MAT437: The Continuum Hypothesis and the Calkin Algebra

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1 Acknowledgements

I would like to thank Professor George Elliott for his invaluable teachings on C*-algebras and K-theory. I would also like to thank Professor Ilijas Farah for his incredibly helpful insights on $\beta\mathbb{N} \setminus \mathbb{N}$. I would lastly like to thank my friend Kai Qi Hao for the many discussions we have had about topics covered in this essay.

2 Introduction

On a Hilbert space $\mathcal{H}$, we denote the space of bounded linear operators by $B(\mathcal{H})$ and the compact operators in $B(\mathcal{H})$ by $\mathcal{K}(\mathcal{H})$. Noting that $\mathcal{K}(\mathcal{H})$ is an ideal in $B(\mathcal{H})$, we define $\mathcal{Q}(\mathcal{H}) := B(\mathcal{H})/\mathcal{K}(\mathcal{H})$. $\mathcal{Q}(\mathcal{H})$ is called the *Calkin algebra* of $\mathcal{H}$. We will briefly review the role of $\mathcal{Q}(\mathcal{H})$ in the context of K-theory as seen in [1]; in particular we compute its K_0 and K_1 groups and discuss Fredholm operators. Then, we will turn our attention towards the role of $\mathcal{Q}(\mathcal{H})$ in set theory. In particular, we will examine a consequence on $\mathcal{Q}(\mathcal{H})$ from assuming the Continuum Hypothesis (denoted CH) analogous to a result from CH on $\beta\mathbb{N} \setminus \mathbb{N}$. We will discuss another result that comes from instead assuming Todorčević's formulation of the Open Coloring Axiom (denoted OCA).

3 The role of $\mathcal{Q}(\mathcal{H})$ in K-theory

3.1 $K_0(\mathcal{Q}(\mathcal{H}))$ and $K_1(\mathcal{Q}(\mathcal{H}))$

Letting $\iota : \mathcal{K}(\mathcal{H}) \to B(\mathcal{H})$ and $\pi : B(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ denote the inclusion and quotient map respectively, the following sequence is short exact:

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \xrightarrow{\iota} \mathcal{B}(\mathcal{H}) \xrightarrow{\pi} \mathcal{Q}(\mathcal{H}) \longrightarrow 0$$

This induces the following six-term exact sequence:

$$\begin{tikzcd}
K_0(\mathcal{K}(\mathcal{H})) \arrow[r] & K_0(B(\mathcal{H})) \arrow[r] & K_0(\mathcal{Q}(\mathcal{H})) \arrow[d] \\
K_1(\mathcal{Q}(\mathcal{H})) \arrow[u] & K_1(B(\mathcal{H})) \arrow[l] & K_1(\mathcal{K}(\mathcal{H})) \arrow[l]
\end{tikzcd}$$

It is shown in [1] that $K_0(B(\mathcal{H})) = 0, K_1(B(\mathcal{H})) = 0, K_1(\mathcal{K}(\mathcal{H})) = 0$ and $K_0(\mathcal{K}(\mathcal{H})) \cong \mathbb{Z}$. Thus from the sequence above it follows that $K_0(\mathcal{Q}(\mathcal{H})) = 0$ and $K_1(\mathcal{Q}(\mathcal{H})) \cong \mathbb{Z}$.

3.2 Fredholm operators

In the context of operator theory, part of the importance of $\mathcal{Q}(\mathcal{H})$ is its role in characterizing Fredholm operators. This is seen through Atkinson's theorem, for which a full proof can be found in [2]:

Theorem 3.2.1 (Atkinson): For $T \in B(\mathcal{H})$, the following are equivalent:

1. $T(H)$ is closed, $dim(ker(T)) < \infty$ and $dim(ker(T^*)) < \infty$
2. There exists $S \in B(\mathcal{H})$ such that $I - ST$ and $I - TS$ are compact operators
3. $\pi(T)$ is invertible in $\mathcal{Q}(\mathcal{H})$

In the finite-dimensional setting, there is a relationship between the image and kernel of an operator given by the rank-nullity theorem. While such a relationship does not generalize to infinite dimensions, a suitable alternative is to establish a relationship between the kernel and cokernel. *Fredholm operators* are operators which satisfy any of the equivalent conditions in Theorem 3.2.1. In particular, Fredholm operators are operators for which there is such a relationship, as given by the following:

Definition 3.2.2: The **Fredholm index** of a Fredholm operator is defined by $index(T) := dim(ker(T)) - dim(ker(T^*))$. Note that $index(T) \in \mathbb{Z}$ since, for a Fredholm T, $dim(ker(T))$, $dim(ker(T^*)) < \infty$.

