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Sensitive Dependence of Gibbs Measures at Low Temperatures

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Abstract The Gibbs measures of an interaction can behave chaotically as the temperature drops to zero. We observe that for some classical lattice systems there are interactions exhibiting a related phenomenon of sensitive dependence of Gibbs measures: An arbitrarily small perturbation of the interaction can produce significant changes in the low-temperature behavior of its Gibbs measures. For some one-dimensional XY models we exhibit sensitive dependence of Gibbs measures for a (nearest-neighbor) interaction given by a smooth function, and for perturbations that are small in the smooth category. We also exhibit sensitive dependence of Gibbs measures for an interaction on a classical lattice system with finite-state space. This interaction decreases exponentially as a function of the distance between sites; it is given by a Lipschitz continuous potential in the configuration space. The perturbations are small in the Lipschitz topology. As a by-product we solve some problems stated by Chazottes and Hochman.

Keywords Low-temperature Gibbs measure · XY models · Lattice system

1 Introduction

The Gibbs measures of an interaction can behave chaotically as the temperature drops to zero. This phenomenon was first exhibited by van Enter and Ruszel for N-vector models [27], and later by Chazottes and Hochman for a classical lattice system with finite-state space [9]. More

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recently, chaotic temperature dependence was exhibited by the authors for a quasi-quadratic map and its geometric potential [12].

In [12], a related phenomenon of "sensitive dependence of geometric Gibbs measures" was found: An arbitrarily small perturbation of the map can produce significant changes in the low-temperature behavior of its geometric Gibbs measures. The purpose of this paper is to show that this phenomenon is also present in some one-dimensional 2-vector models,[1] as well as in some classical lattice systems of arbitrary dimension and finite-state space. Since the methods used here, based partially on the hats-in-hats idea of [27], differ significantly from those of [12], this gives evidence that the sensitive dependence of Gibbs measures phenomenon is robust; it does not depend on the particulars of either setting.

Roughly speaking, for a given interaction "chaotic temperature dependence" is the non-convergence of Gibbs measures along a certain sequence of temperatures going to zero. This concept first arose in the spin-glass literature, where the interactions contain disorder, see for example [23,24]. In contrast, in [9,12,27] and in this note the interactions are deterministic and contain no disorder.

In the chaotic temperature dependence, the non-convergence of Gibbs measures cannot occur for every sequence of temperatures going to zero, due to the compactness of the space of probability measures. In rough terms, the phenomenon of sensitive dependence of Gibbs measures exhibited here, is that the non-convergence can indeed occur along any prescribed sequence of temperatures going to zero, by making an arbitrarily small perturbation of the original interaction.

To be more precise, we introduce the following terminology. An interaction Φ is *chaotic*, if there is a sequence of inverse temperatures $(\beta_\ell)_{\ell\in\mathbb{N}}$ such that $\beta_\ell \to +\infty$ as $\ell \to +\infty$, and such that the following property holds: If for each ℓ in $\mathbb{N}$ we choose an arbitrary Gibbs measure ρ_ℓ for the interaction $\beta_\ell \cdot \Phi$, then the sequence $(\rho_\ell)_{\ell\in\mathbb{N}}$ does not converge.[2] That is, an interaction is chaotic if it displays chaotic temperature dependence in the sense of van Enter and Ruszel [27].

We also say an interaction Φ is *sensitive*, if for every sequence of inverse temperatures $(\beta_\ell)_{\ell\in\mathbb{N}}$ such that $\beta_\ell \to +\infty$ as $\ell \to +\infty$ there is an arbitrarily small perturbation $\widetilde{\Phi}$ of Φ[3] such that the following property holds: If for each ℓ in $\mathbb{N}$ we choose an arbitrary Gibbs measure ρ_ℓ for the interaction $\beta_\ell \cdot \widetilde{\Phi}$, then the sequence $(\rho_\ell)_{\ell\in\mathbb{N}}$ does not converge. Note that in the definition of chaotic interaction the sequence of inverse temperatures $(\beta_\ell)_{\ell\in\mathbb{N}}$ is not arbitrary; in fact, by the compactness of the space of probability measures we can always choose a sequence of inverse temperatures and corresponding Gibbs measures for which we have convergence. In contrast, in the definition of sensitive interaction the sequence of

[1] We thank Aernout van Enter for pointing out that the proof of the sensitive dependence of Gibbs measures for one-dimensional 2-vector models given here extends to N-vector models of arbitrary dimension.

[2] In this definition, it is crucial that the non-convergence holds for an arbitrary choice of Gibbs measure for each ℓ. For example, for the 2-dimensional Ising model and an arbitrary sequence of inverse temperatures going to infinity, there are some choices of Gibbs measures for which we have non-convergence, like choosing the ferromagnetic and the antiferromagnetic phases in an alternate way. However, there is no sensitive dependence because there are some choices for which we do have convergence, like choosing the ferromagnetic phase for each ℓ. See also [14] for further remarks and clarifications about this phenomenon.

[3] The notion of proximity between interactions depends on the setting. For one-dimensional XY models we consider the space of symmetric nearest-neighbor interactions. Such an interaction is determined by a function on the circle, as in the classical XY model. In this setting we use the smooth topology on the space of smooth functions on the circle as a notion of proximity between interactions, see Sect. 1.1. In the case of a classical lattice system with finite-state space, the interactions we consider are determined by a potential defined on the configuration space. In this setting we use the Lipschitz topology, see Sect. 1.2.

inverse temperatures $(\beta_\ell)_{\ell\in\mathbb{N}}$ is arbitrary, but the non-convergence is for Gibbs measures of a perturbation of the original interaction.[4]

For one-dimensional 2-vector (or XY) models, we exhibit a sensitive interaction by modifying the example of van Enter and Ruszel, see Sect. 1.1. In our modification of their example, the (nearest-neighbor) interaction is determined by a smooth function defined on the circle and the perturbations are small in the smooth topology. We show this is in a certain sense best possible: In the analytic category there is no chaotic interaction, and therefore no sensitive interaction.

In the case of a classical lattice system with finite-state space, we exhibit a sensitive interaction that decays exponentially as a function of the distance between sites: The interaction is given by a Lipschitz continuous potential on the configuration space, and the perturbations are small in the Lipschitz topology, see Sect. 1.2. We use a new construction that is very flexible and that allows us to solve some of the problems stated by Chazottes and Hochman in [9].[5]

1.1 One-Dimensional XY Models

Denote the circle by $\mathbb{T}:=\mathbb{R}/\mathbb{Z}$, endowed with the (additive) group structure inherited from $\mathbb{R}$. Given a function $U\colon \mathbb{T}\to\mathbb{R}$, consider the nearest-neighbor interaction Φ_U on $\mathbb{T}^{\mathbb{Z}}$ defined by

$$\Phi_U(\{k,k+1\})\left((\theta_n)_{n\in\mathbb{Z}}\right) := -\,U(\theta_k-\theta_{k+1}).$$

When U is continuous there is a unique Gibbs measure for the interaction Φ_U, and this measure is translation invariant, see for example [26, Theorem III.8.2] or Lemma 2.1. Denote this measure by ρ_U.

A configuration $(\theta_n)_{n\in\mathbb{Z}}$ in $\mathbb{T}^{\mathbb{Z}}$ is *ferromagnetic* (resp. *antiferromagnetic*), if for every n we have $\theta_{n+1}=\theta_n$ (resp. $\theta_{n+1}=\theta_n+\frac{1}{2}$). Note that a ferromagnetic (resp. antiferromagnetic) configuration is completely determined by its value at the site $n=0$. The *ferromagnetic* (resp. *antiferromagnetic*) *phase* is the measure on $\mathbb{T}^{\mathbb{Z}}$ that is evenly distributed on ferromagnetic (resp. antiferromagnetic) configurations.

Theorem A (Sensitive dependence of Gibbs measures on the interaction) *There is a smooth function $U_0\colon\mathbb{T}\to\mathbb{R}$ such that for every sequence of positive numbers $(\widehat{\beta}_\ell)_{\ell\in\mathbb{N}}$ satisfying $\widehat{\beta}_\ell\to+\infty$ as $\ell\to+\infty$, the following property holds: There is an arbitrarily small smooth perturbation U of U_0 such that the sequence of Gibbs measures $\left(\rho_{\widehat{\beta}_\ell\cdot U}\right)_{\ell\in\mathbb{N}}$ accumulates at the same time on the ferromagnetic and the antiferromagnetic phases.*

Using the terminology introduced above, this theorem implies that the interaction Φ_{U_0} is sensitive. We show U_0 can be chosen so that in addition the interaction Φ_{U_0} is chaotic (resp. non-chaotic), see Remark 2.3. More precisely, we show that the function U_0 can be chosen so that the family of Gibbs measures $(\rho_{\beta\cdot U_0})_{\beta>0}$ converges to either the ferromagnetic, or the antiferromagnetic phase as $\beta\to+\infty$; it can also be chosen so that this family of Gibbs measures accumulates at the same time on the ferromagnetic and antiferromagnetic phases as $\beta\to+\infty$.

[4] Note that in the definition of sensitive interaction there is no assertion about the low-temperature behavior of Gibbs measures of the original interaction. In fact, we show that there are sensitive interactions that are chaotic, and that there are sensitive interactions that are non-chaotic.

[5] It is also possible to modify the construction of Chazottes and Hochman in [9] to exhibit a sensitive interaction. The construction introduced here is more qualitative in nature, and somewhat simpler.

The first example of a chaotic interaction was given by van Enter and Ruszel in [27] using a discontinuous function U, see also [2, Sect. 6]. We use a modification of their example that allows us to get smooth functions. The smooth regularity is essentially optimal: In the real analytic category there are no chaotic interactions, and therefore no sensitive ones. In fact, for every real analytic function $U: \mathbb{T} \to \mathbb{R}$, the one-parameter family of Gibbs measures $(\rho_{\beta \cdot U})_{\beta > 0}$ converges as $\beta \to +\infty$, see Remark 2.4. See also [2, 19–21] for other results on the behavior of Gibbs measures as temperature drops to zero.

The following is our main technical result, from which Theorem A follows easily. Throughout this note we endow $\{+, -\}$ with the discrete topology, and $\{+, -\}^{\mathbb{N}}$ with the corresponding product topology. Denote by $\pi : \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}$ the projection defined by

$$\pi \left((\theta_n)_{n \in \mathbb{Z}} \right) := \theta_0 - \theta_1.$$

Main Lemma A *There is a family of smooth functions* $\left(U(\underline{\varsigma}) \right)_{\underline{\varsigma} \in \{+,-\}^{\mathbb{N}}}$ *that is continuous in the* C^∞ *topology, and such that the following property holds. For each integer* $m \geq 1$ put $\beta_m := 2^{(m+10)^3}$. *Then for every* $\underline{\varsigma} = (\varsigma(m))_{m \in \mathbb{N}}$ *in* $\{+, -\}^{\mathbb{N}}$, *every pair of integers* $\widehat{m}$ and m *satisfying*

$$\widehat{m} \geq m \geq 1 \text{ and } \varsigma(m) = \cdots = \varsigma(\widehat{m}),$$

and every β in $[\beta_m, \beta_{\widehat{m}}]$, *the unique Gibbs measure* $\rho_{\beta \cdot U(\underline{\varsigma})}$ *for the interaction* $\Phi_{\beta \cdot U(\underline{\varsigma})}$ *satisfies*

$$\rho_{\beta \cdot U(\underline{\varsigma})} \left(\pi^{-1} \left(\left[-2^{-(m+1)^2}, 2^{-(m+1)^2} \right] \right) \right) \geq 1 - 2^{-m}$$

if $\varsigma(m) = +$, *and*

$$\rho_{\beta \cdot U(\underline{\varsigma})} \left(\pi^{-1} \left(\left[\tfrac{1}{2} - 2^{-(m+1)^2}, \tfrac{1}{2} + 2^{-(m+1)^2} \right] \right) \right) \geq 1 - 2^{-m}$$

if $\varsigma(m) = -$.

The proofs of Theorem A and Main Lemma A are given in Sect. 2.

1.2 Symbolic Space

Let $d \geq 1$ be an integer, and let G be either $\mathbb{Z}^d$ or $\mathbb{N}_0^d$. Given a finite set F containing at least 2 elements, consider the space $\Sigma := F^G$ endowed with the distance dist defined for distinct elements $(\theta_n)_{n \in G}$ and $(\theta'_n)_{n \in G}$ of Σ, by

$$\operatorname{dist} \left((\theta_n)_{n \in G}, (\theta'_n)_{n \in G} \right) := 2^{-\min\{\|n\| : \theta_n \neq \theta'_n\}},$$

where $\| \cdot \|$ is the sup-norm. Denote by σ the action of G on Σ by translations, by $\mathscr{M}$ the space of Borel probability measures on Σ endowed with the weak* topology, and by $\mathscr{M}_\sigma$ the subspace of those that are invariant by σ. For ν in $\mathscr{M}_\sigma$, denote by h_ν the *measure-theoretic entropy of* ν. The *topological pressure of a continuous function* $\varphi : \Sigma \to \mathbb{R}$, is

$$P(\varphi) := \sup \left\{ h_\nu + \int \varphi \, \mathrm{d}\nu : \nu \in \mathscr{M}_\sigma \right\}.$$

A *equilibrium state for the potential* φ is a measure ν at which the supremum above is attained. When φ is Lipschitz continuous and $F = \mathbb{Z}^d$, the set of equilibrium states agrees with the set of translation invariant Gibbs measures, see for instance [13, Theorem 5.3.1]. Moreover, if the dimension d is 1, then there is a unique equilibrium state that we denote by ρ_φ, see for example [6, Theorem 1.22] in the case $G = \mathbb{N}_0$ and [26, Theorem III.8.2] in the case $G = \mathbb{Z}$.

