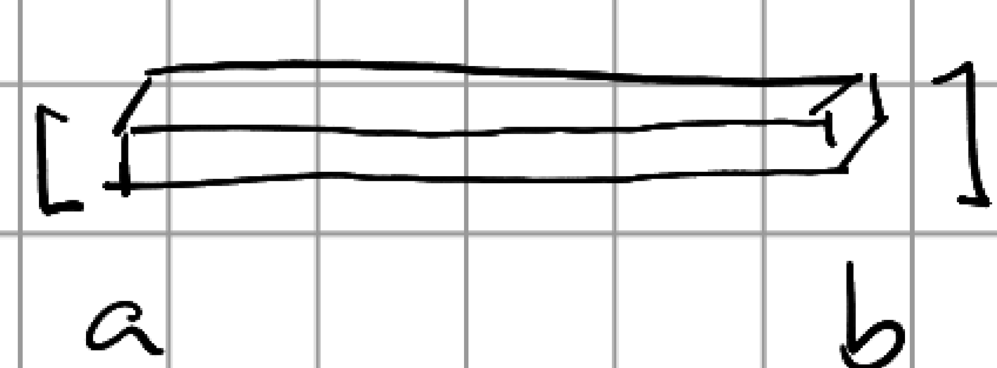
1) Métodos numérico para EDO con valores en la frontera

$y' = f(t,y)$

Ecuación de Calor:

$$\begin{cases} \overbrace{-\dfrac{d}{dx}\left[k(x)\dfrac{du}{dx}\right]}^{\text{difusión}} + \overbrace{b(x)\,u}^{\text{reacción}} = \overbrace{f(x)}^{\text{Fuente}} & x \in (a,b) \\ u(a) = \alpha \\ u(b) = \beta \end{cases}$$

u: es la temperatura en el medio $°C$, $°F$, K



$\longmapsto x$

$k(x)$: conductividad térmica

$k(x) > 0$, $b(x) \geq 0$.

