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Rapid Convergence to Frequency for Substitution Tilings of the Plane

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Abstract: This paper concerns self-similar tilings of the Euclidean plane. We consider the number of occurrences of a given tile in any domain bounded by a Jordan curve. For a large class of self-similar tilings, including many well-known examples, we give estimates of the oscillation of this number of occurrences around its average frequency times the total number of tiles in the domain, which depend only on the Jordan curve.

1. Introduction

Quasicrystals are alloys that show long range order but possess symmetries that prevent them from being crystals. From their spectacular experimental realization in the early 80's [SBGC84] to their very recent discovery as natural objects in the Kamtchatka mountains [BSYL09], quasicrystals have been the subject of very active research, whose application and interest extend far beyond the scope of solid state physics. The first examples observed were rapidly quenched alloys of Aluminum and Manganese exhibiting icosahedral symmetry. In these quasicrystals, atoms are known to appear with a given frequency. This means that for a large ball B of radius R, the ratio of the number N_s of atoms inside B corresponding to a specific atomic element s (for example Aluminum in the first known cases) to the total number N of atoms in the ball B tends to a limit ν_s as R tends to infinity:

$$\lim_{R \to +\infty} \frac{N_s}{N} = \nu_s.$$

For such examples, the following questions arise naturally:

(1) to estimate the oscillation of the ratio N_s/N around its limit ν_s, that is to say, to give upper bounds for the speed of convergence to 0 of the ratio $|N_s/N - \nu_s|$ as R goes to infinity;
(2) to extend these estimates to domains whose closure are not Euclidean balls.

We now make these questions more concrete by using aperiodic tilings of the Euclidean space to model quasicrystals (see [Bel03]), for details and precise statements of the results, see Sect. 2. Consider a tiling $\mathcal{T}$ of d-dimensional Euclidean space $\mathbb{R}^d$ made with isometric copies of a finite set of tiles $\{p_1, \ldots, p_n\}$, where the p_i's are homeomorphic to a closed ball in $\mathbb{R}^d$. Let V be a (large) set in $\mathbb{R}^d$ also homeomorphic to a closed ball and let ∂V be its boundary. For each $i \in \{1, \ldots, n\}$, we let $\mathcal{N}(V, \mathcal{T}, p_i)$ denote the number of isometric copies of p_i in V, $\mathcal{N}(V, \mathcal{T})$ the number of tiles of $\mathcal{T}$, and $\mathcal{L}(\partial V, \mathcal{T})$ the number of tiles of $\mathcal{T}$ that intersect ∂V. If V is a ball, then it is well-known that there are tilings such that for each i in $\{1, \ldots, n\}$, the ratio $\mathcal{N}(V, \mathcal{T}, p_i)/\mathcal{N}(V, \mathcal{T})$ tends to a well-defined limit ν_i as the radius of V tends to infinity (see for instance [GS87,LP03,Sol97]). In fact, this is usually a consequence of the unique ergodicity of the associated dynamical system (see for instance [LMS02]). However, in general, very little is known concerning an upper bound for the quantity $|\mathcal{N}(V, \mathcal{T}, p_i) - \nu_i.\mathcal{N}(V, \mathcal{T})|$, especially when V is not a Euclidean ball (see [APC,LP03] for the case when V equals a ball or box). In particular, for an aperiodic tiling, can we hope to obtain an estimate as strong as the one we can formulate for periodic tilings which reads:

$$(*) \quad |\mathcal{N}(V, \mathcal{T}, p_i) - \nu_i\mathcal{N}(V, \mathcal{T})| \leq K\mathcal{L}(\partial V, \mathcal{T})$$

for some $K > 0$?

In this paper, we answer this question in the affirmative for a large class of self-similar tilings in two dimensions. Self-similar tilings, which are probably the most studied examples of aperiodic tilings (see for instance [AP98,CS06,Rad94,Rob04,Sol97] and references therein, examples are given in Sect. 5), are associated with a (substitution) primitive square matrix M having non-negative integer elements. By Perron-Frobenius theory (see for instance [HJ90,Sen81], see also Subsect. 3.3 for more details), we know that there exists a largest positive real eigenvalue μ, the remainder of the spectrum being in a ball centered at 0 with radius smaller than μ. We denote by $r(M)$ the modulus of the second largest eigenvalue of M, that is,

$$r(M) = \max\{|\eta| \mid \eta \neq \mu \text{ is an eigenvalue of } M\}.$$

The simple condition $r(M) < \sqrt{\mu}$ ensures that the above estimate $(*)$ holds true for the associated self-similar tiling. If $r(M) = \sqrt{\mu}$ and if all eigenvalues with modulus $r(M)$ are semi-simple (their algebraic and geometric multiplicities coincide, see for instance [IJ90] for more details), then the estimate $(*)$ has to be relaxed to

$$(**) \quad |\mathcal{N}(V, \mathcal{T}, p_i) - \nu_i\mathcal{N}(V, \mathcal{T})| \leq K\mathcal{L}(\partial V, \mathcal{T}) \ln \mathcal{L}(\partial V, \mathcal{T})$$

for some $K > 0$. Some explicit examples of self-similar 2-dimensional tilings satisfying all these conditions will be given at the end of the paper.

These results are more explicitly stated in the next section. Ideas for the proofs have been inspired by J. Peyrière's 1986 paper [Pey86] and can be related to B. Adamcezswki's work [Ada04].

2. Definitions and Main Result

2.1. Definitions. Let Λ be a closed subset of the Euclidean plane $\mathbb{R}^2$. A *tiling* of Λ is a countable collection $\mathcal{T} = (t_j)_{j \in J}$ of closed subsets of Λ that cover Λ and have pairwise disjoint interiors. The sets t_j are called *tiles* and, in this paper, all tiles are supposed to be homeomorphic to the closed unit ball in $\mathbb{R}^2$, see Sect. 5 for examples.

Let $\mathcal{E}$ be a group of isometries on the plane that contains the group of translations. A tile q is $\mathcal{E}$-*equivalent* to a tile p (or is a $\mathcal{E}$-*copy* of p) if q is the image of p by an isometry in $\mathcal{E}$. Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a finite collection of tiles. A tiling $\mathcal{T}$ of $\Lambda \subseteq \mathbb{R}^2$ is $\mathcal{E}$-*generated* by $\mathcal{P}$ if every tile in $\mathcal{T}$ is $\mathcal{E}$-equivalent to some tile in $\mathcal{P}$. The set of all tilings of Λ that are $\mathcal{E}$-generated by $\mathcal{P}$ is denoted by $\Omega_{\mathcal{E},\mathcal{P}}(\Lambda)$. When $\Lambda = \mathbb{R}^2$, we write $\Omega_{\mathcal{E},\mathcal{P}}$ instead of $\Omega_{\mathcal{E},\mathcal{P}}(\mathbb{R}^2)$.

