→ Métodos Multipaso (caso escalar)

1) $y' = f(t,y) \quad t \in (a,b)$

$y(a) = y_0$

$\hat{y}_0 = y_0$

$\hat{y}_{i+1} = \hat{y}_i + \phi(h, t_i, f, \hat{y}_i, \hat{y}_{i+1})$ ←

$\hat{y}_{i+1}$ puede depender de los puntos $\hat{y}_{i-1}, \hat{y}_{i-2}, \hat{y}_{i-3}, \dots, \hat{y}_0$

Por ejemplo $\hat{y}_3$ puede utilizar $\hat{y}_2, \hat{y}_1$

2) Los métodos multipaso son métodos que utilizan aproximaciones anteriores a la aproximación anterior

Formalmente: (m+1 - puntos anteriores)

$$\begin{cases} \hat{y}_{i+1} = a_{m-1}\hat{y}_i + a_{m-2}\hat{y}_{i-1} + \dots + a_0 \hat{y}_{i+1-m} \\ \quad + h\left[b_m f(t_{i+1}, \hat{y}_{i+1}) + b_{m-1} f(t_i, \hat{y}_i) + b_{m-2} f(t_{i-1}, \hat{y}_{i-1}) \right. \\ \quad \left. \cong + \dots + b_0 f(t_{i+1-m}, \hat{y}_{i+1-m}) \right] \end{cases}$$

$i = m-1, m, \dots, N-1$, $h = \frac{b-a}{N}$, $a_0, a_1, \dots, a_{m-1}, b_0, b_1, \dots, b_m$

Son constantes. Si $b_m \neq 0$ entonces es un mét. implícito.

Tenemos valores iniciales $\hat{y}_0, \hat{y}_1, \hat{y}_2, \dots, \hat{y}_{m-1}$

3) Ejemplos:

(implícito)

3.i) Mét. Adams-Moulton. Dados: $\hat{y}_0, \hat{y}_1, \hat{y}_2$

$$\hat{y}_{i+1} = \hat{y}_i + \frac{h}{24}\left[9f(t_{i+1}, \hat{y}_{i+1}) + 19 f(t_i, \hat{y}_i) - 5 f(t_{i-1}, \hat{y}_{i-1}) + f(t_{i-2}, \hat{y}_{i-2}) \right] \qquad i = 2, \dots, N$$

Cuando $i=2$ necesitamos para calcular $\hat{y}_3$ $\quad \hat{y}_2, \hat{y}_1, \hat{y}_0$

$t_i = a + ih$, (Recordar)

3.ii) Adams-Beshfort. Dados los valores aproximados $\hat{y}_0, \hat{y}_1, \hat{y}_2, \hat{y}_3$

$$\hat{y}_{i+1} = \hat{y}_i + \frac{h}{24}\left[55 f(t_i, \hat{y}_i) - 59 f(t_{i-1}, \hat{y}_{i-1}) + 37 f(t_{i-2}, \hat{y}_{i-2}) - 9 f(t_{i-3}, \hat{y}_{i-3}) \right]$$

$$i = 3, \dots, N$$

4) Derivación de los métodos multipaso

(*) $\quad y' = f(t, y) \qquad t \in (a, b)$

$y(a) = y_0$

$y' = \frac{dy}{dt}$, integrando en el intervalo $[t_i, t_{i+1}]$

$$y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} \frac{dy}{dt}\, dt = \int_{t_i}^{t_{i+1}} f(t, y(t))\, dt$$

$$\Rightarrow \quad \overline{y(t_{i+1})} = y(t_i) + \int_{t_i}^{t_{i+1}} \overline{f(t, y(t))}\, dt$$

Como no tenemos la solución exacta $y(t)$, no podemos evaluar la integral de forma exacta. La técnica de los métodos multipaso es aproximar $f(t, y(t))$ por un polinomio $P_{m-1}(t) \in \mathbb{P}^{m-1}(\mathbb{R})$

