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Low-temperature phase transitions in the quadratic family

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Abstract

We give the first example of a quadratic map having a phase transition after the first zero of the geometric pressure function. This implies that several dimension spectra and large deviation rate functions associated to this map are not (expected to be) real analytic, in contrast to the uniformly hyperbolic case. The quadratic map we study has a non-recurrent critical point, so it is non-uniformly hyperbolic in a strong sense.

Keywords: Quadratic family; Thermodynamic formalism

1. Introduction

In their pioneer works, Sinaĭ, Bowen and Ruelle [43,4,41] initiated the thermodynamic formalism of smooth dynamical systems. They gave a complete description in the case of a uniformly hyperbolic diffeomorphism and a Hölder continuous potential. In the last decades there has been a substantial progress in extending the theory beyond this setting. A complete picture is emerging in real and complex dimension 1, see [5,19,22,23,33–35] and references therein. See also [42,46,47] and references therein for (recent) results in higher dimensions.

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In this paper we focus in the quadratic family; one of the simplest and yet challenging families of smooth one-dimensional maps. For a real parameter c we consider 2 dynamical systems arising from the complex quadratic polynomial

$$f_c(z) := z^2 + c;$$

the action of f_c on $\mathbb{R}$ and the action of f_c on its complex Julia set. For each of these dynamical systems and for a varying real number t, we consider the pressure of the geometric potential $-t \log|Df_c|$. There are thus 2 pressure functions associated to f_c: One in the real setting and another one in the complex setting. In what follows we use "geometric pressure function" to refer to any of these functions; precise definitions and statements are given in Section 1.1.

Our main interest are "phase transitions" in the statistical mechanics sense: For a real number t_* the map f_c has a *phase transition at* $t = t_*$ if the geometric pressure function is not real analytic at $t = t_*$. In the real case, phase transitions might be caused by lack of transitivity, see for example [10]. Since these phase transitions are well understood, we restrict our discussion to parameters for which the real map is transitive. For $c = -2$ the map f_{-2} is a Chebyshev polynomial and it has a phase transition at $t = -1$.[1] The mechanism behind this phase transition, and of any phase transition in the complex setting that occurs at a negative value of t, was explained by Makarov and Smirnov, see [22, Theorem B].[2] Combining the results of Makarov and Smirnov with recent results of Przytycki and the second named author, it follows that for every real parameter $c \neq -2$ the map f_c has at most 1 phase transition; moreover, if f_c has a phase transition, then it occurs at some $t > 0$. See [34, §A.3] for the complex case and [35] for the real case.

To describe the possible phase transitions for $c \neq -2$, it is useful to distinguish 3 complementary cases: f_c uniformly hyperbolic, f_c satisfying the *Collet–Eckmann condition*:

$$\chi_{\mathrm{crit}}(c) := \liminf_{n \to +\infty} \frac{1}{n} \log\left|Df_c^n(c)\right| > 0, \text{[3]}$$

and the remaining case, when f_c is not uniformly hyperbolic and does not satisfy the Collet–Eckmann condition. The Collet–Eckmann condition is one of the strongest and most studied non-uniform hyperbolicity conditions in dimension 1, see for example [1,2,14,31,37] and references therein. Benedicks and Carleson showed that the set of real parameters c such that f_c satisfies the Collet–Eckmann condition has positive Lebesgue measure, see [2]. Moreover, Avila and Moreira showed that the set of real parameters c such that f_c is not uniformly hyperbolic and does not satisfy the Collet–Eckmann condition has zero Lebesgue measure, see [1] and also [14].

[1] For $c = -2$ the Julia set of f_{-2} is the interval $[-2, 2]$ and both, the real and complex geometric pressure functions of f_{-2} are given by $t \mapsto \max\{-t \log 4, (1-t)\log 2\}$.

[2] Makarov and Smirnov showed this type of phase transition is caused by the existence of a gap in the Lyapunov spectrum; more precisely, they showed that if a complex rational map has a phase transition at some $t < 0$, then there is a finite set of periodic points F such that there is a definite distance separating the Lyapunov exponents of periodic points in F and the Lyapunov exponents of measures that do not charge F. Makarov and Smirnov also showed that this type of phase transition is removable in the following sense: The function obtained by omitting the measures that charge F in the supremum defining the geometric pressure function is real analytic on $(-\infty, 0)$ and coincides with the geometric pressure function up to the phase transition.

[3] In the complex setting the Collet–Eckmann condition is usually formulated in such a way that a uniformly hyperbolic map satisfies the Collet–Eckmann condition by vacuity. Here we use the usual terminology in the real setting, for which a uniformly hyperbolic map does not satisfy the Collet–Eckmann condition.

When f_c is uniformly hyperbolic, the work of Sinaĭ, Bowen and Ruelle can be adapted to show that the geometric pressure function is real analytic at every real number, see for example [36, §6.4]. That is, if f_c is uniformly hyperbolic, then it has no phase transitions.

If f_c is not uniformly hyperbolic and does not satisfy the Collet–Eckmann condition, then the geometric pressure function is non-negative and vanishes for large values of t, see [31, Theorem A] or [39, Corollary 1.3] for the real case and [37, Main Theorem] for the complex case. Thus in this case f_c has a phase transition at the first zero of the geometric pressure function. Note that this phase transition is associated to the lack of (non-uniform) expansion of f_c.

This paper is focused in the remaining case, when f_c satisfies the Collet–Eckmann condition. We show that, contrary to a widespread belief, such a map can have a phase transition at some $t > 0$. As a consequence, several dimension spectra and large deviation rate functions associated to such an f_c are not (expected to be) real analytic, see Remark 1.1. In the complex setting it also follows that the corresponding integral means spectrum is not real analytic either.

Our construction is very flexible. We give the simplest example here, of a "first-order" phase transition: The geometric pressure function is not differentiable at the phase transition. In the companion paper [7] we modify our construction to obtain a "high-order" phase transition: The geometric pressure function is bounded from above and from below by smooth functions that coincide at the phase transition. To the best of our knowledge it is the first example of a (transitive) smooth dynamical system having such an infinite contact-order phase transition. Our construction is also robust: In every sufficiently small perturbation of the quadratic family there is a Collet–Eckmann parameter having a phase transition.

The quadratic maps studied here are largely inspired by the conformal Cantor sets with analogous properties studied by Makarov and Smirnov, see [23, §5]. There are however several important differences. Most notably, the conformal Cantor set studied by Makarov and Smirnov is defined through a map having 2 affine branches, something that cannot be replicated in a complex polynomial or rational map.

These examples show that lack of (non-uniform) expansion is not the only source of phase transitions.[4] In fact, the quadratic maps studied here satisfy a property that is even stronger than the Collet–Eckmann condition: The critical point is non-recurrent.[5] Thus, no slow recurrence condition, such as the one studied by Benedicks and Carleson [2] or by Yoccoz and by Pesin and Senti [33], is sufficient to avoid phase transitions.

1.1. Statements of results

We consider a set of real parameters close to $c = -2$, for which the critical point $z = 0$ is mapped to a certain uniformly expanding set under forward iteration by f_c, see Section 3 for details. For such a parameter c we have $f_c(c) > c$, the interval $I_c := [c, f_c(c)]$ is invariant by f_c, and f_c is topologically exact on this set. We consider both, the interval map $f_c|_{I_c}$ and the complex quadratic polynomial f_c acting on its Julia set J_c.

[4] In some sense, the phase transitions studied here, as those studied by Makarov and Smirnov, are caused by the irregular behavior of the critical orbit.

[5] This is usually called the "Misiurewicz condition" and it is known to imply the Collet–Eckmann condition, see [30] for the real setting and [24] for the complex one.

For a real parameter c denote by $\mathscr{M}_c^{\mathbb{R}}$ the space of probability measures supported on I_c that are invariant by f_c. For a measure μ in $\mathscr{M}_c^{\mathbb{R}}$ denote by $h_\mu(f_c)$ the measure-theoretic entropy of f_c with respect to μ and for each t in $\mathbb{R}$ put

$$P_c^{\mathbb{R}}(t) := \sup\left\{h_\mu(f_c) - t\int \log|Df_c|\,d\mu \;\middle|\; \mu \in \mathscr{M}_c^{\mathbb{R}}\right\},$$

which is finite. The function $P_c^{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ so defined is called the *geometric pressure function of $f_c|_{I_c}$*; it is convex and non-increasing. An invariant probability measure supported on I_c is an *equilibrium state of $f_c|_{I_c}$ for the potential $-t\log|Df_c|$*, if the supremum above is attained at this measure.

Similarly, denote by $\mathscr{M}_c^{\mathbb{C}}$ the space of probability measures supported on J_c that are invariant by f_c and for a measure μ in $\mathscr{M}_c^{\mathbb{C}}$ denote by $h_\mu(f_c)$ the measure-theoretic entropy of f_c with respect to μ. The *geometric pressure function $P_c^{\mathbb{C}} : \mathbb{R} \to \mathbb{R}$ of $f_c|_{J_c}$* is defined by

$$P_c^{\mathbb{C}}(t) := \sup\left\{h_\mu(f_c) - t\int \log|Df_c|\,d\mu \;\middle|\; \mu \in \mathscr{M}_c^{\mathbb{C}}\right\}.$$

An invariant probability measure supported on J_c is an *equilibrium state of $f_c|_{J_c}$ for the potential $-t\log|Df_c|$* if the supremum above is attained at this measure.

Following the usual terminology in statistical mechanics, for a given $t_* > 0$ the map $f_c|_{I_c}$ (resp. $f_c|_{J_c}$) has a *phase transition at t_** if $P_c^{\mathbb{R}}$ (resp. $P_c^{\mathbb{C}}$) is not real analytic at $t = t_*$. As mentioned above, if $c \neq -2$ and if f_c is not uniformly hyperbolic and does not satisfy the Collet–Eckmann condition, then $f_c|_{I_c}$ (resp. $f_c|_{J_c}$) has a phase transition at the first zero of $P_c^{\mathbb{R}}$ (resp. $P_c^{\mathbb{C}}$) and it has no other phase transitions. In accordance with the usual interpretation of $t > 0$ as the inverse of the temperature in statistical mechanics, we say that such a phase transition is of *high temperature*. For a real parameter c and for $t_* > 0$ the map $f_c|_{I_c}$ (resp. $f_c|_{J_c}$) has a *low-temperature phase transition at t_**, if it has a phase transition at t_* and $P_c^{\mathbb{R}}(t_*) < 0$ (resp. $P_c^{\mathbb{C}}(t_*) < 0$). Note that if $f_c|_{I_c}$ (resp. $f_c|_{J_c}$) has a low-temperature phase transition, then f_c satisfies the Collet–Eckmann condition.

Main Theorem. *There is a real parameter c such that the critical point of f_c is non-recurrent and such that both, $f_c|_{I_c}$ and $f_c|_{J_c}$ have a low-temperature phase transition. Furthermore, the parameter c can be chosen so that each of the functions $P_c^{\mathbb{R}}$ and $P_c^{\mathbb{C}}$ is non-differentiable at the phase transition and so that each of the maps $f_c|_{I_c}$ and $f_c|_{J_c}$ has a unique equilibrium state at the phase transition.*

For the parameter c we use to prove the Main Theorem, we show that the equilibrium state at the phase transition is ergodic and mixing and that its measure-theoretic entropy is strictly positive, see Proposition A in Section 4. Combined with results of Young [48] and Gouezël [16, *Théorème* 2.3.1], our estimates imply that the decay of correlations of this measure is (at most) stretch exponential.

In the companion paper [7], we use the results of this paper to show that there is a real parameter c and $t_* > 0$ such that both, $f_c|_{I_c}$ and $f_c|_{J_c}$ have a high-order phase transition at $t = t_*$ and such that the functions $P_c^{\mathbb{R}}$ and $P_c^{\mathbb{C}}$ are bounded from above and from below by smooth functions that coincide at $t = t_*$. In that case there is no equilibrium state at $t = t_*$, see [18, Corollary 1.3].

Remark 1.1. For a parameter c in $\mathbb{C}$ the dimension spectrum for Lyapunov exponents of the complex quadratic polynomial $f_c(z) = z^2 + c$ is essentially the Legendre transform of $P_c^{\mathbb{C}}(t)$, see [15, Theorem 1] for a precise statement and [32] for the general theory. So, for a complex quadratic polynomial as in the Main Theorem the dimension spectrum for Lyapunov exponents is not real analytic.[6] A similar behavior is expected for the dimension spectrum for Lyapunov exponents of an interval map as in the Main Theorem.[7] The Legendre transform of $P_c^{\mathbb{R}}$ (or $P_c^{\mathbb{C}}$) is also related to the dimension spectrum for pointwise dimension and the rate function in certain large deviation principles; see for example [22, §5] for the former and [20, Theorem 1.2 or 1.3] and [34, Corollary B.4] for the latter. So for a map as in the Main Theorem we expect these functions not to be real analytic either. Finally, note that in the complex setting the integral means spectrum associated to f_c is an affine function of $P_c^{\mathbb{C}}$, see [3, Lemma 2]. So for a parameter c as in the Main Theorem the integral means spectrum associated to f_c is not real analytic.

1.2. Organization

After recalling some well-known facts in Section 2, we define in Section 3 the set of parameters $\bigcup_{n=3}^{+\infty} \mathcal{K}_n$, from which we choose the parameter fulfilling the properties in the Main Theorem. In Sections 3, 4.1 we show various combinatorial properties of the corresponding quadratic maps, as well as some distortion bounds and other preliminary estimates. For $n \geqslant 3$ and c in $\mathcal{K}_n$, the integer n indicates the time the forward orbit of c under f_c takes for entering a certain Cantor set Λ_c that is invariant by f_c^3, see Section 3.3 for the definition of Λ_c and some of its properties. The map $f_c^3|_{\Lambda_c}$ is uniformly expanding and conjugated to the shift map acting on $\{0,1\}^{\mathbb{N}_0}$. The set $\mathcal{K}_n$ is such that the function that to each c in $\mathcal{K}_n$ associates the itinerary of $f_c^n(c)$ in Λ_c under $f_c^3|_{\Lambda_c}$, is a bijection (Proposition 3.1). Thus, within $\mathcal{K}_n$, we can uniquely prescribe the itinerary of the postcritical orbit.

For $n \geqslant 3$ and c in $\mathcal{K}_n$, the map f_c^3 has precisely 2 fixed points in Λ_c, denoted by $p(c)$ and $\tilde{p}(c)$. They correspond to the symbols 0 and 1, respectively. For large n and every c in $\mathcal{K}_n$, the derivative of f_c^3 at $p(c)$ is strictly larger than that at $\tilde{p}(c)$, see Appendix A and Proposition 3.1. Similarly as in the example of Makarov and Smirnov, we consider a parameter c such that for every large integer $k \geqslant 1$, the forward orbit of c under f_c up to a time k spends roughly $\sqrt{k}$ of the time in the branch of $f_c^3|_{\Lambda_c}$ corresponding to $p(c)$ (of symbol 0), and the rest of the time in the other branch (of symbol 1). An additional difficulty in our situation is that the map $f_c^3|_{\Lambda_c}$ is non-linear, and thus in our estimates we have to deal with additional distortion terms. We overcome this difficulty, in part, by choosing an itinerary having only large blocks of 0's and 1's, see Lemma 4.4 for the precise definition of the itinerary. Choosing n large also help us to overcome this difficulty. Roughly speaking, in the example of Makarov and Smirnov this last choice corresponds to taking a small critical branch.[8]

A step in proving that for a parameter c as above the geometric pressure function is not real analytic on all of $(0,+\infty)$, is to show that this function is larger than or equal to $t \mapsto -t\chi_{\mathrm{crit}}(c)/2$. We do this by exhibiting a sequence of periodic orbits whose Lyapunov exponents converge

[6] The following argument shows that for c as in the Main Theorem, the Legendre transform of $P_c^{\mathbb{C}}$ is not real analytic. Since $P_c^{\mathbb{C}}$ is not differentiable at the phase transition, there is an interval on which the Legendre transform of $P_c^{\mathbb{C}}$ is affine. So, if the Legendre transform was real analytic, then it would be affine on all of its domain of definition. This can only happen if $P_c^{\mathbb{C}}$ is affine up to the phase transition. But [22, Theorem C] or [34, Theorem D] imply that this is not the case.

[7] More precisely, we expect the dimension spectrum of Lyapunov exponents not to be real analytic at the left end point of the interval A appearing in [19, Theorem A].

[8] This is not entirely accurate, but it is a good first approximation. By choosing n large we are essentially forced to consider the first return map to a smaller neighborhood of the critical point, and thus we have to deal with a larger set of orbits that never enter this set. These extra orbits are not present in the example of Makarov and Smirnov.

to $\chi_{\mathrm{crit}}(c)/2$, see Section 6.3. The bulk of the proof of the Main Theorem, in Sections 4.2–7, is devoted to show that for a large value of $t > 0$ the geometric pressure is less than or equal to $-t\chi_{\mathrm{crit}}(c)/2$. This implies that the geometric pressure is in fact equal to $-t\chi_{\mathrm{crit}}(c)/2$, and therefore that the geometric pressure function coincides with the function $t \mapsto -t\chi_{\mathrm{crit}}(c)/2$ on some (right) half line. Since at $t = 0$ the geometric pressure is equal to the topological entropy and it is therefore strictly positive, it follows that the geometric pressure function cannot be real analytic on all of $(0, +\infty)$.

To prove that for a large value of $t > 0$ the geometric pressure is less than or equal to $-t\chi_{\mathrm{crit}}(c)/2$, we show, as in the example of Makarov and Smirnov, that the pressure function can be estimated using a certain "postcritical series", defined solely in terms of the derivatives of f_c along the forward orbit of c (Proposition D in Section 7). To make this estimate we proceed in a different way than the example of Makarov and Smirnov. We consider the pressure function as defined through the tree of preimages of the critical point. An important step of the proof is to show that the dynamics is sufficiently expanding far away from the critical point (Proposition B in Section 5), and thus that the geometric pressure is governed by those backward orbits that visit a given neighborhood of the critical point. For a conveniently chosen neighborhood V_c of the critical point, we estimate the pressure of the backward orbits of the critical point that visit V_c using the first return map F_c of f_c to V_c, and a certain 2 variables pressure function of F_c. This last pressure function depends on the geometric potential of F_c and the first return time function. The neighborhood V_c and the first return map F_c are defined in Section 6.1, and the 2 variables pressure function of F_c is defined in Section 6.2. The connection between the geometric pressure of f_c and the 2 variables pressure function of F_c is through a Bowen type formula that we state as Proposition C in Section 6.2.

The 2 variables pressure function of F_c is defined through a subadditive sequence in a standard way, see Section 6.2. Thanks to the fact that our distortion bounds are independent of n and of c in $\mathcal{K}_n$, the first term of the subadditive sequence provides an estimate of the 2 variable pressure that is good enough for our purposes. To estimate the first term of the subadditive sequence, in Section 7 we partition the components of the domain of F_c into "levels", according to the first return time to a certain neighborhood of Λ_c. The proof of Proposition D consists of showing that for each integer $k \geqslant 0$, the contribution of the components of the domain of F_c of level k is equal to the k-th term of the postcritical series, up to a multiplicative constant, see Lemma 7.2.

We state a consequence of Proposition D as Proposition A in Section 4, from which we deduce the Main Theorem in Section 4.2. The proof of Proposition A is given in Section 7, after the proof of Proposition D.

2. Preliminaries

We use $\mathbb{N}$ to denote the set of integers that are greater than or equal to 1 and put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For an annulus A contained in $\mathbb{C}$ we use $\mathrm{mod}(A)$ to denote the conformal modulus of A.

2.1. Koebe principle

We use the following version of Koebe distortion theorem that can be found, for example, in [26]. Given an open subset G of $\mathbb{C}$ and a biholomorphic map $f : G \to \mathbb{C}$, the *distortion of f* on a subset C of G is

$$\sup_{x,y \in C} |Df(x)|/|Df(y)|.$$

Koebe Distortion Theorem. *For each $A > 0$ there is a constant $\Delta > 1$ such that for each topological disk $\widehat{W}$ contained in $\mathbb{C}$ and each compact set K contained in $\widehat{W}$ and such that $\widehat{W} \setminus K$ is an annulus of modulus at least A, the following property holds: For each open topological disk U contained in $\mathbb{C}$ and every biholomorphic map $f : U \to \widehat{W}$, the distortion of f on $f^{-1}(K)$ is bounded by Δ.*

2.2. Quadratic polynomials, Green's functions and Böttcher coordinates

In this subsection and the next we recall some basic facts about the dynamics of complex quadratic polynomials, see for instance [6] or [28] for references.

