma)$ is a regular open set. Furthermore, these sets are disjoint and their union is dense in $C^\gamma(\mathbb{R})$. In particular, we have*

$$(3.3) \quad \partial\mathcal{I}(\gamma) = \partial S(\gamma) = C^\gamma(\mathbb{R}) \setminus (\mathcal{I}(\gamma) \cup S(\gamma)).$$

The first assertion of the proposition, that in the case where $\alpha < 1$ and $\gamma > \alpha$ we have $\mathcal{I}(\gamma) = C^\gamma(\mathbb{R})$, was shown in [LRL14a, LRL14b] as a consequence of a more general result. In the case where $\varphi(0) = \varphi(1)$, see also [GIR24, Proposition 11] or [Klo20, Theorem A]. We give a proof specialized for intermittent maps that is significantly shorter.

The proof of Proposition 3.1 is at the end of this section. It relies on the following direct consequence of [IRRL25, Theorem A.2], which is also used several times in the rest of the paper.

Theorem 1. *For every γ in $(0, 1]$, we have*

$$(3.4) \quad \mathcal{I}(\gamma) = \{\varphi \in C^\gamma(\mathbb{R}) : P(\varphi) > \varphi(0)\},$$

the set $\mathcal{I}(\gamma)$ is open in $C^\gamma(\mathbb{R})$, and the pressure function P is real-analytic on $\mathcal{I}(\gamma)$.

For each γ in $(0, +\infty)$, let ω_γ be the function in $C^\gamma(\mathbb{R})$ defined by $\omega_\gamma(x) := -x^\gamma$. In the case where γ is in $(0, 1]$, this function is in $C^\gamma(\mathbb{R})$ and for every ψ in $C^\gamma(\mathbb{R})$ we have

$$(3.5) \quad \psi \geq \psi(0) + |\psi|_\gamma \omega_\gamma.$$

Recall from §1.2 that a HÖLDER continuous potential φ undergoes a phase transition in temperature, if the function $\beta \mapsto P(\beta\varphi)$ fails to be real-analytic on $(0, +\infty)$.

Proposition 3.2 ([CR25, Proposition 2.8]). *For every γ in $(0, +\infty)$, the following properties hold.*

1. *If $\gamma > \alpha$, then for every β in $(0, +\infty)$ we have $P(\beta\omega_\gamma) > 0$ and $\beta\omega_\gamma$ is in $\mathcal{F}(\gamma)$.*
2. *If $\gamma \leq \alpha$, then for every sufficiently large β in $(0, +\infty)$ we have $P(\beta\omega_\gamma) = 0$ and $\beta\omega_\gamma$ is outside $\mathcal{F}(\gamma)$.*

In particular, the potential ω_γ undergoes a phase transition in temperature if and only if $\gamma \leq \alpha$.

We give a different proof of this result. To do this, we introduce some notation. For each j in \mathbb{N} , let y_j be the unique point in J_0 such that $f(y_j) = x_{j-1}$, put $I_j := (y_{j+1}, y_j]$, and let $m: J_0 \rightarrow \mathbb{N}$ be the function so that for each j in \mathbb{N} we have $m^{-1}(j) = I_j$. It is the first return time to J_0 . The first return map of f to J_0 , is the function $F: J_0 \rightarrow J_0$ defined by $F(x) := f^{m(x)}(x)$. Note that for each j in \mathbb{N} , the map F maps I_j diffeomorphically onto J_0 .

Proof of Proposition 3.2. To prove item 1, put

$$(3.6) \quad \zeta := \beta \sum_{j=0}^{+\infty} x_j^\gamma$$

and note that this sum is finite by our hypothesis $\gamma > \alpha$ and (2.4) in Lemma 2.2. It follows that for all k in \mathbb{N}_0 and y in $[0, x_k]$, we have

$$(3.7) \quad S_{k+1}(\beta\omega_\gamma)(y) \geq S_{k+1}(\beta\omega_\gamma)(x_k) \geq -\zeta.$$

Recall from §2.2 that \mathcal{P} is the partition $\{[0, x_1], (x_1, 1]\}$ of $[0, 1]$ and that for each n in \mathbb{N} we denote its n -th refinement $\bigvee_{k=0}^{n-1} \mathcal{P}$ by \mathcal{P}_n . Let

$$(3.8) \quad \iota_n: \mathcal{P}_n \rightarrow \{0, 1\}^{\{0, \dots, n-1\}}$$

be the itinerary map defined in such a way that for all $Q \in \mathcal{P}_n$ and $j \in \{0, \dots, n-1\}$ we have $\iota_n(Q)_j = 1$ if and only if $f^j(Q) \subseteq (x_1, 1]$. Note that ι_n is a bijection. For each $a_0 \cdots a_{n-1}$ in $\{0, 1\}^{\{0, \dots, n-1\}}$, put $|a_0 \cdots a_{n-1}| := \sum_{j=0}^{n-1} a_j$. It follows that for every Q in \mathcal{P}_n , we have

$$(3.9) \quad \sup_Q S_n(\beta\omega_\gamma) \geq -(|\iota_n(Q)| + 1)\zeta.$$

Hence

$$(3.10) \quad \begin{aligned} \sum_{Q \in \mathcal{P}_n} \sup_Q \exp(S_n(\beta\omega_\gamma)) &\geq \exp(-\zeta) \sum_{\underline{a} \in \{0, 1\}^{\{0, \dots, n-1\}}} \exp(-|\underline{a}|\zeta) \\ &= \exp(-\zeta)(1 + \exp(-\zeta))^n. \end{aligned}$$

Taking logarithms, dividing by n , and then taking the limit as $n \rightarrow +\infty$, we obtain by (2.6) in Lemma 2.3

$$(3.11) \quad P(\beta\omega_\gamma) \geq \log(1 + \exp(-\zeta)) > 0.$$

Together with Theorem 1, this implies that $\beta\omega_\gamma$ is in $\mathcal{F}(\gamma)$.

To prove item 2, for every β in $(0, +\infty)$ and each n in \mathbb{N} put

$$(3.12) \quad \xi_n(\beta) := \exp\left(-\beta \sum_{j=1}^n x_j^\gamma\right).$$

In view of (2.4) in Lemma 2.2, our hypothesis $\gamma \leq \alpha$ implies that there is β_0 in $(0, +\infty)$ such that for every β in $[\beta_0, +\infty)$ we have

$$(3.13) \quad \sum_{j=1}^{+\infty} \xi_j(\beta) < 1.$$

Let β in $[\beta_0, +\infty)$ be given. In view of (2.7) in Lemma 2.3, to prove $P(\beta\omega_\gamma) = 0$ it is sufficient to show that the sum

$$(3.14) \quad \sum_{n=1}^{+\infty} \sum_{x \in f^{-n}(1)} \exp(\beta S_n(\omega_\gamma)(x))$$

is finite. Note first that for all j in \mathbb{N} and x in J_j , we have

$$(3.15) \quad \exp(\beta S_j(\omega_\gamma)(x)) \leq \exp\left(-\beta \sum_{k=1}^j x_{k+1}^\gamma\right) = \exp(\beta x_1^\gamma) \xi_{j+1}(\beta).$$

It follows that (3.14) is bounded from above by

$$(3.16) \quad \left(1 + \exp(\beta x_1^\gamma) \sum_{j=1}^{+\infty} \xi_{j+1}(\beta)\right) \left(\sum_{n=1}^{+\infty} \sum_{\substack{x \in f^{-n}(1) \\ x \in J_0}} \exp(\beta S_n(\omega_\gamma)(x))\right).$$

The first factor above is finite by (3.13). To prove that the second factor above is finite, note that all j in \mathbb{N} and y in I_j , the point $f(y)$ is in J_{j-1} , so by (3.15) we have

$$(3.17) \quad \exp(\beta S_j(\omega_\gamma)(y)) \leq \exp(-\beta y^\gamma) \exp(\beta x_1^\gamma) \xi_j(\beta) \leq \xi_j(\beta).$$

Let x be a point of $f^{-n}(1)$ in J_0 and let ℓ in \mathbb{N} be the integer such that

$$(3.18) \quad m(x) + \dots + m(F^{\ell-1}(x)) = n.$$

Applying (3.17) repeatedly, we obtain

$$(3.19) \quad \exp(\beta S_n(\omega_\gamma)(x)) \leq \prod_{k=0}^{\ell-1} \xi_{m(F^k(x))}(\beta).$$

Together with (3.13), this implies

$$(3.20) \quad \sum_{n=1}^{+\infty} \sum_{\substack{x \in f^{-n}(1) \\ x \in J_0}} \exp(\beta S_n(\omega_\gamma)(x)) \leq \sum_{\ell=1}^{+\infty} \left(\sum_{j=1}^{+\infty} \xi_j(\beta_0)\right)^\ell < +\infty.$$

This proves that the second factor in (3.16) is finite and completes the proof of $P(\beta\omega_\gamma) = 0$. It follows that $\beta\omega_\gamma$ is outside $\mathcal{G}(\gamma)$ by Theorem 1. The proof of the lemma is thus complete. \square

Lemma 3.3. *For every γ in $(0, 1]$, we have*

$$(3.21) \quad C^\gamma(\mathbb{R}) \setminus \mathcal{G}(\gamma) = \{\varphi \in C^\gamma(\mathbb{R}) : \delta_0 \text{ is a GIBBS state of } f \text{ for the potential } \varphi\}.$$

Furthermore, this last set is convex and its interior is equal to $S(\gamma)$.

Proof. Putting $\widehat{S}(\gamma) := C^\gamma(\mathbb{R}) \setminus \mathcal{G}(\gamma)$, by Theorem 1 we have

$$(3.22) \quad \begin{aligned} \widehat{S}(\gamma) &= \{\varphi \in C^\gamma(\mathbb{R}) : P(\varphi) = \varphi(0)\} \\ &= \{\varphi \in C^\gamma(\mathbb{R}) : \delta_0 \text{ is a GIBBS state of } f \text{ for the potential } \varphi\}. \end{aligned}$$

In particular, $S(\gamma) \subseteq \text{int}(\widehat{S}(\gamma))$. It also follows that $\widehat{S}(\gamma)$ is convex, because P is convex and $\varphi \mapsto \varphi(0)$ is linear.

