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The cohomological equation over dynamical systems arising from Delone sets

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Abstract. The hull Ω of an aperiodic repetitive Delone set P in $\mathbb{R}^d$ is a compact metric space on which $\mathbb{R}^d$ acts continuously by translation. Let G be $\mathbb{R}^m$ or $\mathbb{T}^m$ and α be a continuous G-cocycle over the dynamical system $(\Omega, \mathbb{R}^d)$. In this paper we study conditions under which the cohomological equation $\alpha(\omega, x) = \psi(\omega - x) - \psi(\omega)$ has continuous solutions. We give a sufficient condition for general continuous G-cocycles and a necessary condition for transversally locally constant G-cocycles. These conditions are given in terms of the set of first return vectors associated with a tower system for Ω. For linearly repetitive Delone sets we give a necessary and sufficient condition for solving the cohomological equation in the class of transversally Hölder G-cocycles.

1. *Introduction*

We recall some basic notions about aperiodic repetitive Delone sets and their hulls. Then, we introduce the cohomological equation in which we are interested and we show in a series of examples that natural questions concerning aperiodic repetitive Delone sets are reduced to solving an equation of this type. Finally, we state the goal of this paper and describe the main results.

1.1. *Aperiodic repetitive Delone sets.* Let $\mathbb{R}^d$ be the Euclidean d-space equipped with its Euclidean norm. A subset P of $\mathbb{R}^d$ is called a Delone set if it is *uniformly discrete*, which means that there is a number $r > 0$ such that every closed ball of radius r intersects P in at most one point; and *relatively dense*, which means that there is a number $R > 0$ such that every closed ball of radius R intersects P in at least one point.

Let P be a Delone set in $\mathbb{R}^d$. For every $x \in \mathbb{R}^d$, we denote by $P - x$ the Delone set $\{p - x \mid p \in P\}$. We say that P is *aperiodic* if it does not have translational symmetries, that is, if for every x in $\mathbb{R}^d \backslash \{0\}$, $P \neq P - x$.

Let us denote by $B_r(x)$ the *closed ball* in $\mathbb{R}^d$ of radius r and centered at x. The *r-patch* of P *centered* at x in P is the set $P \cap B_r(x)$. We consider two notions of long-range order based on local properties of Delone sets. The first one says that a Delone set P has *finite local complexity* if for every $r > 0$ it has a finite number of r-patches up to translation. The second one says that P is *repetitive* if for each $r > 0$ there is a number $M > 0$ such that each closed ball of radius M contains the center of a translated copy of every possible r-patch of P.

This paper concerns repetitive aperiodic Delone sets in $\mathbb{R}^d$. Standard examples are given by the vertices of the Penrose or the octagonal tiling, see for instance Robinson's review [**Rob**]. We work with Delone sets but all results in this paper can be easily adapted to tilings of $\mathbb{R}^d$.

In the study of repetitive aperiodic Delone sets it is useful to associate a topological space with each Delone set instead of studying each Delone set separately. This approach is based on the following standard construction. Let P be an aperiodic Delone set in $\mathbb{R}^d$ with finite local complexity. The *hull* of P, denoted by Ω, is the topological space arising from the completion of $P - \mathbb{R}^d = \{P - x \mid x \in \mathbb{R}^d\}$ equipped with the following metric:

$$\rho(\omega, \omega') = \min\{\tilde{\rho}(\omega, \omega'), 2^{-1/2}\}, \tag{1}$$

where

$$\tilde{\rho}(\omega, \omega') = \inf\{\varepsilon > 0 \mid \exists y \in B_\varepsilon(0), (\omega - y) \cap B_{1/\varepsilon}(0) = \omega' \cap B_{1/\varepsilon}(0)\}.$$

We observe that the factor $2^{-1/2}$ in the definition of the metric ρ is only to verify the triangle inequality, see [**LMS**].

Notice that the chosen metric has nothing canonical. The same topology is induced by any other metric satisfying that two Delone sets are close to each other if and only if they agree in a big ball surrounding 0, up to a small translation. It is well known that the elements of Ω can be viewed as Delone sets whose patches are translated copies of the patches of P (see for instance [**Rob**] for a proof in the context of tilings that can be easily adapted to Delone sets). It is also well known that the finite local complexity of P implies that Ω is totally bounded, that is, for every $\varepsilon > 0$ there is a finite cover of Ω by open balls of radius ε. Since Ω is a complete metric space we have that it is compact. In this way, we see that the geometrical property of finite local complexity of P is reflected in the compactness of Ω.

The $\mathbb{R}^d$-action by translation on $P - \mathbb{R}^d$ extends to a continuous action on Ω and geometrical properties of P are reflected in dynamical properties of the dynamical system $(\Omega, \mathbb{R}^d)$. For example, P is repetitive if and only if the dynamical system $(\Omega, \mathbb{R}^d)$ is minimal, that is, the orbit $\{\omega - x \mid x \in \mathbb{R}^d\}$ of every $\omega \in \Omega$ is dense in Ω. For a proof, see for instance [**LP**]. In another example, P has uniform patch frequency if and only if the dynamical system $(\Omega, \mathbb{R}^d)$ is uniquely ergodic, that is, there is a unique $\mathbb{R}^d$-invariant measure on the Borel σ-algebra of Ω. For a proof, see for instance [**LMS**].

We remark that if P is aperiodic and repetitive, then the $\mathbb{R}^d$-action on Ω is free and minimal.

The hull of an aperiodic repetitive Delone set P actually inherits a richer structure than the one of a compact metric space. It is a laminated space which is locally homeomorphic to the product of $\mathbb{R}^d$ by a Cantor set [**G**]. More precisely, Ω possesses an $\mathbb{R}^d$-solenoid structure that we describe now (see [**BBG, BG**] for details). First, the set

$$\Omega^0 = \{\omega \in \Omega \mid 0 \in \omega\},$$

called the *canonical transversal* of Ω, is a Cantor set. That is, Ω has a countable basis for its topology that consists of closed and open sets (*clopen* sets) and it does not have isolated points.

Let D be a subset of $\mathbb{R}^d$ and C be a clopen set in Ω^0. A map $h : C \times D \to \Omega$ is called a *parametrization* of Ω if it is a homeomorphism onto its image. We use the notation $(h, C \times D)$ for a parametrization h with domain $C \times D \subseteq \Omega^0 \times \mathbb{R}^d$.

An *atlas* of an $\mathbb{R}^d$-solenoid structure on Ω is the data $(h_i, C_i \times D_i)_{i\in I}$ of a collection of parametrizations of Ω such that their images $U_i = h_i(C_i \times D_i)$ form an open cover of Ω and, for all $i, j \in I$, there is $v_{ij} \in \mathbb{R}^d$ such that the *transition map* $h_j^{-1} \circ h_i$ reads where it is defined as

$$h_j^{-1} \circ h_i(\omega, x) = (\omega - v_{ij}, x - v_{ij}). \tag{2}$$

Two atlases are *equivalent* if their union is again an atlas. We say that Ω endowed with an equivalent class of atlas is an $\mathbb{R}^d$*-solenoid*.

We call *canonical parametrization* a parametrization $h : C \times D \to \Omega$ of the form $h(\omega, x) = \omega - x$ for every $(\omega, x) \in C \times D$. Let $(h, C \times D)$ and $(\tilde{h}, \tilde{C} \times \tilde{D})$ be two canonical parametrizations of Ω such that

$$h(C \times D) \cap \tilde{h}(\tilde{C} \times \tilde{D}) \neq \emptyset.$$

It is easy to see that there exists $v \in \mathbb{R}^d$ such that the transition map $\tilde{h}^{-1} \circ h$ reads where it is defined as

$$\tilde{h}^{-1} \circ h(\omega, x) = (\omega - v, x - v). \tag{3}$$

Thus, it is not difficult to show the existence of an atlas of an $\mathbb{R}^d$-solenoid structure on Ω given by canonical parametrizations (see for instance [**BBG, BG**]).

By equation (3), each pair of atlases given by canonical parametrizations are always equivalent. Consider the *maximal atlas* given by the set of all canonical parametrizations. We endow Ω with the equivalent class $\mathcal{L}$ of this maximal atlas and denote by $(\Omega, \mathcal{L})$ the corresponding $\mathbb{R}^d$-solenoid. Observe that each parametrization in an atlas in $\mathcal{L}$ is a canonical parametrization.

The rigidity of the transition maps implies the following properties.

- *Laminated structure.* A *slice* is a set of the form $h(\{\omega\} \times D)$. The *leaves* of the lamination are the smallest connected subsets that contain all the slices they intersect. From equation (2), we see that these leaves are flat d-manifolds isometric to $\mathbb{R}^d$. The leaves of Ω correspond to orbits of the $\mathbb{R}^d$-action.
- *Vertical germ.* A *local vertical* is a set of the form $h(C \times \{x\})$. The transition maps map local verticals onto local verticals. This allows us to define above each point in Ω a germ of local verticals (independently of the h's).

The following definitions are used only in the first and second examples in §1.3 and are not needed for the rest of this paper.

Let $A_k(\mathbb{R}^d, \mathbb{R}^m)$ be the set of alternating k-linear maps from $\mathbb{R}^d$ to $\mathbb{R}^m$. Let D be an open subset of $\mathbb{R}^d$. We call a *k-form on D*, with values in $\mathbb{R}^m$, a smooth map $\omega : D \to A_k(\mathbb{R}^d, \mathbb{R}^m)$.

Let $(h_i, C_i \times D_i)_{i \in I}$ be an atlas of Ω. A *k-form on $U_i = h_i(C_i \times D_i)$* is a family of k-forms on the slices $D_i \times \{\omega\}$ that depends continuously on the vertical parameter ω (on the C^∞ topology).

A k-form on Ω is given by k-forms α_i on the open sets $U_i = h_i(C_i \times D_i)$ which are compatible on the intersections, that is,

$$\text{if } h_j^{-1} \circ h_i(\omega, x) = (\omega - v_{ij}, x - v_{ij}), \text{ then } \alpha_i(\omega, x) = \alpha_j(\omega - v_{ij}, x - v_{ij}).$$

We denote by $\Lambda^k(\Omega, \mathbb{R}^m)$ the set of k-forms on Ω with values in $\mathbb{R}^m$. The differential operator on the leaves defines a differential operator $d : \Lambda^k(\Omega, \mathbb{R}^m) \to \Lambda^{k+1}(\Omega, \mathbb{R}^m)$ and we say that a k-form ω is *closed* if $d\alpha = 0$.

Let V be a real vector space of finite dimension. A continuous function $f : \Omega \to V$ is *transversally locally constant* (t.l.c. for short) if it is constant on small local verticals. This notion can be extended to *t.l.c. k-forms*. A continuous function $f : \mathbb{R}^d \to V$ is *strongly P-equivariant* if there exists $r > 0$, its *range*, such that for all $x, y \in \mathbb{R}^d$, if $(P - x) \cap B_r(0) = (P - y) \cap B_r(0)$, then $f(x) = f(y)$. A function which is a uniform limit of a sequence of strongly P-equivariant functions is called weakly P-equivariant.

Remark 1. The notions of weakly and strongly P-equivariant functions were introduced by Kellendonk [**K**]. These notions are related to continuous and t.l.c. functions on Ω as follows. When restricting a continuous (respectively t.l.c.) function to a leaf of Ω we obtain a weakly (respectively strongly) P-equivariant function. Conversely, a continuous weakly (respectively strongly) P-equivariant function defines a continuous (respectively t.l.c.) function on the orbit of P which extends to all Ω. This remark also applies to k-forms (see [**KP1**]).

1.2. *Cohomological equations.* Cohomological equations have a long history in dynamical systems theory and they also appear naturally in our setting.

Let G be the group $\mathbb{R}^m$ or $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$. A continuous G-cocycle α over the $\mathbb{R}^d$-action $(\Omega, \mathbb{R}^d)$ is a continuous function $\alpha : \Omega \times \mathbb{R}^d \to G$ which satisfies the following property:

$$\alpha(\omega, x + y) = \alpha(\omega, x) + \alpha(\omega - x, y) \quad \text{for all } \omega \in \Omega \text{ and } x, y \in \mathbb{R}^d. \tag{4}$$

It turns to be a natural and important question to know whether the *cohomological equation*

$$\alpha(\omega, x) = \psi(\omega - x) - \psi(\omega) \tag{5}$$

has a measurable, continuous, or even tangential smooth solution $\psi : \Omega \to G$. We say that α is a *coboundary* if there exists a solution to equation (5). This solution is called a *transfer function*.

The aim of this paper is to study the existence of continuous solutions to the cohomological equation (5) for continuous cocycles.

Let us describe a series of examples where natural questions concerning repetitive aperiodic Delone sets boil down to the problem of solving a cohomological equation of the type (5).

We need the following definition that is fundamental in our study. A continuous cocycle α is *transversally locally constant* if it is constant on the local verticals of small enough neighborhoods, that is, if there exist constants $r > 0$, called the *range* of α, and $R > 0$ such that for all $\omega, \omega' \in \Omega$ and $x \in B_R(0)$,

$$\text{if } \omega \cap B_r(0) = \omega' \cap B_r(0), \text{ then } \alpha(\omega, x) = \alpha(\omega', x).$$

1.3. *Examples.* Let P be an aperiodic repetitive Delone set in $\mathbb{R}^d$.

