An Introduction to Descriptive Set Theory

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Table of Contents

Preliminary notions

For the following definitions, we let X be a set equipped with a topology τ.

Definition

We say that $D \subseteq X$ is **dense** in X if $D \cap U \neq \emptyset$ for any $U \in \tau$.

Definition

We say that X is **separable** if it contains a countable dense set. For example, $\mathbb{R}$ with its usual topology is separable since $\mathbb{Q}$ is dense in $\mathbb{R}$.

Preliminary notions cont.

Definition

We say that X is **metrizable** if there exists a metric d on X such that τ is equal to the topology induced by d.

Definition

We say that (X, d) is **complete** if every Cauchy sequence in X converges to a point in X.

Preliminary notions cont.

Definition

We say that Y is G_δ in X if $Y = \bigcap_{n\in\mathbb{N}} U_n$ where $U_n \in \tau$. We say that Y is F_σ in X if $Y = \bigcup_{n\in\mathbb{N}} U_n^c$ where $U_n \in \tau$. Note that the complement of a G_δ set is F_σ and vice versa.

Preliminary notions cont.

Definition

Let $\{X_i\}_{i \in I}$ be a collection of topological spaces and define $X := \prod_{i \in I} X_i$. The **product topology** on X is the smallest topology on X such that all projection maps are continuous.

Basic open sets in X are of the form $\prod_{i \in I} U_i$ where U_i is open in X_i and $U_i = X_i$ for all but finitely many U_i.

What is a Polish space?

Definition

A **Polish space** is a separable, completely metrizable topological space.

Remark

Note that Polish spaces are *completely metrizable*, not complete with respect to a specific metric: i.e $(0, 1)$ as a subspace of $\mathbb{R}$ with the usual topology is not complete, but there exists a complete metric on $(0, 1)$ (ex. any homeomorphism from $(0, 1)$ to $\mathbb{R}$ induces a complete metric on $(0, 1)$).

Examples of Polish spaces

Some simple examples of Polish spaces:

- $\mathbb{R}^n$
- $\mathbb{R}^{\mathbb{N}}$
- $\mathbb{C}^n$
- $\mathbb{C}^{\mathbb{N}}$
- The n-dimensional cube
- Any countable set with the discrete topology

Special Examples of Polish spaces

Definition

The Cantor space, denoted $\mathcal{C}$, is defined by $\mathcal{C} := \{0,1\}^{\mathbb{N}}$ equipped with the product topology.

Definition

The Hilbert cube, denoted $\mathbb{I}^{\mathbb{N}}$, is defined by $\mathbb{I}^{\mathbb{N}} := [0,1]^{\mathbb{N}}$ equipped with the product topology.

Definition

The Baire space, denoted $\mathcal{N}$, is defined by $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$ equipped with the product topology.

Polish subspaces of Polish spaces

Theorem

If X is metrizable and $Y \subseteq X$ is completely metrizable, then Y is G_δ in X. Conversely, if X is completely metrizable and $Y \subseteq X$ is G_δ, then Y is completely metrizable. In particular, a subspace Y of a Polish space X is Polish iff Y is G_δ in X.

Polish spaces and the Hilbert cube

Theorem

Every separable metrizable space is homeomorphic to a subspace of $\mathbb{I}^{\mathbb{N}}$. In particular, Polish spaces are, up to homeomorphism, G_δ subspaces of the Hilbert cube.

Some definitions

Definition

A point x in a topological space is an **isolated point** if $\{x\}$ is open.

Definition

A point x in a topological space is a **limit point** if it isn't an isolated point.

Definition

A space is **perfect** if it contains no isolated points.

Definition

A point x in a topological space is a **condensation point** if every open neighbourhood of x is uncountable.

Some definitions cont.

