MAT437: Connecting C*-Algebras with Model Theory

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1 Acknowledgements

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2 Introduction

In the study of C*-algebras, it is an often practice to restrict one's attention to certain definable sets that exhibit useful properties. For example, in [1] the reader is immediately introduced to the set of projections on unital C*-algebras, which are then used to define the K_0 group of a unital C*-algebra. Our ability to consider definable sets on a C*-algebra is limited by the underlying properties of the C*-algebra (as is discussed extensively in [2]). It then becomes useful to distinguish different classes of C*-algebras by axiomiatizing them. Axiomatization entails the use of formal logic and model theory; in particular, it makes use of continuous model theory. In this essay we will briefly recall notions from "classical" model theory followed by an introduction to notions of continuous model theory. Along the way we will compare the two, noting both similarities and differences. Following this, we will discuss axiomatization with some examples on C*-algebras. Finally, we will discuss an interesting problem presented by Professor Hannes Thiel in a talk he gave at the Fields Institute this September (in [3]). This problem will focus on abstract Cuntz semigroups that are fundamentally connected to C*-algebras.

3 Model Theory

3.1 Classical model theory

We begin with a review of classical model theory. We recall the following definitions from [4]:

Definition 3.1.1: A (first-order) **language** $\mathcal{L}$ is a collection of constant symbols, function symbols, relation symbols, variables, quantifiers, parantheses and logical connectives.

Definition 3.1.2: A **term** of a language $\mathcal{L}$ is a non-empty finite string of symbols of $\mathcal{L}$ that is either a variable, a constant symbol, or of the form $ft_1, ..., t_n$ for some n-ary function symbol in $\mathcal{L}$ where each t_i is a term in $\mathcal{L}$.

Definition 3.1.3: A **formula** of a language $\mathcal{L}$ is a non-empty finite string of symbols of $\mathcal{L}$ that is either of the form $t_i = t_j$ where t_i, t_j are terms in $\mathcal{L}$ or of the form $Rt_1, .., t_n$ where R is an n-ary relation symbol in $\mathcal{L}$ and $t_1, ..., t_n$ are terms in $\mathcal{L}$ or of the form $\neg\alpha$ or $\alpha \vee \beta$ or

$\forall v\alpha$ where α, β are formulas in $\mathcal{L}$ and v is a variable in $\mathcal{L}$.

Definition 3.1.4: A **sentence** in a language $\mathcal{L}$ is a formula in $\mathcal{L}$ with no free variables.

Now that we have defined formulas and sentences, we have the basic building blocks of mathematical statements. However, as defined above, formulas and sentences are just strings of symbols. Without a sense of interpretation, they are not very useful to us. This makes necessary the following definition as discussed in [5]:

Definition 3.1.5: A **model** for a language $\mathcal{L}$ is a pair $\mathfrak{A} := (A, \mathcal{I})$ where A is the "universe" of $\mathfrak{A}$ and $\mathcal{I}$ is an interpretation function that maps constant, function and relation symbols in $\mathcal{L}$ to corresponding elements in A. If φ is a formula in $\mathcal{L}$ we write $\mathfrak{A} \models \varphi$ if $\mathfrak{A} \models \varphi[s]$ for any assignment s of variables in $\mathcal{L}$ to A (this is discussed more thoroughly in [4]). We will use the term $\mathcal{L}$-structure interchangeably.

Definition 3.1.6: A **theory** of a model $\mathfrak{A}$ for a language $\mathcal{L}$ denoted $Th(\mathfrak{A})$ is defined by $Th(\mathfrak{A}) := \{\varphi : \varphi \text{ is a formula in } \mathcal{L} \text{ such that } \mathfrak{A} \models \varphi\}$.

Definition 3.1.7: Let $\mathfrak{B} \subseteq \mathfrak{A}$. We say that $\mathfrak{B}$ is an **elementary submodel** of $\mathfrak{A}$ if for any formula φ and any $a_1, ..., a_n \in B$, $\mathfrak{A} \models \varphi[a_1, ..., a_n] \iff \mathfrak{B} \models \varphi[a_1, ..., a_n]$.

Now that we have recalled the basic definitions of languages and models we can begin to recall some classical non-trivial results in model theory. We will discuss analogues of these theorems in continuous model theory in the next section.

Theorem 3.1.8 (Löwenheim–Skolem): An infinite model of a countable language has a countable elementary submodel.

For a proof of this theorem, see [4].