1.3.1.*Strongly P-equivariant vector fields and 1-forms.* In this example we assume that $d = 3$. Let X be a smooth strongly P-equivariant vector field on $\mathbb{R}^3$ with curl $X = 0$. A natural question is to know whether X is the gradient of a weakly P-equivariant smooth function. By duality, X defines a closed strongly P-equivariant 1-form β and the problem is equivalent to finding a weakly P-equivariant smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\beta = df$.

We show that finding a solution to the equation $\beta = df$ with f weakly P-equivariant is equivalent to finding a solution to a cohomological equation of the type (5).

Using Remark 1, β and f can be respectively extended to a t.l.c. 1-form $\tilde{\beta}$ and a continuous tangential smooth function $\tilde{f}$ defined on Ω which satisfy $\tilde{\beta} = d\tilde{f}$. For every $\omega \in \Omega$ and every $x \in \mathbb{R}^3$, define

$$\beta(\omega, x) = \int_{\omega}^{\omega - x} \tilde{\beta},$$

which satisfies equation (4). Thus, $\tilde{f}$ satisfies the cohomological equation

$$\beta(\omega, x) = \tilde{f}(\omega - x) - \tilde{f}(\omega), \tag{6}$$

for all $\omega \in \Omega$ and $x \in \mathbb{R}^3$. Conversely, a continuous function $\tilde{f}$ which satisfies the cohomological equation (6) will also satisfy the equation $\tilde{\beta} = d\tilde{f}$ on Ω, and thus its restriction f to the leaf that contains P satisfies $\beta = df$.

1.3.2.*Deformation of Delone sets.* Recently, Sadun, Williams, and Clark introduced a theory of deformation of tilings [**CS, CS1, SW**]. Kellendonk [**K1**] provided a formulation of this deformation theory using P-equivariant 1-forms. More precisely, consider a bi-Lipschitz smooth map $\psi : \mathbb{R}^d \to \mathbb{R}^d$. If the closed 1-form $d\psi$ is strongly P-equivariant, then $\psi(P)$ is a Delone set with finite local complexity (see [**K1**]), in which case $d\psi$ or $\psi(P)$ is called a *deformation of P*.

Denote by Id the identity map on $\mathbb{R}^d$ and by $d\mathrm{Id} : \mathbb{R}^d \to A_1(\mathbb{R}^d, \mathbb{R}^d)$ the corresponding 1-form. Kellendonk [**K1**] proved that there is a strong relation between the cohomology class of $d\mathrm{Id}$ and the deformations that give dynamical systems topologically conjugate to $(\Omega, \mathbb{R}^d)$. More precisely, he proved that if a deformation $d\psi$ is cohomologous to $d\mathrm{Id}$, that is, there exists a weakly P-equivariant map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ such that $d\psi - d\mathrm{Id} = d\phi$, then the dynamical system associated with the hull of $\psi(P)$ is a topological factor of $(\Omega, \mathbb{R}^d)$

and the factor map sends P to $\psi(P)$. Moreover, if $d\psi$ is close enough to $d\mathrm{Id}$, then the factor map is actually a topological conjugacy.

We show that finding a solution to the equation $d\psi - d\mathrm{Id} = d\phi$ with ϕ weakly P-equivariant is equivalent to finding a solution to a cohomological equation of the type (5).

Using Remark 1, $d\psi$, $d\mathrm{Id}$, and ϕ can be respectively extended to transversally locally constant 1-forms $\tilde{d\psi}$, $\tilde{d\mathrm{Id}}$, and a continuous tangential smooth map $\tilde{\phi}$. For every $\omega \in \Omega$ and every $x \in \mathbb{R}^d$, we define $\alpha_{d\psi}(\omega, x) = \int_{\omega}^{\omega-x} \tilde{d\psi}$ and $\alpha_{d\mathrm{Id}}(\omega, x) = \int_{\omega}^{\omega-x} \tilde{d\mathrm{Id}} = x$, which satisfy equation (4). Thus, $\tilde{\phi}$ satisfies the cohomological equation

$$\alpha_{d\psi}(\omega, x) = \alpha_{d\mathrm{Id}}(\omega, x) + \tilde{\phi}(\omega - x) - \tilde{\phi}(\omega), \tag{7}$$

for all $\omega \in \Omega$ and $x \in \mathbb{R}^d$. Conversely, a continuous map $\tilde{\phi}$ on Ω which satisfies the cohomological equation (7) will also satisfy the equation $\tilde{d\psi} - \tilde{d\mathrm{Id}} = d\tilde{\phi}$ on Ω, and thus its restriction ϕ to the leaf that contains P satisfies $d\psi - d\mathrm{Id} = d\phi$.

1.3.3.*Eigenvalue problem.* Let us denote by $\langle \cdot, \cdot \rangle$ the usual inner product in $\mathbb{R}^d$ and let μ be an ergodic $\mathbb{R}^d$-invariant measure on Ω. Given a vector $\lambda \in \mathbb{R}^d$, the problem is to know whether there is a measurable function $\psi : \Omega \to S^1 \subseteq \mathbb{C}$ such that μ-almost everywhere in Ω,

$$\psi(\omega - x) = e^{2\pi i \langle \lambda, x \rangle} \psi(\omega) \quad \text{for all } x \in \mathbb{R}^d.$$

If such a function exists, then λ is called a measurable *eigenvalue* of the dynamical system $(\Omega, \mathbb{R}^d)$ and ψ is the corresponding *eigenfunction*. If the function ψ is continuous, then one says that λ is a *continuous eigenvalue*. The map $(\omega, x) \mapsto e^{2\pi i \langle \lambda, x \rangle}$ is a S^1-cocycle or in additive notation a $\mathbb{T}^1$-cocycle. Thus, λ is a (continuous) measurable eigenvalue of $(\Omega, \mathbb{R}^d)$ if and only if there is a (continuous) measurable solution $\psi : \Omega \to \mathbb{T}^1$ to the cohomological equation $\langle \lambda, x \rangle \bmod \mathbb{Z} = \psi(\omega - x) - \psi(\omega)$.

1.3.4.*Rotation-type factors.* In the previous example a continuous eigenvalue gives a topological factor on the 1-torus $\mathbb{T}^1$. We now consider other factors of 'rotation type'. Take $\theta = (\theta_1, \ldots, \theta_d)$ in $\mathbb{R}^d$ and write $[x, \theta] = (x_1\theta_1, \ldots, x_d\theta_d) \bmod \mathbb{Z}^d$. Let $\mathcal{A}_\theta : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d$ be defined by

$$\mathcal{A}_\theta(p, x) = p + [x, \theta],$$

for every $p \in \mathbb{T}^d$ and every $x \in \mathbb{R}^d$. The map $(\omega, x) \mapsto [x, \theta]$ is a $\mathbb{T}^d$-cocycle over $(\Omega, \mathbb{R}^d)$. Thus, by definition $(\mathbb{T}^d, \mathcal{A}_\theta)$ is a topological factor of $(\Omega, \mathbb{R}^d)$ if and only if there exists a continuous solution $\psi : \Omega \to \mathbb{T}^d$ to the cohomological equation

$$[x, \theta] = \psi(\omega - x) - \psi(\omega).$$

1.4. *Goal of the paper.* The aim of this paper is to derive some necessary and sufficient conditions which yield the existence of solutions to the cohomological equation (5). The main results are Theorems 4.1, 4.2, 4.3, and 4.4. Roughly speaking, Theorem 4.1 shows that the existence of continuous solutions to the cohomological equation (5) for continuous G-cocycles is ensured by the convergence of a numerical series. This series is given in terms of the sets of first return vectors (see definition in §3) associated with a tower system of the hull of P. The other theorems show that, under some hypotheses on the cocycles and the tower systems, the convergence of a similar numerical series is a necessary condition for the existence of solutions to the cohomological equation (5). In the class of linearly repetitive Delone sets, Theorems 4.1 and 4.4 give a necessary and sufficient condition on transversally Hölder G-cocycles for the existence of solutions to the cohomological equation (5).

These theorems can be seen as a generalization of Theorem 3.1 in [**CGM**], where a result about rotation-type factors of minimal $\mathbb{Z}^d$-actions on the Cantor set was stated, and also as a generalization of the results in [**BDM, CDHM**] about continuous eigenvalues of minimal $\mathbb{Z}$-actions on the Cantor set where this circle of ideas appeared for the first time. The main differences between [**CGM**, Theorem 3.1] and Theorems 4.1, 4.2, 4.3, and 4.4 in this paper are that in the latter theorems the action is continuous and they apply to a quite general class of continuous G-cocycles.

One can use the results of this paper to construct, in dimension one, examples of systems with different sets of cocycles. See for instance the examples in [**BDM, CDHM**]. The reason for this flexibility comes from the fact that the hull of an aperiodic repetitive Delone set can be described as an inverse limit of branched manifolds (see [**BBG, BG**]) and in dimension one it is quite easy to construct such inverse limits but in greater dimension it is not clear how to construct them. Thus, one way of proceeding in greater dimension is to construct 'good' tower systems for interesting classes of Delone sets and tilings, which is an ongoing research program.

In §2 we start our study of the cohomological equation (5) by reducing the problem of finding a continuous solution to equation (5) to finding solutions to the restriction of this equation to the Cantor set Ω^0. We remark that although the groupoid $G = \{(\omega, x) \in \Omega^0 \times \mathbb{R}^d \mid x \in \omega\}$, which is by definition the restriction of the groupoid $\Omega \times \mathbb{R}^d$ to Ω^0, is not in general a groupoid given by a group action, this restriction enables us to use the ideas in the study of continuous eigenvalues of minimal $\mathbb{Z}$ or $\mathbb{Z}^d$ actions on the Cantor set in [**CDHM, CGM**], respectively, for studying the existence of continuous solutions to the cohomological equation (5). We show in Lemma 2.2 that the existence of solutions to the cohomological equation (5) is completely determined by the behavior of the cocycle on the 'return times'.

In §3, we recall the definition of a tower system introduced in [**BBG, BG**]. We define the base of a box decomposition and define first return vectors to the base, which are our main tool in the study of the cohomological equation (5).

In §4, we state Theorems 4.1, 4.2, 4.3, and 4.4, and we prove Theorem 4.1. In §5, we prove Theorems 4.2, 4.3, and 4.4.

1.5. *Notation.*

- Norm on $\mathbb{R}^m$. We denote by $|\cdot|$ the l^∞-norm on $\mathbb{R}^m$: if $x = (x_1, \dots, x_m)$,

$$|x| = \sup_{1 \le i \le m} |x_i|.$$

- Metric on $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$. We choose the quotient metric on $\mathbb{T}^m$: if x and y are in $\mathbb{T}^m$

and $\tilde{x}$ and $\tilde{y}$ are liftings to $\mathbb{R}^m$, then

$$|||x - y||| = \inf_{p \in \mathbb{Z}^m} |\tilde{x} - \tilde{y} + p|.$$

Write $|||x||| = |||x - 0|||$. We have:
- $|||-x||| = |||x|||$;
- $|||x + y||| \leq |||x||| + |||y|||$.

- When we do not specialize the group $\mathbb{R}^m$ or $\mathbb{T}^m$, we use $|\cdot|$ to denote the distance to 0 in both cases.

2. *A necessary and sufficient condition for the existence of solutions to the cohomological equation in terms of return times*

Let P be an aperiodic repetitive Delone set in $\mathbb{R}^d$. First, we show that the problem of finding continuous solutions to the cohomological equation (5) can be reduced to finding solutions to the restriction of this equation to the Cantor set Ω^0. For every $\omega \in \Omega$, denote by $\mathrm{Orb}(\omega)$ the orbit of ω.

LEMMA 2.1. *Let α be a continuous G-cocycle on $(\Omega, \mathbb{R}^d)$. The following conditions are equivalent.*

(i) *The cocycle α is a continuous coboundary with continuous transfer function.*

(ii) *There exists a continuous function ψ on Ω^0 which satisfies*

$$\alpha(\omega, x) = \psi(\omega - x) - \psi(\omega),$$

for all ω and $\omega - x$ in Ω^0.

(iii) *Fix $\omega_0 \in \Omega^0$. There exists a uniformly continuous function ψ on $\Omega^0 \cap \mathrm{Orb}(\omega_0)$ which satisfies*

$$\alpha(\omega_0 - y, x) = \psi((\omega_0 - y) - x) - \psi(\omega_0 - y),$$

for all $\omega_0 - y$ and $(\omega_0 - y) - x$ in Ω^0.

Proof. We prove that (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i). We remark that (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (ii). Since ψ is uniformly continuous, we can extend it to a unique continuous function on Ω^0 that we also denote by ψ.

