MAT495: The Continuum Hypothesis

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Contents

1 Acknowledgements

I would like to thank Professor Stevo Todorčević for his insights on the topics covered in this essay and for teaching me how to think like a set theorist. I would like to thank my friend and classmate Kai Qi Hao for the many discussions we've had about the topics in this essay.

2 Introduction

It was initially conjectured by Cantor in 1878 that every subset of $\mathbb{R}$ is either countable or has cardinality $\mathfrak{c}$, or equivalently, that there is no set X such that $\aleph_0 < |X| < \mathfrak{c}$. This conjecture is known as the Continuum Hypothesis, which we will denote by CH. Cantor attempted to prove CH, unsurprisingly now, to no avail. It was shown by Gödel in 1938 that

CH is consistent with ZFC and subsequently by Cohen in 1963 that $\neg$ CH is consistent with ZFC, yielding the now renowned result that CH is independent of ZFC. In this essay, we complete a selection of exercises in the seventeenth chapter of Just & Weese's *Discovering Modern Set Theory II: Set-Theoretic Tools for Every Mathematician.* These exercises will focus on applications of CH to Lebesgue measure and Baire category. We follow this with a brief survey of other interesting applications of CH collected from various other texts.

3 Applications of CH in Just & Weese

Note that all results in this section, unless indicated otherwise, are sourced from [1]. Definitions and certain results are omitted from this essay under the assumption that it is read side-by-side with [1].

3.1 Setting the Stage

In this subsection, we begin with a characterization of CH in terms of vertical and horizontal sections. This is followed by the use of CH to show the existence of a Sierpiński set, parallel to the use of CH to show the existence of a Luzin set in [1].

Theorem 17.1 (CH): There exists $A \subseteq [0,1] \times [0,1]$ such that each horizontal section of A is countable and each vertical section of A contains all but countably many reals.

Proof: Assume CH, and equip $[0,1]$ with a binary relation $\leq$ such that $([0,1], \leq)$ is a well-ordering of type ω_1. Then $A =\leq$ is the desired set. $\square$

Exercise 17.1: Show that $A =\leq$ is as required in Theorem 17.1.

Solution: Let $y \in [0,1]$ and consider $A^y := \{x \in \mathbb{R} : (x,y) \in A\}$. Let $y \mapsto y' \in \omega_1$ under the given order isomorphism $([0,1], \leq) \to (\omega_1, \in)$. Now, $A^{y'} := \{x' \in \omega_1 : x' \in y'\}$ is countable by CH, hence A^y is also countable. Now let $x \in [0,1]$ and consider $A_x := \{y \in \mathbb{R} : (x,y) \in A\}$. Let $x \mapsto x'$ under the order isomorphism. Now, $A_{x'} = \{y' \in \omega_1 : x' \in y'\}$ contains all $\alpha \in \omega_1$ except for those satisfying $\alpha \leq x'$ which is a countable set by CH, hence A_x contains all elements of $\mathbb{R}$ except the countably many corresponding to the ordinals $\alpha \leq x'$. $\square$

Exercise 17.2: Show that the converse of Theorem 17.1 is also true.

Solution: Let $B \subseteq \mathbb{R}$ be uncountable. We first show that $\bigcap_{x \in B} A_x = \emptyset$; suppose there is $z \in \bigcap_{x \in B} A_x$ so that $(x, z) \in A$ for each $x \in B$. Then since B is uncountable, A^z is uncountable, yielding a contradiction. It then follows that $\bigcup_{x \in B} A_x^c = \mathbb{R}$, and since A_x^c is countable, it follows that $|B| = \mathfrak{c}$ hence CH holds. $\square$

Exercise 17.3.a: Show that the set A defined in Theorem 17.1 is not Lebesgue measurable.

Solution: Per the given hint, it suffices to apply Fubini's theorem to χ_A and then observe that $\int_0^1 \chi_A dx = 1$ while $\int_0^1 \chi_A dy = 0$, yielding a contradiction. $\square$

Exercise 17.3.b: Conclude that if $([0,1], \leq)$ is a well-ordering of type ω_1 then $\leq$ is not a Borel subset of $[0,1] \times [0,1]$.

Solution: Such a set would not be Lebesgue measurable as shown above, but Borel sets are Lebesgue measurable. $\square$

Exercise 17.4: Show that CH implies the existence of a Sierpiński set.