($\mathbb{P}^{m-1}(\mathbb{R}) = \{$ conjunto de polinomios de hasta grado o igual de $m-1\}$)

que interpola los m-puntos:

$$(t_i, f(t_i, \hat{y}_i)),\ (t_{i-1}, f(t_{i-1}, \hat{y}_{i-1})),\ (t_{i-2}, f(t_{i-2}, \hat{y}_{i-2})) \ldots$$

$$(t_{i+1-m}, f(t_{i+1-m}, \hat{y}_{i+1-m}))$$

y entonces obtenemos la aproximación sig.:

$$y(t_{i+1}) \simeq y(t_i) + \int_{t_i}^{t_{i+1}} P_{m-1}(t)\, dt$$

5) Repaso de interpolación Polinomial

Notación Sea $P_n(x) \in \mathbb{P}^n(\mathbb{R})$

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, \quad a_i \in \mathbb{R} \quad i = 0, 1, \ldots n$$

5.i) Polinomios de interpolación con "diferencias "por atras" "atrasadas" "

Sea $\{p_j\}_{j=1}^{\infty}$ una sucesión. Se introduce el operador de "diferencia atrasada" como $\nabla p_j = p_j - p_{j-1}$ $j \geq 1$ ←

Recursivamente se define $\nabla^k p_j = \nabla(\nabla^{k-1} p_j)$ y $\nabla^0 p_j = p_j$

Por ejemplo: $\nabla^2 p_j = \nabla(\nabla p_j) = \nabla(p_j - p_{j-1})$

$= \nabla p_j - \nabla p_{j-1}$

$= (p_j - p_{j-1}) - (p_{j-1} - p_{j-2})$

$\Rightarrow = p_j - 2p_{j-1} + p_{j-2}$ ←

$\nabla^3 p_j = \nabla(\nabla^2 p_j) = \nabla(p_j - 2p_{j-1} + p_{j-2})$

$= \nabla p_j - 2\nabla p_{j-1} + \nabla p_{j-2}$, $\nabla(a p_j) = a \nabla p_j$

$= p_j - p_{j-1} - 2(p_{j-1} - p_{j-2}) + (p_{j-2} - p_{j-3})$

$= p_j - 3p_{j-1} + 3p_{j-2} - p_{j-3}$ ←

5.ii) Def. Sean los $(n+1)$-puntos ordenados, distintos y equidistantes

$\rightsquigarrow$ $x_n, x_{n-1}, x_{n-2}, \ldots, x_0$, i.e., $x_n - x_{n-1} = x_{n-1} - x_{n-2} = \ldots = x_1 - x_0 := h$

$x_n > x_{n-1} > x_{n-2} \ldots > x_1 > x_0$

Entonces, para $x = x_n + sh$, s es un entero positivo

Dada $f: \mathbb{R} \to \mathbb{R}$, entonces se define

$$P_n(x) = f(x_n) + (-1)^1 \binom{-s}{1} \nabla f(x_n) + (-1)^2 \binom{-s}{2} \nabla^2 f(x_n)$$

$$+ \ldots + (-1)^n \binom{-s}{n} \nabla^n f(x_n) = f(x_n) + \sum_{k=1}^{n-1} (-1)^k \binom{-s}{k} \nabla^k f(x_n)$$

donde $\binom{-s}{k} = \dfrac{(-1)^k s(s+1)(s+2)\cdots(s+k-1)}{k!}$, $k \geq 1$

5.iii) Teorema de interpolación polinomial por diferencias atrasadas

Sean los $(n+1)$-puntos $x_n, x_{n-1}, \ldots, x_1, x_0$ equidistantes, distintos y ordenados. y sea $f \in C^{n+1}[a,b]$. Entonces para

$$x = x_n + sh, \; s \in \mathbb{Z}^+, \; h = x_n - x_{n-1} = \ldots = x_1 - x_0$$

$x \in [a,b]$, existe $\xi \in [a,b]$ tal que

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

6) Regresando al problema de aproximar;

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\, dt$$

Introduciendo la sustitución $t = t_i + sh \Rightarrow dt = h\,ds$.

