→ Métodos de Taylor de orden n

Burden - Faires 5.3

→ Motivación:

(*) $$y' = f(t,y) \,, \quad t \in (a,b) \qquad y(a) = y_0$$

Método de diferencias gral:

$N :=$ # intervalos

$h = \frac{b-a}{N}$

$t_i = a + ih \quad i = 0, \dots N$

(**) $$\begin{cases} \hat{y}_0 = y_0 \\ \hat{y}_i = \hat{y}_{i-1} + h \underbrace{\phi(t_{i-1}, \hat{y}_{i-1})} \end{cases}$$

$\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$

El error de truncamiento del método (**) se define como:

(***) $$\tau_{i+1}(h) = \frac{1}{h}\left[y(t_{i+1}) - (y(t_i) - h\,\phi(t_i, y(t_i))) \right]$$
$$= \frac{y(t_{i+1}) - y(t_i)}{h} - \phi(t_i, y(t_i))$$

* Ejemplo: M. de Euler:

$$\hat{y}_0 = y_0$$
$$\hat{y}_i = \hat{y}_{i-1} + h\, f(t_{i-1}, \hat{y}_{i-1}) \quad i = 0 \dots N$$

$$\phi(t_i, \hat{y}_i) = f(\cdot,\cdot)$$

$$y'(t_i) = \phi(\cdot,\cdot)$$

Teorema de Taylor:

$$y(t_{i+1}) = y(t_i) + h\, \overbrace{f(t_i, y(t_i))}^{y'(t_i)} + \frac{h^2}{2!} y''(\xi_i)$$

$$\xi_i \in (t_i, t_{i+1})$$

$$\Rightarrow \underbrace{\frac{y(t_{i+1}) - y(t_i)}{h} - f(t_i, y(t_i))}_{T_{i+1}(h)} = \frac{h}{2} y''(\xi_i)$$

$$\Rightarrow T_{i+1}(h) = \frac{h}{2} y''(\xi_i) = Ch = O(h^1)$$

$\Rightarrow$ el método de Euler tiene un error de truncamiento de orden **1**

$\#$.

→ Métodos de Taylor de orden n $\boxed{y^{(n)} = \frac{d^n y}{dt^n}}$

⇒ Teorema de Taylor: → $\underline{*}.\ y' = f(t,y) \quad t \in (a,b)$

$$y(a) = y_0$$

Suponemos que la solución $y : \mathbb{R} \to \mathbb{R}$ es continuamente derivable hasta $(n+1)$-veces

Por T. de Taylor expresando t_{i+1} alrededor t_i:

$$\underline{(**)}.\ y(t_{i+1}) = y(t_i) + h\, y'(t_i) + \frac{h^2}{2!} y^{(2)}(t_i) + \dots + \frac{h^n}{n!} y^{(n)}(t_i)$$
$$+ \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i) \quad , \xi_i \in (t_i, t_{i+1})$$

Usando:

$$y^{(j)} = \frac{d^j}{dt^j}(f) := f^j$$

$$y(t_{i+1}) = y(t_i) + h\, f(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \dots$$
$$+ \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

→ El método de Taylor de orden n se obtiene eliminando el residuo:

$$\begin{cases} \hat{y}_0 = y_0 \quad , h = (b-a)/N \, , \ t_i = a + ih \, , i = 0 \dots N \\ \hat{y}_i = \hat{y}_{i-1} + h\, T^{(n)}(t_{i-1}, \hat{y}_{i-1}) \quad i = 1 \dots N \end{cases}$$

* $T^{(n)}(t_{i-1}, \hat{y}_{i-1}) = f(t_{i-1}, \hat{y}_{i-1}) + \frac{h}{2} f'(t_{i-1}, \hat{y}_{i-1}) + \dots$

$$+ \frac{h^{n-1}}{n!} f^{(n-1)}(t_{i-1}, \hat{y}_{i-1})$$

→ El método de Euler es un método de Taylor de orden $n=1$.

→ Ejemplo:

$$y' = y - t^2 + 1 \qquad t \in [0,2)$$

$$y(0) = y_0 = 0.5$$

M. Taylor de orden $n=2$.

$$f(t,y) = y - t^2 + 1$$

$$T^{(2)}(t_{i-1}, \hat{y}_{i-1}) = f(t_{i-1}, \hat{y}_{i-1}) + \frac{h}{2} f'(t_{i-1}, \hat{y}_{i-1})$$

$$\rightarrow f' = \frac{d}{dt} f(t, y(t)) = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} y'$$

$$= (-2t) + 1 \cdot y' = -2t + y'$$

$$= -2t + y - t^2 + 1$$

$$\frac{d}{dt}[f(t, y(t))] = \frac{d}{dt}[y - t^2 + 1]$$

$$= y' - 2t =$$