Networks and hyperbolic geometry

Advanced Topic Post-Euclidean Geometry
Project Presentation

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Thank you. Any questions?

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Networks

What is a network?

A system of interconnected elements (called **nodes** or **vertices**) linked by relationships (called **edges** or **connections**).

Example

- Sites on the Internet
- Social connections ("six degrees of separation")
- Neural networks and genetic or protein interactions
- Road, railway, and air routes
- Food webs in ecosystems
- Citations among academic papers
- International alliances
- Finances among banks and institutions
- Co-purchases on e-commerce platforms
- Microservice dependencies in tech
- Transmission of infectious diseases
- Semantic networks
- Collaborations in arts, music, and film
- Blockchain and cryptocurrency transactions

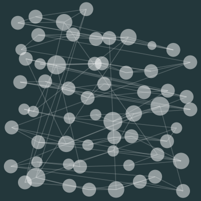
Properties of these networks

What do all of these networks have in common?

They cannot be modelled by completely random networks, where connections are made by uniform random chance. There are two main differences:

- **Heterogeneity**. Some nodes have more connections than others. On social media, most people have a few hundred followers, but a few celebrities have millions.
- **Clustering**. Connections within groups tend to happen more than outside of them. If A and B follow each other, and B and C follow each other, A and C likely follow each other too.

Heterogeneity and clustering

	Low heterogeneity	High heterogeneity
Low clustering		
High clustering		

All networks shown here have approximately 60 nodes and 120 connections.

Scale-free networks

We will look at **scale-free** networks specifically. The degree distribution follows a power law.

The fraction $P(k)$ of nodes having k connections is approximately

$$P(k) \sim k^{-\gamma}$$

for large k.

Typically, $2 < \gamma < 3$. For the Internet, $\gamma \approx 2.1$.[1]

[1] Krioukov et al. 2010.

The solution: hyperbolic geometry

	Low heterogeneity	High heterogeneity
Low clustering	Random graphs[2]	Preferential attachment[3]
High clustering	Small-world[4]	Hyperbolic geometry

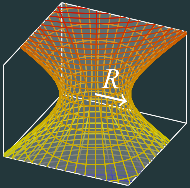
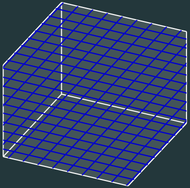
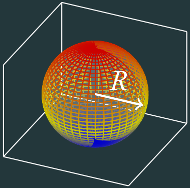
[2] Erdős and Rényi 1959, Connects pairs completely at random.

[3] Barabási and Albert 1999, New nodes prefer to connect to already well-connected hubs.

[4] Watts and Strogatz 1998, Starts with a regular, clustered lattice and rewires a few connections to create long-range shortcuts.

Hyperbolic geometry

The three geometries

	Hyperbolic	Euclidean	Spherical
Model			
Curvature K	$= -1/R^2 < 0$	$= 0$	$= +1/R^2 > 0$
Parallel lines (**geodesics**)	∞	1	0
Sum of angles in triangle	$< \pi$	$= \pi$	$> \pi$
Circumference of a circle	$> 2\pi r$	$= 2\pi r$	$< 2\pi r$

Distance and area behaviour

Consider a circle of radius r on a plane with radius of curvature R.

	Circumference of a circle	Area of a circle
Hyperbolic	$C_{\mathrm{H}}(r) = 2\pi R \sinh\left(\frac{r}{R}\right) \sim e^{r/R}$	$A_{\mathrm{H}}(r) = 2\pi R^2 \left(\cosh\left(\frac{r}{R}\right) - 1\right) \sim e^{r/R}$
Euclidean	$C_{\mathrm{E}}(r) = 2\pi r \sim r$	$A_{\mathrm{E}}(r) = \pi r^2 \sim r^2$
Spherical	$C_{\mathrm{S}}(r) = 2\pi R \sin\left(\frac{r}{R}\right) \to 0$	$A_{\mathrm{S}}(r) = 2\pi R^2 \left(1 - \cos\left(\frac{r}{R}\right)\right) \to 4\pi R^2$

Hyperbolic geometry has “more space”.

The Poincaré disc

How do we represent an infinite, expanding space in a way we can visualize and compute with?

- The **Poincaré disc** is a mapping of a hyperbolic plane (assume onto a unit disc with $K = -R = -1$); it preserves angles but not distances.
- The distance between two points is

$$d(u, v) = \operatorname{arcosh}\left(1 + \frac{2\|u - v\|^2}{(1 - \|u\|^2)(1 - \|v\|^2)}\right)$$

- Geodesics are circular arcs perpendicular to the boundary.

