1) Métodos numérico para EDO con valores en la frontera

$y' = f(t, y)$

Ecuación de Calor:

$$\begin{cases} \overbrace{-\dfrac{d}{dx}\left[k(x)\dfrac{du}{dx} \right]}^{\text{difusión}} + \overbrace{b(x)\,u}^{\text{reacción}} = \overbrace{f(x)}^{\text{Fuente.}} & x \in (a,b) \\ u(a) = \alpha \\ u(b) = \beta \end{cases}$$

u: es la temperatura en el medio °C, °F, K

a b $\longmapsto x$

$k(x)$: conductividad térmica

$k(x) > 0$, $b(x) \geq 0$.

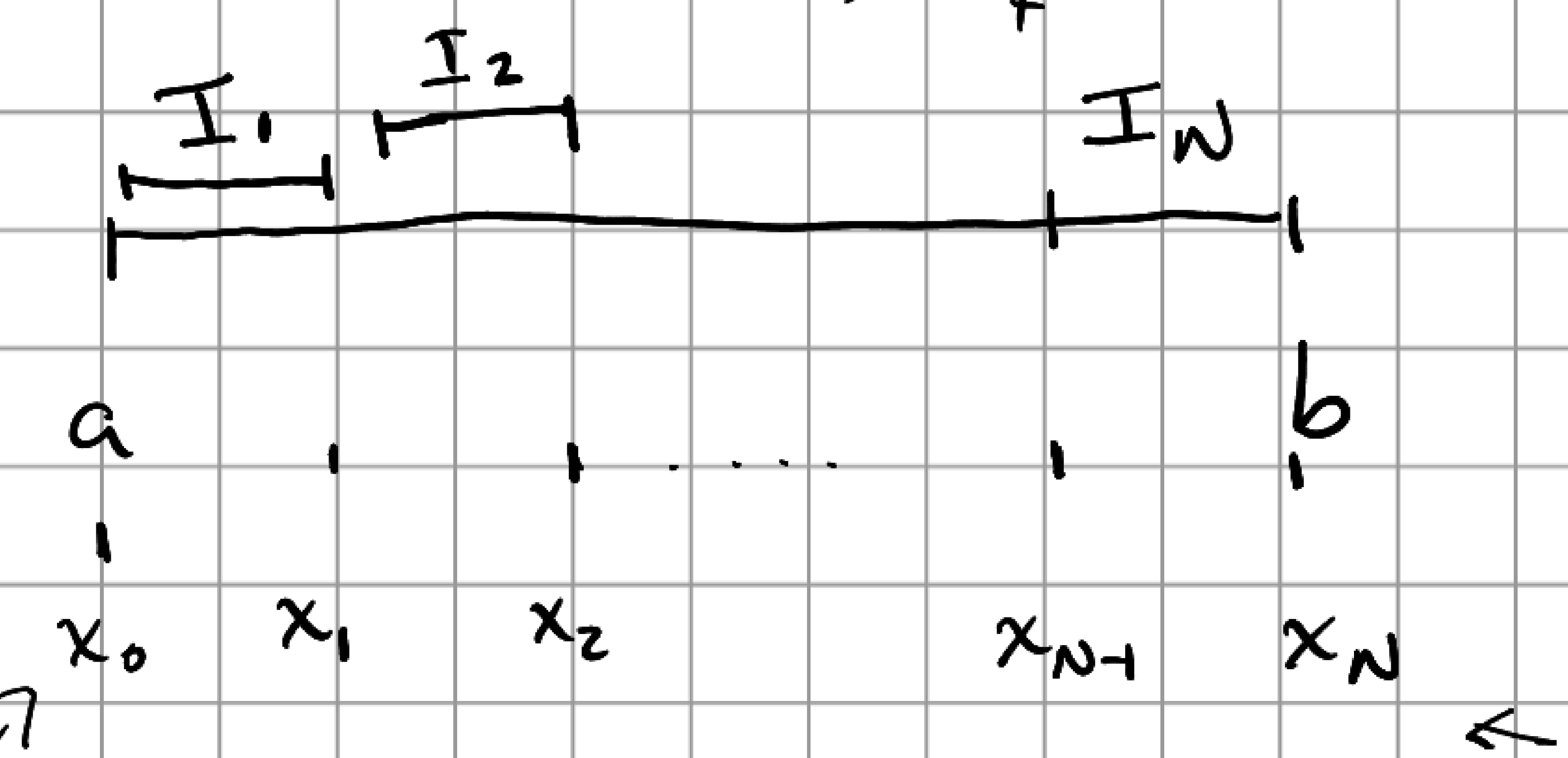
2) Método de diferencias finitas

Suponemos que $b(x) \equiv 0$, $k(x) = k > 0$.