From now on we use "translation invariant Gibbs measure" instead of equilibrium state, even when $G = \mathbb{N}_0^d$.

Theorem B (Sensitive dependence of Gibbs measures on the potential) *There is a Lipschitz continuous potential $\varphi_0 \colon \Sigma \to \mathbb{R}$ and complementary open subsets U^+ and U^- of Σ, such that for every sequence of positive numbers $(\widehat{\beta}_\ell)_{\ell \in \mathbb{N}}$ satisfying $\widehat{\beta}_\ell \to +\infty$ as $\ell \to +\infty$, the following property holds: There is an arbitrarily small Lipschitz continuous perturbation φ of φ_0 such that if for each ℓ we choose an arbitrary translation invariant Gibbs measure ρ_ℓ for the potential $\widehat{\beta}(\ell) \cdot \varphi$, then the sequence $(\rho_\ell)_{\ell \in \mathbb{N}}$ accumulates at the same time on a measure supported on U^+ and on a measure supported on U^-.*

Using the terminology above, this theorem shows the potential φ_0 is sensitive for translation invariant Gibbs measures. As for XY models in Sect. 1.1, the interaction can be chosen to be chaotic, and it can also be chosen to be non-chaotic.

The first examples in a finite-state space of a chaotic interaction for translation invariant Gibbs measures were given in dimensions $d = 1$ and $d \geq 3$ by Chazottes and Hochman in [9]. In dimensions $d \geq 3$, the interactions constructed by Chazottes and Hochman are of finite range. To the best of our knowledge it is open if in dimension $d = 2$ there is a finite-range interaction that is chaotic for translation invariant Gibbs measures.[6] In dimension $d = 1$, Brémont showed that there is no finite-range interaction that is chaotic, see [7] and also [8, 16, 22]. See also [3, 5, 11, 17] and the monograph [4] for recent related results.

Theorem B follows easily from the following.

Main Lemma B *There is a continuous family of Lipschitz continuous potentials $\left(\varphi(\underline{\varsigma})\right)_{\underline{\varsigma} \in \{+,-\}^{\mathbb{N}}}$, complementary open subsets U^+* and *U^-* of Σ, *and an increasing sequence of positive numbers $(\beta_m)_{m \in \mathbb{N}}$ converging to $+\infty$, such that the following property holds: For every $\underline{\varsigma} = (\varsigma(m))_{m \in \mathbb{N}}$* in $\{+,-\}^{\mathbb{N}}$, *every pair of integers m and $\widehat{m}$ such that*

$$\widehat{m} \geq m \geq 1 \text{ and } \varsigma(m) = \cdots = \varsigma(\widehat{m}),$$

every β in $[\beta_m, \beta_{\widehat{m}}]$, *and every translation invariant Gibbs measure ρ for the potential $\beta \cdot \varphi(\underline{\varsigma})$, we have*

$$\rho(U^{\varsigma(m)}) \geq 1 - 2^{-m}.$$

The following is a corollary of (the proof of) Main Lemma B. This corollary is proven in Sect. 3.2. A subset X of Σ is *invariant* if for every g in G we have $\sigma^g(X) = X$.[7]

Corollary 1.1 *Assume that the dimension d is 1. Let X^+ and X^- be disjoint compact subsets of Σ that are invariant, minimal, and uniquely ergodic for σ. Suppose furthermore that $h_{\mathrm{top}}(\sigma|_{X^+}) = h_{\mathrm{top}}(\sigma|_{X^-})$, and let ρ^+ and ρ^- be the unique invariant probability measure supported on X^+ and X^-, respectively. Then there is a Lipschitz continuous potential $\varphi \colon \Sigma \to \mathbb{R}$ such that the one-parameter family of Gibbs measures $\left(\rho_{\beta \cdot \varphi}\right)_{\beta > 0}$ accumulates at the same time on ρ^+ and ρ^- as $\beta \to +\infty$.*

[6] We note that for $d = 2$ the interaction given by Theorem B is chaotic, but it is not of finite range. It seems possible to obtain a similar example by adapting the construction of Chazottes and Hochman in [9]. To be more precise, denote by X the subshift of $\{0,1\}^{\mathbb{N}}$ constructed in [9], and denote by $\widehat{X}$ an invariant subshift of $\{0,1\}^{\mathbb{N}^2}$ obtained by embedding the product of countably many copies of X, as in Sect. 3.2.2. Then it seems possible to adapt the computations in [9], to show that the potential $x \mapsto -\mathrm{dist}(x, \widehat{X})$ is chaotic.

[7] When $G = \mathbb{N}_0^d$ such a set is sometimes called "forward invariant".

The hypothesis $h_{\mathrm{top}}(\sigma|_{X^+}) = h_{\mathrm{top}}(\sigma|_{X^-})$ is necessary, see [1, Corollary 1] or Lemma 3.1.

Combined with the Jewett–Krieger realization theorem, the following is a direct consequence of the previous corollary, see for example [15, Corollary 3.2]. It solves a problem formulated by Chazottes and Hochman in [9, Sect. 4.2], see also Appendix 1.

Corollary 1.2 *Assume that the dimension d is 1. Let μ^+ and μ^- be ergodic measures defined on a Lebesgue space having the same finite entropy. Then, provided the finite set F is sufficiently large, there is a Lipschitz continuous potential $\varphi\colon \Sigma \to \mathbb{R}$ such that the one-parameter family of Gibbs measures $\left(\rho_{\beta\cdot\varphi}\right)_{\beta>0}$ accumulates at the same time on a measure isomorphic to μ^+ and on a measure isomorphic to μ^- as $\beta \to +\infty$.*

The proof of Main Lemma B is given in Sect. 3. The deduction of Theorem B from Main Lemma B is analogous to that of Theorem A from Main Lemma A given in Sect. 2, and we omit it.

2 One-Dimensional XY Models

The purpose of this section is to prove Theorem A and Main Lemma A. After some general considerations on Gibbs measures in Sect. 2.1, the proofs of these results are given in Sect. 2.2.

Throughout this section we use Leb to denote the probability measure on $\mathbb{T}$ induced by the Lebesgue measure on $\mathbb{R}$.

2.1 Gibbs Measures of Symmetric Nearest-Neighbor Interactions

Let $D\colon \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}}$ be the map defined by

$$D\left((\theta_n)_{n\in\mathbb{Z}}\right) := \left((\theta_n - \theta_{n+1})_{n\in\mathbb{Z}}\right),$$

and for each θ in $\mathbb{T}$ denote by $T_\theta : \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}}$ the map defined by

$$T_\theta\left((\theta_n)_{n\in\mathbb{Z}}\right) := \left((\theta_n + \theta)_{n\in\mathbb{Z}}\right).$$

Note that for each θ in $\mathbb{T}$ we have $D\circ T_\theta = D$, and that for each $\underline{\theta}$ in $\mathbb{T}^{\mathbb{Z}}$ we have $D^{-1}(D(\underline{\theta})) = \{T_\theta(\underline{\theta}) : \theta \in \mathbb{T}\}$.

A measure on $\mathbb{T}^{\mathbb{Z}}$ is *symmetric* if for each θ in $\mathbb{T}$ it is invariant by T_θ.

Lemma 2.1 *Let $U : \mathbb{T} \to \mathbb{R}$ be a continuous function. Then for every β in $\mathbb{R}$ there is a unique Gibbs measure $\rho_{\beta\cdot U}$ for the interaction $\Phi_{\beta\cdot U}$. Moreover, $\rho_{\beta\cdot U}$ is characterized as the unique symmetric measure whose image by D is equal to*

$$\bigotimes_{\mathbb{Z}}\left(\frac{\exp(\beta\cdot U)}{\int_{\mathbb{T}}\exp(\beta\cdot U(\theta))\,\mathrm{d}\theta}\right)\mathrm{Leb}.$$

In particular, we have

$$\pi_*\rho_{\beta\cdot U} = \left(\frac{\exp(\beta\cdot U)}{\int_{\mathbb{T}}\exp(\beta\cdot U(\theta))\,\mathrm{d}\theta}\right)\mathrm{Leb}. \tag{2.1}$$

The proof of this lemma is given after the following general lemma.

Lemma 2.2 *For every measure μ on $\mathbb{T}^{\mathbb{Z}}$ there is a unique symmetric measure $\widehat{\mu}$ on $\mathbb{T}^{\mathbb{Z}}$ such that $D_*\widehat{\mu} = \mu$.*

Proof Given a continuous function $f\colon \mathbb{T}^{\mathbb{Z}} \to \mathbb{R}$, put $\widehat{f} := \int f \circ T_\theta \, \mathrm{dLeb}(\theta)$, and note that there is a continuous function $\widetilde{f}\colon \mathbb{T}^{\mathbb{Z}} \to \mathbb{R}$ satisfying $\widehat{f} = \widetilde{f} \circ D$.

Given a measure μ on $\mathbb{T}^{\mathbb{Z}}$, the map $f \mapsto \int \widetilde{f} \, \mathrm{d}\mu$ defines a symmetric measure on $\mathbb{T}^{\mathbb{Z}}$ whose image by D is equal to μ. To prove that this is the only measure with these properties, let $\widehat{\mu}$ be a symmetric measure on $\mathbb{T}^{\mathbb{Z}}$ satisfying $D_*\widehat{\mu} = \mu$, and let $f\colon \mathbb{T}^{\mathbb{Z}} \to \mathbb{R}$ be a continuous function. Then by the change of variable formula we have

$$\int_{\mathbb{T}^{\mathbb{Z}}} f \, \mathrm{d}\widehat{\mu} = \int_{\mathbb{T}} \int_{\mathbb{T}^{\mathbb{Z}}} f \circ T_\theta \, \mathrm{d}\widehat{\mu} \, \mathrm{d}\theta = \int_{\mathbb{T}^{\mathbb{Z}}} \widehat{f} \, \mathrm{d}\widehat{\mu} = \int_{\mathbb{T}^{\mathbb{Z}}} \widetilde{f} \, \mathrm{d}\mu.$$

This proves uniqueness and completes the proof of the lemma. □

Proof of Lemma 2.1 Replacing U by $\beta \cdot U$ if necessary, assume $\beta = 1$.

Denote by $\mathscr{P}_f(\mathbb{Z})$ the collection of finite subsets of $\mathbb{Z}$. For Λ in $\mathscr{P}_f(\mathbb{Z})$ denote by $\pi_\Lambda\colon \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}^{\Lambda}$ the canonical projection, and by $\mathrm{Leb}_\Lambda := \bigotimes_\Lambda \mathrm{Leb}$ the product measure on $\mathbb{T}^{\Lambda}$. Moreover, consider the free boundary condition Hamiltonian $H_\Lambda\colon \mathbb{T}^{\Lambda} \to \mathbb{R}$ defined by

$$H_\Lambda = \sum_{X \in \mathscr{P}_f(\mathbb{Z}) : X \subset \Lambda} \Phi_U(X),$$

and put $Z_\Lambda := \int_{\mathbb{T}^\Lambda} \exp\left(-H_\Lambda(\underline{\theta})\right) \mathrm{dLeb}_\Lambda(\underline{\theta})$.

For each integer $n \geq 1$, put

$$\Lambda_n := \{-n, \ldots, n\} \text{ and } \rho_n := Z_{\Lambda_n}^{-1} \exp\left(-H_{\Lambda_n}\right) \mathrm{Leb}_{\Lambda_n}.$$

A straightforward computation shows that for every pair of integers n and $\widehat{n}$ satisfying $\widehat{n} \geq n \geq 1$, and for every measurable subset A of $\mathbb{T}^{\Lambda_n}$ we have $\rho_{\widehat{n}}\left(A \times \mathbb{T}^{\Lambda_{\widehat{n}} \setminus \Lambda_n}\right) = \rho_n(A)$. Thus, by Kolmogorov's theorem there is a unique measure ρ_∞ on $\mathbb{T}^{\mathbb{Z}}$ so that for every integer $n \geq 1$ we have $(\pi_{\Lambda_n})_* \rho_\infty = \rho_n$. The measure ρ_∞ is clearly translation invariant.

Denote by $\mathscr{M}_U$ the simplex of all Gibbs measures for the interaction Φ_U. In part 1 below we show that ρ_∞ is absolutely continuous with respect to each measure in $\mathscr{M}_U$, and in part 2 we conclude the proof of the lemma using this fact.

(1) For each Λ in $\mathscr{P}_f(\mathbb{Z})$ and each $\underline{\theta}'$ in $\mathbb{T}^{\mathbb{Z} \setminus \Lambda}$ consider the Hamiltonian $H_\Lambda\colon \mathbb{T}^{\Lambda} \times \mathbb{T}^{\mathbb{Z}\setminus\Lambda} \to \mathbb{R}$ defined by

$$H_\Lambda(\underline{\theta}|\underline{\theta}') = \sum_{X \in \mathscr{P}_f(\mathbb{Z}) : X \cap \Lambda \neq \emptyset} \Phi_U(X)(\underline{\theta} \times \underline{\theta}'),$$

and put $Z_\Lambda(\underline{\theta}') := \int_{\mathbb{T}^\Lambda} \exp\left(-H_\Lambda(\underline{\theta}|\underline{\theta}')\right) \mathrm{dLeb}_\Lambda(\underline{\theta})$.

