7) Método del Trapecio.

$$\text{(*)} \quad \begin{cases} \hat{\underline{y}}_0 = \underline{y}_0 \\ \hat{\underline{y}}_{i+1} = \hat{\underline{y}}_i + \dfrac{h}{2}\left[\underline{f}(t_i, \hat{\underline{y}}_i) + \underline{f}(t_{i+1}, \underbrace{\hat{\underline{y}}_{i+1}}) \right] \end{cases} \qquad i = 0, \dots, N$$

Método numérico de la forma siguiente:

$$\text{(**)} \quad \begin{cases} \hat{\underline{y}}_0 = \underline{y}_0 \\ \hat{\underline{y}}_{i+1} = \underline{G}_i(\underline{f}, h, \hat{\underline{y}}_0, \hat{\underline{y}}_1, \dots, \hat{\underline{y}}_i, \hat{\underline{y}}_{i+1}) \end{cases}$$

El error de truncamiento $\underline{T}_{i+1}(h)$ se define como:

$$\underline{T}_{i+1}(h) := \frac{1}{h}\left[\underline{y}(t_{i+1}) - \underline{G}_i(\underline{f}, h, \underline{y}_0, \underline{y}(t_1), \underline{y}(t_2), \dots \underline{y}(t_i), \underline{y}(t_{i+1})) \right]$$

$$= O(h^r)$$

Decimos que una función $\underline{g}$ es de orden $O(h^r)$ si y solo si $\| \underline{g} \| \leq c h^r$

El error de truncamiento es de orden r

8) El error de truncamiento del método del trapecio es de orden $r = 2$.

$$(*1)\quad \underline{y}(t_{i+1}) - G_i(f, h, \underline{y}(t_i), \underline{y}(t_{i+1}))$$

$$\overset{*}{=} \underline{y}(t_{i+1}) - \left\{ \underline{y}(t_i) + \frac{h}{2}\Big[\underbrace{f(t_i, y(t_i))}_{y'(t_i)} + \underbrace{f(t_{i+1}, y(t_{i+1}))}_{y'(t_{i+1})}\Big]\right.$$

$$= \cancel{\underline{y}(t_i)} + \cancel{h\,\underline{y}'(t_i)} + \cancel{\tfrac{1}{2}h^2\,\underline{y}''(t_i)} + O(h^3)$$

$$- \left\{ \cancel{\underline{y}(t_i)} + \frac{h}{2}\Big[\cancel{\underline{y}'(t_i)} + \big(\cancel{\underline{y}'(t_i)} + \cancel{h\,\underline{y}''(t_i)} + O(h^2)\big)\Big] \right\}$$

$$= O(h^3) - \frac{h}{2}O(h^2) = O(h^3) + O(h^3)$$

$$= O(h^3) \longleftarrow$$

$$\tau_{i+1} := \frac{1}{h}\Big[y(t_{i+1}) - G_i(f, h, \underline{y}(t_i), \underline{y}(t_{i+1}))\Big]$$

$$= \frac{1}{h}\left[O(h^3)\right] = O(h^2)$$

$\boxed{\Sigma}$

→ Convergencia del método del Trapecio.

i) $\lim_{h \to 0^+} \max_i \| \underline{e}_i \| = 0$ (convergencia)

→ ii) $\| \underline{e}_i \| \leq c h^2$ $\boxed{\text{si } h < \frac{2}{L}}$

Del análisis anterior:

$$\underline{y}(t_{i+1}) = \underline{y}(t_i) + \frac{h}{2} \left[\underline{f}(t_i, \underline{y}(t_i)) + \underline{f}(t_{i+1}, \underline{y}(t_{i+1})) \right] + O(h^3)$$