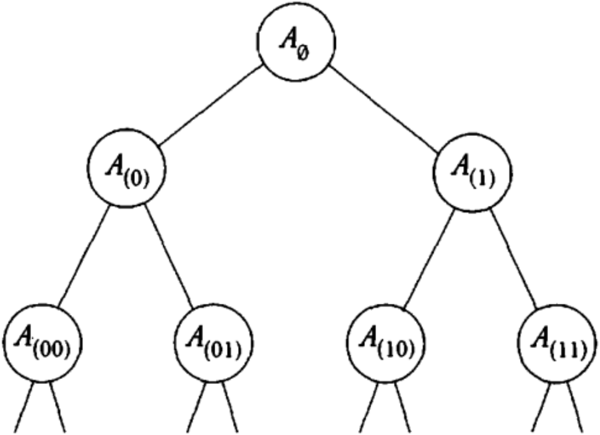
We can think of $\mathcal{C}$ as a tree, which motivates the following definition:

Definition

A **Cantor scheme** on a set X is a collection $(A_s)_{s<2^{\mathbb{N}}}$ of subsets of X satisfying:

- $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$ for $s \in 2^{<\mathbb{N}}$
- $A_{s^\frown i} \subseteq A_s$ for $s \in 2^{<\mathbb{N}}, i \in \{0, 1\}$

Some definitions cont.



Embedding the Cantor space

Theorem

Let X be a non-empty, perfect Polish space. Then there is an embedding of $\mathcal{C}$ into X.

The Cantor-Bendixson theorem

Cantor-Bendixson Theorem

If X is a Polish space, then X can be uniquely written as $X = P \cup C$ where P is perfect in X and C is open and countable.

Corollary

Any uncountable Polish space contains a copy of $\mathcal{C}$ and thus has cardinality $\mathfrak{c}$.

Borel sets

Definition

Given a set X, we say that $M \subseteq P(X)$ is a σ-algebra on X if M is closed under countable unions and complements.

Given $A \subseteq P(X)$, we refer to the smallest σ-algebra containing A as the σ-algebra **generated by** A.

Definition

The class of **Borel sets** of a topological space X is the σ-algebra generated by the open sets of X.

Borel sets (cont.)

Theorem

Let (X, τ) be Polish and let $A \subseteq X$ be Borel. Then there is a Polish topology $\tau_A \supseteq \tau$ such that τ and τ_A generate the same Borel sets and A is clopen in τ_A.

The perfect set theorem for Borel sets

Perfect set theorem for Borel sets

Let (X, τ) be Polish and $A \subseteq X$ be uncountable and Borel. Then A contains a copy of $\mathcal{C}$.

Proof: We extend τ to τ_A in which A is clopen and Borel sets of τ_A are Borel sets of τ. Since A is closed in τ_A, it's G_δ and therefore Polish. Since A is uncountable and Polish, by Cantor-Bendixson A contains a copy of $\mathcal{C}$.

Null and meager sets

Definition

Given $A \subseteq \mathbb{R}$, we say that A has **Lebesgue measure 0** or that A is a **null set** if, for any $\epsilon > 0$, A can be covered by some $\{I_n\}_{n\in\mathbb{N}}$ where I_n is an open interval and $\sum_{n\in\mathbb{N}} \text{length}(I_n) < \epsilon$.

Definition

Given $U \subseteq \mathbb{R}$, we say that U is **nowhere dense** in $\mathbb{R}$ if the closure of U has empty interior. We say that $A \subseteq \mathbb{R}$ is **meager** if $A = \bigcup_{n\in\mathbb{N}} U_n$ where each U_n is nowhere dense in $\mathbb{R}$.

Baire category

Remark

If $A \subseteq \mathbb{R}$ is meager, it is said to be of **first category**. If A is non-meager, it is said to be of **second category**. We will avoid this convention to prevent confusion.

The Baire category theorem

Baire category theorem

If X is completely metrizable, then it satisfies the following equivalent conditions:

- Every non-empty open subset of X is non-meager.
- Every comeager set in X is dense in X.
- The intersection of countably many dense open sets in X is dense.

Proof: Let $\{U_n\}_{n\in\mathbb{N}}$ be dense open and let U be open in X. We will show that $\bigcap_{n\in\mathbb{N}} U_n \cap U \neq \emptyset$.