Let $\lambda > 1$. Given a subset $U \subseteq \mathbb{R}^2$, let $\lambda U := \{\lambda x \mid x \in U\}$. A *substitution rule* (with dilation factor λ) is a collection $\mathcal{S} = (\mathcal{S}_i)_{i=1}^n$, where $\mathcal{S}_i$ belongs to $\Omega_{\mathcal{E},\mathcal{P}}(\lambda p_i)$ for all $i \in \{1, \ldots, n\}$. Let $\mathcal{S}$ be a substitution rule. On one hand, the dilation by λ induces a natural map from $\Omega_{\mathcal{E},\mathcal{P}}(\Lambda)$ to $\Omega_{\mathcal{E},\lambda\mathcal{P}}(\lambda\Lambda)$, where $\lambda\mathcal{P} = \{\lambda p \mid p \in \mathcal{P}\}$. On the other hand, $\mathcal{S}$ induces a map from $\Omega_{\mathcal{E},\lambda\mathcal{P}}(\lambda\Lambda)$ to $\Omega_{\mathcal{E},\mathcal{P}}(\lambda\Lambda)$, which is defined by subdividing the tiles of a tiling in $\Omega_{\mathcal{E},\lambda\mathcal{P}}(\lambda\Lambda)$ according to the substitution rule. For details see SubSect. 3.2. The composition of these two maps yields a map $\mathcal{I}_{\mathcal{S},\Lambda} : \Omega_{\mathcal{E},\mathcal{P}}(\Lambda) \to \Omega_{\mathcal{E},\mathcal{P}}(\lambda\Lambda)$. If $\Lambda = \mathbb{R}^2$, then $\mathcal{I}_\mathcal{S} := \mathcal{I}_{\mathcal{S},\mathbb{R}^2}$ is a self-map of $\Omega_{\mathcal{E},\mathcal{P}}$ and is referred to as the *substitution map*. A tiling $\mathcal{T}$ is called *self-similar* if it is a periodic point for $\mathcal{I}_\mathcal{S}$. A tiling $\mathcal{T}$ in $\Omega_{\mathcal{E},\mathcal{P}}$ is *admissible* for $\mathcal{S}$ if it belongs to

$$\Omega_\mathcal{S} := \bigcap_{k \geq 0} \mathcal{I}_\mathcal{S}^k(\Omega_{\mathcal{E},\mathcal{P}}).$$

The *substitution matrix* associated with $\mathcal{S}$ is the n-by-n matrix $M_\mathcal{S} = (m_{i,j})_{i,j}$, where for each $i, j \in \{1, \ldots, n\}$, the coefficient $m_{i,j}$ is the number of $\mathcal{E}$-copies of p_i in $\mathcal{S}_j$. Recall that a matrix M is *primitive* if there exists $n > 0$ such that all the elements of M^n are positive. The substitution rule $\mathcal{S}$ is *primitive* if its substitution matrix is primitive. In this paper, all the substitution rules considered will be primitive.

2.2. Main result. Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a finite set of tiles, $\mathcal{S}$ a substitution rule with dilation factor $\lambda > 1$ and $\mathcal{T}$ an admissible tiling for $\mathcal{S}$. Given a Jordan curve Γ in $\mathbb{R}^2$ bounding a topological closed disk Δ, we let $\mathcal{L}(\Gamma, \mathcal{T})$ denote the number of tiles of $\mathcal{T}$ that intersect Γ, $\mathcal{N}(\Delta, \mathcal{T})$ be the number of tiles of $\mathcal{T}$ included in Δ, and for each $i \in \{1, \ldots, n\}$, we define $\mathcal{N}(\Delta, \mathcal{T}, p_i)$ to be the number of $\mathcal{E}$-copies of p_i contained in Δ.

The following theorem, which constitutes the main result of this paper, provides estimates on the oscillation of the number of occurrences $\mathcal{N}(\Delta, \mathcal{T}, p_i)$ around an average frequency, which only depends on the Jordan curve Γ bounding Δ.

Theorem 2.1. *Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a finite collection of tiles, $\mathcal{S}$ a substitution rule with dilation factor $\lambda > 1$ and primitive substitution matrix $M_\mathcal{S}$. There exist positive numbers $v_1, \ldots, v_n$ depending only on the substitution matrix $M_\mathcal{S}$ that satisfy:*

(i) *If $r(M_\mathcal{S}) < \lambda$, then there exists $K > 0$ such that, for every Jordan curve Γ bounding a topological closed disk Δ and every tiling $\mathcal{T}$ in $\Omega_\mathcal{S}$, we have:*

$$|\mathcal{N}(\Delta, \mathcal{T}, p_i) - v_i \mathcal{N}(\Delta, \mathcal{T})| \leq K\mathcal{L}(\Gamma, \mathcal{T}).$$

(ii) *If $r(M_\mathcal{S}) = \lambda$ and all the eigenvalues with modulus λ are semi-simple (that is, their algebraic and geometric multiplicities coincide), then there exists $K > 0$ such that, for every Jordan curve Γ bounding a topological closed disk Δ and every tiling $\mathcal{T}$ in $\Omega_\mathcal{S}$, we have:*

$$|\mathcal{N}(\Delta, \mathcal{T}, p_i) - v_i \mathcal{N}(\Delta, \mathcal{T})| \leq K\mathcal{L}(\Gamma, \mathcal{T}) \ln \mathcal{L}(\Gamma, \mathcal{T})).$$

Remark 2.2. Notice that Theorem 2.1 is given for a quite general setting:

(1) On one hand, we do not require the tiles in $\mathcal{P}$ to be polygons, they need not even be disks with piece-wise smooth boundaries. In fact, the boundary of the tiles could be Jordan curves with infinite length and positive Hausdorff dimension, as is the case for Rauzy tilings that we shall discuss later.
(2) On the other hand, we do not require the standard finite pattern condition used in tiling theory (see [KP00]). This allows us to deal with self-similar tilings which have "fault lines" along which tiles can slide past one another. This was the case for the self-affine tilings studied by N. Priebe Frank and L. Sadun [PFS09].

Remark 2.3. For a given substitution, the estimates given by Theorem 2.1 may depend on the group $\mathcal{E}$ being considered. This is the case of Penrose tilings, see Sect. 5 for details.

Remark 2.4. Theorem 2.1 can be easily extended to colored tilings (for an example, see the square tilings in Sect. 5).

3. The Tools

3.1. Jordan curves and locally finite tilings. In this section, we provide some simple combinatorial estimates for the number of tiles of a tiling that intersect a Jordan curve. First we must define some notation. Denote by $B_x(r)$ the closed ball of radius r around x in $\mathbb{R}^2$. Given a subset $U \subseteq \mathbb{R}^2$, define

$$r_{\min}(U) := \sup\{r > 0 \mid \text{there exists } x \in U \text{ s.t. } B_x(r) \subseteq U\}.$$

To a tiling $\mathcal{T}$ of $\mathbb{R}^2$, we associate

$$r_{\mathcal{T}} = \inf\{r_{\min}(t) \mid t \in \mathcal{T}\}$$

and

$$R_{\mathcal{T}} = \sup\{\text{diam}(t) \mid t \in \mathcal{T}\}/2.$$

The following lemma shows that if $r_{\mathcal{T}}$ is positive and $R_{\mathcal{T}}$ is finite, then there is a uniform bound on the number of tiles intersected by balls of a prescribed radius.

Lemma 3.1. *Suppose that $\mathcal{T}$ is a tiling with $R_{\mathcal{T}} < +\infty$ and $r_{\mathcal{T}} > 0$. Then every ball of radius $2R_{\mathcal{T}}$ intersects at most $K_{\mathcal{T}} := \lfloor 16R_{\mathcal{T}}^2 r_{\mathcal{T}}^{-2} \rfloor$ tiles of $\mathcal{T}$, where $\lfloor - \rfloor$ stands for the integer part.*

Proof. Fix $x \in \mathbb{R}^2$. It is clear that each tile of $\mathcal{T}$ intersecting $B_x(2R_{\mathcal{T}})$ is included in $B_x(4R_{\mathcal{T}})$. Since each one of these tiles contains a ball of radius $r_{\mathcal{T}}$, comparing area yields $r_{\mathcal{T}}^2 \pi N \leq 16\pi R_{\mathcal{T}}^2$, where N is the number of tiles intersecting $B_x(2R_{\mathcal{T}})$ and the conclusion follows. □

By virtue of the previous lemma, we say that a tiling $\mathcal{T}$ is *locally finite* if $R_{\mathcal{T}} < +\infty$ and $r_{\mathcal{T}} > 0$. The next result provides an estimate for the diameter of a simple curve in terms of the number of tiles of a locally finite tiling that the curve intersects.