Putting $C := \sup_{\mathbb{T}} |U|$, which is finite since U is continuous, for every integer $n \geq 1$, every $\underline{\theta}$ in $\mathbb{T}^{\Lambda_n}$, and every $\underline{\theta}'$ in $\mathbb{T}^{\mathbb{Z}\setminus\Lambda_n}$, we have

$$|H(\underline{\theta}|\underline{\theta}') - H(\underline{\theta})| \leq 2C.$$

It follows that $Z_{\Lambda_n} \geq \exp(-2C) Z_{\Lambda_n}(\underline{\theta}')$ and, together with the DLR equations, that for every ρ in $\mathscr{M}_U$ and every measurable subset A of $\mathbb{T}^{\Lambda_n}$ we have

$$
\begin{aligned}
&\rho_\infty\left(A \times \mathbb{T}^{\mathbb{Z}\setminus\Lambda_n}\right)\\
&= \int_A Z_{\Lambda_n}^{-1} \exp\left(-H_{\Lambda_n}(\underline{\theta})\right) \mathrm{dLeb}_{\Lambda_n}(\underline{\theta})\\
&= \int_{\mathbb{T}^{\mathbb{Z}\setminus\Lambda_n}} \int_A Z_{\Lambda_n}^{-1} \exp\left(-H_{\Lambda_n}(\underline{\theta})\right) \mathrm{dLeb}_{\Lambda_n}(\underline{\theta})\, \mathrm{d}(\pi_{\mathbb{Z}\setminus\Lambda_n})_*\rho(\underline{\theta}')\\
&\le \exp(4C) \int_{\mathbb{T}^{\mathbb{Z}\setminus\Lambda_n}} \int_A Z_{\Lambda_n}(\underline{\theta}')^{-1} \exp\left(-H_{\Lambda_n}(\underline{\theta}|\underline{\theta}')\right) \mathrm{dLeb}_{\Lambda_n}(\underline{\theta})\, \mathrm{d}(\pi_{\mathbb{Z}\setminus\Lambda_n})_*\rho(\underline{\theta}')\\
&= \exp(4C)\rho\left(A \times \mathbb{T}^{\mathbb{Z}\setminus\Lambda_n}\right).
\end{aligned}
$$

Since n and A are arbitrary, this shows that ρ_∞ is absolutely continuous with respect to ρ.

(2) By, e.g., [26, Corollaries III.2.10 and III.3.10], there is a translation invariant and ergodic measure ρ in $\mathscr{M}_U$. By part 1 the measure ρ_∞ is absolutely continuous with respect to ρ. Since ρ_∞ is also translation invariant, it follows that $\rho = \rho_\infty$, see for example [13, Lemma 2.2.2]. In particular, ρ_∞ is in $\mathscr{M}_U$. Let ρ' and ρ'' be pure states for the interaction Φ_U, *i.e.*, extreme points of $\mathscr{M}_U$. By part 1 the measure ρ_∞ is absolutely continuous with respect to ρ' and with respect to ρ''. This implies that ρ and ρ' are not mutually singular, and therefore that they are equal, see for example [26, Theorem III.5.1(b)]. This proves that $\mathscr{M}_U$ has a unique extreme point, and therefore that $\mathscr{M}_U$ is reduced to $\{\rho_\infty\}$.

Since for each θ in $\mathbb{T}$ the interaction Φ_U is invariant by T_θ, and since ρ_∞ is the unique Gibbs measure for this interaction, it follows that ρ_∞ is symmetric. On the other hand, if for each integer $n \ge 1$ we denote by $D_n\colon \mathbb{T}^{\Lambda_n} \to \mathbb{T}^{\Lambda_n\setminus\{n\}}$ the map defined by

$$
D_n\left((\theta_k)_{k\in\Lambda_n}\right) = (\theta_k - \theta_{k+1})_{k\in\Lambda_n\setminus\{n\}},
$$

then a straightforward computation shows that

$$
\left(\pi_{\Lambda_n\setminus\{n\}}\right)_* D_*\rho_\infty = (D_n)_*\rho_n = \bigotimes_{\Lambda_n\setminus\{n\}} \left(\frac{\exp(U)}{\int_{\mathbb{T}} \exp(U(\theta))\, \mathrm{d}\theta}\right) \mathrm{Leb}\,.
$$

Since this holds for every n, this proves $D_*\rho_\infty = \bigotimes_{\mathbb{Z}} \left(\frac{\exp(U)}{\int_{\mathbb{T}} \exp(U(\theta))\, \mathrm{d}\theta}\right) \mathrm{Leb}$, and together with Lemma 2.2 concludes the proof of the lemma. □

2.2 Proofs of Theorem A and Main Lemma A

Note that, with the notation and terminology in Sect. 2.1, the ferromagnetic (resp. antiferromagnetic) phase is characterized as the unique symmetric measure on $\mathbb{T}^{\mathbb{Z}}$ whose image by D is equal to $\bigotimes_{\mathbb{Z}} \delta_0$ (resp. $\bigotimes_{\mathbb{Z}} \delta_{\frac{1}{2}}$).

For an integer $r \ge 1$ and a function $\varphi\colon \mathbb{T} \to \mathbb{R}$ that is r times continuously differentiable, consider the C^r-norm of φ:

$$
\|\varphi\|_{C^r} := \sum_{\ell=0}^{r} \left\|\varphi^{(\ell)}\right\|_\infty .
$$

Fix a smooth function $\chi\colon \mathbb{R} \to [0,1]$ that is constant equal to 0 on $\mathbb{R}\setminus(-1,1)$ and constant equal to 1 on $\left[-\frac{2}{3}, \frac{2}{3}\right]$. For an interval I of $\mathbb{T}$, denote by $|I| := \mathrm{Leb}(I)$ its length, and let $\chi_I\colon \mathbb{T} \to [0,1]$ be the function that is constant equal to 0 on $\mathbb{T}\setminus I$ and that is defined on I as follows: Let c in $\mathbb{R}$ be such that $c \mod \mathbb{Z}$ is the middle point of I, and for each x in $\left[-\frac{|I|}{2}, \frac{|I|}{2}\right]$ put

$$\chi_I(c + x \mod \mathbb{Z}) := \chi\left(\frac{2x}{|I|}\right).$$

Note that for every integer $\ell \geq 0$ we have $\left\|\chi_I^{(\ell)}\right\|_\infty = \left(\frac{2}{|I|}\right)^\ell \|\chi^{(\ell)}\|_\infty$. Considering that $|I| \leq 1$, this implies

$$\|\chi_I\|_{C^r} \leq \left(\tfrac{2}{|I|}\right)^r \|\chi\|_{C^r}. \tag{2.2}$$

Proof of Main Lemma A For each integer $m \geq 0$, define the following intervals of $\mathbb{T}$:

$$I_m^+ := \left[-2^{-(m+11)^2}, 2^{-(m+11)^2}\right] \text{ and } I_m^- := I_m^+ + \tfrac{1}{2}.$$

Moreover, for $m \geq 1$ put

$$Y_m^+ = I_{m-1}^+ \cup I_m^-, Y_m^- = I_{m-1}^- \cup I_m^+,$$

$$\chi_m^+ := \chi_{I_{m-1}^+} + \chi_{I_m^-}, \text{ and } \chi_m^- := \chi_{I_{m-1}^-} + \chi_{I_m^+}.$$

Note that for each ς in $\{+, -\}$ we have $Y_{m+1}^\varsigma \subset Y_m^\varsigma$. Finally, put $\beta_0 := 2^{10^3}$ and for $\underline{\varsigma} = (\varsigma(m))_{m\in\mathbb{N}}$ in $\{+, -\}^\mathbb{N}$ put

$$U(\underline{\varsigma}) := -\beta_0^{-1} + \sum_{m=1}^{+\infty} \left(\beta_{m-1}^{-1} - \beta_m^{-1}\right) \chi_m^{\varsigma(m)}.$$

Note that for each integer $r \geq 1$ we have by (2.2)

$$\begin{aligned}
\sum_{m=1}^{+\infty} \beta_{m-1}^{-1} \left\|\chi_m^{\varsigma(m)}\right\|_{C^r} &\leq 2\|\chi\|_{C^r} \sum_{m=1}^{+\infty} 2^{-(m+9)^3} \left(\frac{2}{2 \cdot 2^{-(m+11)^2}}\right)^r \\
&\leq 2\|\chi\|_{C^r} \sum_{m=1}^{+\infty} 2^{-((m+9)^3 - r(m+11)^2)} \\
&< +\infty,
\end{aligned}$$

so the series defining $U(\underline{\varsigma})$ converges uniformly with respect to $\|\cdot\|_{C^r}$. It follows that $U(\underline{\varsigma})$ is r times differentiable. Since $r \geq 1$ is arbitrary, this proves that $U(\underline{\varsigma})$ is smooth. To prove that $U(\underline{\varsigma})$ depends continuously on $\underline{\varsigma}$ in the C^∞ topology, let $r \geq 1$ and $m_0 \geq 1$ be given integers and let $\underline{\varsigma} = (\varsigma(m))_{m\in\mathbb{N}}$ and $\underline{\varsigma}' = (\varsigma'(m))_{m\in\mathbb{N}}$ be such that for every k in $\{1, \ldots, m_0\}$ we have $\varsigma(k) = \varsigma'(k)$. Then by (2.2) we have

$$\begin{aligned}
\left\|U\left(\underline{\varsigma}\right) - U\left(\underline{\varsigma}'\right)\right\|_{C^r} &\leq \sum_{m=m_0+1}^{+\infty} \beta_{m-1}^{-1} \left\|\chi_{Y_m^+} - \chi_{Y_m^-}\right\|_{C^r} \\
&\leq 4\|\chi\|_{C^r} \sum_{m=m_0+1}^{+\infty} 2^{-(m+9)^3} \left(\frac{2}{2 \cdot 2^{-(m+11)^2}}\right)^r \\
&\leq 4\|\chi\|_{C^r} \sum_{m=m_0+1}^{+\infty} 2^{-((m+9)^3 - r(m+11)^2)}.
\end{aligned}$$

Since this last sum goes to 0 as $m_0 \to +\infty$, it follows that $U(\underline{\varsigma})$ depends continuously on $\underline{\varsigma}$ in the C^r topology. Since $r \geq 1$ is arbitrary, we conclude that $U(\underline{\varsigma})$ depends continuously on $\underline{\varsigma}$ in the C^∞ topology.

To prove the estimate of the theorem, define for each integer $m \geq 1$ the subsets of $\mathbb{T}$:

$$M_m^+ := \left[-\tfrac{2}{3}2^{-(m+10)^2}, -\tfrac{1}{3}2^{-(m+10)^2}\right] \cup \left[\tfrac{1}{3}2^{-(m+10)^2}, \tfrac{2}{3}2^{-(m+10)^2}\right]$$

and

$$M_m^- := M_m^+ + \tfrac{1}{2}.$$

Note that for each ς in $\{+, -\}$ we have

$$M_m^\varsigma \subset Y_m^\varsigma \setminus \left(Y_{m+1}^+ \cup Y_{m+1}^-\right).$$

Let $\underline{\varsigma} = (\varsigma(m))_{m\in\mathbb{N}}$ be in $\{+, -\}^{\mathbb{N}}$ and let μ and $\widehat{\mu}$ be in $\mathbb{N}$ such that

$$\widehat{\mu} \geq \mu \geq 1 \text{ and } \varsigma(\mu) = \cdots = \varsigma(\widehat{\mu}).$$

Fix β in $\left[\beta_\mu, \beta_{\widehat{\mu}}\right]$, let m_0 be the least integer m in $\{\mu, \ldots, \widehat{\mu}\}$ such that $\beta_m \geq \beta$, and put

$$\rho_\beta := \pi_* \rho_{\beta \cdot U(\underline{\varsigma})} \text{ and } Z(\beta) := \int_{\mathbb{T}} \exp(\beta \cdot U(\theta))\, \mathrm{d}\theta.$$

By (2.1) we have $\rho_\beta = \frac{\exp(\beta \cdot U)}{Z(\beta)}$ Leb.

