Basic Embedding Results in Descriptive Set Theory

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What is a Polish space?

The spaces of greatest interest in descriptive set theory are Polish spaces. We begin with a basic overview of Polish spaces and some incredibly useful results.

Definition 1.1

A **Polish space** is a separable, completely metrizable topological space.

Remark 1.2

Note that Polish spaces are *completely metrizable*, not complete with respect to a specific metric: i.e $(0, 1)$ as a subspace of $\mathbb{R}$ with the usual topology is not complete, but there exists a complete metric on $(0, 1)$ (ex. any homeomorphism from $(0, 1)$ to $\mathbb{R}$ induces a complete metric on $(0, 1)$).

Examples of Polish spaces

Some simple examples of Polish spaces:

- $\mathbb{R}^n$
- $\mathbb{R}^{\mathbb{N}}$
- $\mathbb{C}^n$
- $\mathbb{C}^{\mathbb{N}}$
- The n-dimensional cube
- Any countable set with the discrete topology

Special Examples of Polish spaces

Some important Polish spaces are the following:

Definition 1.3

The Cantor space, denoted $\mathcal{C}$, is defined by $\mathcal{C} := \{0, 1\}^{\mathbb{N}}$ equipped with the product topology.

Definition 1.4

The Hilbert cube, denoted $\mathbb{I}^{\mathbb{N}}$, is defined by $\mathbb{I}^{\mathbb{N}} := [0, 1]^{\mathbb{N}}$ equipped with the product topology.

Definition 1.5

The Baire space, denoted $\mathcal{N}$, is defined by $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$ equipped with the product topology.

Polish subspaces of Polish spaces

Theorem 1.6

If X is metrizable and $Y \subseteq X$ is completely metrizable, then Y is G_δ in X. Conversely, if X is completely metrizable and $Y \subseteq X$ is G_δ, then Y is completely metrizable. In particular, a subspace Y of a Polish space X is Polish iff Y is G_δ in X.

Proposition 1.7

$\mathcal{N}$ is homeomorphic to a G_δ subspace of $\mathcal{C}$.

Proof: We define an embedding $f : \mathcal{N} \to \mathcal{C}$ by $f(x) = 0^{x_0}10^{x_1}10^{x_2}\ldots$ where $x = (x_i)$. Note that $f(\mathcal{N}) = \{x \in \mathcal{C} : x_i = 1 \text{ for infinitely many indices}\}$. If we define $U_n := \{x \in \mathcal{C} : x_i = 1 \text{ for at least } n \text{ many indices}\}$ then $f(\mathcal{N}) = \bigcap_n U_n$ and is hence G_δ.

Polish spaces and the Hilbert cube

Theorem 1.8

Every separable metrizable space is homeomorphic to a subspace of $\mathbb{I}^{\mathbb{N}}$. In particular, Polish spaces are, up to homeomorphism, G_δ subspaces of the Hilbert cube.

Proof: To prove the first statement, let (X, d) be separable metrizable and WLOG let $d \leq 1$.
Let (x_n) be dense in X. Define $f : X \to \mathbb{I}^{\mathbb{N}}$ by $f(x) = (d(x, x_1), d(x, x_2), ...)$. Note then f is continuous and injective, and a sequential continuity argument shows continuity of $f^{-1} \restriction_{f(X)}$.
The second statement then follows by applying Theorem 1.6.

Some definitions

Definition 3.1

A point x in a topological space is an **isolated point** if $\{x\}$ is open.

Definition 3.2

A point x in a topological space is a **limit point** if it isn't an isolated point.

Definition 3.3

A space is **perfect** if it contains no isolated points.

Definition 3.4

A point x in a topological space is a **condensation point** if every open neighbourhood of x is uncountable.

Some definitions cont.

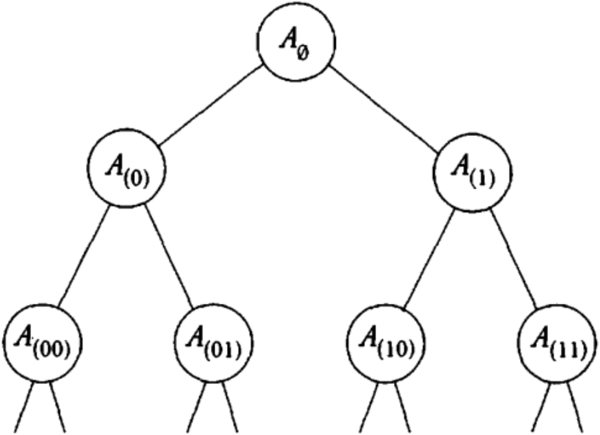
We can think of $\mathcal{C}$ as a tree, which motivates the following definitions:

Definition 3.5

A **Cantor scheme** on a set X is a collection $(A_s)_{s<2^{\mathbb{N}}}$ of subsets of X satisfying:

- $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$ for $s \in 2^{<\mathbb{N}}$
- $A_{s^\frown i} \subseteq A_s$ for $s \in 2^{<\mathbb{N}}, i \in \{0,1\}$

Some definitions cont.



