

Networks and hyperbolic geometry

Advanced Topic Post-Euclidean Geometry
Project Presentation

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Welcome.



Code available on GitHub at
[`https://github.com/danieldanhe/hyperbolic-networks`](https://github.com/danieldanhe/hyperbolic-networks)

Networks

Hyperbolic geometry

The connection between the two

Routing between nodes

Networks

What is a network?

A system of interconnected elements (called **nodes** or **vertices**) linked by relationships (called **edges** or **connections**).

Example

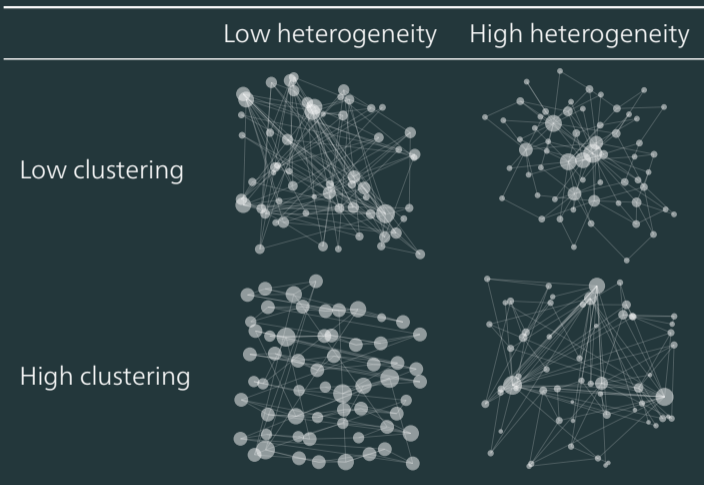
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|---|---|--|
| ● Sites on the Internet | ecosystems | dependencies in tech |
| ● Social connections ("six degrees of separation") | ● Citations among academic papers | ● Transmission of infectious diseases |
| ● Neural networks and genetic or protein interactions | ● International alliances | ● Semantic networks |
| ● Road, railway, and air routes | ● Finances among banks and institutions | ● Collaborations in arts, music, and film |
| ● Food webs in | ● Co-purchases on e-commerce platforms | ● Blockchain and cryptocurrency transactions |
| | ● Microservice | |

What do all of these networks have in common?

They cannot be modelled by completely random networks, where connections are made by uniform random chance. There are two main differences:

- **Heterogeneity**. Some nodes have more connections than others. On social media, most people have a few hundred followers, but a few celebrities have millions.
- **Clustering**. Connections within groups tend to happen more than outside of them. If A and B follow each other, and B and C follow each other, A and C likely follow each other too.

Heterogeneity and clustering



All networks shown here have approximately 60 nodes and 120 connections.

We will look at **scale-free** networks specifically. The degree distribution follows a power law.

The fraction $P(k)$ of nodes having k connections is approximately

$$P(k) \sim k^{-\gamma}$$

for large k .

Typically, $2 < \gamma < 3$. For the Internet, $\gamma \approx 2.1$.¹

¹Krioukov et al. 2010.

The solution: hyperbolic geometry

	Low heterogeneity	High heterogeneity
Low clustering	Random graphs ²	Preferential attachment ³
High clustering	Small-world ⁴	Hyperbolic geometry

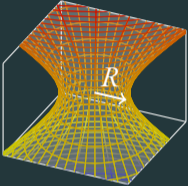
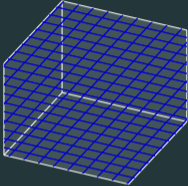
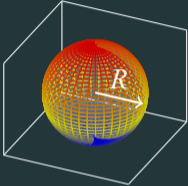
²Erdős and Rényi 1959, Connects pairs completely at random.

³Barabási and Albert 1999, New nodes prefer to connect to already well-connected hubs.

⁴Watts and Strogatz 1998, Starts with a regular, clustered lattice and rewires a few connections to create long-range shortcuts.

Hyperbolic geometry

The three geometries

	Hyperbolic	Euclidean	Spherical
Model			
Curvature K	$= -1/R^2 < 0$	$= 0$	$= +1/R^2 > 0$
Parallel lines (geodesics)	∞	1	0
Sum of angles in triangle	$< \pi$	$= \pi$	$> \pi$
Circumference of a circle	$> 2\pi r$	$= 2\pi r$	$< 2\pi r$

Distance and area behaviour

Consider a circle of radius r on a plane with radius of curvature R .

	Circumference of a circle	Area of a circle
Hyperbolic	$C_H(r) = 2\pi R \sinh\left(\frac{r}{R}\right) \sim e^{r/R}$	$A_H(r) = 2\pi R^2 \left(\cosh\left(\frac{r}{R}\right) - 1 \right) \sim e^{r/R}$
Euclidean	$C_E(r) = 2\pi r \sim r$	$A_E(r) = \pi r^2 \sim r^2$
Spherical	$C_S(r) = 2\pi R \sin\left(\frac{r}{R}\right) \rightarrow 0$	$A_S(r) = 2\pi R^2 \left(1 - \cos\left(\frac{r}{R}\right) \right) \rightarrow 4\pi R^2$

Hyperbolic geometry has “more space”.

The Poincaré disc

How do we represent an infinite, expanding space in a way we can visualize and compute with?

