PHASE TRANSITIONS IN TEMPERATURE FOR INTERMITTENT MAPS

DANIEL CORONEL AND JUAN RIVERA-LETELIER,
WITH AN APPENDIX IN COLLABORATION WITH IRENE INOQUIO-RENTERIA

ABSTRACT. This article characterizes phase transitions in temperature within a specific space of HÖLDER continuous potentials, distinguished by their regularity and asymptotic behavior at zero. We also characterize the phase transitions in temperature that are robust within this space. Our results reveal a connection between phase transitions in temperature and ergodic optimization.

1. INTRODUCTION

Since the last century, the probabilistic approach to studying "complex" dynamical systems has become a significant paradigm. To carry out such a study, one requires a probability measure that is invariant under the dynamics and describes the behavior of most orbits. Thermodynamic formalism has proven to be a powerful tool for selecting such measures in the context of uniformly hyperbolic and expanding maps. For these systems, every sufficiently regular potential φ admits a unique equilibrium state that is fully supported, has positive entropy, and enjoys strong statistical properties. Moreover, the pressure function associated with the one-parameter family of potentials $\beta\varphi$, for $\beta \in (0, +\infty)$, is real-analytic in the parameter β. This property is commonly referred to as the absence of phase transitions in temperature for the potential. However, these results generally hold only when the dynamics are uniformly hyperbolic or expanding, or when the potential is sufficiently regular. Notable examples of phase transitions in temperature include the potentials constructed by HOFBAUER in [Hof77] for the one-sided full shift on two symbols, and the geometric potential for the MANNEVILLE–POMEAU maps [PS92].

The MANNEVILLE–POMEAU family of dynamical systems belongs to the class of intermittent maps, which have been extensively studied in smooth ergodic theory. These maps provide the simplest examples of non-expanding dynamics; see, for instance, [BC23, BLL12, BT16, BTT19, CT13, CV13, GIR22, GIR24, Gou04, Hol05, Klo20, Lop93, LRL14a, LRL14b, LSV99, MT12, PM80, PS92, PW99, Sar01, Sar02, Tha00, You99].

Most previous work has focused on the geometric potential. However, some papers deal with HÖLDER continuous potentials; see, for example, [Klo20, LRL14a, LRL14b]. In this article, we take an intermediate approach. Our goal is to characterize phase transitions in temperature for the MANNEVILLE–POMEAU family within a specific class of HÖLDER continuous potentials, distinguished by its regularity and asymptotic behavior at zero (see Theorem 1 and Corollary 1.1). This class, introduced by SARIG in [Sar01], enables, among other things, the study of the thermodynamic formalism for potentials near the geometric potential. Another advantage of this class is that it allows for a more tractable technical analysis compared to the full class of HÖLDER continuous potentials. More precisely, the additional regularity of potentials in this class implies a form of “bounded variation of ergodic sums." At the same time, the asymptotic behavior at zero allows for a clear description of potentials exhibiting robust phase transitions in temperature (see Theorem 3). Moreover, when the indifferent fixed point is "flatter" (i.e., for $\alpha > 1$), this class of potentials reveals an additional regularity in phase transitions in temperature that is not present in the HÖLDER class (see Theorem 2 and Corollary 1.3). This final result aligns with the philosophy proposed by ISRAEL in [Isr79], which states that, to observe interesting phase diagrams, one must consider smaller interaction spaces. In our setting, interactions correspond to potentials.

A disadvantage of our smaller class of potentials is that it excludes the historically significant class of HÖLDER continuous potentials. We address these in the companion paper [CRL25], providing a complete topological description of their phase diagram.

To state our results more precisely, we begin with some definitions.

1.1. **Phase transitions in temperature.** Let $\alpha > 0$ be given, and let $f\colon [0,1] \to [0,1]$ be defined by

$$f(x) := \begin{cases} x(1+x^\alpha), & \text{if } x(1+x^\alpha) \le 1; \\ x(1+x^\alpha) - 1, & \text{otherwise.} \end{cases} \tag{1.1}$$

Denote by $\mathscr{M}$ the space of BOREL probability measures that are invariant under f. For every measure $\mu \in \mathscr{M}$, we denote by h_μ its entropy. Let φ be a continuous function on $[0,1]$. We call φ a *potential* and define the *pressure* $P(\varphi)$ of the potential φ by

$$P(\varphi) := \sup\left\{ h_\mu + \int \varphi \,\mathrm{d}\mu : \mu \in \mathscr{M} \right\}. \tag{1.2}$$

A measure $\mu \in \mathscr{M}$ that realizes the supremum above is called an *equilibrium state* of f for φ. We say that a measure $\nu \in \mathscr{M}$ is *maximizing for the potential* φ if

$$\int \varphi \,\mathrm{d}\nu = \sup_{\mu \in \mathscr{M}} \int \varphi \,\mathrm{d}\mu. \tag{1.3}$$

We say that a continuous potential φ exhibits a *phase transition in temperature* if there exists $\beta_* \in (0, +\infty)$ such that the function $\beta \mapsto P(\beta\varphi)$ is not real-analytic at β_*. The terminology comes from Statistical Mechanics, where β is interpreted as the inverse temperature. If the potential φ is HÖLDER continuous, then there is at most one point in $(0, +\infty)$ where the function $\beta \mapsto P(\beta\varphi)$ fails to be real-analytic, and if a phase transition occurs at β_*, then for all $\beta \ge \beta_*$, one has $P(\beta\varphi) = \beta\varphi(0)$; see Corollary 1.4 in §1.3. In particular, if a HÖLDER continuous potential φ has a phase transition in temperature at $\beta_* \in (0, +\infty)$, then for every $\beta \ge \beta_*$, the invariant measure δ_0 is an equilibrium state for $\beta\varphi$. Consequently, it is also a maximizing measure for φ. Moreover, δ_0 is the unique maximizing measure for φ (see §1.5).

In Theorem 1 and Corollary 1.1 below, we show that for φ in a suitable class of potentials, the uniqueness of δ_0 as the maximizing measure for φ implies the existence of a phase transition in temperature for φ. These results establish a strong connection between phase transitions in temperature and the theory of Ergodic Optimization [Jen19]. The class of potentials we consider behaves asymptotically like cx^γ near zero, with $c \in \mathbb{R}$ and $\gamma \in (0, +\infty)$. For example, the geometric potential $-\log Df$ has this asymptotic form with $c = -(\alpha + 1)$ and $\gamma = \alpha$. Theorem 1 shows that the occurrence of a phase transition in temperature is not merely a local property depending on the asymptotic behavior of the potential at zero;

it also has a global component that can be characterized via maximizing measures. The dependence of the phase transition on the asymptotic behavior at zero is subtle, as further illustrated by Theorem 1. However, when the exponent of the system α and the exponent of the potential γ coincide, the potentials exhibiting a phase transition in temperature display an additional rigidity, as shown in Theorem 2.

Theorem 1. *Let α be in $(0, +\infty)$ and let f be the* MANNEVILLE–POMEAU *map of parameter α. Let $\varphi : [0, 1] \to \mathbb{R}$ be a continuous potential with continuous derivative on $(0, 1]$ such that there are c in $\mathbb{R}$ and $\gamma > 0$ verifying*

$$c = \lim_{x \to 0^+} \frac{\varphi'(x)}{\gamma x^{\gamma-1}}. \tag{1.4}$$

The following hold.

1. *If $\gamma \leq \alpha$ and $c > 0$, then δ_0 is not a maximizing measure for φ;*
2. *If $\gamma \leq \alpha$, $c < 0$ and δ_0 is the unique maximizing measure for φ, then φ has a phase transition in temperature;*
3. *If φ has a phase transition in temperature, then $\gamma \leq \alpha, c \leq 0$ and δ_0 is the unique maximizing measure for φ.*

The following result is a stronger version of Theorem 1(3) with $\gamma = \alpha$.

Theorem 2. *Let α, f, γ and c be as in Theorem 1. If φ has a phase transition in temperature and $\gamma = \alpha$, then $c < 0$.*

The following corollary is a direct consequence of Theorems 1 and 2.

Corollary 1.1. *Let α, f, γ and c be as in Theorem 1. The following hold.*

1. *If $\gamma \leq \alpha$ and $c \neq 0$, then φ has a phase transition in temperature if and only if δ_0 is the unique maximizing measure for φ.*
2. *If $\gamma = \alpha$, then φ has a phase transition in temperature if and only if $c < 0$ and δ_0 is the unique maximizing measure for φ.*

Notice that every φ as in Theorem 1 is HÖLDER continuous with exponent $\min\{1, \gamma\}$. Also notice that under the hypotheses of Theorem 1, if $\gamma \leq \alpha, c < 0$ and $\varphi < \varphi(0)$ on $(0, 1]$, then φ has a phase transition in temperature. In this case, we give a simpler proof that φ has a phase transition in temperature; see Corollary 4.1(2). An example is the geometric potential (Proposition 4.2).

We remark that Theorem 1(1) is false for $\gamma > \alpha$ as the following example shows. Put $\delta := \gamma - \alpha$ and for every a in $(0, +\infty)$ define the potential $\varphi_a(x) := (\delta/2)x^\gamma(1 - ax)$ for every x in $[0, 1]$. Now, consider the potential $h(x) := -x^\delta$ for every x in $[0, 1]$, and the coboundary $h \circ f - h$. We have that $h \circ f(0) - h(0) = 0$, and for every x in $[0, 1]$ close to 0, we have $h \circ f(x) - h(x)$ is close to $-\delta x^\gamma$. Therefore, by taking a sufficiently large, we can ensure that $\varphi_a + h \circ f - h$ is strictly negative on $(0, 1]$, which implies that δ_0 is the unique maximazing measure for $\varphi_a + h \circ f - h$ and hence also for φ_a.

Note that Theorem 1(2) does not hold for $c = 0$. Indeed, for γ' in $(\gamma, +\infty)$ the potential $\omega_{\gamma'}$ defined by $\omega_{\gamma'}(x) := -x^{\gamma'}$ satisfies

$$\lim_{x \to 0^+} \frac{\omega'_{\gamma'}(x)}{\gamma x^{\gamma-1}} = 0, \tag{1.5}$$

and it has a phase transition in temperature if and only if $\gamma' \le \alpha$, see Proposition 2.5. Thus, if $\gamma < \alpha < \gamma'$ then $\omega_{\gamma'}$ does not have a phase transition in temperature, and if $\gamma < \gamma' < \alpha$ then $\omega_{\gamma'}$ has a phase transition in temperature. In both cases, $\gamma < \alpha$ and δ_0 is the unique maximizing measure for $\omega_{\gamma'}$. The same example shows that Theorem 2 does not hold for $\gamma \neq \alpha$. Since we can take $\gamma < \gamma' < \alpha$ and thus, $\omega_{\gamma'}$ has a phase transition in temperature but $c = 0$ by (1.5).

In [Sar01, Proposition 1(2)], it was stated that under the hypotheses of Theorem 1, if $\gamma \le \alpha$ and $c < 0$, then φ has a phase transition in temperature. However, the following example shows we can not omit the hypothesis that δ_0 is the unique maximizing measure for φ in Theorem 1(2).

Example 1.2. Consider the potential

$$\widehat{\varphi}(x) := -x^{\alpha}(1-x). \tag{1.6}$$

It satisfies (1.4) with $c = -1$ and $\gamma = \alpha$. We have that δ_0 and δ_1 are maximizing measures for $\widehat{\varphi}$. Thus, by Theorem 1(3), the potential $\widehat{\varphi}$ does not exhibit a phase transition in temperature.

We now provide an alternative proof that does not rely on Theorem 1. Let x_2 be the smaller preimage of 1 under f^2, and let X be the maximal invariant set of f in $[x_2, 1]$. Then $f|_X$ is topologically conjugate to a topologically mixing subshift of finite type (the golden mean shift), via a conjugacy that is Hölder continuous. By the theory of Thermodynamic Formalism for Hölder continuous potentials, it follows that for every $\beta > 0$, the potential $\beta\widehat{\varphi}|_X$ has a unique equilibrium state with positive entropy (see, for instance, [Bow08, Theorem 1.25]). Therefore,

$$P(\beta\widehat{\varphi}) \ge P(\beta\widehat{\varphi}|_X) > h_{\delta_1} + \int \beta\widehat{\varphi}|_X \, \mathrm{d}\delta_1 = \beta\widehat{\varphi}(1) = \beta\widehat{\varphi}(0). \tag{1.7}$$

By Corollary 1.4(1) in §1.3, we conclude that the potential $\widehat{\varphi}$ does not exhibit a phase transition in temperature.

1.2. **Robust phase transitions in temperature.** The following results aim to understand when a phase transition in temperature is robust. For this, we introduce a function space containing the potentials introduced in Theorem 1. Denote by $C(\mathbb{R})$ for space of continuous functions on $[0,1]$ endowed with the uniform norm $\|\cdot\|$ and by $C^1_\dagger(\mathbb{R})$ the subspace of $C(\mathbb{R})$ of functions with continuous derivative on $(0,1]$. For every γ in $(0,+\infty)$ we define the space of function $C^{1,\gamma}_\dagger(\mathbb{R})$ by

$$C^{1,\gamma}_\dagger(\mathbb{R}) := \left\{ \varphi \in C^1_\dagger(\mathbb{R}) \colon \lim_{x\to 0^+} \frac{\varphi'(x)}{x^{\gamma-1}} \in \mathbb{R} \right\}. \tag{1.8}$$

For every φ in $C^{1,\gamma}_\dagger(\mathbb{R})$ we call

$$\lim_{x\to 0^+} \frac{\varphi'(x)}{\gamma x^{\gamma-1}} \tag{1.9}$$

the *leading coefficient of* φ *in* $C^{1,\gamma}_\dagger(\mathbb{R})$. The meaning of this number is explained in Lemma 2.7 in §2.1. For every φ in $C^{1,\gamma}_\dagger(\mathbb{R})$ put

$$|\varphi|_{1,\gamma} := \sup_{x\in(0,1]} \frac{|\varphi'(x)|}{\gamma x^{\gamma-1}} \text{ and } \|\varphi\|_{1,\gamma} := \|\varphi\| + |\varphi|_{1,\gamma}. \tag{1.10}$$

The space $C^{1,\gamma}_{\dagger}(\mathbb{R})$ endowed with the norm $\|\cdot\|_{1,\gamma}$ is a BANACH space.

We say that a potential φ in $C^{1,\gamma}_{\dagger}(\mathbb{R})$ has a *robust phase transition in temperature for f in $C^{1,\gamma}_{\dagger}(\mathbb{R})$*, if every potential sufficiently close to φ in $C^{1,\gamma}_{\dagger}(\mathbb{R})$ has a phase transition in temperature. In this case, $\gamma \leq \alpha$ and the leading coefficient of φ in $C^{1,\gamma}_{\dagger}(\mathbb{R})$ is nonpositive by Theorem 1(3). The following result characterizes potentials in $C^{1,\gamma}_{\dagger}(\mathbb{R})$ with a robust phase transition in temperature.

Theorem 3. *Let α be in $(0, +\infty)$ and let f be the MANNEVILLE–POMEAU map of parameter α. Let γ be in $(0, \alpha]$ and let φ be a potential in $C^{1,\gamma}_{\dagger}(\mathbb{R})$. The following are equivalent.*

1. *φ has a robust phase transition in temperature for f in $C^{1,\gamma}_{\dagger}(\mathbb{R})$;*
2. *δ_0 is, robustly in $C^{1,\gamma}_{\dagger}(\mathbb{R})$, the unique maximizing measure for φ;*
3. *φ has a phase transition in temperature and its leading coefficient in $C^{1,\gamma}_{\dagger}(\mathbb{R})$ is strictly negative.*

Observe that for all $\gamma \in (0, \alpha]$, $\gamma' \in (\gamma, \alpha]$, and $\varphi \in C^{1,\gamma'}_{\dagger}(\mathbb{R})$, the leading coefficient of φ in $C^{1,\gamma}_{\dagger}(\mathbb{R})$ is zero. By Theorem 3, no phase transition in temperature of a potential in $C^{1,\gamma'}_{\dagger}(\mathbb{R})$ is robust in $C^{1,\gamma}_{\dagger}(\mathbb{R})$. When $\gamma < \alpha$, not all nonrobust phase transitions in temperature in $C^{1,\gamma}_{\dagger}(\mathbb{R})$ are of this form. For example, the potential $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}(x) := \frac{1}{-x + \log x} x^{\gamma} \quad \text{for } x \in (0,1], \quad \text{and} \quad \widetilde{\varphi}(0) := 0,$$

belongs to $C^{1,\gamma}_{\dagger}(\mathbb{R}) \setminus \bigcup_{\gamma' \in (\gamma, \alpha]} C^{1,\gamma'}_{\dagger}(\mathbb{R})$, has zero leading coefficient in $C^{1,\gamma}_{\dagger}(\mathbb{R})$, and exhibits a phase transition in temperature (see Corollary 4.1(1)). The situation is entirely different in the case $\gamma = \alpha$, as shown in the following result, which is an immediate consequence of Theorems 2 and 3.

Corollary 1.3. *Let α be in $(0, +\infty)$ and let f be the MANNEVILLE–POMEAU map of parameter α. For every φ in $C^{1,\alpha}_{\dagger}(\mathbb{R})$ the following are equivalent.*

1. *φ has a phase transition in temperature;*
2. *φ has a robust phase transition in temperature for f in $C^{1,\alpha}_{\dagger}(\mathbb{R})$.*

When these equivalent conditions hold, the leading coefficient of φ in $C^{1,\alpha}_{\dagger}(\mathbb{R})$ is strictly negative.