Theorem 3.1.9 (Compactness Theorem): If $\sum$ is a set of sentences then $\sum$ has a model if and only if any finite subset of $\sum$ has a model.

Note that this theorem is a corollary of the Completeness and Soundness theorems, which together tell us that if $\sum$ is a set of formulas in $\mathcal{L}$ and φ is a formula in $\mathcal{L}$ then $\sum \models \varphi$ if and only if there is a deduction of φ from $\sum$. See [4] for a full proof.

Definition 3.1.10: Let S be a non-empty set and let $U \subseteq \mathcal{P}(S)$ be such that $\emptyset \notin U$, U is closed under finite intersections and upward inclusion. We say that U is a **filter** on S. If also for any $A \subseteq S$ either $A \in U$ or $A^c \in U$, we say that U is an **ultrafilter** on S.

Ultrafilters are particularly useful because they provide a binary selection between any set and its complement. We use them for the following construction:

Definition 3.1.11: Let S be a non-empty set with an ultrafilter U. Let $\{\mathfrak{A}_x : x \in S\}$ be a collection of models on a language $\mathcal{L}$. Define an equivalence relation $\sim_U$ on $\prod_{x \in S} A_x$ by $f \sim_U g \iff \{x \in S : f(x) = g(x)\} \in U$. Then we obtain a model $\mathfrak{A}$ with universe $A := \prod_{x \in S} A_x / \sim_U$ which we call the **ultraproduct** of $\{\mathfrak{A}_x : x \in S\}$ by U.

This construction satisfies key properties given by the following result:

Theorem 3.1.12 (Łoś' theorem): With the notation used in **Definition 3.1.11**, for any formula φ and $f_1, ..., fn \in \prod_{x \in S} A_x, \mathfrak{A} \models \varphi([f_1], ..., [f_n])$ if and only if $\{x \in S : \mathfrak{A}_x \models \varphi[f_1(x), ..., f_n(x)]\} \in U$, and consequently if ϕ is a sentence then $\mathfrak{A} \models \phi$ if and only if $\{x \in S : \mathfrak{A}_x \models \phi\} \in U$.

This theorem is proven by induction on the complexity of φ; the full proof can be seen in [5]. Note that if we let U be a filter on S (but not an ultrafilter) we can mimic the construction outlined in **Definition 3.1.11** and the resulting model $\mathfrak{A}$ is called a **reduced product**. However, the reduced product does not satisfy the properties outlined in **Theorem 3.1.12**. This is because the binary selection given by the ultrafilter property is vital for the negation case in the induction on φ.

We conclude our review of classical model theory and turn our attention towards continuous model theory.

3.2 Continuous model theory

We will present an abridged version of the subject as seen in [2] with many results and proofs omitted for the sake of brevity. We do so with the goal of developing basic intuition for the subject and its connections to C*-algebras. A complete introduction to the subject can be seen in [2].

Definition 3.2.1: A **metric structure** is a triple $(\mathcal{S}, \mathcal{F}, \mathcal{R})$ satisfying the following:

(i) The collection $\mathcal{S}$, called the **sorts**, is a family of bounded, complete metric spaces.

(ii) The collection $\mathcal{F}$, called the **functions**, is a family of uniformly continuous functions satisfying the condition that the domain of each function is a finite product of sorts and the range of each function is a sort.

(iii) The collection $\mathcal{R}$, called the **relations**, is a family of uniformly continuous functions satisfying the condition that the domain of each relation is a finite product of sorts and the range of each relation is a bounded interval in $\mathbb{R}$.

If we take a C*-algebra A and we consider the collection of balls centered at 0 of radius n with functions given by the restriction of addition, multiplication and scalar multiplication on A, we obtain a metric structure. Now, the construction of a metric structure using sorts, functions and relations is analogous to the construction of a language using constants, functions and relations (as given in **Definition 3.1.1**). Thus, if we abstract and formalize the notion of a metric structure using notions of continuous logic, we can begin to draw connections between C*-algebras and model theory.

Definition 3.2.2: A **language** $\mathcal{L}$ is a triple $(\mathfrak{S}, \mathfrak{F}, \mathfrak{R})$ satisfying the following:

(i) The collection $\mathfrak{S}$, called the **sorts**, is a family of pairs (d_S, M_s) which are symbols intended to be interpreted as a metric with a bound.