Noting that $U(\underline{\varsigma})$ is constant equal to $-\beta_{m_0}^{-1}$ on $M_{m_0}^{\varsigma(m_0)}$, by (2.1) we have

$$\begin{aligned}\rho_\beta\left(M_{m_0}^{\varsigma(m_0)}\right) &= \frac{1}{Z(\beta)} \text{Leb}\left(M_{m_0}^{\varsigma(m_0)}\right) \exp\left(-\beta\beta_{m_0}^{-1}\right)\\ &= \left(\tfrac{2}{3}\frac{1}{Z(\beta)}\right) 2^{-(m_0+10)^2} \exp\left(-\beta\beta_{m_0}^{-1}\right). \qquad (2.3)\end{aligned}$$

On the other hand, noting that $U(\underline{\varsigma})$ is constant equal to $-\beta_0^{-1}$ on $\mathbb{T}\setminus Y_1^{\varsigma(1)}$, by (2.1) we have

$$\begin{aligned}\rho_\beta\left(\mathbb{T}\setminus Y_1^{\varsigma(1)}\right) &= \frac{1}{Z(\beta)} \text{Leb}\left(\mathbb{T}\setminus Y_1^{\varsigma(1)}\right) \exp\left(-\beta\beta_0^{-1}\right)\\ &\leq \frac{1}{Z(\beta)} \exp\left(-\beta\beta_0^{-1}\right).\end{aligned}$$

Combined with (2.3) and $m_0 \geq 1$, this implies

$$\begin{aligned}\frac{\rho_\beta\left(\mathbb{T}\setminus Y_1^{\varsigma(1)}\right)}{\rho_\beta\left(M_{m_0}^{\varsigma(m_0)}\right)} &\leq \tfrac{3}{2}2^{(m_0+10)^2} \exp\left(-\beta\left(\beta_0^{-1} - \beta_{m_0}^{-1}\right)\right)\\ &\leq \tfrac{3}{2}2^{(m_0+10)^2} \exp\left(-\tfrac{9}{10}\beta\beta_0^{-1}\right)\\ &\leq \tfrac{3}{2}2^{(m_0+10)^2 - \beta\beta_0^{-1}}\end{aligned}$$

In the case $m_0 = 1$ we have $\beta = \beta_1$, and we obtain an upper bound of

$$\tfrac{3}{2}2^{11^2 - 2^{11^3 - 10^3}} \leq \tfrac{1}{2^{20}}2^{-m_0}.$$

In the case $m_0 \geq 2$ we have $\beta \geq \beta_{m_0-1}$ and

$$\begin{aligned}\beta\beta_0^{-1} - (m_0+10)^2 &\geq 2^{(m_0+9)^3-10^3} - (m_0+10)^2 \\ &\geq 2^{(m_0-1)\left((m_0+9)^2+100\right)} - (m_0+10)^2 \\ &\geq 2^{100}(m_0+9)^2 - (m_0+10)^2 \\ &\geq 100m_0.\end{aligned}$$

So, in all the cases we obtain

$$\frac{\rho_\beta\left(\mathbb{T}\setminus Y_1^{\varsigma(1)}\right)}{\rho_\beta\left(M_{m_0}^{\varsigma(m_0)}\right)} \leq \frac{1}{2^{20}} 2^{-m_0}. \tag{2.4}$$

Note that for every m in $\mathbb{N}$ we have $U(\underline{\varsigma}) \leq -\beta_m^{-1}$ on

$$\Delta_m := Y_m^{\varsigma(m)} \setminus Y_{m+1}^{\varsigma(m+1)}.$$

So, by (2.1) for every integer $k \geq -(m_0-1)$ we have

$$\begin{aligned}\rho_\beta\left(\Delta_{m_0+k}\right) &\leq \frac{1}{Z(\beta)} \operatorname{Leb}(\Delta_{m_0+k}) \exp\left(-\beta\beta_{m_0+k}^{-1}\right) \\ &\leq \left(4\frac{1}{Z(\beta)}\right) 2^{-(m_0+k+10)^2} \exp\left(-\beta\beta_{m_0+k}^{-1}\right).\end{aligned}$$

Combined with (2.3), we obtain

$$\frac{\rho_\beta(\Delta_{m_0+k})}{\rho_\beta\left(M_{m_0}^{\varsigma(m_0)}\right)} \leq 6 \cdot 2^{(m_0+10)^2-(m_0+k+10)^2} \exp\left(-\beta\left(\beta_{m_0+k}^{-1} - \beta_{m_0}^{-1}\right)\right). \tag{2.5}$$

In the case $k \geq 1$ the right-hand side is bounded from above by

$$(6e)2^{(m_0+10)^2-(m_0+k+10)^2} \leq (3e)2^{-2k(m_0+10)} \leq \frac{3e}{2^{20}} 2^{-km_0} \leq \frac{1}{2^{16}} 2^{-km_0}.$$

On the other hand, in the case $k \leq -2$ we have $m_0 \geq 3$, $\beta \geq \beta_{m_0-1}$,

$$\begin{aligned}\beta\left(\beta_{m_0+k}^{-1} - \beta_{m_0}^{-1}\right) &\geq \frac{9}{10}\beta\beta_{m_0+k}^{-1} \\ &\geq \frac{9}{10} 2^{(m_0+9)^3-(m_0+k+10)^3} \\ &= \frac{9}{10} 2^{(|k|-1)\left((m_0+9)^2+(m_0+9)(m_0+k+10)+(m_0+k+10)^2\right)} \\ &\geq \frac{9}{10} 2^{10|k|(m_0+10)},\end{aligned}$$

and therefore

$$\begin{aligned}\frac{\rho_\beta(\Delta_{m_0+k})}{\rho_\beta\left(M_{m_0}^{\varsigma(m_0)}\right)} &\leq 6 \cdot 2^{2|k|(m_0+10)} \exp\left(-\frac{9}{10} 2^{10|k|(m_0+10)}\right) \\ &\leq 6 \cdot 2^{2|k|(m_0+10)-2^{10|k|(m_0+10)}} \\ &\leq 6 \cdot 2^{-3|k|(m_0+10)} \\ &\leq \frac{1}{2^{20}} 2^{-3|k|m_0}.\end{aligned}$$

So for every k different from 0 and -1 we have

$$\frac{\rho_\beta(\Delta_{m_0+k})}{\rho_\beta\left(M_{m_0}^{\varsigma(m_0)}\right)} \leq \frac{1}{2^{16}} 2^{-|k|m_0}.$$

Put $\widetilde{\Delta}_{m_0} = \Delta_1$ if $m_0 = 1$, and $\widetilde{\Delta}_{m_0} := \Delta_{m_0} \cup \Delta_{m_0-1}$ if $m_0 \geq 2$. Combined with (2.4) and $m_0 \geq 1$, the last estimate implies

$$\begin{aligned}
\frac{\rho_\beta(\mathbb{T}\setminus\widetilde{\Delta}_{m_0})}{\rho_\beta\left(M_{m_0}^{\varsigma(m_0)}\right)} &= \frac{\rho_\beta\left(\mathbb{T}\setminus Y_1^{\varsigma(1)}\right) + \sum_{\substack{k=-(m_0-1)\\ k\neq 0,-1}}^{+\infty} \rho_\beta(\Delta_{m_0+k})}{\rho_\beta\left(M_{m_0}^{\varsigma(m_0)}\right)} \\
&\leq \tfrac{1}{2^{20}} 2^{-m_0} + \tfrac{1}{2^{16}} 2 \cdot \sum_{k=1}^{+\infty} 2^{-km_0} \\
&\leq \tfrac{1}{2^{13}} 2^{-m_0}.
\end{aligned} \tag{2.6}$$

Since $U(\underline{\varsigma}) \leq -\beta_{m_0}^{-1}$ on Δ_{m_0}, by (2.1) we have

$$\begin{aligned}
\rho_\beta\left(\Delta_{m_0}\setminus I_{m_0-1}^{\varsigma(m_0)}\right) &\leq \frac{1}{Z(\beta)} \operatorname{Leb}\left(\Delta_{m_0}\setminus I_{m_0-1}^{\varsigma(m_0)}\right) \exp\left(-\beta\beta_{m_0}^{-1}\right) \\
&\leq \left(2\frac{1}{Z(\beta)}\right) 2^{-(m_0+11)^2} \exp\left(-\beta\beta_{m_0}^{-1}\right).
\end{aligned} \tag{2.7}$$

Combined with (2.3) this implies

$$\frac{\rho_\beta\left(\Delta_{m_0}\setminus I_{m_0-1}^{\varsigma(m_0)}\right)}{\rho_\beta\left(M_{m_0}^{\varsigma(m_0)}\right)} \leq 3 \cdot 2^{(m_0+10)^2-(m_0+11)^2} \leq \tfrac{1}{2^{19}} 2^{-m_0}. \tag{2.8}$$

If $m_0 = 1$, then $\widetilde{\Delta}_{m_0} = \Delta_1$, so the previous estimate combined with (2.6) and the inclusions $M_1^{\varsigma(\mu)} \subset I_0^{\varsigma(\mu)} \setminus I_1^{\varsigma(\mu)} \subset \Delta_1 \cap I_0^{\varsigma(\mu)}$, implies

$$\begin{aligned}
\frac{\rho_\beta\left(\mathbb{T}\setminus\left(\Delta_1 \cap I_0^{\varsigma(\mu)}\right)\right)}{\rho_\beta\left(\Delta_1 \cap I_0^{\varsigma(\mu)}\right)} &= \frac{\rho_\beta\left(\mathbb{T}\setminus\Delta_1\right) + \rho_\beta\left(\Delta_1\setminus I_0^{\varsigma(\mu)}\right)}{\rho_\beta\left(\Delta_1 \cap I_0^{\varsigma(\mu)}\right)} \\
&\leq \tfrac{1}{2^{13}} 2^{-1} + \tfrac{1}{2^{19}} \cdot 2^{-1} \\
&\leq \tfrac{1}{2^{12}} 2^{-1}.
\end{aligned}$$

This proves

$$\rho_\beta\left(I_0^{\varsigma(\mu)}\right) \geq \rho_\beta\left(\Delta_1 \cap I_0^{\varsigma(\mu)}\right) \geq 1 - \tfrac{1}{2^{12}} 2^{-1},$$

and completes the proof of the theorem when $m_0 = 1$.

It remains to consider the case where $m_0 \geq 2$. Suppose $\varsigma(m_0 - 1) \neq \varsigma(\mu)$. Then we have $m_0 = \mu$ and therefore $\beta = \beta_{m_0}$. On the other hand, by (2.5) with $k = -1$ we have

$$\begin{aligned}
\frac{\rho_\beta(\Delta_{m_0-1})}{\rho_\beta\left(M_{m_0}^{\varsigma(m_0)}\right)} &\leq (6e)2^{(m_0+10)^2-(m_0+9)^2} \exp\left(-2^{(m_0+10)^3-(m_0+9)^3}\right) \\
&\leq (6e)2^{2m_0+19} \exp\left(-2^{3m_0+100}\right) \\
&\leq 6 \cdot 2^{2m_0+19} \exp(-(3m_0+100)) \\
&\leq 6 \cdot 2^{-m_0-30} \\
&\leq \tfrac{1}{2^{20}} 2^{-m_0}.
\end{aligned}$$

Combined with (2.6), (2.8), and the inclusions

$$M_{m_0}^{\varsigma(m_0)} \subset I_{m_0-1}^{\varsigma(\mu)} \setminus I_{m_0}^{\varsigma(\mu)} \subset \Delta_{m_0} \cap I_{m_0-1}^{\varsigma(\mu)},$$

we obtain

$$\begin{aligned}
\frac{\rho_\beta\left(\mathbb{T}\setminus\left(\Delta_{m_0} \cap I_{m_0-1}^{\varsigma(\mu)}\right)\right)}{\rho_\beta\left(\Delta_{m_0} \cap I_{m_0-1}^{\varsigma(\mu)}\right)} &= \frac{\rho_\beta\left(\mathbb{T}\setminus\widetilde{\Delta}_{m_0}\right) + \rho_\beta\left(\Delta_{m_0-1}\right)}{\rho_\beta\left(\Delta_{m_0} \cap I_{m_0-1}^{\varsigma(\mu)}\right)} \\
&\quad + \frac{\rho_\beta\left(\Delta_{m_0} \setminus I_{m_0-1}^{\varsigma(\mu)}\right)}{\rho_\beta\left(\Delta_{m_0} \cap I_{m_0-1}^{\varsigma(\mu)}\right)} \\
&\leq \tfrac{1}{2^{13}} 2^{-m_0} + \tfrac{1}{2^{20}} 2^{-m_0} + \tfrac{1}{2^{19}} 2^{-m_0} \\
&\leq \tfrac{1}{2^{12}} 2^{-m_0}.
\end{aligned}$$

This proves that

$$\rho_\beta\left(I_{m_0-1}^{\varsigma(\mu)}\right) \geq \rho_\beta\left(\Delta_{m_0} \cap I_{m_0-1}^{\varsigma(\mu)}\right) \geq 1 - \tfrac{1}{2^{12}} 2^{-m_0},$$

and completes the proof of the theorem when $m_0 \geq 2$ and $\varsigma(m_0 - 1) \neq \varsigma(\mu)$.