We conclude this section by discussing the phase diagram of potentials in $C^{1,\gamma}_{\dagger}(\mathbb{R})$, and the role of Theorem 3 and Corollary 1.3 in its description. Let α be in $(0, +\infty)$ and let f be the MANNEVILLE–POMEAU map of parameter α. Given $\gamma \in (0, +\infty)$, we define the *phase transition in temperature locus* $\mathcal{PT}_T(\gamma)$ as the set of potentials in $C^{1,\gamma}_{\dagger}(\mathbb{R})$ that exhibit a phase transition in temperature at $\beta = 1$. By Theorem 1, the set $\mathcal{PT}_T(\gamma)$ is empty for $\gamma > \alpha$, and for $\gamma \in (0, \alpha]$, by Corollary 1.4, it has empty interior in $C^{1,\gamma}_{\dagger}(\mathbb{R})$. We also define the *robust phase transition in temperature locus* $\mathcal{PT}_{RT}(\gamma)$ as the subset of $\mathcal{PT}_T(\gamma)$ consisting of potentials whose phase transition in temperature is robust. Motivated by the Gibbs phase rule from Statistical Mechanics (see, for instance, [Isr79]), one may ask about the regularity of the sets $\mathcal{PT}_T(\gamma)$ and $\mathcal{PT}_{RT}(\gamma)$ in $C^{1,\gamma}_{\dagger}(\mathbb{R})$. For example, by Theorem 3, one may ask whether $\mathcal{PT}_T(\gamma)$ is a real-analytic manifold with boundary in $C^{1,\gamma}_{\dagger}(\mathbb{R})$, or whether $\mathcal{PT}_{RT}(\gamma)$

is a real-analytic manifold in $C^{1,\gamma}_{\dagger}(\mathbb{R})$. By Corollary 1.3, in the case $\gamma=\alpha$, these questions reduce to: Is $\mathcal{PT}_T(\alpha)$ a real-analytic manifold in $C^{1,\alpha}_{\dagger}(\mathbb{R})$? In the companion paper [CRL25], we answer similar questions in a topological setting and for HÖLDER continuous potentials, but the questions above remain open.

1.3. **Phase transition in temperature for HÖLDER continuous potentials and the Key Lemma.** The main new feature in Theorems 1 and 3 is the role played by the invariant measure δ_0 as a maximizing measure for the potential. To detect that, for a HÖLDER continuous potential with a phase transition in temperature, δ_0 is the unique maximizing measure (Theorem 1(3)), we rely on the following two results.

Corollary 1.4. *Let α be in $(0,+\infty)$ and let f be the MANNEVILLE–POMEAU map of parameter α. For every HÖLDER continuous potential φ, the following dichotomy holds:*

1. *For every $\beta\in(0,+\infty)$, we have $P(\beta\varphi)>\beta\varphi(0)$, and the function $\beta\mapsto P(\beta\varphi)$ is real-analytic on $(0,+\infty)$; or*
2. *There exists $\beta_0>0$ such that $P(\beta_0\varphi)=\beta_0\varphi(0)$. Define*

$$\beta_*:=\inf\{\beta>0\colon P(\beta\varphi)\leq\beta\varphi(0)\}. \tag{1.11}$$

Then $\beta_*>0$; for every $\beta\in(0,\beta_*)$, we have $P(\beta\varphi)>\beta\varphi(0)$, and for every $\beta\geq\beta_*$, we have $P(\beta\varphi)=\beta\varphi(0)$. Furthermore, the function $\beta\mapsto P(\beta\varphi)$ is real-analytic on $(0,+\infty)\setminus\{\beta_*\}$.

Key Lemma. *Let α be in $(0,+\infty)$ and let f be the MANNEVILLE–POMEAU map of parameter α. For each HÖLDER continuous potential φ and each μ in $\mathscr{M}$ distinct from δ_0, we have*

$$P(\varphi)>\int\varphi\,\mathrm{d}\mu. \tag{1.12}$$

The proof of Corollary 1.4 is based on the following result from [IRRL25], which in turn relies on the Spectral Gap Theorem proved by KELLER in [Kel85]: For every HÖLDER continuous potential φ if $P(\varphi)>\varphi(0)$, then the function $t\in\mathbb{R}\mapsto P(\varphi+t\varphi)$ is real-analytic at 0. Now, observe that if there is β_0 in $(0,+\infty)$ such that $P(\beta_0\varphi)=\beta_0\varphi(0)$ then for every $\beta\geq\beta_0$ we have $P(\beta\varphi)=\beta\varphi(0)$. Indeed, by definition of the pressure for every $\beta\geq\beta_0$, we have that

$$\frac{1}{\beta}P(\beta\varphi)\leq\frac{1}{\beta_0}P(\beta_0\varphi), \tag{1.13}$$

and then,

$$\beta\varphi(0)\leq P(\beta\varphi)\leq\frac{\beta}{\beta_0}P(\beta_0\varphi)=\beta\varphi(0). \tag{1.14}$$

Since the pressure $P(\varphi)$ is continuous in the potential and $P(0)=\log 2>0$ (see Lemma 3.3(4) below), we obtain Corollary 1.4.

In [BF23, Theorem C], as in Corollary 1.4, it was shown that for MANNEVILLE–POMEAU-like maps of the circle, HÖLDER continuous potentials exhibit at most one phase transition in temperature.

The Key Lemma was first proved for rational maps in [IRRL12] and then for multimodal maps in [Li15]. Since no one of these proofs applied directly to the setting of this article we have included a detailed and simplified proof of the Key Lemma for Manneville–Pomeau maps in Appendix A.

1.4. **Notes and references.** Although not very detailed, the first proof that the geometric potential for the Manneville–Pomeau map has a phase transition in temperature at 1 appeared in [PS92]. See also [CT13, Theorem 4.3] for a proof when α is in $(0,1)$. We take the opportunity to give a simple proof of this fact in §4.2.

Let α be in $(0,+\infty)$ and let f be the Manneville–Pomeau map of parameter α. From Corollary 1.4 and Proposition 2.5 in §2.2, we deduce that for a Hölder continuous potential φ satisfying $\varphi(0)=0$, if there exists $C>0$ such that $\varphi \leq C\omega_\alpha$ on $[0,1]$, then φ undergoes a phase transition in temperature. This naturally raises the question of whether every Hölder continuous potential φ with $\varphi(0)=0$ that undergoes a phase transition in temperature necessarily satisfies this condition. As the following example shows, the answer is negative. Let x_1 denote the discontinuity point of the map f. Consider the potential ψ defined by $\psi(x) := -x^\alpha(x-x_1)^2$. By Theorem 1(2), this potential exhibits a phase transition in temperature, yet there does not exist $C>0$ such that $\psi \leq C\omega_\alpha$ on $[0,1]$. However, one may still ask whether there are $C>0$ and a bounded measurable function h such that

$$\varphi \leq C\omega_\alpha + h\circ f - h \text{ on } [0,1].$$

In this case, the occurrence of a phase transition in temperature would be equivalent to the latter property.

A naturally related question, for every $\gamma \in (0,\alpha]$, is the following: Let $\varphi \in C^{1,\gamma}_{\dagger}(\mathbb{R})$ be a potential satisfying $\varphi(0)=0$, with a negative leading coefficient in $C^{1,\gamma}_{\dagger}(\mathbb{R})$, and for which δ_0 is the unique maximizing measure. Is φ cohomologous, via a bounded measurable function, to a potential $\phi \in C^{1,\gamma}_{\dagger}(\mathbb{R})$ that is negative on $(0,1]$, satisfies $\phi(0)=0$, and has a negative leading coefficient in $C^{1,\gamma}_{\dagger}(\mathbb{R})$? A positive answer to this question would provide an alternative proof of Theorem 1(2); see Corollary 4.1(2).

1.5. **About the proof of the theorems and the organization.** In §2.1, we state the Main Theorem and use it to prove Theorems 1, 2, and 3 in §2.3. The proofs of these theorems also rely on several auxiliary results. In §2.1, we state a form of bounded variation of ergodic sums (Lemma 2.2). In §2.2, we introduce an inducing scheme, and we state the relation between the pressure of the induced system and that of the original system, formulated as the Bowen-type formula in §2.2. From the Bowen-type formula, we also deduce Proposition 2.5 and Corollary 2.6. The proofs of almost all these additional results are given in the subsequent sections.

The most challenging aspects of the proofs of Theorem 1 and Theorem 3 are Theorem 1(2) and the implication $3 \Rightarrow 1$ in Theorem 3. Both implications follow from the Main Theorem and Remark 2.3 in §2.1. The Main Theorem is the most technical part of the article. Roughly speaking, it shows that if φ is a potential in $C^{1,\gamma}_{\dagger}(\mathbb{R})$ with a negative leading coefficient in $C^{1,\gamma}_{\dagger}(\mathbb{R})$, then φ exhibits a phase transition in temperature if and only if δ_0 is the unique maximizing measure for φ. Moreover, when these equivalent conditions hold, they define an open subset of $C^{1,\gamma}_{\dagger}(\mathbb{R})$.

To prove that δ_0 is the unique maximizing measure for φ in Theorem 1(3), we use Corollary 1.4 and the Key Lemma. Indeed, if δ_0 is not the unique maximizing measure for φ, then the same holds for every $\beta>0$ for the potential $\beta\varphi$. Then, by the Key Lemma, for every $\beta > 0$, we have $P(\beta\varphi) > \beta\varphi(0)$, and thus, by Corollary 1.4, the system does not exhibit a phase transition in temperature.

In §3.1, we provide a lemma concerning the geometry near the indifferent fixed point at 0 (Lemma 3.1), as well as a bounded distortion result (Lemma 3.2). These results are known to specialists, but we include detailed proofs for the reader's convenience. In §3.2, we prove Lemmas 2.2 and 2.7. The latter describes the form of potentials in $C^{1,\gamma}_{\dagger}(\mathbb{R})$ near 0, and clarifies the meaning of the leading coefficient introduced in §1.2. The maps in the MANNEVILLE–POMEAU family are discontinuous. We consider a continuous extension to apply the thermodynamic formalism for continuous maps on compact metric spaces. This is a standard construction for discontinuous expanding maps on compact intervals, known as the doubling construction. In §3.3, we provide a detailed proof of the doubling construction for MANNEVILLE–POMEAU maps. We use this construction to derive formulas for the topological pressure in Lemma 3.6 in §3.4.

In §4, we prove the Bowen-type formula. This is a key ingredient in proving all the theorems, particularly Theorem 2.

In §5, we prove the Main Theorem, and finally, in Appendix A, we prove the Key Lemma.

2. MAIN THEOREM AND PROOF OF THEOREMS 1, 2 AND 3

In this section, we state our principal technical result, the Main Theorem, and prove Theorems 1, 2 and 3. Before writing the Main Theorem, we need some results that will be proved in §3. The proof of the Main Theorem is in §5. For the proof of Theorems 1, 2 and 3, we also need some additional results. The main one is the Bowen-type formula in §2.2, from which we deduce Proposition 2.5 and Corollary 2.6. The proof of these additional results will be given in §4.

We use $\mathbb{N}$ to denote the set of integers greater than or equal to 1 and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Fix α in $(0, +\infty)$ throughout the rest of this paper.

2.1. **Main Theorem.**

Before stating the Main Theorem, we need some results about the geometry around the indifferent fixed point and the ergodic sums.

Denote by x_1 the unique discontinuity of f . Then $f(x_1) = 1$ and the map $f\colon [0, x_1] \to [0, 1]$ is a diffeomorphism. Put $x_0 := 1$ and for each integer j satisfying $j \geq 2$, put $x_j := f|_{[0,x_1]}^{-(j-1)}(x_1)$. Also put $J_0 := (x_1, 1]$.

The following limit is known to the specialists. However, we prefer to state and prove it as part of Lemma 3.1 in §3 for the reader's convenience. We have

$$\lim_{n\to+\infty} n \cdot x_n^{\alpha} = \frac{1}{\alpha}. \tag{2.1}$$

The next lemma is about the control up to an additive constant of the ergodic sums. This is a classical technical requirement in the theory of Thermodynamic Formalism that is satisfied, for example, by the geometric potential.

Definition 2.1. We say that a continuous potential $\varphi : [0, 1] \to \mathbb{R}$ has *bounded variation ergodic sums on J_0 for f*, if there is a constant $C > 0$ such that for every n in $\mathbb{N}$, every connected component J of $f^{-n}(J_0)$ and all x and y in J the following inequality holds:

$$|S_n\varphi(x) - S_n\varphi(y)| \leq C. \tag{2.2}$$

Lemma 2.2. *Let γ be a positive real number. There is a positive constant D such that for every potential φ in $C^{1,\gamma}_{\dagger}(\mathbb{R})$, for every n in $\mathbb{N}$, for every connected component J of $f^{-n}(J_0)$, and all x and y in J the following inequality holds:*

$$|S_n\varphi(x) - S_n\varphi(y)| \le D|\varphi|_{1,\gamma}. \tag{2.3}$$

In particular, φ has bounded variation ergodic sums on J_0 for f with constant $D|\varphi|_{1,\gamma}$.

Now, we present our main technical result. It gives a quantitative version of Theorem 1, which helps study robust phase transitions in temperature.

Main Theorem. *Let γ be in $(0,\alpha]$, put $\theta := 1-\gamma/\alpha$, and let D be the constant in Lemma 2.2 for γ. Let n_0 in $\mathbb{N}$ such that for every integer $n \ge n_0$ we have*

$$\frac{1}{2\alpha^{\frac{\gamma}{\alpha}}} n^{-\frac{\gamma}{\alpha}} < x_n^{\gamma}. \tag{2.4}$$

Let φ be in $C^{1,\gamma}_{\dagger}(\mathbb{R})$ and let c be in $(-\infty, 0)$ such that for every x in $[0, x_{n_0}]$ we have

$$\varphi(x) - \varphi(0) < cx^{\gamma}, \tag{2.5}$$

and let m_0 be the least integer satisfying

$$m_0 > \begin{cases} \left[2(n_0+1)^{\theta} + \frac{4\alpha^{\frac{\gamma}{\alpha}}\theta}{-c}\left(D|\varphi|_{1,\gamma} + 2n_0\|\varphi\|\right)\right]^{\frac{1}{\theta}}, & \text{if } \gamma < \alpha; \\ (n_0+1)^2 \exp\left(\frac{4\alpha}{-c}\left(D|\varphi|_{1,\gamma} + 2n_0\|\varphi\|\right)\right), & \text{if } \gamma = \alpha. \end{cases} \tag{2.6}$$

Then, the following are equivalent:

1. *φ has a phase transition in temperature in $C^{1,\gamma}_{\dagger}(\mathbb{R})$;*
2. *δ_0 is the unique maximizing measure for φ;*
3.

$$\sup\left\{\int \varphi \, \mathrm{d}\nu \colon \nu \in \mathscr{M}, \operatorname{supp}(\nu) \subseteq [x_{m_0}, 1]\right\} < \varphi(0). \tag{2.7}$$

Moreover, conditions (2.5), (2.6), and (2.7) are open in $C^{1,\gamma}_{\dagger}(\mathbb{R})$. When the equivalent conditions 1,2 and 3 hold, we have that φ has a robust phase transition in temperature in $C^{1,\gamma}_{\dagger}(\mathbb{R})$ and that δ_0 is, robustly in $C^{1,\gamma}_{\dagger}(\mathbb{R})$, the unique maximizing measure for φ.

Remark 2.3. By (2.1) and Lemma 2.7 stated in §2.3, if φ is a potential in $C^{1,\gamma}_{\dagger}(\mathbb{R})$ with negative leading coefficient in $C^{1,\gamma}_{\dagger}(\mathbb{R})$, then there is n_0 in $\mathbb{N}$ satisfying Main Theorem (2.4) and (2.5) with c equal to 2 times the leading coefficient of φ.

2.2. **Preliminary results for the proof of Theorems 1, 2 and 3.** This section introduces the inducing scheme and the two-variable pressure function. The main technical result is the Bowen-type formula, from which we obtain two other results, Proposition 2.5 and Corollary 2.6, that we use in the proof of Theorem 1 and 3. The Bowen-type formula, together with Lemma 2.4, is used directly in the proof of Theorem 2.

2.2.1. *Induced map and the two-variables pressure.* We introduce the inducing scheme and the two-variable pressure functions, which are the primary tools for proving the existence of phase transitions in temperature.

For every $n \in \mathbb{N}$ let y_n be the unique point in $(x_1, 1]$ such that $f(y_n) = x_{n-1}$, and put $I_n \coloneqq (y_{n+1}, y_n]$ and $J_n \coloneqq (x_{n+1}, x_n]$. We have that for every n in $\mathbb{N}$, the map f^n sends J_n and I_n diffeomorphically onto $(x_1, 1]$. Define $m : (0, 1] \to \mathbb{N}$ by

$$m^{-1}(n) \coloneqq I_n \cup J_n,$$

$F : J_0 \to J_0$ by

$$F(x) \coloneqq f^{m(x)}(x),$$

and $L : (0, x_1] \to J_0$ by

$$L(x) \coloneqq f^{m(x)}(x).$$

The maps F and L are called *the first return map* and *the first landing map* of f onto J_0, respectively.