To complete the proof of the last assertion, let φ be in $\text{int}(\widehat{S}(\gamma))$ and let β_* in $(0, 1)$ be sufficiently close to 1 so that $\beta_*\varphi$ is in $\widehat{S}(\gamma)$. Then, $P(\beta_*\varphi) = \beta_*\varphi(0)$ and by Corollary 4 in §2.2 with $\beta = 1$ we have that δ_0 is the unique GIBBS state of f for the potential φ . Since this holds for every potential in $\text{int}(\widehat{S}(\gamma))$ and this set is open, we conclude $\text{int}(\widehat{S}(\gamma)) \subseteq S(\gamma)$. The proof of the lemma is thus complete. \square

Lemma 3.4. *Let γ in $(0, \min\{\alpha, 1\}]$ and ψ_0 in $C^\gamma(\mathbb{R}) \setminus \mathcal{G}(\gamma)$ be given. Then, for every τ in $(0, +\infty)$ the potential $\psi_0 + \tau\omega_\gamma$ is in $S(\gamma)$.*

Proof. Let Δ in $C^\gamma(\mathbb{R})$ be such that $|\Delta|_\gamma \leq \tau$. By (3.5) with $\psi = -\Delta$, we have

$$(3.23) \quad \psi_0 + \tau\omega_\gamma + \Delta \leq \psi_0 + \Delta(0).$$

Together with our hypothesis that ψ_0 is outside $\mathcal{G}(\gamma)$ and Theorem 1, this implies

$$(3.24) \quad \begin{aligned} P(\psi_0 + \tau\omega_\gamma + \Delta) &\leq P(\psi_0 + \Delta(0)) = P(\psi_0) + \Delta(0) = (\psi_0 + \Delta)(0) \\ &= h_{\delta_0} + \int \psi_0 + \Delta \, d\delta_0 \leq P(\psi_0 + \tau\omega_\gamma + \Delta). \end{aligned}$$

Hence δ_0 is a GIBBS state for the potential $\psi_0 + \tau\omega_\gamma + \Delta$. In view of Lemma 3.3, we conclude that $\psi_0 + \tau\omega_\gamma$ is in $S(\gamma)$. \square

Proof of Proposition 3.1. To prove that $\mathcal{G}(\gamma)$ contains $\mathbf{0}$, let v_0 in \mathcal{M} be such that $h_{v_0} > 0$ (Lemma 2.4(1)) and note that

$$(3.25) \quad \mathbf{0}(0) = 0 < h_{v_0} + \int \mathbf{0} \, dv_0 \leq P(\mathbf{0}).$$

Together with Theorem 1, this implies that $\mathbf{0}$ is in $\mathcal{G}(\gamma)$.

To prove (3.1), suppose $\alpha < 1$ and $\gamma > \alpha$. Let φ in $C^\gamma(\mathbb{R})$ be given and observe that by (3.5) with $\psi = \varphi$ and Proposition 3.2(1) with $\beta = |\varphi|_\gamma$, we have

$$(3.26) \quad \varphi - \varphi(0) \geq |\varphi|_\gamma \omega_\gamma \text{ and } P(\varphi) - \varphi(0) = P(\varphi - \varphi(0)) \geq P(|\varphi|_\gamma \omega_\gamma) > 0.$$

Together with Theorem 1, this implies that φ is in $\mathcal{G}(\gamma)$.

Henceforth, suppose $\gamma \leq \alpha$. To prove item 1, recall that $\mathcal{G}(\gamma)$ contains $\mathbf{0}$ and let φ in $\mathcal{G}(\gamma)$ be given. By Theorem 1, for every β in $(0, 1)$ we have

$$(3.27) \quad P(\beta\varphi) = \sup_{v \in \mathcal{M}} \left\{ h_v + \beta \int \varphi \, dv \right\} \geq \beta P(\varphi) > \beta\varphi(0).$$

Using Theorem 1 again, we conclude that $\beta\varphi$ is in $\mathcal{G}(\gamma)$.

To prove item 2, note that by Proposition 3.2(2) there is β_* in $(0, +\infty)$ such that $\beta_*\omega_\gamma$ is outside $\mathcal{G}(\gamma)$. Combined with Lemma 3.4 with $\psi_0 = \beta_*\omega_\gamma$, this implies that $S(\gamma)$ is unbounded and therefore nonempty. That $S(\gamma)$ is convex and the last assertion of item 2, both follow from Lemma 3.3.

To prove item 3, first note that $S(\gamma)$ is open by definition, $\mathcal{G}(\gamma)$ is open by Theorem 1, and $\mathcal{G}(\gamma)$ and $S(\gamma)$ are disjoint by definition. On the other hand, $S(\gamma)$ is equal to the interior of $C^\gamma(\mathbb{R}) \setminus \mathcal{G}(\gamma)$ by Lemma 3.3. This implies that $S(\gamma)$ is a regular open set, that

$$(3.28) \quad \partial \mathcal{G}(\gamma) = C^\gamma(\mathbb{R}) \setminus (\mathcal{G}(\gamma) \cup S(\gamma)),$$

and that $\mathcal{G}(\gamma) \cup S(\gamma)$ is dense in $C^\gamma(\mathbb{R})$. To prove the remaining assertions, we use that the set $C^\gamma(\mathbb{R}) \setminus \mathcal{G}(\gamma)$ is convex Lemma 3.3 and that its interior $S(\gamma)$ is nonempty. It follows that $C^\gamma(\mathbb{R}) \setminus \mathcal{G}(\gamma)$ is equal to the closure of its interior and, as a consequence, the open set $\mathcal{G}(\gamma)$ is regular and $\partial \mathcal{G}(\gamma) = \partial S(\gamma)$. This completes the proof of item 3 and of the proposition. \square

3.2. The phase transition locus. In this section we prove Theorem A. The proof is given at the end of this section, after a couple of lemmas.

Lemma 3.5. *Let γ in $(0, \min\{\alpha, 1\}]$ and φ_0 in $C^\gamma(\mathbb{R})$ be given. Then, there is τ_0 in \mathbb{R} such that for every τ in \mathbb{R} we have*

$$(3.29) \quad \varphi_0 + \tau \omega_\gamma \in \begin{cases} \mathcal{G}(\gamma) & \text{if } \tau < \tau_0; \\ \mathcal{PT}(\gamma) & \text{if } \tau = \tau_0; \\ S(\gamma) & \text{if } \tau > \tau_0. \end{cases}$$

Proof. For each τ in \mathbb{R} put $\psi_\tau := \varphi_0 + \tau \omega_\gamma$ and let $g: \mathbb{R} \rightarrow [0, +\infty)$ be defined by

$$(3.30) \quad g(\tau) := P(\psi_\tau) - \psi_\tau(0).$$

Using $\omega_\gamma \leq 0$ and $\omega_\gamma(0) = 0$, for all τ and τ' in \mathbb{R} satisfying $\tau \leq \tau'$ we obtain

$$(3.31) \quad \psi_\tau \geq \psi_{\tau'} \text{ and } g(\tau) = P(\psi_\tau) - \psi_\tau(0) \geq P(\psi_{\tau'}) - \psi_\tau(0) = g(\tau').$$

In particular, g is nonincreasing. On the other hand, using that δ_1 is in \mathcal{M} we obtain that for every sufficiently negative τ in \mathbb{R} we have

$$(3.32) \quad g(\tau) \geq \int \psi_\tau d\delta_1 - \psi_\tau(0) = \varphi_0(1) - \tau - \varphi_0(0) > 0.$$

Our next step is to show that g attains the value 0. To do this, let β_0 in $(0, +\infty)$ be such that $P(\beta_0 \omega_\gamma) = 0$ (Proposition 3.2(2)) and put $\tau_\dagger := \beta_0 + |\varphi_0|_\gamma$. By (3.5) with $\psi = -\varphi_0$, we have

$$(3.33) \quad \psi_{\tau_\dagger} = \varphi_0 + (\beta_0 + |\varphi_0|_\gamma) \omega_\gamma \leq \beta_0 \omega_\gamma + \varphi_0(0)$$

and therefore

$$(3.34) \quad P(\psi_{\tau_\dagger}) \leq P(\beta_0 \omega_\gamma + \varphi_0(0)) = P(\beta_0 \omega_\gamma) + \varphi_0(0) = \varphi_0(0) = h_{\delta_0} + \int \psi_{\tau_\dagger} d\delta_0 \leq P(\psi_{\tau_\dagger}).$$

This implies $g(\tau_\dagger) = 0$, as wanted. It follows that there is τ_0 in \mathbb{R} such that g is strictly positive on $(-\infty, \tau_0)$ and vanishes identically on $[\tau_0, +\infty)$.

In view of Theorem 1 and the definition of $\mathcal{PT}(\gamma)$, we conclude that for every τ in $(-\infty, \tau_0)$ the potential ψ_τ is in $\mathcal{G}(\gamma)$ and that for $\tau = \tau_0$ the potential ψ_{τ_0} is in $\mathcal{PT}(\gamma)$. Using Lemma 3.4 with $\psi_0 = \psi_{\tau_0}$, we obtain that for every τ in $(\tau_0, +\infty)$ the potential ψ_τ is in $S(\gamma)$. \square

Lemma 3.6. *Let γ be in $(0, \min\{\alpha, 1\}]$, let I be an open interval of \mathbb{R} , and let $(\varphi_\tau)_{\tau \in I}$ a real-analytic family of potentials in $C^\gamma(\mathbb{R})$. Suppose that there is a parameter τ_0 in I , such that for every τ in I the potential φ_τ is in $\mathcal{G}(\gamma)$ if $\tau < \tau_0$ and it is outside $\mathcal{G}(\gamma)$ if $\tau \geq \tau_0$. Then, $\tau \mapsto P(\varphi_\tau)$ fails to be real-analytic at τ_0 .*

Proof. Let $p: I \rightarrow [0, +\infty)$ be defined by

$$(3.35) \quad p(\tau) := P(\varphi_\tau) - \varphi_\tau(0).$$

In view of Theorem 1, our hypotheses imply that p is strictly positive on $(-\infty, \tau_0)$ and that it vanishes identically on $[\tau_0, +\infty)$. In particular, p fails to be real-analytic at τ_0 . Since the function $\varphi \mapsto \varphi(0)$ is real-analytic on $C^\gamma(\mathbb{R})$, it follows that $\tau \mapsto P(\varphi_\tau)$ also fails to be real-analytic at τ_0 . \square

Proof of Theorem A. The pressure function P is real-analytic on $\mathcal{G}(\gamma)$ (Theorem 1) and it coincides with the real-analytic function $\varphi \mapsto \varphi(0)$ on $S(\gamma)$. It is thus sufficient to show that for every φ_0 in $\mathcal{PT}(\gamma)$ the pressure function P fails to be real analytic at φ_0 . To do this, consider the real-analytic family $(\varphi_0 + \tau\omega_\gamma)_{\tau \in \mathbb{R}}$ of potentials in $C^\gamma(\mathbb{R})$. By Lemma 3.5 there is τ_0 in \mathbb{R} such that for every τ in \mathbb{R} the potential φ_τ is in $\mathcal{G}(\gamma)$ if $\tau < \tau_0$ and it is outside $\mathcal{G}(\gamma)$ if $\tau \geq \tau_0$. Then the function $\tau \mapsto P(\varphi_\tau)$ fails to be real-analytic at τ_0 by Lemma 3.6 and therefore the pressure function P fails to be real-analytic at φ_0 . \square

3.3. Receding from zero temperature. In this section, we relate the geometry at infinity of the stationary phase and the phase transition locus to the set potentials for which δ_0 is a ground state of f . This is summarized in Proposition 3.7 below.