We prove that ψ satisfies the cohomological equation (5) on Ω^0. First, observe that since P is a Delone set, there exists $r > 0$ such that every closed ball of radius r in $\mathbb{R}^d$ intersects every ω in Ω in at most one point. Consider ω and $\omega - x$ in Ω^0. By density of the orbit of ω_0, there exists a sequence $(\omega_0 - x_n)_{n \geq 0}$ in Ω^0 which converges to ω. Hence, for n large enough,

$$(\omega_0 - x_n) \cap B_{\|x\|}(0) = \omega \cap B_{\|x\|}(0).$$

Since $\omega - x \in \Omega^0$, we deduce that $(\omega_0 - x_n) - x \in \Omega^0$. Thus, for n large enough,

$$\alpha(\omega_0 - x_n, x) = \psi((\omega_0 - x_n) - x) - \psi(\omega_0 - x_n).$$

By continuity of the $\mathbb{R}^d$-action, we have $(\omega_0 - x_n) - x \to \omega - x$ as $n \to \infty$. Therefore,

$$\begin{aligned} \psi(\omega - x) - \psi(\omega) &= \lim_{n \to \infty} \psi((\omega_0 - x_n) - x) - \psi(\omega_0 - x_n) \\ &= \lim_{n \to \infty} \alpha(\omega_0 - x_n, x) = \alpha(\omega, x). \end{aligned}$$

(ii) $\Rightarrow$ (i). We extend ψ to all Ω as follows. Let ω be in Ω and $\omega - x$ be in Ω^0; we define $\psi(\omega)$ by

$$\psi(\omega) = \psi(\omega - x) + \alpha(\omega - x, -x) = \psi(\omega - x) - \alpha(\omega, x).$$

We first prove that ψ is well defined, that is, given $\omega - y$ in Ω^0, we show that

$$\psi(\omega - y) - \alpha(\omega, y) = \psi(\omega - x) - \alpha(\omega, x).$$

Indeed, since $\omega - x$ and $\omega - y$ are in the same orbit, there exists z in $\mathbb{R}^d$ such that $x + z = y$. By hypothesis, $\psi((\omega - x) - z) - \alpha(\omega - x, z) = \psi(\omega - x)$, which implies that

$$\begin{aligned}\psi(\omega - y) - \alpha(\omega, y) &= \psi((\omega - x) - z) - \alpha(\omega, x + z)\\ &= \psi((\omega - x) - z) - \alpha(\omega, x) - \alpha(\omega - x, z)\\ &= \psi(\omega - x) - \alpha(\omega, x).\end{aligned}$$

We show now that ψ satisfies the cohomological equation. Let (ω, x) be in $\Omega \times \mathbb{R}^d$. Since ω and $\omega - x$ are in the same leaf, there exist y and z in $\mathbb{R}^d$ such that $\omega - y$ and $\omega - (x + z)$ are in Ω^0 and $y = x + z$. Thus,

$$\begin{aligned}\alpha(\omega, x) &= \alpha(\omega, y) + \alpha(\omega - y, -z)\\ &= \alpha(\omega, y) - \alpha(\omega - x, z)\\ &= \psi(\omega - y) - \psi(\omega) + \psi(\omega - x) - \psi(\omega - y)\\ &= \psi(\omega - x) - \psi(\omega).\end{aligned}$$

Finally, we prove that ψ is continuous on Ω. Fix $\epsilon > 0$. Let ω be in Ω and x be in $\mathbb{R}^d$. The continuity of ψ on Ω^0 implies that there exists $\delta' > 0$ such that for every ω' in Ω^0,

$$\text{if } \omega \cap B_{\delta'^{-1}}(0) = \omega' \cap B_{\delta'^{-1}}(0), \text{ then } |\psi(\omega) - \psi(\omega')| < \epsilon/2. \tag{8}$$

The continuity of α implies that there exists $\delta'' > 0$ such that for every $(\omega', x') \in \Omega \times \mathbb{R}^d$,

$$\text{if } \max\{\rho(\omega, \omega'), \|x - x'\|\} < \delta'', \text{ then } |\alpha(\omega, x) - \alpha(\omega', x')| < \epsilon/2. \tag{9}$$

Since P is a Delone set, there exists $R > 0$ such that every closed ball of radius R in $\mathbb{R}^d$ intersects every ω in Ω in at least one point.

Choose $\delta < \min\{\min\{\delta', \delta''\}/(1 + \min\{\delta', \delta''\}R), \delta''/2, 2^{-1/2}\}$. For every ω' in Ω, if $\rho(\omega, \omega') < \delta$, then there exist y and y' in $B_\delta(0)$ such that

$$(\omega - y) \cap B_{\delta^{-1}}(0) = (\omega' - y') \cap B_{\delta^{-1}}(0).$$

Since $\delta^{-1} > R$, there exists z in $B_R(0)$ such that $\omega - (y + z)$ and $\omega' - (y' + z)$ are in Ω^0. Moreover, since $\delta^{-1} > \min\{\delta', \delta''\}^{-1} + R$, we deduce that

$$(\omega - (y + z)) \cap B_{\delta'^{-1}}(0) = (\omega' - (y' + z)) \cap B_{\delta'^{-1}}(0)$$

and

$$\rho(\omega - (y + z), \omega' - (y' + z)) < \delta''.$$

As $\|y' - y\| \leq 2\delta < \delta''$, from equations (8) and (9) we conclude that

$$\begin{aligned}|\psi(\omega) - \psi(\omega')| &\leq |\psi(\omega - (y + z)) + \psi(\omega' - (y' + z))| + |\alpha(\omega, y + z) - \alpha(\omega', y' + z)| \\ &< \epsilon. \end{aligned}$$ □

Given a clopen set C in Ω^0, the *set of entrance times* of a Delone set $\omega \in \Omega$ to C is defined as

$$\mathcal{R}_C(\omega) = \{x \in \mathbb{R}^d \mid \omega - x \in C\}. \tag{10}$$

When the Delone set ω belongs to C, the set $\mathcal{R}_C(\omega)$ is called the *set of return times* of ω to C.

The following lemma is the main result of this section.

LEMMA 2.2. *Let P be an aperiodic repetitive Delone set in $\mathbb{R}^d$ and $(C_n)_{n\geq 0}$ be a decreasing sequence of clopen sets in Ω^0 such that $\{\omega_0\} = \bigcap_{n\geq 0} C_n$. The following conditions are equivalent.*

(i) *The cocycle α is a coboundary with continuous transfer function.*

(ii) $\sup_{x\in\mathcal{R}_{C_n}(\omega_0)} |\alpha(\omega_0, x)| \to 0$ *as $n \to \infty$.*

Remark 2. Condition (ii) in Lemma 2.2 implies that $|\alpha(\omega_0, x)|$ is uniformly bounded in $x \in \mathbb{R}^d$, which in the particular case of $\mathbb{R}$-cocycles implies (i). This can be proved by adapting the proof of a classical theorem of Gottschalk and Hedlund; see for instance [**KH**] for a proof.

Proof of Lemma 2.2. We prove that (ii) ⇒ (i). We remark that (i) ⇒ (ii) is immediate and follows from the continuity of the transfer function.

By Lemma 2.1, it is enough to prove that there exists a uniformly continuous function ψ defined on $\Omega^0 \cap \mathrm{Orb}(\omega_0)$ which satisfies

$$\alpha(\omega_0 - y, x) = \psi((\omega_0 - y) - x) + \psi(\omega_0 - y), \tag{11}$$

for all $\omega_0 - y$ and $(\omega_0 - y) - x$ in Ω^0. Define ψ on $\mathrm{Orb}(\omega_0) \cap \Omega^0$ by

$$\psi(\omega_0 - x) = \alpha(\omega_0, x).$$

It is easy to see that ψ satisfies equation (11). We prove now that ψ is uniformly continuous. Fix $\epsilon > 0$. By (ii), there exists $n \geq 0$ such that

$$|\alpha(\omega_0, x)| < \frac{\epsilon}{4}, \tag{12}$$

for all x in $\mathcal{R}_{C_n}(\omega_0)$. By minimality, there is $R > 0$ such that every closed ball of radius R in $\mathbb{R}^d$ intersects $\mathcal{R}_{C_n}(\omega_0)$. The uniform continuity of α on the compact set $\Omega^0 \times B_R(0)$ implies that there exists $\delta' > 0$ such that for all ω and ω' in Ω and x in $B_R(0)$,

$$\text{if } \omega \cap B_{\delta'^{-1}}(0) = \omega' \cap B_{\delta'^{-1}}(0), \text{ then } |\alpha(\omega, x) - \alpha(\omega', x)| < \frac{\epsilon}{2}. \tag{13}$$

Since for every $n \geq 0$, C_n is a clopen set in Ω^0, there is $\eta > 0$ satisfying the following property: for every $\omega \in \Omega^0$, if $\omega \cap B_\eta(0) = \omega' \cap B_\eta(0)$ for some $\omega' \in C_n$, then $\omega \in C_n$.

Choose $\delta < \min\{\delta'/(1+\delta' R), \eta/(1+\eta R), 2^{-1/2}\}$ and consider $\omega_0 - x$ and $\omega_0 - y$ in $\mathrm{Orb}(\omega_0) \cap \Omega^0$. If $\rho(\omega_0 - x, \omega_0 - y) < \delta$, then

$$(\omega_0 - x) \cap B_{\delta^{-1}}(0) = (\omega_0 - y) \cap B_{\delta^{-1}}(0).$$

Since $\delta^{-1} > R$, there exists $v \in (\omega_0 - x) \cap B_R(0)$ such that $x + v$ is in $\mathcal{R}_{C_n}(\omega_0)$. Since $\delta^{-1} > \eta^{-1} + R$,

$$(\omega_0 - (x+v)) \cap B_{\eta^{-1}}(0) = (\omega_0 - (y+v)) \cap B_{\eta^{-1}}(0),$$

which implies that $y + v$ is in $\mathcal{R}_{C_n}(\omega_0)$. Since $\delta^{-1} > \delta'^{-1} + R$,

$$(\omega_0 - (x+v)) \cap B_{\delta'^{-1}}(0) = (\omega_0 - (y+v)) \cap B_{\delta'^{-1}}(0).$$

From equations (12) and (13), we conclude that

$$\begin{aligned}|\psi(\omega_0 - x) - \psi(\omega_0 - y)| \leq\ & |\alpha(\omega_0, x+v) - \alpha(\omega_0, y+v)| \\ & + |\alpha(\omega_0 - (x+v), -v) - \alpha(\omega_0 - (y+v), -v)| \\ <\ & \epsilon. \end{aligned}$$ □

3. *Tower systems for the hull of aperiodic repetitive Delone sets*

In this section we recall the definition of tower systems from [**BBG, BG**]. We introduce the notion of the base of a box decomposition and define first return vectors to the base. We use first return vectors to study the existence of solutions to the cohomological equation (5) in the next section.

3.1. *Boxes and box decompositions.*

Let P be an aperiodic repetitive Delone set in $\mathbb{R}^d$. Let $(h, C \times D)$ be a canonical parametrization in the maximal atlas in $\mathcal{L}$ with $D \subseteq \mathbb{R}^d$ the interior of a finite union of polyhedrons, $0 \in D$, and $C \subseteq \Omega^0$ a clopen set. The image $h(C \times D)$ is called a *box* of Ω. We say that C is the *base* of B (with respect to the parametrization $(h, C \times D)$). A set of the form $h(C \times \{x\})$, with $x \in D$, is called a *vertical* of the box and a set of the form $h(\{\omega\} \times D)$, with $\omega \in C$, is called a *horizontal* of the box.

A *box decomposition* $\mathcal{B}$ of Ω with *base* C is a finite collection of mutually disjoint boxes whose closures cover Ω and the union of the bases of the boxes in $\mathcal{B}$ is C. If each box in $\mathcal{B}$ is parametrized by $C_i \times D_i$, then we use the notation $(C, \{C_i \times D_i\}_{i=1}^t)$ for the box decomposition $\mathcal{B}$ with base C.

Let $s > 0$. If for every vertical V in a box in $\mathcal{B}$, all Delone sets in V agree in the closed ball of center 0 and radius s, then we say that $\mathcal{B}$ is a box decomposition with *size* s. We also use the notation $(C, \{C_i \times D_i\}_{i=1}^t, s)$ for a box decomposition with base C and size s.

Let $B = h(C \times D)$ be a box. Denote by ∂D the boundary of D in $\mathbb{R}^d$. Since h is just the restriction of the $\mathbb{R}^d$-action to $C \times D$, we have that h extends continuously to $C \times \overline{D}$. Since C is a clopen set, we have that the boundary of B is equal to $h(C \times \partial D)$.

Let $\mathcal{B}$ and $\mathcal{B}'$ be two box decompositions of Ω. We say that $\mathcal{B}'$ is *zoomed out* of $\mathcal{B}$ if the following hold.

(Z1) For every box $B' \in \mathcal{B}'$, each vertical in B' is contained in a vertical or the boundary of a box in $\mathcal{B}$.

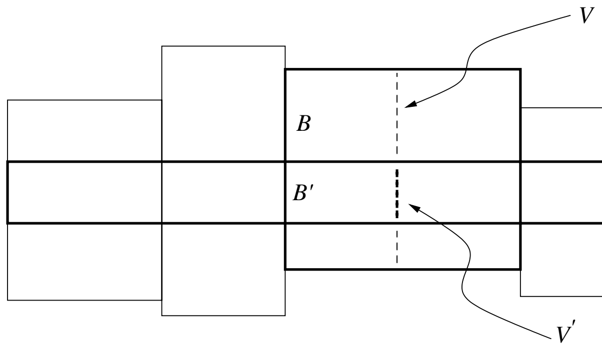

FIGURE 1. The box B' is in the box decomposition $\mathcal{B}'$, which is zoomed out of the box decomposition $\mathcal{B}$. Each vertical V' in B' is contained in a vertical V in the box $B \in \mathcal{B}$.