Let $\mathfrak{D}$ be the partition of J_0 given by the intervals I_n with n in $\mathbb{N}$. For every continuous potential $\varphi : [0, 1] \to \mathbb{R}$, every p in $\mathbb{R}$, and every ℓ in $\mathbb{N}$, we define

$$\begin{aligned} & Z_\ell(\varphi, p) \\ (2.8) \quad & \coloneqq \sum_{J \in \bigvee_{j=0}^{\ell-1} F^{-j}(\mathfrak{D})} \exp\left(\sup_{x \in J} \left(S_{m(x)+\cdots+m(F^{\ell-1}(x))}\varphi(x) - (m(x) + \cdots + m(F^{\ell-1}(x))p \right) \right). \end{aligned}$$

The sequence $(Z_\ell(\varphi, p))_{\ell \in \mathbb{N}}$ is in $(0, +\infty]$ and it is submultiplicative. Thus

$$(\log Z_\ell(\varphi, p))_{\ell \in \mathbb{N}} \tag{2.9}$$

is a subadditive sequence in $\mathbb{R} \cup \{+\infty\}$. Here, we use the convention that $\log(+\infty) = +\infty$. When the limit

$$\lim_{\ell \to +\infty} \frac{1}{\ell} \log Z_\ell(\varphi, p) \tag{2.10}$$

exists in the extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$, we denote it by $\mathscr{P}(\varphi, p)$ and call it *the two-variable pressure function for the potential φ with parameter p.* It is is exactly the pressure for the potential defined on J_0 by

$$S_{m(x)+\cdots+m(F^{\ell-1}(x))}\varphi(x) - (m(x) + \cdots + m(F^{\ell-1}(x))p \tag{2.11}$$

of the induced system (J_0, F) viewed as a full shift on countable many symbols [URM22].

When the sequence $(\log Z_\ell(\varphi, p))_{\ell \in \mathbb{N}}$ is finite, by the Subadditive Lemma, the limit (2.10) exists and is in $\mathbb{R} \cup \{-\infty\}$. In particular, $Z_1(\varphi, p) < +\infty$ implies $\mathscr{P}(\varphi, p) < +\infty$. From Lemma 2.4 below, we have that for potentials φ with bounded variation ergodic sums $\mathscr{P}(\varphi, p) < +\infty$ implies $Z_1(\varphi, p) < +\infty$.

2.2.2. *Bowen-type formula.* In this section, we state the Bowen-type formula relating the pressure of the original system to the two-variable pressure of the induced system for potentials whose ergodic sums have bounded variation.

Bowen-type formula. *Let $\varphi : [0,1] \to \mathbb{R}$ be a continuous potential with bounded bounded variation ergodic sums on J_0 for f. For each p in $\mathbb{R}$, we have*

$$\mathscr{P}(\varphi, p) \begin{cases} > 0 & \text{if } p < P(\varphi); \\ \leq 0 & \text{if } p = P(\varphi); \\ < 0 & \text{if } p > P(\varphi), \end{cases}$$

and

$$P(\varphi) = \inf\left\{p \in \mathbb{R} \colon \mathscr{P}(\varphi, p) \leq 0\right\}. \tag{2.12}$$

Now, we introduce a transfer operator for the induced system, which is helpful to compute the two-variable pressure.

Denote by L^+ the set of functions on J_0 taking values in $[0, +\infty]$. For every function $\varphi : [0,1] \to \mathbb{R}$ and every p in $\mathbb{R}$, define *the transfer operator* $\mathscr{L}_{\varphi,p}$ acting on a function h in L^+ by

$$(\mathscr{L}_{\varphi,p} h)(y) \coloneqq \sum_{x \in F^{-1}(y)} \exp(S_{m(x)}\varphi(x) - m(x)p) h(x). \tag{2.13}$$

The following lemma relates the transfer operator $\mathscr{L}_{\varphi,p}$ and the two-variable pressure $\mathscr{P}(\varphi, p)$.

Lemma 2.4. *Let $\varphi : [0,1] \to \mathbb{R}$ be a continuous potential with bounded variation ergodic sums on J_0 for f with constant $C > 0$. The following properties hold.*

1. *For every p in $\mathbb{R}$ the two-variable pressure $\mathscr{P}(\varphi, p)$ exists and is in $\mathbb{R} \cup \{+\infty\}$.*
2. *For all y in J_0, p in $\mathbb{R}$ and ℓ in $\mathbb{N}$, we have*

$$\exp(-C) Z_\ell(\varphi, p) \leq (\mathscr{L}^\ell_{\varphi,p} \mathbb{1})(y) \leq Z_\ell(\varphi, p), \tag{2.14}$$

and thus,

$$\mathscr{P}(\varphi, p) = \lim_{\ell \to +\infty} \frac{1}{\ell} \log(\mathscr{L}^\ell_{\varphi,p} \mathbb{1})(y), \tag{2.15}$$

where the limit is independent of y in J_0.

3. *For all y in J_0, p in $\mathbb{R}$ and ℓ in $\mathbb{N}$, we have*

$$\begin{aligned} \frac{1}{\ell} \log\left(\frac{(\mathscr{L}^\ell_{\varphi,p} \mathbb{1})(y)}{\exp(C)}\right) \leq \frac{1}{\ell} \log\left(\frac{Z_\ell(\varphi, p)}{\exp(C)}\right) \leq \mathscr{P}(\varphi, p) \\ \leq \frac{1}{\ell} \log Z_\ell(\varphi, p) \leq \frac{1}{\ell} \log\left(\exp(C)(\mathscr{L}^\ell_{\varphi,p} \mathbb{1})(y)\right). \end{aligned} \tag{2.16}$$

In particular, $\mathscr{P}(\varphi, p) < +\infty$ if and only if $Z_1(\varphi, p) < +\infty$.

Finally, we state some consequences of the Bowen-type formula. Recall that for every γ in $(0, +\infty)$, we denote by ω_γ the function from $[0,1]$ to $\mathbb{R}$ defined by $\omega_\gamma(x) \coloneqq -x^\gamma$. Observe that ω_γ is in $C^{1,\gamma}_\dagger(\mathbb{R})$. See also [CRL25, Proposition 3.3], for a proof of this result avoiding the inducing scheme.

Proposition 2.5. *For every γ in $(0, +\infty)$, the potential ω_γ has a phase transition in temperature if and only if $\gamma \leq \alpha$.*

The following corollary, together with [IRRL25, Theorem A.1], extends [Klo20, Theorem C] for Maneville-Pomeau maps.

Corollary 2.6. *Let $\varphi : [0,1] \to \mathbb{R}$ be a HÖLDER continuous potential. If there is δ in $(0,1]$ such that $\varphi \geq \varphi(0)$ on $[0,\delta]$, then φ does not have a phase transition in temperature.*

Proof. Take $\gamma > \alpha$, and put $m := \inf_{x \in [\delta,1]} \varphi(x) - \varphi(0)$ and $c := -m\delta^{-\gamma}$. If $m < 0$, then $\varphi(x) - \varphi(0) \geq c\omega_\gamma(x)$ on $[\delta,1]$ and $\varphi(x) - \varphi(0) \geq 0$ on $[0,\delta]$. Hence, $\varphi(x) - \varphi(0) \geq c\omega_\gamma(x)$ on $[0,1]$. By Corollary 1.4 and Proposition 2.5, we have

$$P(\varphi - \varphi(0)) \geq P(c\omega_\gamma) > c\omega_\gamma(0) = 0. \tag{2.17}$$

Then, $P(\varphi) > \varphi(0)$. Again, by Corollary 1.4, the potential φ does not have a phase transition in temperature. If $m \geq 0$ then $\varphi(x) - \varphi(0) \geq \omega_\gamma(x)$ on $[0,1]$. The same argument shows that φ has no phase transition in temperature, which concludes the proof of the result. □

2.3. **Proof of Theorems 1, 2, and 3.** Before beginning the proofs of the theorems, we state a lemma concerning the expansion at zero for potentials in $C^{1,\gamma}_{\dagger}(\mathbb{R})$. In particular, this lemma clarifies the meaning of the leading coefficient introduced in §1.2. The proof of the lemma is given in §3.2.

Lemma 2.7. *Let γ be in $(0,+\infty)$ and let $\varphi : [0,1] \to \mathbb{R}$ be a function. We have that φ belongs to $C^{1,\gamma}_{\dagger}(\mathbb{R})$ if and only if there are c in $\mathbb{R}$ and a function h in $C^{1}_{\dagger}(\mathbb{R})$ such that for every x in $[0,1]$*

$$\varphi(x) = \varphi(0) + cx^\gamma + h(x)x^\gamma, h(0) = 0, \text{ and } \lim_{x \to 0^+} h'(x)x = 0. \tag{2.18}$$

In particular, c and h are unique with $c = \lim_{x\to 0^+} \frac{\varphi'(x)}{\gamma x^{\gamma-1}}$.

Proof of Theorem 1. Under the assumptions of the theorem, we have that φ belongs to $C^{1,\gamma}_{\dagger}(\mathbb{R})$.

1. Assume that $\gamma \leq \alpha$ and $c > 0$. Replacing φ by $\varphi - \varphi(0)$ if necessary, assume $\varphi(0) = 0$. By (2.1) and Lemma 2.7, there is n_0 in $\mathbb{N}$ such that for every integer $k > n_0$ and every x in $(x_k, x_{k-1}]$ we have

$$\varphi(x) \geq \frac{c}{2}(\alpha k)^{-\gamma/\alpha}. \tag{2.19}$$

For every integer $n \geq 1$, let p_n be the periodic point of f of period n in $(y_{n+1}, y_n]$. We have that for every j in $\{1,\ldots,n-1\}$, $f^j(p_n)$ is in $(x_{n+1-j}, x_{n-j}]$. Together with (2.19) this implies

$$S_n\varphi(p_n) \geq S_{n-1-n_0}\varphi(f(p_n)) - (n_0+1)\|\varphi\| \geq \left(\frac{c}{2}\sum_{k=n_0+1}^{n-1}\frac{1}{(\alpha k)^{\gamma/\alpha}}\right) - (n_0+1)\|\varphi\|. \tag{2.20}$$

Since $\gamma \leq \alpha$ and $c > 0$, for n sufficiently large, we get $S_n\varphi(p_n) > 0$. Thus, the invariant probability measure μ_n supported on the orbit of p_n satisfies $\int \varphi \, \mathrm{d}\mu_n > 0$, which implies that δ_0 is not a maximizing measure for φ.

2. Assume that $\gamma \leq \alpha, c < 0$ and δ_0 is the unique maximizing measure for φ. By the Main Theorem and Remark 2.3, the potential φ has a phase transition in temperature.

3. Assume that φ has a phase transition in temperature. We have $c \leq 0$ by item 1, and by Corollary 1.4 and the Key Lemma, δ_0 is the unique maximizing measure for φ; see §1.5. Suppose we had $\gamma > \alpha$. By Lemma 2.7 there is c' in $(0,+\infty)$ such that for every x in $[0,1]$

we have $\varphi(x) - \varphi(0) \geq c'\omega_\gamma(x)$. By Proposition 2.5 and Corollary 1.4, for every $\beta > 0$ we have

$$P(\beta\varphi) - \beta\varphi(0) = P(\beta\varphi - \beta\varphi(0)) \geq P(\beta c'\omega_\gamma) > 0. \tag{2.21}$$

Then, by Corollary 1.4, the potential φ does not have a phase transition in temperature, which is a contradiction. $\square$

Proof of Theorem 2. By Theorem 1(3), the leading coefficient c of φ in $C^{1,\alpha}_{\dagger}(\mathbb{R})$ is nonpositive. Suppose we had $c = 0$. By Lemma 2.7 there is a continuous function $h : [0, 1] \to \mathbb{R}$ such that $h(0) = 0$ and for every x in $[0, 1]$ we have $\varphi(x) = \varphi(0) + h(x)x^\alpha$. Then, for every β in $(0, +\infty)$ there is n_1 in $\mathbb{N}$ such that for every x in $[0, x_{n_1}]$ we have

$$-\frac{\alpha\beta^{-1}}{2}x^\alpha \leq \varphi(x) - \varphi(0). \tag{2.22}$$

By (2.1), taking n_1 larger, if necessary, for every integer $n \geq n_1$ we have

$$x_n^\alpha \leq \frac{2}{\alpha n}. \tag{2.23}$$

Then, there is a constant $K > 0$ such that for every n in $\mathbb{N}$ we get

$$\frac{K}{n^{\beta^{-1}}} \leq \exp(S_n\varphi(x_n) - n\varphi(0)). \tag{2.24}$$

Thus,

$$(\mathscr{L}_{\beta\varphi, \beta\varphi(0)}\mathbb{1})(1) = +\infty. \tag{2.25}$$

By the Bowen-type formula and Lemma 2.4, $P(\beta\varphi) > \beta\varphi(0)$. Thus, by Corollary 1.4, the potential φ does not have a phase transition in temperature, which is a contradiction and finishes the proof of the theorem. $\square$

Proof of Theorem 3. The implication $1 \Rightarrow 2$ follows from Theorem 1(3). Now we prove $2 \Rightarrow 3$. Theorem 1(1) shows that the leading coefficient c of φ in $C^{1,\gamma}_{\dagger}(\mathbb{R})$ is nonpositive. Let's demonstrate that c is negative. Suppose we had $c = 0$. Then, for every $\varepsilon > 0$ the potential $\varphi - \varepsilon\omega_\gamma$ has positive leading coefficient in $C^{1,\gamma}_{\dagger}(\mathbb{R})$. By Theorem 1(1), δ_0 is not a maximizing measure for $\varphi - \varepsilon\omega_\gamma$, which is a contradiction. Therefore, $c < 0$. Together with Theorem 1(2), this implies that φ has a phase transition in temperature.

Finally, the implication $3 \Rightarrow 1$ follows from the Main Theorem and Remark 2.3, concluding the proof of the theorem. $\square$

3. Preliminaries

3.1. **Indifferent branch and bounded distortion.** Note that the function $\log Df$ is strictly increasing and Hölder continuous of exponent $\min\{1, \alpha\}$. For every j in $\mathbb{N}_0$ note that $f(x_{j+1}) = x_j$ and $f(J_{j+1}) = J_j$.

We start by proving (2.1) and some estimates for Df^n on J_n.

Lemma 3.1. *We have*

$$\lim_{n \to +\infty} n \cdot x_n^\alpha = \frac{1}{\alpha}. \tag{3.1}$$

Moreover, there is ε_0 *in* $(0,1)$ *such that for all* n *in* $\mathbb{N}$ *and* x *in* J_n *we have*

$$(1+\varepsilon_0 n)^{\frac{1}{\alpha}+1} \leq Df^n(x) \leq (1+\varepsilon_0^{-1} n)^{\frac{1}{\alpha}+1}. \tag{3.2}$$

Proof. Following [Fat19, §11], we consider the maps $\widehat{f}$, h, and $\mathscr{F}$ from $(0,+\infty)$ to $(0,+\infty)$ defined by

$$\widehat{f}(x) := x(1+x^\alpha), h(x) := x^{-\alpha}, \text{ and } \mathscr{F} := h \circ \widehat{f} \circ h^{-1}. \tag{3.3}$$

For every X in $(0,+\infty)$, we have

$$\mathscr{F}(X) = X(1+X^{-1})^{-\alpha}. \tag{3.4}$$

It follows that there is C in $(0,+\infty)$, such that for every sufficiently large X we have

$$|\mathscr{F}(X) - (X-\alpha)| \leq C\frac{1}{X}. \tag{3.5}$$

For every n in $\mathbb{N}_0$ put $X_n := h(x_n)$. Since $\widehat{f}$ coincides with f on $[0,x_1]$, for every n in $\mathbb{N}_0$ we have

$$\mathscr{F}(X_{n+1}) = X_n. \tag{3.6}$$

Since the sequence $(x_n)_{n\in\mathbb{N}_0}$ is strictly decreasing and the only fixed point of f on $[0,x_1]$ is 0, we have

$$\lim_{n\to+\infty} x_n = 0 \text{ and } \lim_{n\to+\infty} X_n = +\infty. \tag{3.7}$$

Together with (3.5) and (3.6), this implies that for every sufficiently large n we have $X_{n+1} \geq X_n + 2\alpha/3$. It follows that for every sufficiently large n, we have

$$X_n \geq \alpha n/2. \tag{3.8}$$

Next, define for each n in $\mathbb{N}$ the number $\delta_n := X_n - \alpha n$. By (3.5) and (3.8), for every sufficiently large n we have

$$|\delta_{n+1} - \delta_n| \leq (2C\alpha^{-1})\frac{1}{n}. \tag{3.9}$$

It follows that there is a constant C' in $(0,+\infty)$ such that for every sufficiently large n we have

$$|X_n - \alpha n| = |\delta_n| \leq C' \log n. \tag{3.10}$$

This implies (3.1). To prove (3.2), note that Df is increasing, so for all n in $\mathbb{N}$ and x in J_n we have

$$\begin{aligned}\prod_{j=1}^{n}\left(1+(1+\alpha)\frac{1}{X_{j+1}}\right) &= Df^n(x_{n+1}) \leq Df^n(x) \\ &\leq Df^n(x_n) = \prod_{j=1}^{n}\left(1+(1+\alpha)\frac{1}{X_j}\right).\end{aligned} \tag{3.11}$$

By (3.9), there is C'' in $(0,+\infty)$ such that for every sufficiently large j we have

$$\left|\log\left(1+(1+\alpha)\frac{1}{X_j}\right) - \left(\frac{1}{\alpha}+1\right)\frac{1}{j}\right| \leq C''\frac{\log j}{j^2}. \tag{3.12}$$

Combined with (3.11), this implies that there is C''' in $(1,+\infty)$ such that for every sufficiently large n we have

$$\frac{1}{C'''} n^{\frac{1}{\alpha}+1} \le Df^n(x) \le C''' n^{\frac{1}{\alpha}+1}. \tag{3.13}$$

Together with the fact that for every n in $\mathbb{N}$ we have $Df^n(x_n) > 1$, this implies (3.2). □

Lemma 3.2 (Bounded distortion)**.** *There are ε_1 in $(0,1)$ and C_1 in $(1,+\infty)$, such that the following properties hold. For every n in $\mathbb{N}$ and every connected component J of $f^{-n}(J_0)$, we have*

$$|J| \le \frac{|J_0|}{(1+\varepsilon_1 n)^{\frac{1}{\alpha}+1}} \tag{3.14}$$

and for all x and y in J we have

$$Df^n(x) \ge (1+\varepsilon_1 n)^{\frac{1}{\alpha}+1} \text{ and } C_1^{-1} \le \frac{Df^n(x)}{Df^n(y)} \le C_1. \tag{3.15}$$

Proof. Let ε_0 in $(0,1)$ be from Lemma 3.1 and let ε_1 in $(0,\varepsilon_0]$ be such that for every x in J_0 we have

$$Df(x) \ge (1+\varepsilon_1)^{\frac{1}{\alpha}+1}. \tag{3.16}$$

Put

$$C_1' := |J_0| \cdot |Df|_{\min\{1,\alpha\}} \sum_{m=0}^{\infty} \frac{1}{(1+\varepsilon_1 n)^{\min\{1,\alpha\}\left(\frac{1}{\alpha}+1\right)}} \tag{3.17}$$

and note that C_1' is finite because $\min\{1,\alpha\}\left(\frac{1}{\alpha}+1\right) > 1$.

