

# **Networks and hyperbolic geometry**

Advanced Topic Post-Euclidean Geometry

Project Presentation

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Welcome.



Code available on GitHub at  
<https://github.com/danieldanhe/hyperbolic-networks>

Networks

Hyperbolic geometry

The connection between the two

Routing between nodes

# Networks

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# What is a network?

A system of interconnected elements (called **nodes** or **vertices**) linked by relationships (called **edges** or **connections**).

## Example

- Sites on the Internet
- Social connections ("six degrees of separation")
- Neural networks and genetic or protein interactions
- Road, railway, and air routes
- Food webs in ecosystems
- Citations among academic papers
- International alliances
- Finances among banks and institutions
- Co-purchases on e-commerce platforms
- Microservice dependencies in tech
- Transmission of infectious diseases
- Semantic networks
- Collaborations in arts, music, and film
- Blockchain and cryptocurrency transactions

## Properties of these networks

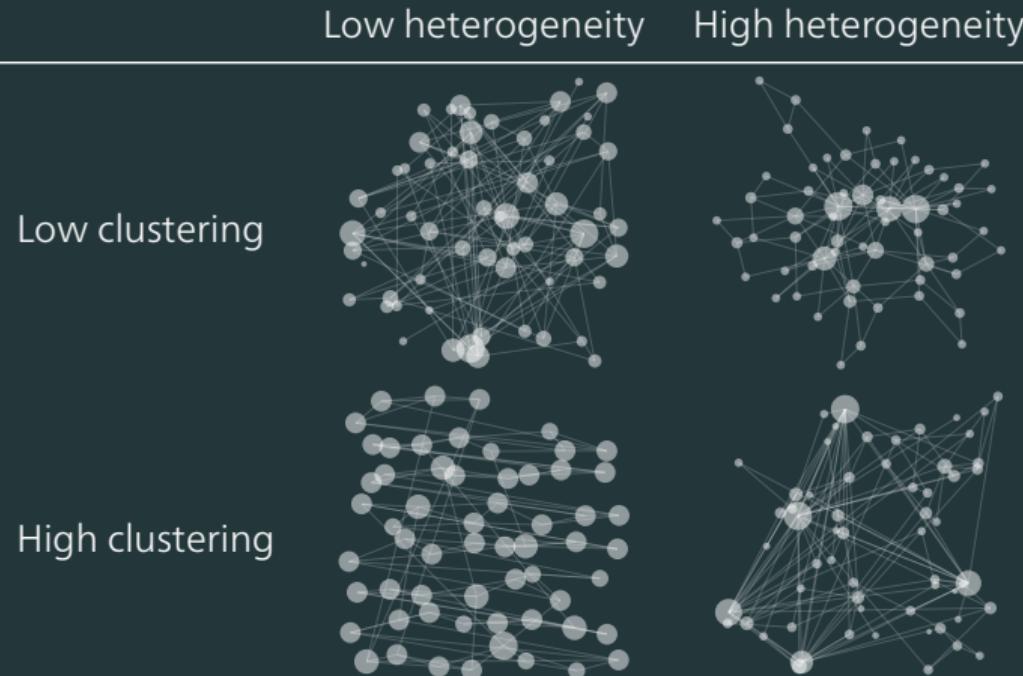
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What do all of these networks have in common?

They cannot be modelled by completely random networks, where connections are made by uniform random chance. There are two main differences:

- **Heterogeneity**. Some nodes have more connections than others. On social media, most people have a few hundred followers, but a few celebrities have millions.
- **Clustering**. Connections within groups tend to happen more than outside of them. If A and B follow each other, and B and C follow each other, A and C likely follow each other too.

## Heterogeneity and clustering



All networks shown here have approximately 60 nodes and 120 connections.

## Scale-free networks

We will look at **scale-free** networks specifically. The degree distribution follows a power law.

The fraction  $P(k)$  of nodes having  $k$  connections is approximately

$$P(k) \sim k^{-\gamma}$$

for large  $k$ .

Typically,  $2 < \gamma < 3$ . For the Internet,  $\gamma \approx 2.1$ .<sup>1</sup>

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<sup>1</sup>Krioukov et al. 2010.

## The solution: hyperbolic geometry

	Low heterogeneity	High heterogeneity
Low clustering	Random graphs <sup>2</sup>	Preferential attachment <sup>3</sup>
High clustering	Small-world <sup>4</sup>	Hyperbolic geometry

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<sup>2</sup>Erdős and Rényi 1959, Connects pairs completely at random.

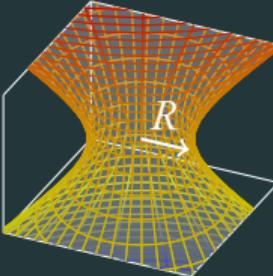
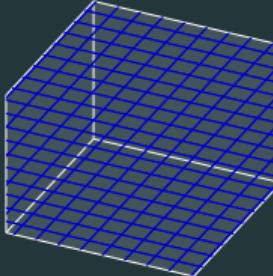
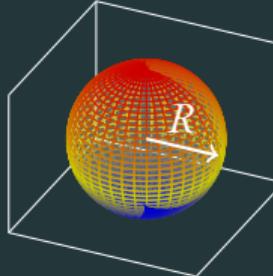
<sup>3</sup>Barabási and Albert 1999, New nodes prefer to connect to already well-connected hubs.

<sup>4</sup>Watts and Strogatz 1998, Starts with a regular, clustered lattice and rewrites a few connections to create long-range shortcuts.