For c in $\mathbb{C}$ denote by f_c the complex quadratic polynomial

$$f_c(z) = z^2 + c,$$

and by K_c the *filled Julia set* of f_c; that is, the set of all points z in $\mathbb{C}$ whose forward orbit under f_c is bounded in $\mathbb{C}$. The set K_c is compact and its complement is the connected set consisting of all points whose orbit converges to infinity in the Riemann sphere. Furthermore, we have $f_c^{-1}(K_c) = K_c$ and $f_c(K_c) = K_c$. The boundary J_c of K_c is the *Julia set of* f_c.

For a parameter c in $\mathbb{C}$, the *Green's function of* K_c is the function $G_c : \mathbb{C} \to [0, +\infty)$ that is identically 0 on K_c and that for z outside K_c is given by the limit

$$G_c(z) = \lim_{n \to +\infty} \frac{1}{2^n} \log\left|f_c^n(z)\right| > 0. \tag{2.1}$$

The function G_c is continuous, subharmonic, satisfies $G_c \circ f_c = 2G_c$ on $\mathbb{C}$, and it is harmonic and strictly positive outside K_c. On the other hand, the critical values are bounded from above by $G_c(0)$ and the open set

$$U_c := \left\{z \in \mathbb{C} \;\middle|\; G_c(z) > G_c(0)\right\}$$

is homeomorphic to a punctured disk. Notice that $G_c(c) = 2G_c(0)$, thus U_c contains c. Moreover, the Green's functions of f_c and $f_{\bar{c}}$ are related by $G_{\bar{c}}(z) = G_c(\bar{z})$.

By Böttcher's Theorem there is a unique conformal representation

$$\varphi_c : U_c \to \left\{z \in \mathbb{C} \;\middle|\; |z| > \exp\bigl(G_c(0)\bigr)\right\},$$

that conjugates f_c to $z \mapsto z^2$. It is called *the Böttcher coordinate of* f_c and satisfies $G_c = \log|\varphi_c|$. Note that $U_{\bar{c}} = \overline{U_c}$ and $\varphi_{\bar{c}} = \overline{\varphi_c}$.

2.3. External rays and equipotentials

Let c be in $\mathbb{C}$. For $v > 0$ the *equipotential* v *of* f_c is by definition $G_c^{-1}(v)$. A *Green's line of* G_c is a smooth curve on the complement of K_c in $\mathbb{C}$ that is orthogonal to the equipotentials of G_c and that is maximal with this property. Given t in $\mathbb{R}/\mathbb{Z}$, the *external ray of angle* t *of* f_c, denoted by $R_c(t)$, is the Green's line of G_c containing

$$\left\{\varphi_c^{-1}\bigl(r \exp(2\pi i t)\bigr) \;\middle|\; \exp\bigl(G_c(0)\bigr) < r < +\infty\right\}.$$

By the identity $G_c \circ f_c = 2G_c$, for each $v > 0$ and each t in $\mathbb{R}/\mathbb{Z}$ the map f_c maps the equipotential v to the equipotential $2v$ and maps $R_c(t)$ to $R_c(2t)$. For t in $\mathbb{R}/\mathbb{Z}$ the external ray $R_c(t)$ *lands at a point* z, if $G_c : R_c(t) \to (0, +\infty)$ is a bijection and if $G_c|_{R_c(t)}^{-1}(v)$ converges to z as v converges to 0 in $(0, +\infty)$. By the continuity of G_c, every landing point is in $J_c = \partial K_c$.

We use the following general lemma several times.

Lemma 2.1. *Let c be a parameter in $\mathbb{C}$, let t be in $\mathbb{R}/\mathbb{Z}$ and suppose that the external ray $R_c(t)$ lands at a point z_0 of K_c different from c; so $f_c^{-1}(z_0)$ consists of 2 distinct points. Then each point of $f_c^{-1}(z_0)$ is the landing point of precisely 1 of the external rays $R_c(t/2)$ or $R_c((t+1)/2)$.*

Proof. Since $f_c^{-1}(z_0)$ consists of 2 distinct points, it is enough to show that each point z of $f_c^{-1}(z_0)$ is the landing point of either $R_c(t/2)$ or $R_c((t+1)/2)$. Since z_0 is different from c, there is an open neighborhood U of z and an open neighborhood U_0 of z_0 such that f_c maps U biholomorphically to U_0. Reducing U and U_0 if necessary, it follows that $f_c^{-1}(R_c(t))$ is contained in an external ray landing at z. It must be either $R_c(t/2)$ or $R_c((t+1)/2)$. □

The *Mandelbrot set* $\mathcal{M}$ is the subset of $\mathbb{C}$ of those parameters c for which K_c is connected. The function

$$\begin{aligned} \Phi &: \mathbb{C} \setminus \mathcal{M} \to \mathbb{C} \setminus \mathrm{cl}(\mathbb{D}), \\ c &\mapsto \Phi(c) := \varphi_c(c) \end{aligned}$$

is a conformal representation, see [13, VIII, *Théorème* 1]. Since for each parameter c in $\mathbb{C}$ we have $\varphi_{\bar{c}} = \overline{\varphi_c}$, it follows that $\Phi(\bar{c}) = \overline{\Phi(c)}$; that is, Φ is real. For $v > 0$ the *equipotential v of $\mathcal{M}$* is by definition

$$\mathcal{E}(v) := \Phi^{-1}\big(\{z \in \mathbb{C} \mid |z| = v\}\big).$$

On the other hand, for t in $\mathbb{R}/\mathbb{Z}$ the set

$$\mathcal{R}(t) := \Phi^{-1}\big(\{r \exp(2\pi i t) \mid r > 1\}\big).$$

is called the *external ray of angle t of $\mathcal{M}$*. We say that $\mathcal{R}(t)$ *lands at a point z* in $\mathbb{C}$ if $\Phi^{-1}(r \exp(2\pi i t))$ converges to z as $r \searrow 1$. When this happens z belongs to $\partial\mathcal{M}$.

2.4. *The wake* 1/2

In this subsection we recall a few facts that can be found for example in [13] or [27].

Both external rays $\mathcal{R}(1/3)$ and $\mathcal{R}(2/3)$ of $\mathcal{M}$ land at the parameter $c = -3/4$ and these are the only external rays of $\mathcal{M}$ that land at this point, see for example [27, Theorem 1.2]. In particular, the complement in $\mathbb{C}$ of the set

$$\mathcal{R}(1/3) \cup \mathcal{R}(2/3) \cup \{-3/4\}$$

has 2 connected components; denote by $\mathcal{W}$ the connected component containing the point $c = -2$ of $\mathcal{M}$.

For each parameter c in $\mathcal{W}$ the map f_c has 2 distinct fixed points; one of the them is the landing point of the external ray $R_c(0)$ and it is denoted by $\beta(c)$; the other one is denoted by $\alpha(c)$. The only external ray landing at $\beta(c)$ is $R_c(0)$. Lemma 2.1 implies that the only external ray landing at $-\beta(c)$ is $R_c(1/2)$.

For the following fact, see for example [27, Theorem 1.2].

Theorem 1. *Let c be a parameter in $\mathcal{W}$. Then the only external rays of f_c landing at $\alpha(c)$ are $R_c(1/3)$ and $R_c(2/3)$.*

For c in $\mathcal{W}$, the complement of $R_c(1/3) \cup R_c(2/3) \cup \{\alpha(c)\}$ in $\mathbb{C}$ has 2 connected components; on containing $-\beta(c)$ and $z = c$, and the other one containing $\beta(c)$ and $z = 0$. On the other hand, the point $\alpha(c)$ has 2 preimages by f_c: Itself and $\tilde{\alpha}(c) := -\alpha(c)$. Together with Lemma 2.1, the theorem above implies that $R_c(1/6)$ and $R_c(5/6)$ are the only external rays landing at $\tilde{\alpha}(c)$.

Theorem 2. *(See [13, VIII, Théorème 2 and XIII, §1].) Let p and q be integers without common factors, with q even. Then the external ray $\mathcal{R}(p/q)$ of $\mathcal{M}$ lands and the landing point c is such that the critical point of f_c is eventually periodic but not periodic and such that the critical value c of f_c is the landing point of the external ray $R_c(p/q)$ of f_c. Conversely, if c is a parameter in $\mathbb{C}$ such that the critical point of f_c is eventually periodic but not periodic, then there are integers p and q without common factors and with q even, such that the critical value c of f_c is the landing point of $R_c(p/q)$; moreover, every external ray of f_c landing at c is of this form. In this case the parameter c is the landing point of the external ray $\mathcal{R}_c(p/q)$ of $\mathcal{M}$.*

Note that for the parameter $c = -2$ we have $c = -\beta(c)$, so the theorem above implies that $\mathcal{R}(1/2)$ is the only external ray of $\mathcal{M}$ that lands at -2.

2.5. *Yoccoz puzzles and para-puzzle*

In this subsection we recall the definitions of Yoccoz puzzle and para-puzzle. We follow [40].

Definition 2.2 *(Yoccoz puzzles).* Fix c in $\mathcal{W}$ and consider the open region $X_c := \{z \in \mathbb{C} \mid G_c(z) < 1\}$. The *Yoccoz puzzle of f_c* is given by the following sequence of graphs $(I_{c,n})_{n=0}^{+\infty}$ defined for $n = 0$ by

$$I_{c,0} := \partial X_c \cup \big(X_c \cap \mathrm{cl}\big(R_c(1/3)\big) \cap \mathrm{cl}\big(R_c(2/3)\big)\big)$$

and for $n \geqslant 1$ by $I_{c,n} := f_c^{-n}(I_{c,0})$. The *puzzle pieces of depth n* are the connected components of $f_c^{-n}(X_c) \setminus I_{c,n}$. The puzzle piece of depth n containing a point z is denoted by $P_{c,n}(z)$.

Note that for a real parameter c, every puzzle piece intersecting the real line is invariant under complex conjugation. Since puzzle pieces are simply-connected, it follows that the intersection of such a puzzle piece with $\mathbb{R}$ is an interval.

Definition 2.3 *(Yoccoz para-puzzles[9]).* Given an integer $n \geqslant 0$ put

$$J_n := \big\{t \in [1/3, 2/3] \,\big|\, 2^n t \pmod 1 \in \{1/3, 2/3\}\big\},$$

let $\mathcal{X}_n$ be the intersection of $\mathcal{W}$ with the open region in the parameter plane bounded by the equipotential $\mathcal{E}(2^{-n})$ of $\mathcal{M}$, and put

$$\mathcal{I}_n := \partial\mathcal{X}_n \cup \left(\mathcal{X}_n \cap \bigcup_{t \in J_n} \mathrm{cl}\big(\mathcal{R}(t)\big)\right).$$

Then the *Yoccoz para-puzzle of $\mathcal{W}$* is the sequence of graphs $(\mathcal{I}_n)_{n=0}^{+\infty}$. The *para-puzzle pieces of depth n* are the connected components of $\mathcal{X}_n \setminus \mathcal{I}_n$. The para-puzzle piece of depth n containing a parameter c is denoted by $\mathcal{P}_n(c)$.

9 In contrast with [40], we only consider para-puzzles contained in $\mathcal{W}$.

Observe that there is only 1 para-puzzle piece of depth 0 and only 1 para-puzzle piece of depth 1; they are bounded by the same external rays but different equipotentials. Both of them contain $c = -2$.

Definition 2.4 *(Holomorphic motion).* Let $\mathcal{C}$ be a complex manifold and fix c_0 in $\mathcal{C}$. Given a subset Z of $\mathbb{C}$, a map

$$h : \mathcal{C} \times Z \to \mathcal{C} \times \mathbb{C}$$

of the form $(c, z) \mapsto (c, h^c(z))$ is a *holomorphic motion based at* c_0 if h^{c_0} is the identity on Z, if for each z in Z its restriction to $\mathcal{C} \times \{z\}$ is holomorphic and if for each $c \in \mathcal{C}$ its restriction to $\{c\} \times Z$ is injective.

See [40] for a reference to the following lemma; the statement here is slightly different from the statement in [40] since we extend the domain of definition of the holomorphic motions, but the proof is the same. For each integer $n \geqslant 1$, put

$$V_n := \left\{ w \in \mathbb{C} \;\middle|\; \log^+ |w| \geqslant 2^{-n} \right\}.$$

Lemma 2.5. *Let $n \geqslant 0$ be an integer and c_0 a parameter contained in a para-puzzle of depth n. Then there exists a holomorphic motion*

$$h_n : \mathcal{P}_n(c_0) \times \left(I_{c_0,n+1} \cup \varphi_{c_0}^{-1}(V_{n+1}) \right) \to \mathcal{P}_n(c_0) \times \mathbb{C},$$
$$(c, z) \mapsto h_n(c, z) := \left(c, h_n^c(z) \right)$$

such that for every c in $\mathcal{P}_n(c_0)$ the function h_n^c is an extension of the restriction of $\varphi_c^{-1} \circ \varphi_{c_0}$ to $I_{c_0,n+1} \cup \varphi_{c_0}^{-1}(V_{n+1})$ that satisfies $I_{c,n+1} = h_n^c(I_{c_0,n+1})$. Moreover, when $n \geqslant 1$ the map h_n coincides with h_{n-1} on $\mathcal{P}_n(c_0) \times (I_{c_0,n} \cup \varphi_{c_0}^{-1}(V_n))$ and for each c in $\mathcal{P}_n(c_0)$ we have $f_c \circ h_n^c = h_{n-1}^c \circ f_{c_0}$ on $I_{c_0,n+1} \cup \varphi_{c_0}^{-1}(V_{n+1})$.

3. Parameters

In this section we study the set of parameters from which we choose the parameter in the Main Theorem and at the same time introduce some notation used in the rest of the paper.

Given an integer $n \geqslant 3$, let $\mathcal{K}_n$ be the set of all those real parameters c such that the following properties hold.

1. We have $c < 0$ and for each j in $\{1, \ldots, n-1\}$ we have $f_c^j(c) > 0$.
2. For every integer $k \geqslant 0$ we have

$$f_c^{n+3k+1}(c) < 0 \quad \text{and} \quad f_c^{n+3k+2}(c) > 0.$$

Note that for a parameter c in $\mathcal{K}_n$ the critical point of f_c cannot be asymptotic to a non-repelling periodic point, see [29, §8]. This implies that all the periodic points of f_c in $\mathbb{C}$ are hyperbolic repelling and therefore that $K_c = J_c$, see [28]. On the other hand, we have $f_c(c) > c$ and the interval $I_c = [c, f_c(c)]$ is invariant by f_c. This implies that I_c is contained in J_c and hence that for every real number t we have $P_c^{\mathbb{R}}(t) \leqslant P_c^{\mathbb{C}}(t)$. Note also that $f_c|_{I_c}$ is not renormalizable, so f_c is topologically exact on I_c, see for example [9, Theorem III.4.1].

Since for c in $\mathcal{K}_n$ the critical point of f_c is not periodic, we can define the sequence $\iota(c)$ in $\{0,1\}^{\mathbb{N}_0}$ for each $k \geqslant 0$ by

$$\iota(c)_k := \begin{cases} 0 & \text{if } f_c^{n+3k}(c) < 0; \\ 1 & \text{if } f_c^{n+3k}(c) > 0. \end{cases}$$

The remainder of this section is devoted to prove the following proposition.

Proposition 3.1. *For each integer $n \geqslant 3$ the set $\mathcal{K}_n$ is a compact subset of*

$$\mathcal{P}_n(-2) \cap (-2, -3/4)$$

and for every sequence $\underline{x}$ in $\{0,1\}^{\mathbb{N}_0}$ there is a unique parameter c in $\mathcal{K}_n$ such that $\iota(c) = \underline{x}$. Finally, for each $\delta > 0$ there is $n_1 \geqslant 3$ such that for each integer $n \geqslant n_1$ the set $\mathcal{K}_n$ is contained in the interval $(-2, -2+\delta)$.

After defining some sequences of puzzle pieces that are important for the rest of this paper in Section 3.1, we study the para-puzzle pieces containing $c = -2$ in Section 3.2 and the maximal invariant set of f_c^3 in $P_{c,1}(0)$ in Section 3.3. The proof of Proposition 3.1 is in Section 3.4.

3.1. First landing domains to $P_{c,1}(0)$

Fix a parameter c in $\mathcal{P}_0(-2)$.

The following are consequences of the facts recalled in Section 2.4. There are precisely 2 puzzle pieces of depth 0: $P_{c,0}(\beta(c))$ and $P_{c,0}(-\beta(c))$. Each of them is bounded by the equipotential 1 and by the closures of the external rays landing at $\alpha(c)$. Furthermore, the critical value c of f_c is contained in $P_{c,0}(-\beta(c))$ and the critical point in $P_{c,0}(\beta(c))$. It follows that the set $f_c^{-1}(P_{c,0}(\beta(c)))$ is the disjoint union of $P_{c,1}(-\beta(c))$ and $P_{c,1}(\beta(c))$, so f_c maps each of the sets $P_{c,1}(-\beta(c))$ and $P_{c,1}(\beta(c))$ biholomorphically to $P_{c,0}(\beta(c))$. Moreover, there are precisely 3 puzzle pieces of depth 1:

$$P_{c,1}\big(-\beta(c)\big), \qquad P_{c,1}(0) \quad \text{and} \quad P_{c,1}\big(\beta(c)\big);$$

$P_{c,1}(-\beta(c))$ is bounded by the equipotential $1/2$ and by the closures of the external rays that land at $\alpha(c)$; $P_{c,1}(\beta(c))$ is bounded by the equipotential $1/2$ and by the closures of the external rays that land at $\tilde{\alpha}(c)$; and $P_{c,1}(0)$ is bounded by the equipotential $1/2$ and by the closures of the external rays that land at $\alpha(c)$ and at $\tilde{\alpha}(c)$. In particular, the closure of $P_{c,1}(\beta(c))$ is contained in $P_{c,0}(\beta(c))$.

Put

$$\phi_c := f_c|_{P_{c,1}(-\beta(c))}^{-1} \quad \text{and} \quad \tilde{\phi}_c := f_c|_{P_{c,1}(\beta(c))}^{-1}.$$

Since the closure of $\tilde{\phi}_c(P_{c,0}(\beta(c))) = P_{c,1}(\beta(c))$ is contained in $P_{c,0}(\beta(c))$, all the iterates of $\tilde{\phi}_c$ are defined on $P_{c,0}(\beta(c))$ and take images in $P_{c,1}(\beta(c))$. Put $\alpha_0(c) := \alpha(c)$, $\tilde{\alpha}_0(c) := \tilde{\alpha}(c)$ and for each integer $n \geqslant 1$ put

$$\tilde{\alpha}_n(c) := \tilde{\phi}_c^n\big(\tilde{\alpha}_0(c)\big), \qquad \alpha_n(c) := \phi_c\big(\tilde{\alpha}_{n-1}(c)\big),$$
$$\widetilde{V}_{c,n} := \tilde{\phi}_c^n\big(P_{c,1}(0)\big), \quad \text{and} \quad V_{c,n} := \phi_c \circ \tilde{\phi}_c^{n-1}\big(P_{c,1}(0)\big).$$

Note that f_c^n maps each of the sets $V_{c,n}$ and $\widetilde{V}_{c,n}$ biholomorphically to $P_{c,1}(0)$; so both, $V_{c,n}$ and $\widetilde{V}_{c,n}$ are puzzle pieces of depth $n+1$. On the other hand, f_c^n maps each of the sets $P_{c,n}(-\beta(c))$ and $P_{c,n}(\beta(c))$ biholomorphically to $P_{c,0}(\beta(c))$. Note moreover that if c is real, then f_c, ϕ_c and $\tilde{\phi}_c$ are all real, so $\alpha(c)$, $\tilde{\alpha}(c)$, $\alpha_n(c)$ and $\tilde{\alpha}_n(c)$ are also real and each of the sets $V_{c,n}$, $\widetilde{V}_{c,n}$, $P_{c,n}(-\beta(c))$ and $P_{c,n}(\beta(c))$ is invariant by complex conjugation and intersects $\mathbb{R}$.

Lemma 3.2. *Let c be a parameter in $\mathcal{P}_0(-2)$. Then for every integer $n \geqslant 0$ the only external rays that land at $\alpha_n(c)$ are $R_c(\frac{3\cdot 2^n-1}{3\cdot 2^{n+1}})$ and $R_c(\frac{3\cdot 2^n+1}{3\cdot 2^{n+1}})$ and the only external rays that land at $\tilde{\alpha}_n(c)$ are $R_c(\frac{1}{3\cdot 2^{n+1}})$ and $R_c(\frac{3\cdot 2^{n+1}-1}{3\cdot 2^{n+1}})$. Furthermore, for each integer $n \geqslant 1$ the following properties hold.*

1. *The only puzzle pieces of depth $n+1$ contained in $P_{c,n}(-\beta(c))$ are $P_{c,n+1}(-\beta(c))$ and $V_{c,n}$. Moreover, the closure of $P_{c,n+1}(-\beta(c))$ is contained in the open set $P_{c,n}(-\beta(c))$.*
2. *The puzzle piece $P_{c,n}(-\beta(c))$ is bounded by the external rays landing at $\alpha_{n-1}(c)$ and the equipotential $1/2^n$; the puzzle piece $P_{c,n}(\beta(c))$ is bounded by the closure of the external rays landing at $\tilde{\alpha}_{n-1}(c)$ and the equipotential $1/2^n$.*

Proof. For an integer $n \geqslant 0$ put $\theta_n := \frac{1}{3\cdot 2^{n+1}}$ and $\theta_n' := 1-\theta_n$.