Let n in $\mathbb{N}$ be given and let J be a connected component of $f^{-n}(J_0)$. For each m in $\{0,\ldots,n-1\}$, let a_m be equal to 1 if $f^m(J) \subseteq J_0$ and to 0 otherwise. Denote by r the number of 1's in the sequence $a_0 \cdots a_m$, by s the number of blocks of 0's, and by $\ell_0, \ldots, \ell_s$ the lengths of the blocks of 0's. We thus have $r + \ell_1 + \cdots + \ell_s = n$. Combining (3.2) and (3.16), we obtain for every x in J

$$Df^n(x) \ge (1+\varepsilon_1)^{r\left(\frac{1}{\alpha}+1\right)} \prod_{k=1}^{s} (1+\varepsilon_1 \ell_k)^{\frac{1}{\alpha}+1} \ge (1+\varepsilon_1 n)^{\frac{1}{\alpha}+1}. \tag{3.18}$$

This proves the first inequality in (3.15) and implies (3.14). Using the chain rule, (3.14), and the definition (3.17) of C_1', we obtain for all x and y in J

$$\begin{aligned} \left|\log \frac{Df^n(x)}{Df^n(y)}\right| &\le \sum_{m=0}^{n-1} |\log Df(f^m(x)) - \log Df(f^m(y))| \\ &\le |Df|_{\min\{1,\alpha\}} \sum_{m=0}^{n-1} |f^m(J)|^{\min\{1,\alpha\}} \le C_1'. \end{aligned} \tag{3.19}$$

This proves (3.15) with $C_1 = \exp(C_1')$. □

3.2. **Bounded variation ergodic sums and expansion at zero for potential in $C^{1,\gamma}_{\dagger}(\mathbb{R})$.**

Proof of Lemma 2.2. Let γ be in $(0,+\infty)$, and let $\varepsilon_1 \in (0,1)$ be given by Lemma 3.2. Put $\gamma_0 := \min\{1,\gamma\}$. By (2.1), there is a constant $K \geq 1$ such that for every n in $\mathbb{N}$ we have

$$x_n^{\gamma_0 - 1} \leq K \frac{1}{n^{\frac{\gamma_0 - 1}{\alpha}}}. \tag{3.20}$$

Put

$$D := K\gamma\varepsilon_1^{-(\alpha^{-1}+1)} \sum_{j=1}^{+\infty} \frac{1}{j^{1+\frac{\gamma_0}{\alpha}}}. \tag{3.21}$$

Fix φ in $C^{1,\gamma}_{\dagger}(\mathbb{R})$, an integer $n \geq 1$, a connected component J of $f^{-n}(J_0)$, and x and y in J. Observe that for every j in $\{0,\ldots,n-1\}$ we have $f^j(J) \subseteq (x_{n-j+1}, 1]$. Together with (3.20), this implies that for j in $\{0,\ldots,n-1\}$ we have

$$\sup_{u \in f^j(J)} |\varphi'(u)| \leq |\varphi|_{1,\gamma} \sup_{u \in f^j(J)} \gamma u^{\gamma-1} \leq |\varphi|_{1,\gamma} \frac{K\gamma}{(1+n-j)^{\frac{\gamma_0-1}{\alpha}}}. \tag{3.22}$$

Using Lemma 3.2(3.14) we get

$$\left(\sup_{u \in f^j(J)} |\varphi'(u)| \right) |f^j(x) - f^j(y)| \leq |\varphi|_{1,\gamma} K\gamma\varepsilon_1^{-(\alpha^{-1}+1)} \frac{1}{(1+n-j)^{1+\frac{\gamma_0}{\alpha}}}. \tag{3.23}$$

Therefore,

$$\begin{aligned} |S_n\varphi(x) - S_n\varphi(y)| &\leq \sum_{j=0}^{n-1} |\varphi(f^j(x)) - \varphi(f^j(y))| \\ &\leq \sum_{j=0}^{n-1} \left(\sup_{u \in f^j(J)} |\varphi'(u)| \right) |f^j(x) - f^j(y)| \\ &\leq D|\varphi|_{1,\gamma}. \end{aligned} \tag{3.24}$$

This concludes the proof of the lemma. $\square$

The following proof is quite simple, and we included it for completeness.

Proof of Lemma 2.7. If φ belongs to $C^{1,\gamma}_{\dagger}(\mathbb{R})$, put $c := \lim_{x\to 0^+} \frac{\varphi'(x)}{\gamma x^{\gamma-1}}$, and define the function $h : (0,1] \to \mathbb{R}$ by

$$h(x) := \frac{\varphi(x) - \varphi(0)}{x^\gamma} - c.$$

By the definition of c, for every $\varepsilon > 0$, there is $\delta > 0$ such that for every x in $(0,\delta)$ we have

$$(c-\varepsilon)\gamma x^{\gamma-1} \leq \varphi'(x) \leq (c+\varepsilon)\gamma x^{\gamma-1}. \tag{3.25}$$

Then,

$$(c-\varepsilon)x^\gamma \leq \varphi(x) - \varphi(0) \leq (c+\varepsilon)x^\gamma. \tag{3.26}$$

It follows that

$$\lim_{x\to 0^+} h(x) = 0, \tag{3.27}$$

so the function h extends continuously to $[0,1]$. Denote the extension by h and note that $h(0)=0$. Together with the definition of h, for every x in $[0,1]$, we get that

$$\varphi(x)=\varphi(0)+cx^{\gamma}+h(x)x^{\gamma}. \tag{3.28}$$

Since h is continuously differentiable on $(0,1]$, we have that h is in $C^1_{\dagger}(\mathbb{R})$. Finally, from (3.27) and (3.28), we get that

$$\lim_{x\to 0^+} h'(x)x=0. \tag{3.29}$$

The reverse implication follows by direct computation. So, the lemma is proved. □

3.3. **Continuous extension and topological exactness.** In this subsection, we construct a continuous extension of the dynamical system $([0,1],f)$, which allows the application of the theory of thermodynamics formalism on compact metric spaces. We introduce the following notation for continuous dynamical systems on compact metric spaces. Given a compact metric space Z and a continuous transformation $T:Z\to Z$, we denote by $\mathscr{M}_T$ the space of Borel probability measures invariant by T. For every μ in $\mathscr{M}_T$ we denote by $h_\mu(T)$ the entropy of μ, and given a continuous potential $\psi:Z\to\mathbb{R}$ we denote by $P_T(\psi)$ the pressure for the potential ψ of T.

Lemma 3.3. *Put $Y:=\bigcup_{n\in\mathbb{N}_0} f^{-n}(x_1)$. There is a totally ordered set X, endowed with the order topology, and a continuous surjective map $\pi: X\to[0,1]$ such that the following hold.*

1. *The sets $X\setminus\pi^{-1}(Y)$ and $[0,1]\setminus Y$ are equal as totally ordered sets, the map π is the identity on $X\setminus\pi^{-1}(Y)$, and $\pi^{-1}(Y)$ consists of two disjoint copies Y^- and Y^+ of Y such that for every y in Y one has $\pi^{-1}(\{y\})=\{y^-,y^+\}$ and $y^-<y^+$.*
2. *The order topology on X is compact and metrizable.*
3. *There is continuous map $\widetilde{f}:X\to X$ such that $\pi\circ\widetilde{f}=f\circ\pi$,*
$$Y^-=\bigcup_{n\in\mathbb{N}}\widetilde{f}^{-n}(1)\setminus\{1\}=\bigcup_{n\in\mathbb{N}}\widetilde{f}^{-n}(x_1^-),\ \text{ and } Y^+=\bigcup_{n\in\mathbb{N}}\widetilde{f}^{-n}(0)\setminus\{0\}=\bigcup_{n\in\mathbb{N}}\widetilde{f}^{-n}(x_1^+).$$
4. *Put $\widetilde{P}_0:=[0,x_1^-]$ and $\widetilde{P}_1:=[x_1^+,1]$. The map $\widetilde{\pi}:X\to\{0,1\}^{\mathbb{N}_0}$ given by $(\widetilde{\pi}(x))_k=i$ if $\widetilde{f}^k(x)\in\widetilde{P}_i$ is a topological conjugacy from $(X,\widetilde{f})$ to the full shift $(\{0,1\}^{\mathbb{N}_0},\sigma)$.*
5. *For every invariant measure ν for f there is unique invariant measure μ for $\widetilde{f}$ such that $\pi_*\mu=\nu$ and the systems $(X,\widetilde{f},\mu)$ and $([0,1],f,\nu)$ are isomorphic in measure. In particular, the map $\pi_*:\mathscr{M}_{\widetilde{f}}\to\mathscr{M}$ is one-to-one.*

The proof of Lemma 3.3 is given after the following couple of lemmas. Put $\mathcal{P}:=\{[0,x_1],(x_1,1]\}$ and denote by g_0 and g_1 the inverse branches of f on $[0,x_1]$ and $(x_1,1]$, respectively. In general, given two partitions $\mathscr{P}$ and $\mathscr{Q}$ of some set Z we denote by $\mathscr{P}\vee\mathscr{Q}$ the refinement of the both partitions defined by

$$\mathscr{P}\vee\mathscr{Q}:=\{P\cap Q\colon P\in\mathscr{P},Q\in\mathscr{Q}\}. \tag{3.30}$$

Lemma 3.4. *For all x in $(0,1]$, n in $\mathbb{N}$ and $y\in f^{-n}(x)$ there is a unique inverse branch ϕ of $f^n|_{(0,1]}$ defined on $(0,1]$ such that $y\in\phi((0,1])$. For every n in $\mathbb{N}$ there is a unique inverse branch ϕ of f^n defined on $[0,1]$ such that $\phi([0,1])=[0,x_n]$. In particular, for every n in $\mathbb{N}$ and every Q in $\bigvee_{k=0}^{n-1}f^{-k}\mathcal{P}$ we have that f^n sends Q diffeomorphically onto $(0,1]$ except when 0 belongs to Q in which case f^n sends Q diffeomorphically onto $[0,1]$.*

Proof. For every n in $\mathbb{N}$, every inverse branch of f^n has the form $g_{i_1} \circ \cdots \circ g_{i_n}$. When at least one of the g_{i_j} is equal to g_1, the domain of the inverse branch is $(0,1]$ and the image is an interval of the form $(a,b]$ with a and b in $\bigcup_{k=0}^{n} f^{-k}(1)$. When all the g_{i_j} are equal to g_0, the domain of the inverse branch is $[0,1]$ and the image is $[0,x_n]$. In particular, the images of two different inverse branches of f^n are disjoint, and the union of the images of all inverse branches of f^n covers $[0,1]$. On the other hand, since $\{0\} = f^{-1}(0)$, for all x in $(0,1]$, n in $\mathbb{N}$ and y in $f^{-n}(x)$ we have $y \neq 0$. Therefore, there is a unique inverse branch ϕ of $f^n|_{(0,1]}$ defined on $(0,1]$ such that $y \in \phi((0,1])$, and there is a unique inverse branch of f^n defined on $[0,1]$ whose image is $[0,x_n]$. $\square$

Let Z be a topological space and $T : Z \to Z$ be a map. One says that T is *topologically exact* if for every z in Z and every neighborhood V of z there is k in $\mathbb{N}$ such that $T^k(V) = Z$.

Lemma 3.5. *We have that*

$$\max_{Q \in \bigvee_{k=0}^{n-1} f^{-k}\mathcal{P}} \operatorname{diam} Q \to 0 \tag{3.31}$$

as n goes to $+\infty$. In particular, the map f is topologically exact on $(0,1]$.

Proof. Notice that by (2.1) there is $C > 0$ such that for every n in $\mathbb{N}$ one has

$$x_n \leq C n^{-\frac{1}{\alpha}}. \tag{3.32}$$

Let n be in $\mathbb{N}$ and let Q be in $\bigvee_{k=0}^{n-1} f^{-k}\mathcal{P}$. By Lemma 3.4, there is $i_1 \cdots i_n$ in $\{0,1\}^n$ such that $Q = g_{i_1} \circ \cdots \circ g_{i_n}((0,1])$ or $Q = g_0^n([0,1]) = [0,x_n]$. If there is an integer $j \in [\frac{n}{2}, n]$ such that $i_j = 1$ then by Lemma 3.2 we have

$$\operatorname{diam} Q \leq |J_0|(1+\varepsilon_1 j)^{-\left(\frac{1}{\alpha}+1\right)} \leq |J_0| \left(1 + \varepsilon_1 \left\lceil \frac{n}{2} \right\rceil\right)^{-\left(\frac{1}{\alpha}+1\right)}. \tag{3.33}$$

If for every integer $j \in [\frac{n}{2}, n]$ we have that $i_j = 0$ then

$$f^{\lfloor \frac{n}{2} \rfloor}(Q) \subseteq [0, x_{n-\lceil \frac{n}{2} \rceil}]. \tag{3.34}$$

By (3.32) and the fact for every x in $[0,1]$ one has $Df(x) \geq 1$ we get that

$$\operatorname{diam} Q \leq \operatorname{diam} f^{\lfloor \frac{n}{2} \rfloor}(Q) \leq x_{n-\lceil \frac{n}{2} \rceil} \leq C \left\lfloor \frac{n}{2} \right\rfloor^{-\frac{1}{\alpha}}. \tag{3.35}$$

Together with (3.33), this implies the first part of the lemma.

Now, we prove that f is topologically exact on $(0,1]$. Observe that, by (3.31), for every open set V contained in $(0,1]$ there are n in $\mathbb{N}$ and Q in $\bigvee_{k=0}^{n-1} f^{-k}\mathcal{P}$ such that $Q \setminus \{0\} \subset V$. By Lemma 3.4, we have $(0,1] = f^n(Q \setminus \{0\}) \subset f^n(V)$, which implies that f is topologically exact on $(0,1]$ and concludes the proof of the lemma. $\square$

Proof of Lemma 3.3. The idea of the construction of the extension is well known. We follow [Kel98, Appendix A.5]. The set X is obtained from $[0,1]$ by doubling the points in Y. Thus, every y in Y is replaced by two points y^- and y^+ and one declares that $y^- < y^+$. This endows X with a total order. The order topology on X is the topology generated by open order intervals and the intervals of the form $[0,b)$ and $(a,1]$ for all a and b in X. Denote by $\pi : X \to [0,1]$ the projection. Observe that π is continuous since the preimages of every open interval in $[0,1]$ is an open order interval in X. This proves the first statement of the lemma and item 1.

Observe that the total order of X has the least upper bound property, so the order topology on X is compact; see, for instance, [Mun00, Theorem 27.1]. Also, observe that the order topology is Hausdorff, which, together with the compactness, implies that X is regular. Since it is second countable, by the Urysohn Metrization Theorem, see, for instance, [Mun00, Theorem 34.1], the order topology on X is metrizable, proving item 2.

The map $\widetilde{f}$ coincides with f on $[0,1] \setminus Y$ and on y^- and y^+ is defined by continuity. Thus $\widetilde{f}(x_1^-) = 1$ and $\widetilde{f}(x_1^+) = 0$, and for every y in Y different from x_1 we have $\widetilde{f}(y^+) = f(y)^+$ and $\widetilde{f}(y^-) = f(y)^-$. Since $\widetilde{f} : [0, x_1^-] \to \widetilde{f}([0, x_1^-])$ and $\widetilde{f} : [x_1^+, 1] \to \widetilde{f}([x_1^+, 1])$ are increasing bijections, $\widetilde{f}$ is continuous. From the definition of X and $\widetilde{f}$, we have that $\pi \circ \widetilde{f} = f \circ \pi$, which finishes the proof of item 3.