It remains to consider the case where $m_0 \geq 2$ and $\varsigma(m_0 - 1) = \varsigma(\mu)$. By (2.1) we obtain, as in (2.3) and (2.7) with m_0 replaced by $m_0 - 1$,

$$\rho_\beta\left(M_{m_0-1}^{\varsigma(m_0-1)}\right) = \left(\tfrac{2}{3} \frac{1}{Z(\beta)}\right) 2^{-(m_0+9)^2} \exp\left(-\beta\beta_{m_0-1}^{-1}\right)$$

and

$$\rho_\beta\left(\Delta_{m_0-1} \setminus I_{m_0-2}^{\varsigma(\mu)}\right) \leq \left(2 \frac{1}{Z(\beta)}\right) 2^{-(m_0+10)^2} \exp\left(-\beta\beta_{m_0-1}^{-1}\right).$$

Therefore,

$$\frac{\rho_\beta\left(\Delta_{m_0-1} \setminus I_{m_0-2}^{\varsigma(\mu)}\right)}{\rho_\beta\left(M_{m_0-1}^{\varsigma(m_0-1)}\right)} \leq 3 \cdot 2^{(m_0+9)^2-(m_0+10)^2} \leq \tfrac{1}{2^{17}} 2^{-m_0}.$$

Combined with (2.6), (2.8), and the inclusion

$$M_{m_0}^{\varsigma(\mu)} \cup M_{m_0-1}^{\varsigma(\mu)} \subset I_{m_0-2}^{\varsigma(\mu)} \setminus I_{m_0}^{\varsigma(\mu)} \subset \widetilde{\Delta}_{m_0} \cap I_{m_0-2}^{\varsigma(\mu)},$$

we obtain

$$\frac{\rho_\beta\left(\mathbb{T}\backslash\left(\widetilde{\Delta}_{m_0}\cap I_{m_0-2}^{\varsigma(\mu)}\right)\right)}{\rho_\beta\left(\widetilde{\Delta}_{m_0}\cap I_{m_0-2}^{\varsigma(\mu)}\right)} = \frac{\rho_\beta\left(\mathbb{T}\backslash\widetilde{\Delta}_{m_0}\right)+\rho_\beta\left(\Delta_{m_0-1}\backslash I_{m_0-2}^{\varsigma(\mu)}\right)}{\rho_\beta\left(\widetilde{\Delta}_{m_0}\cap I_{m_0-2}^{\varsigma(\mu)}\right)}$$
$$+\frac{\rho_\beta\left(\Delta_{m_0}\backslash I_{m_0-1}^{\varsigma(\mu)}\right)}{\rho_\beta\left(\widetilde{\Delta}_{m_0}\cap I_{m_0-2}^{\varsigma(\mu)}\right)}$$
$$\le \tfrac{1}{2^{13}}2^{-m_0}+\tfrac{1}{2^{19}}2^{-m_0}+\tfrac{1}{2^{17}}2^{-m_0}$$
$$\le \tfrac{1}{2^{12}}2^{-m_0}.$$

This proves

$$\rho_\beta\left(I_{m_0-2}^{\varsigma(\mu)}\right)\ge\rho_\beta\left(\widetilde{\Delta}_{m_0}\cap I_{m_0-2}^{\varsigma(\mu)}\right)\ge 1-\tfrac{1}{2^{12}}2^{-m_0},$$

and completes the proof of the theorem when $m_0\ge 2$ and $\varsigma(m_0-1)=\varsigma(\mu)$. The proof of the lemma is thus complete. □

Proof of Theorem A Fix $\underline{\varsigma}_0=(\varsigma_0(m))_{m\in\mathbb{N}}$ in $\{+,-\}^{\mathbb{N}}$, put $U_0:=U(\underline{\varsigma}_0)$, and fix a integer $m_0\ge 1$. Let $(\beta_m)_{m\in\mathbb{N}}$ be the sequence given by Main Lemma A. Replacing $(\widehat{\beta}_\ell)_{\ell\in\mathbb{N}}$ by a subsequence if necessary, assume this sequence is strictly increasing, that $\widehat{\beta}_1\ge\beta_{m_0+1}$, and that for every $m\ge 1$ there is at most 1 value of ℓ such that $\widehat{\beta}_\ell$ is in $[\beta_m,\beta_{m+1}]$. For each ℓ in $\mathbb{N}$ let $m(\ell)$ be the largest integer $m\ge 1$ such that $\beta_m\le\widehat{\beta}_\ell$. Note that $m(1)\ge m_0+1$ and that for every ℓ in $\mathbb{N}$ the number $\widehat{\beta}_\ell$ is in $[\beta_{m(\ell)},\beta_{m(\ell+1)-1}]$.

Let $\underline{\varsigma}=(\varsigma(m))_{m\in\mathbb{N}}$ in $\{+,-\}^{\mathbb{N}}$ be such that for every m in $\{1,\ldots,m_0\}$ we have $\varsigma(m)=\varsigma_0(m)$, and such that for every even (resp. odd) ℓ in $\mathbb{N}$ and every m in $[m(\ell),m(\ell+1)-1]$ we have $\varsigma(m)=+$ (resp. $\varsigma(m)=-$). Then for every ℓ in $\mathbb{N}$ we have

$$\varsigma(m(\ell))=\cdots=\varsigma(m(\ell+1)-1),$$

and by Main Lemma A we have

$$\lim_{k\to+\infty}\pi_*\rho_{\widehat{\beta}_{2k}\cdot U(\underline{\varsigma})}=\delta_0 \text{ and } \lim_{k\to+\infty}\pi_*\rho_{\widehat{\beta}_{2k+1}\cdot U(\underline{\varsigma})}=\delta_{\frac{1}{2}}.$$

In view of Lemma 2.1, this implies that $\rho_{\widehat{\beta}_{2k}\cdot U(\underline{\varsigma})}$ converges to the ferromagnetic phase as $k\to+\infty$, and that $\rho_{\widehat{\beta}_{2k+1}\cdot U(\underline{\varsigma})}$ converges to the antiferromagnetic phase as $k\to+\infty$. Since $m_0\ge 1$ is an arbitrary integer, and since the first m_0 elements of $\underline{\varsigma}$ and $\underline{\varsigma}_0$ coincide, it follows that $U=U(\underline{\varsigma})$ can be chosen arbitrarily close to U_0 in the C^∞ topology. □

Remark 2.3 If in the proof of Theorem A we choose $\underline{\varsigma}_0$ as the constant sequence equal to $+$ (resp. $-$), then by Main Lemma A it follows that the one-parameter family of Gibbs measures $\left(\rho_{\beta\cdot U_0}\right)_{\beta>0}$ converges to the ferromagnetic (resp. antiferromagnetic) phase as $\beta\to+\infty$. In particular, for such $\underline{\varsigma}_0$ the interaction Φ_{U_0} is non-chaotic. On the other hand, if we choose $\underline{\varsigma}_0$ having infinitely many $+$'s and infinitely many $-$'s, then the one-parameter family of Gibbs measures $\left(\rho_{\beta\cdot U_0}\right)_{\beta>0}$ accumulates at the same time on the ferromagnetic and the antiferromagnetic phases as $\beta\to+\infty$. Thus, for such $\underline{\varsigma}_0$ the interaction Φ_{U_0} is chaotic.

Remark 2.4 In the case U is real analytic, the interaction Φ_U is non-chaotic. In fact, in this case the one-parameter family of Gibbs measures $\left(\rho_{\beta\cdot U}\right)_{\beta>0}$ converges as $\beta\to+\infty$ to a measure ρ_∞ described as follows. If U is constant, then for every $\beta>0$ we have $\rho_{\beta\cdot U}=\bigotimes_{\mathbb{Z}}\text{Leb}$, and therefore $\rho_\infty = \bigotimes_{\mathbb{Z}} \mathrm{Leb}$. Assume U is nonconstant, and note that by Lemma 2.1 the measure ρ_∞ is symmetric and $D_*\rho_\infty$ is a product measure. Thus, to describe ρ_∞ we just need to describe its projection by π. Put $U_{\max} := \sup_{\mathbb{T}} U$, and for each c in the finite set $U^{-1}(U_{\max})$ denote by $\ell(c)$ the least integer $\ell \geq 1$ such that $D^\ell U(c) \neq 0$. Note that $\ell(c)$ is even and that

$$\omega(c) := -\frac{D^{\ell(c)}U(c)}{\ell(c)!} > 0.$$

Putting

$$\ell_{\max} := \max\left\{\ell(c) : c \in U^{-1}(U_{\max})\right\},$$

the measure ρ_∞ is uniquely determined by

$$\pi_*\rho_\infty = \frac{\sum_{\substack{c\in U^{-1}(U_{\max})\\ \ell(c)=\ell_{\max}}} \omega(c)^{-\frac{1}{\ell_{\max}}}\delta_c}{\sum_{\substack{c\in U^{-1}(U_{\max})\\ \ell(c)=\ell_{\max}}} \omega(c)^{-\frac{1}{\ell_{\max}}}}. \tag{2.9}$$

To prove this, fix κ in $(0, 1)$ and note that

$$\lim_{\beta\to+\infty} \exp(-\beta U_{\max})\beta^{\frac{1}{\ell_{\max}}} \int_{\mathbb{T}\setminus\bigcup_{c\in U^{-1}(U_{\max})} B\left(c,\beta^{-\frac{\kappa}{\ell(c)}}\right)} \exp(\beta\cdot U(x))\,\mathrm{d}x = 0,$$

and that for every c in $U^{-1}(U_{\max})$

$$\begin{aligned}\lim_{\beta\to+\infty} \exp(-\beta U_{\max})\beta^{\frac{1}{\ell(c)}} &\int_{B\left(c,\beta^{-\frac{\kappa}{\ell(c)}}\right)} \exp(\beta\cdot U(x))\,\mathrm{d}x \\ &= \omega(c)^{-\frac{1}{\ell(c)}}\int_{\mathbb{R}} \exp\left(-y^{\ell(c)}\right)\mathrm{d}y.\end{aligned}$$

These computations prove that

$$\begin{aligned}&\lim_{\beta\to+\infty} \exp(\beta U_{\max})\beta^{-\frac{1}{\ell_{\max}}} \int_{\mathbb{T}} \exp(\beta\cdot U(x))\,\mathrm{d}x \\ &= \left(\int_{\mathbb{R}} \exp\left(-y^{\ell_{\max}}\right)\mathrm{d}y\right) \sum_{\substack{c\in U^{-1}(U_{\max})\\ \ell(c)=\ell_{\max}}} \omega(c)^{-\frac{1}{\ell_{\max}}}\delta_c.\end{aligned}$$

Combined with (2.1), this implies (2.9).

3 Symbolic Space

This section is devoted to the proof of Main Lemma B. As mentioned in the introduction, the deduction of Theorem B from Main Lemma B is analogous to that of Theorem A from Main Lemma A given in Sect. 2, and we omit it.

We first prove the following weaker version of Main Lemma B, whose proof contains some of the main ideas, but is simpler. Corollary 1.1 follows easily from (the proof of) this result.

Main Lemma B' There is a continuous family of Lipschitz continuous potentials $\left(\varphi(\underline{\varsigma})\right)_{\underline{\varsigma}\in\{+,-\}^{\mathbb{N}}}$, complementary open subsets U^+ and U^- of Σ, and an increasing sequence of positive numbers $(\beta_m)_{m\in\mathbb{N}}$ converging to $+\infty$, such that the following property holds: For every $\underline{\varsigma} = (\varsigma(m))_{m\in\mathbb{N}}$ in $\{+,-\}^{\mathbb{N}}$ and every integer $m \geq 1$, every translation invariant Gibbs measure ρ for the potential $\beta_m \cdot \varphi(\underline{\varsigma})$ satisfies

$$\rho(U^{\varsigma(m)}) \geq \tfrac{2}{3}.$$

The proof of Main Lemma B' is given in Sect. 3.1, that of Corollary 1.1 in Sect. 3.2, and that of Main Lemma B in Sect. 3.3.

A subset of Σ is *clopen* if it is at the same time open and closed.

3.1 Proof of Main Lemma B'

For a function $\varphi\colon \Sigma \to \mathbb{R}$, put

$$\|\varphi\|_\infty := \sup\{|\varphi(x)| : x \in \Sigma\},$$

$$|\varphi|_{\mathrm{Lip}} := \sup\left\{\frac{|\varphi(x) - \varphi(x')|}{\mathrm{dist}(x, x')} : x, x' \in \Sigma, x \neq x'\right\},$$

and

$$\|\varphi\|_{\mathrm{Lip}} := \|\varphi\|_\infty + |\varphi|_{\mathrm{Lip}}.$$

A function $\varphi\colon \Sigma \to \mathbb{R}$ is *Lipschitz continuous* if $\|\varphi\|_{\mathrm{Lip}} < +\infty$. Denote by Lip the space of all Lipschitz continuous functions. Then $\|\cdot\|_{\mathrm{Lip}}$ is a norm on Lip, for which Lip is a Banach space.

The following lemma follows easily from a well-known result, see for example [1, Corollary 1] and also [10, Proposition 29(ii)] for the case $G = \mathbb{N}_0$. We provide the short proof for completeness.

Lemma 3.1 *Let X and X' be disjoint compact subsets of Σ, each of which is invariant by σ, and such that $h_{\mathrm{top}}(\sigma|_X) > h_{\mathrm{top}}(\sigma|_{X'})$. Moreover, let $\varphi\colon \Sigma \to \mathbb{R}$ be a Lipschitz continuous function attaining its maximum precisely on $X \cup X'$. Then for every δ in $(0, 1)$ and every neighborhood U of X there is $\beta_0 > 0$ such that for every $\beta \geq \beta_0$ and every translation invariant Gibbs measure ρ for the potential $\beta \cdot \varphi$ we have*

$$\rho(U) \geq 1 - \delta.$$

In the proof of this lemma, as well as in the proof of Lemma 3.2 below, we use the fact that the entropy function $\nu \mapsto h_\nu$ is upper semi-continuous on $\mathscr{M}_\sigma$, see for example [13, Example 4.2.6].