The proof of the first assertion is by induction. When $n = 0$ the assertion is shown in Section 2.4. Given an integer $n \geqslant 0$ assume that the only external rays that land at $\tilde{\alpha}_n(c)$ are those of angles θ_n and θ_n'. Since $f_c^{-1}(\tilde{\alpha}_n(c)) = \{\alpha_{n+1}(c), \tilde{\alpha}_{n+1}(c)\}$, by Lemma 2.1 the only external rays landing at $\alpha_{n+1}(c)$ or $\tilde{\alpha}_{n+1}(c)$ are those of angles $\frac{\theta_n}{2}$, $\frac{\theta_n+1}{2}$, $\frac{\theta_n'}{2}$ and $\frac{\theta_n'+1}{2}$. Since

$$\frac{\theta_n}{2} < \frac{1}{6} < \frac{\theta_n'}{2} < \frac{\theta_n+1}{2} < \frac{5}{6} < \frac{\theta_n'+1}{2}$$

and since $\tilde{\alpha}_{n+1}(c)$ is in $P_{c,1}(\beta(c))$, the external rays of angles $\frac{\theta_n}{2}$ and $\frac{\theta_n'+1}{2}$ land at $\tilde{\alpha}_{n+1}(c)$. By Lemma 2.1 it follows that the external rays of angles $\frac{\theta_n'}{2}$ and $\frac{\theta_n+1}{2}$ land at $\alpha_{n+1}(c)$. This completes the proof of the induction step and of the first assertion of the lemma.

To prove the rest of the assertions, assume $n \geqslant 1$. Since f_c^n maps $P_{c,n}(-\beta(c))$ biholomorphically to $P_{c,0}(\beta(c))$, $V_{c,n}$ biholomorphically to $P_{c,1}(0)$, and $P_{c,n+1}(\beta(c))$ biholomorphically to $P_{c,1}(\beta(c))$, it follows that the only puzzle pieces of depth $n+1$ contained in $P_{c,n}(-\beta(c))$ are $V_{c,n}$ and $P_{c,n+1}(-\beta(c))$. On the other hand, since the closure of $P_{c,1}(\beta(c))$ is contained in $P_{c,0}(\beta(c))$, it follows that the closure of $P_{c,n+1}(-\beta(c))$ is contained in $P_{c,n}(-\beta(c))$. We have thus proved part 1. When $n = 1$ part 2 follows from the considerations above. To prove part 2 when $n \geqslant 2$, recall that f_c^{n-1} maps each of the sets $P_{c,n-1}(-\beta(c))$ and $P_{c,n-1}(\beta(c))$ biholomorphically to $P_{c,0}(\beta(c))$. Since the closure of $P_{c,1}(\beta(c))$ is contained in $P_{c,0}(\beta(c))$ and since $\tilde{\alpha}(c)$ is the only point in the boundary of $P_{c,1}(\beta(c))$ that is in the Julia set of f_c, it follows that $\alpha_{n-1}(c)$ is the only point in the boundary of $P_{c,n}(-\beta(c))$ that is in the Julia set of f_c and that $\tilde{\alpha}_{n-1}(c)$ is the only point in the boundary of $P_{c,n}(\beta(c))$ that is in the Julia set of f_c. This implies part 2 and completes the proof of the lemma. $\square$

3.2. Para-puzzle pieces containing $c = -2$

The purpose of this subsection is to prove the following lemma.

Lemma 3.3. *The following properties hold.*

1. *For every integer $n \geqslant 1$, the para-puzzle piece $\mathcal{P}_n(-2)$ contains the closure of $\mathcal{P}_{n+1}(-2)$.*

2. *For every integer $n \geqslant 0$ and every parameter c in $\mathcal{P}_n(-2)$, the critical value c of f_c is in $P_{c,n}(-\beta(c))$.*

The proof of this lemma is given after the following one.

For each integer $n \geqslant 0$ put

$$t_n := \frac{3 \cdot 2^n - 1}{3 \cdot 2^{n+1}} \quad \text{and} \quad t'_n := \frac{3 \cdot 2^n + 1}{3 \cdot 2^{n+1}}.$$

Lemma 3.4. *Fix an integer $n \geqslant 1$. Then the parameter $c = -2$ is contained in a para-puzzle piece of depth n and there is a unique point $\hat{\alpha}_{n-1}$ in the boundary of $\mathcal{P}_n(-2)$ that is contained in $\mathcal{M}$. Furthermore, $\hat{\alpha}_{n-1}$ is in $\mathbb{R}$, the only the external rays of $\mathcal{M}$ that land at $\hat{\alpha}_{n-1}$ are $\mathcal{R}(t_{n-1})$ and $\mathcal{R}(t'_{n-1})$, and $\mathcal{P}_n(-2)$ is invariant under complex conjugation. In particular, $\mathcal{P}_n(-2)$ is bounded by the equipotential $1/2^n$ and the closures of the external rays $\mathcal{R}(t_{n-1})$ and $\mathcal{R}(t'_{n-1})$ of $\mathcal{M}$.*

Proof. Since $\mathcal{R}(1/2)$ is the only external ray of $\mathcal{M}$ that lands at $c = -2$ and since $t = 1/2$ is not in J_n, it follows that $c = -2$ is contained in a para-puzzle of level n. On the other hand, by Theorem 2 the external ray $\mathcal{R}(t_{n-1})$ of $\mathcal{M}$ lands at a parameter in $\mathcal{M}$, denoted by $\hat{\alpha}_{n-1}$, and the external ray $R_{\hat{\alpha}_{n-1}}(t_{n-1})$ of $f_{\hat{\alpha}_{n-1}}$ lands at the critical value $\hat{\alpha}_{n-1}$ of $f_{\hat{\alpha}_{n-1}}$. By Lemma 3.2 we have that $\alpha_{n-1}(\hat{\alpha}_{n-1}) = \hat{\alpha}_{n-1}$ and that $R_{\hat{\alpha}_{n-1}}(t_{n-1})$ and $R_{\hat{\alpha}_{n-1}}(t'_{n-1})$ are the only external rays of $f_{\hat{\alpha}_{n-1}}$ landing at $\hat{\alpha}_{n-1}$. Using Theorem 2 again, we conclude that $\mathcal{R}(t_{n-1})$ and $\mathcal{R}(t'_{n-1})$ are the only external rays of $\mathcal{M}$ landing at $\hat{\alpha}_{n-1}$. Since Φ is real, we have $\mathcal{R}(t'_n) = \overline{\mathcal{R}(t_n)}$ and therefore $\hat{\alpha}_{n-1}$ is in $\mathbb{R}$. On the other hand, since the interval (t_{n-1}, t'_{n-1}) is disjoint from J_n and contains $1/2$, the closures of the external rays $\mathcal{R}(t_{n-1})$ and $\mathcal{R}(t'_{n-1})$ of $\mathcal{M}$ and the equipotential $1/2^n$ of $\mathcal{M}$ bound a para-puzzle piece of depth n that contains $c = -2$; that is, they bound $\mathcal{P}_n(-2)$. It follows that $\hat{\alpha}_{n-1}$ is the only point in the boundary of $\mathcal{P}_n(-2)$ in $\mathcal{M}$. That $\mathcal{P}_n(-2)$ is invariant under complex conjugation follows from the fact that $\hat{\alpha}_{n-1}$ and Φ are real. $\square$

Proof of Lemma 3.3. Part 1 follows from the descriptions of $\mathcal{P}_n(-2)$ and $\mathcal{P}_{n+1}(-2)$ in terms of external rays and equipotentials given by Lemma 3.4.

To prove part 2, let $n \geqslant 0$ be an integer. We use the following direct consequence of the definitions of the puzzle and the para-puzzle and of Theorem 2: A parameter c in $\mathcal{W}$ is in a para-puzzle piece of depth n if and only if the critical value c of f_c is in a puzzle piece of depth n of f_c. Note that for $c = -2$ the critical value of f_{-2} is equal to $-\beta(-2)$ and hence it is in $P_{-2,n}(-\beta(-2))$. Since $P_{c,n}(-\beta(c))$ depends continuously with c on $\mathcal{P}_n(-2)$ (Lemma 2.5) and since $\mathcal{P}_n(-2)$ is connected by definition, it follows that for each c in $\mathcal{P}_n(-2)$ the critical value c of f_c is in $P_{c,n}(-\beta(c))$. $\square$

3.3. The uniformly expanding Cantor set

Let c be a parameter in $\mathcal{P}_3(-2)$. In this subsection we study the maximal invariant set Λ_c of f_c^3 in $P_{c,1}(0)$.

Note first that the critical value c of f_c is in $P_{c,3}(-\beta(c))$ (part 2 of Lemma 3.3) and hence in $P_{c,2}(-\beta(c))$. Since $\alpha_1(c)$ is in the boundary of $P_{c,2}(-\beta(c))$ and since the only external rays that land at this point are $R_c(5/12)$ and $R_c(7/12)$ (Lemma 3.2), by Lemma 2.1 the external rays $R_c(7/24)$ and $R_c(17/24)$ land at the same point, denoted by $\gamma(c)$, and these are the only

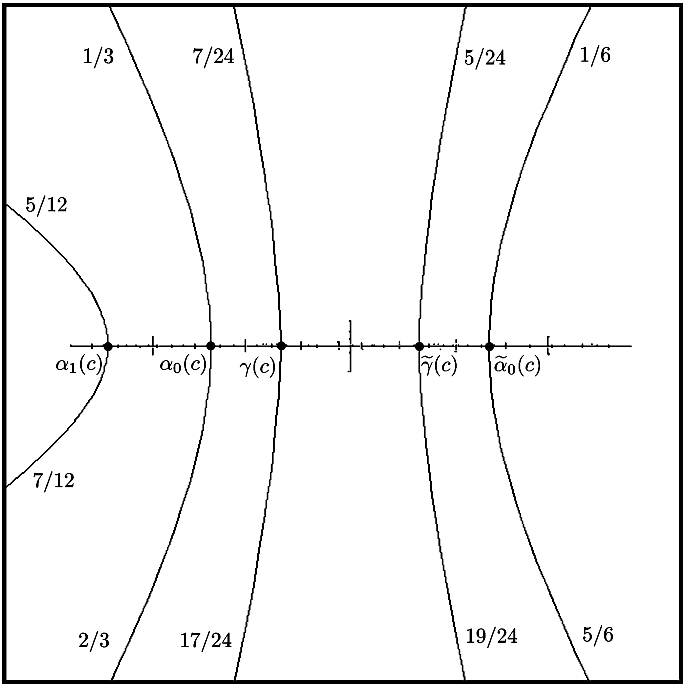

Fig. 1. Rays landing at $\alpha_0(c)$, $\tilde{\alpha}_0(c)$, $\alpha_1(c)$, $\gamma(c)$, and $\tilde{\gamma}(c)$.

external rays that land at this point. Similarly, the external rays $R_c(5/24)$ and $R_c(19/24)$ land at the same point, denoted by $\tilde{\gamma}(c)$, and these are the only external rays that land at $\tilde{\gamma}(c)$, see Fig. 1. Note that if in addition c is real, then the points $\alpha(c)$ and $\alpha_1(c)$ are both real (Section 3.1); together with the fact that c is in $P_{c,2}(-\beta(c))$, this implies $c < \alpha_1(c) < \alpha(c)$ (*cf.*, part 2 of Lemma 3.2). It follows that the points $\gamma(c)$ and $\tilde{\gamma}(c)$ are both real and that the set $P_{c,3}(0) = f_c^{-1}(P_{c,2}(-\beta(c)))$ satisfies

$$P_{c,3}(0) \cap \mathbb{R} = \big(\gamma(c), \tilde{\gamma}(c)\big).$$

Lemma 3.5. *Let c be a parameter in $\mathcal{P}_3(-2)$. Then there are precisely 2 connected components of $f_c^{-3}(P_{c,1}(0))$ contained in $P_{c,1}(0)$: One containing $\gamma(c)$ in its closure, denoted by Y_c, and another one containing $\tilde{\gamma}(c)$ in its closure, denoted by $\widetilde{Y}_c$, see Fig. 2; the map f_c^3 maps each of the sets Y_c and $\widetilde{Y}_c$ biholomorphically to $P_{c,1}(0)$. Moreover, the closures of Y_c and of $\widetilde{Y}_c$ are disjoint and contained in $P_{c,1}(0)$ and the set $Y_c \cup \widetilde{Y}_c$ is contained in $P_{c,3}(0)$ and it is disjoint from $P_{c,4}(0)$. Finally, if c is real, then each of the sets Y_c and $\widetilde{Y}_c$ is invariant by complex conjugation and intersects $\mathbb{R}$.*

Proof. We prove first

$$f_c^{-3}\big(P_{c,1}(0)\big) \cap P_{c,1}(0) = f_c^{-1}(V_{c,2}). \tag{3.1}$$

First notice that, since f_c^2 maps $V_{c,2}$ biholomorphically to $P_{c,1}(0)$, the set $f_c^{-1}(V_{c,2})$ is contained in $f_c^{-3}(P_{c,1}(0))$. On the other hand, the set $f_c(P_{c,1}(0)) = P_{c,0}(-\beta(c))$ contains $V_{c,2}$, so $f_c^{-1}(V_{c,2})$ is contained in $P_{c,1}(0)$. This proves that the set in the right-hand side of (3.1) is

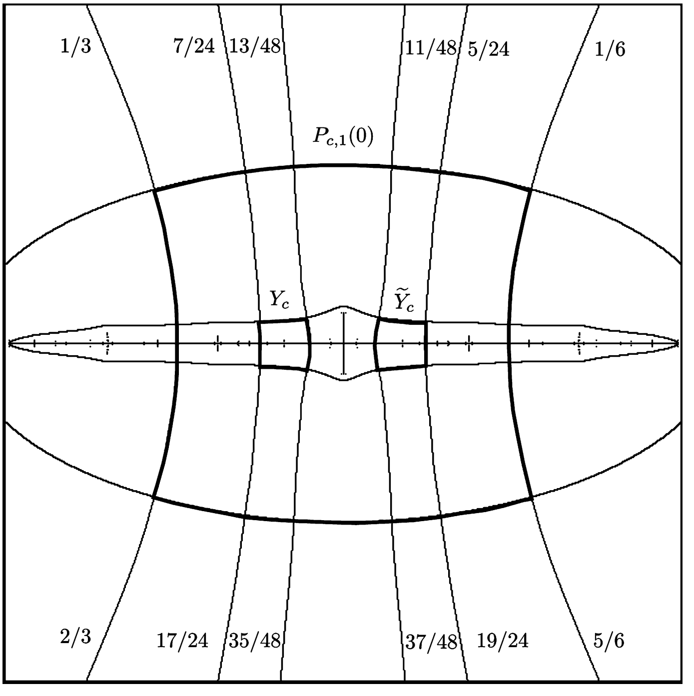

Fig. 2. The puzzle pieces $P_{c,1}(0)$, Y_c, and $\widetilde{Y}_c$.

contained in the set in the left-hand side. To prove the reverse inclusion, let z be a point in $P_{c,1}(0)$ such that $f_c^3(z)$ is in $P_{c,1}(0)$. Then z is in a puzzle piece of depth 4 and $f_c(z)$ is in a puzzle piece of depth 3 contained in $P_{c,1}(-\beta(c))$. This implies $f_c^2(z)$ is in $P_{c,0}(\beta(c))$. On the other hand, $f_c^2(z)$ is in $f_c^{-1}(P_{c,1}(0)) = V_{c,1} \cup \widetilde{V}_{c,1}$ and $V_{c,1}$ is contained in $P_{c,1}(-\beta(c)) \subset P_{c,0}(-\beta(c))$ (part 1 of Lemma 3.2), so we conclude that $f_c^2(z)$ is in $\widetilde{V}_{c,1}$ and hence that $f_c(z)$ is in $f_c^{-1}(\widetilde{V}_{c,1}) = V_{c,2} \cup \widetilde{V}_{c,2}$. Since $f_c(z)$ is in $P_{c,1}(-\beta(c))$ and $\widetilde{V}_{c,2}$ is contained in $P_{c,2}(\beta(c)) \subset P_{c,1}(\beta(c))$ (part 1 of Lemma 3.2), we conclude that $f_c(z)$ is in $V_{c,2}$ and hence that z is in $f_c^{-1}(V_{c,2})$. This completes the proof of (3.1).

To prove the assertions of the lemma, note that by part 2 of Lemma 3.3 the critical value c of f_c is in $P_{c,3}(-\beta(c))$, so it is not in the closure of $V_{c,2}$. This implies that $f_c^{-1}(V_{c,2})$ has 2 connected components whose closures are disjoint. On the other hand, $V_{c,2}$ contains $\alpha_1(c)$ in its closure (*cf.*, parts 1 and 2 of Lemma 3.2), so one of the connected components of $f_c^{-1}(V_{c,2})$ contains $\gamma(c)$ in its closure and the other one contains $\tilde{\gamma}(c)$ in its closure; denote them by Y_c and $\widetilde{Y}_c$, respectively. It follows that f_c^3 maps each of the sets Y_c and $\widetilde{Y}_c$ biholomorphically to $P_{c,1}(0)$. From the fact that $V_{c,2}$ is contained in $P_{c,2}(-\beta(c))$ and that the closure of this last set is contained in $P_{c,1}(-\beta(c))$ (part 1 of Lemma 3.2 with $n = 1$ and $n = 2$), it follows that closures of Y_c and $\widetilde{Y}_c$ are both contained in $P_{c,2}(0) = f_c^{-1}(P_{c,1}(-\beta(c)))$. Note also that $V_{c,2}$ is contained in $P_{c,2}(-\beta(c))$ and it is disjoint from $P_{c,3}(-\beta(c))$ (part 1 of Lemma 3.2), so $Y_c \cup \widetilde{Y}_c$ is contained in $P_{c,3}(0)$ and it is disjoint from $P_{c,4}(0)$. To prove the last statement of the lemma, suppose c is real. Then f_c and $\alpha_1(c)$ are real and $V_{c,2}$ is invariant by complex conjugation (Section 3.1). Since c is in $P_{c,3}(-\beta(c))$ we also have $c < \alpha_1(c)$. It follows that each of the sets Y_c and $\widetilde{Y}_c$ is invariant by complex conjugation and intersects $\mathbb{R}$. This completes the proof of the lemma. $\square$

For a parameter c in $\mathcal{P}_3(-3)$ define

$$g_c : Y_c \cup \widetilde{Y}_c \to P_{c,1}(0),$$
$$z \mapsto g_c(z) := f_c^3(z).$$

Lemma 3.5 implies that g_c maps each of the sets Y_c and $\widetilde{Y}_c$ biholomorphically to $P_{c,1}(0)$ and that

$$\Lambda_c = \bigcap_{n\in\mathbb{N}} g_c^{-n}\big(\mathrm{cl}\big(P_{c,1}(0)\big)\big).$$

In particular, Λ_c is contained in $Y_c \cup \widetilde{Y}_c$. So Lemma 3.5 implies that Λ_c is contained in $P_{c,3}(0)$ and that it is disjoint from $P_{c,4}(0)$. Moreover, Lemma 3.5 also implies that g_c is a Markov map, so Λ_c is a Cantor set and g_c is uniformly expanding on Λ_c, see for instance [8]. In particular, g_c has a unique fixed point in Y_c and a unique fixed point in $\widetilde{Y}_c$. Finally, note that if c is real, then g_c is real and Λ_c is contained in $\mathbb{R}$.