Utilizando la ecuación anterior y la definición del metodo del trapecio:

$$\underbrace{\underline{y}(t_{i+1}) - \underline{\hat{y}}_{i+1}}_{\underline{e}_{i+1}} = \overbrace{\underline{y}(t_i) - \underline{\hat{y}}_i}^{\underline{e}_i} + \frac{h}{2} [\underline{f}(t_i, \underline{y}(t_i)) - \underline{f}(t_i, \underline{\hat{y}}_i)]$$
$$+ \frac{h}{2} [\underline{f}(t_{i+1}, \underline{y}(t_{i+1})) - \underline{f}(t_{i+1}, \underline{\hat{y}}_{i+1})]$$
$$+ O(h^3)$$

$$\| \underline{e}_{i+1} \| \leq \| \underline{e}_i \| + \frac{h}{2} \| \underline{f}(t_i, \underline{y}(t_i)) - \underline{f}(t_i, \underline{\hat{y}}_i) \| + \frac{h}{2} \| \underline{f}(t_{i+1}, \underline{y}(t_{i+1})) - \underline{f}(t_{i+1}, \underline{\hat{y}}_{i+1}) \| + c h^3$$

⇒

$$\| e_{i+1} \| \leq \| \underline{e}_i \| + L \cdot \frac{h}{2} \underbrace{\| \underline{y}(t_i) - \underline{\hat{y}}_i \|}_{\| \underline{e}_i \|} + \frac{L h}{2} \underbrace{\| \underline{y}(t_{i+1}) - \underline{\hat{y}}_{i+1} \|}_{\| e_{i+1} \|}$$

$$\| e_{i+1} \| \leq \| e_i \| \left[1 + \frac{hL}{2} \right] + \underbrace{\frac{hL}{2} \| e_{i+1} \|}_{} + c h^3$$

$$\Rightarrow \quad \|\underline{e}_{i+1}\| \overbrace{\left[1 - \frac{hL}{2}\right]}^{>0} \leq \|\underline{e}_i\| \left[1 + \frac{hL}{2}\right] + ch^3$$

$$(*) \underbrace{1 - \frac{hL}{2} > 0}_{\Rightarrow} \Rightarrow \underline{h < \frac{2}{L}}$$

$$\Rightarrow \|\underline{e}_{i+1}\| \leq \|\underline{e}_i\| \underbrace{\left[\frac{1 + \frac{hL}{2}}{1 - \frac{hL}{2}}\right]}_{1 + w} + \frac{\overbrace{c\,h^3}^{t}}{1 - \frac{hL}{2}}$$

$$\frac{1 + \frac{hL}{2}}{1 - \frac{hL}{2}} = \underbrace{\frac{1}{1 - \frac{hL}{2}}} + \frac{\frac{hL}{2}}{1 - \frac{hL}{2}} \quad , \quad \frac{1}{1 - \frac{hL}{2}} = 1 + z$$

$$= 1 + z + \overbrace{\frac{\frac{hL}{2}}{1 - \frac{hL}{2}}} \quad , \quad z = \frac{1}{1 - \frac{hL}{2}} - 1 = \frac{1 - \left[1 - \frac{hL}{2}\right]}{1 - \frac{hL}{2}}$$

$$= \underbrace{\frac{+\frac{hL}{2}}{1 - \frac{hL}{2}}}$$

$$= 1 + \frac{hL}{1 - \frac{hL}{2}} = 1 + \frac{2hL}{2 - hL}$$

$$\Rightarrow \|\underline{e}_{i+1}\| \leq \|\underline{e}_i\| \left[1 + \frac{2hL}{2 - hL}\right] + \frac{ch^3}{1 - \frac{hL}{2}}$$

Usando Lemma 2 $a_i = \|\underline{e}_i\|$, $a_0 = \|0\| \geq 0 \geq -\frac{t}{s}$ ✓

$$t = \frac{ch^3}{1 - \frac{hL}{2}} \quad , \quad s = \frac{2hL}{2 - hL}$$

$$\frac{t}{s} = \frac{ch^3}{1-\frac{hL}{2}} \cdot \frac{2-hL}{2hL} = \frac{\cancel{2}ch^3}{(\cancel{2-hL})} \cdot \frac{(\cancel{2-hL})}{\cancel{2}hL} = \frac{ch^2}{L}$$

$$\Rightarrow \|\underline{e}_{i+1}\| \leq e^{(i+1)\left[\frac{2hL}{2-hL}\right]} \left[0 + \frac{t}{s}\right] - \frac{t}{s}$$

$$\Leftrightarrow \frac{ch^2}{L}\left[e^{\overline{(i+1)h}\left[\frac{2L}{2-hL}\right]} - 1 \right] \qquad (i+1)h = (i+1)\pm a$$