Recall the short exact sequence introduced at the beginning of Section 3.1:

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \xrightarrow{\iota} \mathcal{B}(\mathcal{H}) \xrightarrow{\pi} \mathcal{Q}(\mathcal{H}) \longrightarrow 0$$

We let δ_1 denote the index map induced by this sequence. Then as shown in [1] we have the following formula for $index(T)$:

Proposition 3.2.3: For a Fredholm operator T on $\mathcal{H}$, $index(T) = (K_0(Tr) \circ \delta_1)([\pi(T)]_1)$.

If $\mathcal{H}$ is finite-dimensional, then it follows from the rank-nullity theorem that $index(T) = 0$. In particular then, it must be the case by Proposition 3.2.3 that $(K_0(Tr) \circ \delta_1)([\pi(T)]_1) = 0$. This can also be seen without appealing to Proposition 3.2.3. Since the unit ball is compact in finite-dimensional spaces, any operator on $\mathcal{H}$ is compact; so we have that $\mathcal{K}(\mathcal{H}) = B(\mathcal{H})$ from which the result follows.

4 The role of $\mathcal{Q}(\mathcal{H})$ in set theory

4.1 Preliminaries

We will briefly review some preliminary notions from set theory and C*-algebra theory. For the remainder of this essay, we will impose the constraint that $\mathcal{H}$ be separable.

Definition 4.1.1: On a C*-algebra A, an **inner automorphism** on A is an automorphism φ on A defined by $\varphi(a) = uau^*$ where $u \in \tilde{A}$ is unitary (note that $\tilde{A}$ denotes the unitization of A). An **outer automorphism** on A is an automorphism on A that is not inner.

Let $Aut(A)$ denote the group of automorphisms on A and $Inn(A)$ denote the inner automorphisms on A. Then $Inn(A)$ is a normal subgroup of A. Note that $K_0(\varphi) = id$ for any $\varphi \in Inn(A)$.

For our purposes we always assume the axiom of choice, which allows for the following formulation of CH:

Definition 4.1.2: The **Continuum Hypothesis** (CH) is the statement that $2^{\aleph_0} = \aleph_1$.

Recall that CH is independent of ZFC. Another statement independent of ZFC which we shall discuss is Todorčević's formulation of the Open Coloring Axiom (often referred to as Todorčević's axiom in certain literature). We present OCA as it appears in [3].

Let X be a separable metric space and define $[X]^2 := \{(x, y) \in X \times X : x \neq y\}$. A coloring $[X]^2 = E_1 \cup E_2$ is *open* if E_0 is open in the product topology. For $E \subseteq [X]^2$, we say that $Y \subseteq X$ is *E-homogeneous* if $[Y]^2 \subseteq E$.

Definition 4.1.3: The **Open Coloring Axiom** is the statement that, if X is a separable metric space and $[X]^2 = E_0 \cup E_1$ is an open coloring, then X contains an uncountable E_0-homogeneous set or X has a countable covering of E_1 homogeneous sets.

Given a topological space X, recall that a compactification of X is a pair (c, Y) where Y

is a compact space and $c : X \to Y$ is an embedding with dense image. In particular, X has a compactification if and only if it's completely regular. The class of compactifications on X is ordered, with $cX \leq dX \iff$ there exists a continuous surjection $f : dX \to cX$ that fixes X. Thus we have the following:

Definition 4.1.4: The **Stone–Čech compactification** of a completely regular space X, denoted βX, is the compactification of X maximal in the ordering $\leq$ defined above.

Let $\mathfrak{A}$ be the space of ultrafilters on $\mathbb{N}$. In particular, basic open sets of $\mathfrak{A}$ are given by any $A \subseteq \mathbb{N}$ and are of the form $\{\mathcal{U} \in \mathfrak{A} : A \in \mathcal{U}\}$. Then $\mathfrak{A} = \beta\mathbb{N}$. A proof of this result and further discussion of $\beta\mathbb{N}$ can be seen in [4].

4.2 $\mathcal{Q}(\mathcal{H})$ and CH

Set theorists often view $\mathcal{Q}(\mathcal{H})$ as a "non-commutative analogue of $\beta\mathbb{N} \setminus \mathbb{N}$" [5]. Consider the following class of homeomorphisms:

Definition 4.2.1 A homeomorphism from $\beta\mathbb{N} \setminus \mathbb{N}$ to itself is **trivial** if it is induced by a pair of cofinite subsets of $\mathbb{N}$ and is **non-trivial** otherwise.

The question of whether non-trivial self-homeomorphisms exist on $\beta\mathbb{N} \setminus \mathbb{N}$ was resolved by W. Rudin, who showed the following in [6]:

Theorem 4.2.2 (W. Rudin): Assuming CH, there is a non-trivial homeomorphism of $\beta\mathbb{N}\setminus\mathbb{N}$ onto itself. In particular there are $2^{\aleph_1}$ non-trivial homeomorphisms of $\beta\mathbb{N}\setminus\mathbb{N}$ onto itself.