Note that $U_1 \cap U \neq \emptyset$ so construct a sufficiently small $B_1 \subseteq U_1 \cap U$ such that $\overline{B_1} \subseteq U_1 \cap U$. Then $B_1 \cap U_2 \neq \emptyset$ so construct a smaller $B_2 \subseteq U_2 \cap B_1$ such that $\overline{B_2} \subseteq B_1 \cap U_2$. Repeat this process inductively.

Let x_i be the centre of B_i. Then (x_i) is a Cauchy sequence and by completeness $x_i \to x \in \bigcap_{n\in\mathbb{N}} B_n \subseteq \bigcap_{n\in\mathbb{N}} U_n \cap U$.

The continuum hypothesis

Definition

We define $\aleph_0 := |\mathbb{N}|$; a set S is **countable** if $|S| \leq \aleph_0$ and **uncountable** otherwise. We denote the first uncountable ordinal by $\aleph_1$.

The continuum hypothesis

The continuum hypothesis (denoted CH) is the statement that $\mathfrak{c} = \aleph_1$, or equivalently that if $A \subseteq \mathbb{R}$ is uncountable then $|A| = \mathfrak{c}$.

It has been shown that CH is independent of ZFC, meaning that we can freely assume either CH or $\neg$ CH without contradiction.

Duality of measure and category

Assume CH. Then, if φ is a statement about subsets of $\mathbb{R}$ and φ^* is the same statement but with all occurrences of "Lebesgue measure 0" replaced with "meager" or vice versa, then $\varphi \iff \varphi^*$. This is thanks to the following result:

Erdős-Sierpiński theorem

Assume CH. Then there exists a bijection $f : \mathbb{R} \to \mathbb{R}$ such that, for any $A \subseteq \mathbb{R}$, A is null iff $f(A)$ is meager and vice versa.

Duality of measure and category cont.

Lemma

There exist disjoint $F, G \subseteq \mathbb{R}$ such that F is null, G is meager and $F \cup G = \mathbb{R}$.

Proof: Enumerate $\mathbb{Q}$ by $\{q_i\}_{i \in \mathbb{N}}$ and for $n \in \mathbb{N}$ define $U_n := \bigcup_{i \in \mathbb{N}} (q_i - \frac{1}{2^{i+n}}, q_i + \frac{1}{2^{i+n}})$. Then $F := \bigcap_{n \in \mathbb{N}} U_n$ is null and $G := F^c$ is meager.

Duality of measure and category cont.

Lemma

If $F, G \subseteq \mathbb{R}$ are null and meager respectively, there exist respectively null and meager $F^+, G^+ \subseteq \mathbb{R}$ such that $F \subseteq F^+, G \subseteq G^+$ and $|F^+ \setminus F| = |G^+ \setminus G| = \mathfrak{c}$.

Proof (sketch): WLOG, G is F_σ. Then G^c is Polish and comeager, and therefore uncountable by the Baire category theorem, so it contains a copy of $\mathcal{C}$ by Cantor-Bendixson, so define $G^+ := G \cup \mathcal{C}$.

WLOG, F is Borel, so F^c is Borel and uncountable, and so by the perfect set theorem it contains a copy of $\mathcal{C}$. Let $\{q_i\}_{i\in\mathbb{N}} \subseteq \mathcal{C}$ be dense and for $n \in \mathbb{N}$ define $U_n := \bigcup_{i\in\mathbb{N}} (q_i - \frac{1}{2^{i+n}}, q_i + \frac{1}{2^{i+n}})$. Then $F^+ := F \cup \bigcap_{n\in\mathbb{N}} U_n$ is our desired null set.

References

- Alexander S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, New York, 1995.
- Winfried Just & Martin Weese, *Discovering Modern Set Theory II. Set-Theoretic Tools for Every Mathematician*, American Mathematical Society, Providence, 1997.
- John C. Oxtoby, *Measure and Category*, Springer-Verlag, New York, 1980.
- James R. Munkres, *Topology*, Prentice Hall, Upper Saddle River, 2000.

Thanks for your attention!