(Z2) The boundaries of the boxes in $\mathcal{B}'$ are contained in the boundaries of boxes in $\mathcal{B}$.
(Z3) Each horizontal in a box in $\mathcal{B}$ is contained in a horizontal of a box in $\mathcal{B}'$.

From properties (Z1)–(Z3), we have that if B' is a box in $\mathcal{B}'$ and B is a box in $\mathcal{B}$, then $B' \cap B$ is also a box whose verticals are verticals of B' and whose horizontals are horizontals of B.

We sketch a box by drawing a rectangle where the clopen set C is in the vertical and D in the horizontal. See Figure 1 for a geometric intuition of the definition of zoomed out.

Let $(\mathcal{B}_n = (C_n, \{C_{n,i} \times D_{n,i}\}_{i=1}^{t_n}, s_n))_{n\geq 0}$ be a sequence of zoomed out box decompositions of Ω. We mostly consider sequences satisfying the following conditions.
(C1) For every $n \geq 0$, $C_{n+1} \subseteq C_n$.
(C2) There exists $\omega_0 \in \Omega^0$ such that $\bigcap_{n\geq 0} C_n = \{\omega_0\}$ and, for every $n \geq 0$, $\omega_0 \in C_{n,1}$.
(C3) $s_n \to \infty$ as $n \to \infty$.

We call a *tower system* for Ω a sequence of zoomed out box decompositions satisfying (C1)–(C3). The following result was proved in [**BG**] (see also [**BBG**]).

THEOREM 3.1. *Let P be an aperiodic repetitive Delone set in $\mathbb{R}^d$. The hull Ω admits a tower system.*

Associated with each tower system $(\mathcal{B}_n)_{n\geq 0}$ is a sequence of non-negative integer matrices $(M_n)_{n\geq 1}$, where $M_n = (m_{ij}^{(n)})$ is the $t_n \times t_{n-1}$ matrix whose coefficient $m_{ij}^{(n)}$ is the number of horizontals in the box $B_{n-1,j}$ that are covered by a horizontal in the box $B_{n,i}$. Since $\mathcal{B}_n$ is zoomed out of $\mathcal{B}_{n-1}$, we have that $m_{ij}^{(n)}$ is well defined, and

$$m_{ij}^{(n)} = \{x \in D_{n,i} \mid C_{n,i} - x \subseteq C_{n-1,j}\}.$$

3.2. *Derived tilings.* It will be useful for describing the combinatorics of the tower systems to consider the tiling of $\mathbb{R}^d$ obtained by intersecting a box decomposition in a tower system with a leaf of Ω. Before defining these objects, we recall basic definitions about tilings. A *tile* T in $\mathbb{R}^d$ is a compact set that is the closure of its interior (not necessarily connected). A *tiling* $\mathcal{T}$ of $\mathbb{R}^d$ is a countable collection of tiles that cover $\mathbb{R}^d$ and have pairwise disjoint interiors. Tiles can be *decorated*: they may have a color and/or be punctured at an interior point. Formally, this means that decorated tiles are tuples (T, i, x), where i is in a finite set of *colors*, and x belongs to the interior of T. Two tiles have the same type if they differ by a translation. When the tiles are punctured, then the translation must also send one puncture to the other, and when they are colored, then they must have the same color.

The *derived tiling* of the box decomposition $\mathcal{B} = (C, \{(C_i \times D_i)\}_{i=1}^t)$ in a point ω in Ω is the tiling of $\mathbb{R}^d$ given by

$$\mathcal{T}_{\mathcal{B}}(\omega) = \{\overline{D}_i + x \mid x \in \mathcal{R}_{C_i}(\omega) \text{ for some } 1 \leq i \leq t\},$$

where we consider each tile $\overline{D}_i + x$ colored by i and punctured in x. Recall that for every $i \in \{1, \dots, t\}$, 0 belongs to D_i; thus, it makes sense to puncture $\overline{D}_i + x$ in x. The set of punctures of $\mathcal{T}_{\mathcal{B}}(\omega)$ is $\mathcal{R}_C(\omega)$.

3.3. *First return vectors.* Let $\mathcal{B} = (C, \{C_i \times D_i\}_{i=1}^t)$ be a box decomposition of Ω with base C. Write B_i for the box in $\mathcal{B}$ parametrized by $C_i \times D_i$.

A *first return vector to* C associated with $\mathcal{B}$ is a vector $v \in \mathbb{R}^d$ with a label (i, j) in $\{1, \dots, t\} \times \{1, \dots, t\}$ such that

$$(C_i - v) \cap C_j \neq \emptyset \quad \text{and} \quad \overline{B}_i \cap \overline{B}_j \neq \emptyset.$$

Formally, a first return vector is a pair $(v, (i, j))$. We denote by $\vec{\mathcal{F}}$ the set of first return vectors to C associated with $\mathcal{B}$.

Now we show some basic properties of first return vectors. We need a standard lemma about return times which we prove by completeness. A similar lemma for $\mathbb{Z}^d$-actions on the Cantor set appears in [**CGM**].

LEMMA 3.1. *Let P be an aperiodic repetitive Delone set in $\mathbb{R}^d$. Let C be a clopen set in Ω^0. Then, for every $\omega \in C$, the set of return times $\mathcal{R}_C(\omega)$ is a repetitive Delone set whose patches depend only on C.*

Proof. *$\mathcal{R}_C(\omega)$ is a Delone set with finite local complexity.* By minimality of the $\mathbb{R}^d$-action, there exists $R > 0$ such that every closed ball of radius R intersects $\mathcal{R}_C(\omega)$. Thus, $\mathcal{R}_C(\omega)$ is relatively dense. On the other hand, $\mathcal{R}_C(\omega) \subseteq \omega$ and, thus, $\mathcal{R}_C(\omega)$ is relatively discrete. Therefore, $\mathcal{R}_C(\omega)$ is a Delone set. Moreover, since every r-patch of $\mathcal{R}_C(\omega)$ is included in some r-patch of ω, which has finite local complexity, one concludes that $\mathcal{R}_C(\omega)$ has finite local complexity.

$\mathcal{R}_C(\omega)$ is repetitive. Fix $r > 0$ and let $p = \mathcal{R}_C(\omega) \cap B_r(x)$ be the r-patch of $\mathcal{R}_C(\omega)$ around $x \in \mathcal{R}_C(\omega)$. Thus, $\omega - x$ is in C. Let V be a small enough clopen neighborhood of $\omega - x$ in C such that every ω' in V agrees with $\omega - x$ in the closed ball of radius r around 0. Since $\mathcal{R}_V(\omega - x)$ is relatively dense, there exists $M > 0$ such that every closed ball of radius M in $\mathbb{R}^d$ intersects $\mathcal{R}_V(\omega - x)$. Since $\mathcal{R}_V(\omega - x) + x \subseteq \mathcal{R}_C(\omega)$, every closed ball of radius $M + r$ contains a translate of p.

$\mathcal{R}_C(\omega)$ and $\mathcal{R}_C(\omega')$ have the same r-patches up to translation. Let p be an r-patch of $\mathcal{R}_C(\omega)$ centered at x. Consider the clopen neighborhood V of $\omega - x$ in C given by

$$V = \{\omega' \in C \mid \mathcal{R}_C(\omega') \cap B_r(0) = p - x\}.$$

By minimality, there exists x' in $\mathcal{R}_C(\omega')$ such that $\omega' - x'$ is in V, which implies that the r-patch $B_r(x') \cap \mathcal{R}_C(\omega')$ is a translated copy of p. $\square$

Let C be a clopen set in Ω^0. By Lemma 3.1, the set of *return vectors* associated with C,

$$\vec{\mathcal{R}} = \mathcal{R}_C(\omega) - \mathcal{R}_C(\omega) = \{x - y \mid (x, y) \in \mathcal{R}_C(\omega) \times \mathcal{R}_C(\omega)\},$$

is well defined.

PROPOSITION 3.1. *Let $\mathcal{B} = (C, \{C_i \times D_i\}_{i=1}^t)$ be a box decomposition of Ω. The set of first return vectors $\vec{\mathcal{F}}$ satisfies the following properties.*

(i) $\vec{\mathcal{F}} = -\vec{\mathcal{F}}$.

(ii) *Every vector in $\vec{\mathcal{R}}$ is a linear combination with non-negative integer coefficients of vectors in $\vec{\mathcal{F}}$.*

(iii) *$\vec{\mathcal{F}}$ is finite.*

Proof. It is immediate from the definition of $\vec{\mathcal{F}}$ that $\vec{\mathcal{F}} = -\vec{\mathcal{F}}$.

Let ω be in C and consider $\mathcal{R}_C(\omega)$. We say that two different points x and y in $\mathcal{R}_C(\omega)$ are neighbors (in $\mathcal{R}_C(\omega)$) with respect to $\mathcal{T}_{\mathcal{B}}(\omega)$ if there are $i, j \in \{1, \ldots, t\}$ such that $\omega - x \in C_i$, $\omega - y \in C_j$, and $(\overline{D}_i + x) \cap (\overline{D}_j + y) \neq \emptyset$. We label the vector $v = y - x$ with (i, j) and denote by $\vec{\mathcal{F}}(\omega)$ the set of these labeled vectors. It is easy to see that $\vec{\mathcal{F}}(\omega) \subseteq \vec{\mathcal{F}}$. By minimality of $(\Omega, \mathbb{R}^d)$, we see that $\vec{\mathcal{F}}(\omega)$ is actually equal to $\vec{\mathcal{F}}$. Thus, $\vec{\mathcal{F}}(\omega)$ is independent of ω in C. Since every vector in $\vec{\mathcal{R}}$ is a linear combination with non-negative integer coefficients of vectors in $\vec{\mathcal{F}}(\omega)$ for some $\omega \in C$, we conclude (ii).

By Lemma 3.1, for every $\omega \in C$ the set $\mathcal{R}_C(\omega)$ is a Delone set with finite local complexity. Hence, $\vec{\mathcal{F}}$ is finite, since it is equal to $\vec{\mathcal{F}}(\omega)$ for every ω in C. □

3.4. *'Good' tower systems.* In this subsection we state some properties of tower systems that we use in the study of the cohomological equation (5).

Let

$$(\mathcal{B}_n = (C_n, \{C_{n,i} \times D_{n,i}\}_{i=1}^{t_n}, s_n))_{n \geq 0}$$

be a sequence of zoomed out box decompositions of Ω. We consider the following properties.

(P1) For every $R > 0$, there exists n such that $B_R(0) \subseteq D_{n,1}$.

(P2) Assume that $(\mathcal{B}_n)_{n \geq 0}$ satisfies condition (C2).

- (i) For every $n \geq 0$ and every first return vector $v \in \vec{\mathcal{F}}_n$ with label (i, j), there are x and x' in $D_{n+1,1}$ such that $\omega_0 - x \in C_{n,i}$, $\omega_0 - x' \in C_{n,j}$, and $v = x' - x$.
- (ii) For all $n \geq 0$, $C_{n+1} \subseteq C_{n,1}$.

(P3) (i) The set $\{M_n \mid n \geq 0\}$ is finite.

- (ii) There is $0 < \gamma < 1$ such that $(s_n^{-1})_{n \geq 0}$ is $O(\gamma^n)$.
- (iii) There is a constant $C > 0$ such that for all $n \geq 0$ and $i \in \{1, \ldots, t_n\}$, $D_{n,i} \subseteq B_{Cs_n}(0)$.

Observe that if $(\mathcal{B}_n)_{n \geq 0}$ satisfies property (P2), then it not only satisfies condition (C2) but also condition (C1).

For every aperiodic repetitive Delone set in $\mathbb{R}^d$, one can construct tower systems satisfying properties (P1) and (P2) by controlling the sequence of clopen sets $(C_n)_{n \geq 0}$. The construction follows the same lines as the constructions of tower systems in [**BBG, BG**].

To construct tower systems satisfying properties (P1)–(P3) is more difficult. In what follows, we show the existence of such tower systems for the hull of linearly repetitive Delone sets. This is based on the construction of special tower systems given in Theorem 3.2 below for linearly repetitive Delone sets. This theorem will appear in [**AC**].

We need to introduce some definitions and notation. Recall that an aperiodic repetitive Delone set P is linearly repetitive with constant $L > 0$ if, for all $r > 0$,

$$M_P(r) \leq Lr, \tag{14}$$

where $M_P(r)$ is the smallest M such that each closed ball of radius M contains the center of a translated copy of every possible r-patch of P. In [**LP1**], it was proved that if P is aperiodic, then $L > 1/3$. Since we only consider aperiodic Delone sets, we always assume that this condition on L holds.