Solution: If we define a Sierpiński set as an uncountable subset of $\mathbb{R}$ that has countable intersection with every Borel null subset of $\mathbb{R}$, then, noting that there are $2^{\aleph_0}$ Borel null subsets of $\mathbb{R}$, it suffices to prove the following lemma, since we may then use it to mimic the proof of Theorem 17.2 for our construction:

Lemma 17.4.1: If $N \subseteq \mathbb{R}$ is null, there exists a Borel null set containing N.

Proof: Let $N \subseteq \mathbb{R}$ be null and let $\epsilon > 0$. Since $\mathbb{R}$ is hereditarily Lindelöf, without loss of generality we can find a set $\{x_i\}_{i \in \omega} \subseteq N$ such that $\mathcal{U} := \{B_{x_i}(\frac{\epsilon}{2^{i+1}})\}$ is an open cover of N. Then, with m denoting Lebesgue measure, we have $m(\mathcal{U}) = \sum_{i=1}^{\infty} \frac{\epsilon}{2^{i+1}} = \frac{\epsilon}{2} < \epsilon$. Thus we can find open covers of N of arbitrarily small Lebesgue measure. Taking a countable, decreasing collection of such covers and then taking their intersection yields a null set containing N which is G_δ and hence Borel. $\square$

3.2 The Duality of Measure and Category

In this subsection, we focus on the main result of the chapter, which is the following theorem:

Theorem 17.3 (Erdős-Sierpiński) Assuming CH, there's a bijection $f : \mathbb{R} \to \mathbb{R}$ that carries meager sets to null sets and vice versa.

This theorem is also discussed extensively in Oxtoby's text on measure and category [2]. The authors of both [1] and [2] note that the result establishes the duality of measure and category, which is that for any statement φ involving solely statements about meager sets, null sets, and purely set theoretical statements, if φ^* is given by interchanging statements about null and meager sets within φ, then if we assume CH, φ holds iff φ^* holds.

The proof of Theorem 17.3 as given in [1] makes use of the following lemmas:

Lemma 17.4: There exists a decomposition of $\mathbb{R}$ into disjoint sets $F_0 \in \mathcal{M}$ and $G_0 \in \mathcal{N}$.

Lemma 17.5 (CH): There exist bases $\{F_\alpha : \alpha < 2^{\aleph_0}\}$ of $\mathcal{M}$ and $\{G_\alpha : \alpha < 2^{\aleph_0}\}$ of $\mathcal{N}$ such that $F_\alpha \subseteq F_\beta, G_\alpha \subseteq G_\beta, |F_{\alpha+1} \setminus F_\alpha| = |G_{\alpha+1} \setminus G_\alpha| = 2^{\aleph_0}$ for all $\alpha < \beta < 2^{\aleph_0}$ and $F_\alpha = \bigcup_{\beta<\alpha} F_\beta, G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ whenever $\alpha > 0$ is a limit ordinal.

We proceed with a string of exercises that are used in proving these lemmas as well as Theorem 17.3.

Exercise 17.5 Show that if f is as in Theorem 17.3, then $f(L)$ is a Sierpiński set whenever L is a Luzin set and $f(S)$ is a Luzin set whenever S is a Sierpiński set.

Solution: Let f be given as in Theorem 17.3 and let L be a Luzin set. Then L is uncountable and so $f(L)$ is also uncountable, and for any nowhere dense $K \subseteq \mathbb{R}, f(L \cap K) = f(L) \cap f(K) = f(L) \cap N$ where N is null, and $f(L) \cap N$ is countable since $L \cap K$ is countable, hence $f(L)$ is a Sierpiński set. We omit the proof that $f(S)$ is Luzin when S is Sierpiński as it is done analogously. $\square$

Exercise 17.6.a: Show that in the proof of Lemma 17.4, the sequence $(q_i)_{i\in\omega}$ can be chosen in such a way that $[0,1]\setminus U_3$ is homeomorphic with the cantor set $\mathcal{C}$.

Solution: We discuss the intuition for this problem instead of a full solution. Enumerating the midpoints from the regular construction of $\mathcal{C}$ and then removing decreasing sequences of intervals about the midpoints produces a Cantor set of positive measure; all that remains is to discard extraneous rationals, which we omit from this solution. $\square$

Exercise 17.6.b: Conclude that the property of being a null set is not a topological property.