$$\Rightarrow \int_{t_i}^{t_{i+1}} f(t, y(t))\, dt = \int_{t_i}^{t_{i+1}} f(t_i, y(t_i))\, dt + \int_{t_i}^{t_{i+1}} \sum_{k=1}^{m-1} (-1)^k \binom{-s}{k} \nabla^k f(t_i, y(t_i))\, dt + \int_{t_i}^{t_{i+1}} \frac{f^{(m)}(\xi_i, y(\xi_i))}{m!} (t-t_i)(t-t_{i-1}) \cdots (t-t_{i+1-m})\, dt$$

$$= h f(t_i, y(t_i)) + \sum_{k=1}^{m-1} (-1)^k \nabla^k f(t_i, y(t_i)) \int_{t_i}^{t_{i+1}} \binom{-s}{k} dt + \frac{f^{(m)}(\xi_i, y(\xi_i))}{m!} \int_{t_i}^{t_{i+1}} (t-t_i)(t-t_{i-1}) \cdots (t-t_{i+1-m})\, dt$$

Hacemos el cambio de variable $t - t_i = sh$

$$t - t_{i-1} = sh + t_i - t_{i-1} = sh + h = h(s+1)$$

$$\begin{cases} t - t_{i-2} = sh + t_i - t_{i-2} \\ = sh + t_i - t_{i-1} + t_{i-1} = t_{i-2} \\ = sh + h + h = h(s+2) \end{cases}$$

$$\rightsquigarrow \int_{t_i}^{t_{i+1}} f(t,y(t))\,dt = h\,f(t_i,y(t_i)) + \sum_{k=1}^{m-1} \nabla^k f(t_i,y(t_i))\,(-1)^k \cdot h \int_0^1 \binom{-s}{k}\,ds$$

$$+ \frac{f^{(m)}(\xi_i,y(\xi_i))}{m!}\, h \int_0^1 (sh)(h(s+1))(h(s+2))\cdots(h(s+m-1))\,ds$$

$$\overset{(*)}{=} h f(t_i,y(t_i)) + \sum_{k=1}^{m-1} \nabla^k f(t_i,y(t_i)) \cdot h\,(-1)^k \int_0^1 \binom{-s}{k} ds$$

$$+ \frac{h^{m+1} f^{(m)}(\xi_i,y(\xi_i))}{m!} \int_0^1 s\cdot(s+1)\cdot(s+2)\cdots(s+m-1)\,ds$$

$$\leq c h^{m+1}$$

que evaluer

Terems $\quad T_k = (-1)^k \int_0^1 \binom{-s}{k} ds \quad , k \geq 1$

$$\binom{-s}{k} = \frac{(-1)^k}{k!}\, s\cdot(s+1)(s+2)\cdots(s+k-1)$$

$k=1 \quad T_1 = \int_0^1 s\,ds = \frac{1}{2}$

$\sim$

$k=2 \quad T_2 = \frac{1}{2}\int_0^1 s(s+1)\,ds = \frac{1}{2}\int_0^1 (s^2+s)\,ds = \frac{1}{2}\left[\frac{1}{3}+\frac{1}{2}\right]$

$= \frac{5}{12}$

Tabla del cálculo

k	$(-1)^k \int_0^1 \binom{-s}{k} ds$
1	1/2
2	5/12
3	3/8
4	251/720
5	95/288

7) Ejemplo: Para $m=3$ tenemos el método de Adams-Bashforth (3-niveles).

$$y(t_{i+1}) \simeq y(t_i) + h\left[f(t_i, y(t_i)) + \frac{1}{2}\nabla f(t_i, y(t_i)) + \frac{5}{12}\nabla^2 f(t_i, y(t_i))\right]$$

$$\simeq y(t_i) + h\left[f(t_i, \hat{y}_i) + \frac{1}{2}\left(f(t_i, y(t_i)) - f(t_{i-1}, \hat{y}_{i-1})\right) + \frac{5}{12}\left(f(t_i, y(t_i)) - 2f(t_{i-1}, y(t_{i-1})) + f(t_{i-2}, y(t_{i-2}))\right)\right]$$