(ii) The collection $\mathfrak{F}$, called the **functions**, is a family of function symbol f, intended to be interpreted as a uniformly continuous function, with domain being a finite sequence in $\mathfrak{S}$ and range an element in $\mathfrak{S}$. Additionally, if $S_1, ..., S_n \in \mathfrak{S}$ corresponds to the domain of f, for each i we introduce corresponding function symbols δ_i called **uniform continuity moduli** which help capture the notion of intended uniform continuity.

(iii) The collection $\mathfrak{R}$, called the **relations**, is a family of relation symbols with domain being a finite sequence in $\mathfrak{S}$ and range a compact interval in $\mathbb{R}$. We also associate uniform continuity moduli with relations.

Definition 3.2.3: A **term** of a language $\mathcal{L}$ is either a variable with domain and range being a sort or of the form $f\mathcal{T}_1, ..., \mathcal{T}_n$ for some n-ary $f \in \mathfrak{F}$ where each $\mathcal{T}_i$ is a term in $\mathcal{L}$.

Definition 3.2.4: A **formula** of a language $\mathcal{L}$ is a non-empty finite string of symbols of $\mathcal{L}$ that is either of the form $R\mathcal{T}_1, ..., \mathcal{T}_n$ where $\mathcal{T}_1, ..., \mathcal{T}_n$ are terms in $\mathcal{L}$ or of the form $f\varphi_1, ..., \varphi_n$ where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $\varphi_1, ..., \varphi_n$ are formulas or of the form $inf_{x\in S}\varphi, sup_{x\in S}\varphi$ where x is a variable on $S \in \mathfrak{S}$ and φ is a formula.

Note that sentences in continuous logic are defined precisely as in **Definition 3.1.4**. We will let $Sent_{\mathcal{L}}$ denote the set of sentences in $\mathcal{L}$.

It is clear how terms in continuous logic are defined analogously to those in classical logic by direct inspection of **Definition 3.1.2** and **Definition 3.2.4**. The correspondence of the definitions of formula is a bit more subtle. In the case of **Definition 3.2.4**, taking a formula over the form of $f\mathcal{T}_1, ..., \mathcal{T}_n$ is analogous to building a formula with logical connectives, and taking the supremum or infimum over a sort of a formula is analogous to quantification.

Example 3.2.5: Let A be a C*-algebra and let $S := (d, 1) \in \mathfrak{S}$ correspond to the unit ball in A. Then note that x is a variable on S and is therefore a term in $\mathcal{L}$. Similarly, x^* and x^2 are functions on S and are therefore terms in $\mathcal{L}$. Then $\varphi(x) := max(d(x^2, x), (x, x^*))$ is an $\mathcal{L}$-formula in the free variable x.

In classical logic, we define the satisfaction relation and develop the definition of a theory using this relation. However in continuous logic we do the converse. We define the notion of a theory as follows:

Definition 3.2.6: Let $\mathfrak{A}$ be a model of $\mathcal{L}$. We let $\varphi^{\mathfrak{A}}$ denote the interpretation of φ in $\mathfrak{A}$. We define the **theory** of $\mathfrak{A}$ to be a functional $Th(\mathfrak{A}) : Sent_{\mathcal{L}} \to \mathbb{R}$ defined by $Th(\mathfrak{A})(\varphi) := \varphi^{\mathfrak{A}}$.

Now if $\sum$ is a set of sentences we define the satisfaction relation by $\mathfrak{A} \models \sum$ if and only if $\sum \subseteq ker(Th(\mathfrak{A}))$. This is well-defined since $Th(\mathfrak{A})$ is determined by its kernel, and so we may identify $Th(\mathfrak{A})$ with its kernel. In doing so, we obtain $Th(\mathfrak{A}) = \{\varphi : \varphi \text{ is a formula in } \mathcal{L} \text{ such that } \varphi^{\mathfrak{A}} = 0\} = \{\varphi : \varphi \text{ is a formula in } \mathcal{L} \text{ such that } \mathfrak{A} \models \varphi\}$ which precisely corresponds to **Definition 3.1.6**.

Definition 3.2.7: Let $\mathfrak{B} \subseteq \mathfrak{A}$. We say that $\mathfrak{B}$ is an **elementary submodel** of $\mathfrak{A}$ if for any formula φ and any $b \in \mathfrak{B}, \varphi^{\mathfrak{B}}(b) = \varphi^{\mathfrak{A}}(b)$.

Now we will turn out attention towards the continuous analogues of the theorems recalled in the previous section. We will not provide proofs of these theorems, but they can be seen in [2].