To simplify notation, it is convenient to identify curves with their images. Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a simple curve. The identification induces an order on the image: Let y, y' in γ with $y = \gamma(s)$ and $y' = \gamma(t)$. If γ is open, then $y \leq y'$ if and only if $0 \leq s \leq t \leq 1$; if γ is closed, then $y \leq y'$ if and only if $0 \leq s \leq t < 1$.
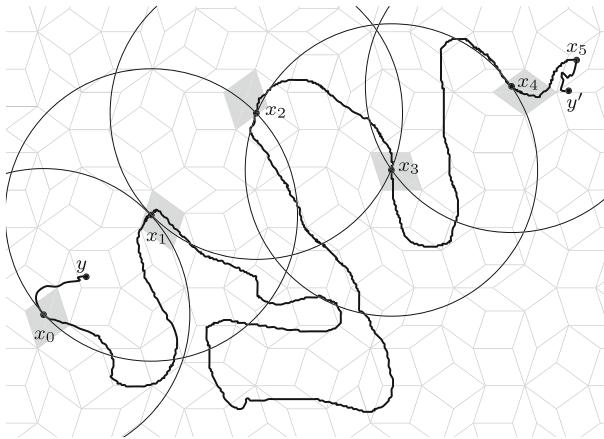
Fig. 1. Proof of Lemma 3.2: Construction of the sequence $x_1, \ldots, x_l$ and $t_{j(1)}, \ldots, t_{j(l)}$

Lemma 3.2. *Let $\mathcal{T}$ be a locally finite tiling and γ be a simple curve. Then,*

$$\mathrm{diam}(\gamma) \leq 2R_{\mathcal{T}} \mathcal{L}(\gamma, \mathcal{T}),$$

where $\mathcal{L}(\gamma, \mathcal{T})$ denotes the number of tiles that γ intersects.

Proof. By compactness, there exist $x, x' \in \gamma$ with $x \leq x'$ and $d(x, x') = \mathrm{diam}(\gamma)$. We construct a sequence of points $(x_i)_{i=0}^{\ell}$ as follows. Fix $\varepsilon > 0$ and set $x_0 = x$. To construct x_1, if x' belongs to $B_{x_0}(2R_{\mathcal{T}} + \varepsilon)$, then set $\ell = 1$ and $x_1 = x'$. If x' does not belong to $B_{x_0}(2R_{\mathcal{T}} + \varepsilon)$, then define x_1 by

$$x_1 := \max\{z \in \gamma \mid z \in B_{x_0}(2R_{\mathcal{T}} + \varepsilon)\}.$$

It is clear that $d(x_1, x_0) = 2R_{\mathcal{T}} + \varepsilon$. This construction can be extended inductively to obtain a sequence $x = x_0 < x_1 < \cdots < x_{\ell-1} < x_\ell = x'$ in γ such that for all $k \in \{1, \ldots, \ell - 1\}$:

- $d(x_{k-1}, x_k) = 2R_{\mathcal{T}} + \varepsilon$,
- $\{z \in \gamma \mid x_k \leq z \leq x'\} \cap \cup_{m=0}^{k-1} B_{x_m}(2R_{\mathcal{T}} + \varepsilon) = \{x_k\}$,

and such that $d(x_{\ell-1}, x_\ell) \leq 2R_{\mathcal{T}} + \varepsilon$ (see Fig. 1).

For each $k \in \{0, \ldots, \ell-1\}$, choose a tile $t_{j(k)}$ in $\mathcal{T}$ that contains x_k. Since $d(x_k, x_{k'}) \geq 2R_{\mathcal{T}} + \varepsilon$ for all k, k' in $\{0, \ldots, \ell - 1\}$, it follows that $t_{j(k)}$ is not the same tile as $t_{j(k')}$ unless $k = k'$. Hence, the number $\mathcal{L}(\gamma, \mathcal{T})$ of tiles in $\mathcal{T}$ that intersect γ is at least ℓ (see Fig. 1). Moreover, it is easy to check that $\ell(2R_{\mathcal{T}} + \varepsilon) \geq d(x, x') = \mathrm{diam}(\gamma)$. Thus, using the estimate for $\mathcal{L}(\gamma, \mathcal{T})$ we get

$$\mathrm{diam}(\gamma) \leq (2R_{\mathcal{T}} + \varepsilon)\mathcal{L}(\gamma, \mathcal{T}),$$

and the conclusion can be obtained by letting ε go to 0. □

Given a Jordan curve Γ and two tilings $\mathcal{T}$ and $\mathcal{T}'$, the next lemma compares the number of tiles of $\mathcal{T}$ that Γ intersects with the number of tiles of $\mathcal{T}'$ that Γ intersects.

Lemma 3.3. *Let Γ be a Jordan curve and $\mathcal{T}$ be a locally finite tiling. Then, for every locally finite tiling $\mathcal{T}'$ such that $R_{\mathcal{T}'} > R_{\mathcal{T}}$ and $K_{\mathcal{T}'} < \mathcal{L}(\Gamma, \mathcal{T}') \leq \mathcal{L}(\Gamma, \mathcal{T})$, we have:*

$$\mathcal{L}(\Gamma, \mathcal{T}') \leq (2K_{\mathcal{T}'} + 1) \frac{R_{\mathcal{T}}}{R_{\mathcal{T}'}} \mathcal{L}(\Gamma, \mathcal{T}),$$

where $K_{\mathcal{T}'}$ is the constant defined in Lemma 3.1.

Proof. For every point $y \in \mathbb{R}^2$, we define

$$C_y := \{t \in \mathcal{T}' \mid t \cap B_{2R'_{\mathcal{T}}}(y) \neq \emptyset\} \quad \text{and} \quad \widehat{C}_y := \cup_{t \in C_y} t.$$

We construct a collection $\mathcal{Y} = \{y_i\}_{i=1}^p$ (with p to be determined) of points in Γ such that the collection $\{\widehat{C}_{y_i}\}_{i=1}^p$ covers Γ. Fix any point of Γ as y_1. Now suppose that $y_1, \ldots, y_j$ have been chosen such that, for all $i \in \{1, \ldots, j\}$ and $k \in \{1, \ldots, i-1\}$, the point y_i does not belong to $\widehat{C}_{y_k}$. There are two cases to consider. Either the sets $\{\widehat{C}_{y_i}\}_{i=0}^j$ cover Γ, in which case we set $p = j$ and the construction is completed; or they do not cover Γ, in which case we choose any point of $\Gamma \setminus \cup_{i=0}^j \cup_{t \in C_{y_i}} t$ to be y_{j+1} and continue iterating the construction. Observe that $\mathcal{L}(\Gamma, \mathcal{T}')$ is finite because $\mathcal{T}'$ is locally finite and Γ is compact, and in each iteration, we add at least one tile of $\mathcal{T}'$ that intersects Γ to the area covered by $\{\widehat{C}_{y_i}\}_{i=0}^j$. Hence, the construction stops in finitely many steps.

On one hand, by Lemma 3.1, each C_{y_i} contains at most $K_{\mathcal{T}'}$ tiles of $\mathcal{T}'$. It follows that

$$\mathcal{L}(\Gamma, \mathcal{T}') \leq p K_{\mathcal{T}'}, \tag{1}$$

and since $K_{\mathcal{T}'} < \mathcal{L}(\Gamma, \mathcal{T}')$, we deduce that $p \geq 2$.

