BDF : Backward Differentation

→ Métodos BDF

explícitos

implícitos ←

1) Dados m-aproximaciones $\hat{y}_i, \hat{y}_{i-1}, \dots, \hat{y}_{i+1-m}$

queremos obtener el valor $\hat{y}_{i+1}$ considerando el

polinomio de interpolación que interpola los puntos

$$\{ (t_{i+1}, \hat{y}_{i+1}), (t_i, \hat{y}_i), \dots, (t_{i+1-m}, \hat{y}_{i+1-m}) \}$$

$s \in \mathbb{R}$

$$(*) \quad q(t) = q(t_i + sh) = \sum_{j=0}^{m} (-1)^j \binom{-s+1}{j} \nabla^j \hat{y}_{i+1}$$

Donde $\binom{-s+1}{0} = 1$, $(-1)^j \binom{-s+1}{j} = \frac{1}{j!}(s-1)s(s+2)\dots(s+j-2)$

$j \geq 1$.

→ La incógnita $\hat{y}_{i+1}$ se obtiene haciendo que el

polinomio $q(t)$ debe satisfacer la EDO en un punto

$$(**) \quad q'(t_{i+1-r}) = h f(t_{i+1-r}, \hat{y}_{i+1-r})$$ para un

número entero r.

2) BDF-explícitos. Se obtienen haciendo $r=1$ en (*a)

$\Rightarrow$ (**). $q'(t_i) = h f(t_i, \hat{y}_i)$

Observar que $q'(t_i) = \frac{d}{ds}\left[q(t_i+sh)\right]_{s=0}$

$q(t) = q(t_i+sh)$

Utilizando (*)

$$\frac{d}{ds}\left[q(t_i+sh)\right]_{s=0} = \sum_{j=0}^{m} \nabla^j \hat{y}_{i+1} \frac{d}{ds}\Big[\overbrace{(-1)^j \binom{-s+1}{j}}^{\delta_j^*}\Big]_{s=0}$$

$$= \sum_{j=0}^{m} \nabla^j \hat{y}_{i+1} \cdot \delta_j^* \Big|_{s=0}$$

$$\delta_j^* = \frac{d}{ds}\left[(-1)^j \binom{-s+1}{j}\right] = \frac{d}{ds}\left[\frac{1}{j!}(s-1)s\cdot(s+1)\cdots(s+j-2)\right]$$

$j \geq 1$

2.i) $m=1$ (**) $q'(t_i) = h f(t_i, \hat{y}_i)$

$$q'(t_i) = \sum_{j=0}^{1} \nabla^j \hat{y}_{i+1} \cdot \delta_j^* \Big|_{s=0} \qquad \delta_0^* = \frac{d}{ds}[1] = 0.$$

$\Rightarrow \delta_0^* \big|_{s=0} = 0. \qquad \delta_1^* = \frac{d}{ds}\left[\frac{1}{1}\cdot(s-1)\right] = \underline{1}.$

$\Rightarrow \delta_1^* \big|_{s=0} = \underline{1}$

$$\Rightarrow \quad q'(t_i) = \sum_{j=0}^{1} \nabla^j \hat{y}_{i+1} \cdot \delta_j^* \big|_{s=0} \qquad \nabla^0 \hat{y}_{i+1} = \hat{y}_{i+1}$$

$$= \cancel{\hat{y}_{i+1} \cdot \delta_0^*} + (\hat{y}_{i+1} - \hat{y}_i) \cdot 1$$

$$\Rightarrow (**) \qquad \hat{y}_{i+1} - \hat{y}_i = h f(t_i, \hat{y}_i)$$

método Euler. explícito.

= BDF-1 explícito.

2.iii) $m = 2$.

$$\Rightarrow (*) \quad q'(t_i) = \sum_{j=0}^{2} \nabla^j \hat{y}_{i+1} \cdot \delta_j^* \big|_{s=0}.$$

$$\delta_0^* \big|_{s=0} = 0, \quad \delta_1^* \big|_{s=1} = 1.$$

$$\delta_2^* = \frac{d}{ds}\left[\frac{1}{2!}(s-1)s\right] = \frac{1}{2}\left[(s-1) + s\right] = \underline{\underline{\frac{1}{2}(2s-1)}}$$

$$\Rightarrow \quad \delta_2^* \big|_{s=0} = -\frac{1}{2}.$$

$$\Rightarrow \quad q'(t_i) = (\hat{y}_{i+1} - \hat{y}_i) + \left(\frac{-1}{2}\right) \nabla^2 (\hat{y}_{i+1})$$

$$= (\hat{y}_{i+1} - \cancel{\hat{y}_i}) + \left(-\frac{1}{2}\right)\left[\hat{y}_{i+1} - \cancel{2\hat{y}_i} + \hat{y}_{i-1}\right]$$

$$= \frac{1}{2}\hat{y}_{i+1} - \frac{1}{2}\hat{y}_{i-1}$$

$\Rightarrow$ BDF-2 (explícito) : $\quad \frac{1}{2}\hat{y}_{i+1} - \frac{1}{2}\hat{y}_{i-1} = h f(t_i, \hat{y}_i)$