3.4. *Proof of Proposition 3.1*

Lemma 3.6. *There is a constant $\Delta_1 > 1$ such that for each parameter c in $\mathcal{P}_2(-2)$ the following properties hold for each integer $k \geqslant 2$: We have*

$$\Delta_1^{-1}\big|Df_c\big(\beta(c)\big)\big|^{-k} \leqslant \mathrm{diam}\big(P_{c,k}\big(-\beta(c)\big)\big) \leqslant \Delta_1\big|Df_c\big(\beta(c)\big)\big|^{-k}$$

and for each point y in $P_{c,k}(-\beta(c))$ or in $P_{c,k}(\beta(c))$ we have

$$\Delta_1^{-1}\big|Df_c\big(\beta(c)\big)\big|^{k} \leqslant \big|Df_c^k(y)\big| \leqslant \Delta_1\big|Df_c\big(\beta(c)\big)\big|^{k}.$$

Proof. Since $P_{c,1}(\beta(c))$ depends continuously with c on $\mathcal{P}_0(-2)$ (*cf.*, Lemma 2.5) and since $\mathcal{P}_0(-2)$ contains the closure of $\mathcal{P}_2(-2)$ (part 1 of Lemma 3.3), we have

$$\Xi_1 := \sup_{c\in\mathcal{P}_2(-2)} \sup_{z\in P_{c,1}(\beta(c))} \big|Df_c(z)\big| < +\infty,$$
$$\Xi_2 := \inf_{c\in\mathcal{P}_2(-2)} \inf_{z\in P_{c,1}(\beta(c))} \big|Df_c(z)\big| > 0,$$
$$\Xi_3 := \sup_{c\in\mathcal{P}_2(-2)} \mathrm{diam}\big(P_{c,1}\big(\beta(c)\big)\big) < +\infty,$$

and

$$\Xi_4 := \inf_{c\in\mathcal{P}_2(-2)} \mathrm{diam}\big(P_{c,1}\big(\beta(c)\big)\big) > 0.$$

On the other hand, since for each c in $\mathcal{P}_0(-2)$ the set $P_{c,0}(\beta(c))$ contains the closure of $P_{c,1}(\beta(c))$ (*cf.*, Section 3.1), we have

$$\Xi_5 := \inf_{c\in\mathcal{P}_2(-2)} \mathrm{mod}\big(P_{c,0}\big(\beta(c)\big) \setminus \mathrm{cl}\big(P_{c,1}\big(\beta(c)\big)\big)\big) > 0.$$

Let $\Delta > 1$ be the constant given by Koebe Distortion Theorem with $A = \Xi_5$.

Let c be a parameter in $\mathcal{P}_2(-2)$ and let $k \geqslant 2$ be an integer. Since f_c^{k-1} maps each of the sets $P_{c,k-1}(\beta(c))$ and $P_{c,k-1}(-\beta(c))$ biholomorphically to $P_{c,0}(\beta(c))$, the distortion of f_c^{k-1} on $P_{c,k}(\beta(c))$ is bounded by Δ. So for each y in $P_{c,k}(-\beta(c))$ or in $P_{c,k}(\beta(c))$ we have

$$\Delta^{-1}\big|Df_c\big(\beta(c)\big)\big|^{k-1} \leqslant \big|Df_c^{k-1}(y)\big| \leqslant \Delta\big|Df_c\big(\beta(c)\big)\big|^{k-1}.$$

This implies the first assertion of the lemma with $\Delta_1 = \Delta \max\{\Xi_1 \Xi_3, \Xi_2^{-1} \Xi_4^{-1}\}$ and second with $\Delta_1 = \Delta \Xi_1 \Xi_2^{-1}$. $\square$

Proof of Proposition 3.1. By the monotonicity of the kneading invariant, the set $\mathcal{K}_n$ is contained in $(-2, \hat{\alpha}_{n-1})$, see [29, Theorem 13.1]. Combined with Lemma 3.4 this implies that $\mathcal{K}_n$ is contained in $\mathcal{P}_n(-2)$. Since $\hat{\alpha}_1 = -3/4$, we also have $\mathcal{K}_n \subset (-2, -3/4)$. To prove that $\mathcal{K}_n$ is compact, just observe that from the definitions we have

$$\mathcal{K}_n = \left\{ c \in [-2, \hat{\alpha}_{n-1}] \;\middle|\; f_c^n(c) \in \Lambda_c \right\}.$$

For a given $\underline{x}$ in $\{0,1\}^{\mathbb{N}_0}$ the existence and uniqueness of c in $\mathcal{K}_n$ such that $\iota(c) = \underline{x}$ is a direct consequence of general results of Milnor and Thurston and of Yoccoz, see for example [29,9] and [17].

To prove the last statement of the proposition we show that $\operatorname{diam}(\mathcal{P}_n(-2)) \to 0$ as $n \to +\infty$. To do this, let $\Delta_1 > 1$ be given by Lemma 3.6, put

$$\Xi := \inf_{c \in \mathcal{P}_2(-2)} \left| Df_c\big(\beta(c)\big) \right| > 1,$$

and let $\tau : \mathcal{P}_0(-2) \to \mathbb{C}$ be the holomorphic function defined by $\tau(c) := c + \beta(c)$. A direct computation shows that $c = -2$ is the only zero of τ and that $\tau'(-2) \neq 0$. Since the closure of $\mathcal{P}_2(-2)$ is contained in $\mathcal{P}_0(-2)$ (part 1 of Lemma 3.3), there is a constant $C > 0$ such that for every c in $\mathcal{P}_2(-2)$ we have

$$\left| c - (-2) \right| \leqslant C \frac{|\tau(c)|}{|\tau'(-2)|}. \tag{3.2}$$

Let $n \geqslant 2$ be an integer and c a parameter in $\mathcal{P}_n(-2)$. By part 2 of Lemma 3.3 we have $c \in P_{c,n}(-\beta(c))$. So by Lemma 3.6 with $k = n$ and the definition of Ξ we have,

$$\left|\tau(c)\right| = \left| c - \big(-\beta(c)\big) \right| \leqslant \Delta_1 \left| Df_c\big(\beta(c)\big)\right|^{-n} \leqslant \Delta_1 \Xi^{-n}.$$

Combining this inequality with (3.2), we conclude that $\operatorname{diam} \mathcal{P}_n(-2) \to 0$ as $n \to +\infty$. This completes the proof of the proposition. $\square$

4. Reduced statement

The purpose of this section is to state a sufficient criterion for a quadratic map corresponding to a parameter in $\bigcup_{n=3}^{+\infty} \mathcal{K}_n$ to have a low-temperature phase transition (Proposition A). The rest of this section is devoted to prove the Main Theorem using this criterion. The proof of Proposition A occupies Sections 5, 6 and 7.

Recall that for a real parameter c,

$$\chi_{\mathrm{crit}}(c) = \liminf_{m \to +\infty} \frac{1}{m} \log \left| Df_c^m(c) \right|.$$

Proposition A. *There is $n_0 \geqslant 3$ and a constant $C_0 > 1$ such that for every integer $n \geqslant n_0$ and every parameter c in $\mathcal{K}_n$ the following property holds. Suppose that for every $t > 0$ sufficiently large the sum*

$$\sum_{k=0}^{+\infty} \exp\big((n+3k) t \chi_{\mathrm{crit}}(c)/2\big) \left| Df_c^{n+3k}(c) \right|^{-t/2}$$

is less than or equal to C_0^{-t} *and that for some* $t_0 \geqslant 3$ *the sum above with* $t = t_0$ *is finite and greater than or equal to* $C_0^{t_0}$. *Then there is* $t_* > t_0$ *such that* $f_c|_{I_c}$ *(resp.* $f_c|_{J_c}$*) has a low-temperature phase transition at* $t = t_*$. *If in addition the sum*

$$\sum_{k=0}^{+\infty} k \cdot \exp\bigl((n+3k)t_* \chi_{\mathrm{crit}}(c)/2\bigr)\bigl|Df_c^{n+3k}(c)\bigr|^{-t_*/2}$$

is finite, then $P_c^{\mathbb{R}}$ *(resp.* $P_c^{\mathbb{C}}$*) is not differentiable at* $t = t_*$ *and there is a unique equilibrium state of* $f_c|_{I_c}$ *(resp.* $f_c|_{J_c}$*)++ for the potential* $-t_* \log|Df_c|$. *Furthermore, this measure is ergodic, mixing, and its measure-theoretic entropy is strictly positive.*

After making a uniform distortion bound in Section 4.1, we give the proof of the Main Theorem in Section 4.2.

4.1. Uniform distortion bound

In this subsection we prove a uniform distortion bound, stated as Lemma 4.3 below. We start with some preparatory lemmas. Recall that for a parameter c in $\mathcal{P}_2(-2)$ the external rays $R_c(7/24)$ and $R_c(17/24)$ land at the point $\gamma(c)$ in $P_{c,1}(0)$, see Section 3.3.

Lemma 4.1. *For every parameter* c *in* $\mathcal{P}_2(-2)$ *the following properties hold.*

1. *The open disk* $\widehat{U}_c$ *containing* $-\beta(c)$ *that is bounded by the equipotential* 2 *and by*
$$R_c(7/24) \cup \bigl\{\gamma(c)\bigr\} \cup R_c(17/24), \tag{4.1}$$
contains the closure of $P_{c,0}(-\beta(c))$.
2. *The open set* $\widehat{W}_c := f_c^{-1}(\widehat{U}_c)$ *contains the closure of* $P_{c,1}(0)$ *and it depends continuously with* c *on* $\mathcal{P}_3(-2)$.

Proof. 1. Since the puzzle piece $P_{c,0}(-\beta(c))$ is bounded by the equipotential 1 and by $R_c(1/3) \cup \{\alpha(c)\} \cup R_c(2/3)$ (Theorem 1 and Section 3.1) and since $7/24 < 1/3 < 2/3 < 17/24$, we deduce that $\widehat{U}_c$ contains the closure of $P_{c,0}(-\beta(c))$.

2. That $\widehat{W}_c$ contains the closure of $P_{c,1}(0) = f_c^{-1}(P_{c,0}(-\beta(c)))$ is a direct consequence of part 1. To show that $\widehat{W}_c$ depends continuously with c on $\mathcal{P}_3(-2)$, it is enough to show that $\partial \widehat{W}_c$ depends continuously with c on $\mathcal{P}_3(-2)$. This last assertion follows directly from Lemma 2.5. □

For the following lemma, see Fig. 3.

Lemma 4.2. *Let* $n \geqslant 3$ *be an integer and let* c *be a parameter in* $\mathcal{K}_n$. *Then for every integer* $j \geqslant 1$ *the point* $f_c^j(0)$ *is contained in*

$$P_{c,1}\bigl(-\beta(c)\bigr) \cup \Lambda_c \cup P_{c,1}\bigl(\beta(c)\bigr). \tag{4.2}$$

Moreover, this set is disjoint from $\widehat{U}_c \setminus P_{c,0}(-\beta(c))$ *and from* $P_{c,4}(0)$.

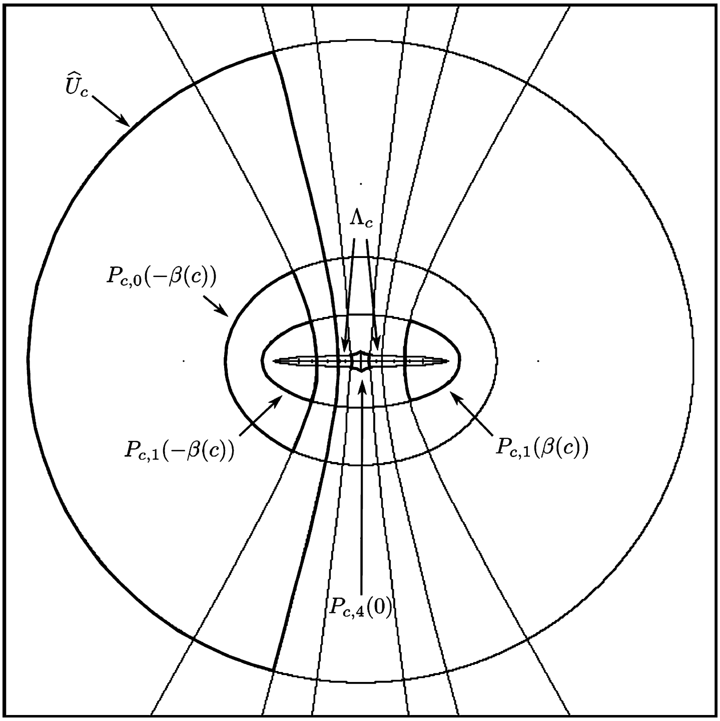

Fig. 3. The sets $\widehat{U}_c$ and Λ_c, and the puzzle pieces $P_{c,0}(-\beta(c))$, $P_{c,1}(-\beta(c))$, $P_{c,1}(\beta(c))$, and $P_{c,4}(0)$.

Proof. By part 2 of Lemma 3.3, the critical value c of f_c is in $P_{c,n}(-\beta(c))$. Thus for each j in $\{1, \dots, n-1\}$ the point $f_c^j(c)$ belongs to $P_{c,n-j}(\beta(c)) \subset P_{c,1}(\beta(c))$. Using the hypothesis that c is in $\mathcal{K}_n$, we conclude that for every integer $k \geqslant 0$ the point $f_c^{n+3k}(c)$ belongs to Λ_c, that $f_c^{n+3k+1}(c)$ belongs to $V_{c,2} \subset P_{c,1}(-\beta(c))$ and that $f_c^{n+3k+2}(c)$ belongs to $\widetilde{V}_{c,1} \subset P_{c,1}(-\beta(c))$. This proves the first part of the lemma.

To prove the last assertion of the lemma, note that $P_{c,4}(0)$ is disjoint from Λ_c (Section 3.3). On the other hand, $P_{c,4}(0)$ is contained in $P_{c,1}(0)$ and it is therefore disjoint from $P_{c,1}(-\beta(c)) \cup P_{c,1}(\beta(c))$. It remains to prove that (4.2) is disjoint from $\widehat{U}_c \setminus P_{c,0}(-\beta(c))$. This last set is disjoint from $P_{c,1}(-\beta(c))$. To complete the proof, observe that the set (4.1) separates $\mathbb{C}$ into 2 connected components: One containing $-\beta(c)$, denoted by H, and another one containing $\beta(c)$, denoted by $\widetilde{H}$. Clearly $\widehat{U}_c$ is contained in H. On the other hand, part 2 of Lemma 3.2 implies that $P_{c,1}(\beta(c))$ is contained in $\widetilde{H}$. Finally, note that Λ_c is contained in $P_{c,3}(0)$ (Section 3.3) and that this last set is contained in $\widetilde{H}$, see the beginning of Section 3.3. This shows that Λ_c and $P_{c,1}(\beta(c))$ are both disjoint from $\widehat{U}_c$, and hence from $\widehat{U}_c \setminus P_{c,0}(-\beta(c))$. This completes the proof of the lemma. □

Lemma 4.3 *(Uniform distortion bound). There is $\Delta_2 > 1$ such that for each integer $n \geqslant 4$ and each parameter c in $\mathcal{K}_n$ the following properties hold: For each integer $m \geqslant 1$ and each connected component W of $f_c^{-m}(P_{c,1}(0))$ on which f_c^m is univalent, f_c^m maps a neighborhood of W biholomorphically to $\widehat{W}_c$ and the distortion of this map on W is bounded by Δ_2.*

Proof. Recall that for each parameter c in $\mathcal{P}_3(-2)$ the set $\widehat{W}_c$ contains the closure of $P_{c,1}(0)$ and that these sets depend continuously with c on $\mathcal{P}_3(-2)$ (*cf.*, part 2 of Lemma 4.1 and Lemma 2.5). As the closure of $\mathcal{P}_4(-2)$ is contained in $\mathcal{P}_3(-2)$ (part 1 of Lemma 3.3), we have

$$A := \inf_{c \in \mathcal{P}_4(-2)} \operatorname{mod}\bigl(\widehat{W}_c \setminus \operatorname{cl}\bigl(P_{c,1}(0)\bigr)\bigr) > 0.$$

Then the desired assertion follows from Lemma 4.2 and Koebe Distortion Theorem for this choice of the constant A. $\square$

4.2. *Proof of Main Theorem assuming Proposition A*

The following elementary lemma describes the itinerary of the postcritical orbit, for the parameter c for which we show there is a low-temperature phase transition.

Lemma 4.4. *Let $N \geqslant 1$ and $\ell_0 \geqslant 1$ be given integers satisfying $2\ell_0 \geqslant N$. Define $(a_k)_{k=0}^{+\infty}$ as the sequence in $\{0,1\}^{\mathbb{N}_0}$ such that for k in $\mathbb{N}_0$ we have $a_k = 0$ if and only if there is an integer $\ell \geqslant \ell_0$ such that*

$$\ell^2 \leqslant k \leqslant \ell^2 + N - 1.$$

Moreover, let $N : \mathbb{N} \to \mathbb{N}_0$ be the function defined for $k \geqslant 1$ by

$$N(k) := \#\bigl\{j \in \{0, \dots, k-1\} \bigm| a_j = 0\bigr\}$$

and let $B : \mathbb{N} \to \mathbb{N}$ be the function defined by $B(1) = 1$ and for $k \geqslant 2$ by

$$B(k) := 1 + \#\bigl\{j \in \{0, \dots, k-2\} \bigm| a_j \neq a_{j+1}\bigr\}.$$

Then for every k in $\{1, \dots, \ell_0^2\}$ we have $N(k) = 0$ and $B(k) = 1$ and for every $k \geqslant \ell_0^2 + 1$ we have

$$B(k) \leqslant 2(\sqrt{k} - \ell_0) + 3 \quad \textit{and} \quad N \cdot (\sqrt{k} - \ell_0) \leqslant N(k) \leqslant N\sqrt{k}. \tag{4.3}$$

Proof. The assertions for k in $\{1, \dots, \ell_0^2\}$ and the upper bound of $B(k)$ are straight forward consequences of the definitions. Let $k \geqslant \ell_0^2 + 1$ be a given integer. If there is an integer $\ell \geqslant \ell_0$ such that $\ell^2 \leqslant k - 1 \leqslant \ell^2 + N - 1$, then

$$N(k) = N \cdot (\ell - \ell_0) + k - \ell^2$$

and therefore

$$\begin{aligned} N \cdot (\sqrt{k} - \ell_0) + (\sqrt{k} - \ell)(\sqrt{k} + \ell - N) = N(k) & \\ \leqslant N\sqrt{k} + N \cdot (\ell + 1 - \sqrt{k} - \ell_0). & \end{aligned}$$

Using $N \leqslant 2\ell$ and $\ell + 1 - \sqrt{k} \leqslant 1$, we obtain the estimates for $N(k)$ in (4.3). Suppose there is an integer $\ell \geqslant \ell_0$ such that

$$\ell^2 + N \leqslant k - 1 \leqslant (\ell + 1)^2 - 1.$$

Then $N(k) = N \cdot (\ell - \ell_0 + 1)$ and we also get the bounds for $N(k)$ in (4.3). $\square$

Proof of the Main Theorem. Let $n_0 \geqslant 3$ and $C_0 > 1$ be given by Proposition A and let $\Delta_1 > 1$ and $\Delta_2 > 1$ be given by Lemmas 3.6 and 4.3, respectively.

For a given parameter c in $\mathcal{P}_3(-2)$ denote by $p(c)$ the unique fixed point of $g_c = f_c^3|_{Y_c \cup \widetilde{Y}_c}$ in Y_c and by $\tilde{p}(c)$ the unique fixed point of g_c in $\widetilde{Y}_c$, see Section 3.3. Each of the functions

$$p : \mathcal{P}_3(-2) \to \mathbb{C} \quad \text{and} \quad \tilde{p} : \mathcal{P}_3(-2) \to \mathbb{C}$$

so defined is holomorphic and real. By Lemma A.1 in Appendix A there is $\delta > 0$ such that for each parameter c in the interval $(-2, -2 + \delta)$ we have

$$\eta_c := \frac{|Dg_c(p(c))|}{|Dg_c(\tilde{p}(c))|} > 1.$$

Since for $c = -2$ we have

$$\left|Dg_{-2}\big(\tilde{p}(-2)\big)\right|^{1/3} = 2 \quad \text{and} \quad \left|Df_{-2}\big(\beta(-2)\big)\right| = 4,$$

taking $\delta > 0$ smaller if necessary we assume that for each c in $(-2, -2 + \delta)$ we have

$$2/3 > \left|Dg_c\big(\tilde{p}(c)\big)\right|^{1/3} / \left|f_c\big(\beta(c)\big)\right| > 1/3. \tag{4.4}$$

By Proposition 3.1 there is $n_1 \geqslant 3$ such that for each integer $n \geqslant n_1$ the set $\mathcal{K}_n$ is contained in $(-2, -2 + \delta)$.

Fix a sufficiently large integer $n \geqslant \max\{n_0, n_1\}$ such that

$$\Delta_1^{1/2} \Delta_2^{3/2} (2/3)^{n/2} < C_0^{-1}/2. \tag{4.5}$$

Since $\mathcal{K}_n$ is compact, we have

$$\eta := \inf\{\eta_c \mid c \in \mathcal{K}_n\} > 1.$$

Let $N \geqslant 1$ be sufficiently large so that $\Delta_2^2 \eta^{-N} < 1$ and let $\ell_0 \geqslant 1$ be a sufficiently large integer so that

$$2\ell_0 \geqslant N \quad \text{and} \quad \ell_0^2 > C_0^3 \big(\Delta_1 \Delta_2 3^n\big)^{1/2}.$$

By Proposition 3.1 there is a unique parameter c_0 in $\mathcal{K}_n$ such that $\iota(c_0)$ is given by the sequence $(a_k)_{k=0}^{+\infty}$ defined in Lemma 4.4 for these choices of N and ℓ_0. To prove the Main Theorem we just need to show that the hypotheses of Proposition A are satisfied for this choice of n and for $c = c_0$ and $t_0 = 3$.