Solution: From the previous exercise, we observe that $[0,1]\setminus U_3$ is homeomorphic to $\mathcal{C}$, yet $m([0,1]\setminus U_3)=\frac{1}{2}$ while $m(\mathcal{C})=0$. $\square$

Exercise 17.7.a Show that if F is a meager subset of $\mathbb{R}$ and G is a null subset of $\mathbb{R}$, there exists a meager set $F^+\supseteq F$ and a null set $G^+\supseteq G$ such that $|F^+\setminus F|=|G^+\setminus G|=2^{\aleph_0}$.

Solution: We first consider the following theorems and corollary, the proofs for which may be found in Kechris' text on descriptive set theory [3]:

Theorem 17.7.a.1: Let X be a non-empty perfect Polish space. Then there is an embedding of the Cantor space $\mathcal{C}$ into X.

Theorem 17.7.a.2 (Cantor-Bendixson): If X is a Polish space then X can be uniquely written as $X=P\cup C$ where P is a perfect subset of X and C is countable and open.

Corollary 17.7.a.3: Any uncountable Polish space contains a homeomorphic copy of $\mathcal{C}$ and thus has cardinality $2^{\aleph_0}$.

Theorem 17.7.a.4 The Perfect Set Theorem for Borel Sets (Alexandrov, Hausdorff): Let X be Polish and $A\subseteq X$ be Borel. Then either A is countable or it contains a homeomorphic copy of $\mathcal{C}$.

We first show that $\mathcal{C}$ can be embedded in F^c and in G^c; in particular, by the above theorems, we must first show that F^c and G^c are Polish.

Since the closure of a nowhere dense set is nowhere dense, we may assume without loss of generality that F is F_σ, hence F^c is G_δ and is therefore Polish (as G_δ subsets of Polish spaces are Polish). Now, note that category is preserved by homeomorphism, and that $\mathcal{C}$ is meager and has cardinality $2^{\aleph_0}$; then, because the union of two meager sets is meager, we may take F^+ to be the union of F with our copy of $\mathcal{C}$. We use a similar approach for construction of G^+, but we must account for the fact that homeomorphism does not preserve measure (as was seen in a previous exercise).

By Lemma 17.1.4, we may assume without loss of generality that G is Borel. Now, if G is countable, then it's F_σ since singletons are closed in $\mathbb{R}$ so that G^c is G_δ and therefore Polish. If G is uncountable, then since G^c is also Borel, it contains a homemomorphic copy of $\mathcal{C}$ by Theorem 17.7.a.4. Now as a subspace of $\mathbb{R}$, our copy of $\mathcal{C}$ is separable and therefore contains a countable dense set which we enumerate as $\{q_i\}_{i\in\omega}$. Now we mimic the proof of Lemma 17.4. For $i \in \omega$ define $U_i := \bigcup_{n\in\omega}(q_n - 2^{-(i+n)}, q_n + 2^{-(i+n)})$ and define $V := \bigcap_{i\in\omega} U_i$. Now, note that V is an uncountable Borel set and thus $|V| = 2^{\aleph_0}$ by Theorem 17.7.4.a. Note also that V is null by construction. Hence we define $G^+ := V \cup G$. $\square$

Exercise 17.7b Show that the ideal of meager sets and the ideal of null sets have bases of size $2^{\aleph_0}$.

Solution: We showed in Lemma 17.4.1 that any null set is contained in a Borel null set. Additionally, any meager set is contained in its closure, which is also meager. Hence it suffices to show that there are $2^{\aleph_0}$ Borel null sets and $2^{\aleph_0}$ closed meager sets.

Now, singletons are closed in $\mathbb{R}$ and are clearly meager hence there are at least $2^{\aleph_0}$ closed meager subsets of $\mathbb{R}$. Singletons are null sets and are G_δ (for $x \in \mathbb{R}$ we have $x = \bigcap_{n\in\omega}(x - \frac{1}{2^n}, x + \frac{1}{2^n})$ and hence Borel, so there are at least $2^{\aleph_0}$ Borel null subsets of $\mathbb{R}$. Now it suffices to show that there are at most $2^{\aleph_0}$ Borel of $\mathbb{R}$ (note that closed meager subsets are Borel because they are closed, hence this also gives a bound on the number of closed meager sets). We note that there are $2^{\aleph_0}$ open intervals in $\mathbb{R}$ (since the set of open intervals may be regarded as $\mathbb{R}^2$), and we obtain our upper bound by induction on the number of σ-algebra operations on open intervals. $\square$

Exercise 17.8: Where in the proof of Lemma 17.5 is CH used?