$\Rightarrow$ el método es

$$\hat{y}_{i+1} = \hat{y}_i + h\left[\frac{23}{12} f(t_i, \hat{y}_i) - \frac{16}{12} f(t_{i-1}, \hat{y}_{i-1}) + \frac{5}{12} f(t_{i-2}, \hat{y}_{i-2})\right]$$

8) Def.: Error de truncamiento para un método multipaso

sea el mét. multipaso definido por:

$$\hat{y}_{i+1} = a_{m-1}\hat{y}_i + a_{m-2}\hat{y}_{i-2} + \dots + a_0\hat{y}_{i+1-m}$$
$$+h\left[b_m f(t_{i+1},\hat{y}_{i+1}) + b_{m-1} f(t_i,\hat{y}_i) + b_{m-2} f(t_i,\hat{y}_{i-1})\right.$$
$$\left. + \dots + b_0 f(t_{i+1-m},\hat{y}_{i+1-m})\right],$$

entonces el error de truncamiento local está definido por

$$\tau_{i+1}(h) = \frac{1}{h}\left[y(t_{i+1}) - a_{m-1}y(t_i) - a_{m-2}y(t_{i-1}) - \dots - a_0 y(t_{i+1-m})\right]$$
$$-\left[b_m f(t_{i+1}, y(t_{i+1})) + \dots + b_0 f(t_{i+1-m}, y(t_{i+1-m}))\right]$$

9) Por ejemplo para el met. de Adams-Bashfort Explícito

$m=3$ (3 niveles)

$$y(t_{i+1}) = y(t_i) + \frac{h}{12}\left[23 f(t_i, y(t_i)) - 16 f(t_{i-1}, y(t_{i-1}))\right.$$
$$\left. + 5 f(t_{i-2}, y(t_{i-2}))\right] + ch^4 \qquad (*)$$

$$\Rightarrow \frac{y(t_{i+1}) - y(t_i)}{h} - \frac{1}{12}\left[23 f(t_i, y(t_i)) - 16 f(t_{i-1}, y(t_{i-1}))\right.$$
$$\left. + 5 f(t_{i-2}, y(t_{i-2}))\right] = ch^3$$

$a_2 = 1$, $a_1 = c_0 = 0$

$b_2 = \frac{23}{12}$, $b_1 = \frac{-16}{12}$, $b_0 = \frac{5}{12}$

$\Rightarrow \quad \tau_{i+1}(h) = ch^3$

Burden - Faires

1) métodos Adams-Bashforth (explícitos) de
niveles = 2, 3, 4, 5

$$(*)\quad \begin{cases} y' = f(t,y) & t \in (a,b) \\ y(a) = y_0 \end{cases}$$

$$\Rightarrow \quad y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} y'(t)\,dt = \int_{t_i}^{t_{i+1}} f(t, y(t))\,dt$$

$$\Rightarrow \quad y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\,dt$$

$$y(t_{i+1}) \simeq y(t_i) + \int_{t_i}^{t_{i+1}} P_{m-1}(t)\,dt$$

donde $P_{m-1}(t) \in \mathbb{P}^{m-1}(\mathbb{R})$ interpola los m-puntos:

$$(t_i, f(t_i, y(t_i))),\ (t_{i-1}, f(t_{i-1}, y(t_{i-1}))),\ \ldots,$$
$$(t_{i+1-m}, f(t_{i+1-m}, y(t_{i+1-m})))$$

$$(**) \quad \begin{cases} y(t_{i+1}) = y(t_i) + f(t_i, y(t_i))\, h \int_0^1 ds \\ \qquad + \sum_{k=1}^{m-1} \nabla^k f(t_i, y(t_i))\, h\, (-1)^k \int_0^1 \binom{-s}{k} ds \\ \qquad + c h^{m+1} \end{cases}$$

Tablas.