- The **Poincaré disc** is a mapping of a hyperbolic plane (assume onto a unit disc with $K = -R = -1$); it preserves angles but not distances.
- The distance between two points is

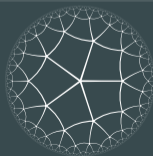
$$d(u, v) = \operatorname{arcosh}\left(1 + \frac{2\|u - v\|^2}{(1 - \|u\|^2)(1 - \|v\|^2)}\right)$$

- Geodesics are circular arcs perpendicular to the boundary.

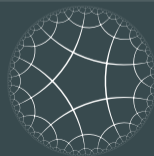
Example



$\{3, 7\}$



$\{4, 5\}$



$\{5, 4\}$

The connection between the two

Network generation in hyperbolic space

- There are N nodes. Each node i is assigned coordinates (r_i, θ_i) .
 - r_i is drawn from the probability density function

$$\rho(r) = \frac{\alpha \sinh(\alpha r)}{\cosh(\alpha L) - 1} \sim e^{\alpha r}$$

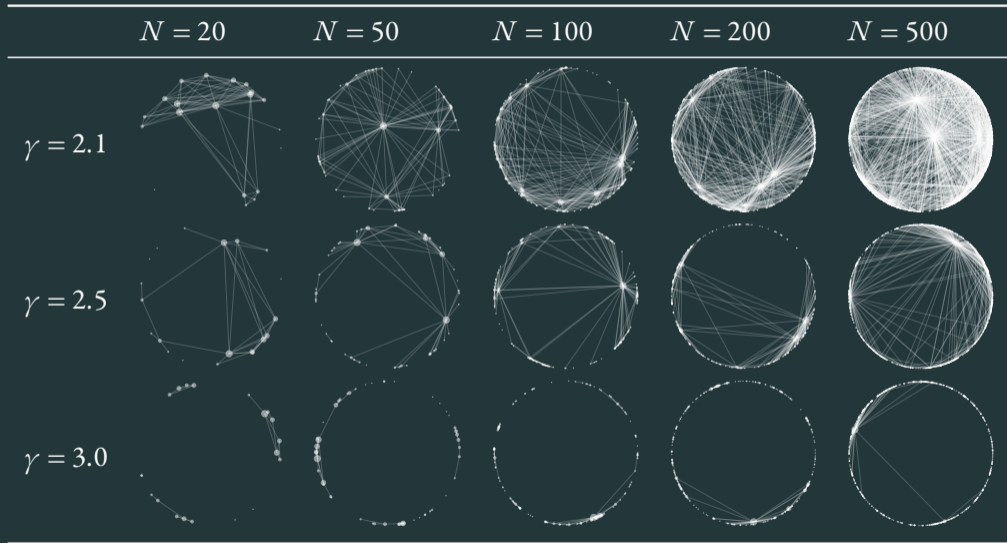
where α and L are chosen such that $\bar{k} = \int_0^L \rho(r) \bar{k}(r) dr \approx (8/\pi) N e^{-L/2}$ and $\gamma = 2\alpha + 1$.

- θ_i is drawn uniformly from $[0, 2\pi)$.
- For any two nodes i, j , find the hyperbolic distance between them. For large L , r_i, r_j , this is approximately

$$x_{i,j} = r_i + r_j + 2 \ln \left(\frac{\min(|\theta_i - \theta_j|, 2\pi - |\theta_i - \theta_j|)}{2} \right)$$

Connect with the connection probability $p(x_{i,j})$. The simplest choice is the step function; define $p(x_{i,j}) = 1$ for $x_{i,j} \leq L$ and $p(x_{i,j}) = 0$ otherwise.

Visualizing the model



The resulting network has heterogeneity and clustering.⁵

The converse is also true; a network with heterogeneity and clustering can also be modelled by this model. It is possible to use statistical techniques to infer (r, θ) for each node using **network embedding**.⁶

Why does this matter? It is possible to use a **greedy algorithm** to route from one node to another. A greedy algorithm makes the locally optimal choice at each stage.

⁵Krioukov et al. 2010.

⁶Krioukov et al. 2010.

Routing between nodes

Greedy routing algorithm

We will route from node m to n . Assume we know the coordinates of node m , its neighbours, and its destination n .

Original Greedy Forwarding (OGF):

- If $m = n$, the problem is solved.
- If, out of all its neighbours, m itself is the closest to n , it is a local minimum and the algorithm fails.
- Otherwise, find the node k whose hyperbolic distance to n is least. Route from k to n .

Modified Greedy Forwarding (MGF): The algorithm will not exclude the current node. However, it will fail if the best neighbour is the previous node, thus causing an infinite loop.

With $\gamma = 2.1$, OGF and MGF have a success rate of 99.92 % and 99.99 %, respectively.⁷

⁷Krioukov et al. 2010.

References



A.-L. Barabási and R. Albert (15 October 1999).

"Emergence of scaling in random networks".

In: *Science* 286(5439), pp. 509–512. ISSN: 0036-8075. DOI: 10.1126/science.286.5439.509.



P. Erdős and A. Rényi (1959).

"On random graphs I".

In: *Publicationes Mathematicae Debrecen* 6, pp. 290–297. ISSN: 0033-3883. DOI: 10.5486/PMD.1959.6.3–4.12.



D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat, and M. Boguñá (9 September 2010).

"Hyperbolic geometry of complex networks".

In: *Physical Review E* 82(3), 036106. ISSN: 1550-2376. DOI: 10.1103/PhysRevE.82.036106.



D. J. Watts and S. H. Strogatz (4 June 1998).

"Collective dynamics of 'small-world' networks".

In: *Nature* 393(6684), pp. 440–442. ISSN: 0028-0836. DOI: 10.1038/30918.

Thank you. Any questions?



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