Definition 3.2.8: In a metric structure, the structure's **density character** is the infimum over cardinalities of dense subsets of the structure. Note that separability is characterized by having density character at most $\aleph_0$.

Theorem 3.2.9 (Löwenheim–Skolem): Given a separable language $\mathcal{L}$ and a separable subset X of an $\mathcal{L}$-structure $\mathfrak{B}$ such that $\mathfrak{B}$ has density character $\lambda, \mathfrak{B}$ has a separable elementary submodel $\mathfrak{A}$ containing X.

In comparison with **Theorem 3.1.8**, we focus on cardinalities of density characters instead of on cardinalities of the models themselves. Dense subsets are of particular interest here because we require that our functions be continuous, and continuous functions are determined on dense subsets.

Theorem 3.2.10 (Compactness Theorem): Given a set of sentences $\sum, \sigma$ has a model if and only if every finite subset of $\sum$ has a model if and only if every finite subset of $\{|\varphi| - \epsilon : \varphi \in \sum, \epsilon > 0\}$ has a model.

Not only does **Theorem 3.1.9** hold here, but the continuous structure also allows us to satisfy finite collections of approximations of sentences.

Definition 3.2.11: Let I be a non-empty set with an ultrafilter U. Let $\{\mathfrak{A}_x : x \in I\}$ be a collection of models on a language $\mathcal{L}$. Now consider $\prod_{x \in I} S_x$ where $(S_x, d_x) \in \mathfrak{S}_x$ and let $\mathfrak{A}$ be the structure obtained from taking the quotient of this product by $d := lim_{x \to U} d_x^{S_x}$. This structure is the continuous analogue of the **ultraproduct**. Note that if our metric structure is a C*-algebra this construction coincides with the usual construction of the ultraproduct.

Theorem 3.2.12 (Łoś' theorem): With the notation used in **Definition 3.1.11**, if φ is a formula in $\mathcal{L}$ and $a \in A$ then $\varphi^M(a) = lim_{x \to U} \varphi^{\mathfrak{A}_x}(\pi_x(a))$. Like its classical analogue, this theorem is proven by induction on the complexity of φ.

This concludes our survey of continuous model theory. With the logical preliminaries out of the way, we can begin to discuss axiomatization.

4 Axiomatization of C*-Algebras

4.1 Definition of axiomatization

Definition 4.1.1: Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. We say that $\mathcal{C}$ is **axiomatizable** if $Mod(T) = \mathcal{C}$ for some theory T, where $Mod(T)$ is the class of models of T.

While this definition intuitively captures the notion of assigning axioms to a class, it is incredibly abstract and a priori not applicable. The following theorem, which is in part a consequence of **Theorem 3.2.10** and **Theorem 3.2.12** will give us a more concrete way to determine if a class is axiomatizable.

Definition 4.1.2: We say that a class of $\mathcal{L}$-structures $\mathcal{C}$ is **closed under ultraroots** if $A^U \in \mathcal{C} \implies A \in \mathcal{C}$ where A is some structure and U is an ultrafilter.

Theorem 4.1.3: A class $\mathcal{C}$ of $\mathcal{L}$-structures is axiomatizable if and only if it is closed under isomorphisms, ultraproducts and elementary submodels, if and only if it is closed under isomorphisms, ultraproducts and ultraroots.

Example 4.1.4: Let A be a C*-algebra satisfying the condition that, when $x, y \in A$ are positive and $\epsilon > 0$ satisfies $||xy||^2 < \epsilon$ then there is a projection p satisfying $||p(x) < \epsilon$ and $||(1-p)y|| < \epsilon$. Now **Theorem 4.1.3** can be used to deduce that A is axiomatizable. Note that such a choice of A satisfies the condition that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements. So, in particular, this tells us that C*-algebras satisfying this weaker condition are axiomatizable. (Note that this example is discussed in further detail in [2]).

Classes of C*-algebras are axiomatizable in different ways depending on the behavior of certain collections of sentences in elements of the class. This idea is made concrete by the following definition:

Definition 4.1.5: Let $\mathcal{C}$ be a class of C*-algebras and $\sum$ be the set of $sup_x\varphi$ where φ ranges over quantifier-free $\mathbb{R}^+$ valued formulas. Then, if there is $\psi \subseteq \sum$ such that $A \in \mathcal{C} \iff \varphi^A = 0$ for all $\varphi \in \psi$ then we say that A is **universally axiomatizable**. If we replace $\sum$ with the set of $inf_x\varphi$ then A is said to be **existentially axiomatizable**. If we replace $\sum$ with the set of $\sup_x \inf_y \varphi$ then A is said to be **∀∃-axiomatizable**.