Let V be a real vector space. A subset C of V is a *cone*, if for all v in C and β in $(0, +\infty)$ the vector βv is in C . In this case, C is *convex* if for all v and v' in C the vector $v + v'$ is in C and it is *salient* if $C \cap -C$ is reduced to the zero vector of V . On the other hand, the *recession cone* of a subset S of V , is the convex cone $\text{recc}(S)$ defined by

$$(3.36) \quad \text{recc}(S) := \{v \in V: \text{ for all } s \text{ in } S \text{ and } t \text{ in } (0, +\infty), s + tv \in S\}.$$

Moreover, the *lineality space* of S is the vector subspace $\text{recc}(S) \cap -\text{recc}(S)$ of V .

Recall from (1.8) that for each γ in $(0, 1]$ we denote by $\mathcal{G}(\gamma)$ the set of potentials in $C^\gamma(\mathbb{R})$ for which δ_0 is a ground state of f . We also consider the closed vector subspace $\mathcal{B}(\gamma)$ of $C^\gamma(\mathbb{R})$, defined by

$$(3.37) \quad \mathcal{B}(\gamma) := \left\{ \varphi \in C^\gamma(\mathbb{R}): \text{ for every } v \text{ in } \mathcal{M}, \int \varphi \, dv = \varphi(0) \right\}.$$

Proposition 3.7. *For every γ in $(0, \min\{\alpha, 1\}]$, the following properties hold.*

1. *We have*

$$(3.38) \quad \text{recc}(S(\gamma)) = \text{recc}(S(\gamma) \cup \mathcal{PT}(\gamma)) = \mathcal{G}(\gamma).$$

2. *The lineality spaces of $S(\gamma)$ and $S(\gamma) \cup \mathcal{PT}(\gamma)$ are both equal to $\mathcal{B}(\gamma)$. Moreover, this vector subspace of $C^\gamma(\mathbb{R})$ is of infinite co-dimension.*

Proof. To prove item 1, we begin showing

$$(3.39) \quad \mathcal{G}(\gamma) \subseteq \text{recc}(S(\gamma)).$$

Let φ in $\mathcal{G}(\gamma)$ and φ_0 in $S(\gamma)$ be given and let ν_0 be a GIBBS state of f for the potential $\varphi_0 + \varphi$. Then we have

$$(3.40) \quad P(\varphi_0 + \varphi) = h_{\nu_0} + \int \varphi_0 + \varphi \, d\nu_0 \leq P(\varphi_0) + \int \varphi \, d\nu_0 \leq P(\varphi_0) + \varphi(0) = (\varphi_0 + \varphi)(0)$$

and therefore $\varphi_0 + \varphi$ is outside $\mathcal{G}(\gamma)$ by Theorem 1. Since this holds for every φ_0 in $\mathcal{S}(\gamma)$, we conclude

$$(3.41) \quad \mathcal{S}(\gamma) + \varphi \subseteq C^\gamma(\mathbb{R}) \setminus \mathcal{G}(\gamma).$$

Using that $\mathcal{S}(\gamma)$ is open and Proposition 3.1(3), we obtain

$$(3.42) \quad \mathcal{S}(\gamma) + \varphi \subseteq \text{int}(C^\gamma(\mathbb{R}) \setminus \mathcal{G}(\gamma)) = \mathcal{S}(\gamma).$$

Since $\mathcal{S}(\gamma)$ is convex by Proposition 3.1(2), this yields that φ is in $\text{recc}(\mathcal{S}(\gamma))$ and proves (3.39).

The inclusion $\text{recc}(\mathcal{S}(\gamma)) \subseteq \text{recc}(\mathcal{S}(\gamma) \cup \mathcal{PT}(\gamma))$ being trivial, to complete the proof of item 1 it is sufficient to show

$$(3.43) \quad \text{recc}(\mathcal{S}(\gamma) \cup \mathcal{PT}(\gamma)) \subseteq \mathcal{G}(\gamma).$$

To do this, let φ in $\text{recc}(\mathcal{S}(\gamma) \cup \mathcal{PT}(\gamma))$ be given, let ν_\dagger in \mathcal{M} be such that

$$(3.44) \quad \int \varphi \, d\nu_\dagger = \sup_{\nu \in \mathcal{M}} \int \varphi \, d\nu,$$

and choose an arbitrary $\varphi_\mathcal{S}$ in $\mathcal{S}(\gamma)$. Then, for every t in $(0, +\infty)$ the potential $\varphi_\mathcal{S} + t\varphi$ is in $\mathcal{S}(\gamma) \cup \mathcal{PT}(\gamma)$. Together with Proposition 3.1(3) and Theorem 1, this implies

$$(3.45) \quad (\varphi_\mathcal{S} + t\varphi)(0) = P(\varphi_\mathcal{S} + t\varphi) \geq \int \varphi_\mathcal{S} + t\varphi \, d\nu_\dagger = \int \varphi_\mathcal{S} \, d\nu_\dagger + t \sup_{\nu \in \mathcal{M}} \int \varphi \, d\nu.$$

Dividing by t and rearranging, we obtain

$$(3.46) \quad \varphi(0) \geq \sup_{\nu \in \mathcal{M}} \int \varphi \, d\nu - \frac{1}{t} \varphi_\mathcal{S}(0) + \frac{1}{t} \int \varphi_\mathcal{S} \, d\nu_\dagger \geq \sup_{\nu \in \mathcal{M}} \int \varphi \, d\nu - \frac{2}{t} \|\varphi_\mathcal{S}\|.$$

Taking the limit as $t \rightarrow +\infty$, we conclude that φ is in $\mathcal{G}(\gamma)$. This completes the proof of (3.43) and therefore of (3.38) and item 1.

To prove item 2, note first that for every φ in $\mathcal{G}(\gamma) \cap -\mathcal{G}(\gamma)$ we have

$$(3.47) \quad \sup_{\nu \in \mathcal{M}} \int \varphi \, d\nu = \varphi(0) = \inf_{\nu \in \mathcal{M}} \int \varphi \, d\nu,$$

so for every ν in \mathcal{M} we have $\int \varphi \, d\nu = \varphi(0)$ and therefore φ is in $\mathcal{G}(\gamma)$. Conversely, every φ in $\mathcal{G}(\gamma)$ satisfies (3.47) and is therefore in $\mathcal{G}(\gamma) \cap -\mathcal{G}(\gamma)$.

To prove the last statement of item 2, denote by \mathcal{M}' the infinite subset of \mathcal{M} of those measures supported on a single periodic orbit of f . Moreover, for each ν in \mathcal{M}' let ℓ_ν be the linear functional of $C^\gamma(\mathbb{R})$ defined by

$$(3.48) \quad \ell_\nu(\varphi) := \varphi(0) - \int \varphi \, d\nu.$$

Then

$$(3.49) \quad \mathcal{G}(\gamma) \subseteq \bigcap_{\nu \in \mathcal{M}'} \text{Ker}(\ell_\nu).$$

On the other hand, the set of linear functionals $\{\ell_\nu : \nu \in \mathcal{M}'\}$ is linearly independent, because the supports of the measures in \mathcal{M}' are pairwise disjoint. This implies that the right side of (3.49) has infinite co-dimension in $C^\gamma(\mathbb{R})$ and therefore that the same holds for $\mathcal{G}(\gamma)$. This completes the proof of item 2 and of the proposition. \square

3.4. Structure of the phase transition locus. This section establishes fundamental properties of the phase transition locus. These are summarized in the proposition below, from which we derive Theorem B at the end of this section.

Note that for every γ in $(0, \min\{\alpha, 1\}]$, the GIBBS states of f for 2 potentials that differ by an element of $\mathcal{G}(\gamma)$, coincide. Thus, each of the sets $\mathcal{I}(\gamma)$, $\mathcal{S}(\gamma)$, and $\mathcal{PT}(\gamma)$ is invariant under translations by elements of $\mathcal{G}(\gamma)$. In particular, $\mathcal{PT}(\gamma)$ contains a translate of $\mathcal{G}(\gamma)$.

Proposition 3.8. *For every γ in $(0, \min\{\alpha, 1\}]$, the following properties hold.*

1. *The phase transition locus $\mathcal{PT}(\gamma)$ is linear homeomorphic to the graph of a real-valued convex function. More precisely, there is a co-dimension 1 vector subspace \mathcal{K} of $C^\gamma(\mathbb{R})$, a convex function $\tau: \mathcal{K} \rightarrow (0, +\infty)$, and a linear homeomorphism from $C^\gamma(\mathbb{R})$ onto $\mathcal{K} \times \mathbb{R}$ that is the identity on \mathcal{K} and that maps $\mathcal{PT}(\gamma)$ onto the graph of τ .*
2. *Every affine space contained in $\mathcal{PT}(\gamma) \cup \mathcal{S}(\gamma)$ is a translate of a vector subspace of $\mathcal{G}(\gamma)$ and it is of infinite co-dimension in $C^\gamma(\mathbb{R})$.*

As mentioned above, for every γ in $(0, \min\{\alpha, 1\}]$ each of the sets $\mathcal{I}(\gamma)$, $\mathcal{S}(\gamma)$, and $\mathcal{PT}(\gamma)$ is invariant under translations by elements of $\mathcal{G}(\gamma)$. That is,

$$(3.50) \quad \mathcal{I}(\gamma) + \mathcal{G}(\gamma) = \mathcal{I}(\gamma), \mathcal{S}(\gamma) + \mathcal{G}(\gamma) = \mathcal{S}(\gamma), \text{ and } \mathcal{PT}(\gamma) + \mathcal{G}(\gamma) = \mathcal{PT}(\gamma).$$

Thus, the phase diagram for the intermittent map f descends naturally to the quotient space $C^\gamma(\mathbb{R})/\mathcal{G}(\gamma)$. Proposition 3.8(2) implies that the (convex and unbounded) set

$$(3.51) \quad (\mathcal{PT}(\gamma) \cup \mathcal{S}(\gamma))/\mathcal{G}(\gamma)$$

contains no nontrivial affine subspace of $C^\gamma(\mathbb{R})/\mathcal{G}(\gamma)$. Equivalently, the recession cone of this set is salient.