Ecuación de calor: $\begin{cases} -k \dfrac{d^2u}{dx^2} = f(x) & x \in (a,b) \\ u(a) = \alpha \\ u(b) = \beta . \end{cases}$

$$\underline{\underline{(*)}}. \quad k \frac{d^2u}{dx^2} = -f(x)$$

Creamos un malla para $[a,b]$. Sea $\underline{\underline{N}}$ el # de sub-intervalos (uniformes) que divide a $[a,b]$



$x_i = a + ih$

$i = 0, \ldots, N$

$h = |I_i|$ $i = 1, \ldots, N$. Observar que tenemos $\underline{\underline{N+1}}$ puntos en la malla.

Como $x_0 = a \Rightarrow u(x_0) = \alpha$ y similarmente $u(x_N) = \beta$.

Observer que para $x_i \in (a,b)$

$$(*) \qquad k\, u''(x_i) = -f(x_i) \qquad \text{(Dif. finitos)}$$

3) Usando el teorema de Taylor con residuo para $u(x_{i+1})$ y $u(x_{i-1})$, respectivamente, alrededor del punto $u(x_i)$ obtenemos:

$$u(x_{i+1}) = u(x_i+h) = u(x_i) + h\cancel{u'(x_i)} + \frac{h^2}{2}u''(x_i) + \frac{h^3}{6}\cancel{u'''(x_i)} + ch^4$$

$$u(x_{i-1}) = u(x_i-h) = u(x_i) - h\cancel{u'(x_i)} + \frac{h^2}{2}u''(x_i) - \frac{h^3}{6}\cancel{u'''(x_i)} + ch^4$$

$$\Rightarrow \quad u(x_{i+1}) + u(x_{i-1}) = 2u(x_i) + h^2 u''(x_i) + ch^4$$

$$u''(x_i) = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} + ch^2$$

Formula de diferencia central de orden 2 para $u''(x_i)$

$$\Rightarrow \quad (*) \quad \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} = -\frac{1}{k} f(x_i) + ch^2$$

$$\hat{u}_i \sim u(x_i)$$

$$\Rightarrow \begin{cases} \hat{u}_{i+1} - 2\hat{u}_i + \hat{u}_{i-1} = -\dfrac{h^2}{k} f(x_i) & i = 1, \dots, N-1 \\ \hat{u}_0 = \alpha \;(= u(x_0) = u(a)) \\ \hat{u}_N = \beta \;(= u(x_N) = u(b)) \end{cases}$$

$$x_i \in (a,b)$$

$$(\hat{u}_1, \dots, \hat{u}_{N-1})$$

Este es un sistema lineal $A\underline{u} = f$

$$\begin{cases} i=1 \\ \hat{u}_2 - 2\hat{u}_1 + \hat{u}_0 = -\dfrac{h^2}{k} f(x_1) \\ \hat{u}_2 - 2\hat{u}_1 = -\dfrac{h^2}{k} f(x_1) - \alpha \\ i = N-1 \\ \hat{u}_N - 2\hat{u}_{N-1} + \hat{u}_{N-2} = -\dfrac{h^2}{k} f(\hat{u}_{N-1}) \\ -2\hat{u}_{N-1} + \hat{u}_{N-2} = -\dfrac{h^2}{k} f(\hat{u}_{N-1}) - \beta \\ i = 2, \dots, N-2 \\ \hat{u}_{i+1} - 2\hat{u}_i + \hat{u}_{i-1} = -\dfrac{h^2}{k} f(x_i). \end{cases}$$

4) En forma matricial

$$\hat{\tilde{\underline{u}}} = [\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_{N-1}]^t \in \mathbb{R}^{N-1}$$

$$\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \vdots \\ \hat{u}_{N-2} \\ \hat{u}_{N-1} \end{bmatrix} = \begin{bmatrix} -\frac{h^2}{k} f(x_1) - \alpha \\ -\frac{h^2}{k} f(x_2) \\ -\frac{h^2}{k} f(x_3) \\ \\ -\frac{h^2}{k} f(x_{N-2}) \\ -\frac{h^2}{k} f(x_{N-1}) - \beta \end{bmatrix}$$

$$\tilde{\underline{A}} \qquad \hat{\tilde{\underline{u}}} = \tilde{\underline{f}}$$

$$\tilde{\underline{A}} \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}, \quad \tilde{\underline{f}} \in \mathbb{R}^{N-1}$$

$$\hat{\tilde{\underline{u}}} = \tilde{A}^{-1} \cdot \tilde{\underline{f}}$$