On the other hand, fix $i \in \{1, \ldots, p\}$. Hence, there exists $j \in \{1, \ldots, p\}$ such that y_j, which belongs to Γ, does not belong to $B_{y_i}(R_{\mathcal{T}'} - R_{\mathcal{T}})$. Suppose that $y_i < y_j$ (the other case is analogous), then there is a point $y_i' > y_i$ in Γ with $d(y_i, y_i') = R_{\mathcal{T}'} - R_{\mathcal{T}}$ such that the arc $[y_i, y_i']_\Gamma = \{z \in \Gamma \mid y_i \leq z \leq y_i'\}$, which joins y_i and y_i', is contained by the ball $B_{y_i}(R_{\mathcal{T}'} - R_{\mathcal{T}})$. Since i was arbitrary, using Lemma 3.2 yields

$$R_{\mathcal{T}'} - R_{\mathcal{T}} \leq \text{diam}([y_i, y_i']_\Gamma) \leq 2R_{\mathcal{T}} \mathcal{L}([y_i, y_i']_\Gamma, \mathcal{T})$$

for all $i \in \{1, \ldots, p\}$. Combining all these inequalities, we obtain

$$p(R_{\mathcal{T}'} - R_{\mathcal{T}}) \leq 2R_{\mathcal{T}} \sum_{i=1}^p \mathcal{L}([y_i, y_i']_\Gamma, \mathcal{T}). \tag{2}$$

From the construction of $\mathcal{Y}$, it is clear that $d(y_i, y_j) > 2R_{\mathcal{T}'}$ for all $i < j$, and, in particular, the collection $\mathcal{B} = \{B_{y_i}(R_{\mathcal{T}'} - R_{\mathcal{T}})\}_{i=1}^p$ is pairwise disjoint. Moreover, the distance between two different balls in $\mathcal{B}$ is greater than $2R_{\mathcal{T}}$. It follows that no tile of $\mathcal{T}$ may intersect more than one ball in $\mathcal{B}$. Hence, no tile of $\mathcal{T}$ may intersect more than one arc $[y_i, y_i']_\Gamma$. Thus, combining (2) and (1) we get

$$\mathcal{L}(\Gamma, \mathcal{T}') \leq 2K_{\mathcal{T}'} \frac{R_{\mathcal{T}}}{R_{\mathcal{T}'} - R_{\mathcal{T}}} \mathcal{L}(\Gamma, \mathcal{T}). \tag{3}$$

To finish the proof, fix $c > 0$ arbitrarily and consider the following two cases. First, suppose that $R_{\mathcal{T}'}/R_{\mathcal{T}} \le 1 + c$. Since $\mathcal{L}(\Gamma, \mathcal{T}') \le \mathcal{L}(\Gamma, \mathcal{T})$, it follows that

$$\mathcal{L}(\Gamma, \mathcal{T}') \le (1 + c)\frac{R_{\mathcal{T}}}{R_{\mathcal{T}'}}\mathcal{L}(\Gamma, \mathcal{T}). \tag{4}$$

Now suppose that $R_{\mathcal{T}'}/R_{\mathcal{T}} > 1 + c$. It is not difficult to check that

$$(1 + c)(R_{\mathcal{T}'} - R_{\mathcal{T}}) > c R_{\mathcal{T}'}.$$

Replacing this inequality in (3) we get

$$\mathcal{L}(\Gamma, \mathcal{T}') \le 2K_{\mathcal{T}'}\frac{R_{\mathcal{T}}}{R_{\mathcal{T}'}}\left(1 + \frac{1}{c}\right)\mathcal{L}(\Gamma, \mathcal{T}). \tag{5}$$

Combining (5) and (4) yields

$$\mathcal{L}(\Gamma, \mathcal{T}') \le \max\left\{1 + c, 2K_{\mathcal{T}'}\left(1 + \frac{1}{c}\right)\right\}\frac{R_{\mathcal{T}}}{R_{\mathcal{T}'}}\mathcal{L}(\Gamma, \mathcal{T}).$$

An easy computation shows that the last bound is optimal when $c = 2K_{\mathcal{T}'}$ and the conclusion follows. □

3.2. Hierarchical sequences. In this subsection, we recall the concept of "hierarchical sequence" of a tiling. This notion has been used in the context of self-similar tilings satisfying the standard finite pattern condition by many authors, see for instance [KP00]. Here, we extend its use to admissible tilings.

Let $\mathcal{S}$ be a substitution rule with dilation factor λ. For each $l \in \mathbb{N}$, let $\lambda^l \mathcal{P} = \{\lambda^l p \mid p \in \mathcal{P}\}$ and consider the map $D_l : \Omega_{\mathcal{E},\lambda^{l-1}\mathcal{P}} \to \Omega_{\mathcal{E},\lambda^l\mathcal{P}}$ defined by $D_l(\mathcal{T}) = \lambda\mathcal{T}$. Also consider the *decomposition* maps $S_l : \Omega_{\mathcal{E},\lambda^l\mathcal{P}} \to \Omega_{\mathcal{E},\lambda^{l-1}\mathcal{P}}$ defined by

$$S_l(\mathcal{T}) = \{g(\lambda^{l-1}\mathcal{S}_i) \mid g \in \mathcal{E}, i \in \{1, \ldots, n\}, g(\lambda^l p_i) \in \mathcal{T}\}$$

for each $l \in \mathbb{N}^*$. By definition, $\mathcal{I}_{\mathcal{S}} = S_1 \circ D_1$.

It is not difficult to check that $\mathcal{I}_{\mathcal{S}}$ is onto when restricted to $\Omega_{\mathcal{S}}$. This implies that for each admissible tiling $\mathcal{T}$, there is a sequence $(\mathcal{T}^l)_{l\in\mathbb{N}}$ of tilings, called a *hierarchical sequence* of $\mathcal{T}$, such that $\mathcal{T}^0 = \mathcal{T}$ and $\mathcal{T}^{l-1} = S_l(\mathcal{T}^l)$ for all $l \in \mathbb{N}$, that is, each tile of $\mathcal{T}^l$ can be decomposed into tiles of $\mathcal{T}^{l-1}$ according to the substitution rule $\lambda^{l-1}\mathcal{S}$. This sequence is constructed as follows: Set $\mathcal{T}^0 = \mathcal{T}$ and, for each $l > 0$, inductively choose $\bar{\mathcal{T}}^l \in I_{\mathcal{S}}^{-1}(\bar{\mathcal{T}}^{l-1})$ and then set $\mathcal{T}^l = \lambda^l\bar{\mathcal{T}}^l$. It is not difficult to check that $\lambda^{l-1}S_1(\lambda^{1-l}\mathcal{T}') = S_l(\mathcal{T}')$ for every $\mathcal{T}'$ in $\Omega_{\mathcal{E},\lambda^l\mathcal{P}}$. It follows that $\mathcal{T}^{l-1} = S_l(\mathcal{T}^l)$ for all $l \in \mathbb{N}$.

Remark 3.4. Notice that all tilings in $\Omega_{\mathcal{S}}$ are locally finite and that for each tiling $\mathcal{T}$ and each $l > 0$, $\mathcal{T}^l$ is also locally finite and satisfies:

$$r_{\mathcal{T}^l} = \lambda^l r_{\mathcal{T}}, \quad R_{\mathcal{T}^l} = \lambda^l R_{\mathcal{T}}, \text{ and thus } K_{\mathcal{T}^l} = K_{\mathcal{T}}.$$

Proposition 3.5. *Let $\mathcal{T}$ be an admissible tiling for $\mathcal{S}$. For every topological closed disk Δ there is a finite collection $\Delta_0, \ldots, \Delta_{m-1}$ of closed subsets of Δ such that*:

(i) $\cup_{l=0}^{l=m-1}\Delta_i = \cup_{j\in J_\Delta}t_j$ *where* $J_\Delta = \{j \mid t_j \subset \Delta\}$.