Proof of Proposition 3.8. Fix ν in $\mathcal{M} \setminus \{\delta_0\}$ and let $\ell: C^\gamma(\mathbb{R}) \rightarrow \mathbb{R}$ be the continuous linear functional defined by

$$(3.52) \quad \ell(\psi) := \psi(0) - \int \psi \, d\nu.$$

Our hypothesis $\nu \neq \delta_0$ implies

$$(3.53) \quad \ell(\omega_\gamma) = \int x^\gamma \, d\nu(x) > 0.$$

We can thus define the linear projection $P: C^\gamma(\mathbb{R}) \rightarrow \text{Ker}(\ell)$ by

$$(3.54) \quad P(\varphi) := \varphi - \frac{\ell(\varphi)}{\ell(\omega_\gamma)} \omega_\gamma$$

and a linear map $C^\gamma(\mathbb{R}) \rightarrow \text{Ker}(\ell) \times \mathbb{R}$ by

$$(3.55) \quad \varphi \mapsto \left(P(\varphi), \frac{\ell(\varphi)}{\ell(\omega_\gamma)} \right).$$

This last map is a homeomorphism with inverse $(\psi, t) \mapsto \psi + t\omega_\gamma$.

For each φ in $\text{Ker}(\ell)$, denote by $\tau(\varphi)$ the real number τ_0 given by Lemma 3.5 with $\varphi_0 = \varphi$. This lemma implies that the homeomorphism (3.55) maps $\mathcal{PT}(\gamma) \cup \mathcal{S}(\gamma)$ onto the epigraph of the function $\tau: \text{Ker}(\ell) \rightarrow \mathbb{R}$ so defined. Since $\mathcal{PT}(\gamma) \cup \mathcal{S}(\gamma)$ is convex by Proposition 3.1(2, 3), it follows that the function τ is convex and therefore continuous.

By construction, the homeomorphism (3.55) maps $\mathcal{PT}(\gamma)$ onto the graph of τ . Hence, to complete the proof of the desired assertion with $\mathcal{K} = \text{Ker}(\ell)$, it only remains to show that the function τ takes values in $(0, +\infty)$. To do this, let φ in $\text{Ker}(\ell)$ be given and recall that $\varphi + \tau(\varphi)\omega_\gamma$ is in $\mathcal{PT}(\gamma)$ by the definition of $\tau(\varphi)$. Using Theorem 1, the hypothesis that φ is in $\text{Ker}(\ell)$, and the Key Lemma in §2.2, we obtain

$$(3.56) \quad P(\varphi + \tau(\varphi)\omega_\gamma) = (\varphi + \tau(\varphi)\omega_\gamma)(0) = \varphi(0) = \int \varphi \, dv < P(\varphi).$$

Together with $\omega_\gamma \leq 0$, this implies $\tau(\varphi) > 0$.

To prove item 2, let \mathcal{A} be an affine subspace of $C^\gamma(\mathbb{R})$ contained in $\mathcal{PT}(\gamma) \cup S(\gamma)$. The vector subspace of $C^\gamma(\mathbb{R})$ parallel to \mathcal{A} is contained in the lineality space of $\mathcal{G}(\gamma)$, which is equal to $\mathcal{G}(\gamma)$ and has infinite co-dimension in $C^\gamma(\mathbb{R})$ by Proposition 3.7(2). It follows that \mathcal{A} also has infinite co-dimension in $C^\gamma(\mathbb{R})$. \square

Proof of Theorem B. Let \mathcal{K} and τ be given by Proposition 3.8(1) and denote by

$$(3.57) \quad L: C^\gamma(\mathbb{R}) \rightarrow \mathcal{K} \times \mathbb{R}$$

the linear homeomorphism given by this result. The map $H: \mathcal{K} \times \mathbb{R} \rightarrow \mathcal{K} \times \mathbb{R}$ defined by

$$(3.58) \quad H(\varphi_0, h) := (\varphi_0, h - \tau(\varphi_0))$$

is a homeomorphism mapping the graph of τ onto $\mathcal{K} \times \{0\}$. It follows from Proposition 3.8(1) that $L^{-1} \circ H \circ L$ is a homeomorphism from $C^\gamma(\mathbb{R})$ onto itself mapping $\mathcal{PT}(\gamma)$ onto \mathcal{K} . This implies that $\mathcal{PT}(\gamma)$ is a submanifold of co-dimension 1 of $C^\gamma(\mathbb{R})$.

Finally, note that if $\mathcal{PT}(\gamma)$ were an affine subspace of $C^\gamma(\mathbb{R})$, then it would be of co-dimension 1. This would contradict Proposition 3.8(2), so $\mathcal{PT}(\gamma)$ cannot be an affine subspace of $C^\gamma(\mathbb{R})$. The proof of the theorem is thus complete. \square

4. ON THE PERSISTENCE OF PHASE TRANSITIONS IN TEMPERATURE

This section investigates the persistence of phase transitions in temperature. In §4.1, we relate, for each γ in $(0, \min\{\alpha, 1\}]$, the stationary phase $S(\gamma)$ to the interior of the cone $\mathcal{G}(\gamma)$ (Proposition 4.1) and prove Theorem C. In §4.2, we state and prove our results in the case where $\gamma < \alpha$ (Proposition 4.4). In §4.3, we give several characterizations of the potentials that undergo a phase transition in temperature in the remaining case, where $\alpha \leq 1$ and $\gamma = \alpha$ (Theorem D' in §4.3). This characterization is the main ingredient in the proof of Theorem D, which is given in §4.4.

4.1. From zero to low temperatures. In this section, we prove Theorem C. The main new ingredient is the following proposition.

Proposition 4.1. *For every γ in $(0, \min\{\alpha, 1\}]$, we have*

$$(4.1) \quad \text{int}(\mathcal{G}(\gamma)) = \bigcup_{\beta \in (0, +\infty)} \beta S(\gamma).$$

In particular,

$$(4.2) \quad S(\gamma) \subseteq \text{int}(\mathcal{G}(\gamma)) \text{ and } \mathcal{PT}(\gamma) \subseteq \text{cl}(S(\gamma)) \subseteq \mathcal{G}(\gamma).$$

Proof. For all β in $(0, +\infty)$ and φ in $\beta S(\gamma)$, we have

$$(4.3) \quad \beta^{-1}\varphi(0) = P(\beta^{-1}\varphi) \geq \sup_{v \in \mathcal{M}} \int \beta^{-1}\varphi \, dv \geq \beta^{-1}\varphi(0).$$

This implies that φ is in $\mathcal{G}(\gamma)$ and $\beta S(\gamma) \subseteq \mathcal{G}(\gamma)$. Since $S(\gamma)$ is open and this holds for every β in $(0, +\infty)$, we obtain

$$(4.4) \quad \bigcup_{\beta \in (0, +\infty)} \beta S(\gamma) \subseteq \text{int}(\mathcal{G}(\gamma)).$$

To prove the reverse inclusion, let φ in $\text{int}(\mathcal{G}(\gamma))$ be given and choose an arbitrary φ_\bullet in $S(\gamma)$. If φ and φ_\bullet are linearly dependent, then there is β in $(0, +\infty)$ such that φ is in $\beta S(\gamma)$. Suppose φ and φ_\bullet are linearly independent, denote by \mathcal{P} the plane in $C^\gamma(\mathbb{R})$ generated by φ and φ_\bullet , and put

$$(4.5) \quad \mathcal{G} := \mathcal{G}(\gamma) \cap \mathcal{P}.$$

The set \mathcal{G} is a closed convex cone and therefore there are linear functionals ℓ and ℓ' of \mathcal{P} , so that

$$(4.6) \quad \mathcal{G} = \{\psi \in \mathcal{P} : \ell(\psi) \geq 0, \ell'(\psi) \geq 0\}.$$

Since φ is in the interior of $\mathcal{G}(\gamma)$, we have $\ell(\varphi) > 0$ and $\ell'(\varphi) > 0$. Let t in $(0, +\infty)$ be sufficiently large so that

$$(4.7) \quad t\ell(\varphi) \geq \ell(\varphi_\bullet) \text{ and } t\ell'(\varphi) \geq \ell'(\varphi_\bullet),$$

and put $\hat{\varphi} := t\varphi - \varphi_\bullet$. Then $\ell(\hat{\varphi}) \geq 0$, $\ell'(\hat{\varphi}) \geq 0$, and $\hat{\varphi}$ is in \mathcal{G} by (4.6). In particular, $\hat{\varphi}$ is in $\mathcal{G}(\gamma)$ and therefore in $\text{recc}(S(\gamma))$ by (3.38) in Proposition 3.7(1). Since φ_\bullet is in $S(\gamma)$ and $t\varphi = \varphi_\bullet + \hat{\varphi}$, we conclude that $t\varphi$ is in $S(\gamma)$. This proves that φ is in $t^{-1}S(\gamma)$ and completes the proof of (4.1).

The first inclusion in (4.2) is a direct consequence of (4.1), the second is a direct consequence of the definition of $\mathcal{PT}(\gamma)$, and the last is a direct consequence of the first inclusion. The proof of the proposition is thus complete. \square

The proof of Theorem C also relies on the following lemma.

Lemma 4.2. *Let γ in $(0, \min\{\alpha, 1\}]$ and φ in $\text{int}(\mathcal{G}(\gamma))$ be given. Then φ undergoes a phase transition in temperature and, if we denote by β_* the phase transition parameter of φ , then for every β in $(\beta_*, +\infty)$ the potential $\beta\varphi$ is in $S(\gamma)$.*

Proof. By Proposition 4.1 there is β_0 in $(0, +\infty)$ such that $\beta_0 S(\gamma)$ contains φ . It follows that φ undergoes a phase transition in temperature by Corollary 1 in §1.2. Denote by β_* the phase transition parameter of φ and note that $\beta_*\varphi$ is in $\partial\mathcal{G}(\gamma)$ and that $\beta_* < \beta_0^{-1}$ by Corollary 1 in §1.2. It follows that $\beta_*\varphi$ is in $\partial S(\gamma)$ by Proposition 3.1(3). On the other hand, φ is in $\text{recc}(S(\gamma))$ by Proposition 3.7(1) and therefore for every β in $[\beta_0^{-1}, +\infty)$ the potential $\beta\varphi$ is in $S(\gamma)$.