Proof of Lemma 3.1 It is enough to show that for every family of translation invariant Gibbs measures $(\rho_\beta)_{\beta>0}$ for the potentials $\beta \cdot \varphi$, every accumulation measure as $\beta \to +\infty$ is supported on X. Let $(\beta_\ell)_{\ell=1}^{+\infty}$ be a sequence of positive numbers such that $\beta_\ell \to +\infty$ as $\ell \to +\infty$ and such that $\rho_\ell := \rho_{\beta_\ell}$ converges to a measure ρ as $\ell \to +\infty$. Putting $m := \sup\{\varphi(x) : x \in \Sigma\}$, for every ν in $\mathscr{M}_\sigma$ that is supported on $X \cup X'$ we have

$$\int \varphi \,\mathrm{d}\rho = \lim_{\ell\to+\infty}\left(\frac{h_{\rho_\ell}}{\beta_\ell} + \int \varphi \,\mathrm{d}\rho_\ell\right) \geq \lim_{\ell\to+\infty}\left(\frac{h_\nu}{\beta_\ell} + \int \varphi \,\mathrm{d}\nu\right) = \int \varphi \,\mathrm{d}\nu = m.$$

It follows that ρ is supported on $X \cup X'$. On the other hand, for every $\ell \geq 1$ we have

$$h_{\rho_\ell} + \beta_\ell m \geq h_{\rho_\ell} + \beta_\ell \int \varphi \,\mathrm{d}\rho_\ell \geq h_\nu + \beta_\ell \int \varphi \,\mathrm{d}\nu = h_\nu + \beta_\ell m,$$

so $h_{\rho_\ell} \geq h_\nu$. Since the entropy function is upper semi-continuous, it follows that $h_\rho \geq h_\nu$. Since this holds for every invariant probability measure ν supported on $X \cup X'$, and by hypothesis $h_{\text{top}}(\sigma|_X) > h_{\text{top}}(\sigma|_{X'})$, we conclude that $h_\rho = h_{\text{top}}(\sigma|_X)$ and that ρ is supported on X. □

Lemma 3.2 *Let $\varphi_0 \colon \Sigma \to \mathbb{R}$ be a Lipschitz continuous function and let $\beta_0 \geq 0$ be given. Then for every $\delta > 0$ and every continuous function $\psi \colon \Sigma \to \mathbb{R}$ there is $\varepsilon > 0$ such that for every Lipschitz continuous function $\varphi \colon \Sigma \to \mathbb{R}$ satisfying $\|\varphi - \varphi_0\|_{\text{Lip}} \leq \varepsilon$ and every translation invariant Gibbs measure ρ for the potential $\beta_0 \cdot \varphi$ there is a translation invariant Gibbs measure ν for the potential $\beta_0 \cdot \varphi_0$ such that*

$$\left| \int \psi \, \mathrm{d}\rho - \int \psi \, \mathrm{d}\nu \right| < \delta.$$

Moreover, if the dimension d is 1, then for every β in $[0, \beta_0]$ we have

$$\left| \int \psi \, \mathrm{d}\rho_{\beta \cdot \varphi} - \int \psi \, \mathrm{d}\rho_{\beta \cdot \varphi_0} \right| < \delta.$$

The proof of Lemma 3.3 is after the proof of the following lemma.

Lemma 3.3 *Let $\varphi \colon \Sigma \to \mathbb{R}$ a Lipschitz continuous function and $(\varphi_\ell)_{\ell \in \mathbb{N}}$ be a sequence in* Lip *converging to φ. For every sequence of translation invariant Gibbs measures $(\rho_\ell)_{\ell \in \mathbb{N}}$ for the potentials in the sequence $(\varphi_\ell)_{\ell \in \mathbb{N}}$, every accumulation point is an translation invariant Gibbs measure for the potential φ.*

Proof To prove this, we use the fact that the pressure $P(\varphi)$ depends continuously on φ in Lip, see for example [13, Theorem 4.1.10b)]. Let assume that ρ_ℓ converges to a measure ρ as $\ell \to +\infty$. Using that the entropy function is upper semi-continuous on $\mathscr{M}_\sigma$, we have

$$P(\varphi) = \lim_{\ell \to +\infty} P(\varphi_\ell) = \lim_{\ell \to +\infty} \left(h_{\rho_\ell} + \int \varphi_\ell \, \mathrm{d}\rho_\ell \right) \leq h_\rho + \int \varphi \, \mathrm{d}\rho \leq P(\varphi),$$

so $h_\rho + \int \varphi \, \mathrm{d}\rho = P(\varphi)$ and therefore ρ is a translation invariant Gibbs measure for φ.

Proof of Lemma 3.2 By contradiction. Assume that there are $\delta > 0$, a continuous function $\psi \colon \Sigma \to \mathbb{R}$, a sequence of Lipschitz continuous functions $(\varphi_\ell)_{\ell \in \mathbb{N}}$ converging to φ_0 in Lip and a sequence of translation invariant Gibbs measures $(\rho_\ell)_{\ell \in \mathbb{N}}$ for the sequence of potential $\beta_0 \cdot \varphi_\ell$ such that for every translation invariant Gibbs measure ν for the potential $\beta_0 \cdot \varphi_0$ one has

$$\left| \int \psi \, \mathrm{d}\rho_\ell - \int \psi \, \mathrm{d}\nu \right| \geq \delta.$$

By compactness the sequence of measures $(\rho_\ell)_{\ell \in \mathbb{N}}$ has an accumulation point which is a translation invariant Gibbs measure for the potential $\beta_0 \cdot \varphi_0$ (Lemma 3.3). This is in contradiction with the previous inequality.

In dimension 1, the same argument together with the uniqueness of the translation invariant Gibbs measures gives the uniformity in β, as β varies in $[0, \beta_0]$. □

Proof of Main Lemma B' Given a compact subset Y of Σ, denote by $\chi_Y \colon \Sigma \to \mathbb{R}$ the function defined by $\chi_Y(x) := -\operatorname{dist}(x, Y)$. A straightforward computation shows that $\|\chi_Y\|_{\text{Lip}} \leq 2$.

Let $(X_m^+)_{m \in \mathbb{N}_0}$ and $(X_m^-)_{m \in \mathbb{N}_0}$ be decreasing sequences of compact subsets of Σ that are invariant by σ, such that X_0^+ and X_0^- are disjoint, and such that for every $m \geq 0$ we have

$$h_{\mathrm{top}}\left(\sigma|_{X^-_{m+1}}\right) < h_{\mathrm{top}}\left(\sigma|_{X^+_m}\right) \text{ and } h_{\mathrm{top}}\left(\sigma|_{X^+_{m+1}}\right) < h_{\mathrm{top}}\left(\sigma|_{X^-_m}\right).^{8} \tag{3.1}$$

Let U^+ be a clopen neighborhood of X_0^+ that is disjoint from X_0^-, and put $U^- := \Sigma \setminus U^+$. Finally, for each integer $m \geq 1$, put

$$Y_m^+ := X_{m-1}^+ \cup X_m^- \text{ and } Y_m^- := X_{m-1}^- \cup X_m^+,$$

and for ς in $\{+, -\}$ put $\chi_m^{\varsigma} := \mathbb{1}_{\Sigma} + \chi_{Y_m^{\varsigma}}$.

(1) Define inductively sequences of positive numbers $(\beta_m)_{m\in\mathbb{N}}$ and $(\varepsilon_m)_{m\in\mathbb{N}}$, as follows. Put $\varepsilon_1 := 1$, and let $m \geq 1$ be an integer such that $\varepsilon_1, \ldots, \varepsilon_m$ are already defined. In the case $m \geq 2$, assume that $\beta_1, \ldots, \beta_{m-1}$ are already defined. Given $\underline{\varsigma} = (\varsigma(k))_{k=1}^m$ in $\{+, -\}^{\{1,\ldots,m\}}$, put

$$\varphi(\underline{\varsigma}) := \frac{1}{8} \sum_{k=1}^{m} \varepsilon_k \chi_k^{\varsigma(k)}, \tag{3.2}$$

and note that $\varphi(\underline{\varsigma})$ attains it maximum precisely on $Y_m^{\varsigma(m)}$. Let $\beta(\underline{\varsigma})$ be the number β_0 given by Lemma 3.1 with $\varphi = \varphi(\underline{\varsigma})$, $X = X_{m-1}^{\varsigma(m)}$, $X' = Y_m^{\varsigma(m)} \setminus X_{m-1}^{\varsigma(m)}$, $\delta = \frac{1}{6}$, and $U = U^{\varsigma(m)}$. Define

$$\beta_m := \max\left\{\beta(\underline{\varsigma}) : \underline{\varsigma} \in \{+, -\}^{\{1,\ldots,m\}}\right\}.$$

In the case $m \geq 2$, replace β_m by $\beta_{m-1} + m$ if necessary, so that $\beta_{m+1} > \max\{\beta_m, m\}$.

To define ε_{m+1}, for each $\underline{\varsigma} = (\varsigma_k)_{k=1}^m$ in $\{+, -\}^{\{1,\ldots,m\}}$ let $\varepsilon(\underline{\varsigma})$ be given by Lemma 3.2 with $\varphi_0 = \varphi(\underline{\varsigma})$, $\beta_0 = \beta_m$, $\delta = \frac{1}{6}$, and $\psi = \mathbb{1}_{U^+}$. Put

$$\varepsilon_{m+1} := \min\left\{\varepsilon(\underline{\varsigma}) : \underline{\varsigma} \in \{+, -\}^{\{1,\ldots,m\}}\right\}.$$

Replacing ε_{m+1} by $\varepsilon_m/2$ if necessary, assume $\varepsilon_{m+1} \leq \varepsilon_m/2$.

This completes the definition of $(\varepsilon_m)_{m\in\mathbb{N}}$ and $(\beta_m)_{m\in\mathbb{N}}$. Note that for every integer $m \geq 0$ we have $\varepsilon_{m+1} \leq \varepsilon_m/2$ and $\beta_{m+1} > \max\{\beta_m, m\}$, so the sequence $(\beta_m)_{m\in\mathbb{N}}$ is strictly increasing and $\beta_m \to +\infty$ as $m \to +\infty$.

(2) Given $(\varsigma(m))_{m\in\mathbb{N}}$ in $\{+, -\}^{\mathbb{N}}$, put

$$\varphi\left((\varsigma(m))_{m\in\mathbb{N}}\right) := \frac{1}{8} \sum_{m=1}^{+\infty} \varepsilon_m \chi_m^{\varsigma(m)}.$$

To prove that $\left(\varphi(\underline{\varsigma})\right)_{\underline{\varsigma}\in\{+,-\}^{\mathbb{N}}}$ is continuous in Lip, let $m_0 \geq 1$ be an integer, and let $(\varsigma(m))_{m\in\mathbb{N}}$ and $(\varsigma'(m))_{m\in\mathbb{N}}$ in $\{+, -\}^{\mathbb{N}}$ be such that for every k in $\{1, \ldots, m_0\}$ we have $\varsigma(k) = \varsigma'(k)$. Then

$$\begin{aligned} \left\|\varphi\left((\varsigma(m))_{m\in\mathbb{N}}\right) - \varphi\left((\varsigma'(m))_{m\in\mathbb{N}}\right)\right\|_{\mathrm{Lip}} &\leq \frac{1}{8} \sum_{m=m_0+1}^{+\infty} \varepsilon_m \left\|\chi_{Y_m^+} - \chi_{Y_m^-}\right\|_{\mathrm{Lip}} \\ &\leq \frac{1}{2} \sum_{m=m_0+1}^{+\infty} \varepsilon_m \\ &\leq \varepsilon_{m_0+1}. \end{aligned}$$

[8] There are various choices for $(X_m^+)_{m\in\mathbb{N}_0}$ and $(X_m^-)_{m\in\mathbb{N}_0}$, see Sect. 3.2 for a couple of them.

To complete the proof of the theorem, let $\underline{\varsigma} = (\varsigma(k))_{k\in\mathbb{N}}$ in $\{+,-\}^{\mathbb{N}}$ and fix m in $\mathbb{N}$. Put $\underline{\widetilde{\varsigma}}:=(\varsigma(k))_{k=1}^{m}$ and let $\varphi(\underline{\widetilde{\varsigma}})$ be defined by (3.2) with $\underline{\varsigma}$ replaced by $\underline{\widetilde{\varsigma}}$. By our choice of β_m, for every $\beta \geq \beta_m$ and every equilibrium state ν for the potential $\beta \cdot \varphi(\underline{\widetilde{\varsigma}})$ we have

$$\nu\left(U^{\varsigma(m)}\right) \geq \frac{5}{6}. \tag{3.3}$$

On the other hand,

$$\begin{aligned}\|\varphi(\underline{\varsigma}) - \varphi(\underline{\widetilde{\varsigma}})\|_{\mathrm{Lip}} &\leq \frac{1}{8}\sum_{k=m+1}^{+\infty} \varepsilon_k \left\|\chi_k^{\varsigma(k)}\right\|_{\mathrm{Lip}} \\ &\leq \frac{1}{8}\sum_{k=m+1}^{+\infty} \varepsilon_k \left(1 + \left\|\chi_{Y_k^{\varsigma(k)}}\right\|_{\mathrm{Lip}}\right) \\ &\leq \frac{3}{8}\sum_{k=m+1}^{+\infty} \varepsilon_l \\ &< \varepsilon_{m+1},\end{aligned}$$

so by our choice of ε_{m+1} it follows that for every translation invariant Gibbs measure ρ for the potential $\beta_m \cdot \varphi(\underline{\varsigma})$ there is an equilibrium state ν for the potential $\beta_m \cdot \varphi(\underline{\widetilde{\varsigma}})$ such that

$$\left|\rho(U^{\varsigma(m)}) - \nu(U^{\varsigma(m)})\right| < \frac{1}{6}.$$

Together with (3.3) with $\beta = \beta_m$ this gives the desired conclusion. □

3.2 Ground States

In this section we describe a couple of ways to choose the sequences $(X_m^+)_{m\in\mathbb{N}_0}$ and $(X_m^-)_{m\in\mathbb{N}_0}$ in the proof of Main Lemma B'. We discuss the case of dimension 1 in Sect. 3.2.1, where we also give the proof of Corollary 1.1, and the case of dimension larger than 1 in Sect. 3.2.2.

3.2.1 Dimension 1

We use the following lemma.

Lemma 3.4 *Let X be a compact subset of Σ that is invariant and transitive for σ, and that is not a subshift of finite type. Then there is a decreasing sequence $(X_m)_{m\in\mathbb{N}}$ of compact subsets of Σ that are invariant by σ, such that*

$$\bigcap_{m\in\mathbb{N}} X_m = X \text{ and } \lim_{m\to+\infty} h_{\mathrm{top}}\left(\sigma|_{X_m}\right) = h_{\mathrm{top}}\left(\sigma|_X\right),$$

and such that for every m in $\mathbb{N}$ we have $h_{\mathrm{top}}\left(\sigma|_{X_{m+1}}\right) < h_{\mathrm{top}}\left(\sigma|_{X_m}\right)$.