(ii) *All Δ_l's have pairwise disjoint interiors.*

(iii) *For each $l \in \{0, \ldots, m-1\}$, Δ_l is a union of tiles in $\mathcal{T}^l$, which does not contain a tile in $\mathcal{T}^{l+1}$.*

(iv) *Δ does not contain a tile of $\mathcal{T}^m$.*

The collection $\Delta_0, \ldots, \Delta_{m-1}$ is called a hierarchical decomposition of the closed disk Δ. Moreover, if Γ is the Jordan curve bounding Δ, then

$$\lambda^{m-l-1} \leq \frac{R_{\mathcal{T}}}{r_{\mathcal{T}}} \mathcal{L}(\Gamma, \mathcal{T}^l) \quad and \quad \mathcal{N}(\Delta_l, \mathcal{T}^l) \leq \|M\|_1 \mathcal{L}(\Gamma, \mathcal{T}^{l+1}) \tag{6}$$

for all $l \in \{0, \ldots, m-1\}$, where $\|M\|_1$ is the maximum absolute column sum of the substitution rule M.

Proof. Choose m to be the smallest integer such that no tile in $\mathcal{T}^m$ is included in Δ and set

$$P_{m-1} = \{t \in \mathcal{T}^{m-1} \mid t \subseteq \Delta\} \quad \text{and} \quad \widehat{\Delta}_{m-1} = \cup_{t \in P_{m-1}} t.$$

Applying the recursion

$$P_{l-1} = \{t \in \mathcal{T}^{l-1} \mid t \subseteq \overline{\Delta \setminus \widehat{\Delta}_l}\} \quad \text{and} \quad \widehat{\Delta}_{l-1} = \widehat{\Delta}_l \cup (\cup_{t \in P_{l-1}} t),$$

for $l \in \{1, \ldots, m-1\}$, we obtain a sequence of sets $P_0, \ldots, P_{m-1}$ and $\widehat{\Delta}_0, \ldots, \widehat{\Delta}_{m-1}$, and it is straightforward to check that the sequence $\Delta_l = \cup_{t \in P_l} t$ for all $l \in \{0, \ldots, m-1\}$ satisfies properties (i) to (iv) (Fig. 2).

To check the first inequality in (6), observe that, on one hand, Lemma 3.2 implies $\mathrm{diam}(\Delta) \leq 2R_{\mathcal{T}^l}\mathcal{L}(\Gamma, \mathcal{T}^l)$. On the other hand, since Δ contains a tile of $\mathcal{T}^{m-1}$, it follows that $2r_{\mathcal{T}^m} \leq \mathrm{diam}(\Delta)$. The inequality is now obtained by replacing $R_{\mathcal{T}^l} = \lambda^l R_{\mathcal{T}}$ and $r_{\mathcal{T}^{m-1}} = \lambda^{m-1} r_{\mathcal{T}}$ in the previous inequalities.

It remains to check the second inequality in (6). From the construction of the P_l's, it is clear that each tile in P_l is contained in a tile of $\mathcal{T}^{l+1}$ that meets Γ (which is the boundary of Δ). Since each tile of $\mathcal{T}^{l+1}$ is subdivided in tiles of $\mathcal{T}^l$ according to the substitution rule, it follows that

$$\mathcal{N}(\Delta_l, \mathcal{T}^l) \leq \|M\|_1 \mathcal{L}(\Gamma, \mathcal{T}^{l+1}).$$

3.3. Perron-Frobenius theory. We now recall the basic Perron-Frobenius theory that we will need in the sequel. For proofs see [HJ90, Chap. 8]. Let $M = M_{\mathcal{S}}$ be the substitution matrix associated with $\mathcal{S}$. Since $\mathcal{S}$ is primitive, the matrix M is a primitive matrix with non-negative integer coefficients. The Perron-Frobenius Theorem states that the largest real eigenvalue $\mu > 0$ of M, which is called the *Perron eigenvalue*, is simple and greater than one. Moreover, there exists a right eigenvector $v^T = (v_1, \ldots, v_n)$ and a left eigenvector $w = (w_1, \ldots, w_n)$ such that v and w have positive coefficients and $\langle v, w \rangle = 1$. We denote by $r(M)$ the modulus of the second largest eigenvalue of M, that is,

$$r(M) = \max\{|\eta| \mid \eta \neq \mu \text{ is an eigenvalue of } M\}.$$

Recall that an eigenvalue is called *semi-simple* if its algebraic multiplicity is equal to its geometric multiplicity. The following proposition, or more precisely the corollary below, will be crucial for the proof of our main result. The first part is a well-known consequence of the Perron-Frobenius Theorem, see for instance [HJ90, Theorem 8.5.1].

The second part is less well-known, but can be easily deduced from the first part by using the Jordan Canonical Form of M after recalling that the Jordan blocks associated with semi-simple eigenvalues are diagonal. For $l > 0$ and $i, j \in \{1, \ldots, n\}$, $m^l_{i,j}$ denotes the (i, j)-element of the matrix M^l.

Proposition 3.6. *For every $\rho > r(M)$ there exists $K = K(\rho) > 0$ such that*

$$|m^l_{i,j} - v_i \mu^l w_j| \le K\rho^l, \tag{7}$$

for all $l > 0$. Moreover, if the eigenvalues of modulus $r(M)$ are semi-simple, then there exists $K > 0$ such that (7) holds with $\rho = r(M)$.

Corollary 3.7. *Let $\nu_i = v_i / \sum_j v_j$ for all $i \in \{1, \ldots, n\}$. Then, for all $\rho > r(M)$, there exists a constant $K > 0$ such that for all $i \in \{1, \ldots, n\}$,*

$$|m^l_{i,j} - \nu_i \sum_{k=1}^n m^l_{k,j}| \le K\rho^l, \tag{8}$$

for all $l > 0$. Moreover, if all the eigenvalues with modulus $r(M)$ are semi-simple, then there exists a constant $K > 0$ such that (8) holds with $\rho = r(M)$.

Proof. An easy computation shows

$$|m^l_{i,j} - \nu_i \sum_{k=1}^n m^l_{k,j} \le |m^l_{i,j} - v_i \mu^l w_j| + \frac{v_i}{\sum_j v_j} \sum_{k=1}^n |v_k \mu^l w_j - m^l_{k,j}|.$$

The conclusion now follows from applying Proposition 3.6 twice.   □
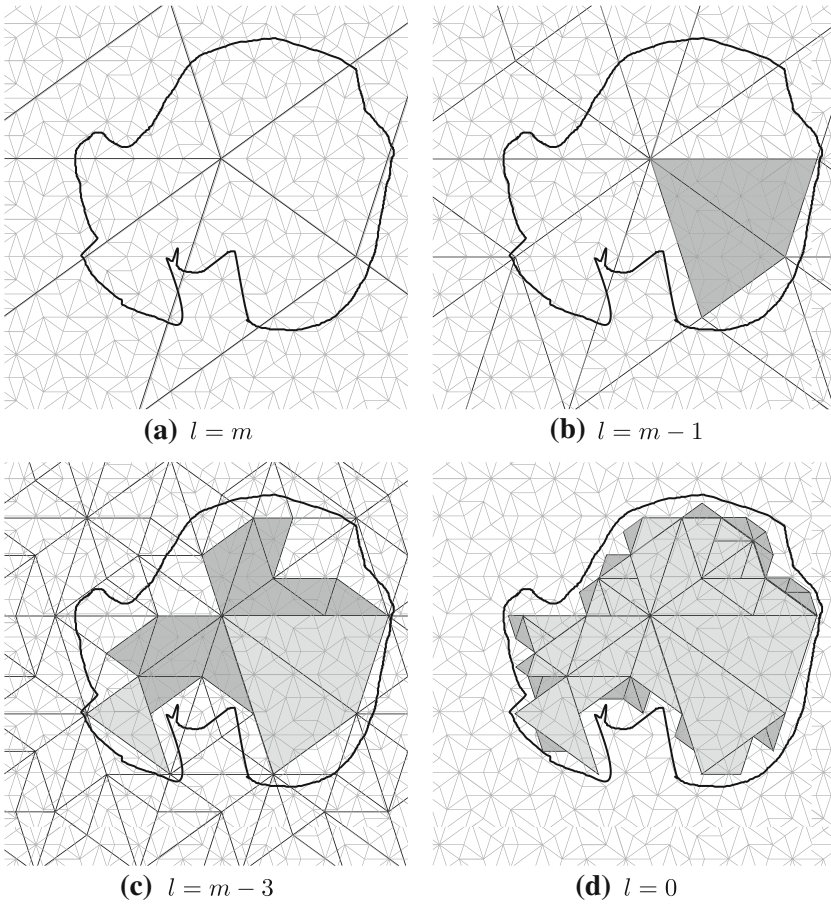
4. Proof of the Main Result

Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a finite set of tiles and $\mathcal{S}$ a primitive substitution rule with dilation factor $\lambda > 1$. Suppose that $\mathcal{T}$ is an admissible tiling for $\mathcal{S}$ and Γ is a Jordan curve in $\mathbb{R}^2$ bounding a closed disk Δ.

The idea of the proof is as follows. Fix $i \in \{1, \ldots, n\}$ and let ν_i be defined as in Corollary 3.7. First, it is not difficult to check that the number of $\mathcal{E}$-copies of t_j after applying the substitution l times to t_i is exactly $m^l_{i,j}$. It follows that the density of the tile t_i is ν_i. Second, we consider a hierarchical sequence $(\mathcal{T}^l = (t^l_j)_{j \ge 0})_{l \ge 0}$ of $\mathcal{T}$ and the hierarchical decomposition $\{\Delta_0, \ldots, \Delta_{m-1}\}$ of Δ constructed in Proposition 3.5. We use the decomposition of Δ to estimate $\mathcal{N}(\Delta, \mathcal{T}, p_i)$ and $\mathcal{N}(\Delta, \mathcal{T})$. Since the sets Δ_l have disjoint interiors, we have

$$\mathcal{N}(\Delta, \mathcal{T}, p_i) = \sum_{l=0}^{m-1} \mathcal{N}(\Delta_l, \mathcal{T}, p_i) = \sum_{l=0}^{m-1} \sum_{t^l_j \subseteq \Delta_l} \mathcal{N}(t^l_j, \mathcal{T}, p_i) \tag{9}$$

and

$$\mathcal{N}(\Delta, \mathcal{T}) = \sum_{l=0}^{m-1} \sum_{t^l_j \subseteq \Delta_l} \mathcal{N}(t^l_j, \mathcal{T}). \tag{10}$$

Fig. 2. Construction of the hierarchical sequence for a given Δ. The tiling in gray $\mathcal{T}^0$ is a Penrose tiling (see Sect. 5). The tiling in black is $\mathcal{T}^l$, the dark-gray region is Δ_l and the light-gray one is $\widehat{\Delta}_{l+1}$. In this example, $m = 5$

Next, multiplying (10) by ν_i and then subtracting (9), we get

$$\mathcal{N}(\Delta, \mathcal{T}, p_i) - \nu_i(\mathcal{N}(\Delta, \mathcal{T})) = \sum_{l=0}^{m-1} \sum_{t_j^l \subseteq \Delta_l} \mathcal{N}(t_j^l, \mathcal{T}, p_i) - \nu_i \mathcal{N}(t_j^l, \mathcal{T}). \quad (11)$$

Suppose that t_j^l is a $\mathcal{E}$-copy of $\lambda^l p_k$ for some $k \in \{1, \ldots, n\}$. Then $\mathcal{N}(t_j^l, \mathcal{T}, p_i) = m_{i,k}^l$ and $\mathcal{N}(t_j^l, \mathcal{T}) = \sum_i m_{i,k}^l$. Therefore, applying Corollary 3.7 to (11), we get

$$|\mathcal{N}(\Delta, \mathcal{T}, p_i) - \nu_i \mathcal{N}(\Delta, \mathcal{T})| \leq \sum_{l=0}^{m-1} \sum_{t_j^l \subseteq \Delta_l} K\rho^l \leq \sum_{l=0}^{m-1} \mathcal{N}(\Delta_l, \mathcal{T}^l) K\rho^l,$$

where $K > 0$, and either $\lambda > \rho > r(M)$ if $r(M) < \lambda$ or $\rho = \lambda$ if $r(M) = \lambda$ and the eigenvalues of modulus $r(M)$ are semi-simple. Applying Proposition 3.5 to the last

inequality, we obtain:

$$|\mathcal{N}(\Delta, \mathcal{T}, p_i) - v_i \mathcal{N}(\Delta, \mathcal{T})| \leq K \|M\|_1 \sum_{l=0}^{m-1} \mathcal{L}(\Gamma, \mathcal{T}^{l+1}) \rho^l. \qquad (12)$$

We want to apply Lemma 3.3 to give an upper bound of $\mathcal{L}(\Gamma, \mathcal{T}^l)$ in terms of $\mathcal{L}(\Gamma, \mathcal{T})$ for all $l \in \{0, \ldots, m\}$. Since for each $l \geq 0$, the tiles of $\mathcal{T}^{l+1}$ are tiled by tiles of $\mathcal{T}^l$, it follows that $\mathcal{L}(\Gamma, \mathcal{T}^{m-1}) \leq \cdots \leq \mathcal{L}(\Gamma, \mathcal{T}^1) \leq \mathcal{L}(\Gamma, \mathcal{T})$. It is easy to check that $K_{\mathcal{T}^l} = K_{\mathcal{T}}$ for all $l \geq 0$.

Next, we define $l_0 \in \{0, \ldots, m\}$ as follows. If $\mathcal{L}(\Gamma, \mathcal{T}^{m-1}) > K_{\mathcal{T}}$, then $\ell_0 := m$. Otherwise, we define l_0 to be the minimal $l \in \{0, \ldots, m-1\}$ such that $\mathcal{L}(\Gamma, \mathcal{T}^{l_0}) \leq K_{\mathcal{T}}$. Observe that in both cases $\mathcal{L}(\Gamma, \mathcal{T}^{l_0-1}) > K_{\mathcal{T}}$ for all $l \in \{0 \ldots, l_0 - 1\}$.