Note for each integer $k \geqslant 1$ the number $N(k)$ is equal to the number of 0's in the sequence $(a_j)_{j=0}^{k-1}$ and that $B(k)$ is the number of blocks of 0's or 1's in this sequence. Let k be an integer satisfying $k \geqslant \ell_0^2 + 1$. Applying Lemma 4.3 to each block of 0's or 1's in $(a_j)_{j=0}^{k-1}$, we obtain by (4.3) and by the definition of η,

$$\begin{aligned}
\Delta_2^{2(\sqrt{k} - \ell_0) + 3} \left(\frac{|Dg_{c_0}(p(c_0))|}{|Dg_{c_0}(\tilde{p}(c_0))|}\right)^{N\sqrt{k}} &\geqslant \frac{|Dg_{c_0}^k(f_{c_0}^n(c_0))|}{|Dg_{c_0}(\tilde{p}(c_0))|^k} \\
&\geqslant \Delta_2^{-2(\sqrt{k} - \ell_0) - 3} \left(\frac{|Dg_{c_0}(p(c_0))|}{|Dg_{c_0}(\tilde{p}(c_0))|}\right)^{N \cdot (\sqrt{k} - \ell_0)} \\
&\geqslant \Delta_2^{-3} \big(\Delta_2^2 \eta^{-N}\big)^{-(\sqrt{k} - \ell_0)}.
\end{aligned} \tag{4.6}$$

This implies that

$$\begin{aligned}\chi_{\mathrm{crit}}(c_0) &= \lim_{m\to+\infty}\frac{1}{m}\log\left|Df_{c_0}^m(c_0)\right| \\ &= \frac{1}{3}\lim_{k\to+\infty}\frac{1}{k}\log\left|Dg_{c_0}^k\left(f_{c_0}^n(c_0)\right)\right| \\ &= \log\left|Dg_c\left(\tilde{p}(c_0)\right)\right|^{1/3}, \end{aligned} \tag{4.7}$$

and, by Lemma 3.6 with $k=n$ and $y=c_0$ and by (4.4) and (4.5), that for each integer $k \geqslant \ell_0^2+1$ we have

$$\exp\left((n+3k)\chi_{\mathrm{crit}}(c_0)/2\right)\left|Df_{c_0}^{n+3k}(c_0)\right|^{-1/2} \leqslant \left(C_0^{-1}/2\right)\left(\Delta_2^2\eta^{-N}\right)^{(\sqrt{k}-\ell_0)/2}. \tag{4.8}$$

This implies that for every $t>0$ the sum

$$\sum_{k=0}^{+\infty} k\cdot\exp\left(t(n+3k)\chi_{\mathrm{crit}}(c_0)/2\right)\left|Df_{c_0}^{n+3k}(c_0)\right|^{-t/2}$$

is finite and hence that the last hypothesis of Proposition A is automatically satisfied when the other ones are.

To prove that the rest of the hypotheses of Proposition A are satisfied, observe that by (4.7), by Lemma 4.3, by Lemma 3.6 with $k=n$ and $y=c_0$ and by (4.4) and (4.5), for each integer k in $\{0,1,\dots,\ell_0^2\}$ we have

$$\begin{aligned}\Delta_1^{-1/2}\Delta_2^{-1/2}(1/3)^{n/2} &\leqslant \exp\left((n+3k)\chi_{\mathrm{crit}}(c_0)/2\right)\left|Df_{c_0}^{n+3k}(c_0)\right|^{-1/2} \\ &\leqslant \Delta_1^{1/2}\Delta_2^{1/2}(2/3)^{n/2} \\ &< C_0^{-1}/2. \end{aligned} \tag{4.9}$$

So, if for each $t>0$ we put

$$S(t) := \sum_{k=\ell_0^2+1}^{+\infty}\left(\Delta_2^2\eta^{-N}\right)^{t(\sqrt{k}-\ell_0)/2},$$

then by (4.8) we have

$$\sum_{k=0}^{+\infty}\exp\left(t(n+3k)\chi_{\mathrm{crit}}(c_0)/2\right)\left|Df_{c_0}^{n+3k}(c_0)\right|^{-t/2} < \frac{C_0^{-t}}{2^t}\left(\ell_0^2+1+S(t)\right).$$

By our choice of N we have $\Delta_2^2\eta^{-N}<1$ and hence $S(t)\to 0$ as $t\to+\infty$. Together with the fact that $C_0>1$, this proves that the sum above converges to 0 as $t\to+\infty$. Finally, note that by (4.9) and our choice of ℓ_0, the sum above with $t=3$ is greater than C_0^3. This completes the proof that the hypotheses of Proposition A are satisfied with $c=c_0$ and $t_0=3$ and thus completes the proof of the Main Theorem. □

5. Expansion away from the critical point

For an integer $n\geqslant 4$ and a parameter c in $\mathcal{P}_n(-2)$ put $V_c := P_{c,n+1}(0)$ and denote by D_c' the set of all those points z in $\mathbb{C}\setminus V_c$ for which there is an integer $m\geqslant 1$ such that $f^m(z)$ is in V_c;

for such z denote by $m_c(z)$ the least integer m with this property and call it the *first landing time of z to V_c*. The *first landing map to V_c* is the map $L_c : D'_c \to V_c$ defined by $L_c(z) = f_c^{m_c(z)}(z)$.

The purpose of this section is to prove the following proposition.

Proposition B. *There is a constant $C_1 > 1$ such that for each $\varepsilon > 0$ there is $n_2 \geqslant 4$ such that the following property holds: For each integer $n \geqslant n_2$, each parameter c in $\mathcal{K}_n$, and each z in $L_c^{-1}(V_c)$ we have*

$$\left|DL_c(z)\right| \geqslant C_1^{-1} 2^{m_c(z)(1-\varepsilon)}.$$

To prove this proposition we first show that the restriction of L_c to each connected component of its domain admits a univalent extension onto $P_{c,1}(0)$ (Lemma 5.1). The proof of Proposition B is given in Section 5.2, after some derivative estimates stated as Lemmas 5.3 and 5.4.

5.1. Univalent pull-back property

Note that the domain D'_c of L_c is a disjoint union of puzzle pieces, so each connected component of D'_c is a puzzle piece. Furthermore, for each connected component W of D'_c the first landing time to V_c of all points in W is the same; denote the common value by $m_c(W)$. So L_c maps W biholomorphically to V_c.

Lemma 5.1 *(Univalent pull-back property). For every integer $\tilde{n} \geqslant 0$ and every parameter c in $\mathcal{P}_{\tilde{n}}(-2)$, the following property holds. Let $m \geqslant 1$ be an integer and z a point in $f_c^{-m}(P_{c,1}(0))$ such that for each j in $\{0, \ldots, m-1\}$ we have $f_c^j(z) \notin P_{c,\tilde{n}+1}(0)$. Then the puzzle piece P of depth $m+1$ containing z is such that for every j in $\{0, \ldots, m-1\}$ the set $f_c^j(P)$ is disjoint from $P_{c,\tilde{n}+1}(0)$ and f_c^m maps P biholomorphically to $P_{c,1}(0)$. If in addition c is real, then P intersects the real line.*

The proof of this lemma is below, after the following one.

Lemma 5.2. *Let c be a parameter in $\mathcal{P}_0(-2)$, let $\ell \geqslant 1$ be an integer, and let z be a point in $f_c^{-\ell}(P_{c,1}(0))$ such that for each j in $\{0, \ldots, \ell-1\}$ the point $f_c^j(z)$ is not in $P_{c,1}(0)$. Then z is in $V_{c,\ell}$ or in $\widetilde{V}_{c,\ell}$.*

Proof. We proceed by induction in ℓ. The case $\ell = 1$ follows from

$$f_c^{-1}\big(P_{c,1}(0)\big) = \phi_c\big(P_{c,1}(0)\big) \cup \tilde{\phi}_c\big(P_{c,1}(0)\big) = V_{c,1} \cup \widetilde{V}_{c,1}.$$

Let $\ell \geqslant 2$ be an integer and suppose the desired assertion holds with ℓ replaced by $\ell - 1$. If z is as in the lemma, then z is not in $P_{c,1}(0)$ and $G_c(z) \leqslant 1/2^\ell \leqslant 1/2$. Therefore z is in either $P_{c,1}(-\beta(c))$ or $P_{c,1}(\beta(c))$; in both cases $f_c(z)$ is in $P_{c,0}(\beta(c))$. Applying the induction hypothesis to $f_c(z)$ we conclude that $f_c(z)$ is in $\widetilde{V}_{c,\ell-1}$ and therefore that z is in

$$f_c^{-1}(\widetilde{V}_{c,\ell-1}) = V_{c,\ell} \cup \widetilde{V}_{c,\ell}.$$

This completes the proof of the induction step and of the lemma. □

Proof of Lemma 5.1. We proceed by induction in m. Since $f_c^{-1}(P_{c,1}(0)) = V_{c,1} \cup \widetilde{V}_{c,1}$, the desired assertions clearly hold for $m = 1$. Given an integer $m \geqslant 2$, suppose by induction that the desired assertions hold for every integer less than or equal to $m-1$. Given z as in the statement of the lemma, let P be the puzzle piece of f_c of depth $m+1$ containing z, so that $f_c^m(P) = P_{c,1}(0)$.

First, we prove that f_c^m maps P biholomorphically to $P_{c,1}(0)$. Let $\ell \geqslant 1$ be the least integer such that $f_c^\ell(P)$ is contained in $P_{c,1}(0)$; we have $\ell \leqslant m$. If P is not contained in $P_{c,1}(0)$, then it is contained in $V_{c,\ell}$ or $\widetilde{V}_{c,\ell}$ (Lemma 5.2); in both cases P is contained in a puzzle piece of depth $\ell+1$ that is mapped biholomorphically to $P_{c,1}(0)$ by f_c^ℓ. If P is contained in $P_{c,1}(0)$, then $\ell \geqslant 2$, $f_c(P)$ is contained in $V_{c,\ell-1}$ (Lemma 5.2) and hence in $P_{c,\ell-1}(-\beta(c))$ (part 1 of Lemma 3.2). Since our hypotheses imply that $f_c(P)$ is not contained in $P_{c,\tilde{n}}(c)$ and since this last puzzle piece is equal to $P_{c,\tilde{n}}(-\beta(c))$ (part 2 of Lemma 3.3), we conclude that $P_{c,\tilde{n}}(-\beta(c))$ is strictly contained in $P_{c,\ell-1}(-\beta(c))$ and therefore that $\ell - 1 < \tilde{n}$. So the puzzle piece of f_c of depth $\ell+1$ containing P is mapped biholomorphically to $V_{c,\ell-1}$ by f_c; this proves that in all the cases f_c^ℓ maps the puzzle piece of depth $\ell+1$ containing P biholomorphically to $P_{c,1}(0)$ and shows the inductive step in the case where $\ell = m$. If $\ell \leqslant m-1$, then by the induction hypothesis applied to $m-\ell$ instead of m and with $f_c^\ell(z)$ instead of z, we conclude that $f_c^{m-\ell}$ maps $f_c^\ell(P)$ biholomorphically to $P_{c,1}(0)$. This completes the proof that f_c^m maps P biholomorphically to $P_{c,1}(0)$.

Now we prove the other assertions of the lemma. For each j in $\{0,\dots,m-1\}$ we have $f_c^j(z) \notin P_{c,\tilde{n}+1}(0)$. Let P' be the puzzle piece of depth m containing $z' := f_c(z)$. By our induction hypothesis we just need to prove that P is disjoint from $P_{c,\tilde{n}+1}(0)$, and if c is real, that P intersects $\mathbb{R}$. Suppose z is not in $P_{c,1}(0)$. Then by Lemma 5.2 there is an integer $\ell \geqslant 1$ such that z belongs to $V_{c,\ell}$ or $\widetilde{V}_{c,\ell}$. Then $m \geqslant \ell$ and P is contained in one of these sets; it follows that P is disjoint from $P_{c,1}(0)$ and hence from $P_{c,\tilde{n}+1}(0)$. Suppose c is real. Then the maps ϕ_c and $\tilde{\phi}_c$ are both real and by our induction hypothesis P' intersects $\mathbb{R}$. Since P is equal to either $\phi_c(P')$ or $\tilde{\phi}_c(P')$, it follows that P also intersects $\mathbb{R}$. It remains to consider the case where z belongs to $P_{c,1}(0)$. Since by hypothesis z is not in $P_{c,\tilde{n}+1}(0)$, the point z' is not in $P_{c,\tilde{n}}(-\beta(c))$. So there is an integer $\ell \leqslant m-1$ in $\{1,\dots,\tilde{n}-1\}$ such that z' is in $V_{c,\ell}$ (Lemma 5.2). It follows that P' is contained in $V_{c,\ell}$ and that it is therefore disjoint from $P_{c,\tilde{n}}(-\beta(c))$; this implies that P is disjoint from $P_{c,\tilde{n}+1}(0)$. If c is real, then by the induction hypothesis P' intersects $\mathbb{R}$. Since P' is contained in $V_{c,\ell}$ and $\ell \leqslant \tilde{n}-1$, it follows that $P' \cap \mathbb{R}$ is contained in $f_c(\mathbb{R})$. This implies that P intersects $\mathbb{R}$ and completes the proof of the induction step and of the lemma. □

5.2. *Derivatives estimates*

Lemma 5.3. *There is a constant $C_2 > 1$ such that for every $\varepsilon > 0$ and every integer $m_1 \geqslant 1$ there is $n_3 \geqslant 3$ such that the following property holds for each integer $n \geqslant n_3$ and each parameter c in $\mathcal{K}_n$: For every integer $m \geqslant 1$ and every point z in $f_c^{-m}(P_{c,1}(0))$ such that for each j in $\{0,\dots,m-1\}$ we have $f_c^j(z) \notin P_{c,m_1}(0)$, we have*

$$\left|Df_c^m(z)\right| \geqslant C_2^{-1} 2^{m(1-\varepsilon)}.$$

If in addition z is in $P_{c,1}(0)$, then

$$\left|Df_c^m(z)\right| \leqslant C_2 2^{m(1+\varepsilon)}.$$

Proof. Let $\Delta_2 > 1$ be the constant given by Lemma 4.3. Given a parameter c in $[-2, 1/4)$ consider the smooth homeomorphism

$$h_c : [0,1] \to \bigl[-\beta(c), \beta(c)\bigr],$$
$$\theta \mapsto h_c(\theta) := \beta(c)\cos(\pi\theta);$$

it depends smoothly on c and when $c = -2$ we have

$$\inf_{\substack{x \in [-\beta(-2), \beta(-2)], \\ y \in [\alpha(-2), \tilde{\alpha}(-2)]}} \bigl|Dh_{-2}\bigl(h_{-2}^{-1}(y)\bigr)\bigr| / \bigl|Dh_{-2}\bigl(h_{-2}^{-1}(x)\bigr)\bigr| > 0.$$

So there is $\delta_0 \in (0, 9/4)$ such that

$$\kappa := \inf_{c \in [-2, -2+\delta_0]} \inf_{\substack{x \in [-\beta(c), \beta(c)], \\ y \in [\alpha(c), \tilde{\alpha}(c)]}} \bigl|Dh_c\bigl(h_c^{-1}(y)\bigr)\bigr| / \bigl|Dh_c\bigl(h_c^{-1}(x)\bigr)\bigr| > 0$$

and

$$\hat{\kappa} := \inf_{c \in [-2, -2+\delta_0]} \inf_{\substack{x \in [\alpha(c), \tilde{\alpha}(c)], \\ y \in [\alpha(c), \tilde{\alpha}(c)]}} \bigl|Dh_c\bigl(h_c^{-1}(y)\bigr)\bigr| / \bigl|Dh_c\bigl(h_c^{-1}(x)\bigr)\bigr| < +\infty.$$

Let $\varepsilon > 0$ and let $m_1 \geqslant 1$ be given. Taking m_1 larger if necessary, we assume $m_1 \geqslant 4$. Since $P_{c,m_1}(0)$ depends continuously with c on $\mathcal{P}_{m_1-1}(-2)$ (*cf.*, Lemma 2.5) and since this last set contains the closure of $\mathcal{P}_{m_1}(-2)$ (part 1 of Lemma 3.3), there is $\tau > 0$ such that for each parameter c in $\mathcal{P}_{m_1}(-2) \cap [-2, 1/4)$ we have

$$[1/2 - \tau, 1/2 + \tau] \subset h_c^{-1}\bigl(P_{c,m_1}(0) \cap \bigl[-\beta(c), \beta(c)\bigr]\bigr).$$

For each parameter c in $[-2, 1/4)$ let $T_c : [0,1] \to [0,1]$ be the map defined by $T_c = h_c^{-1} \circ f_c \circ h_c$. When $c = -2$ the map T_{-2} is the tent map given by $T_{-2}(\theta) = 2\theta$ on $[0, 1/2]$ and $T_{-2}(\theta) = 2 - 2\theta$ on $[1/2, 1]$. When c is not equal to -2, the map T_c is smooth on $[0,1]$. A direct computation shows that there is δ_1 in $(0, \delta_0)$ such that for each parameter c in $[-2, -2+\delta_1]$ and each θ in $[0,1]$ satisfying $|\theta - 1/2| \geqslant \tau$, we have

$$2^{1-\varepsilon} \leqslant \bigl|DT_c(\theta)\bigr| \leqslant 2^{1+\varepsilon}.$$

Let n_1 be given by Proposition 3.1 with $\delta = \delta_1$.

Fix an integer $n \geqslant \max\{n_1, m_1\}$ and a parameter c in $\mathcal{K}_n$. By Proposition 3.1 we have $\mathcal{K}_n \subset (-2, -2+\delta_1)$. Let $m \geqslant 1$ be an integer, z a point in $f_c^{-m}(P_{c,1}(0))$ such that for each j in $\{0, \ldots, m-1\}$ we have $f_c^j(z) \notin P_{c,m_1}(0)$ and let P be the puzzle piece of f_c of depth $m+1$ that contains z. By Lemma 5.1 with n replaced by $m_1 - 1$ there is a real point x in P and for every j in $\{0, \ldots, m-1\}$ the point $f_c^j(x)$ of $f_c^j(P)$ is not in $P_{c,m_1}(0)$; by our choice of τ it follows that $h_c^{-1}(f_c^j(x))$ is not in $[1/2 - \tau, 1/2 + \tau]$. On the other hand, f_c^m maps P biholomorphically to $P_{c,1}(0)$ and by Lemma 4.3 the distortion of f_c^m on P is bounded by Δ_2. Since x is in $[-\beta(c), \beta(c)]$ and

$$f_c^m(x) \in f_c^m(P) \cap \mathbb{R} = P_{c,1}(0) \cap \mathbb{R} = \bigl(\alpha(c), \tilde{\alpha}(c)\bigr),$$

by the considerations above we have by the definition of κ,

$$\begin{aligned}
\bigl|Df_c^m(z)\bigr| &\geqslant \Delta_2^{-1}\bigl|Df_c^m(x)\bigr| \\
&= \Delta_2^{-1}\bigl|Dh_c\bigl(h_c^{-1}\bigl(f_c^m(x)\bigr)\bigr)\bigr| \cdot \bigl|DT_c^m\bigl(h_c^{-1}(x)\bigr)\bigr| / \bigl|Dh_c\bigl(h_c^{-1}(x)\bigr)\bigr| \\
&\geqslant \Delta_2^{-1}\kappa 2^{m(1-\varepsilon)}.
\end{aligned}$$

If in addition z is in $P_{c,1}(0)$, then x belongs to $(\alpha(c), \tilde{\alpha}(c))$ and by the considerations above we have by the definition of $\hat{\kappa}$,

$$\begin{aligned}\left|Df_c^m(z)\right| &\leqslant \Delta_2\left|Df_c^m(x)\right| \\ &= \Delta_2\left|Dh_c\left(h_c^{-1}\left(f_c^m(x)\right)\right)\right| \cdot \left|DT_c^m\left(h_c^{-1}(x)\right)\right| / \left|Dh_c\left(h_c^{-1}(x)\right)\right| \\ &\leqslant \Delta_2 \hat{\kappa}^{-1} 2^{m(1+\varepsilon)}.\end{aligned}$$

This proves the lemma with $n_2 = \max\{n_1, m_1\}$ and $C_2 = \Delta_2^{-1} \max\{\kappa^{-1}, \hat{\kappa}\}$. $\square$

Lemma 5.4. *There is $C_3 > 1$ such that for each integer $n \geqslant 4$ and each parameter c in $\mathcal{K}_n$ the following properties hold for each integer $q \geqslant 1$.*

1. *For each open set W that is mapped biholomorphically to $P_{c,1}(0)$ by f_c^q and each x in W we have*

$$\left|Df_c(x)\right| \geqslant C_3^{-1}\left|Df_c^{q-1}\left(f_c(x)\right)\right|^{-1/2}.$$

2. *If $q - 1 \neq n$, then for each point x in $f_c^{-1}(V_{c,q-1})$ we have*

$$\left|Df_c^q(x)\right| \geqslant C_3^{-1}\left|Df_c\left(\beta(c)\right)\right|^{q/2}.$$

Proof. Let $\Delta_1 > 1$ and $\Delta_2 > 1$ be the constants given by Lemmas 3.6 and 4.3, respectively.