Proof For each integer $\ell \geq 1$, let Σ_ℓ be the subshift of finite type of all words in Σ with the property that every subword of length ℓ is a subword of a word in X. Clearly, $\bigcap_{\ell\in\mathbb{N}} \Sigma_\ell = X$, so $\lim_{\ell\to+\infty} h_{\mathrm{top}}(\sigma|_{\Sigma_\ell}) = h_{\mathrm{top}}(\sigma|_X)$, see for example [18, Proposition 4.4.6]. Our hypothesis that σ is transitive on X implies that for every ℓ in $\mathbb{N}$ the map σ is transitive on Σ_ℓ. On the other hand, our hypothesis that X is not a subshift of finite type implies that for every ℓ there is $\ell' \geq \ell + 1$ such that $\Sigma_{\ell'}$ is strictly contained in Σ_ℓ. Since Σ_ℓ and $\Sigma_{\ell'}$ are both subshifts of finite type and Σ_ℓ is transitive, it follows that $h_{\mathrm{top}}(\Sigma_{\ell'}) < h_{\mathrm{top}}(\Sigma_\ell)$, see for example [18, Corollary 4.4.9]. So we can extract a subsequence $(X_m)_{m\in\mathbb{N}_0}$ of $(\Sigma_\ell)_{\ell\in\mathbb{N}}$ satisfying the desired properties. □

We now explain a way to choose the sequences $(X_m^+)_{m\in\mathbb{N}_0}$ and $(X_m^-)_{m\in\mathbb{N}_0}$ in the proof of Main Lemma B' when the dimension is 1. Let X^+ and X^- be disjoint and infinite compact subsets of Σ that are invariant and minimal for σ, and such that $h_{\text{top}}(\sigma|_{X^+}) = h_{\text{top}}(\sigma|_{X^-})$. Since X^+ (resp. X^-) is infinite and minimal for σ, it follows that it is not a subshift of finite type. So X^+ and X^- satisfy the hypotheses of Lemma 3.4. Let $(X_m^+)_{m\in\mathbb{N}_0}$ (resp. $(X_m^-)_{m\in\mathbb{N}_0}$) be the sequence $(X_m)_{m\in\mathbb{N}_0}$ given by Lemma 3.4 with $X = X^+$ (resp. $X = X^-$). Replacing $(X_m^+)_{m\in\mathbb{N}_0}$ and $(X_m^-)_{m\in\mathbb{N}_0}$ by subsequences if necessary, assume X_0^+ and X_0^- are disjoint. These sequences satisfy all the requirements, with the possible exception of (3.1) which is easy to satisfy by taking subsequences.

Proof of Corollary 1.1 In the case ρ^+ (resp. ρ^-) is not purely atomic, the set X^+ (resp. X^-) is infinite and therefore it is not a subshift of finite type. So in this case X^+ (resp. X^-) satisfies the hypotheses of Lemma 3.4. In the case ρ^+ (resp. ρ^-) is purely atomic, the set X^+ (resp. X^-) is a periodic orbit of σ. We enlarge X^+ (resp. X^-) to a set satisfying the hypotheses of Lemma 3.4, as follows. Suppose first $G = \mathbb{Z}$, and let x_0 be a point in Σ that is not in X^+ (resp. X^-) and that differs with some point in X^+ (resp. X^-) only at finitely many positions. Then the orbit of x_0 is forward and backwards asymptotic to X^+ (resp. X^-), and the invariant set

$$X^+ \cup \{\sigma^g(x_0) : g \in \mathbb{Z}\}, (\text{resp.} X^+ \cup \{\sigma^g(x_0) : g \in \mathbb{Z}\})$$

is compact and transitive. Furthermore, this set is not a subshift finite type and ρ^+ (resp. ρ^-) is the only invariant measure supported on this set. Suppose now $G = \mathbb{N}_0$, let x_0 be a point in $\sigma^{-1}(X^+)\backslash X^+$ (resp. $\sigma^{-1}(X^-)\backslash X^-$), and let $(x_j)_{j\in\mathbb{N}}$ be a sequence in Σ that is asymptotic to X^+ (resp. X^-) and such that for every j we have $\sigma(x_j) = x_{j-1}$. Then the invariant set

$$X^+ \cup \{x_j : j \in \mathbb{N}\} \ (\text{resp.} X^- \cup \{x_j : j \in \mathbb{N}\})$$

is compact and transitive. Furthermore, this set is not a subshift finite type and ρ^+ (resp. ρ^-) is the only invariant measure supported on this set. In all the cases the (enlarged) sets X^+ and X^- satisfy the hypotheses of Lemma 3.4, and ρ^+ and ρ^- are the only invariant probability measures supported on X^+ and X^-, respectively.

We now explain how to modify the proof of Main Lemma B' to obtain the desired statement. Choose sequences $(X_m^+)_{m\in\mathbb{N}_0}$ and $(X_m^-)_{m\in\mathbb{N}_0}$ as explained above for our choices of X^+ and X^-. For each integer m in $\mathbb{N}$ choose disjoint clopen neighborhoods U_m^+ and U_m^- of X_m^+ and X_{m-1}^-, respectively, so that

$$\bigcap_{m\in\mathbb{N}_0} U_m^+ = X^+ \text{ and } \bigcap_{m\in\mathbb{N}_0} U_m^- = X^-.$$

With this notation, the only changes in the proof of Main Lemma B' are the following:

- Apply Lemma 3.1 with $U = U_m^{\varsigma(m)}$ and $\delta = 2^{-m}$ instead of $U = U^{\varsigma(m)}$ and $\delta = \frac{1}{6}$;
- Apply Lemma 3.2 with $\delta = 2^{-m}$ instead of $\delta = \frac{1}{6}$.

Then we obtain the following: For every $(\varsigma(m))_{m\in\mathbb{N}}$ in $\{+,-\}^{\mathbb{N}}$, every ς in $\{+,-\}$, and every sequence of integers $(m_\ell)_{\ell\in\mathbb{N}}$ in $\mathbb{N}$ such that $m_\ell \to +\infty$ as $\ell \to +\infty$ and such that for every ℓ such that $\varsigma(m_\ell) = \varsigma$, we have

$$\rho_{\beta_{m_\ell}\cdot\varphi(\underline{\varsigma})} \to \rho^\varsigma \text{ as } \ell \to +\infty.$$

So for every $(\varsigma(m))_{m\in\mathbb{N}}$ in $\{+,-\}^{\mathbb{N}}$ having infinitely many $+$'s and infinitely many $-$'s, the Lipschitz continuous function $\varphi = \varphi(\underline{\varsigma})$ satisfies the desired property.

Remark 3.5 By construction, the function $\varphi(\underline{\varsigma})$ in the proof of Corollary 1.1 attains its maximum precisely on the set $X^+ \cup X^-$. It follows that ρ^+ and ρ^- are the only invariant and ergodic probability measures ρ on Σ maximizing the integral $\int \varphi \, \mathrm{d}\rho$.

3.2.2 Dimension Larger than 1

We show 2 ways to choose the sequences $(X_m^+)_{m\in\mathbb{N}_0}$ and $(X_m^-)_{m\in\mathbb{N}_0}$ in the proof of Main Lemma B'. The first one is a general way to construct a multidimensional subshift from a one-dimensional subshift, preserving the topological entropy. The second way is based in some results of [25] on multidimensional subshifts of finite type.

We start with a construction of a multidimensional subshift from a one-dimensional subshift. Fix $d \geq 2$ and recall that $\Sigma = F^G$, with G equal to $\mathbb{N}_0^d$ or $\mathbb{Z}^d$. Put $G_0 := \mathbb{N}_0$ or $\mathbb{Z}$, accordingly. Given a one-dimensional subshift X of F^{G_0}, put

$$\widehat{X} := \left\{ (x_n)_{n\in G} \in \Sigma : \text{ for all } n_2, \ldots, n_d \in G_0, \left(x_{(i,n_2,n_3,\ldots,n_d)} \right)_{i\in G_0} \in X \right\}.$$

Notice that $\widehat{X}$ can be naturally identified with $X^{G_0^{d-1}}$. Since X is invariant by translation and closed in the product topology, the set $\widehat{X}$ is invariant by the action of G and closed in the product topology. Thus, $\widehat{X}$ is a multidimensional subshift, and it is easy to see that the topological entropies of X and $\widehat{X}$ are equal. Applying this construction to any pair of sequences used to prove Main Lemma B' in dimension 1, for example those constructed in the previuos subsection, we obtain the desired sequences in dimension d.

For the second construction, we use the following lemma which follows from some results in [25].[9] For the definition of strongly irreducible subshift we refer the reader to [25].

Lemma 3.6 *Let $d \geq 2$. Let X be a strongly irreducible subshift of finite type in Σ with at least two elements. Then there is a decreasing sequence $(X_m)_{m\in\mathbb{N}_0}$ of strongly irreducible subshifts of finite type in Σ starting with X and such that for every m we have $h_{\mathrm{top}}\left(\sigma|_{X_{m+1}}\right) < h_{\mathrm{top}}\left(\sigma|_{X_m}\right)$.*

Proof Put $X_0 = X$. By [25, Lemma 4.11], X_0 has positive entropy. By [25, Theorem 1.2, Lemma 9.2], if one removes a sufficiently large pattern of X_0 then the resulting subshift of finite type X_1 is strongly irreducible, has at least 2 elements, and its entropy is strictly smaller than the entropy of X_0. Again by [25, Lemma 4.11], X_1 has positive entropy. Continuing in this way one can construct a decreasing sequence of strongly irreducible subshifts of finite type with the desired property.

Now we show a way to choose the sequences $(X_m^+)_{m\in\mathbb{N}_0}$ and $(X_m^-)_{m\in\mathbb{N}_0}$ in the proof of Main Lemma B' when the dimension d is larger than 1. Recall that the alphabet F has at least 2 symbols, say 0 and 1. Let X_0 be the subshift of finite type of Σ that is contained in $\{0,1\}^G$ and whose set of forbidden patterns consist of two-site patterns with two consecutive 1's (in each direction, including the diagonals). Clearly, X_0 is a strongly irreducible subshift and

[9] Although the results in [25] are stated for $G = \mathbb{Z}^d$, they extend without change to the case $G = \mathbb{N}_0^d$ by remarking that a subshift of finite type of $F^{\mathbb{N}_0^d}$ and its natural extension have the same topological entropy. Recall that the natural extension of a subshift of finite type X of $F^{\mathbb{N}_0^d}$ is the set of configurations in $F^{\mathbb{Z}^d}$ having all of its patterns appearing in some configuration in X.

by Lemma 3.6 there is a decreasing sequence of subshifts $(X_m)_{m\in\mathbb{N}}$ with strictly decreasing entropy. For every integer $m \geq 1$ put $X_m^+ := X_m$ and let X_m^- be the subshift obtained by exchanging 0's and 1's in X_m. These last two sequences verify the desired properties.

3.3 Proof of Main Lemma B

The following is a variant of Lemma 3.1, with a similar proof. We include it for completeness.

Lemma 3.7 *Let X and X' (resp. $\widetilde{X}$ and $\widetilde{X}'$) be disjoint compact subsets of Σ, each of which is invariant by σ, and such that*

$$\widetilde{X} \subset X,\ \widetilde{X}' \subset X',\ h_{\text{top}}(\sigma|_X) > h_{\text{top}}(\sigma|_{X'}),\ \text{ and } h_{\text{top}}\left(\sigma|_{\widetilde{X}}\right) > h_{\text{top}}\left(\sigma|_{\widetilde{X}'}\right).$$

Moreover, let $\varphi\colon \Sigma \to \mathbb{R}$ and $\widetilde{\varphi}\colon \Sigma \to \mathbb{R}$ be Lipschitz continuous functions attaining its maximum precisely on $X \cup X'$ and $\widetilde{X} \cup \widetilde{X}'$, respectively. Then for every $\varepsilon_0 > 0$, every δ in $(0, 1)$, and every neighborhood U of X there is $\beta_0 > 0$ such that for every $\beta \geq \beta_0$, every ε in $[0, \varepsilon_0]$ and every translation invariant Gibbs measure ρ for the potential $\beta \cdot (\varphi + \varepsilon\widetilde{\varphi})$ we have

$$\rho(U) \geq 1 - \delta.$$

Proof It is enough to show that for every sequence $(\varepsilon_\ell)_{\ell\in\mathbb{N}}$ in $[0, \varepsilon_0]$ and every sequence of positive numbers $(\beta_\ell)_{\ell\in\mathbb{N}}$ such that $\beta_\ell \to +\infty$ and such that every sequence of translation invariant Gibbs measures $(\rho_\ell)_{\ell\in\mathbb{N}}$ for the potentials $\beta_\ell\cdot(\varphi+\varepsilon_\ell\widetilde{\varphi})$ that converges to a measure ρ as $\ell \to +\infty$, the measure ρ is supported on X. Note that ρ is in $\mathscr{M}_\sigma$. Taking a subsequence if necessary, assume $(\varepsilon_\ell)_{\ell\in\mathbb{N}}$ converges to a number ε in $[0, \varepsilon_0]$.