We split the sum in (12) into two parts

$$\sum_{l=0}^{l_0-1} \mathcal{L}(\Gamma, \mathcal{T}^l) \rho^l + \sum_{l=l_0}^{m-1} \mathcal{L}(\Gamma, \mathcal{T}^l) \rho^l, \qquad (13)$$

and deal with each part separately. For the first sum, applying Lemma 3.3, we get

$$\mathcal{L}(\Gamma, \mathcal{T}^l) \leq (2K_{\mathcal{T}^l} + 1) \frac{R_{\mathcal{T}}}{R_{\mathcal{T}^l}} \mathcal{L}(\Gamma, \mathcal{T})$$

for all $l \in \{0, \ldots, l_0 - 1\}$. Since $R_{\mathcal{T}}/R_{\mathcal{T}^l} = \lambda^{-l}$ for every $l \geq 0$, it follows that

$$\sum_{l=0}^{l_0-1} \mathcal{L}(\Gamma, \mathcal{T}^l) \rho^l \leq (2K_{\mathcal{T}} + 1) \mathcal{L}(\Gamma, \mathcal{T}) \sum_{l=0}^{l_0-1} \left(\frac{\rho}{\lambda}\right)^l. \qquad (14)$$

To estimate the second sum in (13), we suppose that $l_0 < m$ (otherwise, the sum is zero). Since $\rho \leq \lambda$ and $\mathcal{L}(\Gamma, \mathcal{T}^l)$ is decreasing, we have

$$\sum_{l=l_0}^{m-1} \mathcal{L}(\Gamma, \mathcal{T}^l) \rho^l \leq \mathcal{L}(\Gamma, \mathcal{T}^{l_0}) \sum_{l=l_0}^{m-1} \lambda^l = \mathcal{L}(\Gamma, \mathcal{T}^{l_0}) \left(\frac{\lambda^m - \lambda^{l_0}}{\lambda - 1}\right).$$

From Lemma 6 and the fact that $\mathcal{L}(\Gamma, \mathcal{T}_{l_0}) < K_{\mathcal{T}}$, we get $\lambda^{m-1-l_0} \leq K_{\mathcal{T}} R_{\mathcal{T}}/r_{\mathcal{T}}$. Denote $N_{\mathcal{T}} := (\lambda K_{\mathcal{T}} R_{\mathcal{T}}/r_{\mathcal{T}} - 1)/(\lambda - 1)$. It follows that

$$\sum_{l=l_0}^{m-1} \mathcal{L}(\Gamma, \mathcal{T}^l) \rho^l \leq N_{\mathcal{T}} \lambda^{l_0} \mathcal{L}(\Gamma, \mathcal{T}^{l_0}). \qquad (15)$$

If $l_0 > 0$, then from Lemma 3.3 and (15), we get

$$\sum_{l=l_0}^{m-1} \mathcal{L}(\Gamma, \mathcal{T}^l) \rho^l \leq (2K_{\mathcal{T}} + 1) N_{\mathcal{T}} \mathcal{L}(\Gamma, \mathcal{T}). \qquad (16)$$

It is clear that (16) also holds when $l_0 = 0$, since $(2K_{\mathcal{T}} + 1) > 1$. Combining (14) and (16), we get

$$|\mathcal{N}(\Delta, \mathcal{T}, p_i) - v_i \mathcal{N}(\Delta, \mathcal{T})| \leq (2K_{\mathcal{T}} + 1) \mathcal{L}(\Gamma, \mathcal{T}) \left(\sum_{l=0}^{l_0-1} \left(\frac{\rho}{\lambda}\right)^l + N_{\mathcal{T}}\right). \qquad (17)$$

Finally, we deal with the different cases for ρ. In the first case, we have $r(M_{\mathcal{S}}) < \rho < \lambda$ and it follows from (17) that

$$|\mathcal{N}(\Delta, \mathcal{T}, p_i) - v_i \mathcal{N}(\Delta, \mathcal{T})| \le (2K_{\mathcal{T}} + 1) \left(\frac{\lambda}{\lambda - \rho} + N_{\mathcal{T}} \right) \mathcal{L}(\Gamma, \mathcal{T}),$$

which finishes part (i) of the main result, since $R_{\mathcal{T}}, r_{\mathcal{T}}$ and $K_{\mathcal{T}}$ are constant in $\Omega_{\mathcal{S}}$. In the second case, we have $\rho = \lambda$ and then from (17) and the estimation of $m \ge \ell_0$ given in Proposition 3.5, we obtain

$$|\mathcal{N}(\Delta, \mathcal{T}, p_i) - v_i \mathcal{N}(\Delta, \mathcal{T})| \le (2K_{\mathcal{T}} + 1)\mathcal{L}(\Gamma, \mathcal{T}) \left(\log_\lambda \mathcal{L}(\Gamma, \mathcal{T}) + \tilde{N}_{\mathcal{T}} \right),$$

where $\tilde{N}_{\mathcal{T}} = N_{\mathcal{T}} + 1 + \log_\lambda(R_{\mathcal{T}}/r_{\mathcal{T}})$. Thus, there exists $K > 0$ such that

$$|\mathcal{N}(\Delta, \mathcal{T}, p_i) - v_i \mathcal{N}(\Delta, \mathcal{T})| \le K\mathcal{L}(\Gamma, \mathcal{T}) \ln \mathcal{L}(\Gamma, \mathcal{T}),$$

as soon as Γ meets at least 2 tiles. This completes the proof of part (ii) of the main result.

5. Examples

5.1. Penrose tilings. Penrose tilings are among the most known examples of self-similar tilings, see for instance [AP98, GS87, Pen80, Sen95]. Here, we follow the construction of Anderson and Putnam [AP98]. Consider two isosceles triangles t_1 and t_2 in $\mathbb{R}^2$, where the vertices of t_1 have coordinates $(\sin(\pi/10), 0)$, $(-\sin(\pi/10), 0)$ and $(0, \cos(\pi/10))$ and those of t_2 have coordinates $(\sin(3\pi/10), 0)$, $(-\sin(3\pi/10), 0)$ and $(0, \cos(3\pi/10))$. Both triangles are equipped with decorations (arrows) on their edges as shown in Fig. 3.

Let $\mathcal{E}$ denote the group of translations in $\mathbb{R}^2$ and $\mathcal{P}_{\text{Penrose}}$ be the set of 40 triangles obtained by rotation of one of the two triangles t_1 or t_2 with its prescribed decoration by an angle $k\pi/10$, where k is in $\{0, 1, \ldots, n-1\}$, and let Ω_{Penrose} be the set of tilings of $\mathbb{R}^2$ made with translated copies of triangles in $\mathcal{P}_{\text{Penrose}}$ such that their tiles meet full-edge to full-edge and the decorations on overlapping edges coincide. Elements of Ω_{Penrose} are called Penrose tilings. They can also be constructed by using the substitution $\mathcal{S}_{\text{Penrose}}$ described in Fig. 3. The dilation factor of $\mathcal{S}_{\text{Penrose}}$ is the golden mean $\lambda = (1 + \sqrt{5})/2$.

The substitution matrix M_{Penrose} is a 40×40 non negative primitive matrix (see [AP98] for details). It is well-known that the Perron eigenvalue of M_{Penrose} is equal to $\lambda^2 = (3 + \sqrt{5})/2$, and that λ and $-\lambda$ are the only eigenvalues of modulus $r(M_{\text{Penrose}})$.
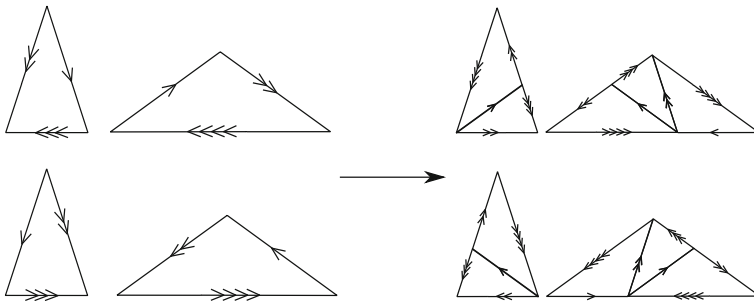

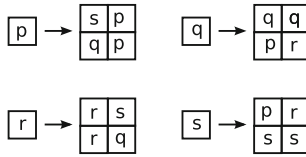**Fig. 3.** Local and substitution rules to construct Penrose tilings

Fig. 4. Square table substitution

Straightforward computations show that both of them are semi-simple with multiplicity 3. Thus, we can apply Theorem 2.1 (part (ii)), and obtain:

Each triangle p_i in $\mathcal{P}_{\text{Penrose}}$ has a well-defined frequency $\nu_i > 0$, and there exists $K > 0$ such that, for every Jordan curve Γ and every tiling $\mathcal{T}$ in Ω_{Penrose}, we have:

$$|\mathcal{N}(\Delta, \mathcal{T}, p_i) - \nu_i \mathcal{N}(\Delta, \mathcal{T})| \leq K\mathcal{L}(\Gamma, \mathcal{T}) \ln(\mathcal{L}(\Gamma, \mathcal{T}),$$

where Δ is the closed disk bounded by Γ.