Example

$\{3, 7\}$ tilings/tiling-3-7.pdf $\{4, 5\}$ tilings/tiling-4-5.pdf $\{5, 4\}$ tilings/tiling-5-4.pdf

The connection between the two

Network generation in hyperbolic space

- There are N nodes. Each node i is assigned coordinates (r_i, θ_i).
  - r_i is drawn from the probability density function

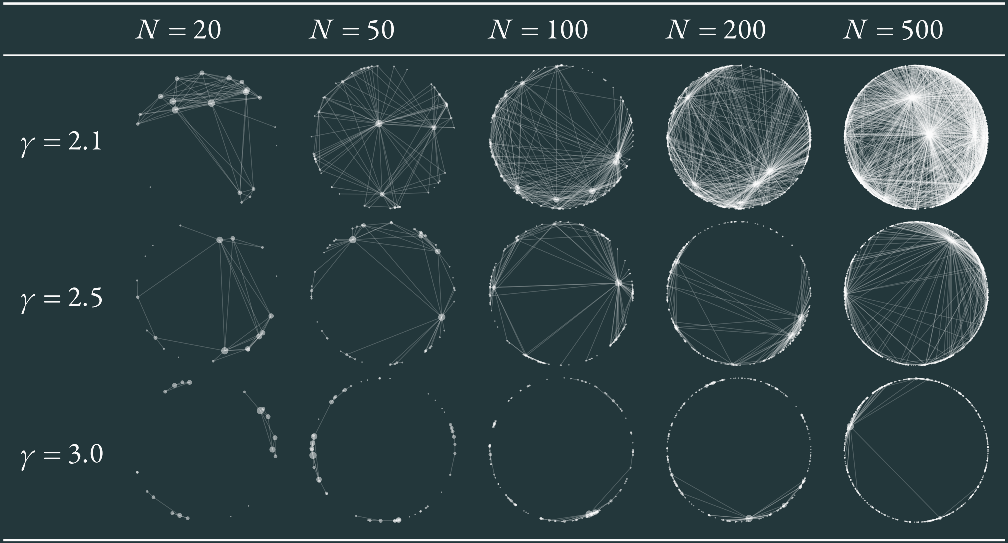
$$\rho(r) = \frac{\alpha \sinh(\alpha r)}{\cosh(\alpha L) - 1} \sim e^{\alpha r}$$

    where α and L are chosen such that $\bar{k} = \int_0^L \rho(r)\bar{k}(r)\,\mathrm{d}r \approx (8/\pi)Ne^{-L/2}$ and $\gamma = 2\alpha + 1$.
  - θ_i is drawn uniformly from $[0, 2\pi)$.
- For any two nodes i, j, find the hyperbolic distance between them. For large L, r_i, r_j, this is approximately

$$x_{i,j} = r_i + r_j + 2\ln\left(\frac{\min(|\theta_i - \theta_j|, 2\pi - |\theta_i - \theta_j|)}{2}\right)$$

  Connect with the connection probability $p(x_{i,j})$. The simplest choice is the step function; define $p(x_{i,j}) = 1$ for $x_{i,j} \leq L$ and $p(x_{i,j}) = 0$ otherwise.

Visualizing the model



The result and its converse

The resulting network has heterogeneity and clustering.[5]

The converse is also true; a network with heterogeneity and clustering can also be modelled by this model. It is possible to use statistical techniques to infer (r, θ) for each node using **network embedding**.[6]

Why does this matter? It is possible to use a **greedy algorithm** to route from one node to another. A greedy algorithm makes the locally optimal choice at each stage.

[5] Krioukov et al. 2010.
[6] Krioukov et al. 2010.

Routing between nodes

Greedy routing algorithm

We will route from node m to n. Assume we know the coordinates of node m, its neighbours, and its destination n.

Original Greedy Forwarding (OGF):

- If $m = n$, the problem is solved.
- If, out of all its neighbours, m itself is the closest to n, it is a local minimum and the algorithm fails.
- Otherwise, find the node k whose hyperbolic distance to n is least. Route from k to n.

Modified Greedy Forwarding (MGF): The algorithm will not exclude the current node. However, it will fail if the best neighbour is the previous node, thus causing an infinite loop.

With $\gamma = 2.1$, OGF and MGF have a success rate of $99.92\,\%$ and $99.99\,\%$, respectively.[7]

[7] Krioukov et al. 2010.

References

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