Let C be a clopen set in Ω^0. By Lemma 3.1, the following quantities are well defined:

$$r(C) = \tfrac{1}{2} \inf\{\|x - y\| \mid x,\, y \in \mathcal{R}_C(\omega),\ x \neq y\}$$

and

$$R(C) = \inf\{R > 0 \mid \mathcal{R}_C(\omega) \cap B_R(y) \neq \emptyset,\ \forall y \in \mathbb{R}^d\}.$$

Let ω be in Ω^0 and R be a positive number. We denote by $C_{\omega,R}$ the clopen set $\{\omega' \in \Omega \mid \omega' \cap B_R(0) = \omega \cap B_R(0)\}$. The *recognition radius* of C, denoted by $\operatorname{rec}(C)$, is defined as

$$\operatorname{rec}(C) = \inf\{R > 0 \mid C_{\omega,R} \subseteq C,\ \forall \omega \in C\}.$$

By compactness, every clopen set $C \subseteq \Omega^0$ can be written as a finite union of disjoint sets of the form C_{ω_i,R_i}. Thus, $\operatorname{rec}(C)$ is always finite. The motivation for defining the recognition radius is that to decide whether $\omega \in \Omega^0$ belongs or not to C, it suffices to look at the $\operatorname{rec}(C)$-patch of ω centered at 0. Of course, if $C = C_{\omega,R}$, then its recognition radius is smaller than R.

Given a box decomposition $\mathcal{B} = \{C_i \times D_i\}_{i=1}^t$, define its external and internal radii by

$$R_{\text{ext}}(\mathcal{B}) = \max_{1\leq i\leq t} \inf\{R > 0 \mid B_R(0) \supseteq D_i\},$$
$$r_{\text{int}}(\mathcal{B}) = \min_{1\leq i\leq t} \sup\{r > 0 \mid B_r(0) \subseteq D_i\},$$

respectively. Define also its *recognition radius* by

$$\operatorname{rec}(\mathcal{B}) = \max_{1\leq i\leq t} \operatorname{rec}(C_i).$$

THEOREM 3.2. *Let P be a linearly repetitive Delone set with constant $L > 1/3$ and ω be arbitrary in Ω^0. Let $r_0 > 0$ and $K \geq 6L(L+1)^2$. Set $r_n = K^n r_0$ for every $n \geq 0$ and consider the sequence of clopen sets $(C_n)_{n\geq 0}$ in Ω^0 defined by $C_n = C_{\omega,r_n}$. Then, there exists a tower system $(\mathcal{B}_n = (C_n, \{C_{n,i} \times D_{n,i}\}_{i=1}^{t_n}))_{n\geq 0}$ for Ω that satisfies the following additional properties.*

(i) *For every $n \geq 0$, $C_{n+1} \subseteq C_{n,1}$.*

(ii) *For every $n \geq 1$,*

$$r_{\text{int}}(\mathcal{B}_n) \geq r(C_n) - R_{\text{ext}}(\mathcal{B}_{n-1}), \tag{15}$$
$$R_{\text{ext}}(\mathcal{B}_n) \leq R(C_n) + R_{\text{ext}}(\mathcal{B}_{n-1}). \tag{16}$$

Moreover, there exist constants

$$K_1 := \frac{1}{2(L+1)} - \frac{L}{K-1} \quad \textit{and} \quad K_2 := \frac{LK}{K-1}$$

such that $0 < K_1 < 1 < K_2$ *and, for every* $n \geq 0$,

$$K_1 r_n \leq r_{\text{int}}(\mathcal{B}_n) < R_{\text{ext}}(\mathcal{B}_n) \leq K_2 r_n. \tag{17}$$

(iii) *For every* $n \geq 0$, *every* $i \in \{1, \ldots, t_n\}$, *and every* $\omega \in C_{n,i}$,

$$C_{n,i} = C_{\omega, 2R(C_n)+r_n}.$$

In particular,

$$\text{rec}(\mathcal{B}_n) \leq 2R(C_n) + r_n. \tag{18}$$

(iv) *The set* $\{M_n \mid n \geq 0\}$ *is finite, and each matrix* M_n *has positive coefficients.*

PROPOSITION 3.2. *Let* P *be a linearly repetitive Delone set in* $\mathbb{R}^d$. *For* K *large enough, the tower system given by Theorem 3.2 satisfies properties (P1)–(P3).*

Proof. Let $(\mathcal{B}_n = \{C_{n,i} \times D_{n,i}\}_{i=1}^{t_n})_{n\geq 0}$ be the tower system of Ω given by Theorem 3.2.

Property (P1) follows directly from Theorem 3.2(ii) and the definition of the sequence $(r_n)_{n\geq 0}$.

Now we prove that $(\mathcal{B}_n)_{n\geq 0}$ satisfies property (P2). We start by proving property (P2) part (i). Let ω_0 be the unique Delone set in $\bigcap_{n\geq 0} C_n$. Let v be a first return vector in $\vec{\mathcal{F}}_n$ with label (i, j). We have to show that there are x and x' in $D_{n+1,1}$ such that $\omega_0 - x \in C_{n,i}$, $\omega_0 - x' \in C_{n,j}$, and $v = x' - x$. Denote by $p_{n,i}$ the $(2R(C_n) + r_n)$-patch centered at 0 that defines the clopen set $C_{n,i}$. Since for knowing whether $\omega_0 - x \in C_{n,i}$ and $\omega_0 - x' \in C_{n,j}$ it is enough to look at a $\text{rec}(\mathcal{B}_n)$-patch centered at 0 of $\omega_0 - x$ and $\omega_0 - x'$, respectively, then to prove that such x and x' exist in $D_{n+1,1}$ we have to show that some translation of the patch $p_{n,i} \cup (p_{n,j} + v)$ appears in $D_{n+1,1}$.

We have that $\|v\| \leq 2R_{\text{ext}}(\mathcal{B}_n)$. Thus, $p_{n,i} \cup (p_{n,j} + v)$ is included in some $2(\text{rec}(\mathcal{B}_n) + R_{\text{ext}}(\mathcal{B}_n))$-patch centered at 0. By repetitivity of P, every closed ball of radius $M_P(2(\text{rec}(\mathcal{B}_n) + R_{\text{ext}}(\mathcal{B}_n)))$ contains the center of a $2(\text{rec}(\mathcal{B}_n) + R_{\text{ext}}(\mathcal{B}_n))$-patch, in particular the center of a translated copy of $p_{n,i} \cup (p_{n,j} + v)$. Thus, it is enough to prove that

$$M_P(2\text{rec}(\mathcal{B}_n) + 2R_{\text{ext}}(\mathcal{B}_n)) + 2(\text{rec}(\mathcal{B}_n) + R_{\text{ext}}(\mathcal{B}_n)) \leq r_{\text{int}}(\mathcal{B}_{n+1}), \tag{19}$$

for showing that a translated copy of $p_{n,i} \cup (p_{n,j} + v)$ is included in $D_{n+1,1}$.

By linear repetitivity, and properties (ii) and (iii) in Theorem 3.2, we have that

$$\begin{aligned}
&M_P(2\text{rec}(\mathcal{B}_n) + 2R_{\text{ext}}(\mathcal{B}_n)) + 2(\text{rec}(\mathcal{B}_n) + R_{\text{ext}}(\mathcal{B}_n)) \\
&\quad \leq L(2\text{rec}(\mathcal{B}_n) + 2R_{\text{ext}}(\mathcal{B}_n)) + 2(\text{rec}(\mathcal{B}_n) + R_{\text{ext}}(\mathcal{B}_n)) \\
&\quad \leq (L+1)(2R(C_n) + r_n + 2K_2 r_n) \\
&\quad \leq (L+1)(2Lr_n + r_n + 2K_2 r_n) \\
&\quad = \frac{(L+1)(2L+1+2K_2)}{K} r_{n+1}.
\end{aligned}$$

By the form of K_2, taking K large enough, we have that $(L+1)(2L+1+2K_2)/K < 1$, which implies that (19) holds.

Property (P2) part (ii) is exactly property (ii) in Theorem 3.2.

Finally, we show that property (P3) holds. Part (i) follows from property (iv) in Theorem 3.2. Before we prove parts (ii) and (iii) in property (P3), we show that for every $n \geq 0$ the box decomposition $\mathcal{B}_n$ has size $R(C_n) + r(C_n)/2$. Indeed, by (iii) in Theorem 3.2, all Delone sets in a vertical of a box in $\mathcal{B}_n$ agree in the closed ball of center 0 and radius $2R(C_n) + r_n - R_{\text{ext}}(\mathcal{B}_n)$. By inequalities (16) and (17),

$$\begin{aligned} 2R(C_n) + r_n - R_{\text{ext}}(\mathcal{B}_n) &\geq 2R(C_n) + r_n - (R(C_n) + R_{\text{ext}}(\mathcal{B}_{n-1})) \\ &\geq R(C_n) + r_n - K_2 r_{n-1} \\ &\geq R(C_n) + (1 - K_2/K) r_n. \end{aligned}$$

By the form of K_2, we have that

$$2R(C_n) + r_n - R_{\text{ext}}(\mathcal{B}_n) \geq R(C_n) + r(C_n)/2.$$

Thus, $\mathcal{B}_n$ is a box decomposition with $s_n = R(C_n) + r(C_n)/2$. Now, part (ii) in property (P3) follows from inequality (17) and the definitions of s_n and r_n. Part (iii) in property (P3) follows from equation (17). □

4. *Conditions for the existence of solutions to the cohomological equation in terms of first return vectors*

In this section we state some conditions for the existence of solutions to the cohomological equation (5). First, we give a sufficient condition for the existence of solutions. Then, we state three theorems about a necessary condition for cocycles with different (transversal) regularity. These three theorems show that there is a trade-off between the regularity of the cocycle and the complexity of the Delone set.

Let P be an aperiodic repetitive Delone set in $\mathbb{R}^d$ and α be a continuous G-cocycle over $(\Omega, \mathbb{R}^d)$. Let $\mathcal{B} = (C, \{C_i \times D_i\}_{i=1}^{t})$ be a box decomposition of Ω with base C. Let $\vec{\mathcal{F}}$ be the set of first return vectors to C associated with $\mathcal{B}$. Given a first return vector $v \in \vec{\mathcal{F}}$ with label (i, j), set $C(v) = C_i \cap (C_j + v)$. Observe that if $\omega \in C(v)$, then $\omega \in C_i$ and $\omega - v \in C_j$. We define the *almost period* of α over $v \in \vec{\mathcal{F}}$ as the real number

$$|(\alpha, v)| = \sup\{|\alpha(\omega, v)| \mid \omega \in C(v)\}.$$

Remark 3. Let α be a continuous transversally locally constant G-cocycle over $(\Omega, \mathbb{R}^d)$ with range $r > 0$. Consider ω and ω' in Ω. If there is a continuous path $\gamma : [0, 1] \to \mathbb{R}^d$ from 0 to x such that, for every $t \in [0, 1]$, the Delone sets $\omega - \gamma(t)$ and $\omega' - \gamma(t)$ agree on the ball $B_r(0)$, then $\alpha(\omega, x) = \alpha(\omega', x)$.

Remark 4. Let $\mathcal{B}$ be a box decomposition with size $s > 0$. By Remark 3, if α is a transversally locally constant G-cocycle with range $r \geq s > 0$, then, for every $v \in \vec{\mathcal{F}}$, $\alpha(\omega, v)$ is independent of $\omega \in C(v)$. Thus, $|(\alpha, v)| = |\alpha(\omega, v)|$ for every $\omega \in C(v)$.

Let $(\mathcal{B}_n = (C_n, \{C_{n,i} \times D_{n,i}\}_{i=1}^{t_n}))_{n\geq 0}$ be a sequence of zoomed out box decompositions of Ω satisfying condition (C2). Let $\{\omega_0\}$ be the unique point in $\bigcap_{n\geq 0} C_n$. To simplify notation, we write $\mathcal{R}_n$, $\vec{\mathcal{R}}_n$, and $\mathcal{T}_n$ instead of $\mathcal{R}_{C_n}(\omega_0)$, $\vec{\mathcal{R}}_{C_n}$, and $\mathcal{T}_{\mathcal{B}_n}(\omega_0)$. We denote by $\vec{\mathcal{F}}_n$ the set of first return vectors associated with $\mathcal{B}_n$.

Let $A = (a_{ij})$ be a matrix. Denote by $\|A\|$ the L^1-norm of A, that is, $\|A\| = \sum_{i,j} |a_{ij}|$.

The following theorem gives a sufficient condition on cocycles to be a coboundary with continuous transfer function in terms of a summability condition on the almost periods.

THEOREM 4.1. *Let P be an aperiodic repetitive Delone set in $\mathbb{R}^d$, $(\mathcal{B}_n = (C_n, \{C_{n,i} \times D_{n,i}\}_{i=1}^{t_n}))_{n\geq 0}$ be a sequence of zoomed out box decompositions of Ω satisfying conditions (C1) and (C2) and property (P1), G be $\mathbb{R}^m$ or $\mathbb{T}^m$, and α be a continuous G-cocycle over $(\Omega, \mathbb{R}^d)$. If the series*

$$\sum_{n\geq 0} \|M_{n+1}\| \max_{v\in\vec{\mathcal{F}}_n} |(\alpha, v)|$$

converges, then α is a coboundary with continuous transfer function.