Solution: By CH, $\alpha < 2^{\aleph_0}$ is countable so that F_α remains meager for any any limit ordinal $\alpha < 2^{\aleph_0}$ (and analogously G_α remains null for any limit ordinal $\alpha < 2^{\aleph_0}$). $\square$

Exercise 17.9: Verify that f is a bijection that satisfies conditions (i) and (ii) of Theorem 17.3.

Solution: We first verify that f is a bijection. Let $y \in \mathbb{R}$, so that $y \in F_0$ or $y \in G_0$. In the former case, we immediately have that $y \in rng(\tilde{f})$ and in the latter case we immediately have that $y \in rng(\tilde{f}^{-1})$, hence $y \in rng(f)$. Now let $x_1 \neq x_2 \in \mathbb{R}$ which resolves without loss of generality into three cases. In the first case we have $x_1, x_2 \in G_0$, so that $x_1 \in f_\alpha, x_2 \in f_\beta$ for some $\alpha, \beta < 2^{\aleph_0}$. If $\alpha \neq \beta$ then clearly $f(x_1) \neq f(x_2)$. If $\alpha = \beta$ then $f(x_1) \neq f(x_2)$ by injectivity of f_α. In the second case we have $x_1, x_2 \in F_0$; this case is analogous to the previous case. In the third case we have $x_1 \in G_0, x_2 \in F_0$ in which case we immediately have that $f(x_1) \neq f(x_2)$ by noting that $(x_1, f(x_1)) \in \tilde{f}$ and $(x_2, f(x_2)) \in \tilde{f}^{-1}$.

Now we verify that $f(A) \in \mathcal{M} \iff A \in \mathcal{N}$ (we omit the proof that $f(A) \in \mathcal{N} \iff A \in \mathcal{M}$ as it is done analogously). Let $A \in \mathcal{N}$ so that $A \subseteq G_\alpha$ for some $\alpha < 2^{\aleph_0}$; then $f(A) = \tilde{f}^{-1}(A) = f_\beta^{-1}(A) \subseteq F_\alpha \in \mathcal{M}$ where $\alpha = \beta + 1$. The proof for the converse is symmetric to this proof. $\square$

3.3 Weakening our Assumption

In this final subsection, we discuss results in the previous subsections under the assumption that $add(\mathcal{M}) = add(\mathcal{N}) = 2^{\aleph_0}$, a weaker assumption than CH. We conclude with a brief discussion of sets of strong measure zero.

Exercise 17.10.a: Show that CH implies that $add(\mathcal{M}) = add(\mathcal{N}) = 2^{\aleph_0}$.

Solution: Assuming CH, the desired result follows from noting that countable unions of null sets are null and countable unions of meager sets are meager and that $\mathfrak{c}$ unions of null or meager sets need not be null or meager (for example, $\mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x\}$ is neither null nor meager). $\square$

Exercise 17.10.b: Show that Theorem 17.3 remains valid if we replace CH by the assumption that $add(\mathcal{M}) = add(\mathcal{N}) = 2^{\aleph_0}$.

Solution: Note that CH is required in the proof of Theorem 17.3 to ensure that F_α is meager and G_α is null for limit ordinals $\alpha < 2^{\aleph_0}$ (since countable unions of meager/null sets are meager/null), but $add(\mathcal{M}) = add(\mathcal{N}) = 2^{\aleph_0}$ gives us this result by definition. $\square$

Exercise 17.11 (modified): Show that if $add(\mathcal{M}) \neq add(\mathcal{N})$ then the bijection $f : \mathbb{R} \to \mathbb{R}$ constructed in the proof of Theorem 17.3 does not satisfy the desired conditions.

Solution: Suppose without loss of generality that $add(\mathcal{M}) = \kappa_1 < \kappa_2 = add(\mathcal{N}) \leq 2^{\aleph_0}$. Now let $\{A_\alpha\}_{\alpha<\kappa_1} \subseteq add(\mathcal{N})$ so that $\bigcup_{\alpha<\kappa_1} A_\alpha \in \mathcal{N}$ but $f(\bigcup_{\alpha<\kappa_1} A_\alpha) = \bigcup_{\alpha<\kappa_1} f(A_\alpha) \notin \mathcal{M}$. $\square$

Exercise 17.12: Show that if $add(\mathcal{M}) = add(\mathcal{N}) = 2^{\aleph_0}$ then there exist generalized Luzin sets and generalized Sierpiński sets.