To show item 4, we first define a distance on X compatible with the topology. Define an atomic measure λ by

$$\lambda := \sum_{n=0}^{+\infty} \sum_{y \in f^{-n}(x_1)} 4^{-n} \delta_y, \tag{3.36}$$

and the increasing maps $\iota^-, \iota^+ : [0,1] \to \mathbb{R}$ by

$$\iota^-(x) := x + \lambda([0,x)) \text{ and } \iota^+(x) := x + \lambda([0,x]). \tag{3.37}$$

Observe that

$$\begin{aligned}
&\iota^+(x) < \iota^-(x'), \text{ for } x < x', \\
&\iota^-(x) = \iota^+(x), \text{ for } x \in [0,1] \setminus Y, \\
&\iota^+(y) = \iota^-(y) + 4^{-n}, \text{ for } y \in f^{-n}(x_1).
\end{aligned}$$

We define $\iota : X \to \mathbb{R}$ by

$$\iota(x) = \begin{cases} \iota^+(x), & \text{if } x \in X \setminus Y^-, \\ \iota^-(x), & \text{if } x \in Y^-. \end{cases} \tag{3.38}$$

Observe that $\iota(X)$ is closed in $\mathbb{R}$, so its order topology coincides with the induced topology from $\mathbb{R}$. Furthermore, $\iota : X \to \iota(X)$ is an increasing bijection and thus a homeomorphism. For all x and x' in X define the distance

$$d(x, x') := |\iota(x) - \iota(x')|. \tag{3.39}$$

Put $\widetilde{Y} := \pi^{-1}(Y)$. Now, we prove that the map $\widetilde{\pi}$ in item 4 is a conjugacy. For every n in $\mathbb{N}$, and every word w on the alphabet $\{0,1\}$ of length n denote by $[w]$ the cylinder in $\{0,1\}^{\mathbb{N}_0}$ that has the word w in the first n coordinates. We have that $\widetilde{\pi}^{-1}([w])$ is an interval $\widetilde{J}$ of the form $[a^+, b^-], [0, b^-]$ or $[a^+, 1]$ with a and b in Y. Since $[a^+, b^-] = (a^-, b^+), [0, b^-] = [0, b^+)$ and $[a^+, 1] = (a^-, 1]$ we get that $\widetilde{\pi}$ is continuous. By Lemma 3.4, each of the intervals $(a, b], (0, b]$ or $(a, 1]$ is the image of $(0, 1]$ by an inverse branch of $f^n|_{(0,1]}$, and denote by J any of these intervals. Since there is no other point of $\pi^{-1}(\bigcup_{k=0}^{n-1} f^{-k}(x_1))$ in $\widetilde{J}$ we have that

$$\operatorname{diam}(\widetilde{J}) \leq \operatorname{diam}(J) + 2^{-n}. \tag{3.40}$$

By Lemma 3.5, we have that $\operatorname{diam}(J) \to 0$ as the length of w goes to $+\infty$. Then, for every ω in $\{0,1\}^{\mathbb{N}_0}$ we have that there is a unique x in X such that $\widetilde{\pi}(x) = \omega$. Since X is compact, we get that $\widetilde{\pi}$ is a homeomorphism, and since from the definition of $\widetilde{\pi}$ we have $\widetilde{\pi} \circ \widetilde{f} = \sigma \circ \widetilde{\pi}$, we conclude that the map $\widetilde{\pi}$ is a conjugacy.

Observe that a subset B of X is the same as the set $\pi(B)$ in $[0,1]$ with the points in $\pi(B) \cap Y$ duplicated. Then the BOREL σ-algebra of $X \setminus \widetilde{Y}$ is the same than the BOREL σ-algebra of $[0,1] \setminus Y$. Since the set Y does not support any invariant BOREL probability for f, the same holds for $\widetilde{Y}$ and $\widetilde{f}$. We deduce that for every invariant BOREL probability measure μ for $\widetilde{f}$ we have that $\pi_* \mu$ is equal to the extension to the BOREL σ-algebra of $[0,1]$ of the restriction of μ to the BOREL σ-algebra of $X \setminus \widetilde{Y}$, which coincides with the BOREL σ-algebra of $[0,1] \setminus Y$. On the other hand, for every invariant BOREL probability measure ν for f, there is a unique μ in $\mathscr{M}_{\widetilde{f}}$ such that $\pi_* \mu = \nu$. Therefore, the dynamical systems $(X, \widetilde{f}, \mu)$ and $([0,1], f, \nu)$ are measurably isomorphic. This finishes the proof of item 5 and the proof of the lemma. □

3.4. **Topological and tree pressures.** Let $\psi : [0,1] \to \mathbb{C}$ be a function. For every n in $\mathbb{N}$, put $S_n \psi \coloneqq \sum_{k=0}^{n-1} \psi \circ f^k$.

Lemma 3.6. *For every continuous potential $\varphi : [0,1] \to \mathbb{R}$, the following properties hold.*

1. *We have*

$$P(\varphi) = \lim_{n \to +\infty} \frac{1}{n} \log \sum_{Q \in \bigvee_{k=0}^{n-1} f^{-k} \mathcal{P}} \sup_{x \in Q} \exp(S_n \varphi(x)). \tag{3.41}$$

   In particular, $P(0) = \log 2$.

2. *For every $y \in (0,1]$ we have*

$$P(\varphi) = \lim_{n \to +\infty} \frac{1}{n} \log \sum_{x \in f^{-n}(y)} \exp(S_n \varphi(x)). \tag{3.42}$$

Proof. Let $X, \pi, \widetilde{f}, Y, Y^-$ and Y^+ be as in Lemma 3.3 and put $\widetilde{\varphi} \coloneqq \varphi \circ \pi$.

1. From Lemma 3.3(5) for every continuous potential $\varphi : [0,1] \to \mathbb{R}$ and $\widetilde{\varphi} \coloneqq \varphi \circ \pi$ we have

$$P_{\widetilde{f}}(\widetilde{\varphi}) = \sup_{\mu \in \mathscr{M}_{\widetilde{f}}} h_\mu(\widetilde{f}) + \int \widetilde{\varphi} \, \mathrm{d}\mu = \sup_{\nu \in \mathscr{M}} h_\nu + \int \varphi \, \mathrm{d}\nu = P(\varphi). \tag{3.43}$$

Let $\widetilde{P}_0$ and $\widetilde{P}_1$ be the sets in Lemma 3.3(4). Observe that $\mathscr{C} = \{\widetilde{P}_0, \widetilde{P}_1\}$ is the open cover of X corresponding to the open cover $\{[0],[1]\}$ in $\{0,1\}^{\mathbb{N}_0}$. Thus, by the Variational Principle (see for instance [Bow08, 2.17. Variational Principle and Lemma 1.20]), we get that

$$P_{\widetilde{f}}(\widetilde{\varphi}) = \lim_{n \to +\infty} \frac{1}{n} \log \sum_{P \in \bigvee_{k=0}^{n-1} \widetilde{f}^{-k} \mathscr{C}} \sup_{x \in P} \exp(S_n \widetilde{\varphi}(x)). \tag{3.44}$$

Notice that by the equation $\pi \circ \widetilde{f} = f \circ \pi$ in item 3 in Lemma 3.3, for every n in $\mathbb{N}$, we have $S_n \widetilde{\varphi}(x) = S_n \varphi(\pi(x))$. Also observe that every Q in $\bigvee_{k=0}^{n-1} f^{-k} \mathcal{P}$ is of the form $[0,b], (a,b]$ or $(a,1]$, and that $[0,b^-], [a^+,b^-]$ or $[a^+,1]$ is in $\bigvee_{k=0}^{n-1} \widetilde{f}^{-k} \mathscr{C}$. Together with the continuity of $\widetilde{\varphi}$ and $\widetilde{f}$ we get

$$\sup_{x \in [a^+,b^-]} \exp(S_n \widetilde{\varphi}(x)) = \sup_{x \in (a^+,b^-]} \exp(S_n \widetilde{\varphi}(x)) = \sup_{z \in (a,b]} \exp(S_n \varphi(z)),$$

and the same holds for Q of the form $[0,b]$ or $(a,b]$. Then,

$$\lim_{n\to+\infty}\frac{1}{n}\log\sum_{P\in\bigvee_{k=0}^{n-1}\widetilde{f}^{-k}\mathscr{C}}\sup_{x\in P}\exp(S_n\widetilde{\varphi}(x))=\lim_{n\to+\infty}\frac{1}{n}\log\sum_{Q\in\bigvee_{k=0}^{n-1}f^{-k}\mathcal{P}}\sup_{x\in Q}\exp(S_n\varphi(x)). \tag{3.45}$$

Together with (3.43) and (3.44), this implies (3.41). Also from (3.43) and Lemma 3.3(4), we get that $P(0)=P_{\widetilde{f}}(0)=\log 2$.

2. Fix y in $(0,1]$. By Lemma 3.5, the diameter of the elements in $\bigvee_{k=0}^{n-1}f^{-k}\mathcal{P}$ tends to zero as n goes to $+\infty$. Then, since φ is uniformly continuous on $[0,1]$, for every $\varepsilon>0$ there is $n_0\in\mathbb{N}$ such that for every integer $n\geq n_0$, every $Q\in\bigvee_{k=0}^{n-1}f^{-k}\mathcal{P}$ and every $x_Q\in Q\cap f^{-n}(\{y\})$ we have

$$\sup_{z\in Q}|S_{n-n_0}\varphi(z)-S_{n-n_0}\varphi(x_Q)|\leq(n-n_0)\varepsilon.$$

Then,

$$\exp(S_n\varphi(x_Q))\leq\sup_{z\in Q}\exp(S_n\varphi(z))\leq\exp((n-n_0)\varepsilon+n_0\|\varphi\|)\exp(S_n\varphi(x_Q)), \tag{3.46}$$

which implies

$$\begin{aligned}\sum_{x\in f^{-n}(y)}\exp(S_n\varphi(x))\leq&\sum_{Q\in\bigvee_{k=0}^{n-1}f^{-k}\mathcal{P}}\sup_{x\in Q}\exp(S_n\varphi(x))\\ \leq&\exp((n-n_0)\varepsilon+n_0\|\varphi\|)\sum_{x\in f^{-n}(y)}\exp(S_n\varphi(x)).\end{aligned} \tag{3.47}$$

Therefore, by item 1, we have

$$\limsup_{n\to+\infty}\frac{1}{n}\log\sum_{x\in f^{-n}(y)}\exp(S_n\varphi(x))\leq P(\varphi)\leq\varepsilon+\liminf_{n\to+\infty}\frac{1}{n}\log\sum_{x\in f^{-n}(y)}\exp(S_n\varphi(x)), \tag{3.48}$$

which implies (3.42). $\square$

4. Proof of the Bowen-type formula and applications

In this section, we prove the Bowen-type formula and all the other results used in the proof of Theorems 1, 2 and 3 in §2. Like, Lemma 2.4 and Proposition 2.5. We also derive some additional consequences as Corollary 4.1 and Proposition 4.2.

4.1. **Proof of the Bowen-type formula and Lemma 2.4.** We start with the proof of Lemma 2.4, which is used in the proof of the Bowen-type formula.

Proof of Lemma 2.4. First, observe that by the bounded distortion property for every p in $\mathbb{R}$, and all k and ℓ in $\mathbb{N}$, we have

$$(\exp(-C)Z_\ell(\varphi,p))^k\leq Z_{k\ell}(\varphi,p)\leq Z_\ell(\varphi,p)^k. \tag{4.1}$$

Then, for $Z_1(\varphi,p)$ finite, the sequence $(\log Z_\ell(\varphi,p))_{\ell\in\mathbb{N}}$ is finite and subadditive. Thus, by the Subadditive Lemma $\mathscr{P}(\varphi,p)$ exists and belongs to $\{-\infty\}\cup\mathbb{R}$. But, by (4.1), we have that $\mathscr{P}(\varphi,p)$ is in $\mathbb{R}$. For $Z_1(\varphi,p)=+\infty$, by (4.1) for every ℓ in $\mathbb{N}$ we have $Z_\ell(\varphi,p)=+\infty$. Then, $\mathscr{P}(\varphi,p)$ exists, and it is equal to $+\infty$. This proves item 1.

For all y in J_0, p in $\mathbb{R}$ and ℓ in $\mathbb{N}$ we have

$$(4.2)\quad (\mathscr{L}_{\varphi,p}^{\ell}h)(y) = \sum_{x\in F^{-\ell}(y)} \exp(S_{m(x)+\cdots+m(F^{\ell-1}(x))}\varphi(x) - (m(x)+\cdots+m(F^{\ell-1}(x)))p)h(x).$$

Together with the bounded distortion property of φ, this implies

$$\exp(-C)Z_\ell(\varphi,p) \leq (\mathscr{L}_{\varphi,p}^{\ell}\mathbb{1})(y). \tag{4.3}$$

Since the other inequality is always true, (2.14) holds, which implies (2.15) and proves item 2.

Finally, from (2.15) and (4.1) we get

$$\frac{1}{\ell}\log\frac{Z_\ell(\varphi,p)}{\exp(C)} \leq \mathscr{P}(\varphi,p) \leq \frac{1}{\ell}\log Z_\ell(\varphi,p). \tag{4.4}$$

Together with (2.14), this implies (2.16), finishing the proof of the lemma. □

Proof of Bowen-type formula. Let $C>0$ be the distortion constant of φ. For every y in $(x_1,1]$ put

$$L_p(y) \coloneqq 1 + \sum_{z\in L^{-1}(y)} \exp(S_{m(z)}\varphi(z) - m(z)p). \tag{4.5}$$

Notice that, if we put $M \coloneqq \exp(\|\varphi\|+p)$, then

$$1 + M^{-1}\mathscr{L}_{\varphi,p}\mathbb{1}(y) \leq L_p(y) \leq 1 + M\mathscr{L}_{\varphi,p}\mathbb{1}(y). \tag{4.6}$$

The proof of the proposition is divided into several parts.

1. For every p_0 such that $\mathscr{P}(\varphi,p_0)>0$, we prove $P(\varphi)>p_0$. First we prove that there is $p>p_0$ such that $\mathscr{P}(\varphi,p)>0$. If $\mathscr{P}(\varphi,p_0)$ is finite, then the function $p\mapsto\mathscr{P}(\varphi,p)$ is finite, continuous, and strictly decreasing on $[p_0,+\infty)$. It follows that there is $p>p_0$ such that $\mathscr{P}(\varphi,p)>0$. If $\mathscr{P}(\varphi,p_0)=+\infty$, then $(\mathscr{L}_{\varphi,p_0}\mathbb{1})(1)=+\infty$ by Lemma 2.4(2.16). By the Monotone Convergence Theorem, for every decreasing sequence $(p_n)_{n\in\mathbb{N}_0}$ in $(p_0,+\infty)$ converging to p_0, we have that $(\mathscr{L}_{\varphi,p_n}\mathbb{1})(1)$ converges to $(\mathscr{L}_{\varphi,p_0}\mathbb{1})(1)$ as n goes to $+\infty$. It follows that there is $p>p_0$ such that $(\mathscr{L}_{\varphi,p}\mathbb{1})(1)>\exp(C)$, and therefore $\mathscr{P}(\varphi,p)>0$ by Lemma 2.4(2.16). This proves that in all of the cases, there is $p>p_0$ satisfying $\mathscr{P}(\varphi,p)>0$. Since for each integer $\ell\geq1$, every point of $F^{-\ell}(1)$ is a preimage of 1 by an iterate of f, we have by Lemma 2.4(2.16)

$$\begin{aligned}
&\sum_{m=1}^{+\infty}\exp(-mp)\sum_{y\in f^{-m}(1)}\exp(S_m\varphi(y)) \\
&\qquad\geq \sum_{\ell=1}^{+\infty}\sum_{y\in F^{-\ell}(1)}\exp(S_{m(f^{\ell-1}(y))+\cdots+m(y)}\varphi(y)-(m(f^{\ell-1}(y))+\cdots+m(y))p) \\
&\qquad= \sum_{\ell=1}^{+\infty}(\mathscr{L}_{\varphi,p}^{\ell}\mathbb{1})(1) \\
&\qquad= +\infty.
\end{aligned} \tag{4.7}$$

Together with (3.42) in Lemma 3.6(2), this implies $P(\varphi) \geq p > p_0$.

2. For every p_0 such that $\mathscr{P}(\varphi, p_0) < 0$, we prove $p_0 \geq P(\varphi)$. Note that by Lemma 2.4(2.14) and (4.6) there is $\Delta_0 > 1$ such that for every y in $(x_1, 1]$ we have $L_{p_0}(y) \leq \Delta_0 L_{p_0}(1)$. By Lemma 2.4(2.16) , (4.6) and $\mathscr{P}(\varphi, p_0) < 0$ we get that $L_{p_0}(1)$ is finite. Using $\mathscr{P}(\varphi, p_0) < 0$ and Lemma 2.4(2.16) again, we obtain

$$\begin{aligned}
&\sum_{m=1}^{+\infty} \exp(-mp_0) \sum_{y \in f^{-m}(1)} \exp(S_m\varphi(y)) \\
&\overset{(4.8)}{=} L_{\beta,p_0}(1) + \sum_{\ell=1}^{+\infty} \sum_{y \in F^{-\ell}(1)} L_{p_0}(y) \exp(\beta S_{m(f^{\ell-1}(y))+\cdots+m(y)}\varphi(y)) - (m(f^{\ell-1}(y)) + \cdots + m(y))p_0 \\
&\leq \Delta_0 L_{p_0}(1) \left(1 + \sum_{\ell=1}^{+\infty} \left(\mathscr{L}^{\ell}_{\varphi,p_0} \mathbb{1}\right)(1)\right) \\
&< +\infty.
\end{aligned}$$

In view of (3.42) in Lemma 3.6(2), we obtain $P(\varphi) \leq p_0$.

3. Put

$$p_* := \inf \{p \in \mathbb{R} \colon \mathscr{P}(\varphi, p) \leq 0\} .$$

We prove $p_* = P(\varphi)$ and $\mathscr{P}(\varphi, P(\varphi)) \leq 0$. For every $p < p_*$ we have $\mathscr{P}(\varphi, p) > 0$, and therefore $P(\varphi) > p$ by part 1. We conclude that $P(\varphi) \geq p_*$. To prove the reverse inequality, note that from the fact that $p \mapsto \mathscr{P}(\varphi, p)$ is nonincreasing on $\mathbb{R}$ and strictly decreasing on the set where it is finite, we have that this function is finite and strictly decreasing on $(p_*, +\infty)$. It follows that for every p in $(p_*, +\infty)$ we have $\mathscr{P}(\varphi, p) < 0$, and therefore $p \geq P(\varphi)$ by item 2. We conclude $p_* \geq P(\varphi)$, and therefore $p_* = P(\varphi)$. Finally, observe that by item 1 we must have $\mathscr{P}(\varphi, P(\varphi)) \leq 0$.