# Hyperbolic geometry

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# The three geometries

	Hyperbolic	Euclidean	Spherical
Model			
<b>Curvature <math>K</math></b>	$= -1/R^2 < 0$	$= 0$	$= +1/R^2 > 0$
Parallel lines (geodesics)	$\infty$	1	0
Sum of angles in triangle	$< \pi$	$= \pi$	$> \pi$
Circumference of a circle	$> 2\pi r$	$= 2\pi r$	$< 2\pi r$

## Distance and area behaviour

Consider a circle of radius  $r$  on a plane with radius of curvature  $R$ .

	Circumference of a circle	Area of a circle
Hyperbolic	$C_H(r) = 2\pi R \sinh\left(\frac{r}{R}\right) \sim e^{r/R}$	$A_H(r) = 2\pi R^2 \left(\cosh\left(\frac{r}{R}\right) - 1\right) \sim e^{r/R}$
Euclidean	$C_E(r) = 2\pi r \sim r$	$A_E(r) = \pi r^2 \sim r^2$
Spherical	$C_S(r) = 2\pi R \sin\left(\frac{r}{R}\right) \rightarrow 0$	$A_S(r) = 2\pi R^2 \left(1 - \cos\left(\frac{r}{R}\right)\right) \rightarrow 4\pi R^2$

Hyperbolic geometry has “more space”.

# The Poincaré disc

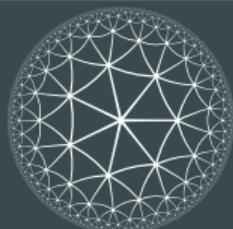
How do we represent an infinite, expanding space in a way we can visualize and compute with?

- The **Poincaré disc** is a mapping of a hyperbolic plane (assume onto a unit disc with  $K = -R = -1$ ); it preserves angles but not distances.
- The distance between two points is

$$d(\mathbf{u}, \mathbf{v}) = \text{arcosh}\left(1 + \frac{2\|\mathbf{u} - \mathbf{v}\|^2}{(1 - \|\mathbf{u}\|^2)(1 - \|\mathbf{v}\|^2)}\right)$$

- Geodesics are circular arcs perpendicular to the boundary.

## Example



{3, 7}



{4, 5}



{5, 4}

**The connection between the two**

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## Network generation in hyperbolic space

- There are  $N$  nodes. Each node  $i$  is assigned coordinates  $(r_i, \theta_i)$ .

- $r_i$  is drawn from the probability density function

$$\rho(r) = \frac{\alpha \sinh(\alpha r)}{\cosh(\alpha L) - 1} \sim e^{\alpha r}$$

where  $\alpha$  and  $L$  are chosen such that  $\bar{k} = \int_0^L \rho(r) \bar{k}(r) dr \approx (8/\pi) N e^{-L/2}$  and  $\gamma = 2\alpha + 1$ .

- $\theta_i$  is drawn uniformly from  $[0, 2\pi)$ .
- For any two nodes  $i, j$ , find the hyperbolic distance between them. For large  $L$ ,  $r_i, r_j$ , this is approximately

$$x_{i,j} = r_i + r_j + 2 \ln\left(\frac{\min(|\theta_i - \theta_j|, 2\pi - |\theta_i - \theta_j|)}{2}\right)$$

Connect with the connection probability  $p(x_{i,j})$ . The simplest choice is the step function; define  $p(x_{i,j}) = 1$  for  $x_{i,j} \leq L$  and  $p(x_{i,j}) = 0$  otherwise.

## Visualizing the model

$N = 20$

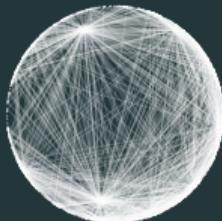
$N = 50$

$N = 100$

$N = 200$

$N = 500$

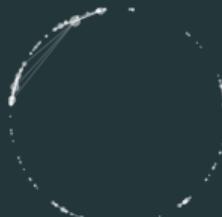
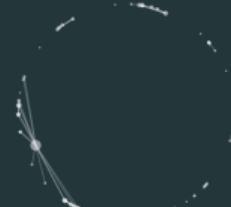
$\gamma = 2.1$



$\gamma = 2.5$



$\gamma = 3.0$



## The result and its converse

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The resulting network has heterogeneity and clustering.<sup>5</sup>

The converse is also true; a network with heterogeneity and clustering can also be modelled by this model. It is possible to use statistical techniques to infer  $(r, \theta)$  for each node using **network embedding**.<sup>6</sup>

Why does this matter? It is possible to use a **greedy algorithm** to route from one node to another. A greedy algorithm makes the locally optimal choice at each stage.

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<sup>5</sup>Krioukov et al. 2010.

<sup>6</sup>Krioukov et al. 2010.

## Routing between nodes

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## Greedy routing algorithm

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We will route from node  $m$  to  $n$ . Assume we know the coordinates of node  $m$ , its neighbours, and its destination  $n$ .

### Original Greedy Forwarding (OGF):

- If  $m = n$ , the problem is solved.
- If, out of all its neighbours,  $m$  itself is the closest to  $n$ , it is a local minimum and the algorithm fails.
- Otherwise, find the node  $k$  whose hyperbolic distance to  $n$  is least. Route from  $k$  to  $n$ .

**Modified Greedy Forwarding (MGF):** The algorithm will not exclude the current node. However, it will fail if the best neighbour is the previous node, thus causing an infinite loop.

With  $\gamma = 2.1$ , OGF and MGF have a success rate of 99.92 % and 99.99 %, respectively.<sup>7</sup>

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<sup>7</sup>Krioukov et al. 2010.

# References

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Thank you. Any questions?



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