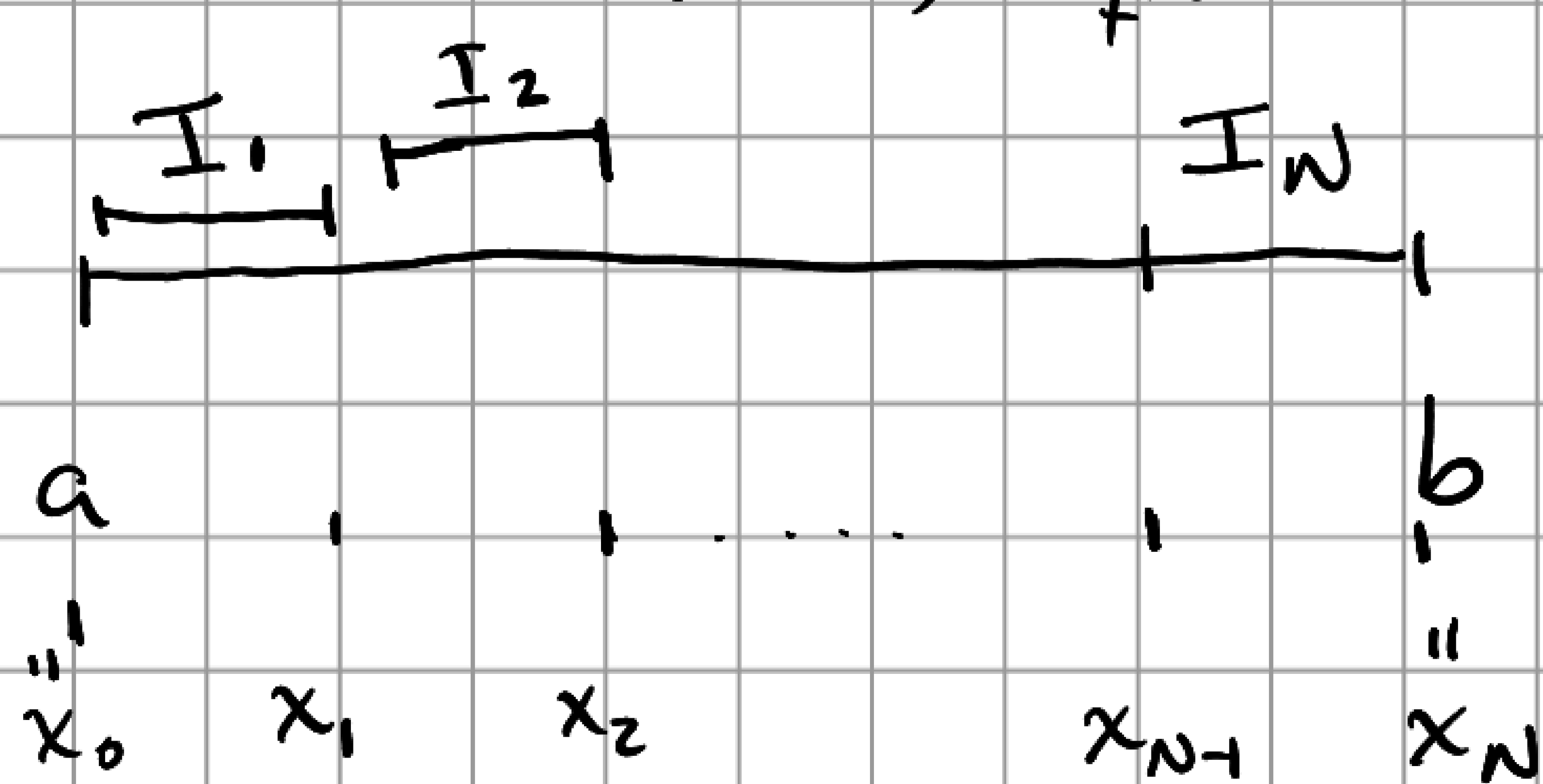
2) Método de diferencias finitos

Suponemos que $b(x) \equiv 0$, $k(x) = k > 0$.

Ecuación de calor: $\begin{cases} -k \dfrac{d^2u}{dx^2} = f(x) & \underline{x \in (a,b)} \\ \left. \begin{array}{l} u(a) = \alpha \\ u(b) = \beta . \end{array} \right] & \end{cases}$

$\underline{\underline{(*)}}.$ $k \dfrac{d^2u}{dx^2} = -f(x)$

Creamos un malla para $[a,b]$. Sea $\underline{\underline{N}}$ el # de sub-intervalos (uniformes) que divide a $[a,b]$



$x_i = a + ih$

$i = 0, \dots, N$

$h = |I_i| \quad i = 1, \dots, N$. Observar que tenemos $\underline{\underline{N+1}}$ puntos en la malla.

Como $x_0 = a \Rightarrow u(x_0) = \alpha$ y similarmente $u(x_N) = \beta$.

Observer que para $x_i \in (a,b)$

$$(*) \qquad k\, u''(x_i) = -f(x_i) \qquad \text{(Dif. finitos)}$$

3) Usando el teorema de Taylor con residuo para $u(x_{i+1})$ y $u(x_{i-1})$, respectivamente, alrededor del punto $u(x_i)$ obtenemos:

$$u(x_{i+1}) = u(x_i+h) = u(x_i) + \cancel{h u'(x_i)} + \frac{h^2}{2} u''(x_i) + \cancel{\frac{h^3}{6} u'''(x_i)} + ch^4$$

$$u(x_{i-1}) = u(x_i-h) = u(x_i) - \cancel{h u'(x_i)} + \frac{h^2}{2} u''(x_i) - \cancel{\frac{h^3}{6} u'''(x_i)} + ch^4$$

$$\Rightarrow \quad u(x_{i+1}) + u(x_{i-1}) = 2u(x_i) + h^2 u''(x_i) + ch^4$$

$$u''(x_i) = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} + ch^2$$

Formula de diferencia central de orden 2 para $u''(x_i)$

$$\Rightarrow \quad (*) \quad \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} = -\frac{1}{k} f(x_i) + ch^2$$

$$\hat{u}_i \sim u(x_i)$$

$$\Rightarrow \begin{cases} \hat{u}_{i+1} - 2\hat{u}_i + \hat{u}_{i-1} = \dfrac{-h^2}{k} f(x_i) & i = 1, \dots, N-1 \\ \begin{cases} \hat{u}_0 = \alpha \; (= u(x_0) = u(a)) \\ \hat{u}_N = \beta \; (= u(x_N) = u(b)) \end{cases} \end{cases}$$

$x_i \in (a,b)$

$(\hat{u}_1, \dots, \hat{u}_{N-1})$

Este es un sistema lineal $\rightsquigarrow A\underline{\hat{u}} = f$

$$\begin{cases} i=1 \\ \quad \hat{u}_2 - 2\hat{u}_1 + \hat{u}_0 = -\dfrac{h^2}{k} f(x_1) \\ \Rightarrow \hat{u}_2 - 2\hat{u}_1 = \dfrac{-h^2}{k} f(x_1) - \alpha \\ i = N-1 \\ \quad \underline{\underline{\hat{u}_N}} - 2\hat{u}_{N-1} + \hat{u}_{N-2} = \dfrac{-h^2}{k} f(\hat{u}_{N-1}) \\ \quad -2\hat{u}_{N-1} + \hat{u}_{N-2} = \dfrac{-h^2}{k} f(\hat{u}_{N-1}) - \beta \\ i = 2, \dots, N-2 \\ \quad \hat{u}_{i+1} - 2\hat{u}_i + \hat{u}_{i-1} = \dfrac{-h^2}{k} f(x_i). \end{cases}$$