4. If $p_0 < P(\varphi)$ then, by part 2, $\mathscr{P}(\varphi, p_0) \geq 0$. But if $\mathscr{P}(\varphi, p_0) = 0$, by (2.12), we get that $p_0 \geq P(\varphi)$, which is a contradiction. Therefore, $\mathscr{P}(\varphi, p_0) > 0$. If $p_0 = P(\varphi)$ then, by part 1, $\mathscr{P}(\varphi, p_0) \leq 0$. Finally, assume $p_0 > P(\varphi)$. By part 3, $\mathscr{P}(\varphi, P(\varphi)) \leq 0$. Since the function $p \mapsto \mathscr{P}(\varphi, p)$ is strictly decreasing on the set where it is finite, we have that it is strictly decreasing on $(P(\varphi), +\infty)$. Therefore, $\mathscr{P}(\varphi, p_0) < 0$. This finishes the proof of the proposition. □

4.2. **Applications.** This section proves Proposition 2.5. We use Proposition 2.5 to study phase transitions in temperature for several potentials already present in the literature. We start with the proof of Corollary 4.1, another consequence of Proposition 2.5. Then, we apply these results to the study of the phase transition in temperature of the geometric potential in Proposition 4.2. This is well known, but we will take the opportunity to provide simple proof using the tools developed in this section. The proof of Proposition 2.5 is at the end of the section.

The following corollary gives some simple conditions on Hölder continuous potentials for having a phase transition in temperature.

Corollary 4.1. *Let $\varphi : [0, 1] \to \mathbb{R}$ be a Hölder continuous potential such that $\varphi - \varphi(0)$ is strictly negative on $(0, 1]$. The following hold.*

1. *If there are $c > 0$ and $\delta > 0$ such that for every x in $[0, \delta]$,*

$$\varphi(x) - \varphi(0) \leq c\omega_\alpha(x), \tag{4.9}$$

then φ has a phase transition in temperature.

2. *For every γ in $(0, \alpha]$, if φ is a potential in $C^{1,\gamma}_\dagger(\mathbb{R})$ with a nonzero leading coefficient in $C^{1,\gamma}_\dagger(\mathbb{R})$, then φ has a phase transition in temperature.*

Proof. **1.** From the asumption that $\varphi - \varphi(0)$ is strictly negative on $(0, 1]$ we get that there is $c' > 0$ such that for every x in $[\delta, 1]$ one has $\varphi(x) - \varphi(0) \leq c'\omega_\alpha(x)$. Together with (4.9), this implies that for $\beta_0 := \min\{c, c'\}$ we have $\varphi - \varphi(0) \leq \beta_0\omega_\alpha$ on $[0, 1]$. But by Proposition 2.5, the potential ω_α has a phase transition in temperature, and thus, by Corollary 1.4, for $\beta > 0$ sufficiently large, one has that $P(\beta\omega_\alpha) = 0$. Then,

$$P(\beta\beta_0^{-1}\varphi) \leq P(\beta\omega_\alpha) + \beta\beta_0^{-1}\varphi(0) = \beta\beta_0^{-1}\varphi(0). \tag{4.10}$$

Again, by Corollary 1.4, the potential φ has a phase transition in temperature.

2. From Lemma 2.7 and the fact that $\varphi - \varphi(0)$ is strictly negative on $(0, 1]$ we get that the leading coefficient of φ is nonpositive. Thus, by hypothesis, it should be negative. Again, by Lemma 2.7, there are $c > 0$ and $\delta > 0$ such that (4.9) holds. From item 1, we conclude that φ has a phase transition in temperature. □

The geometric potential $-\log Df$ is a typical example of a potential that exhibits a phase transition in temperature. It is known that this phase transition occurs at 1. As an application of the Bowen-type formula, we provide a simple proof of this fact here. An alternative proof, which avoids the inducing scheme, is given in [CRL25, Proposition 4.5].

Proposition 4.2 (The geometric potential)**.** *The geometric potential $-\log Df$ has a phase transition in temperature at* 1*.*

Proof. From Corollary 4.1(2), we have that the geometric potential has a phase transition in temperature. Now, we prove this phase transition in temperature occurs at $\beta = 1$.

First, observe that on $(\alpha/(\alpha + 1), +\infty)$ the function $\beta \mapsto \mathscr{P}(-\beta \log Df, 0)$ is finite and thus strictly decreasing. By Lemma 3.2, there is a constant $C_1 > 1$ such that for every n in $\mathbb{N}$ and for every J in $\bigvee_{j=0}^{n-1} F^{-j}(\mathfrak{D})$ we have

$$\inf_{x \in J} DF^n(x) \leq \frac{|J_0|}{|J|} \leq \sup_{x \in J} DF^n(x) \leq C_1 \inf_{x \in J} DF^n(x). \tag{4.11}$$

Observe that for every ℓ in $\mathbb{N}$ we have $\sum_{J \in \bigvee_{j=0}^{\ell-1} F^{-j}(\mathfrak{D})} |J| = |J_0|$. Since,

$$\mathscr{P}(-\log Df, 0) = \lim_{\ell \to +\infty} \frac{1}{\ell} \log \sum_{J \in \bigvee_{j=0}^{\ell-1} F^{-j}(\mathfrak{D})} \sup_{x \in J} \exp(-\log DF^\ell(x)) \tag{4.12}$$

we get

$$
\begin{aligned}
0 = \lim_{\ell\to+\infty}\frac{1}{\ell}\log\left(\frac{C_1}{|J_0|}\sum_{J\in\bigvee_{j=0}^{\ell-1}F^{-j}(\mathfrak{D})}|J|\right) &\geq \mathscr{P}(-\log Df,0) \\
&\geq \lim_{\ell\to+\infty}\frac{1}{\ell}\log\left(\frac{1}{|J_0|}\sum_{J\in\bigvee_{j=0}^{\ell-1}F^{-j}(\mathfrak{D})}|J|\right)\geq 0.
\end{aligned}
\tag{4.13}
$$

This proves $\mathscr{P}(-\log Df,0)=0$. Together with the fact that $\beta\mapsto\mathscr{P}(-\beta\log Df,0)$ is strictly decreasing on $(\alpha/(\alpha+1),+\infty)$, we get that for every β in $(\alpha/(\alpha+1),1)$, one has $\mathscr{P}(-\beta\log Df,0)>0$, and for every $\beta\geq 1$, one has $\mathscr{P}(-\beta\log Df,0)\leq 0$. Then, by Proposition 2.2.2, if $\mathscr{P}(-\beta\log Df,0)>0$ then $P(-\beta\log Df)\neq 0$ and $P(-\beta\log Df)\geq 0$, and thus, $P(-\beta\log Df)>0$. Again, by Proposition 2.2.2, if $\mathscr{P}(-\beta\log Df,0)\leq 0$ then $P(-\beta\log Df)\leq 0$. Therefore, for every β in $(\alpha/(\alpha+1),1)$, one has $P(-\beta\log Df)>0$, and for every $\beta\geq 1$, one has $P(-\beta\log Df)\leq 0$. Since $P(-\beta\log Df)$ is always non-negative we get that for every $\beta\geq 1$, one has $P(-\beta\log Df)=0$. By Corollary 1.4, we conclude that $-\log Df$ has a phase transition in temperature at 1. □

In [CRL25, Proposition 3.3], we provide an alternative proof of Proposition 2.5 that does not depend on the inducing scheme.

Proof of Proposition 2.5. For every integer $n\geq 2$ we have

$$
S_n\omega_\gamma(y_n)=-y_n^\gamma-\sum_{j=1}^{n-1}x_j^\gamma \text{ and } S_n\omega_\gamma(y_{n+1})=-y_{n+1}^\gamma-\sum_{j=2}^{n}x_j^\gamma. \tag{4.14}
$$

Since ω_γ is decreasing, for every integer $n\geq 1$ and every y in $(y_{n+1},y_n]$ we have

$$
S_n\omega_\gamma(y_n)\leq S_n\omega_\gamma(y)\leq S_n\omega_\gamma(y_{n+1}). \tag{4.15}
$$

By (2.1) and (4.14) there are positive constants C and C' such that for all integer $n\geq 2$ and y in $(y_{n+1},y_n]$ we have

$$
-1-C\sum_{j=1}^{n-1}\frac{1}{j^{\frac{\gamma}{\alpha}}}\leq S_n\omega_\gamma(y_n)\leq S_n\omega_\gamma(y)\leq S_n\omega_\gamma(y_{n+1})\leq -C'\sum_{j=2}^{n}\frac{1}{j^{\frac{\gamma}{\alpha}}}. \tag{4.16}
$$

Observe that for every β in $(0,+\infty)$ we have

$$
(\mathscr{L}_{\beta\omega_\gamma,0}\mathbb{1})(1)=\sum_{x\in F^{-1}(1)}\exp(\beta S_{m(x)}\omega_\gamma(x))=\sum_{n=1}^{+\infty}\exp\left(\beta S_n\omega_\gamma(y_n)\right). \tag{4.17}
$$

By (4.15) we have

$$
Z_1(\beta\omega_\gamma,0)\leq\sum_{n=1}^{+\infty}\exp\left(\beta S_n\omega_\gamma(y_{n+1})\right). \tag{4.18}
$$

If $\gamma > \alpha$, then

$$\sum_{j=1}^{+\infty} \frac{1}{j^{\frac{\gamma}{\alpha}}} < +\infty. \tag{4.19}$$

By (4.16) and (4.17), for every β in $(0, +\infty)$ we have

$$(\mathscr{L}_{\beta\omega_\gamma,0}\mathbb{1})(1) = +\infty. \tag{4.20}$$

By Lemma 2.4(2.16), we get $\mathscr{P}(\beta\omega_\gamma, 0) = +\infty$ and from the Bowen-type formula and Lemma 2.2 we conclude $P(\beta\omega_\gamma) > 0$. By Corollary 1.4 the potential ω_γ does not have a phase transition in temperature.

Now, we assume that $\gamma \leq \alpha$ and prove that ω_γ has a phase transition in temperature. First, observe that it is enough to prove the result for $\gamma = \alpha$ because $\omega_\gamma \leq \omega_\alpha$. Since the potential ω_α is strictly negative on $(0, 1]$, by (4.15), for every n in $\mathbb{N}$ and every y in $(y_{n+1}, y_n]$ we have

$$\exp(S_n\omega_\alpha(y)) \leq \exp(S_n\omega_\alpha(y_{n+1})) < 1. \tag{4.21}$$

Together with (4.16) and (4.18) this implies that for $\beta > 0$ sufficiently large we have

$$Z_1(\beta\omega_\alpha, 0) < 1. \tag{4.22}$$

By Lemma 2.4(2.16), we get $\mathscr{P}(\beta\omega_\alpha, 0) < 0$, and thus, by the Bowen-type formula we deduce that $P(\beta\omega_\alpha) = 0$. From Corollary 1.4, we obtain that ω_α has a phase transition in temperature. □

5. Proof of the Main Theorem

The proof of $1 \Rightarrow 2$ follows from Corollary 1.4 and the Key Lemma, and the proof of $2 \Rightarrow 3$ is direct by compactness. We prove $3 \Rightarrow 1$. Let n_0 be in $\mathbb{N}$ satisfying (2.4), let φ be a potential in $C^{1,\gamma}_{\dagger}(\mathbb{R})$, and let c be in $(-\infty, 0)$ verifying (2.5). Let m_0 be as in the theorem's statement and notice that (2.6) implies $m_0 > n_0$. We assume that (2.7) holds.

Put

$$\eta := \sup\left\{\int \varphi \, \mathrm{d}\nu - \varphi(0) \colon \nu \in \mathscr{M}, \operatorname{supp}(\nu) \subseteq [x_{m_0}, 1]\right\}. \tag{5.1}$$

By (2.7), we have

$$\eta < 0. \tag{5.2}$$

Put

$$C' := \max\left\{\frac{c}{4\alpha^{\frac{\gamma}{\alpha}}\theta}, \eta\right\} \text{ and } C'' := \max\left\{\frac{c}{4\alpha}, \frac{\eta}{\log 2}\right\}. \tag{5.3}$$

We prove that for every $\beta > 0$ and every ℓ in $\mathbb{N}$, both sufficiently large, we have

$$Z_\ell(\beta\varphi, \beta\varphi(0)) < 1. \tag{5.4}$$

Thus, by Lemma 2.4(2.16), we get $\mathscr{P}(\beta\varphi, \beta\varphi(0)) < 0$, and then, by the Bowen-type formula, we obtain $\beta\varphi(0) \geq P(\beta\varphi)$. Since the inequality $P(\beta\varphi) \geq \beta\varphi(0)$ is always true, Corollary 1.4 implies that φ has a phase transition in temperature.

To prove (5.4), it is enough to show that for each $\ell \in \mathbb{N}$ sufficiently large every summand in $Z_\ell(\varphi, \varphi(0))$ is strictly less than 1, and that for each β in $(0, +\infty)$ sufficiently large

$$Z_\ell(\beta\varphi, \beta\varphi(0)) < +\infty. \tag{5.5}$$

From the definition of $Z_\ell(\varphi, \varphi(0))$ in (2.8), the first statement is equivalent to prove that for $\ell \in \mathbb{N}$ sufficiently large and every J in $\bigvee_{j=0}^{\ell-1} F^{-j}(\mathfrak{D})$ we have

$$\sup_{x \in J} \left(S_{m(x)+\cdots+m(F^{\ell-1}(x))}\varphi(x) - (m(x) + \cdots + m(F^{\ell-1}(x))\varphi(0) \right) < 0. \tag{5.6}$$

Before proving this inequality, we provide two preliminary estimates—particular instances of (5.6)—which will be used to establish the general case.

Observe that for every ℓ in $\mathbb{N}$, every J in $\bigvee_{j=0}^{\ell-1} F^{-j}(\mathfrak{D})$, and every x in J the sum of the return times $m(x) + \cdots + m(F^{\ell-1}(x))$ is constant.

For the first preliminary estimate, let ℓ' be in $\mathbb{N}$ and let J' be in $\bigvee_{j=0}^{\ell'-1} F^{-j}(\mathfrak{D})$. For z in J' put $n' := m(z) + \ldots + m(F^{\ell'-1}(z))$. Observe that $f^{n'}|_{J'} = F^{\ell'}|_{J'}$ and denote by p the unique periodic point of f of period n' in J'. Assume that the orbit segment $(f^j(z))_{j=0}^{n'}$ is included in $[x_{m_0}, 1]$. Thus, the orbit of p is also included in $[x_{m_0}, 1]$. By (5.1), and since φ has bounded distortion on J_0 for f, by Lemma 2.2, for every z in J' we have

$$S_{n'}\varphi(z) - n'\varphi(0) \leq D|\varphi|_{1,\gamma} + S_{n'}\varphi(p) - n'\varphi(0) \leq D|\varphi|_{1,\gamma} + n'\eta. \tag{5.7}$$

For the second preliminary estimate, let y be in J_0 and assume that $m(y) \geq m_0$. By (2.5),

$$\begin{aligned} S_{m(y)}\varphi(y) - m(y)\varphi(0) &= \varphi(y) - \varphi(0) + S_{m(y)-1}\varphi(f(y)) - (m(y) - 1)\varphi(0) \\ &\leq 2n_0\|\varphi\| + c \sum_{j=n_0+1}^{m(y)} x_j^\gamma. \end{aligned} \tag{5.8}$$

Together with (2.4) and (2.6), for $\gamma < \alpha$, we get

$$\begin{aligned} S_{m(y)}\varphi(y) - m(y)\varphi(0) &\leq 2n_0\|\varphi\| - \frac{c}{2\alpha^{\frac{\gamma}{\alpha}}\theta}(n_0 + 1)^\theta + \frac{c}{2\alpha^{\frac{\gamma}{\alpha}}\theta}(m(y) + 1)^\theta \\ &\leq -D|\varphi|_{1,\gamma} + \frac{c}{4\alpha^{\frac{\gamma}{\alpha}}\theta}(m(y) + 1)^\theta, \end{aligned} \tag{5.9}$$

and for $\gamma = \alpha$, we get

$$\begin{aligned} S_{m(y)}\varphi(y) - m(y)\varphi(0) &\leq 2n_0\|\varphi\| - \frac{c}{2\alpha}\log(n_0 + 1) + \frac{c}{2\alpha}\log(m(y) + 1) \\ &\leq -D|\varphi|_{1,\gamma} + \frac{c}{4\alpha}\log(m(y) + 1). \end{aligned} \tag{5.10}$$

Now we prove (5.6). Let ℓ be in $\mathbb{N}$ and let J be in $\bigvee_{j=0}^{\ell-1} F^{-j}(\mathfrak{D})$. For each x in J put $n(x) := m(x) + \cdots + m(F^{\ell-1}(x))$. Fix x in J and denote by s the cardinality of the set

$$\mathscr{D} := \{j \in \{0, \ldots, \ell - 1\} : m(F^j(x)) \geq m_0\}. \tag{5.11}$$

If $s = \ell$, from (5.9), (5.10) and the fact that the function $r \mapsto r^\theta$ is subadditive on $[0, +\infty)$ when $\gamma < \alpha$, we get

$$S_{n(x)}\varphi(x) - n(x)\varphi(0) \leq \begin{cases} -D|\varphi|_{1,\gamma}\ell + \frac{c}{4\alpha^{\frac{\gamma}{\alpha}}\theta}n(x)^\theta, & \text{if } \gamma < \alpha; \\ -D|\varphi|_{1,\gamma}\ell + \frac{c}{4\alpha}\log n(x), & \text{if } \gamma = \alpha. \end{cases} \tag{5.12}$$

Suppose $s < \ell$ and put

$$\mathscr{S} := \{0, \ldots, \ell - 1\} \setminus \mathscr{D}. \tag{5.13}$$

Then $\mathscr{S}$ is nonempty; it can be decomposed into blocks of consecutive numbers. Let $\ell'_1 \cdots \ell'_k$ be one of these blocks for k in $\mathbb{N}$. Put $n' := m(F^{\ell'_1}(x)) + \cdots + m(F^{\ell'_k}(x))$. By definition of $\mathscr{S}$, for every i in $\{1, \ldots, k\}$ one has that $m(F^{\ell'_i}(x)) < m_0$. Then, the orbit segment $(f^j(F^{\ell'_1}(x)))_{j=0}^{n'-1}$ is included in $[x_{m_0}, 1]$. By (5.7) we get

$$S_{n'}\varphi(F^{\ell_1}(x)) - n'\varphi(0) \leq D|\varphi|_{1,\gamma} + n'\eta. \tag{5.14}$$