3) Los métodos BDF-explícitos para $m \geq 3$

son inestables. $\#$

4) BDF-implícitos $\gamma = 0$ en $(**)$

$\Rightarrow (**) \quad q'(t_{i+1}) = h f(t_{i+1}, \hat{y}_{i+1})$

$$q'(t_{i+1}) = \frac{d}{ds}\left[q(t_i + sh)\right]_{s=1}$$

4.i) $m=1$

$$q'(t_i) = \sum_{j=0}^{1} \nabla^j \hat{y}_{i+1} \cdot \delta_j^* |_{s=1}$$

$$\delta_0^* = 0, \quad \delta_1^* = 1 \Rightarrow \delta_1^* |_{s=1} = 1.$$

$$\Rightarrow q'(t_i) = \hat{y}_{i+1} - \hat{y}_i = h f(t_{i+1}, \hat{y}_{i+1})$$

BDF1 = Método Euler implícito

4.ii) $m=2$ BDF2.

$$q'(t_i) = \sum_{j=0}^{2} \nabla^j \hat{y}_{i+1} \cdot \delta_j^* |_{s=1}$$

$$\delta_2^* |_{s=1} = \frac{1}{2}$$

$\Rightarrow \quad q(t_i) = 1(\hat{y}_{i+1} - \hat{y}_i) + \frac{1}{2}(\hat{y}_{i+1} - 2\hat{y}_i + \hat{y}_{i-1})$

$= \frac{3}{2}\hat{y}_{i+1} - \frac{4}{2}\hat{y}_i + \frac{1}{2}\hat{y}_{i-1}$

$\Rightarrow$ BDF2- implícito :

$$\frac{3}{2}\hat{y}_{i+1} - \frac{4}{2}\hat{y}_i + \frac{1}{2}\hat{y}_{i-1} = h\,f(t_{i+1}, \hat{y}_{i+1})$$

4.iii) $m=3$ BDF3 - implícito

$$\frac{11}{6}\hat{y}_{i+1} - 3\hat{y}_i + \frac{3}{2}\hat{y}_{i-1} - \frac{1}{3}\hat{y}_{i-2} = h\,f(t_{i+1}, \hat{y}_{i+1})$$

4.iv) Los métodos BDF-implícitos son inestables para $m \geq 7$.

5) Error de truncamiento de BDF-2 (implícito) es de $O(h^2)$.

Método:
$$\frac{3}{2}\hat{y}_{i+1} - \frac{4}{2}\hat{y}_i + \frac{1}{2}\hat{y}_{i-1} = h\,f(t_{i+1}, \hat{y}_{i+1})$$

Forma grcl.:
$$\hat{y}_{i+1} = \sum_{s=1}^{m} a_{m-s}\,\hat{y}_{i+1-s} + h\sum_{s=0}^{m} b_{m-s}\,f(t_{i+1}, y_{i+1-s})$$

Para BDF2
$$\hat{y}_{i+1} = \frac{4}{3}\hat{y}_i - \frac{1}{3}\hat{y}_{i-1} + \frac{2}{3}h\,f(t_{i+1}, \hat{y}_{i+1})$$

$$a_1 = \frac{4}{3},\ a_0 = -\frac{1}{3},\ b_2 = \frac{2}{3},\ b_1 = b_0 = 0.$$

Condiciones:
$$\text{i)}\quad \sum_{s=1}^{2} a_{2-s} = 1$$

$$\text{ii)}\quad 2^k - \sum_{s=1}^{2}(2-s)^k a_{2-s} - k\sum_{s=0}^{2}(2-s)^{k-1} b_{2-s} = 0$$

$k=1,2$ convención $0^0 = 1$.

$$\text{i)}\quad \sum_{s=1}^{2} a_{2-s} = \frac{4}{3} - \frac{1}{3} = 1 \checkmark.$$

$$\text{ii)}\quad k=1:\ 2 - a_1 - \cancel{0 a_0} - b_2 - \cancel{b_1} - \cancel{b_0}$$
$$= 2 - \frac{4}{3} - \frac{2}{3} = 2 - \frac{6}{3} = 2 - 2 = 0 \checkmark.$$

$$k=2:\ 4 - a_1 - \cancel{0\cdot a_0} - 2^2 b_2 = 4 - \frac{4}{3} - 4\cdot\frac{2}{3}$$
$$= 4 - \frac{4}{3} - \frac{8}{3} = 4 - \frac{12}{3} = 4 - 4 = 0 \checkmark.$$