To complete the proof of the lemma, let β in (β_*, β_0^{-1}) be given and put

$$(4.8) \quad \eta_* := \frac{\beta_0^{-1} - \beta}{\beta_0^{-1} - \beta_*}, \eta_0 := \frac{\beta - \beta_*}{\beta_0^{-1} - \beta_*}, \text{ and } \mathcal{U} := \eta_*(\beta_*\varphi) + \eta_0\beta_0^{-1}S(\gamma).$$

Note that η_* and η_0 are both in $(0, 1)$ and that we have

$$(4.9) \quad \eta_* + \eta_0 = 1 \text{ and } \eta_*\beta_* + \eta_0\beta_0^{-1} = \beta.$$

In particular, \mathcal{U} contains $\beta\varphi$. On the other hand, since $\beta_*\varphi$ is in $\partial\mathcal{S}(\gamma)$, $\beta_0^{-1}\varphi$ is in $\mathcal{S}(\gamma)$, and $\text{cl}(\mathcal{S}(\gamma))$ is convex by Proposition 3.1(2), we have $\mathcal{U} \subseteq \text{cl}(\mathcal{S}(\gamma))$. But \mathcal{U} is open, so $\mathcal{U} \subseteq \mathcal{S}(\gamma)$ and therefore $\beta\varphi$ is in $\mathcal{S}(\gamma)$. The proof of the lemma is thus complete. \square

Proof of Theorem C. The last assertion is a direct consequence of Corollary 1 in §1.2 and Corollary 4 in §2.2.

To prove the implication $1 \Rightarrow 2$, suppose that φ undergoes a persistent phase transition in temperature in $C^\gamma(\mathbb{R})$ and let \mathcal{U} be an open neighborhood of φ in $C^\gamma(\mathbb{R})$ of potentials undergoing a phase transition in temperature. By Corollary 1 in §1.2, we have $\mathcal{U} \subseteq \mathcal{G}(\gamma)$. But \mathcal{U} is open, so $\mathcal{U} \subseteq \text{int}(\mathcal{G}(\gamma))$ and therefore φ is in $\text{int}(\mathcal{G}(\gamma))$.

To prove the implication $2 \Rightarrow 1$, suppose that φ is in $\text{int}(\mathcal{G}(\gamma))$ and let η in $(0, 1)$ be given. By Lemma 4.2 the potential φ undergoes a phase transition in temperature. Denote by β_* the phase transition parameter of φ and put $\varphi_* := \beta_*\varphi$. Then $\eta\varphi_*$ is in $\mathcal{F}(\gamma)$ by Corollary 1 in §1.2 and $\eta^{-1}\varphi_*$ is in $\mathcal{S}(\gamma)$ by Lemma 4.2. Let \mathcal{U}_* be a sufficiently small neighborhood of φ_* , such that the open neighborhood $\beta_*^{-1}\mathcal{U}_*$ of φ is contained in $\text{int}(\mathcal{G}(\gamma))$ and we have

$$(4.10) \quad \eta\mathcal{U}_* \subseteq \mathcal{F}(\gamma) \text{ and } \eta^{-1}\mathcal{U}_* \subseteq \mathcal{S}(\gamma).$$

Thus, every $\tilde{\varphi}$ in $\beta_*^{-1}\mathcal{U}_*$ undergoes a phase transition in temperature by Lemma 4.2 and satisfies

$$(4.11) \quad (\eta\beta_*)\tilde{\varphi} \in \mathcal{F}(\gamma) \text{ and } (\eta^{-1}\beta_*)\tilde{\varphi} \in \mathcal{S}(\gamma).$$

Together with Corollary 1 in §1.2, this implies that the phase transition parameter $\tilde{\beta}$ of $\tilde{\varphi}$ satisfies

$$(4.12) \quad \eta\beta_* < \tilde{\beta} < \eta^{-1}\beta_*.$$

This proves that φ undergoes a persistent phase transition in temperature in $C^\gamma(\mathbb{R})$ and completes the proof the theorem. \square

4.2. Nonpersistent phase transitions in temperature when $\gamma < \alpha$. In this section, we state and prove our results for γ in $(0, 1]$ satisfying $\gamma < \alpha$ (Proposition 4.4). We begin with the following result, which is also used in the proof of Theorem D in §4.4.

Proposition 4.3. *For every γ in $(0, \min\{\alpha, 1\}]$, the function $\beta_{\text{crit},\gamma}$ is lower semicontinuous and satisfies*

$$(4.13) \quad \beta_{\text{crit},\gamma}^{-1}(1) \subseteq \mathcal{PT}(\gamma) \text{ and } \beta_{\text{crit},\gamma}^{-1}(1) \cap \text{int}(\mathcal{G}(\gamma)) = \mathcal{PT}(\gamma) \cap \text{int}(\mathcal{G}(\gamma)).$$

Proof. To prove that $\beta_{\text{crit},\gamma}$ is lower semicontinuous, let φ in $C^\gamma(\mathbb{R})$ and β in $(0, \beta_{\text{crit},\gamma}(\varphi))$ be given. By Corollary 1 in §1.2 the potential $\beta\varphi$ is in $\mathcal{F}(\gamma)$. Let \mathcal{V} be a neighborhood of φ in $C^\gamma(\mathbb{R})$ such that $\beta\mathcal{V} \subseteq \mathcal{F}(\gamma)$ and let $\tilde{\varphi}$ be in \mathcal{V} . If $\tilde{\varphi}$ undergoes a phase transition in temperature, then we have $\beta_{\text{crit},\gamma}(\tilde{\varphi}) \geq \beta$ by Corollary 1 in §1.2. Otherwise, $\beta_{\text{crit},\gamma}(\varphi) = +\infty$. In all of the cases, $\beta_{\text{crit},\gamma}(\tilde{\varphi}) \geq \beta$.

The first inclusion in (4.13) is a direct consequence of Corollary 1 in §1.2, the definition of $\beta_{\text{crit},\gamma}$, and the equality $\partial\mathcal{F}(\gamma) = \mathcal{PT}(\gamma)$. The second one is given by (4.2) in Proposition 4.1.

To prove the equality in (4.13), let φ in $\text{int}(\mathcal{G}(\gamma))$ be given. If $\beta_{\text{crit},\gamma}(\varphi) = 1$, then φ is in $\mathcal{PT}(\gamma)$ by the first inclusion in (4.13). On the other hand, φ is in $\mathcal{F}(\gamma)$ if $\beta_{\text{crit},\gamma}(\varphi) > 1$ by

Corollary 1 in §1.2 and it is in $\mathcal{S}(\gamma)$ if $\beta_{\text{crit},\gamma}(\varphi) < 1$ by Lemma 4.2. In both of these cases φ is outside $\mathcal{PT}(\gamma)$. \square

The following proposition gathers our results for $\gamma < \alpha$. Recall that the geometric potential $-\log Df$ undergoes a phase transition in temperature, and $\beta_{\text{crit},\gamma}(-\log Df) = 1$, see Proposition 4.5 below.

Proposition 4.4. *For every γ in $(0, 1]$ satisfying $\gamma < \alpha$, the following properties hold.*

1. *The phase transition in temperature of $-\log Df$ is nonpersistent in $C^\gamma(\mathbb{R})$ and $\beta_{\text{crit},\gamma}$ is discontinuous at $-\log Df$.*
2. *The potential $-\log Df$ is in $\partial\mathcal{G}(\gamma)$ and for each β in $(0, +\infty)$ the potential $-\beta \log Df$ is in $\mathcal{PT}(\gamma)$ if and only if β is in $[1, +\infty)$.*
3. *The phase transition locus $\mathcal{PT}(\gamma)$ fails to be a co-dimension 1 real-analytic subset of $C^\gamma(\mathbb{R})$ at $-\log Df$.*

In particular, the set $\mathcal{PT}(\gamma)$ is not contained in $\text{int}(\mathcal{G}(\gamma))$, the function $\beta_{\text{crit},\gamma}$ is discontinuous, and $\beta_{\text{crit},\gamma}^{-1}(1)$ is strictly contained in $\mathcal{PT}(\gamma)$.

To prove this proposition, we rely on the following well-known properties of the geometric potential. We provide a proof for completeness, see also [PS92] and [CR25, Proposition 4.2], as well as [CT13, Theorem 4.3] in the case where α is in $(0, 1)$.

Proposition 4.5. *For every β in $(0, +\infty)$, we have $P(-\beta \log Df) \geq 0$ with equality if and only if β is in $[1, +\infty)$. In particular, $-\log Df$ undergoes a phase transition in temperature and*

$$(4.14) \quad \beta_{\text{crit},\gamma}(-\log Df) = 1$$

The proof of Proposition 4.5 relies on the the following lemma.

Lemma 4.6 ([CR25, Lemma 3.2]). *There are ε_1 and C_1 in $(0, +\infty)$, such that the following properties hold. For every n in \mathbb{N} and every connected component J of $f^{-n}(J_0)$, we have*

$$(4.15) \quad |J| \leq \frac{|J_0|}{(1 + \varepsilon_1 n)^{\frac{1}{\alpha} + 1}}$$

and for all x and y in J we have

$$(4.16) \quad Df^n(x) \geq (1 + \varepsilon_1 n)^{\frac{1}{\alpha} + 1} \text{ and } C_1^{-1} \leq \frac{Df^n(x)}{Df^n(y)} \leq C_1.$$

Proof of Proposition 4.5. First note that for every β in \mathbb{R} , we have

$$(4.17) \quad P(-\beta \log Df) \geq h_{\delta_0} - \beta \int \log Df \, d\delta_0 = 0.$$

On the other hand, if C_1 is the constant from Lemma 4.6, then for every n in \mathbb{N} we have

$$(4.18) \quad 1 \geq \sum_{\substack{J \text{ connected} \\ \text{component of } f^{-n}(J_0)}} |J| \geq C_1^{-1} |J_0| \sum_{x \in f^{-n}(1)} Df^n(x)^{-1}$$

and therefore

$$(4.19) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in f^{-n}(1)} Df^n(x)^{-1} \leq 0.$$

By (2.7) in Lemma 2.3 with $\varphi = -\log Df$ and $x = 1$, the left side of the previous inequality is equal to $P(-\log Df)$. Thus, for every β in $[1, +\infty)$ we have

$$(4.20) \quad P(-\beta \log Df) \leq P(-\log Df) \leq 0$$

and therefore $P(-\beta \log Df) = 0$. To complete the proof of item 1, it remains to prove that for every β in $(0, 1)$ we have $P(-\beta \log Df) > 0$. For each n in \mathbb{N} , denote by K_n the maximal invariant set of f on $[x_n, 1]$. The map f is uniformly expanding and topologically exact on K_n . So, by BOWEN's formula, for every β in $(0, \text{HD}(K_n))$ we have

$$(4.21) \quad P(-\beta \log Df) \geq P(f|_{K_n}, -\beta \log Df) > 0,$$

see, for example, [PU10, Theorem 9.1.6 and Corollary 9.1.7]. Thus, to complete the proof of item 1 it is sufficient to show

$$(4.22) \quad \sup_{n \in \mathbb{N}} \text{HD}(K_n) = 1.$$

This is given, for example, by [URM22, Theorem 19.6.4] applied to the "CGDMS" generated by the first return map of f to J_0 . \square

The proof of Proposition 4.3 also relies on the following lemma. Recall that for γ in $(0, +\infty)$ we denote by ω_γ the function in $C^\gamma(\mathbb{R})$ given by $\omega_\gamma(x) = -x^\gamma$.