Denote by t the number of maximal blocks of consecutive numbers in $\mathscr{S}$. Observe that

$$t \leq s + 1. \tag{5.15}$$

Denote by $\ell_1, \ldots, \ell_s$ the numbers in $\mathscr{D}$ and for every maximal block of consecutive numbers $\ell'_1(j) \cdots \ell'_{k_j}(j)$ in $\mathscr{S}$, for $j \in \{1, \ldots, t\}$, set

$$n'_j := m(F^{\ell'_1(j)}(x)) + \cdots + m(F^{\ell'_{k_j}(j)}(x)). \tag{5.16}$$

Put

$$n'' := \sum_{j=1}^{t} n'_j \tag{5.17}$$

and notice that

$$n(x) = n'' + m(F^{\ell_1}(x)) + \cdots + m(F^{\ell_s}(x)). \tag{5.18}$$

For $\gamma < \alpha$, using (5.2),(5.3),(5.9),(5.14) and (5.15), and the subadditivity of the function $r \mapsto r^\theta$ on $[0, +\infty)$ we get that

$$\begin{aligned} & S_{n(x)}\varphi(x) - n(x)\varphi(0) \\ & \quad \leq -D|\varphi|_{1,\gamma}s + \frac{C}{4\alpha^{\frac{\gamma}{\alpha}}\theta} \sum_{j=1}^{s} (m(F^{\ell_j}(x)) + 1)^\theta + D|\varphi|_{1,\gamma}t + \eta n'' \\ & \quad \leq D|\varphi|_{1,\gamma} + \frac{C}{4\alpha^{\frac{\gamma}{\alpha}}\theta} \left(\sum_{j=1}^{s} (m(F^{\ell_j}(x)) + 1) \right)^\theta + \eta (n'')^\theta \\ & \quad \leq D|\varphi|_{1,\gamma} + C' n(x)^\theta. \end{aligned} \tag{5.19}$$

For $\gamma = \alpha$, from (5.2),(5.3),(5.10),(5.14) and (5.15), and using that for each j in $\{1, \ldots, s\}$ one has that $m(F^{\ell_j}(x)) > 1$ we get that

$$
\begin{aligned}
&S_{n(x)}\varphi(x) - n(x)\varphi(0) \\
&\quad \leq -D|\varphi|_{1,\gamma} s + \frac{c}{4\alpha}\sum_{j=1}^{s} \log(m(F^{\ell_j}(x)) + 1) + D|\varphi|_{1,\gamma} t + n''\eta \\
&\quad \leq D|\varphi|_{1,\gamma} + \frac{c}{4\alpha}\log\left(\prod_{j=1}^{s}(m(F^{\ell_j}(x)) + 1)\right) + \frac{\eta}{\log 2}\log(n'' + 1) \qquad (5.20)\\
&\quad \leq D|\varphi|_{1,\gamma} + C''\log\left((n'' + 1)\prod_{j=1}^{s}(m(F^{\ell_j}(x)) + 1)\right) \\
&\quad \leq D|\varphi|_{1,\gamma} + C''\log n(x).
\end{aligned}
$$

Observe that by (5.2) and the fact that $c < 0$, the constants C' and C'' are strictly negative. Since $n(x) \geq \ell$, by (5.12), (5.19), and (5.20) there is ℓ_0 in $\mathbb{N}$ such that for every integer $\ell \geq \ell_0$ the inequality (5.6) holds. On the other hand, by (5.9) and (5.10), for $\beta > 0$ sufficiently large $Z_1(\beta\varphi, \beta\varphi(0))$ is finite. Thus, for every ℓ in $\mathbb{N}$ we have that $Z_\ell(\beta\varphi, \beta\varphi(0))$ is also finite. Therefore, for $\beta > 0$ and ℓ in $\mathbb{N}$, both sufficiently large, we get (5.4), which prove that φ has a phase transition in temperature.

For the second part of the proposition, notice that by conditions (2.6) and (2.7) there is ε in $(0, \min\{-c, -\eta\})$ such that

$$
m_0 > \begin{cases} \left[2(n_0+1)^\theta + \frac{4\alpha^{\frac{\gamma}{\alpha}}\theta}{-(c+\varepsilon)}\left(D(|\varphi|_{1,\gamma} + \varepsilon) + 2n_0(\|\varphi\| + \varepsilon)\right)\right]^{\frac{1}{\theta}}, & \text{if } \gamma < \alpha; \\ (n_0+1)^2 \exp\left(\frac{4\alpha}{-(c+\varepsilon)}\left(D(|\varphi|_{1,\gamma} + \varepsilon) + 2n_0(\|\varphi\| + \varepsilon)\right)\right), & \text{if } \gamma = \alpha. \end{cases} \qquad (5.21)
$$

Put $\widetilde{c} \coloneqq c + \varepsilon$, and observe that $\widetilde{c} < 0$. By (2.5), if $|\varphi - \widetilde{\varphi}|_{1,\gamma} < \varepsilon$, then for every x in $[0, x_{n_0}]$ we have $\widetilde{\varphi}(x) - \widetilde{\varphi}(0) < \widetilde{c}x^\gamma$, which implies that (2.5) is an open condition in $C^{1,\gamma}_\dagger(\mathbb{R})$. Now, notice that for every $\widetilde{\varphi}$ in $C^{1,\gamma}_\dagger(\mathbb{R})$ such that $\|\widetilde{\varphi} - \varphi\|_{1,\gamma} < \varepsilon$ we have $|\widetilde{\varphi}|_{1,\gamma} < |\varphi|_{1,\gamma} + \varepsilon$ and $\|\widetilde{\varphi}\| < \|\varphi\| + \varepsilon$. Then, by (5.21) we have (2.6) with φ replaced by $\widetilde{\varphi}$. Thus, condition (2.6) is open in $C^{1,\gamma}_\dagger(\mathbb{R})$. Finally, since $\varepsilon < -\eta$ for every $\widetilde{\varphi}$ satisfying $\|\varphi - \widetilde{\varphi}\| < \varepsilon/2$, we have

$$
\sup\left\{\int \widetilde{\varphi}\, \mathrm{d}\nu - \widetilde{\varphi}(0) \colon \nu \in \mathscr{M}, \operatorname{supp}(\nu) \subseteq [x_{m_0}, 1]\right\} \leq \eta + \varepsilon < 0. \qquad (5.22)
$$

Showing that condition (2.7) is also open in $C^{1,\gamma}_\dagger(\mathbb{R})$, which finishes the Main Theorem.

Appendix A. The Key Lemma for Intermittent maps

This appendix aims to prove the Key Lemma stated in §1.3. Recall that we have fixed α in $(0, +\infty)$, that f denotes the Manneville–Pomeau map of parameter α defined in §1.1, that $\mathscr{M}$ denotes the space of Borel probability measures invariant by f, that x_1 is the discontinuity point of f and that J_0 is the interval $(x_1, 1]$.

Let μ be in $\mathscr{M}$ different from δ_0. Denote by $\operatorname{supp}(\mu)$ the topological support of μ. When $0 \notin \operatorname{supp}(\mu)$ one can use that for Hölder continuous potentials on topologically mixing subshift of finite type the equilibrium states have positive entropy (see for instance [Bow08, Theorem 1.25]) to prove that $P(\varphi) > \int \varphi \,\mathrm{d}\mu$. For the general case, we use an iterated function system.

By the ergodic decomposition theorem, we can assume that μ is ergodic. Let Σ be the set of all infinite words in the alphabet $\mathbb{N}$ and for every n in $\mathbb{N}$ put

$$\Sigma_n := \mathbb{N}^n \text{ and } \Sigma^* := \bigcup_{n\in\mathbb{N}} \mathbb{N}^n. \tag{A.1}$$

For every n in $\mathbb{N}$ and every $\underline{\ell}$ in Σ_n the *length of* $\underline{\ell}$ is n and it is denoted by $|\underline{\ell}|$. An infinite sequence of pairwise distinct functions $(\phi_\ell)_{\ell\in\mathbb{N}}$ from $(x_1, 1]$ into $(x_1, 1]$ is called an *Iterated Function System (IFS).* For every n in $\mathbb{N}$ and every finite word $\ell_1 \cdots \ell_n$ in Σ_n put

$$\phi_{\ell_1\cdots\ell_n} := \phi_{\ell_1} \circ \cdots \circ \phi_{\ell_n}. \tag{A.2}$$

We say that the IFS is *free* if for all $\underline{\ell}$ and $\underline{\ell}'$ in Σ^* with $\underline{\ell} \neq \underline{\ell}'$ we have that $\phi_{\underline{\ell}}$ is different from $\phi_{\underline{\ell}'}$. We say that the IFS is *generated by* f if for every ℓ in $\mathbb{N}$ there is m_ℓ in $\mathbb{N}$ such that $f^{m_\ell} \circ \phi_\ell$ is the identity on $(x_1, 1]$. We say that $(m_\ell)_{\ell\in\mathbb{N}}$ is the *time sequence* of $(\phi_\ell)_{\ell\in\mathbb{N}}$. For every n in $\mathbb{N}$ and every finite word $\ell_1 \cdots \ell_n$ in Σ_n put

$$m_{\ell_1\cdots\ell_n} := m_{\ell_1} + \cdots + m_{\ell_n}. \tag{A.3}$$

We say that $(\phi_\ell)_{\ell\in\mathbb{N}}$ is *hyperbolic with respect to* f, if there are constants $C > 0$ and $\lambda > 1$ such that for every $x \in (x_1, 1]$, for every $\underline{\ell} \in \Sigma^*$ and for every $j \in \{1, \ldots, m_{\underline{\ell}}\}$ one has

$$|Df^j(f^{m_{\underline{\ell}}-j}(\phi_{\underline{\ell}}(x)))| \geq C\lambda^j. \tag{A.4}$$

Proposition A.1. *For each* HÖLDER *continuous potential* φ *on* $[0, 1]$ *and for each ergodic measure* μ *in* $\mathscr{M}$ *distinct from* δ_0*, the following holds. There are a constant* $C > 0$ *and a free hyperbolic IFS* $(\phi_\ell)_{\ell\in\mathbb{N}}$ *generated by* f *with strictly increasing time sequence* $(m_\ell)_{\ell\in\mathbb{N}}$ *such that*

$$\inf_{z\in(x_1,1]} S_{m_\ell}\varphi(\phi_\ell(z)) \geq m_\ell \int \varphi \,\mathrm{d}\mu - C. \tag{A.5}$$

The proof of Proposition A.1 is in §A.2. Now, we assume the result and prove the remaining Key Lemma.

A.1. **Proof of the Key Lemma assuming Proposition A.1.** Assume that μ is an ergodic distinct from δ_0. Let $(\phi_\ell)_{\ell\in\mathbb{N}}$ be the IFS given by Proposition A.1 with time sequence $(m_\ell)_{\ell\in\mathbb{N}}$ and constant C. Fix z_0 in $(x_1, 1]$. For every N in $\mathbb{N}$ put

$$\Lambda_N := \sum_{\underline{\ell}\in\Sigma^*, m_{\underline{\ell}}=N} \exp(S_N\varphi(\phi_{\underline{\ell}}(z_0))). \tag{A.6}$$

Since the IFS $(\phi_\ell)_{\ell\in\mathbb{N}}$ is free, by (3.41) we have

$$P(\varphi) \geq \limsup_{N\to+\infty} \frac{1}{N} \log \Lambda_N. \tag{A.7}$$

Now consider the following generating function

$$\Xi(s) := \sum_{N\in\mathbb{N}} \Lambda_N s^N. \tag{A.8}$$

The radius of convergence R of $\Xi(s)$ is at least $\exp(-P(\varphi))$. Observe that

$$\Xi(s)=\sum_{\underline{\ell}\in\Sigma^*}\exp(S_{m_{\underline{\ell}}}\varphi(\phi_{\underline{\ell}}(z_0)))s^{m_{\underline{\ell}}}. \tag{A.9}$$

By (A.5) in Proposition A.1 for every $\underline{\ell}$ in Σ^* we have that

$$S_{m_{\underline{\ell}}}\varphi(\phi_{\underline{\ell}}(z_0))\geq m_{\underline{\ell}}\int\varphi\,\mathrm{d}\mu-|\underline{\ell}|C. \tag{A.10}$$

Then, defining

$$\Phi(s):=\sum_{\ell=1}^{+\infty}\exp\left(m_\ell\int\varphi\,\mathrm{d}\mu-C\right)s^{m_\ell} \tag{A.11}$$

we get that the power series in s

$$\Phi(s)+\Phi(s)^2+\Phi(s)^3+\cdots \tag{A.12}$$

has coefficients smaller than or equal to the corresponding coefficients of $\Xi(s)$. Observe that, since $(m_\ell)_{\ell\in\mathbb{N}}$ is strictly increasing, the radius of convergence of $\Phi(s)$ is $\widehat{R}=\exp(-\int\varphi\,\mathrm{d}\mu)$ and that

$$\lim_{s\to\widehat{R}^-}\Phi(s)=+\infty. \tag{A.13}$$

Then, there is s_0 in $(0,\widehat{R})$ such that $\Phi(s_0)$ is finite and $\Phi(s_0)\geq 1$, and this implies that the radius of convergence of the series (A.12) is strictly smaller than s_0. Then

$$\exp(-P(\varphi))\leq R<s_0<\widehat{R}=\exp\left(-\int\varphi\,\mathrm{d}\mu\right), \tag{A.14}$$

finishing the proof of the lemma.

A.2. **Proof of Proposition A.1.** Before proving Proposition A.1, we demonstrate two lemmas that establish that an IFS is free and has bounded distortion, along with a well-known but folklore result in abstract Ergodic Theory, for which we provide proof for the reader's convenience.

Lemma A.2. *Let $(z_n)_{n\in\mathbb{N}_0}$ be a sequence in $[0,1]$ such that z_0 is in J_0 and for every n in $\mathbb{N}_0$ we have that $z_n=f(z_{n+1})$. Let $M\geq 1$ be an integer and let $(n_\ell)_{\ell\in\mathbb{N}}$ be a strictly increasing sequence of positive integers such that $n_{\ell+1}\geq n_\ell+M$. For every ℓ in $\mathbb{N}$, let x_ℓ be a point of J_0 in $f^{-M}(z_{n_\ell})$ different from $z_{n_\ell+M}$, and let ϕ_ℓ be the inverse branch of $f^{n_\ell+M}$ from J_0 into itself such that $\phi_\ell(z_0)=x_\ell$. Then, the IFS $(\phi_\ell)_{\ell\in\mathbb{N}}$ generated by f is free with time sequence $(n_\ell+M)_{\ell\in\mathbb{N}}$.*

Proof. Let $\underline{\ell}=\ell_1\cdots\ell_n$ and $\underline{\ell}'=\ell_1'\cdots\ell_k'$ be in Σ^* with $\underline{\ell}\neq\underline{\ell}'$. Assume that $m_{\underline{\ell}}\neq m_{\underline{\ell}'}$. Without loss of generality we assume that $m_{\underline{\ell}}<m_{\underline{\ell}'}$. Suppose we had $\phi_{\underline{\ell}}=\phi_{\underline{\ell}'}$ then $f^{m_{\underline{\ell}'}}\circ\phi_{\underline{\ell}}=\mathrm{Id}|_{J_0}$, which implies that $f^{m_{\underline{\ell}'}-m_{\underline{\ell}}}=\mathrm{Id}|_{J_0}$ and thus, $m_{\underline{\ell}}=m_{\underline{\ell}'}$ giving a contradiction. Now assume that $m_{\underline{\ell}}=m_{\underline{\ell}'}$. We also assume that $\ell_n\neq\ell_k'$, the general case, can be reduced to this one. Without loss of generality, we assume that $\ell_n<\ell_k'$. In particular, we have $m_{\ell_n}<m_{\ell_k'}$. Suppose we had $\phi_{\underline{\ell}}=\phi_{\underline{\ell}'}$ then $f^{m_{\underline{\ell}}-m_{\ell_n}}\circ\phi_{\underline{\ell}}=f^{m_{\underline{\ell}'}-m_{\ell_n}}\circ\phi_{\underline{\ell}'}$ and thus,

$$\phi_{\ell_n}=f^{m_{\ell_k'}-m_{\ell_n}}\circ\phi_{\ell_k'}=f^{n_{\ell_k'}-n_{\ell_n}}\circ\phi_{\ell_k'}.$$

Evaluating this last equality at z_0 and using that $n_{\ell'_k} - n_{\ell_n} \geq M$ we get that

$$x_{\ell_n} = f^{n_{\ell'_k} - n_{\ell_n}}(x_{\ell'_k}) = f^{n_{\ell'_k} - n_{\ell_n} - M}(z_{n_{\ell'_k}}) = z_{n_{\ell_n}+M}.$$

However, by hypothesis, these two points are different. Therefore, $\phi_{\underline{\ell}} \neq \phi_{\underline{\ell}'}$ and this concludes the proof of the lemma. □

Lemma A.3. *For each HÖLDER continuous potential $\varphi : [0,1] \to \mathbb{R}$ and for each IFS $(\phi_\ell)_{\ell \in \mathbb{N}}$ generated by f, hyperbolic with respect to f and with time sequence $(m_\ell)_{\ell \in \mathbb{N}}$ the following holds. There is a constant $\Delta' > 0$ such that for every $\underline{\ell}$ in Σ^* and all x and y in J_0 we have that*

$$|S_{m_{\underline{\ell}}}\varphi(\phi_{\underline{\ell}}(x)) - S_{m_{\underline{\ell}}}\varphi(\phi_{\underline{\ell}}(y))| \leq \Delta'. \tag{A.15}$$