Proof. The idea of the proof is to bound the values of the cocycle on the set of return times using almost periods, and then apply Lemma 2.2. We do that by decomposing a return vector of level m into first return vectors of the previous levels using as many vectors of the biggest levels as possible. The norm of a transition matrix of level $n+1$ bounds how many first return vectors of level n are needed in this decomposition. Therefore, the series $\sum_{n\geq 0} \|M_{n+1}\| \max_{v\in\vec{\mathcal{F}}_n} |(\alpha, v)|$ is a bound for the value of the cocycle over a return vector of level m.

Fix $\epsilon > 0$. There exists $n_0 \geq 0$ such that for every $n \geq n_0$,

$$\sum_{i\geq n} \|M_{i+1}\| \max_{v\in\vec{\mathcal{F}}_i} |(\alpha, v)| < \epsilon. \tag{20}$$

Fix $n \geq n_0$ and consider $x \in \mathcal{R}_n \setminus \{0\}$. By conditions (C1) and (C2), $\mathrm{diam}(C_n) \searrow 0$ as $n \to \infty$. Then, there exists $n_1 \geq n$ such that x is in $\mathcal{R}_{n_1}$ and is not in $\mathcal{R}_{n_1+1}$. By property (P1), for every $R > 0$ there is m such that $B_R(0) \subseteq D_{m,1}$. Let m be the smallest positive integer such that x is in the tile of $\mathcal{T}_m$ punctured at 0. Observe that $m > n_1$. Define $x_m = 0$ and, for every $n_1 \leq i < m$, define x_i as the point in $\mathcal{R}_i$ such that x is in the tile of $\mathcal{T}_n$ punctured at x_i. Since $x \in \mathcal{R}_{n_1}$, we have $x_{n_1} = x$. For every $n_1 \leq i < m$, $x_i - x_{i+1} \in \vec{\mathcal{R}}_i$ and it is also in some tile of the tiling $\mathcal{T}_{i+1}$. Thus, there exists a path of vectors in $\vec{\mathcal{F}}_i$ given by a finite sequence $(v_j^i)_{j=1}^{a_i}$ in $\vec{\mathcal{F}}_i$ such that:

- $x_{i+1} + \sum_{j=1}^{l} v_j^i \in \mathcal{R}_i$, for all $1 \leq l \leq a_i$;
- $a_i \leq \|M_{i+1}\|$.

By a computation and equation (20), one obtains that

$$\begin{aligned} |\alpha(\omega_0, x)| &\leq \sum_{i=n_1}^{m-1} \sum_{l=1}^{a_i} \left| \alpha\left(\omega_0 - \left(x_{i+1} + \sum_{j=1}^{l-1} v_j^i\right), v_l^i\right) \right| \\ &\leq \sum_{i=n_1}^{m-1} \|M_{i+1}\| \max_{v\in\vec{\mathcal{F}}_i} |(\alpha, v)| < \epsilon. \end{aligned}$$

□

Now we state three theorems about a necessary condition for solving the cohomological equation (5). Their proofs will be given in the next section.

The first theorem gives a necessary condition on continuous homomorphisms from $\mathbb{R}^d$ to $\mathbb{T}^m$ for solving the cohomological equation in terms of a summability condition similar to that of Theorem 4.1, but without the norms of the matrices M_n.

THEOREM 4.2. *Let P be an aperiodic repetitive Delone set in $\mathbb{R}^d$, α be a continuous homomorphism from $\mathbb{R}^d$ to $\mathbb{T}^m$, and $(\mathcal{B}_n = (C_n, \{C_{n,i} \times D_{n,i}\}_{i=1}^{t_n}))_{n\geq 0}$ be a sequence of zoomed out box decompositions of Ω satisfying conditions (C1) and (C2) and property (P2).*

If α is a coboundary with continuous transfer function, then the series $\sum_{n\geq 0} \max_{v\in\vec{\mathcal{F}}_n} |||\alpha(v)|||$ converges.

The second theorem is similar to the previous one. It applies to a more general class of cocycles but it requires condition (C3) on the sequence of zoomed out box decompositions, that is, the sequence should be a tower system.

THEOREM 4.3. *Let P be an aperiodic repetitive Delone set in $\mathbb{R}^d$, G be $\mathbb{R}^m$ or $\mathbb{T}^m$, α be a continuous G-cocycle over $(\Omega, \mathbb{R}^d)$, and $(\mathcal{B}_n = (C_n, \{C_{n,i} \times D_{n,i}\}_{i=1}^{t_n}, s_n))_{n\geq 0}$ be a tower system for Ω satisfying property (P2).*

If α is a transversally locally constant coboundary with continuous transfer function, then the series $\sum_{n\geq 0} \max_{v\in\vec{\mathcal{F}}_n} |(\alpha, v)|$ converges.

The last theorem applies for an even more general class of cocycles but it requires a stronger hypothesis on the tower system. Before stating this theorem, we need to introduce the following class of cocycles. We say that the continuous cocycle α is a *transversally Hölder* cocycle if there exist constants $K > 0$ and $\delta \in (0, 1)$ such that for all $r > 0$, $\omega, \omega' \in \Omega$, and $x \in B_r(0)$,

$$\text{if } \omega \cap B_r(0) = \omega' \cap B_r(0), \text{ then } |\alpha(\omega, x) - \alpha(\omega', x)| \leq K r^{-\delta}.$$

THEOREM 4.4. *Let P be an aperiodic repetitive Delone set in $\mathbb{R}^d$, G be $\mathbb{R}^m$ or $\mathbb{T}^m$, α be a continuous G-cocycle over $(\Omega, \mathbb{R}^d)$, and $(\mathcal{B}_n = (C_n, \{C_{n,i} \times D_{n,i}\}_{i=1}^{t_n}, s_n))_{n\geq 0}$ be a tower system for Ω satisfying properties (P2) and (P3). If α is a transversally Hölder coboundary with continuous transfer function, then the series $\sum_{n\geq 0} \max_{v\in\vec{\mathcal{F}}_n} |(\alpha, v)|$ converges.*

In §3.4, we prove that every linearly repetitive Delone set P in $\mathbb{R}^d$ admits a tower systems for Ω satisfying properties (P1), (P2), and (P3). Thus, we obtain the following corollary.

COROLLARY 4.1. *Let P be a linearly repetitive Delone set in $\mathbb{R}^d$, G be $\mathbb{R}^m$ or $\mathbb{T}^m$, and α be a continuous G-cocycle over $(\Omega, \mathbb{R}^d)$. Then, there exists a tower system such that α is a transversally Hölder coboundary with continuous transfer function if and only if the series $\sum_{n\geq 0} \max_{v\in\vec{\mathcal{F}}_n} |(\alpha, v)|$ converges.*

5. *Proof of Theorems 4.2, 4.3, and 4.4*

In this section we give the proofs of Theorems 4.2, 4.3, and 4.4. The three proofs follow the same strategy and we do all of them almost simultaneously. We divide the proofs into three steps. First, we state a key combinatorial lemma. Then, we describe the common strategy of the proofs and, finally, we conclude the proof of each theorem separately.

5.1. *The key lemma and some corollaries.* In this subsection we state and prove the key lemma. It is a consequence of property (P2) and roughly speaking it says that given a finite set of first return vectors $v_{n_l} \in \vec{\mathcal{F}}_{n_l}$ with $n_k < n_{k-1} < \cdots < n_1$, one can always find two return times such that their difference is equal to the sum of these first return vectors. Then, we deduce some corollaries that we use to bound the sum of almost periods $\sum_{l=1}^{k} |(\alpha, v_{n_l})|$ in the last subsection.

Let P be an aperiodic repetitive Delone set in $\mathbb{R}^d$. Consider a sequence of zoomed out box decompositions $(\mathcal{B}_n = (C_n, \{C_{n,i} \times D_{n,i}\}_{i=1}^{t_n}))_{n\geq 0}$ of Ω that satisfies property (P2). Let ω_0 be the unique point in $\bigcap_{n\geq 0} C_n$.

Let N_1 and N_0 be two integers with $0 \leq N_1 < N_0$, and $n_k < n_{k-1} < \cdots < n_1$ be an ordered sequence of even or odd integers between N_1 and N_0. For every $1 \leq l \leq k$, let v_{n_l} be a first return vector in $\vec{\mathcal{F}}_{n_l}$ with label (i_{n_l}, j_{n_l}).

LEMMA 5.1. *There exist three sequences $(x_l)_{l=0}^{k}$, $(x'_l)_{l=0}^{k}$, and $(y_l)_{l=0}^{k}$ in $\mathcal{R}_{C_0}(\omega_0)$ with $x_0 = x'_0 = y_0 = 0$ such that for every $1 \leq l \leq k$:*

(a) (i) $x_l - x_{l-1} \in D_{n_l+1,1} \cap \mathcal{R}_{C_{n_l,j_{n_l}}}(\omega_0 - x_{l-1})$;
(ii) $x'_l - x_{l-1} \in D_{n_l+1,1} \cap \mathcal{R}_{C_{n_l,i_{n_l}}}(\omega_0 - x_{l-1})$; *and*
(iii) $v_{n_l} = x_l - x'_l$;

(b) $y_l - y_{l-1} = x'_l - x_{l-1}$;

(c) $x_l - y_l = \sum_{a=1}^{l} v_{n_a}$;

(d) $\omega_0 - x_l \in C_{n_l,j_{n_l}}$ *and* $\omega_0 - x'_l \in C_{n_l,i_{n_l}}$. *In particular,* $\omega_0 - x'_l \in C_{v_{n_l}}$;

(e) $x_l, x'_l, y_l \in \mathcal{R}_{n_l}$.

Proof. We define the sequences $(x_l)_{l=0}^{k}$, $(x'_l)_{l=0}^{k}$, and $(y_l)_{l=0}^{k}$ inductively.

Set $x_0 = x'_0 = y_0 = 0$. By (i) in property (P2), there exist

$$x_1 \in D_{n_1+1,1} \cap \mathcal{R}_{C_{n_1,j_{n_1}}}(\omega_0) \quad \text{and} \quad x'_l \in D_{n_1+1,1} \cap \mathcal{R}_{C_{n_1,i_{n_1}}}(\omega_0)$$

such that $v_{n_1} = x_1 - x'_1$. We define $y_1 = x'_1$. It is easy to see that the sequences $(x_j)_{j=0}^{1}$, $(x'_j)_{j=0}^{1}$, and $(y_j)_{j=0}^{1}$ are $\mathcal{R}_{C_0}(\omega_0)$ and they satisfy properties (a)–(e).

Assume that for $1 < l < k$ we have defined the sequences $(x_j)_{j=0}^{l-1}$, $(x'_j)_{j=0}^{l-1}$, and $(y_j)_{j=0}^{l-1}$ in $\mathcal{R}_{C_0}(\omega_0)$ and that they satisfy properties (a)–(e). We define x_l, x'_l, and y_l.

First we show that $\omega_0 - x_{l-1} \in C_{n_l+1,1}$. By (ii) in property (P2), $C_{n_{l-1}} \subseteq C_{n_{l-1}-1,1}$. Since $n_l + 2 \leq n_{l-1}$, we obtain that $C_{n_{l-1}} \subseteq C_{n_l+1,1}$ and, by property (e), $\omega_0 - x_{l-1} \in C_{n_{l-1}}$. Thus, we get that $\omega_0 - x_{l-1} \in C_{n_l+1,1}$.

By (i) in property (P2), there exist

$$\bar{x}_l \in D_{n_l+1,1} \cap \mathcal{R}_{C_{n_l,j_{n_l}}}(\omega_0 - x_{l-1}), \quad \text{and} \quad \bar{x}'_l \in D_{n_l+1,1} \cap \mathcal{R}_{C_{n_l,i_{n_l}}}(\omega_0 - x_{l-1})$$

such that $v_{n_l} = \bar{x}_l - \bar{x}'_l$. We define

$$x_l = x_{l-1} + \bar{x}_l, \quad x'_l = x_{l-1} + \bar{x}'_l \quad \text{and} \quad y_l = y_{l-1} + x'_l - x_{l-1}.$$

Now we check that the sequences $(x_j)_{j=0}^l$, $(x'_j)_{j=0}^l$, and $(y_j)_{j=0}^l$ satisfy properties (a)–(e). We see that properties (a) and (b) are direct from the definition of x_l, x'_l, and y_l. From properties (a) and (b), we have that

$$\begin{aligned} x_l - y_l &= x_{l-1} - y_{l-1} + x_l - x_{l-1} - (y_l - y_{l-1}) \\ &= x_{l-1} - y_{l-1} + x_l - x'_l + x'_l - x_{l-1} - (y_l - y_{l-1}) \\ &= x_{l-1} - y_{l-1} + v_{n_l} \\ &= \sum_{a=1}^{l} v_{n_a}. \end{aligned}$$

Then, property (c) holds. From property (a), we have that

$$\omega_0 - x_l = (\omega_0 - x_{l-1}) - (x_l - x_{l-1}) \in C_{n_l, j_{n_l}}$$

and

$$\omega_0 - x'_l = (\omega_0 - x_{l-1}) - (x'_l - x_{l-1}) \in C_{n_l, i_{n_l}}.$$

Then, property (d) also holds. We see that property (d) implies that $x_l \in \mathcal{R}_{n_l}$ and x'_l is in $\mathcal{R}_{n_l}$. It remains to check that $y_l \in \mathcal{R}_{n_l}$. Since

$$\omega_0 - y_{l-1} \in C_{n_{l-1}}, \quad C_{n_{l-1}} \subseteq C_{n_l+1,1}, \quad \text{and} \quad C_{n_l+1,1} - \bar{x}' \subseteq C_{n_l, j_{n_l}} \subseteq C_{n_l},$$

we deduce that

$$\omega_0 - y_l = (\omega_0 - y_{l-1}) - (x'_l - x_{l-1}) = (\omega_0 - y_{l-1}) - \bar{x}' \in C_{n_l}.$$

Then, property (e) also holds. □

Let α be a continuous coboundary with continuous transfer function $\psi : \Omega \to G$. By the continuity of α and the compactness of $C(v_{n_l})$, there is $\omega_{n_l} \in C(v_{n_l})$ such that $|\alpha(\omega_{n_l}, v_{n_l})| = |(\alpha, v_{n_l})|$. Let $(x_l)_{l=0}^k$, $(x'_l)_{l=0}^k$, and $(y_l)_{l=0}^k$ be the three sequences given by Lemma 5.1.