Solution: It suffices to show that there is just a generalized Luzin set as we may then obtain a generalized Sierpiński set using the bijection from Theorem 17.3. Note that L as constructed in Theorem 17.2 is a generalized Luzin set; $add(\mathcal{M}) = 2^{\aleph_0}$ implies that $\bigcup_{\xi<\eta} K_\xi \cup \{x_\xi : \xi < \eta\} \in \mathcal{M}$ so that $\mathbb{R} \setminus \bigcup_{\xi<\eta} K_\xi \cup \{x_\xi : \xi < \eta\} \neq \emptyset$. It is easy to see that $|L \cap K| < 2^{\aleph_0}$ and $|L| = 2^{\aleph_0}$. $\square$

Exercise 17.13.a: Show that no uncountable closed subset of $\mathbb{R}$ has strong measure zero.

Solution: Such a set is Borel and therefore contains a homeomorphic copy of $\mathcal{C}$ by Theorem 17.7.a.4. Then note that the Cantor function $f : \mathcal{C} \to [0, 1]$ is uniformly continuous and surjective; the uniformly continuous image of a set of strong measure zero has strong measure zero as discussed in [4], but $[0, 1]$ is clearly not of strong measure zero. $\square$

Exercise 17.13.b: Deduce from (a) that no uncountable Borel set of reals has strong measure zero.

Solution: It suffices to note that Borel subsets of $\mathbb{R}$ are generated by closed intervals (although if we wanted to approach this problem without using part a) explicitly, note that the approach we used to prove part a) may be used identically here). $\square$

Exercise 17.14: Assuming CH, show that every Luzin set has strong measure zero.

Solution: Let $L \subseteq \mathbb{R}$ be Luzin (note that such an L exists by CH); let $\{q_n\}_{n\in\omega} \subseteq L$ be dense in L and let $\{\epsilon_n\}_{n\in\omega} \subseteq \mathbb{R}^+$. Then for $n \in \omega$ define $I_n := \{q_n - \frac{\epsilon_n}{2^n}, q_n + \frac{\epsilon_n}{2^n}\}$. If $\bigcup_{n\in\omega} I_n$ covers L then our proof is complete; if not, note then that $L \setminus \bigcup_{n\in\omega} I_n$ is closed and nowhere dense in L and is therefore countable and has strong measure zero. $\square$

An interesting observation is that if instead of assuming CH we assume $add(\mathcal{M}) = add(\mathcal{N}) = 2^{\aleph_0}$ then the exact same proof shows that generalized Luzin sets have strong measure zero.

4 Other Applications of CH

In this section we survey a small collection of interesting applications of CH gathered from various other sources. The sole purpose of this section is exposition; proofs of these results may be found in the sources from which they have been gathered.

Theorem 4.1 (M.E Rudin, 1974 [5]): Assuming CH, the box product of countably many σ-compact ordinals is paracompact.

Theorem 4.2 (Steen & Seebach, 1978 [6]): Assuming CH, an uncountable space with the cofinite topology is arc-connected.

The following theorem is not an application of CH, but is both interesting and relevant enough to be worthy of mention:

Theorem 4.3 (Steen & Seebach, 1978 [6]): The existence of an uncountable hereditarily G_δ set depends on the denial of CH (i.e $\neg$ CH).

This next theorem by Ulam is also not an application of CH, but together with CH it implies the corollary that follows it:

Theorem 4.4 (Oxtoby, 1980 [2]): A finite measure μ defined for all subsets of a set X of power $\aleph_1$ vanishes identically if it is zero for every point.

Corollary 4.5 (Oxtoby, 1980 [2]): Assuming CH, a finite measure μ defined for all subsets of a set X of power $\mathfrak{c}$ vanishes identically if it is zero for every point.

Theorem 4.6 (Pelc & Prikry, 1983 [7]) Assuming CH, there exist countably generated σ-algebras $\varepsilon_1, \varepsilon_2$ on $[0, 1)$ with respective probability measures μ_1, μ_2 such that $\varepsilon_1, \varepsilon_2$ contain all Borel sets and are translation invariant, μ_1, μ_2 extend the Lebesgue measure and are translation invariant, and there is no non-atomic probability measure on any σ-algebra containing $\varepsilon_1 \cup \varepsilon_2$.

5 References

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