In 1977, it was asked in [7] by Brown, Doughlas and Fillmore if there are outer automorphisms on $\mathcal{Q}(\mathcal{H})$. This problem turns out to be a suitable analogue to that solved by Rudin, thanks to the following result by Phillips and Weaver in [5]:

Theorem 4.2.3 (Phillips, Weaver): Assuming CH, there is an outer automorphism of $\mathcal{Q}(\mathcal{H})$. In particular, there are $2^{\aleph_1}$ outer automorphisms of $\mathcal{Q}(\mathcal{H})$.

As one might expect, there is an analogous use of CH between these two analogous results. Some extra care is needed for Theorem 4.2.3, however. We present a very brief summary of the intuition provided in the introduction of [5].

Let $\mathcal{C}$ denote the algebra of continuous functions on $\beta\mathbb{N} \setminus \mathbb{N}$. Assume CH, and then write $\mathcal{C} = \bigcup_{\alpha<\aleph_1} A_\alpha$ where $A_\alpha \subseteq A_{\alpha+1}$ and A_α is a separable unital C*-subalgebra. Then, extend a $*$-monomorpishm $A_\alpha \to \mathcal{C}$ to a $*$-monomorphism $A_{\alpha+1} \to \mathcal{C}$ in two different ways which produces two different homeomorphisms of $\beta\mathbb{N} \setminus \mathbb{N}$ onto itself. Repeating this process for each $\alpha < 2^{\aleph_1}$ gives $2^{\aleph_1}$ such homeomorphisms; but there are only $\aleph_1$ trivial homeomorphisms of $\beta\mathbb{N} \setminus \mathbb{N}$ onto itself, which gives the result of Theorem 4.2.2.

As pointed out by Ilijas Farah, the following argument can be used to show that such extensions of $*$-monomorphisms exist. Letting X, Y denote compact metrizable spaces and $f : X \to Y$ a continuous surjection, we can show that every surjection $\beta\mathbb{N} \setminus \mathbb{N} \to Y$ filters through X and then combine this fact with the Gelfand-Naimark duality. The former fact can be shown explicitly using Theorem 1.2.5 in [8].

The problem with using this approach to prove Theorem 4.2.3 is that $\mathcal{Q}(\mathcal{H})$ does not have this same extension property for $*$-monomorphisms. Consider the following counterexample from [5]:

Let $\mathcal{S} \in \mathcal{Q}(\mathcal{H})$ denote the image of the unilateral shift operator. Let $A = C([0,1])$ and $B = C(S^1)$. Let u be the standard unitary generator of B and find a homomorphism $\varphi : B \to \mathcal{Q}(\mathcal{H})$ with $\varphi(u) = \mathcal{S}$. Define $\iota : B \to A$ by $\iota(f)(x) = f(e^{2\pi i x})$. Then no homomorphism $\psi : A \to \mathcal{Q}(\mathcal{H})$ is lifted to φ by ι.

Now, consider the following lemma proven in [9]:

Lemma 4.2.4 (Voiculescu): A seperable C*-subalgebra of $\mathcal{Q}(\mathcal{H})$ that contains the identity is equal to its bi-commutant.

The remedy to this problem, as discussed in [5], is to construct a sequence of unitary elements $u_\alpha \in \mathcal{Q}(\mathcal{H})$ for $\alpha < 2^{\aleph_1}$ which induce automorphisms on A_α, and then use Lemma 4.2.4 to provide distinct extensions to $A_{\alpha+1}$.

We recall that OCA is independent of ZFC. Additionally, OCA is not consistent with

CH; in particular, ZFC + OCA $\implies \neg$ CH. While CH gives an affirmative answer to the question posed in [7], OCA gives a negative answer. The following result is proven in [3]:

Theorem 4.2.5: OCA implies that all automorphisms of $\mathcal{Q}(\mathcal{H})$ are inner.

Let E be a partition of $\mathbb{N}$ into finite intervals $(E_n)_{n\in\mathbb{N}}$ and let $\mathcal{D}[E]$ be the von Neumann algebra of $T \in B(\mathcal{H})$ such that $\overline{span}\{e_i : i \in \mathbb{N}\}$ is T-invariant. The key difference in the use of CH and OCA is that CH can be used to construct an outer automorphism φ on $\mathcal{Q}(\mathcal{H})$ such that $\varphi|_{\mathcal{D}[E]}$ is inner, whereas OCA is used to show that for any such φ, $\varphi|_{\mathcal{D}[E]}$ is not inner for some E. This is the intuition provided in [3], which also provides a full proof of Theorem 4.2.5 and a modified proof of Theorem 4.2.3.

5 References

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