Since the sets $P_{c,1}(\beta(c))$ and $P_{c,1}(0)$ are disjoint and depend continuously with c on $\mathcal{P}_0(-2)$ (*cf.*, Section 2.5 and Lemma 2.5) and since $\mathcal{P}_0(-2)$ contains the closure of $\mathcal{P}_4(-2)$ (part 1 of Lemma 3.3), we have

$$\Xi_1 := \inf_{c \in \mathcal{P}_4(-2)} \operatorname{diam}\left(P_{c,1}(0)\right) > 0 \quad \text{and} \quad \Xi_2 := \sup_{c \in \mathcal{P}_4(-2)} \left|Df_c\left(\beta(c)\right)\right| < +\infty.$$

On the other hand, for each c in $\mathcal{P}_3(-2)$ the closure of $P_{c,1}(0)$ is contained in $\widehat{W}_c$ and $\widehat{W}_c$ depends continuously with c on $\mathcal{P}_3(-2)$ (part 2 Lemma 4.1); so

$$\Xi_3 := \inf_{c \in \mathcal{P}_4(-2)} \operatorname{mod}\left(\widehat{W}_c \setminus \operatorname{cl}\left(P_{c,1}(0)\right)\right) > 0.$$

Let $n \geqslant 4$ be a integer and c a parameter in $\mathcal{K}_n$.

1. Note that f_c^q maps a neighborhood $\widehat{W}$ of W biholomorphically to $\widehat{W}_c$ (Lemma 4.3). So if we put $\widehat{W}' := f_c(\widehat{W})$, then c is not in $\widehat{W}'$ and f_c^{q-1} maps $\widehat{W}'$ biholomorphically to $\widehat{W}_c$; in particular we have

$$\operatorname{mod}\left(\widehat{W}' \setminus \operatorname{cl}\left(f_c(W)\right)\right) = \operatorname{mod}\left(\widehat{W}_c \setminus \operatorname{cl}\left(P_{c,1}(0)\right)\right) \geqslant \Xi_3.$$

Thus there is a constant $A_1 > 0$ independent of n, c and q such that for every x in W, we have

$$\left|f_c(x) - c\right| \geqslant \operatorname{dist}\left(f_c(W), c\right) \geqslant \operatorname{dist}\left(f_c(W), \partial \widehat{W}'\right) \geqslant A_1 \operatorname{diam}\left(f_c(W)\right)$$

(*cf.*, [21, Teichmüller's module theorem, §II.1.3]). Thus, if we put $A_2 := 2(A_1 \Delta_2^{-1} \Xi_1)^{1/2}$, then by Lemma 4.3 with $m = q - 1$ and with W replaced by $f_c(W)$ we have

$$\left|Df_c(x)\right| \geqslant 2A_1^{1/2} \operatorname{diam}\left(f_c(W)\right)^{1/2} \geqslant A_2\left|Df_c^{q-1}\left(f_c(x)\right)\right|^{-1/2}.$$

This proves part 1 with constant $C_3 = A_2^{-1}$.

2. Since $f_c(x)$ is in $V_{c,q-1}$ and this last set is contained in $P_{c,q-1}(-\beta(c))$ (part 1 of Lemma 3.2), by Lemma 3.6 with $y = f_c(x)$ and $k = q - 1$, we have

$$\left|Df_c^{q-1}\left(f_c(x)\right)\right| \geqslant \Delta_1^{-1}\left|Df_c\left(\beta(c)\right)\right|^{q-1}. \tag{5.1}$$

Our assumption $q - 1 \neq n$ implies that $f_c(0) = c$ is not in $V_{c,q-1}$, so f_c^q maps the connected component W of $f_c^{-1}(V_{c,q-1})$ containing x biholomorphically to $P_{c,1}(0)$. So the desired assertion with C_3 replaced by $C_3(\Delta_1 \Xi_2)^{1/2}$ follows from (5.1) and from part 1. $\square$

Proof of Proposition B. Let C_2 and C_3 be the constants given by Lemmas 5.3 and 5.4, respectively. Let $m_1 \geqslant 2$ be sufficiently large so that

$$2^{(m_1-1)\varepsilon/2} \geqslant C_2 C_3$$

and let n_3 be given by Lemma 5.3 for this choice of m_1. Notice that for $c = -2$ we have $Df_{-2}(\beta(-2)) = 4$. So, in view of Proposition 3.1, we can take n_3 larger if necessary and assume that for each parameter c in $\mathcal{P}_{n_3}(-2)$ we have

$$\left|Df_c\left(\beta(c)\right)\right|^{1/2} \geqslant 2^{1-\varepsilon/2}.$$

We prove the desired assertion with $n_2 = n_3$ and $C_1 = C_2$. To do this, let $n \geqslant n_3$ be an integer, c a parameter in $\mathcal{K}_n$, and let z be a point in $L_c^{-1}(V_c)$. If for every j in $\{0, \dots, m_c(z) - 1\}$ we have $f_c^j(z) \notin P_{c,m_1}(0)$, then the desired assertion follows from Lemma 5.3 with $m = m_c(z)$. So we assume that there is ℓ in $\{0, \dots, m_c(z) - 1\}$ such that $f_c^\ell(z)$ belongs to $P_{c,m_1}(0)$. Let $k \geqslant 1$ be the number of all such integers, let $\ell_1 < \ell_2 < \cdots < \ell_k$ be the increasing sequence of all of these numbers, and put $\ell_{k+1} := m_c(z)$. Given s in $\{1, \dots, k\}$ let ℓ'_s be the least integer $\ell \geqslant \ell_s + 1$ such that $f_c^\ell(z)$ is in $P_{c,1}(0)$. Then $\ell'_s \leqslant \ell_{s+1}$, $\ell'_s - \ell_s \geqslant m_1 - 1$, and the point $f_c^{\ell_s+1}(z)$ belongs to $V_{c,\ell'_s-\ell_s-1}$ (Lemma 5.2). By our choice of z, the point $f_c^{\ell_s}(z)$ does not belong to $V_c = P_{c,n+1}(0) = f_c^{-1}(V_{c,n})$, so $\ell'_s - \ell_s - 1 \neq n$ and by part 2 of Lemma 5.4 with $q = \ell'_s - \ell_s$ and $x = f_c^{\ell_s}(z)$ and by our choice of n_3 and m_1 we have

$$\begin{aligned}\left|Df_c^{\ell'_s-\ell_s}\left(f_c^{\ell_s}(z)\right)\right| &\geqslant C_3^{-1}\left|Df_c\left(\beta(c)\right)\right|^{(\ell'_s-\ell_s)/2}\\ &\geqslant C_3^{-1}2^{(\ell'_s-\ell_s)(1-\varepsilon/2)}\\ &\geqslant C_2 2^{(\ell'_s-\ell_s)(1-\varepsilon)}.\end{aligned} \tag{5.2}$$

When $\ell'_s = \ell_{s+1}$ we obtain

$$\left|Df_c^{\ell_{s+1}-\ell_s}\left(f_c^{\ell_s}(z)\right)\right| \geqslant 2^{(\ell_{s+1}-\ell_s)(1-\varepsilon)}. \tag{5.3}$$

In the case where $\ell'_s \leqslant \ell_{s+1} - 1$, the point $f_c^{\ell'_s}(z)$ belongs to $P_{c,1}(0)$ but not to $P_{c,m_1}(0)$; so (5.2) together with Lemma 5.3 with $m = \ell_{s+1} - \ell'_s$ and with z replaced by $f_c^{\ell'_s}(z)$ implies, by our choice of n_3, that

$$\left|Df_c^{\ell_{s+1}-\ell_s}\left(f_c^{\ell_s}(z)\right)\right| \geqslant \left|Df_c^{\ell_{s+1}-\ell'_s}\left(f_c^{\ell'_s}(z)\right)\right| C_2 2^{(\ell'_s-\ell_s)(1-\varepsilon)} \geqslant 2^{(\ell_{s+1}-\ell_s)(1-\varepsilon)}.$$

So in all the cases we obtain (5.3) and therefore

$$\left|Df_c^{m_c(z)-\ell_1}\left(f_c^{\ell_1}(z)\right)\right| = \prod_{s=1}^{k}\left|Df_c^{\ell_{s+1}-\ell_s}\left(f_c^{\ell_s}(z)\right)\right| \geqslant 2^{(m_c(z)-\ell_1)(1-\varepsilon)}. \tag{5.4}$$

This proves the desired inequality in the case where $\ell_1 = 0$. If $\ell_1 \geqslant 1$, then by Lemma 5.3 with $m = \ell_1$ we have

$$\left|Df_c^{\ell_1}(z)\right| \geqslant C_2^{-1} 2^{\ell_1(1-\varepsilon)}.$$

Together with (5.4) this implies the desired inequality and completes the proof of the proposition. □

6. Induced map

In this section, for a parameter c in $\mathcal{P}_4(-2)$ we use the first return map F_c of f_c to V_c to study $P_c^{\mathbb{R}}$ and $P_c^{\mathbb{C}}$. After some basic considerations in Section 6.1, we show that $P_c^{\mathbb{R}}$ and $P_c^{\mathbb{C}}$ are related to a 2 variables pressure function of F_c through a Bowen type formula, see Proposition C in Section 6.2 and compare with [44] and [34]. We do this by analyzing the convergence properties of a suitable Poincaré series (Lemma 6.5). In the proof of Proposition C we use a lower bound for $P_c^{\mathbb{C}}$ (Proposition 6.2 in Section 6.3) that is used again in the next section.

6.1. Induced map

Let $n \geqslant 4$ be an integer and c a parameter in $\mathcal{K}_n$. Throughout the rest of this section put $\widehat{V}_c := P_{c,4}(0)$. Since the critical value c of f_c is in $P_{c,n}(-\beta(c))$ (part 2 of Lemma 3.3), the closure of $V_c = P_{c,n+1}(0) = f_c^{-1}(P_{c,n}(-\beta(c)))$ is contained in $\widehat{V}_c = f_c^{-1}(P_{c,3}(-\beta(c)))$ (*cf.*, part 1 of Lemma 3.2).

Let D_c be the set of all those points z in V_c for which there is an integer $m \geqslant 1$ such that $f_c^m(z)$ is in V_c. For z in D_c denote by $m_c(z)$ the least integer m with this property and call it the *first return time of z to V_c*. The *first return map to V_c* is defined by

$$\begin{aligned} F_c : D_c &\to V_c, \\ z &\mapsto F_c(z) := f_c^{m_c(z)}(z). \end{aligned}$$

It is easy to see that D_c is a disjoint union of puzzle pieces; so each connected component of D_c is a puzzle piece. Note furthermore that in each of these puzzle pieces W, the return time function m_c is constant; denote the common value of m_c on W by $m_c(W)$.

Lemma 6.1 *(Uniform bounded distortion). There is a constant $\Delta_3 > 1$ such that for each integer $n \geqslant 5$ and each parameter c in $\mathcal{K}_n$ the following property holds: For every connected component W of D_c the map $F_c|_W$ is univalent and its distortion is bounded by Δ_3. Furthermore, the inverse of $F_c|_W$ admits a univalent extension to $\widehat{V}_c$ taking images in V_c. In particular, F_c is uniformly expanding with respect to the hyperbolic metric on $\widehat{V}_c$.*

Proof. Recall that for each parameter c in $\mathcal{P}_4(-2)$ the critical value c of f_c is in $P_{c,4}(-\beta(c))$ (part 2 of Lemma 3.3), so set $P_{c,4}(0) = f_c^{-1}(P_{c,3}(-\beta(c)))$ contains the closure of $P_{c,5}(0) = f_c^{-1}(P_{c,4}(-\beta(c)))$ (*cf.*, part 1 of Lemma 3.2) and that these sets depend continuously with c on $\mathcal{P}_4(-2)$ (*cf.*, Lemma 2.5). Since $\mathcal{P}_4(-2)$ contains the closure of $\mathcal{P}_5(-2)$ (part 1 of Lemma 3.3) we have

$$A := \inf_{c \in \mathcal{P}_5(-2)} \operatorname{mod}\bigl(P_{c,4}(0) \setminus \operatorname{cl}\bigl(P_{c,5}(0)\bigr)\bigr) > 0.$$

Let Δ_3 be the constant Δ given by Koebe Distortion Theorem for this value of A.

Since $\widehat{V}_c$ is disjoint from the forward orbit of 0 (Lemma 4.2), for each connected component W of D_c the map $f_c^{m_c(W)}$ maps a neighborhood $\widehat{W}$ of W biholomorphically to $\widehat{V}_c$. By Koebe Distortion Theorem the distortion of $f_c^{m_c(W)}$ on W is bounded by Δ_3. Note that $\widehat{W}$ is a puzzle piece intersecting the puzzle piece V_c. Thus, these puzzle pieces are either equal or one is strictly contained in the other. Since $\widehat{W}$ does not contain 0, it follows that $\widehat{W}$ is strictly contained in V_c. Thus $(f_c^{m_c(W)}|_{\widehat{W}})^{-1}$ is an extension of $F_c|_W^{-1}$ to $\widehat{V}_c$ taking images in V_c. $\square$

6.2. *Pressure function of the induced map*

Let $n \geqslant 4$ be an integer and let c be a parameter in $\mathcal{K}_n$. In this subsection we state a Bowen type formula relating $P_c^{\mathbb{R}}$ (resp. $P_c^{\mathbb{C}}$) to a certain 2 variables pressure of F_c (Proposition C) that is shown in Section 6.4.

Denote by $\mathfrak{D}_c$ the collection of connected components of D_c and by $\mathfrak{D}_c^{\mathbb{R}}$ the sub-collection of $\mathfrak{D}_c$ of those sets intersecting $\mathbb{R}$. For each W in $\mathfrak{D}_c$ denote by $\phi_W : \widehat{V}_c \to V_c$ the extension of $F_c|_W^{-1}$ given by Lemma 6.1. Given an integer $\ell \geqslant 1$ denote by $E_{c,\ell}$ (resp. $E_{c,\ell}^{\mathbb{R}}$) the set of all words of length ℓ in the alphabet $\mathfrak{D}_c$ (resp. $\mathfrak{D}_c^{\mathbb{R}}$). So, for each integer $\ell \geqslant 1$ and each word $W_1 \cdots W_\ell$ in $E_{c,\ell}$ the composition

$$\phi_{W_1 \cdots W_\ell} = \phi_{W_1} \circ \cdots \circ \phi_{W_\ell}$$

is defined on $\widehat{V}_c$. Put

$$m_c(W_1 \cdots W_\ell) = m_c(W_1) + \cdots + m_c(W_\ell).$$

For t, p in $\mathbb{R}$ and an integer $\ell \geqslant 1$ put

$$Z_{c,\ell}^{\mathbb{R}}(t,p) := \sum_{\underline{W} \in E_{c,\ell}^{\mathbb{R}}} \exp\bigl(-m_c(\underline{W})p\bigr)\bigl(\sup\bigl\{\bigl|D\phi_{\underline{W}}(z)\bigr| \bigm| z \in V_c\bigr\}\bigr)^t$$

and

$$Z_{c,\ell}^{\mathbb{C}}(t,p) := \sum_{\underline{W} \in E_{c,\ell}} \exp\bigl(-m_c(\underline{W})p\bigr)\bigl(\sup\bigl\{\bigl|D\phi_{\underline{W}}(z)\bigr| \bigm| z \in V_c\bigr\}\bigr)^t.$$

For a fixed t and p in $\mathbb{R}$ the sequence

$$\left(\frac{1}{\ell}\log Z_{c,\ell}^{\mathbb{R}}(t,p)\right)_{\ell=1}^{+\infty} \quad \left(\text{resp. } \left(\frac{1}{\ell}\log Z_{c,\ell}^{\mathbb{C}}(t,p)\right)_{\ell=1}^{+\infty}\right)$$

converges to the pressure function of $F_c|_{D_c \cap \mathbb{R}}$ (resp. F_c) for the potential $-t\log|DF_c| - pm_c$; denote it by $\mathscr{P}_c^{\mathbb{R}}(t,p)$ (resp. $\mathscr{P}_c^{\mathbb{C}}(t,p)$). On the set where it is finite, the function $\mathscr{P}_c^{\mathbb{R}}$ (resp. $\mathscr{P}_c^{\mathbb{C}}$) so defined is strictly decreasing in each of its variables.

Proposition C. *There is $n_4 \geqslant 4$ such that for every integer $n \geqslant n_4$ and every parameter c in $\mathcal{K}_n$, we have for each $t \geqslant 3$*

$$P_c^{\mathbb{R}}(t) = \inf\bigl\{p \bigm| \mathscr{P}_c^{\mathbb{R}}(t,p) \leqslant 0\bigr\} \quad \bigl(\textit{resp. } P_c^{\mathbb{C}}(t) = \inf\bigl\{p \bigm| \mathscr{P}_c^{\mathbb{C}}(t,p) \leqslant 0\bigr\}\bigr).$$

The proof of this proposition is given in Section 6.4, after we give a lower bound on the pressure function in the next subsection.

6.3. Critical line

The purpose of this subsection is to prove the following proposition.

Proposition 6.2. *For every integer $n \geqslant 5$ and every parameter c in $\mathcal{K}_n$ we have*

$$\chi_{\inf}^{\mathbb{R}}(c) := \inf\left\{ \int \log|Df_c|\, d\mu \;\middle|\; \mu \in \mathscr{M}_c^{\mathbb{R}} \right\} \leqslant \chi_{\mathrm{crit}}(c)/2.$$

In particular, for each $t > 0$ we have

$$P_c^{\mathbb{C}}(t) \geqslant P_c^{\mathbb{R}}(t) \geqslant -t\chi_{\mathrm{crit}}(c)/2.$$

The proof of this proposition is given after the following lemma.

Lemma 6.3. *There is a constant $C_4 > 0$ such that for each integer $n \geqslant 5$ and each parameter c in $\mathcal{K}_n$, the following property holds: For every integer $k \geqslant 0$ there is a connected component W of D_c contained in $P_{c,n+3k+1}(0)$, that intersects $\mathbb{R}$ and such that $m_c(W) = n + 3k + 3$ and*

$$\sup_{z \in W} \left| DF_c(z) \right| \leqslant C_4 \left| Df_c^{n+3k}(c) \right|^{1/2}.$$

Proof. Let $\Delta_2 > 1$ and $\Delta_3 > 1$ be the constants given by Lemmas 4.3 and 6.1, respectively. Since the set $P_{c,1}(0)$ depends continuously with c on $\mathcal{P}_0(-2)$ (*cf.*, Lemma 2.5) and since this last set contains the closure of $\mathcal{P}_4(-2)$ (part 1 of Lemma 3.3), we have

$$\Xi_0 := \sup_{c \in \mathcal{P}_4(-2)} \operatorname{diam}\big(P_{c,1}(0)\big) < +\infty$$

and

$$\Xi_1 := \sup_{c \in \mathcal{P}_4(-2)} \sup_{z \in P_{c,1}(0)} \left| Df_c^2(z) \right| < +\infty.$$

Fix an integer $n \geqslant 5$, a parameter c in $\mathcal{K}_n$, and an integer $k \geqslant 0$. Then the parameter c is real, so $\alpha(c)$ and $\tilde{\alpha}(c)$ are both real (Section 3.1) and the set Λ_c is contained in the interval

$$P_{c,3}(0) \cap \mathbb{R} = \big(\gamma(c), \tilde{\gamma}(c)\big),$$

see Section 3.3. On the other hand, the point $\alpha_1(c)$ is real (Section 3.1) and c is in $P_{c,n}(-\beta(c))$ (part 2 of Lemma 3.3), so $c < \alpha_1(c)$. Note moreover that $V_{c,1}$ is invariant by complex conjugation and that $V_{c,1} \cap \mathbb{R} = (\alpha_1(c), \alpha(c))$ (Section 3.1). It follows that $f_c^{-1}(V_{c,1})$ is the disjoint union of 2 puzzle pieces X_c and $\widetilde{X}_c$, such that

$$X_c \cap \mathbb{R} = \big(\alpha(c), \gamma(c)\big) \quad \text{and} \quad \widetilde{X}_c \cap \mathbb{R} = \big(\tilde{\gamma}(c), \tilde{\alpha}(c)\big).$$

Moreover, each of the puzzle pieces X_c and $\widetilde{X}_c$ is a connected component of D_c' and $m_c(X_c) = m_c(\widetilde{X}_c) = 2$.