Note that the class of C*-algebras with invertible self-adjoint elements dense in the set of self-adjoint elements, as discussed in **Example 4.1.4**, is ∀∃-axiomatizable. We will now observe simple examples of universally and existentially axiomatizable classes.

Example 4.1.6: The class of abelian C*-algebras is universally axiomatizable, as it is axiomatized by the sentence $sup_x sup_y||[x, y]||$.

Example 4.1.7: The class of non-abelian C*-algebras is exisentially axiomatizable. If A is a non-abelian C*-algebra then there is a non-zero nilpotent $a \in A$, i.e a satisfies $a^m = 0$ for some minimal m. Then $\frac{1}{||a||^{m-1}}a^{m-1}$ satisfies the sentence $1 - sup_x(||x||^2 - ||x^2||)$. Note that this sentence is always equal to 1 in an abelian C*-algebra since all elements x in an abelian C*-algebra are normal and therefore satisfy $||x||^2 = ||x^2||$.

4.2 An interesting problem

Our goal now is to discuss the problem presented by Professor Hannes Thiel in [3] which draws connections between domains, C*-algebras and model theory. We begin by defining the notion of a domain and Cuntz semigroup.

Definition 4.2.1: In a partially ordered set S, we say that x is **way-below** y (denoted $x << y$ if for any increasing $(z_n)_n \in S$ with $y \leq sup_n z_n$ there is N such that $x \leq z_N$.

Definition 4.2.2: A partially ordered set S is called a **domain** if every increasing sequence in S has a supremum in S and every element of S is the supremum of some way-below increasing sequence in S.

We recall the following definitions from [6]:

Definition 4.2.3: Let A be a C*-algebra. For $a, b \in A$, we say that a is **Cuntz subequivalent** to b if for any $\epsilon > 0$ there is $r \in A$ such that $||a - rbr^*|| < \epsilon$. If a is Cuntz subequivalent to b and vice versa we say that a and b are **Cuntz equivalent**.

Note that Cuntz equivalence is an equivalence relation which we will denote by $\sim$. We then obtain the following construction.

Definition 4.2.4: If A is a C*-algebra, the **Cuntz semigroup** of A, denoted $Cu(A)$ is defined by $Cu(A) := (A \otimes \mathcal{K})/\sim$ where $A \otimes \mathcal{K}$ is the stabilization of A.

The following result allows us to connect domains with C*-algebras:

Theorem 4.2.5 (Coward-Elliott-Ivanescu): Given a C*-algebra $A, Cu(A)$ is a domain.

This result motivates the following abstraction of **Definition 4.2.4**:

Definition 4.2.6: An **abstract Cuntz semigroup** is a domain with a compatible abelian semigroup structure.

For the sake of readability, we will refer to abstract Cuntz semigroups as Cuntz semigroups (unless otherwise indicated).

Morphisms in the category of Cuntz semigroups are maps between Cuntz semigroups that preserve addition, order, way-below ordering, and suprema of increasing sequences. Note also that there is a natural functor $A \mapsto Cu(A)$ between the category of C*-algebras and that of Cuntz semigroups.

Definition 4.2.7: Given a Cuntz semigroup S, we say that $T \subseteq S$ is a **sub Cuntz semigroup** of S if T is a Cuntz semigroup with the inherited order and the inclusion map $\iota : T \to S$ is a morphism in the category of Cuntz semigroups.

Definition 4.2.8: We say that a Cuntz semigroup S is **separable** if there is a countable $B \subseteq S$ such that any $x \in S$ is the supremum of an increasing seuqence in B.

Definition 4.2.9: Let P be a property of Cuntz semigroups. We say that P satisfies the **Löwenheim–Skolem condition** if, for any Cuntz semigroup satisfying P, there is a σ-complete and club collection of separable sub Cuntz semigroups of S.

A simple example is the property of weak cancellation; i.e if $x + z << y + z$ then $x << y$.

The ultimate question proposed in [3] that was left unanswered was whether there is a language in which properties satisfying the Löwenheim–Skolem condition become axiomatizable. It's worth noting that the key result that links C*-algebras to this question is **Theorem 4.2.5**. Since the Cuntz semigroup of a C*-algebra is indeed a domain, this same question can be asked of Cuntz semigroups of C*-algebras.

5 References

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