$$= t_{i+1} - a$$

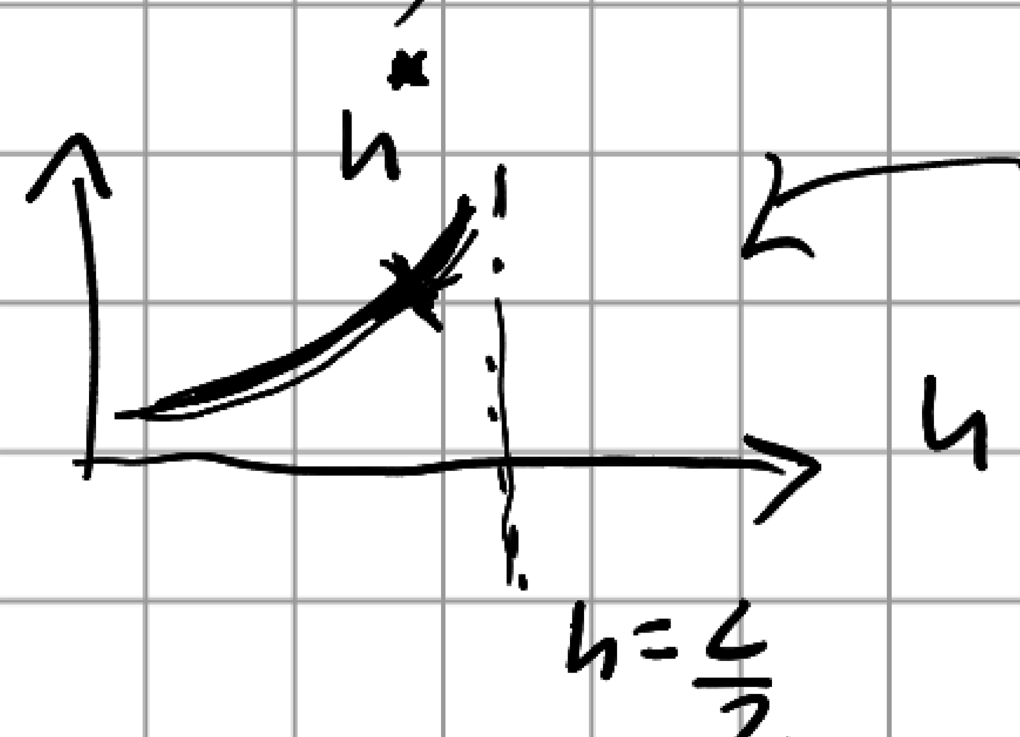
([illegible]) $\underline{\underline{}}$

$$\leq \frac{ch^2}{L} \cdot \left[\underbrace{\overbrace{e^{(b-a)\frac{2L}{2-hL}}}} - 1 \right] \qquad t_{i+1} - a \leq b-a$$

porque $t_{i+1} \in [a,b]$

$$\lim_{h\to 0^+} \|\underline{e}_{i+1}\| \leq \left(\lim_{h\to 0^+} \frac{ch^2}{L}\right)\left[\lim_{h\to 0^+} e^{(b-a)\frac{2L}{2-hL}} - 1\right]$$

$$\lim_{h\to 0^+} \left(\frac{2L}{2-hL}\right) = 0$$



$$\overline{h < \frac{L}{2}}$$

$$\lim_{h\to 0^+} e^{(b-a)\frac{2L}{2-hL}} = 1$$

$$\lim_{h\to 0^+} \|\underline{e}_{i+1}\| \leq 0 \cdot 0 = 0. \qquad i) \checkmark$$

De $h < \frac{L}{2} \Rightarrow h \leq h^* < \frac{L}{2}$

$\Rightarrow$

$$e^{\frac{2L}{2-Lh}} \leq e^{\frac{2L}{2-Lh^*}} = cte := c$$

$$\Rightarrow \|\underline{e}_{i+1}\| \leq \frac{ch^2}{L}\left[e^{(b-c)c} - 1 \right]$$

$$\Leftarrow) \quad ch^2[\,c-1\,] = ch^2$$

$\boxtimes$