Since f_c^{n+3k+1} maps $P_{c,n+3k+2}(0)$ properly onto $P_{c,1}(0)$, it follows that f_c^{n+3k+1} maps the end points of the interval $P_{c,n+3k+2}(0) \cap \mathbb{R}$ into $\partial P_{c,1}(0) \cap \mathbb{R} = \{\alpha(c), \tilde{\alpha}(c)\}$. Since $f_c^{n+3k+1}(0)$ is in $\Lambda_c \subset Y_c \cup \widetilde{Y}_c$, it follows that the interval $f_c^{n+3k+1}(P_{c,n+3k+2}(0) \cap \mathbb{R})$ contains either $X_c \cap \mathbb{R}$ or $\widetilde{X}_c \cap \mathbb{R}$. This proves that there is a connected component W of D_c contained in $P_{n+3k+2}(0)$, that intersects $\mathbb{R}$ and such that $m_c(W) = n + 3k + 3$. Let z_W be the unique point in W such that $f_c^{n+3k+3}(z_W) = 0$. Then $f_c^{n+3k+1}(z_W)$ belongs to $P_{c,1}(0)$, so by definition of Ξ_0 we have

$$\left|f_c^{n+3k+1}(z_W) - f_c^{n+3k}(c)\right| \leqslant \operatorname{diam}\bigl(P_{c,1}(0)\bigr) \leqslant \Xi_0. \tag{6.1}$$

Since f_c^n maps $V_{c,n} = P_{c,n+1}(c)$ biholomorphically to $P_{c,1}(0)$ and $f_c^n(c) \in \Lambda_c$, it follows that f_c^{n+3k} maps $P_{c,n+3k+1}(c)$ biholomorphically to $P_{c,1}(0)$; so the distortion of f_c^{n+3k} on $P_{c,n+3k+1}(c)$ is bounded by Δ_2 (Lemma 4.3) and for each point y in $P_{c,n+3k+1}(c)$ we have

$$\Delta_2^{-1}\left|Df_c^{n+3k}(c)\right| \leqslant \left|Df_c^{n+3k}(y)\right| \leqslant \Delta_2\left|Df_c^{n+3k}(c)\right|. \tag{6.2}$$

Together with (6.1) this implies that,

$$\left|f_c(z_W) - c\right| \leqslant \Delta_2 \Xi_0 \left|Df_c^{n+3k}(c)\right|^{-1}$$

and therefore that,

$$\left|Df_c(z_W)\right| \leqslant 2\Delta_2^{1/2}\Xi_0^{1/2}\left|Df_c^{n+3k}(c)\right|^{-1/2}.$$

Combined with (6.2) with $y = f_c(z_W)$, this implies

$$\left|Df_c^{n+3k+1}(z_W)\right| \leqslant 2\Delta_2^{3/2}\Xi_0^{1/2}\left|Df_c^{n+3k}(c)\right|^{1/2}.$$

Putting $C_4 := 2\Delta_3\Xi_1\Delta_2^{3/2}\Xi_0^{1/2}$, we get by Lemma 6.1

$$\begin{aligned}\sup_{z\in W}\left|DF_c(z)\right| &\leqslant \Delta_3\left|Df_c^{n+3k+3}(z_W)\right| \\ &\leqslant \Delta_3\Xi_1\left|Df_c^{n+3k+1}(z_W)\right| \\ &\leqslant C_4\left|Df_c^{n+3k}(c)\right|^{1/2}. \qquad \square\end{aligned}$$

Proof of Proposition 6.2. Let C_4 be given by Lemma 6.3 and for each integer $k \geqslant 0$ let W_k be the element W of $\mathfrak{D}_c$ given by the same lemma. Since W_k intersects $\mathbb{R}$ and $F_c|_{W_k} = f_c^{n+3k+3}|_{W_k}$ maps W_k biholomorphically to V_c, we have $f_c^{n+3k+3}(W_k \cap \mathbb{R}) = V_c \cap \mathbb{R}$. On the other hand, since $W_k \subset V_c$, there is a periodic point p_k of f_c of period $n + 3k + 3$ in the closure of $W_k \cap \mathbb{R}$. Denoting by μ_k the invariant probability measure supported on the orbit of p_k, we have by Lemma 6.3 that for each $t > 0$

$$\begin{aligned}\chi_{\inf}^{\mathbb{R}}(c) &\leqslant \int \log|Df_c|\, d\mu_k \\ &= \frac{1}{n+3k+3}\log\left|Df_c^{n+3k+3}(p_k)\right| \\ &\leqslant \frac{1}{n+3k+3}\left(\log C_4 + \log\left|Df_c^{n+3k}(c)\right|^{1/2}\right).\end{aligned}$$

We obtain the desired inequality by letting $k \to +\infty$. $\square$

6.4. *Proof of Proposition C*

For future reference, the following lemma is stated in a stronger form than what is needed for this paper.

Lemma 6.4. *There are $n_5 \geqslant 4$ and $C_5 > 1$ such that for every integer $n \geqslant n_5$ and every parameter c in $\mathcal{K}_n$ the following property holds: For each $t \geqslant 3$, $p \geqslant -t\chi_{\mathrm{crit}}(c)/2 - \frac{1}{10}\log 2$, and y in V_c, we have*

$$L_{t,p}(y) := 1 + \sum_{z \in L_c^{-1}(y)} \exp\bigl(-m_c(z)p\bigr)\bigl|DL_c(z)\bigr|^{-t} \leqslant C_5^t.$$

Moreover, for every integer $\tilde{m} \geqslant 1$, we have

$$\sum_{\substack{z \in L_c^{-1}(y),\\ m_c(z) \geqslant \tilde{m}}} \exp\bigl(-m_c(z)p\bigr)\bigl|DL_c(z)\bigr|^{-t} \leqslant C_5^t 2^{-\frac{t}{30}\tilde{m}}.$$

Proof. Put $\varepsilon_0 := \frac{1}{45}$, let C_1 and n_2 be given by Proposition B with $\varepsilon = \varepsilon_0$, and let n_3 be given by Lemma 5.3 with $\varepsilon = \varepsilon_0$ and $m_1 = 4$. We prove the lemma with $n_5 := \max\{n_2, n_3\}$ and $C_5 := C_1(1 - 2^{-1/10})^{-1/3}$.

Let n, c, t, p and y be as in the statement of the lemma. By Lemma 5.3 with $z = f_c^n(c)$, we have

$$\chi_{\mathrm{crit}}(c) = \liminf_{m \to +\infty} \frac{1}{m} \log\bigl|Df_c^m(c)\bigr| \leqslant (1 + \varepsilon_0)\log 2.$$

On the other hand, for each integer $m \geqslant 1$ the set $\{z \in L_c^{-1}(y) \mid m_c(z) = m\}$ is contained in $f_c^{-m}(y)$ and therefore it contains at most 2^m points. So by Proposition B and the definition of C_5, for every integer $\tilde{m} \geqslant 1$ we have

$$\sum_{\substack{z \in L_c^{-1}(y),\\ m_c(z) \geqslant \tilde{m}}} \exp\bigl(-m_c(z)p\bigr)\bigl|DL_c(z)\bigr|^{-t} \leqslant C_1^t \sum_{m=\tilde{m}}^{+\infty} 2^{m(1-\frac{11}{30}t)} \leqslant C_5^t 2^{-\frac{t}{30}\tilde{m}}. \qquad \square$$

Lemma 6.5. *Given an integer $n \geqslant 5$ and a parameter c in $\mathcal{K}_n$, the following property holds for every $t > 0$ and every real number p: If $\mathscr{P}_c^{\mathbb{R}}(t,p) > 0$ (resp. $\mathscr{P}_c^{\mathbb{C}}(t,p) > 0$), then the series*

$$\sum_{j=1}^{+\infty} \exp(-jp) \sum_{y \in f_c|_{I_c}^{-j}(0)} \bigl|Df_c^j(y)\bigr|^{-t} \quad \left(\text{resp. } \sum_{j=1}^{+\infty} \exp(-jp) \sum_{y \in f_c^{-j}(0)} \bigl|Df_c^j(y)\bigr|^{-t}\right) \tag{6.3}$$

diverges. On the other hand, there is $n_6 \geqslant 5$ such that if in addition $n \geqslant n_6$, then for every $t \geqslant 3$ and

$$p \geqslant P_c^{\mathbb{R}}(t) - t\frac{1}{10}\log 2 \quad \left(\text{resp. } p \geqslant P_c^{\mathbb{C}}(t) - t\frac{1}{10}\log 2\right)$$

satisfying $\mathscr{P}_c^{\mathbb{R}}(t,p) < 0$ (resp. $\mathscr{P}_c^{\mathbb{C}}(t,p) < 0$), the series above converges.

Proof. We prove the assertions concerning $f_c|_{J_c}$; the arguments apply without change to $f_c|_{I_c}$. Let $\Delta_3 > 1$ be the constant given by Lemma 6.1.

Suppose first $\mathscr{P}_c^{\mathbb{C}}(t,p) > 0$. Since for each integer $\ell \geqslant 1$ every point of $F_c^{-\ell}(0)$ is a preimage of 0 by an iterate of f_c, by Lemma 6.1 the series (6.3) is bounded from below by

$$\begin{aligned}
&\sum_{\ell=1}^{+\infty} \sum_{y \in F_c^{-\ell}(0)} \exp\bigl(-\bigl(m_c\bigl(F_c^{\ell-1}(y)\bigr) + \cdots + m_c(y)\bigr)p\bigr)\bigl|DF_c^{\ell}(y)\bigr|^{-t} \\
&\quad \geqslant \Delta_3^{-t} \sum_{\ell=1}^{+\infty} Z_{c,\ell}^{\mathbb{C}}(t,p) = +\infty.
\end{aligned}$$

To prove the last part of the lemma, let n_5 and $C_5 > 1$ be given by Lemma 6.4. We prove the desired assertion with $n_6 = n_5$. Suppose in addition we have $n \geqslant n_5$ and let

$$t \geqslant 3 \quad \text{and} \quad p \geqslant P_c^{\mathbb{C}}(t) - t\frac{1}{10}\log 2$$

be such that $\mathscr{P}_c^{\mathbb{C}}(t,p) < 0$. By Proposition 6.2 we have $p \geqslant -t(\chi_{\mathrm{crit}}(c) + \frac{1}{5}\log 2)/2$, so t and p satisfy the hypotheses of Lemma 6.4. Given an integer $m \geqslant 1$ and a point z in $f_c^{-m}(0)$ denote by $\ell(z)$ the number of those j in $\{0, \ldots, m-1\}$ such that $f_c^j(z)$ is in V_c. In the case where z is not in V_c, this point is in the domain of L_c and we have $\ell(z) = 0$ if and only if $L_c(z) = 0$. Moreover, if z is not in V_c and $\ell(z) \geqslant 1$, then $L_c(z)$ is in the domain of $F_c^{\ell(z)}$ and $F_c^{\ell(z)}(L_c(z)) = 0$. So, if z is not in V_c we have in all the cases

$$\bigl|Df_c^m(z)\bigr| = \bigl|DF_c^{\ell(z)}\bigl(L_c(z)\bigr)\bigr| \cdot \bigl|DL_c(z)\bigr|.$$

Then Lemma 6.4 implies that the series (6.3) is bounded from above by

$$\begin{aligned}
&L_{t,p}(0) + \sum_{\ell=1}^{+\infty} \sum_{y \in F_c^{-\ell}(0)} L_{t,p}(y) \exp\bigl(-\bigl(m_c\bigl(F_c^{\ell-1}(y)\bigr) + \cdots + m_c(y)\bigr)p\bigr)\bigl|DF_c^{\ell}(y)\bigr|^{-t} \\
&\quad \leqslant C_5^t\left(1 + \sum_{\ell=0}^{+\infty} Z_{c,\ell}^{\mathbb{C}}(t,p)\right) < +\infty. \qquad \square
\end{aligned}$$

Proof of Proposition C. We prove the assertion for $f_c|_{J_c}$; the arguments apply without change to $f_c|_{I_c}$. Let $\Delta_3 > 1$ be given by Lemma 6.1 and n_6 by Lemma 6.5. Let $n \geqslant n_6$ be an integer and let c be a parameter in $\mathcal{K}_n$. We use that fact that for each $t > 0$ we have

$$P_c^{\mathbb{C}}(t) = \limsup_{m \to +\infty} \frac{1}{m} \log \sum_{y \in f_c^{-m}(0)} \bigl|Df_c^m(y)\bigr|^{-t}, \tag{6.4}$$

see for example [45] or [38].

Fix $t \geqslant 3$. We use the fact that the function $p \mapsto \mathscr{P}_c^{\mathbb{C}}(t,p)$ is strictly decreasing where it is finite, see Section 6.2. In particular, for each p satisfying $p < p_0 := \inf\{p\colon \mathscr{P}_c^{\mathbb{C}}(t,p) \leqslant 0\}$ we have $\mathscr{P}_c^{\mathbb{C}}(t,p) > 0$. Lemma 6.5 implies that for such p the series (6.3) diverges and by (6.4) we have $P_c^{\mathbb{C}}(t) \geqslant p > p_0$. To prove the reverse inequality, suppose by contradiction $p_0 < P_c^{\mathbb{C}}(t)$ and let p be in the interval $(p_0, P_c^{\mathbb{C}}(t))$ satisfying $p \geqslant P_c^{\mathbb{C}}(t) - t\frac{1}{10}\log 2$. Then $\mathscr{P}_c^{\mathbb{C}}(t,p) < 0$ and by Lemma 6.5 the series (6.3) converges. Then (6.4) implies $P_c^{\mathbb{C}}(t) \leqslant p$ and we obtain a contradiction that completes the proof of the proposition. $\square$

7. Estimating the geometric pressure function

The purpose of this section is to prove the following proposition. The proof of Proposition A, at the end of this section, is based on this proposition, together with Propositions C and 6.2.

Recall that for a real parameter c,

$$\chi_{\mathrm{crit}}(c) = \liminf_{m \to +\infty} \frac{1}{m} \log \left| Df_c^m(c) \right|.$$

Proposition D. *There are $n_7 \geqslant 5$ and $C_6 > 1$ such that for every integer $n \geqslant n_7$ and every parameter c in $\mathcal{K}_n$ the following properties hold for each $t \geqslant 3$.*

1. *For p in $[-t\chi_{\mathrm{crit}}(c)/2, 0)$ satisfying*

$$\sum_{k=0}^{+\infty} \exp\bigl(-(n+3k)p\bigr) \left| Df_c^{n+3k}(c) \right|^{-t/2} \geqslant C_6^t,$$

we have $\mathscr{P}_c^{\mathbb{R}}(t,p) > 0$ and $P_c^{\mathbb{R}}(t) \geqslant p$. If in addition the sum above is finite, then $\mathscr{P}_c^{\mathbb{C}}(t,p)$ is finite and $P_c^{\mathbb{R}}(t) > p$.

2. *For $p \geqslant -t\chi_{\mathrm{crit}}(c)/2$ satisfying*

$$\sum_{k=0}^{+\infty} \exp\bigl(-(n+3k)p\bigr) \left| Df_c^{n+3k}(c) \right|^{-t/2} \leqslant C_6^{-t},$$

we have $\mathscr{P}_c^{\mathbb{C}}(t,p) < 0$ and $P_c^{\mathbb{C}}(t) \leqslant p$.

3. *For $p \geqslant -t\chi_{\mathrm{crit}}(c)/2$ satisfying*

$$\sum_{k=0}^{+\infty} k \cdot \exp\bigl(-(n+3k)p\bigr) \left| Df_c^{n+3k}(c) \right|^{-t/2} < +\infty,$$

we have

$$\sum_{W \in \mathfrak{D}_c} m_c(W) \cdot \exp\bigl(-m_c(W)p\bigr) \sup_{z \in W} \left| DF_c(z) \right|^{-t} < +\infty.$$

The proof of Proposition D is given after Lemma 7.1, below, which is used in the proof. The proof of Proposition A is given after the proof of Proposition D.

Let $n \geqslant 4$ be an integer and c a parameter in $\mathcal{K}_n$. Since the critical point $z = 0$ does not belong to D_c (*cf.*, Lemma 4.2), for each integer $\ell \geqslant 1$, each connected component of D_c intersecting $P_{c,\ell}(0)$ is contained in $P_{c,\ell}(0)$. We define the *level* of a connected component W of D_c as the largest integer $k \geqslant 0$ such that W is contained in $P_{c,n+3k+2}(0)$. Given an integer $k \geqslant 0$ denote by $\mathfrak{D}_{c,k}$ the collection of all connected components of D_c of level k; we have $\mathfrak{D}_c = \bigcup_{k=0}^{+\infty} \mathfrak{D}_{c,k}$.

For future reference, the following lemma is stated in a stronger form than what is needed for this paper.

Lemma 7.1. *There is $C_7 > 0$ such that for each integer $n \geqslant 5$, each parameter c in $\mathcal{K}_n$, each integer $k \geqslant 0$, and each pair of real numbers $t > 0$ and p, we have*

$$\sum_{W \in \mathfrak{D}_{c,k}} \exp\bigl(-m_c(W)p\bigr) \sup_{z \in W} \bigl|DF_c(z)\bigr|^{-t}$$
$$\leqslant 2C_7^t \exp\bigl(-(n+3k+1)p\bigr)\bigl|Df_c^{n+3k}(c)\bigr|^{-t/2}$$
$$\cdot \Bigl(1 + \sum_{w \in L_c^{-1}(0) \text{ in } P_{c,1}(0)} \exp\bigl(-m_c(w)p\bigr)\bigl|DL_c(w)\bigr|^{-t}\Bigr).$$

Moreover, for every integer $\tilde{m} \geqslant 1$, we have

$$\sum_{\substack{W \in \mathfrak{D}_{c,k},\\ m_c(W) \geqslant \tilde{m}+n+3k+1}} \exp\bigl(-m_c(W)p\bigr) \sup_{z \in W} \bigl|DF_c(z)\bigr|^{-t}$$
$$\leqslant 2C_7^t \exp\bigl(-(n+3k+1)p\bigr)\bigl|Df_c^{n+3k}(c)\bigr|^{-t/2}$$
$$\cdot \Bigl(\sum_{\substack{w \in L_c^{-1}(0) \text{ in } P_{c,1}(0),\\ m_c(w) \geqslant \tilde{m}}} \exp\bigl(-m_c(w)p\bigr)\bigl|DL_c(w)\bigr|^{-t}\Bigr).$$

Proof. Let Δ_2, C_3, and Δ_3 be the constants given by Lemmas 4.3, 5.4 and 6.1, respectively.