Putting $m := \sup\{\varphi(x) + \varepsilon\widetilde{\varphi}(x) : x \in \Sigma\}$, for every ν in $\mathscr{M}_\sigma$ that is supported on $\widetilde{X} \cup \widetilde{X}'$ we have

$$\begin{aligned}
\int \varphi + \varepsilon\widetilde{\varphi}\,\mathrm{d}\rho &= \lim_{\ell\to+\infty} \left(\frac{h_{\rho_\ell}}{\beta_\ell} + \int \varphi + \varepsilon_\ell\widetilde{\varphi}\,\mathrm{d}\rho_\ell \right)\\
&\geq \lim_{\ell\to+\infty} \left(\frac{h_\nu}{\beta_\nu} + \int \varphi + \varepsilon_\ell\widetilde{\varphi}\,\mathrm{d}\nu \right)\\
&= \int \varphi + \varepsilon\widetilde{\varphi}\,\mathrm{d}\nu\\
&= m.
\end{aligned}$$

It follows that ρ is supported on $\widetilde{X} \cup \widetilde{X}'$ if $\varepsilon > 0$, and on $X \cup X'$ if $\varepsilon = 0$. Let ν' in $\mathscr{M}_\sigma$ be supported on $\widetilde{X}$ if $\varepsilon > 0$, and on X if $\varepsilon = 0$. Then for every ℓ we have

$$h_{\rho_\ell} + \beta_\ell m \geq h_{\rho_\ell} + \beta_\ell \int \varphi + \varepsilon_\ell\widetilde{\varphi}\,\mathrm{d}\rho_\ell \geq h_{\nu'} + \beta_\ell \int \varphi + \varepsilon_\ell\widetilde{\varphi}\,\mathrm{d}\nu' = h_{\nu'} + \beta_\ell m,$$

and therefore $h_{\rho_\ell} \geq h_\nu$. Since the entropy function is upper semi-continuous, it follows that $h_\rho \geq h_{\nu'}$. Since this holds for every ν' in $\mathscr{M}_\sigma$ supported on $\widetilde{X}$ if $\varepsilon > 0$, and on X if $\varepsilon = 0$, and since by hypothesis

$$h_{\text{top}}\left(\sigma|_{\widetilde{X}}\right) > h_{\text{top}}\left(\sigma|_{\widetilde{X}'}\right) \text{ and } h_{\text{top}}(\sigma|_X) > h_{\text{top}}\left(\sigma|_{\widetilde{X}'}\right),$$

we conclude that $h_\rho = h_{\text{top}}\left(\sigma|_{\widetilde{X}}\right)$ if $\varepsilon > 0$, and that $h_\rho = h_{\text{top}}(\sigma|_X)$ if $\varepsilon = 0$. It follows that ρ is supported on $\widetilde{X}$ in the former case, and on X in the latter case. This completes the proof of the lemma. □

Lemma 3.8 *Let $\varphi_0 \colon \Sigma \to \mathbb{R}$ be a Lipschitz continuous function and let $\beta_0 \geq \beta_0' \geq 0$ be given. Let U be a clopen subset of Σ and let $\delta > 0$ be such that for every β in $[\beta_0', \beta_0]$ and every translation invariant Gibbs measure ρ_0 for the potential $\beta \cdot \varphi_0$, we have*

$$\rho_0(U) \geq 1 - \delta.$$

Then there is $\varepsilon > 0$ such that for every Lipschitz continuous function $\varphi \colon \Sigma \to \mathbb{R}$ satisfying $\|\varphi - \varphi_0\|_{\mathrm{Lip}} \leq \varepsilon$, for every β in $[\beta_0', \beta_0]$, and every translation invariant Gibbs measure ρ for the potential $\beta \cdot \varphi$, we have

$$\rho(U) \geq 1 - 2\delta.$$

Proof By contradiction. Let $(\varphi_\ell)_{\ell \in \mathbb{N}}$ be a sequence converging to φ_0 in Lip, and let $(\beta_\ell)_{\ell \in \mathbb{N}}$ be a sequence in $[\beta_0', \beta_0]$, such that the following holds. For every ℓ in $\mathbb{N}$ there is a translation invariant Gibbs measure ρ_ℓ for the potential $\beta_\ell \cdot \varphi_\ell$ such that

$$\rho_\ell(U) \leq 1 - 2\delta.$$

By the compactness of the set of probability measures in the weak* topology and Lemma 3.3, there is a translation invariant Gibbs measure ρ_0 for the potential $\beta \cdot \varphi_0$ such that $\rho_0(U) \leq 1 - 2\delta$, which is a contradiction. □

Proof of Main Lemma B Let $(X_m^\pm)_{m \in \mathbb{N}_0}$, $U^\pm$, $Y_m^\pm$, and $\chi_m^\pm$ be as in the proof of Main Lemma B'. Define inductively sequences of positive numbers $(\beta_m)_{m \in \mathbb{N}}$ and $(\varepsilon_m)_{m \in \mathbb{N}}$ in the same way as in part 1 of the proof of Main Lemma B', except that $\beta(\underline{\varsigma})$ is now defined as the number β_0 given by Lemma 3.7 with

$$X = X_{m-1}^{\varsigma(m)}, X' = Y_m^{\varsigma(m)} \setminus X_{m-1}^{\varsigma(m)}, \widetilde{X} = X_m^{\varsigma(m)}, \widetilde{X}' = Y_{m+1}^{\varsigma(m)} \setminus X_m^{\varsigma(m)},$$

$$\varphi = \varphi(\underline{\varsigma}), \widetilde{\varphi} = \chi_{m+1}^{\varsigma(m)}, \varepsilon_0 = \varepsilon_m, \delta = 2^{-(m+1)}, \text{ and } U = U^{\varsigma(m)},$$

and that ε_{m+1} is defined as follows: Given ς in $\{+, -\}$, let $\varepsilon_{m+1}(\varsigma)$ be the number ε given by Lemma 3.8 with $U = U^\varsigma$, $\delta = 2^{-m}$, and $\beta_0 = \beta_0' = \beta_1$ if $m = 1$ and $\beta_0 = \beta_m$, $\beta_0' = \beta_{m-1}$ if $m \geq 2$, and put

$$\varepsilon_{m+1} := \min\{\varepsilon_m, \varepsilon_{m+1}(+), \varepsilon_{m+1}(-)\}.$$

For $\underline{\varsigma}$ in $\{+, -\}^{\mathbb{N}}$ define $\varphi(\underline{\varsigma})$ as in part 1 of the proof of Main Lemma B'; the proof that $\left(\varphi(\underline{\varsigma})\right)_{\underline{\varsigma} \in \{+,-\}^{\mathbb{N}}}$ is continuous in Lip is the same as that in part 2 of Main Lemma B' and we omit it.

To prove the estimate of the theorem, let $\underline{\varsigma} = (\varsigma(m))_{m \in \mathbb{N}}$ in $\{+, -\}^{\mathbb{N}}$ be given, let m and $\widehat{m}$ be integers such that

$$\widehat{m} \geq m \geq 1 \text{ and } \varsigma(m) = \cdots = \varsigma(\widehat{m}),$$

and fix β in $[\beta_m, \beta_{\widehat{m}}]$. Enlarging m if necessary, assume m is the largest integer j such that $\beta_j \leq \beta$. The proof in the case $\beta = \beta_m$ is similar to the proof of Main Lemma B' and we omit it. Suppose $\beta > \beta_m$, and note that $\beta < \beta_{m+1}$, $\widehat{m} \geq m + 1$, and $\varsigma(m + 1) = \varsigma(m)$. Put $\underline{\widetilde{\varsigma}} := (\varsigma(k))_{k=1}^{m+1}$ and let $\varphi(\underline{\widetilde{\varsigma}})$ be defined by (3.2) with $\underline{\varsigma}$ replaced by $\underline{\widetilde{\varsigma}}$. By our choice of β_m and our assumption $\beta \geq \beta_m$, for every translation invariant Gibbs measure ρ_0 for the potential $\beta \cdot \varphi(\underline{\widetilde{\varsigma}})$ we have

$$\rho_0(U^{\varsigma(m)}) \geq 1 - 2^{-(m+1)}.$$

On the other hand,

$$
\begin{aligned}
\|\varphi(\underline{\varsigma}) - \varphi(\underline{\widetilde{\varsigma}})\|_{\mathrm{Lip}} &\leq \tfrac{1}{8} \sum_{k=m+2}^{+\infty} \varepsilon_k \left\| \chi_k^{\varsigma(k)} \right\|_{\mathrm{Lip}} \\
&\leq \tfrac{1}{8} \sum_{k=m+2}^{+\infty} \varepsilon_k \left(1 + \left\| \chi_{Y_k^{\varsigma(k)}} \right\|_{\mathrm{Lip}} \right) \\
&\leq \tfrac{3}{8} \sum_{k=m+2}^{+\infty} \varepsilon_l \\
&< \varepsilon_{m+2},
\end{aligned}
$$

so by our choice of ε_{m+2} and the inequality $\beta < \beta_{m+1}$, for every translation invariant Gibbs measure ρ for the potential $\beta \cdot \varphi(\underline{\varsigma})$ we have

$$\rho(U^{\varsigma(m)}) \geq 1 - 2^{-m}.$$

This completes the proof of the lemma. □

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Appendix 1: Zero-Temperature Convergence and Marginal Entropy

In their example of a Lipschitz continuous potential whose Gibbs measures do not converge as the temperature goes to zero, Chazottes and Hochman considered a minimal set supporting 2 distinct ergodic probability measures, see [9, Sects. 3, 4.1]. As explained in Sect. 4.3 of that paper, a key property of their example is that at certain scales the marginal entropies of these measures are sufficiently different. They asked whether such a connection between the convergence of Gibbs measures at zero temperature and marginal entropies exists in general, see Problem 4.1 below for a precise formulation. The purpose of this appendix is to exhibit examples for which this is not the case, thus answering the question of Chazottes and Hochman in the negative.

To formulate the question of Chazottes and Hochman more precisely, put $F := \{0, 1\}$, $\Sigma = \{0, 1\}^{\mathbb{N}_0}$, and let $\varphi \colon \Sigma \to \mathbb{R}$ be a Lipschitz continuous potential. Denote by $\mathscr{M}_\sigma(\varphi)$ the space of invariant probability measures ρ on Σ that are invariant by σ and that maximize $\int \varphi \, \mathrm{d}\rho$. For each integer $n \geq 1$ let $\mathscr{M}_n^*$ be the set of marginal distributions obtained by restricting a measure in $\mathscr{M}_\sigma(\varphi)$ to $\{0, 1\}^n$, *i.e.*, if we identify $\{0, 1\}^{\{0,\ldots,n-1\}}$ with $\{0, 1\}^n$ and denote by $\pi_n \colon \Sigma \to \{0, 1\}^n$ canonical projection, then

$$\mathscr{M}_n^* = \{(\pi_n)_* \mu : \mu \in \mathscr{M}_\sigma(\varphi)\}.$$

Since the entropy function defined on $\mathscr{M}_n^*$ is strictly concave, it attains its maximum at a unique point μ_n^*; put

$$\mathscr{M}_n := \left\{ \mu \in \mathscr{M}_\sigma(\varphi) : (\pi_n)_* \mu = \mu_n^* \right\}.$$

Note that if for each n we choose a measure μ_n in $\mathscr{M}_n$, then the set of accumulation measures of the sequence $((\pi_n)_*\mu_n)_{n\in\mathbb{N}}$ is independent of $(\mu_n)_{n\in\mathbb{N}}$. In the case the limit $\lim_{n\to+\infty}(\pi_n)_*\mu_n$ exists, we denote it by $\lim_{n\to+\infty}\mathscr{M}_n$.

Problem 4.1 [9, Sect. 4.3] *Is the existence of* $\lim_{\beta\to+\infty}\rho_{\beta\cdot\varphi}$ *equivalent to the existence of* $\lim_{n\to+\infty}\mathscr{M}_n$*?*

Denote by $\overline{0}$ (resp. $\overline{1}$) the constant sequence in Σ equal to 0 (resp. 1), and let φ be the Lipschitz continuous function given by (the proof of) Corollary 1.1 with $X^+ = \{\overline{0}\}$ and $X^- = \{\overline{1}\}$. Then the Gibbs measures $\left(\rho_{\beta\cdot\varphi}\right)_{\beta>0}$ accumulate at the same time on $\delta_{\overline{0}}$ and on $\delta_{\overline{1}}$ as $\beta\to+\infty$, so the limit $\lim_{\beta\to+\infty}\rho_{\beta\cdot\varphi}$ does not exist. We show below that $\lim_{n\to+\infty}\mathscr{M}_n$ exists. This answers negatively the question in Problem 4.1.

To prove that $\lim_{n\to+\infty}\mathscr{M}_n$ exists, we use that

$$\mathscr{M}_\sigma(\varphi) = \left\{\alpha_0\delta_{\overline{0}} + \alpha_1\delta_{\overline{1}} : \alpha_0 \geq 0, \alpha_1 \geq 0, \alpha_0 + \alpha_1 = 1\right\}, \tag{4.1}$$

see Remark 3.5. Thus, if we denote by $\overline{0}_n$ (resp. $\overline{1}_n$) the constant sequence in $\{0,1\}^n$ equal to 0 (resp. 1), then we have

$$\mathscr{M}_n^* = \left\{\alpha_0\delta_{\overline{0}_n} + \alpha_1\delta_{\overline{1}_n} : \alpha_0 \geq 0, \alpha_1 \geq 0, \alpha_0 + \alpha_1 = 1\right\},$$

and therefore

$$\mu_n^* = \tfrac{1}{2}\left(\delta_{\overline{0}_n} + \delta_{\overline{1}_n}\right) \text{ and } \mathscr{M}_n = \left\{\tfrac{1}{2}\left(\delta_{\overline{0}} + \delta_{\overline{1}}\right)\right\}.$$

It follows that $\lim_{n\to+\infty}\mathscr{M}_n = \frac{1}{2}\left(\delta_{\overline{0}} + \delta_{\overline{1}}\right)$.

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