1.i) $m=2$. Dados $\hat{y}_0, \hat{y}_1$

$$\hat{y}_{i+1} = \hat{y}_i + \frac{h}{2}\left[3 f(t_i, \hat{y}_i) - f(t_{i-1}, \hat{y}_{i-1})\right]$$

$i = 1, \ldots, N$

$$\tau_{i+1}(h) = ch^2$$

1.ii) $m=3$. Dados $\hat{y}_0, \hat{y}_1, \hat{y}_2$

$$\hat{y}_{i+1} = \hat{y}_i + \frac{h}{12}\left[23 f(t_i, \hat{y}_i) - 16 f(t_{i-1}, \hat{y}_{i-1}) + 5 f(t_{i-2}, \hat{y}_{i-2})\right] \quad i = 2, \ldots, N$$

$$\tau_{i+1}(h) = ch^3$$

1.iii) $m=4$. Dados $\hat{y}_0, \hat{y}_1, \hat{y}_2, \hat{y}_3$

$$\hat{y}_{i+1} = \hat{y}_i + \frac{h}{24}\left[55 f(t_i, \hat{y}_i) - 59 f(t_{i-1}, \hat{y}_{i-1}) + 37 f(t_{i-2}, \hat{y}_{i-2}) - 9 f(t_{i-3}, \hat{y}_{i-3})\right]$$

$$\tau_{i+1}(h) = ch^4$$

1.iv) $m=5$. Dados $\hat{y}_0, \ldots, \hat{y}_4$

$$\hat{y}_{i+1} = \hat{y}_i + \frac{h}{720}\left[1901 f(t_i, \hat{y}_i) - 2774 f(t_{i-1}, \hat{y}_{i-1}) + 2616 f(t_{i-2}, \hat{y}_{i-2}) - 1274 f(t_{i-3}, \hat{y}_{i-3}) + 251 f(t_{i-4}, \hat{y}_{i-4})\right]$$

$$\tau_{i+1}(h) = ch^5$$

2) Métodos de multipaso implícitos de Adams-Moulton

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\,dt$$

$$= y(t_i) + \int_{t_i}^{t_{i+1}} P_m(t)\,dt$$

$P_m(t) \in \mathbb{P}^m(\mathbb{R})$ que interpola $(m+1)$-puntos

$(t_{i+1}, f(t_{i+1}, y(t_{i+1}))), (t_i, f(t_i, y(t_i))), \ldots$

$(t_{i+1-m}, f(t_{i+1-m}, y(t_{i+1-m})))$.

2.i) $m=2$. Dados $\hat{y}_0, \hat{y}_1$

$$\hat{y}_{i+1} = \hat{y}_i + \frac{h}{12}\left[5 f(t_{i+1}, \hat{y}_{i+1}) + 8 f(t_i, \hat{y}_i) - f(t_{i-1}, \hat{y}_{i-1})\right] \quad i = 1, \ldots, N$$

$$T_{i+1}(h) = ch^3$$

2.ii) $m=3$. Dados $\hat{y}_0, \ldots, \hat{y}_2$

$$\hat{y}_{i+1} = \hat{y}_i + \frac{h}{24}\left[9 f(t_{i+1}, \hat{y}_{i+1}) + 19 f(t_i, \hat{y}_i) - 5 f(t_{i-1}, \hat{y}_{i-1}) + f(t_{i-2}, \hat{y}_{i-2})\right]$$

$$\tau_{i+1} = ch^4$$

2.iii) $m=4$. Dados $\hat{y}_0, \ldots, \hat{y}_3$

$$\hat{y}_{i+1} = \hat{y}_i + \frac{h}{720}\Big[251 f(t_{i+1}, \hat{y}_{i+1}) + 646 f(t_i, \hat{y}_i) - 264 f(t_{i-1}, \hat{y}_{i-1}) + 106 f(t_{i-2}, \hat{y}_{i-2}) - 19 f(t_{i-3}, \hat{y}_{i-3}) \Big]$$

$$\tau_{i+1}(h) = ch^5$$