Fix an integer $n \geqslant 5$, a parameter c in $\mathcal{K}_n$, and an integer $k \geqslant 0$. Note that there are precisely 2 elements W' of $\mathfrak{D}_{c,k}$ such that $m_c(W') = n + 3k + 1$; denote them by W_0 and W_0'. Indeed, these sets are the connected components of the preimage under f_c of the set $(f_c^{n+3k}|_{P_{c,n+3k+1}(c)})^{-1}(V_c)$. For a connected component W of D_c of level k denote by z_W the unique point in W such that $F_c(z_W) = 0$. If W is different from W_0 and W_0', then $z' := f_c^{n+3k+1}(z_W)$ is different from 0 and it is in the domain of definition D_c' of the first landing map L_c to V_c. So, denoting by W' the connected component of D_c' containing z', there is a unique point w_W in W' such that $L_c(w_W) = 0$ and we have

$$m_c(W) = n + 3k + 1 + m_c(w_W).$$

Since f_c^n maps $V_{c,n} = P_{c,n+1}(c)$ biholomorphically to $P_{c,1}(0)$ and $f_c^n(c)$ is in Λ_c, it follows that f_c^{n+3k} maps $P_{c,n+3k+1}(c)$ biholomorphically to $P_{c,1}(0)$; so the distortion of f_c^{n+3k} on $P_{c,n+3k+1}(c)$ is bounded by Δ_2 (Lemma 4.3) and for each point y in $P_{c,n+3k+1}(c)$ we have

$$\bigl|Df_c^{n+3k}(y)\bigr| \geqslant \Delta_2^{-1}\bigl|Df_c^{n+3k}(c)\bigr|.$$

On the other hand, by part 1 of Lemma 5.4 with $x = z_W$ and $q = n + 3k + 1$, we have

$$\bigl|Df_c^{n+3k+1}(z_W)\bigr| \geqslant C_3^{-1}\bigl|Df_c^{n+3k}\bigl(f_c(z_W)\bigr)\bigr|^{1/2} \geqslant C_3^{-1}\Delta_2^{-1/2}\bigl|Df_c^{n+3k}(c)\bigr|^{1/2}.$$

Together with the inequality,

$$\bigl|Df_c^{m_c(w_W)}\bigl(f_c^{n+3k+1}(z_W)\bigr)\bigr| \geqslant \Delta_2^{-1}\bigl|DL_c(w_W)\bigr|$$

given by Lemmas 4.3 and 5.1, this implies that if we put $C_0 := C_3^{-1}\Delta_2^{-3/2}$, then

$$\bigl|DF_c(z_W)\bigr| = \bigl|Df_c^{m_c(W)}(z_W)\bigr| \geqslant C_0\bigl|Df_c^{n+3k}(c)\bigr|^{1/2}\bigl|DL_c(w_W)\bigr|.$$

Since the distortion of $F_c|_W$ is bounded by Δ_3 (Lemma 6.1), for each $p > 0$ and $t > 0$, we have

$$\begin{aligned}
&\exp\bigl(-m_c(W)p\bigr)\sup_{z\in W}\bigl|DF_c(z)\bigr|^{-t}\\
&\quad\leqslant \Delta_3^t C_0^{-t}\exp\bigl(-(n+3k+1)p\bigr)\bigl|Df_c^{n+3k}(c)\bigr|^{-t/2}\\
&\qquad\cdot\exp\bigl(-m_c(w_W)p\bigr)\bigl|DL_c(w_W)\bigr|^{-t}. \qquad (7.1)
\end{aligned}$$

To prove the desired inequality, observe that for each point w of $L_c^{-1}(0)$ in $P_{c,1}(0)$ there are precisely 2 connected components W of D_c in $\mathfrak{D}_{c,k}$ such that $w_W = w$; in fact for each such W the set $f_c(W)$ is uniquely determined as the preimage by the univalent map $f_c^{n+3k}|_{P_{c,n+3k+1}(c)}$ of the connected component of D'_c containing w_W. Thus, the desired inequalities follow from (7.1) with $C_7 = \Delta_3 C_0^{-1}$. □

Lemma 7.2. *There are $n_8 \geqslant 5$ and $C_8 > 1$ such that for every integer $n \geqslant n_8$ and every parameter c in $\mathcal{K}_n$, the following properties hold for each $t \geqslant 3$ and each integer $k \geqslant 0$:*

1. *For each $p < 0$, we have*

$$\begin{aligned}
&\sum_{W\in\mathfrak{D}_{c,k}\cap\mathfrak{D}_c^{\mathbb{R}}}\exp\bigl(-m_c(W)p\bigr)\inf_{z\in W}\bigl|DF_c(z)\bigr|^{-t}\\
&\quad > C_8^{-t}\exp\bigl(-(n+3k)p\bigr)\bigl|Df_c^{n+3k}(c)\bigr|^{-t/2}.
\end{aligned}$$

2. *For each $p \geqslant -t\chi_{\mathrm{crit}}(c)/2 - t\frac{1}{10}\log 2$, we have*

$$\begin{aligned}
&\sum_{W\in\mathfrak{D}_{c,k}}\exp\bigl(-m_c(W)p\bigr)\sup_{z\in W}\bigl|DF_c(z)\bigr|^{-t}\\
&\quad < C_8^{t}\exp\bigl(-(n+3k)p\bigr)\bigl|Df_c^{n+3k}(c)\bigr|^{-t/2}.
\end{aligned}$$

Proof. Let C_2 and n_3 be given by Lemma 5.3 with $m_1 = 4$ and $\varepsilon = \frac{1}{10}$, let n_5 and $C_5 > 0$ be given by Lemma 6.4, and let C_4 and C_7 be given by Lemmas 6.3 and 7.1, respectively. We prove the lemma with $n_8 := \max\{n_3, n_4, n_5\}$. To do this, fix an integer $n \geqslant n_8$, a parameter c in $\mathcal{K}_n$, $t \geqslant 3$, and an integer $k \geqslant 0$.

To prove part 1, let W_k be the component W of D_c given Lemma 6.3. Then for each $p < 0$, we have

$$\begin{aligned}
&\sum_{W\in\mathfrak{D}_{c,k}\cap\mathfrak{D}_c^{\mathbb{R}}}\exp\bigl(-m_c(W)p\bigr)\inf_{z\in W}\bigl|DF_c(z)\bigr|^{-t}\\
&\quad\geqslant \exp\bigl(-m_c(W_k)p\bigr)\inf_{z\in W_k}\bigl|DF_c(z)\bigr|^{-t}\\
&\quad\geqslant C_4^{-t}\exp\bigl(-(n+3k+3)p\bigr)\bigl|Df_c^{n+3k}(c)\bigr|^{-t/2}\\
&\quad > C_4^{-t}\exp\bigl(-(n+3k)p\bigr)\bigl|Df_c^{n+3k}(c)\bigr|^{-t/2}.
\end{aligned}$$

This proves part 1 of the lemma with $C_8 = C_4$.

To prove part 2, let $p \geqslant -t\chi_{\mathrm{crit}}(c)/2 - t\frac{1}{10}\log 2$ be given. By Lemma 5.3 with $z = f_c^n(c)$, we have

$$\chi_{\mathrm{crit}}(c) = \liminf_{m\to+\infty} \frac{1}{m} \log\left|Df_c^m(c)\right| \leqslant \frac{11}{10}\log 2.$$

Thus $p \geqslant -t\frac{13}{20}\log 2$ and therefore $2\exp(-p) < 2^t$. Combined with Lemmas 6.4 and 7.1, we obtain part 2 of the lemma with $C_8 = 2C_7C_5$. □

Proof of Proposition D. Let n_4 be given by Proposition C and let n_8 and C_8 be given by Lemma 7.2. To prove the proposition, fix an integer $n \geqslant \max\{n_4, n_8\}$, a parameter c in $\mathcal{K}_n$, and $t \geqslant 3$.

To prove part 1, let p be in $[-t\chi_{\mathrm{crit}}(c)/2, 0)$. By part 1 of Lemma 7.2, if the sum

$$\sum_{k=0}^{+\infty} \exp\bigl(-(n+3k)p\bigr)\left|Df_c^{n+3k}(c)\right|^{-t/2} \tag{7.2}$$

is greater than or equal to C_8^t, then $\mathscr{P}_c^{\mathbb{R}}(t,p) > 0$ and by Proposition C we have $P_c^{\mathbb{R}}(t) \geqslant p$. This proves the first part of part 1 with $C_6 = C_8$. To complete the proof of part 1, suppose (7.2) is finite and greater than or equal to $(2C_8)^t$. Then there is $p' > p$ such that (7.2) with p replaced by p' is greater than or equal to C_8^t. As shown above, this implies $P_c^{\mathbb{R}}(t) \geqslant p' > p$. On the other hand, by part 2 of Lemma 7.2 the sum

$$\sum_{W\in\mathfrak{D}_c} \exp\bigl(-m_c(W)p\bigr) \sup_{z\in W}\left|DF_c(z)\right|^{-t}$$

is finite, so $\mathscr{P}_c^{\mathbb{R}}(t,p)$ is also finite. This completes the proof of part 1 with $C_6 = 2C_8$.

To prove part 2, let $p \geqslant -t\chi_{\mathrm{crit}}(c)/2$ be given. By part 2 of Lemma 7.2, if (7.2) is less than or equal to C_8^{-t}, then $\mathscr{P}_c^{\mathbb{C}}(t,p) < 0$ and by Proposition C we have $P_c^{\mathbb{C}}(t) \leqslant p$. This proves part 2 of the proposition with $C_6 = C_8$.

To prove part 3, let $p \geqslant -t\chi_{\mathrm{crit}}(c)/2$ be given and put $p' := p - t\frac{1}{10}\log 2$. By part 2 of Lemma 7.2 with $k = 0$, the sum

$$\sum_{W\in\mathfrak{D}_{c,0}} \exp\bigl(-m_c(W)p\bigr) \sup_{z\in W}\left|DF_c(z)\right|^{-t}$$

is finite. Let $A > 0$ be a constant such that for every pair of integers $k \geqslant 1$ and $m \geqslant 3k+1$, we have

$$m \leqslant Ak2^{t(m-3k)/10}.$$

Applying part 2 of Lemma 7.2 with p replaced by p', we obtain that for each integer $k \geqslant 1$ we have

$$\begin{aligned}
&\sum_{W\in\mathfrak{D}_{c,k}} m_c(W)\cdot\exp\bigl(-m_c(W)p\bigr) \sup_{z\in W}\left|DF_c(z)\right|^{-t} \\
&\quad\leqslant \sum_{W\in\mathfrak{D}_{c,k}} Ak2^{t(m_c(W)-3k)/10} \exp\bigl(-m_c(W)p\bigr) \sup_{z\in W}\left|DF_c(z)\right|^{-t} \\
&\quad= Ak2^{-3kt/10} \sum_{W\in\mathfrak{D}_{c,k}} \exp\bigl(-m_c(W)p'\bigr) \sup_{z\in W}\left|DF_c(z)\right|^{-t} \\
&\quad\leqslant \bigl(AC_8^t 2^{nt/10}\bigr)k\cdot\exp\bigl(-(n+3k)p\bigr)\left|Df_c^{n+3k}(c)\right|^{-t/2}.
\end{aligned}$$

Summing over $k \geqslant 0$ we obtain the desired assertion. □

Proof of Proposition A. We give the proof for $f_c|_{J_c}$; the proof for $f_c|_{I_c}$ is analogous. Let n_4 be given by Proposition C and let n_7 and C_6 be given by Proposition D. Put

$$n_0 := \max\{5, n_4, n_7\} \quad \text{and} \quad C_0 := C_6$$

and let $n \geqslant n_0$ be an integer and c a parameter in $\mathcal{K}_n$ for which the hypotheses of the proposition are satisfied.

The first hypothesis of the proposition together with part 2 of Proposition D with $p = -t\chi_{\text{crit}}(c)/2$ imply that for every sufficiently large $t > 0$, we have

$$P_c^{\mathbb{C}}(t) \leqslant -t\chi_{\text{crit}}(c)/2.$$

From Proposition 6.2 we deduce that for such t we have equality. The second hypothesis of the proposition together with part 1 of Proposition D with $t = t_0$ and $p = -t_0\chi_{\text{crit}}(c)/2$, imply that f_c has a phase transition at some $t_* > t_0$ satisfying

$$P_c^{\mathbb{C}}(t_*) = -t_*\chi_{\text{crit}}(c)/2 < 0.$$

This proves the first part of the proposition.

To prove the second part of the proposition, we first prove that there exists an equilibrium state of f_c for the potential $-t_* \log|Df_c|$. Our additional hypothesis together with part 3 of Proposition D with $t = t_*$ and $p = -t_*\chi_{\text{crit}}(c)/2$, imply that

$$\sum_{W \in \mathfrak{D}_c} m_c(W) \exp\bigl(m_c(W) t_* \chi_{\text{crit}}(c)/2\bigr) \sup_{z \in W} \bigl|DF_c(z)\bigr|^{-t_*} < +\infty. \tag{7.3}$$

The rest of the argument is now standard; we refer to [34, §4] for precisions, see also Remark 7.3 below. Since $\mathscr{P}_c^{\mathbb{C}}(t_*, -t_*\chi_{\text{crit}}(c)/2) = 0$, there is a $(t_*, -t_*\chi_{\text{crit}}(c)/2)$-conformal measure μ for f_c that assigns positive measure to the maximal invariant set of F_c, see [34, Theorem A in §4 and Proposition 4.3]. Standard considerations imply that there is an invariant probability measure ρ for F_c that is absolutely continuous with respect to μ. Thus (7.3) together with the bounded distortion property of F_c (Lemma 6.1) imply that the sum $\sum_{W \in \mathfrak{D}_c} m_c(W)\rho(W)$ is finite. Therefore the measure

$$\hat{\rho} := \sum_{W \in \mathfrak{D}_c} \sum_{j=0}^{m_c(W)-1} \bigl(f_c^j\bigr)_*(\rho|_W)$$

is finite. This measure is invariant by f_c and it is absolutely continuous with respect to μ. To prove that the probability measure proportional to $\hat{\rho}$ is an equilibrium state of f_c for the potential $-t\log|Df_c|$, first remark that ρ is an equilibrium state of F_c for the potential $-t_* \log|DF_c| + (t_*\chi_{\text{crit}}(c)/2)m_c$ and that the measure-theoretic entropy of this measure is strictly positive, see for example [25]. Then the generalized Abramov formula [49, Proposition 5.1] implies that the measure-theoretic entropy of $\hat{\rho}$ is strictly positive and that the probability measure proportional to $\hat{\rho}$ is an equilibrium state of f_c for the potential $-t_* \log|Df_c|$. That this measure is exact, and hence ergodic and mixing, is shown for example in [48]. Finally, the uniqueness of the equilibrium state is given by Ruelle's inequality and by [12, Theorem 6] in the real setting and [11, Theorem 8] in the complex setting.

The non-differentiability of $P_c^{\mathbb{C}}$ at $t = t_*$ follows from the existence of an equilibrium state of f_c for the potential $-t_* \log|Df_c|$, see [18, Corollary 1.3]. □

Remark 7.3. For completeness we show that for a parameter c as in the proof of Proposition A we have $\chi_{\inf}^{\mathbb{R}}(c) = \chi_{\mathrm{crit}}(c)/2$ and

$$\chi_{\inf}^{\mathbb{C}}(c) := \inf\left\{ \int \log|Df_c|\, d\mu \;\middle|\; \mu \in \mathscr{M}_c^{\mathbb{C}} \right\} = \chi_{\mathrm{crit}}(c)/2,$$

although this is not needed in the proof. For each invariant probability measure μ of f_c supported on J_c and every sufficiently large $t > 0$ we have

$$\int \log|Df_c|\, d\mu \geqslant \left(-P_c^{\mathbb{C}}(t) + h_\mu(f_c)\right)/t = \chi_{\mathrm{crit}}(c)/2 + h_\mu(f_c)/t \geqslant \chi_{\mathrm{crit}}(c)/2.$$

This proves $\chi_{\inf}^{\mathbb{C}}(c) \geqslant \chi_{\mathrm{crit}}(c)/2$. Together with Proposition 6.2 and with the inequality $\chi_{\inf}^{\mathbb{R}}(c) \geqslant \chi_{\inf}^{\mathbb{C}}(c)$, this implies $\chi_{\inf}^{\mathbb{R}}(c) = \chi_{\inf}^{\mathbb{C}}(c) = \chi_{\mathrm{crit}}(c)/2$.

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Appendix A. Multipliers of periodic orbits of period 3

This appendix is devoted to prove the following lemma, used in Section 4.2. The functions p and $\tilde{p}$ appearing in the following lemma are defined in Section 4.2.

Lemma A.1. *We have,*

$$\frac{\partial}{\partial c}\left|Df_c^3\big(p(c)\big)\right|\Big|_{c=-2} > \frac{\partial}{\partial c}\left|Df_c^3\big(\tilde{p}(c)\big)\right|\Big|_{c=-2}.$$

Proof. Notice that for $c = -2$

$$\mathcal{O} := \left\{2\cos(2\pi/7), 2\cos(4\pi/7), 2\cos(6\pi/7)\right\}$$

and

$$\widetilde{\mathcal{O}} := \left\{2\cos(2\pi/9), 2\cos(4\pi/9), 2\cos(8\pi/9)\right\}$$

are the only periodic orbits of minimal period 3 of f_{-2}. Since,

$$P_{-2,1}(0) \cap \mathbb{R} = \left[\alpha(-2), \tilde{\alpha}(-2)\right] = [-1, 1],$$

it follows that $x = 2\cos(4\pi/7)$ and $x = 2\cos(4\pi/9)$ are the only periodic points of period 3 of f_{-2} in $P_{-2,1}(0)$. On the other hand, the inequalities

$$2\cos(4\pi/7) < 0 < 2\cos(4\pi/9)$$

imply that $p(-2) = 2\cos(4\pi/7)$ and $\tilde{p}(-2) = 2\cos(4\pi/9)$ and that

$$Df_{-2}^3\big(p(-2)\big) > 0 > Df_{-2}^3\big(\tilde{p}(-2)\big).$$

Since both functions p and $\tilde{p}$ are real, the desired assertion is equivalent to,

$$\frac{\partial}{\partial c} Df_c^3\big(p(c)\big)\bigg|_{c=-2} > -\frac{\partial}{\partial c} Df_c^3\big(\tilde{p}(c)\big)\bigg|_{c=-2}. \tag{A.1}$$

Let π_0 be either one of the functions p, $f_c \circ p$, $f_c^2 \circ p$, $\tilde{p}$, $f_c \circ \tilde{p}$, or $f_c^2 \circ \tilde{p}$ and put

$$\pi_1(c) = f_c \circ \pi_0(c) \quad \text{and} \quad \pi_2(c) = f_c \circ \pi_1(c).$$

Then $f_c(\pi_2(c)) = \pi_0(c)$ and a direct computation shows that

$$D\pi_0 = -\frac{1 + 2\pi_2 + 4\pi_1\pi_2}{8\pi_0\pi_1\pi_2 - 1}.$$

Therefore, for each c in $\mathcal{P}_3(-2)$ we have,

$$\begin{aligned}&\big((D\pi_0)\pi_1\pi_2\big)(c)\\ &\quad = -\frac{\pi_1(c)\pi_2(c) + 2\pi_0(c)\pi_1(c) + 4\pi_0(c)\pi_2(c) - 2c\pi_1(c) - 4c\pi_0(c) - 4c\pi_2(c) + 4c^2}{8\pi_0(c)\pi_1(c)\pi_2(c) - 1}.\end{aligned}$$

Using the formula above and the formula above with π_0 replaced by π_1 and then by π_2, we obtain

$$\begin{aligned}\frac{\partial}{\partial c} Df_c^3\big(\pi_0(c)\big) &= 8D(\pi_0\pi_1\pi_2)(c)\\ &= -8\bigg(\frac{7(\pi_0(c)\pi_1(c) + \pi_1(c)\pi_2(c) + \pi_2(c)\pi_0(c))}{8\pi_0(c)\pi_1(c)\pi_2(c) - 1}\\ &\quad + \frac{-10c(\pi_0(c) + \pi_1(c) + \pi_2(c)) + 12c^2}{8\pi_0(c)\pi_1(c)\pi_2(c) - 1}\bigg).\end{aligned}$$

Thus, if for each j in $\{1, 2, 3\}$ we denote by σ_j (resp. $\tilde{\sigma}_j$) the elementary symmetric function of degree j in the elements of $\mathcal{O}$ (resp. $\widetilde{\mathcal{O}}$), then by the above equation with $\pi_0 = p$ (resp. $\pi_0 = \tilde{p}$) and $c = -2$ we obtain,

$$\begin{aligned}&\frac{\partial}{\partial c} Df_c^3\big(p(c)\big)\bigg|_{c=-2} = -8\frac{7\sigma_2 + 20\sigma_1 + 48}{8\sigma_3 - 1}\\ &\quad\bigg(\text{resp. } \frac{\partial}{\partial c} Df_c^3\big(\tilde{p}(c)\big)\bigg|_{c=-2} = -8\frac{7\tilde{\sigma}_2 + 20\tilde{\sigma}_1 + 48}{8\tilde{\sigma}_3 - 1}\bigg).\end{aligned}$$

To calculate these numbers, for a given an integer $n \geqslant 2$ let T_n be the n-th Chebyshev polynomial, so that for every real number θ we have

$$T_n\big(\cos(\theta)\big) = \cos(n\theta).$$

Notice that the zeros of the polynomial $T_4(x/2) - T_3(x/2)$ different from $x = 2$ are precisely the elements of $\mathcal{O}$. We thus have the identity

$$\frac{2T_4(x/2) - 2T_3(x/2)}{x - 2} = x^3 + x^2 - 2x - 1 = x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3.$$

So $\sigma_1 = -1$, $\sigma_2 = -2$, $\sigma_3 = 1$ and by the above

$$\frac{\partial}{\partial c} Df_c^3\big(p(c)\big)\Big|_{c=-2} = -16.$$

On the other hand, the zeros of the polynomial $T_5(x/2) - T_4(x/2)$ different from $x = 2$ and $x = -1$ are precisely the elements of $\widetilde{\mathcal{O}}$. Therefore we have the identity

$$\frac{2T_5(x/2) - 2T_4(x/2)}{(x-2)(x+1)} = x^3 - 3x + 1 = x^3 - \tilde{\sigma}_1 x^2 + \tilde{\sigma}_2 x - \tilde{\sigma}_3.$$

So $\tilde{\sigma}_1 = 0$, $\tilde{\sigma}_2 = -3$, $\tilde{\sigma}_3 = -1$ and

$$\frac{\partial}{\partial c} Df_c^3\big(\tilde{p}(c)\big)\Big|_{c=-2} = 24.$$

This proves (A.1) and completes the proof of the lemma. □

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