Proof. Let γ be in $(0,1]$ such that potential φ is in $C^\gamma(\mathbb{R})$. By (A.4) there are constants $C > 0$ and $\lambda > 1$ such that for every j in $\{1, \ldots, m_{\underline{\ell}}\}$ we have that

$$|f^{m_{\underline{\ell}}-j}(\phi_{\underline{\ell}}(x)) - f^{m_{\underline{\ell}}-j}(\phi_{\underline{\ell}}(y))| \leq C\lambda^{-j}.$$

Then,

$$|S_{m_{\underline{\ell}}}\varphi(\phi_{\underline{\ell}}(x)) - S_{m_{\underline{\ell}}}\varphi(\phi_{\underline{\ell}}(y))| \leq |\varphi|_\gamma C^\gamma \sum_{j=1}^{m_{\underline{\ell}}} (\lambda^\gamma)^{-j}. \tag{A.16}$$

Taking $\Delta' := |\varphi|_\gamma C^\gamma \sum_{j=1}^{+\infty} (\lambda^\gamma)^{-j}$ we finish the proof of the lemma. □

Lemma A.4. *Let $(X, \mathscr{B}, \mu)$ be a probability space and let $T : X \to X$ be an ergodic measure-preserving map. Then, for each integrable function $\varphi : X \to \mathbb{R}$ with null integral, there is a full measure set of x in X such that*

$$\limsup_{n \to +\infty} S_n\varphi(x) \geq 0. \tag{A.17}$$

Proof. It is enough to prove that for every $\varepsilon > 0$ and every k in $\mathbb{N}$ the set

$$A := \{x \in X\colon \text{ for every } n \in \mathbb{N} \text{ such that } n \geq k \text{ one has } S_n\varphi \leq -\varepsilon\} \tag{A.18}$$

has measure 0. Suppose we had that there are $\varepsilon > 0$ and k in $\mathbb{N}$ such that $\mu(A) > 0$. By the Birkhoff Ergodic Theorem there is x in A such that

$$\lim_{n \to +\infty} \frac{1}{n} S_n\varphi(x) = 0, \tag{A.19}$$

and

$$\lim_{n \to +\infty} \frac{1}{n} S_n \mathbf{1}_A(x) = \mu(A). \tag{A.20}$$

Denote by $(n_\ell)_{\ell \in \mathbb{N}_0}$ the sequence of return times of x to A with $n_0 = 0$. For every ℓ in $\mathbb{N}_0$ we have $n_\ell \geq \ell$ and since $\mu(A) > 0$ we also have that there is P in $[1, +\infty)$ such that for every ℓ in $\mathbb{N}_0$,

$$n_\ell \leq P\ell. \tag{A.21}$$

Since for every m in $\mathbb{N}$ we have $n_{(m+1)k} \geq n_{mk} + k$, by (A.21), for every m in $\mathbb{N}$ we get

$$S_{n_{mk}}\varphi(x) = \sum_{j=0}^{m-1} S_{n_{(j+1)k} - n_{jk}}\varphi(T^{n_{jk}}x) \leq m(-\varepsilon) \leq \frac{n_{mk}}{Pk}(-\varepsilon). \tag{A.22}$$

Together with (A.19) this implies

$$0 = \lim_{m\to+\infty} \frac{1}{n_{mk}} S_{n_{mk}}\varphi(x) \le \frac{1}{Pk}(-\varepsilon) < 0, \tag{A.23}$$

which gives a contradiction and finishes the proof of the lemma. □

Put $I := [0,1]$ and let μ be an ergodic measure in $\mathscr{M}$ distinct from δ_0. Denote by $(\widehat{I}, \widehat{f})$ the natural extension of (I, f). That is, $\widehat{I}$ is the set of sequences $(z_n)_{n\in\mathbb{N}_0}$ in I such that for every n in $\mathbb{N}_0$ one has $z_n = f(z_{n+1})$, and $\widehat{f}$ is the bijective map from $\widehat{I}$ onto $\widehat{I}$ defined for every $(z_n)_{n\in\mathbb{N}_0}$ in $\widehat{I}$ by

$$\widehat{f}((z_n)_{n\in\mathbb{N}_0}) = (f(z_0), z_0, z_1, z_2, \ldots, z_n, \ldots). \tag{A.24}$$

The space $\widehat{I}$ inherits a Borel σ-algebra as a subset of the product space $\Pi_{n=0}^{+\infty} I$ and the map $\widehat{f}$ is a measurable isomorphism. Denote by $\Pi : \widehat{I} \to I$ the projection onto the zeroth coordinate. We have that $\Pi \circ \widehat{f} = f \circ \Pi$, and that there is a unique invariant probability measure ν for $\widehat{f}$ such that $\Pi_*\nu = \mu$. Since μ is ergodic, the measure ν is also ergodic for $\widehat{f}$ and $\widehat{f}^{-1}$. Observe that since μ is different from δ_0 we have $\mu(J_0) > 0$ and $\nu(\Pi^{-1}(J_0)) > 0$. Put $\widehat{I}' := \Pi^{-1}(J_0)$. Fix $\underline{z}$ in $\widehat{I}'$ such that $\underline{z}$ is generic for the Ergodic Theorem for $\widehat{f}^{-1}$, ν and the bounded measurable function $\log Df \circ \Pi$. Then, we have

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log Df \circ \Pi(\widehat{f}^{-j}\underline{z}) = \chi_\mu(f) > 0. \tag{A.25}$$

By Lemma A.4, we can choose $\underline{z}$ so that, in addition, we have

$$\limsup_{n\to+\infty} \sum_{j=0}^{n-1} \left(\varphi \circ \Pi(\widehat{f}^{-j}\underline{z}) - \int \varphi \, \mathrm{d}\mu \right) \ge 0. \tag{A.26}$$

By Lemma 3.4 for every ℓ in $\mathbb{N}$ there is a unique inverse branch $\widetilde{\phi}_\ell$ of f^ℓ defined on $(0,1]$ verifying $\Pi(\widehat{f}^{-\ell}\underline{z}) \in \widetilde{\phi}_\ell((0,1])$. By Lemma 3.2 and (A.25) there are $C' > 0$ and $\lambda > 1$ such that for every n in $\mathbb{N}$ and every z in $(x_1, 1]$ we have

$$Df^n(\widetilde{\phi}_n(z)) \ge C'\lambda^n. \tag{A.27}$$

Then, there is a strictly increasing sequence $(n_\ell)_{\ell\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$S_{n_\ell}\varphi(z_{n_\ell}) \ge n_\ell \int \varphi \, \mathrm{d}\mu - 1, n_{\ell+1} \ge n_\ell + 3, C'\lambda^{\frac{n_1}{2}} > \lambda^{\frac{3}{2}}, \tag{A.28}$$

and the sequence $(z_{n_\ell})_{\ell\in\mathbb{N}}$ converges to some w in I.

Now, we distinguish two cases. First recall that y_2 is the unique point in $(x_1, 1]$ that satisfies $x_1 = f(y_2)$, and observe that f^2 maps $(x_1, y_2]$ and $(y_2, 1]$ bijectively onto $(0,1]$. Let y' and y'' be the preimage by f^2 of x_1 in $(x_1, y_2]$ and $(y_2, 1]$, respectively. We have that $y' < y_2 < y''$ and that f^3 maps each of the intervals $(x_1, y']$ and $(y'', 1]$ bijectively onto $(0,1]$. The first case is when $w \in [0, y_2]$. By passing to a subsequence, we can assume that

$$(z_{n_\ell})_{\ell\in\mathbb{N}} \text{ is in } (0, y'']. \tag{A.29}$$

For every $\ell \in \mathbb{N}$ we put

$$\phi_\ell := (f^3|_{(y'',1]})^{-1} \circ \widetilde{\phi}_{n_\ell}|_{J_0}. \tag{A.30}$$

The second case is when $w \in (y_2, 1]$. Again, by passing to a subsequence, we can assume that

$$(z_{n_\ell})_{\ell\in\mathbb{N}} \text{ is in } (y', 1]. \tag{A.31}$$

For every $\ell \in \mathbb{N}$ we put

$$\phi_\ell \coloneqq (f^3|_{(x_1,y']})^{-1} \circ \widetilde{\phi}_{n_\ell}|_{J_0}. \tag{A.32}$$

In both cases, put $m_\ell \coloneqq n_\ell + 3$. We have that $(\phi_\ell)_{\ell\in\mathbb{N}}$ is an IFS generated by f with time sequence $(m_\ell)_{\ell\in\mathbb{N}}$.

By (A.29), (A.30), (A.31) and (A.32), we have in all of the cases that for every ℓ in $\mathbb{N}$, the point $x_\ell \coloneqq \phi_\ell(z_0)$ is in $f^{-3}(z_{n_\ell})$ and it is different from z_{m_ℓ}. Thus, by Lemma A.2 and the second inequality in (A.28), we get that the IFS $(\phi_\ell)_{\ell\in\mathbb{N}}$ is free.

Now we prove that the IFS $(\phi_\ell)_{\ell\in\mathbb{N}}$ is hyperbolic. By (A.27), the second and third inequalities in (A.28), for every z in J_0 and for every $j \in \{0,1,2\}$ we have in all of the cases that

$$Df^{m_\ell - j}(f^j(\phi_\ell(z))) \geq \lambda^{\frac{m_\ell - j}{2}}. \tag{A.33}$$

Again together with (A.27) this implies that for every z in J_0, for every $\underline{\ell} \in \Sigma^*$ and for every $j \in \{1, \ldots, m_{\underline{\ell}}\}$ one has

$$Df^j(f^{m_{\underline{\ell}} - j}(\phi_{\underline{\ell}}(z))) \geq C'\lambda^{\frac{j}{2}}, \tag{A.34}$$

and thus, the IFS $(\phi_\ell)_{\ell\in\mathbb{N}}$ is hyperbolic with constants $C' > 0$ and $\lambda^{\frac{1}{2}} > 1$.

It remains to prove (A.5). Put

$$C_1 \coloneqq -\inf_{x\in[0,1]} \varphi \text{ and } C'' \coloneqq 1 + 3C_1 + 3\int \varphi \, \mathrm{d}\mu. \tag{A.35}$$

By the first inequality in (A.28), we have

$$\begin{aligned} S_{m_\ell}\varphi(\phi_\ell(z_0)) &= S_3\varphi(\phi_\ell(z_0)) + S_{n_\ell}\varphi(\widetilde{\phi}_{n_\ell}(z_0)) \\ &\geq -3C_1 + n_\ell \int \varphi \, \mathrm{d}\mu - 1 \\ &\geq m_\ell \int \varphi \, \mathrm{d}\mu - C''. \end{aligned} \tag{A.36}$$

Together with Lemma A.3, this finishes the proof of (A.5) with $C = C'' + \Delta'$, concluding the proof of Proposition A.1.

References

[BC23] Thiago Bomfim and Victor Carneiro. Thermodynamical and spectral phase transition for local diffeomorphisms in the circle. *Nonlinearity*, 36(12):6513–6545, 2023.

[BF23] Thiago Bomfim and Afonso Fernandes. Phase transitions for transitive local diffeomorphism with break points on the circle and holder continuous potentials. arXiv:2310.07034, 2023.

[BLL12] Alexandre Baraviera, Renaud Leplaideur, and Artur O. Lopes. The potential point of view for renormalization. *Stoch. Dyn.*, 12(4):1250005, 34, 2012.

[Bow08] Rufus Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, revised edition, 2008. With a preface by David Ruelle, Edited by Jean-René Chazottes.

[BT16] Viviane Baladi and Mike Todd. Linear response for intermittent maps. *Comm. Math. Phys.*, 347(3):857–874, 2016.

[BTT19] Henk Bruin, Dalia Terhesiu, and Mike Todd. The pressure function for infinite equilibrium measures. *Israel J. Math.*, 232(2):775–826, 2019.

[CRL25] D. Coronel and J. Rivera-Letelier. Gibbs phase rule for intermittent maps. In preparation, 2025.

[CT13] Vaughn Climenhaga and Daniel J. Thompson. Equilibrium states beyond specification and the Bowen property. *J. Lond. Math. Soc. (2)*, 87(2):401–427, 2013.

[CV13] A. Castro and P. Varandas. Equilibrium states for non-uniformly expanding maps: decay of correlations and strong stability. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 30(2):225–249, 2013.

[Fat19] P. Fatou. Sur les équations fonctionnelles. *Bull. Soc. Math. France*, 47:161–271, 1919.

[GIR22] Eduardo Garibaldi and Irene Inoquio-Renteria. A Ruelle-Perron-Frobenius theorem for expanding circle maps with an indifferent fixed point. *Lett. Math. Phys.*, 112(4):Paper No. 76, 25, 2022.

[GIR24] Eduardo Garibaldi and Irene Inoquio-Renteria. Exponential rate of decay of correlations of equilibrium states associated with non-uniformly expanding circle maps. *Nonlinearity*, 37(3):Paper No. 035018, 12, 2024.

[Gou04] Sébastien Gouëzel. Central limit theorem and stable laws for intermittent maps. *Probab. Theory Related Fields*, 128(1):82–122, 2004.

[Hof77] Franz Hofbauer. Examples for the nonuniqueness of the equilibrium state. *Trans. Amer. Math. Soc.*, 228:223–241, 1977.

[Hol05] Mark Holland. Slowly mixing systems and intermittency maps. *Ergodic Theory Dynam. Systems*, 25(1):133–159, 2005.

[IRRL12] Irene Inoquio-Renteria and Juan Rivera-Letelier. A characterization of hyperbolic potentials of rational maps. *Bull. Braz. Math. Soc. (N.S.)*, 43(1):99–127, 2012.

[IRRL25] I. Inoquio-Rentería and J. Rivera-Letelier. -. In preparation, 2025.

[Isr79] Robert B. Israel. *Convexity in the theory of lattice gases*. Princeton University Press, Princeton, N.J., 1979. Princeton Series in Physics, With an introduction by Arthur S. Wightman.

[Jen19] Oliver Jenkinson. Ergodic optimization in dynamical systems. *Ergodic Theory Dynam. Systems*, 39(10):2593–2618, 2019.

[Kel85] Gerhard Keller. Generalized bounded variation and applications to piecewise monotonic transformations. *Z. Wahrsch. Verw. Gebiete*, 69(3):461–478, 1985.

[Kel98] Gerhard Keller. *Equilibrium states in ergodic theory*, volume 42 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1998.

[Klo20] Benoît R. Kloeckner. An optimal transportation approach to the decay of correlations for non-uniformly expanding maps. *Ergodic Theory Dynam. Systems*, 40(3):714–750, 2020.

[Li15] Huaibin Li. Equivalent characterizations of hyperbolic Hölder potential for interval maps. *Proc. Amer. Math. Soc.*, 143(5):2129–2141, 2015.

[Lop93] A. O. Lopes. The zeta function, nondifferentiability of pressure, and the critical exponent of transition. *Adv. Math.*, 101(2):133–165, 1993.

[LRL14a] Huaibin Li and Juan Rivera-Letelier. Equilibrium states of interval maps for hyperbolic potentials. *Nonlinearity*, 27(8):1779–1804, 2014.

[LRL14b] Huaibin Li and Juan Rivera-Letelier. Equilibrium states of weakly hyperbolic one-dimensional maps for Hölder potentials. *Comm. Math. Phys.*, 328(1):397–419, 2014.

[LSV99] Carlangelo Liverani, Benoît Saussol, and Sandro Vaienti. A probabilistic approach to intermittency. *Ergodic Theory Dynam. Systems*, 19(3):671–685, 1999.

[MT12] Ian Melbourne and Dalia Terhesiu. Operator renewal theory and mixing rates for dynamical systems with infinite measure. *Invent. Math.*, 189(1):61–110, 2012.

[Mun00] James R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, second edition, 2000.

[PM80] Yves Pomeau and Paul Manneville. Intermittent transition to turbulence in dissipative dynamical systems. *Comm. Math. Phys.*, 74(2):189–197, 1980.

[PS92] Thomas Prellberg and Joseph Slawny. Maps of intervals with indifferent fixed points: thermodynamic formalism and phase transitions. *J. Statist. Phys.*, 66(1-2):503–514, 1992.

[PW99] Mark Pollicott and Howard Weiss. Multifractal analysis of Lyapunov exponent for continued fraction and Manneville-Pomeau transformations and applications to Diophantine approximation. *Comm. Math. Phys.*, 207(1):145–171, 1999.

[Sar01] Omri M. Sarig. Phase transitions for countable Markov shifts. *Comm. Math. Phys.*, 217(3):555–577, 2001.

[Sar02] Omri Sarig. Subexponential decay of correlations. *Invent. Math.*, 150(3):629–653, 2002.

[Tha00] Maximilian Thaler. The asymptotics of the Perron-Frobenius operator of a class of interval maps preserving infinite measures. *Studia Math.*, 143(2):103–119, 2000.

[URM22] Mariusz Urbański, Mario Roy, and Sara Munday. *Non-invertible dynamical systems. Vol. 2. Finer thermodynamic formalism—distance expanding maps and countable state subshifts of finite type, conformal GDMSs, Lasota-Yorke maps and fractal geometry*, volume 69.2 of *De Gruyter Expositions in Mathematics*. De Gruyter, Berlin, [2022] ©2022.

[You99] Lai-Sang Young. Recurrence times and rates of mixing. *Israel J. Math.*, 110:153–188, 1999.

Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Campus San Joaquín, Avenida Vicuña Mackenna 4860, Santiago, Chile.

Email address: acoronel@uc.cl

Department of Mathematics, University of Rochester. Hylan Building, Rochester, NY 14627, U.S.A.

Email address: riveraletelier@gmail.com

URL: http://rivera-letelier.org/