COROLLARY 5.1. *Set $u_l = y_l - y_{l-1} = x'_l - x_{l-1}$. For every $1 \le l \le k$,*

$$\begin{aligned} \sum_{a=1}^{l} \alpha(\omega_{n_a}, v_{n_a}) = \psi(\omega_0 - x_l) - \psi(\omega_0 - y_l) \\ + \sum_{a=1}^{l} \alpha(\omega_{n_a}, v_{n_a}) - \alpha(\omega_0 - x'_a, v_{n_a}) \\ + \sum_{a=1}^{l} \alpha(\omega_0 - y_{a-1}, u_a) - \alpha(\omega_0 - x_{a-1}, u_a). \end{aligned} \tag{21}$$

Proof. We have that

$$\begin{aligned} \alpha(\omega_0, x_l) &= \sum_{a=1}^{l} \alpha(\omega_0 - x_{a-1}, x_a - x_{a-1}) \\ &= \sum_{a=1}^{l} \alpha(\omega_0 - x_{a-1}, x_a - x'_a + x'_a - x_{a-1}) \\ &= \sum_{a=1}^{l} \alpha(\omega_0 - x_{a-1}, u_a) + \alpha(\omega_0 - x_a v_{n_a}) \end{aligned}$$

and

$$\alpha(\omega_0, y_l) = \sum_{a=1}^{l} \alpha(\omega_0 - y_{a-1}, u_a).$$

Hence,

$$\begin{aligned}\psi(\omega_0 - x_l) - \alpha(\omega_0 - y_l) &= \alpha(\omega_0, x_l) - \alpha(\omega_0, y_l) \\ &= \sum_{a=1}^{l} \alpha(\omega_0 - x_{a-1}, u_a) - \alpha(\omega_0 - y_{a-1}, u_a) \\ &\quad + \sum_{a=1}^{l} \alpha(\omega_0 - x_a, v_{n_a}).\end{aligned}$$

Adding $\sum_{a=1}^{l} \alpha(\omega_{n_a}, v_{n_a})$ to both sides of the last equation and rearranging terms, we conclude the proof. □

COROLLARY 5.2. *If α is a continuous homomorphism from $\mathbb{R}^d$ to $\mathbb{T}^m$, then, for every $1 \leq l \leq k$,*

$$\sum_{a=1}^{l} \alpha(v_{n_a}) = \psi(\omega_0 - x_l) - \psi(\omega_0 - y_l).$$

Proof. The proof follows directly from Corollary 5.1. Since the cocycle induced by α does not depend on Ω, we have that both sums on the right-hand side of equation (21) are zero. □

COROLLARY 5.3. *Assume that $(\mathcal{B}_n = (C_n, \{C_{n,i} \times D_{n,i}\}, s_n))_{n \geq 0}$ is a tower system. If α is a transversally locally constant G-cocycle, then there is $n_\alpha \geq 0$ such that if $N_1 \geq n_\alpha$, then, for every $1 \leq l \leq k$,*

$$\sum_{a=1}^{l} \alpha(\omega_{n_a}, v_{n_a}) = \psi(\omega_0 - x_l) - \psi(\omega_0 - y_l).$$

Proof. Denote by r the range of α. Since $s_n \nearrow \infty$ as $n \to \infty$, there exists $n_\alpha \geq 0$ such that for all $n \geq n_\alpha$, $s_n \geq r$.

Fix $a \in \{1, \ldots, l\}$. By Remark 3, for all $n \geq n_\alpha$ and $v \in \vec{\mathcal{F}}_n$ we get that $\alpha(\omega, v)$ is independent of ω in $C(v)$. By Lemma 5.1(d), $\omega_0 - x'_a \in C(v_{n_a})$. Thus,

$$\alpha(\omega_{n_a}, v_{n_a}) = \alpha(\omega_0 - x'_a, v_{n_a}).$$

We prove that $\alpha(\omega_0 - x_{a-1}, u_a) = \alpha(\omega_0 - y_{a-1}, u_a)$. By $n_a + 1 < n_{a-1}$ and (ii) in property (P2),

$$\omega_0 - x_{a-1}, \quad \omega_0 - y_{a-1} \in C_{n_{a-1}} \subseteq C_{n_a+1,1}.$$

By the definition of the sequence $(y_l)_{l=0}^{k}$,

$$y_a - y_{a-1} = x'_a - x_{a-1} \in D_{n_a+1,1} \cap \mathcal{R}_{C_{n_a, i_{n_a}}}(\omega_0),$$

which is included in $D_{n_a+1,1}$. For every $x \in D_{n_a+1,1}$, the Delone sets in $C_{n_a+1,1} - x$ agree in the closed ball of center 0 and radius s_{n_a+1}. From Remark 3, we deduce that

$$\alpha(\omega_0 - x_{a-1}, x'_a - x_{a-1}) = \alpha(\omega_0 - y_{a-1}, y_a - y_{a-1}).$$

By Corollary 5.1, we conclude the proof. □

COROLLARY 5.4. *Assume that $(\mathcal{B}_n = (C_n, \{C_{n,i} \times D_{n,i}\}, s_n))_{n\geq 0}$ is a tower system satisfying property (P3). If α is a transversally Hölder G-cocycle, then there are constants $K' > 0$ and $\delta > 0$ such that for every $1 \leq l \leq k$,*

$$\left|\sum_{a=1}^{l} \alpha(\omega_{n_a}, v_{n_a})\right| \leq |\psi(\omega_0 - x_l) - \psi(\omega_0 - y_l)| + K' \sum_{N_1 \leq n \leq N_0} s_n^{-\delta}.$$

Proof. Assume that α is transversally Hölder. Then, there exist constants $K > 0$ and $\delta \in (0, 1)$ such that for all $s > 0$, ω, $\omega' \in \Omega$, and $x \in B_s(0)$,

$$\text{if } \omega \cap B_s(0) = \omega' \cap B_s(0), \text{ then } |\alpha(\omega, x) - \alpha(\omega', x)| \leq K s^{-\delta}.$$

The idea of the proof is to use Lemma 5.1 to bound $|\sum_{a=1}^{l} \alpha(\omega_{n_a}, v_{n_a})|$. Fix $1 \leq a \leq l$. We bound

$$|\alpha(\omega_{n_a}, v_{n_a}) - \alpha(\omega_0 - x'_a, v_{n_a})| \quad \text{and} \quad |\alpha(\omega_0 - y_{a-1}, u_a) - \alpha(\omega_0 - x_{a-1}, u_a)|,$$

where $u_a = y_a - y_{a-1} = x'_a - x_{a-1}$.

First, we bound $|\alpha(\omega_{n_a}, v_{n_a}) - \alpha(\omega_0 - x'_a, v_{n_a})|$. We have that $v_{n_a} = x_a - x'_a$ is in $\vec{\mathcal{R}}_{n_a}$, which is included in $\vec{\mathcal{R}}_{n_a-1}$. By Proposition 3.1, there is a path of k_a vectors in $\vec{\mathcal{F}}_{n_a-1}$ from x'_a to x_a such that $k_a \leq \|M_{n_a}\|$. We enumerate the vectors in this path, $\{u_b\}_{b=1}^{k_a}$, using the order in which they appear from x'_l to x_l. By Lemma 5.1(c), we see that $\omega_0 - x'_a$ is in $C_{v_{n_a}}$. For every $1 \leq b \leq k_a$, the Delone sets $\omega_{n_a} - \sum_{c=1}^{b-1} u_c$ and $(\omega_0 - x'_a) - \sum_{c=1}^{b-1} u_c$ agree in the closed ball of center 0 and radius s_{n_a}. From (iii) in property (P3), there is $C > 0$ such that u_b is in $B_{2Cs_{n_a}}(0)$, for every $1 \leq b \leq k_{n_a}$. By the transversally Hölder property, we get

$$\begin{aligned}
&|\alpha(\omega_{n_a}, v_{n_a}) - \alpha(\omega_0 - x'_a, v_{n_a})| \\
&\quad \leq \sum_{b=1}^{k_a} \left|\alpha\left(\omega_{n_a} - \sum_{c=1}^{b-1} u_c, u_b\right) - \alpha\left((\omega_0 - x'_a) - \sum_{c=1}^{b-1} u_c, u_b\right)\right| \leq 2K k_a s_{n_a}^{-\delta}. \quad (22)
\end{aligned}$$

Now we bound $|\alpha(\omega_0 - y_{a-1}, u_a) - \alpha(\omega_0 - x_{a-1}, u_a)|$. We have that $\omega_0 - x_{a-1}$ and $\omega_0 - y_{a-1}$ agree in the closed ball of center 0 and radius $s_{n_{a-1}}$. By Lemma 5.1(a), $x'_a - x_{a-1}$ and $y_a - y_{a-1}$ are in $D_{n_a+1,1}$. By (ii) in property (P3),

$$D_{n_a+1,1} \subseteq B_{Cs_{n_a+2}}(0) \subseteq B_{Cs_{n_{a-1}}}(0).$$

By the transversally Hölder property, we get that

$$|\alpha(\omega_0 - x_{a-1}, u_a) - \alpha(\omega_0 - y_{a-1}, u_a)| \leq K s_{n_{a-1}}^{-\delta}. \quad (23)$$

From Corollary 5.1 and inequalities (22) and (23), we deduce that

$$\begin{aligned}
\left|\sum_{a=1}^{l} \alpha(\omega_{n_a}, v_{n_a})\right| &\leq |\psi(\omega_0 - x_l) - \psi(\omega_0 - y_l)| + K \sum_{a=1}^{l} 2k_a s_{n_a}^{-\delta} + s_{n_{a-1}}^{-\delta} \\
&\leq |\psi(\omega_0 - x_l) - \psi(\omega_0 - y_l)| + 2K \sum_{N_1 \leq n \leq N_0} \|M_{n+1}\| s_n^{-\delta}.
\end{aligned}$$

By (i) in property (P3), there exists a constant $L > 0$ such that $\|M_n\| \leq L$ for all $n \geq 1$. Set $K' = 2LK$. Thus,

$$\left|\sum_{a=1}^{l} \alpha(\omega_{n_a}, v_{n_a})\right| \leq |\psi(\omega_0 - x_l) - \psi(\omega_0 - y_l)| + K' \sum_{N_1 \leq n \leq N_0} s_n^{-\delta}. \qquad \square$$

5.2. *Proofs of Theorems 4.2, 4.3, and 4.4.* In this subsection, we start with the proofs of Theorems 4.2, 4.3, and 4.4.

Let α be a continuous coboundary with continuous transfer function $\psi : \Omega \to G$. For each $n \geq 0$, let v_n be in $\vec{\mathcal{F}}_n$ such that

$$|(\alpha, v_n)| = \max_{v \in \vec{\mathcal{F}}_n} |(\alpha, v)| = \max_{v \in \vec{\mathcal{F}}_n} \sup\{|\alpha(\omega, v)| \mid \omega \in C(v)\}.$$

By the continuity of α and the compactness of $C(v_n)$, there is $\omega_n \in C(v_n)$ such that $|\alpha(\omega_n, v_n)| = |(\alpha, v_n)|$. We have to prove that $\sum_{n\geq 0} |\alpha(\omega_n, v_n)|$ converges. We decompose the non-negative integers into 2^m sets and we prove that the restriction of the series $\sum_{n\geq 0} |\alpha(\omega_n, v_n)|$ on each one of these sets converges.

First, we decompose $\mathbb{R}^m$ into 2^m sectors $S_{\epsilon_1,\ldots,\epsilon_m}$, with ϵ_i in $\{-1, 1\}$ for $i \in \{1, \ldots, m\}$, defined by

$$S_{\epsilon_1,\ldots,\epsilon_m} = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \cdot \epsilon_i \geq 0, \forall i \in \{1, \ldots, m\}\}.$$

Let B be the open ball on $\mathbb{T}^m$ of center 0 and radius 1/4, and $\pi : B \longrightarrow\,]-1/4, 1/4[^m$ be the canonical isometry of the ball B with the open hypercube $]-1/4, 1/4[^m$.

Now, we decompose the non-negative integers. Define $I_{\epsilon_1,\ldots,\epsilon_m}$ as the set of integers n such that $\alpha(\omega_n, v_n)$ is in:

- $S_{\epsilon_1,\ldots,\epsilon_m}$, if G is $\mathbb{R}^m$;
- $\pi^{-1}(S_{\epsilon_1,\ldots,\epsilon_m} \cap\,]-1/4, 1/4[^m)$, if G is $\mathbb{T}^m$.