Embedding the Cantor space

Theorem 3.6

Let X be a non-empty, perfect Polish space. Then there is an embedding of $\mathcal{C}$ into X.

Proof: Suppose we define a Cantor scheme $(U_s)_{s\in 2^{<\mathbb{N}}}$ satisfying:

- U_s is open and non-empty
- $\operatorname{diam}(U_s) \leq 2^{-\operatorname{length}(s)}$
- $\overline{U_{s^\frown i}} \subseteq U_s$ for $s \in 2^{<\mathbb{N}}, i \in \{0,1\}$

Then we define an embedding $f : \mathcal{C} \to X$ by $f(x) = \bigcap_{n\in\mathbb{N}} \overline{U_{x|n}}$ (note that such intersections are singletons by the completeness of X, hence f is well-defined).

Embedding the Cantor space cont.

Now by induction we define a Cantor scheme satisfying those conditions. For $s = \emptyset$, let U_s be arbitrary satisfying the first two conditions. Now let U_s be given, and since X is perfect, pick distinct $x, y \in U_s$. Then let $U_{s^\frown 0}, U_{s^\frown 1}$ be sufficiently small open balls around x, y.

Corollary 3.7

If X is a non-empty perfect Polish space, then $|X| = \mathfrak{c}$.

The Cantor-Bendixson theorem

Theorem 3.8

If X is a Polish space, then X can be uniquely written as $X = P \cup C$ where P is perfect in X and C is open and countable.

Corollary 3.9

Any uncountable Polish space contains a copy of $\mathcal{C}$ and thus has cardinality $\mathfrak{c}$.

The Cantor-Bendixson theorem cont.

Proof: Define $P := \{x \in X : \text{x is a condsensation point of } X\}$, and define $C := X \setminus P$. Note that if $\{U_n\}_{n\in\mathbb{N}}$ is a basis for X then C is the union of those U_n which are countable and is therefore open and countable.

Now let $x \in P$ and let U be an open neighbourhood of x. Then U is uncountable. If $U \cap P$ were countable, then we would have that $U \cap C$ is uncountable, contradicting the countability of C. So $U \cap P$ is uncountable and open in P, hence P is perfect in X.

Borel sets

Definition 3.10

The class of **Borel sets** of a topological space X is the σ-algebra generated by the open sets of X.

Theorem 3.11

Let (X, τ) be Polish and let $A \subseteq X$ be Borel. Then there is a Polish topology $\tau_A \supseteq \tau$ such that τ and τ_A generate the same Borel sets and A is clopen in τ_A.

The Perfect Set theorem for Borel sets

Theorem 3.14

Let (X, τ) be Polish and $A \subseteq X$ be uncountable and Borel. Then A contains a copy of $\mathcal{C}$.

Proof: By Theorem 3.11, we extend τ to τ_A in which A is clopen and Borel sets of τ_A are Borel sets of τ. Since A is closed in τ_A, it's G_δ and therefore Polish by Theorem 1.6. Since A is uncountable and Polish, Corollary 3.9 tells us that A contains a copy of $\mathcal{C}$.

The Baire category theorem

Theorem 3.15

If X is completely metrizable, then it satisfies the following equivalent conditions:

- Every non-empty open subset of X is non-meager.
- Every comeager set in X is dense in X.
- The intersection of countably many dense open sets in X is dense.

A useful application

Context: if we assume CH, then there exists a bijection from $\mathbb{R}$ to $\mathbb{R}$ that carries null sets to meager sets and vice versa (Erdős-Sierpiński duality theorem). The proof of this uses the following result:

Proposition 3.16

If $F \subseteq \mathbb{R}$ is meager and $G \subseteq \mathbb{R}$ is null, there exists a meager $F^+ \supseteq F$ and a null $G^+ \supseteq G$ such that $|G^+ \setminus G| = |F^+ \setminus F| = \mathfrak{c}$.

A useful application cont.

Proof: A union of meager sets is meager and union of null sets is null so it suffices to find a meager set and null set in F^c and G^c respectively that satisfy the cardinality condition.

The closure of a nowhere dense set is nowhere dense and every null set is contained in a Borel null set, so WLOG we can assume that F is F_σ and G is Borel.

Then F^c is G_δ and therefore Polish by Theorem 1.6. Now, F^c is comeager, and therefore uncountable by the Baire category theorem. So by Corollary 3.9 it contains a copy of $\mathcal{C}$ which is meager (note that category is preserved by homeomorphism).

A useful application cont.

Now, since G is Borel, so is G^c. G^c is not null and therefore uncountable, so by Theorem 3.14, G^c contains a copy of $\mathcal{C}$. Measure is not preserved by homeomorphism, so we're not done yet.

Let $\{q_i\}_{i\in\mathbb{N}}$ be a dense subset of $\mathcal{C}$ and for $i \in \mathbb{N}$ define $U_i := \bigcup\limits_{n\in\mathbb{N}} (q_n - 2^{-(i+n)}, q_n + 2^{-(i+n)})$. Then $\bigcap_{i\in\mathbb{N}} U_i$ is our desired null set.