Lemma 4.7. *For every γ in $(0, 1]$ satisfying $\gamma < \alpha$ and every τ in $(-\infty, 0)$, the potential $\tau\omega_\gamma - \log Df$ is outside $\mathcal{G}(\gamma)$ and satisfies*

$$(4.23) \quad \beta_{\text{crit}, \gamma}(\tau\omega_\gamma - \log Df) = +\infty.$$

Proof. Fix τ in $(-\infty, 0)$, put

$$(4.24) \quad \varphi_0 := \tau\omega_\gamma - \log Df,$$

and let ε in $(0, -\tau)$ be given. By our hypothesis $\gamma < \alpha$, there is N in \mathbb{N} such that for every x in $[0, x_N]$ we have $\varphi_0(x) \geq \varepsilon x^\gamma$. On the other hand, by (2.4) in Lemma 2.2 there is C_\bullet in $(0, +\infty)$ such that for every i in \mathbb{N}_0 we have

$$(4.25) \quad x_i^\gamma \geq \frac{C_\bullet}{i^{\frac{\gamma}{\alpha}}}.$$

Thus, if for a given n in \mathbb{N} satisfying $n \geq N + 1$ we denote by p_n the unique periodic point of f of period n in J_{n-1} , then we have

$$(4.26) \quad S_{n-N}(\varphi_0)(p_n) \geq \varepsilon \sum_{j=0}^{n-N-1} f^j(p_n)^\gamma \geq \varepsilon \sum_{i=N+1}^n x_i^\gamma \geq \varepsilon C_\bullet \sum_{i=N+1}^n \frac{1}{i^{\frac{\gamma}{\alpha}}}.$$

Hence,

$$(4.27) \quad S_n(\varphi_0)(p_n) \geq \varepsilon C_\bullet \sum_{i=N+1}^n \frac{1}{i^{\frac{\gamma}{\alpha}}} + N \inf_{[0,1]} \varphi_0 \text{ and } \lim_{n \rightarrow +\infty} S_n(\varphi_0)(p_n) = +\infty.$$

This proves that for every sufficiently large n , we have

$$(4.28) \quad \frac{1}{n} S_n(\varphi_0)(p_n) > 0 = \varphi_0(0)$$

and therefore δ_0 is not a ground state of f for the potential φ_0 . That is, φ_0 is outside $\mathcal{G}(\gamma)$. Together with Corollary 2 in §1.3, this yields (4.23). \square

Proof of Proposition 4.4. Item 1 is a direct consequence of the equality $\beta_{\text{crit},\gamma}(-\log Df) = 1$ given by Proposition 4.5 and of (4.23) in Lemma 4.7.

To prove item 2, note that the equality $\beta_{\text{crit},\gamma}(-\log Df) = 1$ and Corollary 1 in §1.2 imply that $-\log Df$ is in $\mathcal{G}(\gamma)$. Together with Lemma 4.7, this implies that $-\log Df$ is in $\partial\mathcal{G}(\gamma)$. To prove the last assertion, note that by the equality $\beta_{\text{crit},\gamma}(-\log Df) = 1$ and Corollary 1 in §1.2, for every β in $(0, 1)$ the potential $-\beta \log Df$ is in $\mathcal{I}(\gamma)$ and for every β in $[1, +\infty)$ the potential $-\beta \log Df$ is outside $\mathcal{I}(\gamma)$. On the other hand, Corollaries 1 in §1.2 and (4.23) in Lemma 4.7 imply that for all β in $(0, +\infty)$ and τ in $(-\infty, 0)$ we have

$$(4.29) \quad \beta_{\text{crit},\gamma}(\tau\omega_\gamma - \beta \log Df) = \beta^{-1} \beta_{\text{crit},\gamma} \left(\frac{\tau}{\beta} \omega_\gamma - \log Df \right) = +\infty$$

and therefore the potential $\tau\omega_\gamma - \beta \log Df$ is in $\mathcal{I}(\gamma)$ by Corollary 1 in §1.2. Fixing β in $[1, +\infty)$ and letting $\tau \rightarrow 0$, we conclude that $-\beta \log Df$ is in $\partial\mathcal{I}(\gamma)$. This last set is equal to $\mathcal{PT}(\gamma)$, so $-\beta \log Df$ is in $\mathcal{PT}(\gamma)$.

To prove item 3, recall that $\mathcal{PT}(\gamma)$ is closed by definition and note the function $\mathbb{R} \mapsto C^\gamma(\mathbb{R})$ given by $\beta \mapsto -\beta \log Df$ is real-analytic. If $\mathcal{PT}(\gamma)$ were a co-dimension 1 real-analytic subset of $C^\gamma(\mathbb{R})$, then combining item 2 with Lemma 2.1 would lead to a contradiction. This completes the proof of item 3.

To prove the last assertion of the proposition, note that for every β in $(1, +\infty)$ the potential $-\beta \log Df$ is in $\mathcal{PT}(\gamma)$ by item 2 and we have

$$(4.30) \quad \beta_{\text{crit},\gamma}(-\beta \log Df) = \beta \beta_{\text{crit},\gamma}(-\log Df) = \beta > 1$$

by Proposition 4.5. Together with Proposition 4.3 this proves that $\beta_{\text{crit},\gamma}^{-1}(1)$ is strictly contained in $\mathcal{PT}(\gamma)$. The remaining assertions are a direct consequence of items 1 and 2 and of Proposition 4.3. \square

Example 4.8. Fix β_0 in $(1, +\infty)$ and let $(\tilde{\varphi}_\tau)_{\tau \in \mathbb{R}}$ be the real-analytic family of potentials in $C^\gamma(\mathbb{R})$, defined by

$$(4.31) \quad \tilde{\varphi}_\tau := \tau\omega_\gamma - \beta_0 \log Df.$$

We show that the function $\tau \mapsto P(\tilde{\varphi}_\tau)$ is real-analytic on $\mathbb{R} \setminus \{0\}$, but fails to be real-analytic at 0. Since for $\tau = 0$ we have

$$(4.32) \quad \beta_{\text{crit},\gamma}(\tilde{\varphi}_0) = \beta_0^{-1} \beta_{\text{crit},\gamma}(-\log Df) = \beta_0^{-1} < 1$$

by Proposition 4.5, this provides an example of a phase transition at a potential where no phase transition in temperature takes place. Note first that for every τ in $(-\infty, 0)$, we have

$$(4.33) \quad \beta_{\text{crit},\gamma}(\tilde{\varphi}_\tau) = \beta_0^{-1} \beta_{\text{crit},\gamma} \left(\frac{\tau}{\beta_0} \omega_\gamma - \log Df \right) = +\infty$$

by Lemma 4.7 and therefore the potential $\tilde{\varphi}_\tau$ is in $\mathcal{I}(\gamma)$ by Corollary 1 in §1.2. On the other hand, for $\tau = 0$ the potential $\tilde{\varphi}_0$ is in $\mathcal{PT}(\gamma)$ by Proposition 4.4(2). It follows that for every τ in $(0, +\infty)$ the potential φ_τ is in $\mathcal{S}(\gamma)$ by Lemma 3.5. Combined with Theorem A, this implies that $\tau \mapsto P(\tilde{\varphi}_\tau)$ is real-analytic on $\mathbb{R} \setminus \{0\}$. On the other hand, the function $\tau \mapsto P(\tilde{\varphi}_\tau)$ fails to be real-analytic at 0 by Lemma 3.6.

4.3. Persistence of phase transitions in temperature when $\gamma = \alpha$. In this section, we prove the following characterization of potentials undergoing a phase transition in temperature in the case where $\alpha \leq 1$ and $\gamma = \alpha$. This result is the main ingredient in the proof of Theorem D, which is given in §4.4.

Theorem D'. *If α is in $(0, 1]$, then for every φ in $C^\alpha(\mathbb{R})$ the following properties are equivalent.*

1. *The potential φ is in $\text{int}(\mathcal{G}(\alpha))$.*
2. *The potential φ undergoes a phase transition in temperature.*
3. *The measure δ_0 is the unique ground state of f for the potential φ and there are C and δ in $(0, +\infty)$, such that for every n in \mathbb{N} we have*

$$(4.34) \quad \exp(S_n(\varphi)(x_n) - n\varphi(0)) \leq \frac{C}{n^\delta}.$$

4. *There are constants \tilde{C} and $\tilde{\delta}$ in $(0, +\infty)$, such that the following property holds. For every $\tilde{\varphi}$ in $C^\alpha(\mathbb{R})$ close to φ and all n in \mathbb{N} and x in $(0, 1]$ such that $f^n(x)$ is in J_0 , we have*

$$(4.35) \quad \exp(S_n(\tilde{\varphi})(x) - n\tilde{\varphi}(0)) \leq \frac{\tilde{C}}{n^{\tilde{\delta}}}.$$

The proof of this proposition is at the end of this section, after a couple of lemmas.

Lemma 4.9. *If α is in $(0, 1]$, then there is D in $(0, +\infty)$ such that for every φ in $C^\alpha(\mathbb{R})$ the following property holds. For every n in \mathbb{N} , every connected component J of $f^{-n}(J_0)$, and all x and x' in J , we have*

$$(4.36) \quad |S_n(\varphi)(x) - S_n(\varphi)(x')| \leq D|\varphi|_\alpha.$$

Proof. Let ε_1 be the constant from Lemma 4.6 and put

$$(4.37) \quad D := |J_0|^\alpha \sum_{k=1}^{+\infty} \frac{1}{(1 + \varepsilon_1 k)^{1+\alpha}}.$$

Let n , x , x' , and J be as in the statement of the lemma. By Lemma 4.6, for every j in $\{0, \dots, n-1\}$ we have

$$(4.38) \quad |f^j(J)| \leq \frac{|J_0|}{(1 + \varepsilon_1(n-j))^{\frac{1}{\alpha}+1}}.$$

Hence

$$(4.39) \quad |S_n(\varphi)(x) - S_n(\varphi)(x')| \leq |\varphi|_\alpha \sum_{j=0}^{n-1} |f^j(x) - f^j(x')|^\alpha \leq |\varphi|_\alpha \sum_{j=0}^{n-1} |f^j(J)|^\alpha \leq D|\varphi|_\alpha. \quad \square$$

Lemma 4.10. *Suppose α is in $(0, 1]$ and let D be the constant from Lemma 4.9. If φ in $C^\alpha(\mathbb{R})$ satisfies*

$$(4.40) \quad \sum_{n=1}^{+\infty} \exp(S_n(\varphi)(x_n) - n\varphi(0)) > \exp(D|\varphi|_\alpha),$$

then $P(\varphi) > \varphi(0)$.