The following lemma motivates such decomposition.

LEMMA 5.2. *For every $k \geq 1$:*

(a) *if $x_1, \ldots, x_k \in S_{\epsilon_1,\ldots,\epsilon_m}$, then $\sum_{l=1}^k |x_l| \leq m|\sum_{l=1}^k x_l|$;*

(b) *if $x_1, \ldots, x_k \in \pi^{-1}(S_{\epsilon_1,\ldots,\epsilon_m} \cap\,]-1/4, 1/4[^m)$ and, for every $1 \leq l \leq k$, $\sum_{a=1}^l x_a \in \pi^{-1}(S_{\epsilon_1,\ldots,\epsilon_m} \cap\,]-1/4, 1/4[^m)$, then $\sum_{l=1}^k \|\!|x_l|\!\| \leq m \|\!|\sum_{l=1}^k x_l|\!\|$.*

Write $l_n = |\alpha(\omega_n, v_n)|$. Fix $\epsilon_1, \ldots, \epsilon_m \in \{-1, 1\}^m$. We use Lemma 5.1 for proving that the series $\sum_{n\in I_{\epsilon_1,\ldots,\epsilon_m}} l_n$ converges. We need to split this sum into two parts:

$$\sum_{n\in I_{\epsilon_1,\ldots,\epsilon_m}} l_n = \sum_{n\in I_{\epsilon_1,\ldots,\epsilon_m},\text{even}} l_n + \sum_{n\in I_{\epsilon_1,\ldots,\epsilon_m},\text{odd}} l_n.$$

Here, we only prove that

$$\sum_{n\in I_{\epsilon_1,\ldots,\epsilon_m},\text{even}} l_n < \infty; \tag{24}$$

a similar proof works for the other case.

5.3. *Conclusion of the proofs.* Assume that the intersection of $I_{\epsilon_1,\ldots,\epsilon_m}$ with the even numbers is infinite. Let N_1 and N_0 be two positive integers with $N_1 < N_0$. Consider the ordered sequence $n_k < n_{k-1} < \cdots < n_1$ of even integers between N_1 and N_0 that belong to $I_{\epsilon_1,\ldots,\epsilon_m}$. We have that

$$\sum_{\substack{n\in I_{\epsilon_1,\ldots,\epsilon_m},\text{even}\\ N_1\leq n\leq N_0}} l_n = \sum_{l=1}^k |\alpha(\omega_{n_l}, v_{n_l})|.$$

We use the corollaries of Corollary 5.1 to bound $|\sum_{l=1}^{k} \alpha(\omega_{n_l}, v_{n_l})|$ and, then, we use Lemma 5.2 to put the norm inside the sum.

5.3.1.*Proof of inequality (24) under the hypothesis of Theorem 4.2.* Let $\alpha : \Omega \to \mathbb{T}^m$ be a continuous homomorphism.

By the continuity of ψ, there exists $N \geq 0$ such that $|l_n| < 1/4$ for all $n \geq N$ and $|||\psi(\omega_0 - x) - \psi(\omega_0 - y)||| < 1/4$ for all $x, y \in \mathcal{R}_N$.

Take $N_1 > N$. By Corollary 5.2, we have that for every $1 \leq l \leq k$,

$$\left|\left|\left|\sum_{a=1}^{l} \alpha(v_{n_a})\right|\right|\right| = |||\psi(\omega_0 - x_l) - \psi(\omega_0 - y_l)||| < 1/4.$$

By Lemma 5.2(b), we have $\sum_{l=1}^{k} |||\alpha(v_{n_l})||| \leq m|||\sum_{l=1}^{k} \alpha(v_{n_l})|||$. We conclude that for every $N_0 > N_1$,

$$\sum_{\substack{n \in I_{\epsilon_1,\ldots,\epsilon_m}, \text{even} \\ N_1 \leq n \leq N_0}} l_n = \sum_{l=1}^{k} |||\alpha(v_{n_l})||| \leq m1/4,$$

which implies inequality (24). □

5.3.2.*Proof of inequality (24) under the hypothesis of Theorem 4.3.* Let α be a transversally locally constant G-cocycle. We separate the cases $G = \mathbb{R}^m$ and $G = \mathbb{T}^m$.

$G = \mathbb{R}^m$. Let n_α be as in Corollary 5.3 and take $N_1 \geq n_\alpha$. By Corollary 5.3, we have that

$$\left|\sum_{l=1}^{k} \alpha(\omega_{n_l} v_{n_l})\right| = |\psi(\omega_0 - x_k) - \psi(\omega_0 - y_k)|.$$

By Lemma 5.2(a), we have $\sum_{l=1}^{k} |\alpha(\omega_{n_l} v_{n_l})| \leq m|\sum_{l=1}^{k} \alpha(\omega_{n_l} v_{n_l})|$. By the continuity of ψ, we conclude that for every $N_0 > N_1$,

$$\sum_{\substack{n \in I_{\epsilon_1,\ldots,\epsilon_m}, \text{even} \\ N_1 \leq n \leq N_0}} l_n = \sum_{l=1}^{k} |\alpha(\omega_{n_l} v_{n_l})| < \infty,$$

which implies inequality (24). □

$G = \mathbb{T}^m$. By the continuity of ψ, there exists $N \geq 0$ such that $|l_n| < 1/4$ for all $n \geq N$ and $|||\psi(\omega_0 - x) - \psi(\omega_0 - y)||| < 1/4$ for all $x, y \in \mathcal{R}_N$.

Let n_α be as in Corollary 5.3 and take $N_1 > \max\{N, n_\alpha\}$. By Corollary 5.3, we have that for every $1 \leq l \leq k$,

$$\left|\left|\left|\sum_{a=1}^{l} \alpha(\omega_{n_a} v_{n_a})\right|\right|\right| = |||\psi(\omega_0 - x_l) - \psi(\omega_0 - y_l)||| < 1/4.$$

By Lemma 5.2(b), we have $\sum_{l=1}^{k} |||\alpha(v_{n_l})||| \leq m|||\sum_{l=1}^{k} \alpha(v_{n_l})|||$. We conclude that for every $N_0 > N_1$,

$$\sum_{\substack{n \in I_{\epsilon_1,\ldots,\epsilon_m}, \text{even} \\ N_1 \leq n \leq N_0}} l_n = \sum_{l=1}^{k} |||\alpha(v_{n_l})||| \leq m1/4,$$

which implies inequality (24). □

5.3.3.*Proof of inequality (24) under the hypothesis of Theorem 4.4.* Let α be a transversally Hölder G-cocycle. We separate the cases $G = \mathbb{R}^m$ and $G = \mathbb{T}^m$.

$G = \mathbb{R}^m$. By Corollary 5.4, there are constants $K' > 0$ and $\delta > 0$ such that

$$\left|\sum_{l=1}^{k} \alpha(\omega_{n_l}, v_{n_l})\right| \leq |\psi(\omega_0 - x_k) - \psi(\omega_0 - y_k)| + K' \sum_{N_1 \leq n \leq N_0} s_n^{-\delta}.$$

By Lemma 5.2(a), we have $\sum_{l=1}^{k} |\alpha(\omega_{n_l} v_{n_l})| \leq m|\sum_{l=1}^{k} \alpha(\omega_{n_l} v_{n_l})|$. From (ii) in property (P3), we have that the series $\sum_{n\geq 0} s_n^{-\delta}$ converges. By the continuity of ψ, we conclude that for every $N_0 > N_1$,

$$\sum_{\substack{n \in I_{\epsilon_1,\ldots,\epsilon_m},\text{even} \\ N_1 \leq n \leq N_0}} l_n = \sum_{l=1}^{k} |\alpha(\omega_{n_l} v_{n_l})| < \infty,$$

which implies inequality (24). □

$G = \mathbb{T}^m$. By the continuity of ψ, there exists $N \geq 0$ such that $|l_n| < 1/4$ for all $n \geq N$ and $|||\psi(\omega_0 - x) - \psi(\omega_0 - y)||| < 1/4$ for all $x, y \in \mathcal{R}_N$.

By Corollary 5.4, there are constants $K' > 0$ and $\delta > 0$ such that for every $1 \leq l \leq k$,

$$\left|\left|\left|\sum_{a=1}^{l} \alpha(\omega_{n_a}, v_{n_a})\right|\right|\right| \leq |||\psi(\omega_0 - x_l) - \psi(\omega_0 - y_l)||| + K' \sum_{N_1 \leq n \leq N_0} s_n^{-\delta}.$$

From (ii) in property (P3), we have that the series $\sum_{n\geq 0} s_n^{-\delta}$ converges. Thus, taking N_1 large enough, we obtain that for every $1 \leq l \leq k$, $|||\sum_{a=1}^{l} \alpha(\omega_{n_a}, v_{n_a})||| < 1/4$. By Lemma 5.2(b), we have $\sum_{l=1}^{k} |||\alpha(v_{n_l})||| \leq m|||\sum_{l=1}^{k} \alpha(v_{n_l})|||$. We conclude that for every $N_0 > N_1$,

$$\sum_{\substack{n \in I_{\epsilon_1,\ldots,\epsilon_m},\text{even} \\ N_1 \leq n \leq N_0}} l_n = \sum_{l=1}^{k} |||\alpha(v_{n_l})||| \leq m1/4,$$

which implies inequality (24). □

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REFERENCES

[AC] J. Aliste-Prieto and A. D. Coronel. Tower systems for linearly repetitive Delone sets. *Preprint*, 2010, arXiv:1003.4309.

[BBG] J. Bellissard, R. Benedetti and J.-M. Gambaudo. Spaces of tilings, finite telescopic approximations and gap-labeling. *Comm. Math. Phys.* **261**(1) (2006), 1–41.

[BG] R. Benedetti and J.-M. Gambaudo. On the dynamics of $\mathbb{G}$-solenoids. Applications to Delone sets. *Ergod. Th. & Dynam. Sys.* **23**(3) (2003), 673–691.

[BDM] X. Bressaud, F. Durand and A. Maass. Necessary and sufficient conditions to be an eigenvalue for linearly recurrent dynamical Cantor systems. *J. Lond. Math. Soc.* (2) **72**(3) (2005), 799–816.

[CS] A. Clark and L. Sadun. When size matters: subshifts and their related tiling spaces. *Ergod. Th. & Dynam. Sys.* **23**(4) (2003), 1043–1057.

[CS1] A. Clark and L. Sadun. When shape matters: deformations of tiling spaces. *Ergod. Th. & Dynam. Sys.* **26**(1) (2006), 69–86.

[CDHM] M. I. Cortez, F. Durand, B. Host and A. Maass. Continuous and measurable eigenfunctions of linearly recurrent dynamical Cantor systems. *J. Lond. Math. Soc.* (2) **67**(3) (2003), 790–804.

[CGM] M. I. Cortez, J.-M. Gambaudo and A. Maass. Rotation topological factors of minimal $\mathbb{Z}^d$ actions on the Cantor set. *Trans. Amer. Math. Soc.* **359**(5) (2007), 2305–2315.

[G] E. Ghys. Laminations par surfaces de Riemann. *Dynamique et géométrie complexes (Lyon, 1997) (Panoramas et Synthèses, 8)*. Société Mathématique de France, Paris, 1999, pp. 49–95, ix, xi.

[KH] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems. With a Supplementary Chapter by Anatole Katok and Leonardo Mendoza (Encyclopedia of Mathematics and its Applications, 54)*. Cambridge University Press, Cambridge, 1995.

[K] J. Kellendonk. Pattern-equivariant functions and cohomology. *J. Phys. A* **36**(21) (2003), 5765–5772.

[K1] J. Kellendonk. Pattern equivariant functions, deformations and equivalence of tiling spaces. *Ergod. Th. & Dynam. Sys.* **28**(4) (2008), 1153–1176.

[KP1] J. Kellendonk and I. F. Putnam. The Ruelle–Sullivan map for actions of $\mathbb{R}^n$. *Math. Ann.* **334**(3) (2006), 693–711.

[LP1] J. C. Lagarias and P. A. B. Pleasants. Local complexity of Delone sets and crystallinity. Dedicated to Robert V. Moody. *Canad. Math. Bull.* **45**(4) (2002), 634–652.

[LP] J. C. Lagarias and P. A. B. Pleasants. Repetitive Delone sets and quasicrystals. *Ergod. Th. & Dynam. Sys.* **23**(3) (2003), 831–867.

[LMS] J.-Y. Lee, R. V. Moody and B. Solomyak. Pure point dynamical and diffraction spectra. *Ann. Henri Poincaré* **3**(5) (2002), 1003–1018.

[Rob] E. A. Robinson Jr. Symbolic dynamics and tilings of $\mathbb{R}^d$. *Symbolic Dynamics and its Applications (Proceedings of Symposia in Applied Mathematics, 60)*. American Mathematical Society, Providence, RI, 2004, pp. 81–119.

[SW] L. Sadun and R. F. Williams. Tiling spaces are Cantor set fiber bundles. *Ergod. Th. & Dynam. Sys.* **23**(1) (2003), 307–316.