More definitions

Definition 4.1

A **Lusin scheme** on a set X is a collection $(A_s)_{s<\mathbb{N}^{\mathbb{N}}}$ of subsets of X satisfying:

- $A_{s^\frown i} \cap A_{s^\frown j} = \emptyset$ for $s \in \mathbb{N}^{<\mathbb{N}}, i \neq j$
- $A_{s^\frown i} \subseteq A_s$ for $s \in \mathbb{N}^{<\mathbb{N}}, i \in \mathbb{N}$

Definition 4.2

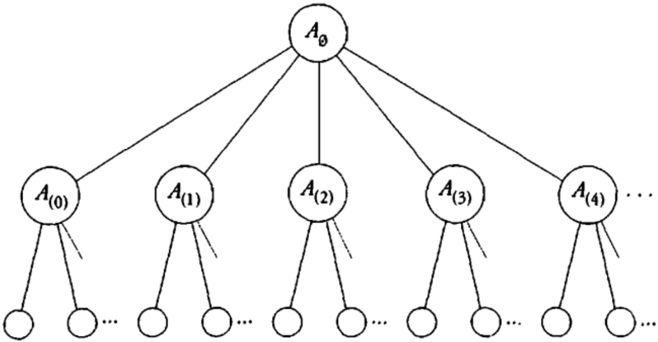
If (X, d) is a metric space with a Lusin scheme $(A_s)_{s<\mathbb{N}^{\mathbb{N}}}$, we say that $(A_s)_{s<\mathbb{N}^{\mathbb{N}}}$ has **vanishing diameter** if $\lim_n \operatorname{diam}(A_{x|n}) = 0$.

More definitions cont.

Definition 4.3

On a metric space (X, d) with Lusin scheme $(A_s)_{s<\mathbb{N}^{\mathbb{N}}}$ with vanishing diameter, define $D := \{x \in \mathcal{N} : \bigcap_n A_{x|n} \neq \emptyset\}$ and define $f : D \to X$ by $f(x) = \bigcap_n A_{x|n}$; we call f the **associated map**.

More definitions cont.



The Baire space and Polish spaces

Lemma 4.4

If (X, d) is a metric space with Lusin scheme $(A_s)_{s<\mathbb{N}^{\mathbb{N}}}$ that has vanishing diameter and associated map $f : D \to X$ then the following conditions hold:

- f is injective and continuous
- If (X, d) is complete and A_s is closed then D is closed
- If A_s is open then f is an embedding

The Baire space and Polish spaces cont.

Theorem 4.5

Every zero-dimensional, separable metrizable space can be embedded into $\mathcal{N}$ and $\mathcal{C}$. Every zero-dimensional Polish space is homeomorphic to a closed subspace of $\mathcal{N}$ and a G_δ subspace of $\mathcal{C}$.

Proof: To prove the statements for $\mathcal{N}$, let (X, d) be zero-dimensional and separable such that $d \leq 1$. Now we can easily construct a Lusin scheme on X such that:

- $A_\emptyset = X$
- A_s is clopen
- $A_s = \bigcup_i A_{s^\frown i}$
- $\mathrm{diam}(A_s) \leq 2^{-\mathrm{length}(s)}$

The Baire space and Polish spaces cont.

Now let $f : D \to X$ be the associated map. Then, $f(D) = X$ so that f is a homeomorphism between D and X by Lemma 4.4; in particular, if X is Polish then WLOG d is complete so that D is closed.

The results on $\mathcal{C}$ follow from combining what we've just shown on $\mathcal{N}$ with Proposition 1.7 ($\mathcal{N}$ is homeomorphic to a G_δ subset of $\mathcal{C}$).

The Baire space and Polish spaces cont.

Lemma 4.6

$\mathcal{N}$ is not σ-compact.

Proof: Note that compact subsets of $\mathcal{N}$ are nowhere dense. So if $\mathcal{N}$ is σ-compact then it's meager, contradicting the Baire category theorem.

Theorem 4.7

If X is Polish, then it contains a closed subspace homeomorphic to $\mathcal{N}$ iff X is not σ-compact.

The Baire space and Polish spaces cont.

Proof: If X contains a closed subspace homeomorphic to $\mathcal{N}$, then X is not σ-compact by Lemma 4.6.

Conversely, if X is not σ-compact, let X have a complete metric $d \leq 1$. We can use the fact that X is not σ-compact to construct a Lusin scheme on X such that the following conditions hold:

- $A_\emptyset = X, A_s \neq \emptyset$
- A_s is closed
- A_s is not σ-compact
- for each $n \in \mathbb{N}$ and $x \in X$ there is an open neighbourhood U of x so that $A_s \cap U \neq \emptyset$ for at most one $s \in \mathbb{N}^n$
- $\operatorname{diam}(A_s) \leq 2^{-\operatorname{length}(s)}$

The Baire space and Polish spaces cont.

Let $f : D \to X$ be the associated map. Note that $D = \mathcal{N}$ and $f(D)$ is closed; it then suffices to note that A_s is open in $f(D)$ so that f is a homeomorphism by Lemma 4.4.

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Thanks for your attention!