Now, Let $\mathcal{E}$ be the group of all isometries (direct and indirect) of $\mathbb{R}^2$. Up to these isometries, there are now only 2 types of Penrose tiles (the fat and thin triangles). The associated substitution matrix reads

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

and $(3 + \sqrt{5})/2$ and $(3 - \sqrt{5})/2$ are its eigenvalues. Thus, we can apply Theorem 2.1 (part (i)) to obtain:

$$|\mathcal{N}(\Delta, \mathcal{T}, p_i) - \nu_i \mathcal{N}(\Delta, \mathcal{T})| \leq K\mathcal{L}(\Gamma, \mathcal{T}).$$

5.2. The square and table tilings. Squares and table tilings were studied by Robinson [Rob99,Rob04]. First, we consider the square tilings. Let $\mathcal{P}_{\text{square}}$ be the set of four unit squares respectively decorated with symbols p, q, r and s. The square substitution rule $\mathcal{S}_{\text{square}}$ is described in Fig. 4. The dilation factor is $\lambda = 2$.

In this case, the substitution matrix M_{square} reads

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}.$$

The eigenvalues of M_{square} are λ^2, λ and 0. It is easy to check that λ is semi-simple with multiplicity 2. Applying Theorem 2.1 (part (ii)) (extended to deal with the decorated case), it follows that each square $t \in \mathcal{P}_{\text{square}}$ has a well-defined frequency $\nu_t > 0$ and there exists $K > 0$ such that for any Jordan curve Γ bounding a close disk Δ and any tiling $\mathcal{T}$ in $\Omega_{\mathcal{S}_{\text{square}}}$, we have:

$$|\mathcal{N}(\Delta, \mathcal{T}, t) - \nu_t \mathcal{N}(\Delta, \mathcal{T})| \leq K\mathcal{L}(\Gamma, \mathcal{T}) \ln(\mathcal{L}(\Gamma, \mathcal{T})).$$

Now we consider the table tilings. Let $\mathcal{P}_{\text{table}} = \{p_1, p_2\}$ where p_1 is a vertical domino and p_2 is a horizontal domino. The substitution rule $\mathcal{S}_{\text{table}}$ is described in Fig. 5. In this case, the dilation factor λ is 2 and the associated substitution matrix reads

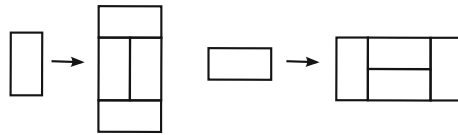$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$
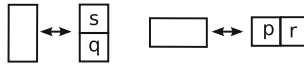
Fig. 5. The table substitution

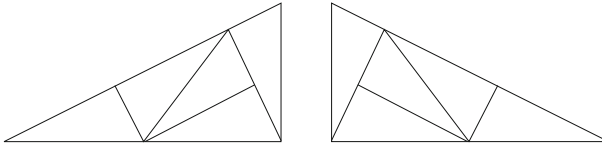**Fig. 6.** Identification of table and square tilings

Fig. 7. The pinwheel substitution

Its eigenvalues are λ^2 and 0. Applying Theorem 2.1 (part (i)), it follows that each domino p_i has a well-defined frequency $\nu_i > 0$, and there exists $K > 0$ such that, for any Jordan curve Γ bounding a close disk Δ and any tiling T in $\Omega_{\mathcal{S}_{\text{table}}}$, we have:

$$|\mathcal{N}(\Delta, T, p_i) - \nu_i \mathcal{N}(\Delta, T)| \leq K\mathcal{L}(\Gamma, T).$$

The square and table tilings can be identified (see [Rob04] for details) by identifying tiles as described in Fig. 6.

This identification can be used to count tiles in the different tilings. Consider for instance the unit square marked with a p. In every tiling in $\Omega_{\mathcal{S}_{\text{square}}}$, the squares marked with p and r always appear as in Fig. 6. Thus, for any closed Jordan curve Γ bounding a close disk Δ we have: $\mathcal{N}(\Delta, T, p) - \mathcal{N}(\Delta, T, p_1)$ is smaller than $\mathcal{L}(\Gamma, T)$, where p_1 is the horizontal domino in $\mathcal{P}_{\text{table}}$. Hence, using the estimates for the table substitution gives better estimates than the estimates for the square substitution. It is natural to wonder why. The answer simply relies on the fact that the square substitution system is less efficient than the table one in the sense that the estimates of the error for the nth substitution of the square p, which is a square of size 2^n, yields an error term of size $\lambda^n = 2^n$. This error term is actually going to be canceled by terms corresponding to the other squares of size 2^n. In our computation, the error term for a collection of squares of size 2^n is bounded from above by the sum of the error terms for each square of size 2^n, and thus we cannot see these cancellations which actually occur.

5.3. The pinwheel tilings. Pinwheel tilings were introduced by Conway and Radin [Rad94]. Let $\mathcal{E}$ be the group $\mathcal{I}^+$ of direct isometries of $\mathbb{R}^2$, and $\mathcal{P}_{\text{pinwheel}}$ be the set of two right triangles which are isometric (but with inverse orientation). Each triangle has a hypotenuse of length 1 and sharp angle α satisfying $\tan \alpha = 1/2$. The substitution rule $\mathcal{S}_{\text{pinwheel}}$ is described in Fig. 7 and has a dilation factor of $\sqrt{5}$.

The substitution matrix is the 2×2 matrix $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. It has two eigenvalues $\lambda^2 = 5$ and -1. Applying Theorem 2.1, it follows that each triangle t_i in $\mathcal{P}_{\text{pinwheel}}$ has a

well-defined frequency $\nu_i > 0$, and there exists $K > 0$ such that, for any Jordan curve Γ bounding a close disk Δ and any tiling T in $\Omega_{\mathcal{S}_{\text{pinwheel}}}$:

$$|\mathcal{N}(\Delta, \mathcal{T}, p_i) - \nu_i \mathcal{N}(\Delta, \mathcal{T})| \leq K\mathcal{L}(\Gamma, \mathcal{T}).$$

5.4. Rauzy tilings. Rauzy tilings (see [Rau82] for details) are tilings made from a set $\mathcal{P}_{\text{rauzy}}$ of three (topological) disks r_1, r_2 and r_3, where r_1 is a disk whose boundary is a Jordan curve (with positive Hausdorff dimension), $r_2 = \lambda r_1$ and $r_2 = \lambda^2 r_3$, where $\lambda \simeq 1.84$ is the unique real root of the polynomial $X^3 - X^2 - X - 1$ (see [Fog02]). The substitution matrix is the 3×3 matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The largest eigenvalue of the substitution matrix is λ^2. All other eigenvalues have modulus smaller than 1. Applying Theorem 2.1, it follows that each Rauzy tile p_i has a well-defined frequency ν_i and there exists $K > 0$ such that for every Jordan curve Γ bounding a close disk Δ and every tiling $\mathcal{T}$ in $\Omega_{\mathcal{S}}$, we have:

$$|\mathcal{N}(\Delta, \mathcal{T}, p_i) - \nu_i \mathcal{N}(\Delta, \mathcal{T})| \leq K\mathcal{L}(\Gamma, \mathcal{T}).$$

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