Proof. Consider the formal power series Ξ and Φ in the variable s , defined by

$$(4.41) \quad \Xi(s) := \sum_{n=1}^{+\infty} \sum_{x \in f^{-n}(1)} \exp(S_n(\varphi)(x) - n\varphi(0)) s^n$$

and

$$(4.42) \quad \Phi(s) := \exp(-D|\varphi|_\alpha) \sum_{n=1}^{+\infty} \exp(S_n(\varphi)(x_n) - n\varphi(0)) s^n.$$

By (2.7) in Lemma 2.3, the convergence radius R of $\Xi(s)$ satisfies

$$(4.43) \quad R = \exp(\varphi(0) - P(\varphi)).$$

To prove that R is strictly less than 1, note that $f|_{J_0}^{-1}$ induces a bijection between the iterated preimages of 1 by f and the iterated preimages of 1 by F . Furthermore, for every n in \mathbb{N} and x in $f^{-n}(1)$ we have the following estimate. Put $y := f|_{J_0}^{-1}(x)$, let k in \mathbb{N} be such that

$$(4.44) \quad \sum_{j=0}^{k-1} m(F^j(y)) = n + 1,$$

and note that $F^k(y) = 1$. Applying Lemma 4.9 repeatedly, we obtain

$$(4.45) \quad S_n(\varphi)(x) \geq S_{n+1}(\varphi)(y) - \sup_{J_0} \varphi \geq \sum_{j=0}^{k-1} (S_{m(F^j(y))}(\varphi)(x_{m(F^j(y))}) - D|\varphi|_\alpha) - \sup_{J_0} \varphi.$$

We conclude that the power series in s

$$(4.46) \quad \exp\left(\varphi(0) - \sup_{J_0} \varphi\right) (\Phi(s) + \Phi(s)^2 + \dots)$$

has coefficients smaller than or equal to the corresponding coefficients of $\Xi(s)$. In particular, the convergence radius \tilde{R} of (4.46) satisfies $R \leq \tilde{R}$. On the other hand, our hypothesis (4.40) implies that evaluating $\Phi(s)$ at $s = 1$, we obtain $\Phi(1) > 1$. It thus follows that there is s_0 in $(0, 1)$, such that $\Phi(s_0)$ is finite and $\Phi(s_0) \geq 1$. Together with (4.43), this implies

$$(4.47) \quad \exp(\varphi(0) - P(\varphi)) = R \leq \tilde{R} \leq s_0 < 1 \text{ and } P(\varphi) > \varphi(0). \quad \square$$

Proof of Theorem D'. The implication $1 \Rightarrow 2$ is a direct consequence of Theorem C. The proofs of the implications $2 \Rightarrow 3$, $3 \Rightarrow 4$, and $4 \Rightarrow 1$ are below. Let D be the constant from Lemma 4.9 and note that by (2.4) in Lemma 2.2, there is C_\bullet in $(0, +\infty)$ such that for every i in \mathbb{N}_0 we have

$$(4.48) \quad x_i^\alpha \geq \frac{C_\bullet}{i}.$$

Given a continuous potential $\psi: [0, 1] \rightarrow \mathbb{R}$ and n in \mathbb{N} , write

$$(4.49) \quad \zeta_n(\psi) := \exp(S_n(\psi)(x_n) - n\psi(0)).$$

$2 \Rightarrow 3$. Suppose φ undergoes a phase transition in temperature. Then there is β in $(0, +\infty)$ such that $P(\beta\varphi) = \beta\varphi(0)$ (Corollary 1 in §1.2) and δ_0 is the unique ground state of f for the

potential φ (Corollary 4 in §2.2). So, by Lemma 4.10 with φ replaced by $\beta\varphi$ we have

$$(4.50) \quad \sum_{n=1}^{+\infty} \zeta_n(\varphi)^\beta \leq \exp(D\beta|\varphi|_\alpha).$$

On the other hand, by (3.5) with $\gamma = \alpha$ and $\psi = \varphi$ and by (4.48), for all k and ℓ in \mathbb{N} we have

$$(4.51) \quad S_\ell(\varphi)(x_{k+\ell}) - \ell\varphi(0) \geq -|\varphi|_\alpha \sum_{j=0}^{\ell-1} x_{k+\ell-j}^\alpha \geq -|\varphi|_\alpha C_\bullet \sum_{i=k+1}^{k+\ell} \frac{1}{i} \geq -|\varphi|_\alpha C_\bullet \log \left(\frac{k+\ell}{k} \right).$$

Assuming in addition $\ell \leq k$, we obtain

$$(4.52) \quad \zeta_{k+\ell}(\varphi) \geq \exp(-(C_\bullet \log 2)|\varphi|_\alpha) \zeta_k(\varphi).$$

Combined with (4.50), this yields

$$(4.53) \quad \exp(\beta D|\varphi|_\alpha) \geq \sum_{n=1}^{+\infty} \zeta_n(\varphi)^\beta \geq \sum_{n=k+1}^{2k} \zeta_n(\varphi)^\beta \geq k \exp(-(\beta C_\bullet \log 2)|\varphi|_\alpha) \zeta_k(\varphi)^\beta$$

and therefore

$$(4.54) \quad \zeta_k(\varphi) \leq \frac{\exp((D + C_\bullet \log 2)|\varphi|_\alpha)}{k^{\frac{1}{\beta}}}.$$

This completes the proof of item 3.

3 \Rightarrow 4. Suppose item 3 holds, put

$$(4.55) \quad \tilde{\delta} := \frac{\delta}{3}, \tilde{D} := \exp \left(D(|\varphi|_\alpha + C_\bullet^{-1} \tilde{\delta}) \right),$$

and let m_0 in \mathbb{N} be sufficiently large so that for every m in \mathbb{N} satisfying $m \geq m_0$ we have

$$(4.56) \quad \exp \left(|\varphi|_\alpha + C_\bullet^{-1} \tilde{\delta} \right) \tilde{D} \frac{C \exp(\tilde{\delta})}{(m-1)^{2\tilde{\delta}}} \leq \frac{\tilde{D}^{-1}}{(m+1)^{\tilde{\delta}}}.$$

Furthermore, put

$$(4.57) \quad \iota := \sup \left\{ \int \varphi \, d\nu : \nu \in \mathcal{M}, \text{supp}(\nu) \subseteq [x_{m_0}, 1] \right\} \text{ and } \kappa := \exp(\iota - \varphi(0)).$$

The supremum above is attained, so our hypothesis that δ_0 is the unique ground state of f for the potential φ implies

$$(4.58) \quad \iota < \varphi(0) \text{ and } \kappa < 1.$$

Choose $\tilde{\kappa}$ in $(\kappa, 1)$ and let \tilde{K} in $(0, +\infty)$ be sufficiently large, so that for every m in \mathbb{N} we have

$$(4.59) \quad \tilde{\kappa}^m \leq \frac{\tilde{K}}{(m+1)^{\tilde{\delta}}}.$$

Let Δ in $C^\alpha(\mathbb{R})$ be such that

$$(4.60) \quad |\Delta|_\alpha < \min \left\{ \log \frac{\tilde{\kappa}}{\kappa}, C_\bullet^{-1} \tilde{\delta} \right\}$$

and put $\tilde{\varphi} := \varphi + \Delta$. Note that for every measure ν in \mathcal{M} satisfying $\text{supp}(\nu) \subseteq [x_{m_0}, 1]$, we have

$$(4.61) \quad \exp \left(\int \tilde{\varphi} \, d\nu - \tilde{\varphi}(0) \right) \leq \kappa \exp \left(\int \Delta \, d\nu - \Delta(0) \right) \leq \tilde{\kappa}.$$

On the other hand, by the definition of \tilde{D} in (4.55) we have

$$(4.62) \quad \exp(D|\tilde{\varphi}|_\alpha) \leq \tilde{D}.$$

Before proving (4.35), we establish a couple of preliminary estimates. Let x' in J_0 and k in \mathbb{N} be such that for every ℓ in $\{0, \dots, k-1\}$ we have $m(F^\ell(x')) \leq m_0$. Put $n' := \sum_{\ell=0}^{k-1} m(F^\ell(x'))$ and note that $f^{n'}(x') = F^k(x')$. Then, for every i in $\{0, \dots, n'-1\}$ the connected component J of $f^{-n'}(J_0)$ containing x' satisfies

$$(4.63) \quad f^i(J) \subseteq \bigcup_{m=0}^{m_0-1} J_m = (x_{m_0}, 1].$$

In particular, if p is the unique periodic point of f of period n' in J , then the orbit of p is contained in $(x_{m_0}, 1]$. Thus, by Lemma 4.9, (4.61), and (4.62) we have

$$(4.64) \quad \exp(S_{n'}(\tilde{\varphi})(x') - n'\tilde{\varphi}(0)) \leq \exp(D|\tilde{\varphi}|_\alpha) \exp(S_{n'}(\tilde{\varphi})(p) - n'\tilde{\varphi}(0)) \leq \tilde{D}\tilde{\kappa}^{n'}.$$

To establish the second preliminary estimate, note that by (4.48) for every m in \mathbb{N} we have

$$(4.65) \quad S_m(\omega_\alpha)(x_m) = \sum_{j=0}^{m-1} x_{m-j}^\alpha \geq -C_\bullet \sum_{i=1}^m \frac{1}{i} \geq -C_\bullet(1 + \log m).$$

Together with (3.5) with $\gamma = \alpha$ and $\psi = \Delta$ and with (4.34) and (4.60), this implies

$$(4.66) \quad \zeta_m(\tilde{\varphi}) = \zeta_m(\varphi)\zeta_m(\Delta) \leq \frac{C}{m^\delta} \exp(-|\Delta|_\alpha S_m(\omega_\alpha)(x_m)) \leq \frac{C \exp(C_\bullet |\Delta|_\alpha)}{m^{\delta-C_\bullet |\Delta|_\alpha}} < \frac{C \exp(\tilde{\delta})}{m^{2\tilde{\delta}}}.$$