4) En forma matricial (Forma óptima)

$$\rightsquigarrow \hat{\tilde{\underline{u}}} = [\hat{u}_1, \hat{u}_2, \dots, \hat{u}_{N-1}]^t \in \mathbb{R}^{N-1}$$

$$\underbrace{\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \vdots \\ \hat{u}_{N-2} \\ \hat{u}_{N-1} \end{bmatrix}}_{\hat{\tilde{\underline{u}}}} = \underbrace{\begin{bmatrix} -\frac{h^2}{k} f(x_1) - \alpha \\ -\frac{h^2}{k} f(x_2) \\ -\frac{h^2}{k} f(x_3) \\ \vdots \\ -\frac{h^2}{k} f(x_{N-2}) \\ -\frac{h^2}{k} f(x_{N-1}) - \beta \end{bmatrix}}_{\tilde{\underline{f}}}$$

$$\tilde{\underline{A}} \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}, \quad \tilde{\underline{f}} \in \mathbb{R}^{N-1}$$

$$\hat{\tilde{\underline{u}}} = \tilde{A}^{-1} \cdot \tilde{\underline{f}}$$

5) Forma Matricial (Forma conveniente)

$$\underline{\hat{u}} = [\,\hat{u}_0 \quad \underbrace{\hat{u}_1 \quad \hat{u}_2 \; \cdots \; \hat{u}_{N-1}}_{\text{Incógnitas del sistema}} \quad \hat{u}_N\,]^t \in \mathbb{R}^{N+1}$$

$\hat{u}_0 = \alpha$ (Valor en la frontera)

$\hat{u}_N = \beta$ (Valor en la frontera)

$$\begin{bmatrix} 1 & \underline{0} & & & & \\ 1 & -2 & 1 & & \underline{0} & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & \underline{0} & & 1 & -2 & 1 \\ & & & \underline{0} & & 1 \end{bmatrix} \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_{N-1} \\ \hat{u}_N \end{bmatrix} = \begin{bmatrix} \alpha \\ -\frac{h^2}{k} f(x_1) \\ -\frac{h^2}{k} f(x_2) \\ \vdots \\ -h^2/k \, f(x_{N-1}) \\ \beta \end{bmatrix}$$

$$\hat{u}_0 = \alpha, \quad \hat{u}_N = \beta$$

$$\hat{u}_0 - 2\hat{u}_1 + \hat{u}_2 = -\frac{h^2}{k} f(x_1)$$

$$A\underline{\hat{u}} = \underline{f} \qquad A \in \mathbb{R}^{(N+1)\times(N+1)}$$

$$f \in \mathbb{R}^{(N+1)}$$

6) Ecuación Parabólica:

$$(*)\quad \begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2} = f(t,x) & (t,x) \in (0,T) \times (a,b) \\ u(0,x) = u_0(x) \\ u(t,a) = \alpha \\ u(t,b) = \beta \end{cases}$$

N_x : # subintervalos que divide (a,b), h_x

N_t : ✓ que divide $(0,T)$, h_t

$$(*) \quad \frac{\partial u}{\partial t}(t,x_i) - k\frac{\partial^2 u}{\partial x^2}(t,x_i) = f(t,x_i)$$

$(t,x_i) \in (0,T) \times (a,b)$

$$\frac{\partial^2 u}{\partial x^2}(t,x_i) = \frac{u(t,x_{i+1}) - 2(t,x_i) + u(t,x_{i-1})}{h_x} + c(t)\cdot h_x^2$$

$$(**) \quad \frac{\partial \tilde{u}}{\partial t}(t,x_i) = \frac{k}{h_x^2}\left[\tilde{u}(t,x_{i+1}) - 2\tilde{u}(t,x_i) + \tilde{u}(t,x_{i-1})\right] + f(t,x_i) \qquad i = 1,\dots,N-1$$

$$\tilde{u}(t,x_i) \sim u(t,x)$$

$$\rightsquigarrow \quad \underline{\tilde{u}}' = F(\tilde{u}, t, \hat{x}, k, h_x)$$