## Solutions in $\mathbb{Z}[i]$ of $A^5 + B^5 = C^5 \pm 1$

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**Introduction** In [1], it is shown how we can derive infinitely many integral solutions of  $A^3 + B^3 = C^3 \pm 1$  starting from Ramanujan's identity:

$$(a^2 + 7ab - 9b^2)^3 + (2a^2 - 4ab + 12b^2)^3 = (2a^2 + 10b^2)^3 + (a^2 - 9ab - b^2)^3$$

We will do something similar but for the equation  $A^5 + B^5 = C^5 \pm 1$ ; the price to pay will be that the proposed solutions are Gaussian integers and not natural integers. With  $i^2 = -1$ , we consider:

$$g(x) := (x^2 + 2ax - 2a^2)^5 + (ix^2 - 2ax + 2ia^2)^5$$

One gets directly g(x) = g(-x) so g is even and this means:

$$(x^{2} + 2ax - 2a^{2})^{5} + (ix^{2} - 2ax + 2ia^{2})^{5} = (x^{2} - 2ax - 2a^{2})^{5} + (ix^{2} + 2ax + 2ia^{2})^{5}$$
(1)

With  $(a, x) \in \mathbb{Z}^2$ , one gets Gaussian integers solutions of  $A^5 + B^5 = C^5 + D^5$  which is a good starting point.

Using classical linear recurrence relations Let's consider the following linear recurrence,  $\forall n \in \mathbb{N}$ :

$$F_{n+2} = -2F_{n+1} + 2F_n$$
  $F_0 := 0, F_1 := 1$  (2)

This can be solved using classical linear algebra:

$$F_n = \frac{\sqrt{3}}{6} \left( \left( -1 + \sqrt{3} \right)^n - \left( -1 - \sqrt{3} \right)^n \right) \qquad \forall n \in \mathbb{N}$$
 (3)

From this, we compute by direct calculation:

$$F_{n+1}^2 = \frac{2^n}{6} \left( \left( 2 + \sqrt{3} \right)^{n+1} + \left( 2 - \sqrt{3} \right)^{n+1} + 2(-1)^n \right)$$

$$F_{n+1}F_n = \frac{1}{12} \left( \left( -1 + \sqrt{3} \right)^{2n+1} + \left( -1 - \sqrt{3} \right)^{2n+1} - (-2)^{n+1} \right)$$

$$F_{n+2}F_n = \frac{2^n}{6} \left( \left( 2 + \sqrt{3} \right)^{n+1} + \left( 2 - \sqrt{3} \right)^{n+1} + 4(-1)^{n+1} \right)$$

From this, we get, in particular,

$$F_{n+1}^2 - F_n F_{n+2} = 2^n (-1)^n (4)$$

Let's take  $F_{n+1} := x$  and  $F_n := a$  and replace this in equation (1), one gets:

$$A_{n} := F_{n+1}^{2} - 2F_{n+1}F_{n} - 2F_{n}^{2}$$

$$= \frac{2^{n}}{3} \left( (1 + \sqrt{3})(2 + \sqrt{3})^{n} + (1 - \sqrt{3})(2 - \sqrt{3})^{n} + (-1)^{n} \right)$$

$$B_{n} := iF_{n+1}^{2} + 2F_{n+1}F_{n} + 2iF_{n}^{2}$$

$$= \frac{2^{n}}{6} \left( (-1 + \sqrt{3})(2 - \sqrt{3})^{n} - (1 + \sqrt{3})(2 + \sqrt{3})^{n} + 2(-1)^{n} \right) + i\frac{2^{n}}{6} \left( (3 - \sqrt{3})(2 - \sqrt{3})^{n} + (3 + \sqrt{3})(2 + \sqrt{3})^{n} \right)$$
(6)

$$C_n := iF_{n+1}^2 - 2F_{n+1}F_n + 2iF_n^2$$

$$= \frac{2^n}{6} \left( (1 + \sqrt{3})(2 + \sqrt{3})^n + (1 - \sqrt{3})(2 - \sqrt{3})^n - 2(-1)^n \right) + i\frac{2^n}{6} \left( (3 - \sqrt{3})(2 - \sqrt{3})^n + (3 + \sqrt{3})(2 + \sqrt{3})^n \right)$$
(7)

Thus, equation (1) becomes  $A_n^5 + B_n^5 = C_n^5 + d_n^5$ . Now,

$$d_n = F_{n+1}^2 + 2F_{n+1}F_n - 2F_n^2 = F_{n+1}^2 - F_n \left(-2F_{n+1} + 2F_n\right)$$

$$= F_{n+1}^2 - F_n F_{n+2} \quad \text{using equation (2)}$$

$$= 2^n (-1)^n \quad \text{using equation (4)}$$

With  $z \in \mathbb{C}$ , |z| < 1, it's easy enough to calculate the following, from equations (5), (6) and (7):

$$\sum_{n\geq 0} A_n z^n = \frac{4z^2 + 1}{(2z+1)(4z^2 - 8z + 1)}$$

$$\sum_{n\geq 0} B_n z^n = \frac{-4z}{(2z+1)(4z^2 - 8z + 1)} + i\left(\frac{1-2z}{4z^2 - 8z + 1}\right)$$

$$\sum_{n\geq 0} C_n z^n = \frac{4z}{(2z+1)(4z^2 - 8z + 1)} + i\left(\frac{1-2z}{4z^2 - 8z + 1}\right)$$

And this satisfies  $A_n^5 + B_n^5 = C_n^5 + 2^{5n}(-1)^n \implies \left(\frac{A_n}{2^n}\right)^5 + \left(\frac{B_n}{2^n}\right)^5 = \left(\frac{C_n}{2^n}\right)^5 + (-1)^n$ . Let's take x := 2z and  $\forall n \in \mathbb{N}$ ,  $a_n := \frac{A_n}{2^n}$ ,  $b_n := \frac{B_n}{2^n}$  and  $c_n := \frac{C_n}{2^n}$ . It follows that

$$\sum_{n\geq 0} a_n x^n = \frac{x^2 + 1}{(x+1)(x^2 - 4x + 1)}$$

$$\sum_{n\geq 0} b_n x^n = \frac{-2x}{(x+1)(x^2 - 4x + 1)} + i\left(\frac{1-x}{x^2 - 4x + 1}\right)$$

$$\sum_{n\geq 0} c_n x^n = \frac{2x}{(x+1)(x^2 - 4x + 1)} + i\left(\frac{1-x}{x^2 - 4x + 1}\right)$$

$$\implies a_n^5 + b_n^5 = c_n^5 + (-1)^n$$

For example,

$$n := 1 \implies 3^5 + (-2 + 3i)^5 = (2 + 3i)^5 - 1$$
  

$$n := 2 \implies 13^5 + (-6 + 11i)^5 = (6 + 11i)^5 + 1$$
  

$$n := 3 \implies 47^5 + (-24 + 41i)^5 = (24 + 41i)^5 - 1$$

## References

[1] Michael D. Hirschhorn, An Amazing Identity of Ramanujan, *Mathematics Magazine*, Vol. 68, No 3 (June 1995), pp.199-201.