We now proceed to the proof of (4.35). Let n in \mathbb{N} and x in $(0, 1]$ be such that $f^n(x)$ is in J_0 . Suppose first that x is in J_0 . Let k in \mathbb{N} be such that $\sum_{\ell=0}^{k-1} m(F^\ell(x)) = n$, note that $F^k(x) = f^n(x)$, and for each ℓ in $\{0, \dots, k-1\}$ put

$$(4.67) \quad z_\ell := F^\ell(x) \text{ and } m_\ell := m(z_\ell).$$

If for every ℓ in $\{0, 1, \dots, k-1\}$ we have $m_\ell \leq m_0$, then by (4.59) and (4.64) we have

$$(4.68) \quad \exp(S_n(\tilde{\varphi})(x) - n\tilde{\varphi}(0)) \leq \tilde{D}\tilde{\kappa}^n \leq \tilde{D} \frac{\tilde{K}}{(n+1)^{\tilde{\delta}}}.$$

Suppose there is ℓ in $\{0, 1, \dots, k-1\}$ satisfying $m(F^\ell(x)) \geq m_0 + 1$. Let $(\ell_j)_{j=1}^s$ be the increasing sequence of all such. Then, for each j in $\{1, \dots, s\}$ the point $f(z_{\ell_j})$ is in $J_{m_{\ell_j}-1}$ and therefore by Lemma 4.9, (4.56) with $m = m_{\ell_j}$, (4.60), (4.62), and (4.66) with $m = m_{\ell_j} - 1$, we have

$$(4.69) \quad \begin{aligned} \exp(S_{m_{\ell_j}}(\tilde{\varphi})(z_{\ell_j}) - m_{\ell_j}\tilde{\varphi}(0)) &\leq \exp(\tilde{\varphi}(z_{\ell_j}) - \tilde{\varphi}(0)) \tilde{D}\zeta_{m_{\ell_j}-1}(\tilde{\varphi}) \\ &\leq \exp(|\tilde{\varphi}|_\alpha) \tilde{D} \frac{C \exp(\tilde{\delta})}{(m_{\ell_j} - 1)^{2\tilde{\delta}}} \leq \frac{\tilde{D}^{-1}}{(m_{\ell_j} + 1)^{\tilde{\delta}}}. \end{aligned}$$

If $s = k$, then for every ℓ in $\{0, \dots, k-1\}$ we have $m_\ell \geq m_0 + 1$ and

$$(4.70) \quad \exp(S_n(\tilde{\varphi})(x) - n\tilde{\varphi}(0)) = \prod_{\ell=0}^{k-1} \exp(S_{m_\ell}(\tilde{\varphi})(z_\ell) - m_\ell \tilde{\varphi}(0)) \leq \tilde{D}^{-k} \prod_{\ell=0}^{k-1} \frac{1}{(m_\ell + 1)^{\tilde{\delta}}} \\ \leq \tilde{D}^{-k} \frac{1}{(n+1)^{\tilde{\delta}}}.$$

Suppose $s < k$, put $\ell_0 := -1$ and $\ell_{s+1} := k$, and let $(j(i))_{i=1}^t$ be the increasing sequence of all those j in $\{0, \dots, s\}$ such that $\ell_{j+1} \geq \ell_j + 2$. For each i in $\{1, \dots, t\}$ put

$$(4.71) \quad M_i := \sum_{\ell=\ell_{j(i)}+1}^{\ell_{j(i)+1}-1} m_\ell$$

and note that for every ℓ in $\{\ell_{j(i)} + 1, \dots, \ell_{j(i)+1} - 1\}$ we have $m_\ell \leq m_0$. Furthermore,

$$(4.72) \quad t \leq s + 1 \text{ and } \sum_{i=1}^t M_i = n - \sum_{j=1}^s m_{\ell_j}.$$

Thus, by (4.64) for every i in $\{1, \dots, t\}$ we have

$$(4.73) \quad \exp(S_{M_i}(\tilde{\varphi})(z_{\ell_{j(i)+1}}) - M_i \tilde{\varphi}(0)) \leq \tilde{D} \tilde{\kappa}^{M_i}.$$

Together with (4.59), (4.69), and (4.72), this implies

$$(4.74) \quad \exp(S_n(\tilde{\varphi})(x) - n\tilde{\varphi}(0)) \\ = \prod_{j=1}^s \exp(S_{m_{\ell_j}}(\tilde{\varphi})(z_{\ell_j}) - m_{\ell_j} \tilde{\varphi}(0)) \cdot \prod_{i=1}^t \exp(S_{M_i}(\tilde{\varphi})(z_{\ell_{j(i)+1}}) - M_i \tilde{\varphi}(0)) \\ \leq \tilde{D}^{t-s} \left(\prod_{j=1}^s \frac{1}{(m_{\ell_j} + 1)^{\tilde{\delta}}} \right) \tilde{\kappa}^{\sum_{i=1}^t M_i} \\ \leq \tilde{D} \tilde{K} \frac{1}{\left(\sum_{j=1}^s m_{\ell_j} + 1 \right)^{\tilde{\delta}}} \cdot \frac{1}{\left(\sum_{i=1}^t M_i + 1 \right)^{\tilde{\delta}}} \\ \leq \tilde{D} \tilde{K} \frac{1}{(n+1)^{\tilde{\delta}}}.$$

Together with (4.68) and (4.70) this completes the proof that in the case where x is in J_0 , we have

$$(4.75) \quad \exp(S_n(\tilde{\varphi})(x) - n\tilde{\varphi}(0)) \leq \frac{\max\{\tilde{D}\tilde{K}, \tilde{D}^{-1}\}}{(n+1)^{\tilde{\delta}}}.$$

To treat the general case, put

$$(4.76) \quad \hat{C} := \tilde{D}C \exp(\tilde{\delta}) \max\{\tilde{D}\tilde{K}, \tilde{D}^{-1}\},$$

suppose x is not J_0 , and let m in \mathbb{N} be such that J_m contains x . Then $m \leq n$, $f^m(x)$ is in J_0 , and by Lemma 4.9 with n replaced by m and with $x' = x_m$, (4.62), and (4.66) we have

$$(4.77) \quad \exp(S_m(\tilde{\varphi})(x) - m\tilde{\varphi}(0)) \leq \tilde{D}\zeta_m(\tilde{\varphi}) \leq \frac{\tilde{D}C \exp(\tilde{\delta})}{m^{2\tilde{\delta}}}.$$

In the case where $n > m$, by (4.75) with n replaced by $n - m$ and x replaced by $f^m(x)$ this implies

$$(4.78) \quad \exp(S_n(\tilde{\varphi})(x) - n\tilde{\varphi}(0)) \leq \frac{\tilde{D}C \exp(\tilde{\delta})}{m^{2\tilde{\delta}}} \exp(S_{n-m}(\tilde{\varphi})(f^m(x)) - (n-m)\tilde{\varphi}(0)) \\ \leq \frac{\hat{C}}{m^{2\tilde{\delta}} \cdot (n-m+1)^{\tilde{\delta}}} \leq \frac{\hat{C}}{n^{\tilde{\delta}}}.$$

Together with (4.75) and (4.77), this completes the proof of item 4 in all of the cases.

4 \Rightarrow 1. Suppose there are \tilde{C} and $\tilde{\delta}$ in $(0, +\infty)$ such that (4.35) holds for every $\tilde{\varphi}$ in $C^\alpha(\mathbb{R})$ close to φ . Let ν be an ergodic measure in \mathcal{M} different from δ_0 . Then, ν charges J_0 and by the BIRKHOFF ergodic theorem there is x_\bullet in J_0 such that

$$(4.79) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} S_n(\tilde{\varphi})(x_\bullet) = \int \tilde{\varphi} d\nu$$

and such that there are arbitrarily large n for which $f^n(x_\bullet)$ is in J_0 . For every such n , we have by (4.35) with $x = x_\bullet$

$$(4.80) \quad S_n(\tilde{\varphi})(x_\bullet) - n\tilde{\varphi}(0) \leq -\tilde{\delta} \log n + \log \tilde{C}.$$

Together with (4.79), this implies

$$(4.81) \quad \int \tilde{\varphi} d\nu \leq \tilde{\varphi}(0).$$

That is, δ_0 is a ground state of f for the potential $\tilde{\varphi}$ and therefore $\tilde{\varphi}$ is in $\mathcal{G}(\alpha)$. This proves that $\mathcal{G}(\alpha)$ contains a neighborhood of φ in $C^\alpha(\mathbb{R})$ and therefore φ is in $\text{int}(\mathcal{G}(\alpha))$. \square

4.4. Proof of Theorem D. That every phase transition in temperature is persistent follows from the equivalence $1 \Leftrightarrow 2$ in Theorem D' in §4.3.

To prove (1.10), let φ in $\mathcal{PT}(\alpha)$ be given. Then $P(\varphi) = \varphi(0)$ by Theorem 1 and φ undergoes a phase transition in temperature by Corollary 1 in §1.2. This phase transition in temperature is persistent in $C^\alpha(\mathbb{R})$, so Theorem C implies that φ is in $\text{int}(\mathcal{G}(\alpha))$. This proves $\mathcal{PT}(\alpha) \subseteq \text{int}(\mathcal{G}(\alpha))$. Together with Proposition 4.3, this inclusion implies

$$(4.82) \quad \mathcal{PT}(\alpha) = \mathcal{PT}(\alpha) \cap \text{int}(\mathcal{G}(\alpha)) = \{\varphi \in \text{int}(\mathcal{G}(\alpha)) : \beta_{\text{crit},\alpha}(\varphi) = 1\}.$$

Thus, to complete the proof of (1.10), it is sufficient to show that $\beta_{\text{crit},\alpha} = +\infty$ on $C^\alpha(\mathbb{R}) \setminus \text{int}(\mathcal{G}(\alpha))$. To do this, let φ be in $C^\alpha(\mathbb{R}) \setminus \text{int}(\mathcal{G}(\alpha))$ and note that for every β in $(0, +\infty)$ the potential $\beta\varphi$ is outside $\mathcal{S}(\alpha)$ by Proposition 4.1 and it is outside $\mathcal{PT}(\alpha)$ because $\mathcal{PT}(\alpha) \subseteq \text{int}(\mathcal{G}(\alpha))$. Hence $\beta\varphi$ is in $\mathcal{I}(\gamma)$. Since this holds for every β in $(0, +\infty)$, by Corollary 1 in §1.2 the potential φ does not undergo a phase transition in temperature and we have $\beta_{\text{crit},\alpha}(\varphi) = +\infty$ by definition. The proof of (1.10) is thus complete.

It remains to show that $\beta_{\text{crit},\alpha}$ is continuous. Since $\beta_{\text{crit},\alpha} = +\infty$ on $C^\alpha(\mathbb{R}) \setminus \text{int}(\mathcal{G}(\alpha))$, this follows from Corollary 2 in §1.3 and the lower semicontinuity of $\beta_{\text{crit},\alpha}$ given by Proposition 4.3.

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